PH 403 ESE CL113

Quantum Mechanics - I : Prof. Shankarnarayan Subramanium.

Tutorial Participation : 10, 2 Quizzes - 20 Mks., Midsem - 30. Endsem - 40.

[Quiz 1 — 1st September
Quiz 2 - 27th October.]

Q-Mech : (i) Mathematics is Simple
(ii) Linearity, Principle of Superposition, Unitarity.

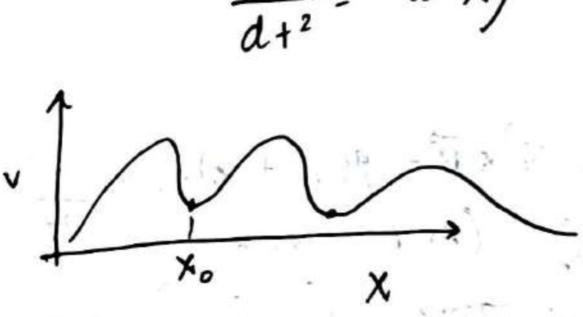
DE vector Spaces.

* (Weather in London - Linear Theory.)

Linear DE's : $\frac{dx}{dt} = \alpha x$

: $\frac{d^2x}{dt^2} = -\omega^2 x$

Linear in Dependant Function.



0 about a minima.

$$V(x) = V(x_0) + \cancel{\frac{dV}{dx}\Big|_{x=x_0}(x-x_0)} + \frac{1}{2}\frac{d^2V}{dx^2}\Big|_{x=x_0}(x-x_0)^2$$

$x(t) \rightarrow \{1, 2, 3, \dots\} \rightarrow$ Satisfies ① Addition ② Multiplication — Diluctation.

✓ Then, $f(x_1+y_1, x_2+y_2, \dots, x_n+y_n) = f(x_1, x_2, \dots, x_n) + f(y_1, y_2, \dots, y_n)$

✓ And $f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda f(x_1, x_2, \dots x_n)$.

$\boxed{\text{Linear functions}}$

• $f_1(x) = \alpha x$ / $f_2(x) = \beta x^2$

$\rightarrow f_2(x+y) = \beta(x+y)^2$ ✗

$\neq f_2(x) + f_2(y) = \beta x^2 + \beta y^2$.

$f_2(\lambda x) = \beta(\lambda x)^2 = \lambda^2(\beta x^2) \neq \lambda f_2(x)$

✓

$\rightarrow f_1(x+y) = \cancel{\alpha x + \alpha y}\ \alpha(x+y)$

$= \cancel{\alpha(x+y)}\ \alpha x + \alpha y$.

$= f_1(x) + f_1(y)$.

$\rightarrow f_1(\lambda x) = \alpha(\lambda x) = \lambda(\alpha x) = \lambda f_1(x)$.

Q. Mechanics & Linearity — DE's which are Linear.

↳ superposition — if $f_1(x)$, $f_2(x)$ are sol^{n}s $\Rightarrow a f_1(x) + b f_2(x)$ is also a solution.

Wave Equation : PDE - linear. $\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2}$

$\rightarrow y(x,t) = T(t) + X(x)$

$\rightarrow \frac{d^2 X}{dx^2} = \alpha^2 X(x)$ —— $X(x) = X_0 e^{i\alpha x} + X_1 e^{-i\alpha x}$

$\rightarrow \frac{d^2 T}{dt^2} = \alpha^2 v^2 T(t)$ —— $T(t) = T_0 e^{-i\alpha v t} + T_1 e^{i\alpha v t}$

extraterm.

• $\cancel{\frac{\partial^2 (T(t) + X(x))}{\partial x^2} = \frac{1}{v^2}\left(\frac{\partial (T(t)+X(x))}{\partial t}\right)^2 = \frac{1}{v^2}\left(\frac{\partial T(t)}{\partial t}\right)^2 + \frac{1}{v^2}\left(\frac{\partial X(x)}{\partial t}\right)^2 + \frac{2}{v^2}\frac{\partial T(t)}{\partial t}\cdot\frac{\partial X(x)}{\partial t}}$

• Fourier Transforms, Laplace Transforms only work for LDE's.

Electrodynamics → polarization of light.

$\vec{\nabla}\cdot\vec{E} = \frac{\rho(t,\vec{r})}{\varepsilon_0}$, $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} = \mu_0 j(t,x) + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$.

(Gauss Law) (Faraday's Law) (Gauss Law For Magnetism) (Ampere's Law)

• $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$ // Gauge — $\vec{A} = \vec{A}' + \nabla\psi$ // $\vec{B} = \nabla\times\vec{A}' = \nabla\times\vec{A}$

• $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{A}) = -\vec{\nabla}\times\frac{\partial\vec{A}}{\partial t} \Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$

↳ $E + \frac{\partial\vec{A}}{\partial t} = -\nabla\phi \Rightarrow \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$

$\rightarrow \vec{A} = \vec{A}' + \nabla\psi \quad \rightarrow \vec{E} = -\nabla\phi - \frac{\partial\vec{A}'}{\partial t} - \nabla\frac{\partial\psi}{\partial t}$

$= -\nabla\left(\phi + \frac{\partial\psi}{\partial t}\right) - \frac{\partial\vec{A}'}{dt} \Rightarrow \boxed{\phi' = \phi + \frac{\partial\psi}{\partial t}}$

Gauss's Law → $-\nabla^2\phi - \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial t} = \frac{\rho}{\varepsilon_0}$

$$\rightarrow \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{\vec{J}(\vec{t}, x)}{\varepsilon c^2} \qquad \boxed{c^2 = \frac{1}{\mu_0 \varepsilon_0}}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0.$$

Thus ~~Given~~ Solutions ~~Arise~~

Required Mathematics

(i) Divergence

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

(ii) Curl

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

(iii) Gradient

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(t, \vec{V})}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Revision :

1) Divergence of curl is 0: $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$ Magnetic vector potential

Gauge Fixing. : A mathematical procedure for coping with redundant degrees of freedom in field variables.

2) curl of Grad is also zero:

$\rightarrow [\vec{A}' = \vec{A} + \nabla \psi] \rightarrow$ B will remain unchanged.

$$\text{Since } [\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}']$$

3) $\rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$

$$\hookrightarrow \vec{\nabla} \times \left(\vec{E} + \frac{d\vec{A}}{dt}\right) = 0 \rightarrow \boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi}$$

$\rightarrow$ Putting $\vec{A} = A' + \nabla \psi] \rightarrow \vec{E} + \frac{\partial \vec{A}'}{\partial t} + \nabla \frac{\partial \psi}{\partial t} = -\nabla \phi] \quad \boxed{\phi' = \phi + \frac{\partial \psi}{\partial t}}$

$$\vec{E} = -\nabla \phi - \frac{\partial A}{\partial t}] \xrightarrow[\text{Law}]{\text{Gauss's}} -\nabla^2 \phi - \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t} = \frac{\rho}{\varepsilon_0}$$

$$\text{So, } \boxed{\phi = \phi_0 e^{i(kx - \omega t)}}, \boxed{A = \vec{A}_0 e^{i(kx - \omega t)}}$$

$$\rightarrow \vec{\nabla} \cdot \left(\vec{E}_0 e^{i(\vec{k}\vec{x} - \omega t)}\right) = 0$$

$$\Rightarrow \vec{E}_0 \cdot i \cdot \vec{k}\, e^{i(\vec{k}\vec{x} - \omega t)} = 0 \quad \Big] \quad \vec{E}_0 \cdot \vec{k} = 0, \quad \vec{B}_0 \cdot \vec{k} = 0$$

$$\rightarrow \boxed{\vec{k} \perp \vec{E}_0 \perp \vec{B}_0} \rightarrow \text{EM Wave free space Maxwell's equations}$$

$$[\vec{K}_z \perp \vec{E}_{xy} \perp \vec{B}_{xy}] \qquad B_y \Leftrightarrow E_x, \quad B_x \Leftrightarrow E_y.$$

Concentrate on $\vec{E}$, $\underline{\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}}$ ✓

$$\vec{K}\cdot\vec{E}_0 = 0$$
$$\vec{K}\cdot\vec{B}_0 = 0$$
$$\vec{K}\times\vec{B}_0 = -\frac{\omega}{c}\vec{E}_0$$
$$\vec{K}\times\vec{E}_0 = \frac{\omega}{c}\vec{B}$$

And $\vec{E}_0 = E_0^{(x)}\hat{i} + E_0^{(y)}\hat{j}$

$\Rightarrow E_0^{(x)} = |E_0^{(x)}| e^{i\phi_x}$, $E_0^{(y)} = |E_0^{y}| e^{i\phi_y}$

$$\rightarrow \vec{E} = |E_0^{(x)}|\left(\hat{i} + \frac{\hat{j}\,|E_0^{(y)}|}{|E_0^{(x)}|} e^{i(\phi_y-\phi_x)}\right) e^{i(kz-\omega t+\phi_x)}$$

ϕ_x, ϕ_y → Phase factor of $E_x^{(x)}, E_y^{(y)}$

$$\rightarrow \boxed{\phi = (\phi_y - \phi_x)} : \vec{E} = |E_0^{(x)}|\left(\hat{i} + \frac{\hat{j}\,|E_0^{(y)}|}{|E_0^{(x)}|} e^{i\phi}\right) e^{i(kz-\omega t+\phi_x)}$$

$\vec{E}_0$ → polarization vector → const. vector

<u>Linear Polarization</u> : $\boxed{\phi = 0}$ ✓

$$\vec{E} = \left(|E_x^{(0)}|\hat{i} + |E_y^{(0)}|\hat{j}\right) e^{i(kz-\omega t)}$$

<u>x-polarization</u> ÷ $|E_y^{(0)}| = 0$, $|E_x^{(0)}| \neq 0$

<u>y-polarization</u> ÷ $|E_y^{(0)}| \neq 0$, $|E_x^{(0)}| = 0$

★★ ★

• $\phi = -\frac{\pi}{2}$, $|E_x^{(0)}| = |E_y^{(0)}| = c_0$] $\vec{E} = (c_0\hat{i} + i c_0\hat{j}) e^{i(kz-\omega t)}$

<u>Polarization</u> ÷ Jone's vector

$$\vec{E} = (E_x\hat{i} + E_y\hat{j}) e^{i(kz-\omega t)}$$

Intensity $= |E|^2 = |E_x|^2 + |E_y|^2$

normalized E vector ÷ $(a\hat{i} + b e^{-i\phi}\hat{j}) e^{i(kz-\omega t)}$

$a = \frac{|E_x^{(0)}|}{\sqrt{I}}$, $b = \frac{|E_y^{(0)}|}{\sqrt{I}}$

$$\boxed{\phi = \phi_x - \phi_y}$$

$$\vec{E}_J = a\hat{i} + b e^{-i\phi}\hat{j}$$

$$\vec{E}_J = \begin{pmatrix} a \\ b e^{-i\phi} \end{pmatrix} : \{\underline{a^2 + b^2 = 1}\}$$

x polarized light : $E_J^{(x)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

y polarized light : $E_J^{(y)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Left circular polarized light : $E_J^{(L)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ $\boxed{\phi = \frac{-\pi}{2}}$ → counter clockwise

Right circular polarized light : $E_J^{(R)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ $\boxed{\phi = \frac{\pi}{2}}$ → Clockwise

Properties of Jones vectors :

1) Jones vectors are orthonormal basis vectors

$\therefore$ $(1\ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ $\quad \frac{1}{2}(1\ \ -i)\begin{pmatrix} 1 \\ -i \end{pmatrix} = 0$

→ $\frac{LCP + RCP}{\sqrt{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: x polarized light.

→ $\frac{LCP - RCP}{\sqrt{2}\,i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: y polarized light.

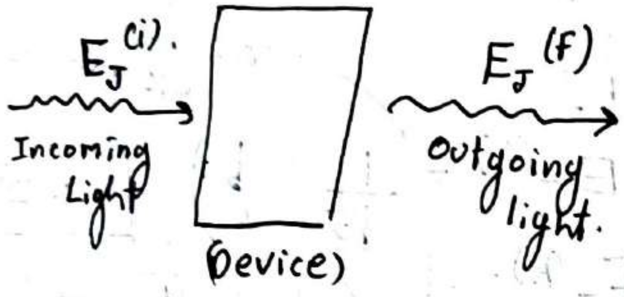
2) We can obtain obtain other vectors from these orthonormal vectors.

- ~~P~~ Polarized light is the combination ~~ation~~ of oscillation of light on the x and y axis. → combination of two perpendicular SHM.
- Phase acts as complex coefficient. Positive imaginary Part : (Counterclockwise Rotation)
- "Linear polarization", "circular polarization", "Elliptical Polarization"
- Eqn of Polarized Light : $\boxed{\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2}\cos\phi = \sin^2\phi}$

Jones Vector : $E_J = \begin{pmatrix} a \\ b e^{-i\phi} \end{pmatrix}$ $I = \vec{E}^* \vec{E} = 1.$

$\boxed{a^2 + b^2 = 1}$

x → Polarized light] → orthonormal vector.
y
RCP/LCP] add two vector, another vector.



$$E_J^{(f)} = J_{2\times 2} \; E_J^{(i)}$$

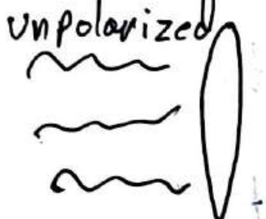
$$\begin{pmatrix} a_f \\ b_f e^{-i\phi_f} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}}_{} \begin{pmatrix} a_i \\ b_i e^{-i\phi_i} \end{pmatrix}$$

constants are defined by the device we are using.

$\boxed{\alpha, \beta, \gamma, \eta \in \mathbb{C}}$

J → encapsulates the properties of the optical device.
↓
[Jone's Matrix].

$\rightarrow a_f = \alpha a_i + \beta b_i e^{-i\phi_i}, \; b_f e^{-i\phi_f} = \gamma a_i + \eta b_i e^{-i\phi_i}$



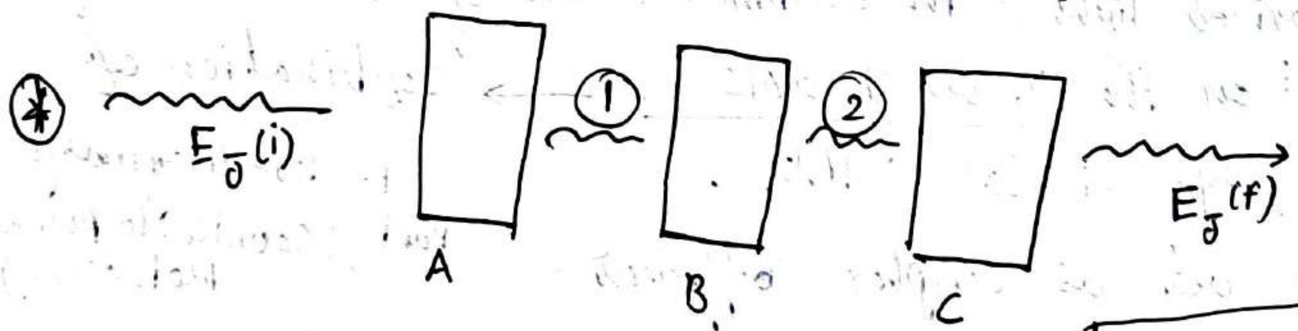
x-polarized light

$\begin{pmatrix} a \\ b e^{-i\phi} \end{pmatrix} J_{x-p}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$J_{x-p} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$J_{y-p} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\rightarrow J_{QWP} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \; J_{RWP} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$\boxed{E_J^{(f)} = (J_C \cdot J_B \cdot J_A) \, E_J^{(i)}}$ ✓

$J_{45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(I) $E_\delta^{(i)}$ unpolarized light → x-polarizer → x-polarized light → y-polarizer → $E_\delta^{(f)}$

$$J_{net} = J_2 J_1 = \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

→ no output from y-polarizer.

(II) $E_\delta^{(i)}$ → x-polarizer → x polarized light → 45° Polarizer → $E_\delta^{(f)}$

$$J_{45}\cdot J_x = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\1&0\end{pmatrix}$$

→ output from 45° polarizer is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ → Reduced intensity.

(III) $E_\delta^{(i)}$ → x-p → 45°-p → y-p → $E_\delta^{(f)}$

$$J_{net} = \begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1&1\\1&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$= \begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\1&0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

output will be $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$

* $J_y\cdot J_{45} \neq J_{45} J_y$.

(IV) $E_\delta^{(i)}$ → x-p → y-p → 45°-p → $E_\delta^{(f)}$

output is 0

✪ Don't commute

Newtonian Mechanics : The way we look at particles in the macroscopic world shouldn't be same as that as the microscopic world.

↓

Since we use Real Numbers → commutative

In Real no.'s commuting is allowed irrespective of what we do.

→ we cannot use Real numbers to describe things in the ~~macro~~ microscopic world. We need Vectors and Matrices. ← Q Mechanics

Q.) Since we described light as such ? why don't we just say it's a good representation for light? Why can we claim it applies to all particles?

If something is true for some particle then it must be true for all Particles since we cannot call any particle 'special' in Physics.

→ Such a Mathematical structure should be developed for any physical sit. Possible.

└ Vectors / Matrices. ✓.

{most of these are now Exp. verified and no longer postulates.}

Postulates of Quantum Mechanics.

• 1. Physical System that is being observed is represented by a State vector.

State vector notation ⟶ $|\ \rangle$ Ket.

Ex: $|E\rangle = \begin{pmatrix} E_x \\ E_y \end{pmatrix} \rightarrow$ 2-D vector

$|X\rangle$ = (infinite dimensional vector) $-\infty \rightarrow +\infty$

So, $\boxed{|E_f\rangle = J_{2\times2}\,|E_i\rangle.}$

2. Any Operation on the system is represented by a Matrix. The operation matrix is reffered to as the operator.

So, Jones Matrix ⟶ $\hat{J}$ ⟶ operator.

Usually, in most cases, no two operators will commute with each other.

3. Every measurement process introduces a sizeable disturbance to the state of the system. $|i\rangle \neq |f\rangle$

Usually, $\frac{\Delta E}{E} << 1$ but for microscopic systems, $\Delta E \sim E$

09/08/23 Hilbert Space and Opertators :

Polarization vector : $|\vec{E}\rangle = \begin{pmatrix} \alpha \\ \beta e^{i\theta} \end{pmatrix}$ Jones s.t. $\underline{\alpha^2+\beta^2=1}$

"In 3D/ cartesion + time" $\to \left[\vec{E} = E_{\delta} e^{i(kx-\omega t)}\right]$

Physically reside on any space. → Represent this vector ~~space which~~ in a "fictious vector space" which we have more ~~[illegible]~~ convinience (2D-vector space).

can be represented as $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ ← Nothing to do with real space ←

- Even if it is a real vector, it's properties can be defined in an artificial vector space.

- $\vec{E} = \alpha|x\rangle + \beta|y\rangle$
  $= \alpha'|R\rangle + \beta'|L\rangle$
  $= \tilde{\alpha}|R_E\rangle + \tilde{\beta}|L_E\rangle$

R: Right circularlly polarized light
L: Left circularly polarized light.

most mathematical simplicity.

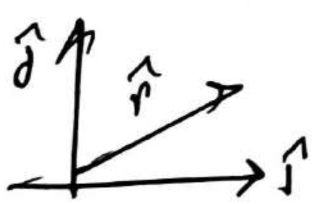
→ we must rewrite E in terms of any vector space we know to get it's compontents.

(Good if all coeffs are real)

Advantage of Moving out of Cartesian : we can represent vectors that are primarily complex in nature.

- So to overcome this : State resides in an ~~[illegible]~~ abstract vector space called Hilbert Space ($\mathcal{H}$) → ($\mathcal{H}$).
  → Hilbert space has complex basis vectors and the coefficients can also be ~~[illegible]~~ complex.

In 3D space.



$\vec{r} = a\hat{i} + b\hat{j}$

$\underline{a, b \in R}$

$|E_{\delta}\rangle = \alpha|x\rangle + \beta|y\rangle$

Lie in the Hibert space.

Hence any conbition (supperposition) of 2 also lies in an Hilbert space.

• The dimension of the Hilbert space varies depending on the physical system.

• We will deal with 2D Hilbert spaces.

Properties of Hilbert Space :-

i) arbitrary vectors: $|a\rangle$ and $|b\rangle$ $\in$ Hilbert then $|c\rangle = |a\rangle + |b\rangle$. $\in$ Hilbert.

ii) if $|a\rangle + |b\rangle \in$ Hilbert then $|b\rangle + |a\rangle$ lie in hilbert.

(iii) $(|a\rangle + |b\rangle) + |d\rangle = |a\rangle + (|b\rangle + |d\rangle)$.

(iv) If $|a\rangle \in$ Hilbert. Then $|a\rangle + |N\rangle = |a\rangle$.

(v) For Every $|a\rangle \in$ Hilbert, we have an inverse $|-a\rangle \in$ Hilbert s.t. $|a\rangle + |-a\rangle = |N\rangle$.

(vi) $\mu, \nu \in \mathbb{C}$ and $|a\rangle, |b\rangle \in$ Hilbert.

→ $\mu(|a\rangle + |b\rangle) = \mu|a\rangle + \mu|b\rangle$.

→ $(\mu + \nu)|a\rangle = \mu|a\rangle + \nu|a\rangle$.

→ $\mu\nu|a\rangle = \nu\mu|a\rangle$

→ $1|a\rangle = |a\rangle$.

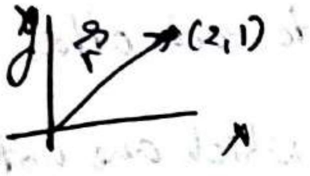
(vii) conjugate of a vector: $(\mu|a\rangle + \nu|b\rangle)^\dagger = \langle a|\bar{\mu} + \langle b|\bar{\nu}$

$= \bar{\mu}\langle a| + \bar{\nu}\langle b|$

* $\langle a|$ → adjoint vector / bra vector.

$\langle a \mid b \rangle$

bra ↙ ↘ Ket.



distance $= \sqrt{x^2+y^2} = |\vec{r}| = \sqrt{4+1} = \sqrt{5}$ ✓.

↓

Distance is defined by the dot product : Distance $= |\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}}$.

row vector ← → column vector

For this to be a Matrix operation.

=> $|\vec{r}|^2 = \vec{r}^{\,T}\cdot\vec{r}$ where $\vec{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so, $|\vec{r}|^2 = (2\ \ 1)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

↓

(Dual space)
(no physical meaning)
(Artificial space).

→ The Dual space helped us figuere out the distance.

* we want a system to get a ~~mea~~ number out of a vector cause they are the only things we observe in measurement.

* thus we define an inner product.

$$\langle R|R\rangle = \frac{1}{2}(1\ \ -i)\begin{pmatrix} 1 \\ i \end{pmatrix} = 1.$$

so, $(|a\rangle)^T = \langle a|$

Hermetian conjugate

conjugate transpose of $|a\rangle$

The Reason for defining a dual space is so that we understand the notion of length of vectors.

- $|\phi\rangle$ and $|\psi\rangle$ in Hilbert space.

=> $\left[\langle\phi|\psi\rangle = (\langle\psi|\phi\rangle)^*\right]$ → Property of inner Product.

Ex: $\langle Y|R\rangle = \frac{1}{\sqrt{2}}(0\ \ 1)\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{i}{\sqrt{2}}$ and $\langle R|Y\rangle = \frac{1}{\sqrt{2}}(1\ \ -i)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}$.

-> $\boxed{\langle\phi|\phi\rangle \in \mathbb{R} \geq 0}$ → Distance in Abstract space.

→ $\langle\psi|\mu|\phi\rangle = \mu\langle\psi|\phi\rangle$ in the case that μ is a scalar

$$|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}$$

Q.) why can't we write,

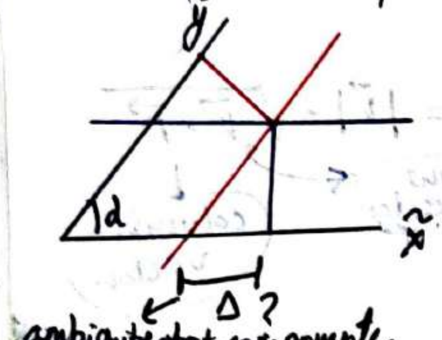
$|\vec{E}\rangle = \alpha_N|x\rangle + \beta_N|R\rangle$?

OR

$|\vec{E}\rangle = \tilde{\alpha}_N|y\rangle + \tilde{\beta}_N|L\rangle$?

P.T.O

Let us see 2D space again, $|x\rangle, |y\rangle$ are orthogonal to each other



If we construct $|\tilde{x}\rangle, |\tilde{y}\rangle$ which are not orthogonal to each other, we will run into Problems.

$\rightarrow \langle x|x\rangle = \langle y|y\rangle = \langle L|L\rangle = \langle R|R\rangle = 1.$ $\neq 0$ is a problem.

orthogonal Also $\langle L|R\rangle = \langle x|y\rangle = 0$ // Here $\langle \tilde{x}|\tilde{y}\rangle = \cos\alpha$!!!

- I can always write an polarization vector $(\vec{E})$ in terms of,

$$\rightarrow \boxed{|\vec{E}\rangle = \sum_{i=1}^{2} d_i\,|i\rangle} \rightarrow \text{basis vectors}$$

Think of this as a fourier transform with orthonormal functions.

$\vec{f(x)} = \sum_{i=1}^{n} a_i \sin(K_i x)$ where $\int_0^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2n\pi x}{L}\right) dx = 0.$

"(Good)" **Basis Vectors** : (i) Any pair of two vectors must be orthogonal to each other

(ii) They must be normal (normalized to unity).

Q.) What if we don't know the dimension of our Hilbert space?

$|\vec{E}\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle + a_3|\psi_3\rangle \rightarrow$ start with trial basis states

$\hookrightarrow (a_i \neq 0).$

If we get $a_i = 0 \Rightarrow$ error in Judgement of Hilbert dimension

State vector $\rightarrow$ $\left(|\psi\rangle = \sum C_x |i\rangle \right)$

$$|\psi\rangle \qquad \langle\phi| \rightarrow (|\phi\rangle)$$

$$\downarrow \qquad \downarrow$$

$$\mathcal{H}^{(2)} \qquad \mathcal{H}^{(2)}$$

$$\langle\phi|\psi\rangle$$

$\rightarrow C_x = \langle i|\psi\rangle$

$|\psi\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ $\langle\phi| = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$.

$$\boxed{\rightarrow \langle\phi|\psi\rangle = (\langle\psi|\phi\rangle)^*}$$

Outer product : in $\mathbb{R}^2$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\rightarrow$ Inner product $= \vec{x}\cdot\vec{y} = x^T y = (x_1\ x_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$

$\rightarrow$ outer product $= \vec{x}\, y^T = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(y_1\ y_2) = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$

* All operators we are interested in can be constructed in the form of outer product.

We have a physical system $\rightarrow$ $\mathcal{H}^{(2)}$, $|i\rangle$ (Basis vectors)

$|\psi\rangle = \sum_{n=1} C_n |i\rangle$

$\hat{O} \rightarrow \sum_{n=1}^{N} C_n |i\rangle\langle i|$

$$\boxed{\begin{aligned} I_{N\times N} &= \sum_i |i\rangle\langle i| \\ &= |x\rangle\langle x| + |y\rangle\langle y| \\ &= |R\rangle\langle R| + |L\rangle\langle L| \end{aligned}}$$

Operators : Any operator on a system can be represented by a Matrix.

① what physical quantities are associated with the operator?

② what are the properties of the operator.?

$\rightarrow$ All observables have an associated operator but all operators need not be associated with an observable.

P.T.O

Linear Operators

$$\hat{A}(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha\hat{A}|\psi\rangle + \beta\hat{A}|\phi\rangle = \alpha|\psi'\rangle + \beta|\phi'\rangle$$

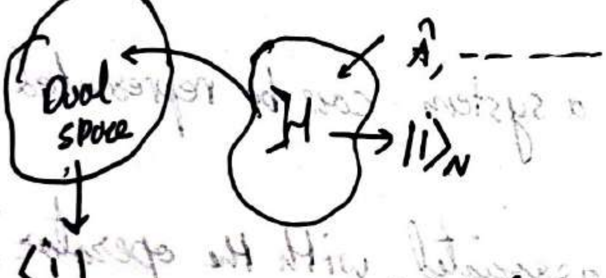
Polarisation : $|E\rangle = \alpha|X\rangle + \beta|Y\rangle$.

$\rightarrow$ $J_x|E\rangle = J_x(\alpha|X\rangle + \beta|Y\rangle) = \alpha J_x|X\rangle + \beta J_x|Y\rangle$
$= \alpha J_x|X\rangle + 0$.

- From now on, operator implies linear operator.

Properties of operators

1) $\hat{A}, \hat{B} \longrightarrow (\mathcal{H}) \rightarrow$ If $\hat{A} = \hat{B}$, $\forall$ vectors $|\psi\rangle$ in $\mathcal{H}$, $\boxed{\hat{A}|\psi\rangle = \hat{B}|\psi\rangle}$
2) unit operator $\rightarrow$ $\hat{\mathbb{1}}|\psi\rangle = |\psi\rangle$.
3) Zero operator $\rightarrow$ $\hat{O}|\psi\rangle = 0$.
4) sum of operators $\rightarrow$ $(\hat{A}+\hat{B})|\psi\rangle = \hat{A}|\psi\rangle + \hat{B}|\psi\rangle$.
   $\hookrightarrow$ $[(\hat{A}+\hat{B})+\hat{C}]|\psi\rangle = [\hat{A}+(\hat{B}+\hat{C})]|\psi\rangle$.
5) ussually, $\hat{A}\hat{B}|\psi\rangle \neq \hat{B}\hat{A}|\psi\rangle$
   $\rightarrow$ $[\hat{A}\ \hat{B}]|\psi\rangle = (\hat{A}\hat{B} - \hat{B}\hat{A})|\psi\rangle$. (Two operators for the same system.)
   (Commutator)



How do we define operators operating on the dual space.

- $\hat{J}_M|E_i\rangle = |E_f\rangle$ — Define a dual vector $\langle E_i|$ $\longrightarrow$ $\langle E_f| = \langle E_i|\hat{J}_?$

We are interested in how $\hat{J}_?$ is related to $\hat{J}_M$.

$\rightarrow |E_i\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\rightarrow \hat{J}_M = \hat{J}_R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$|E_f\rangle = \hat{J}|E_i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\rightarrow \langle E_i| = (0\ \ 1)$, $\langle E_f| = \frac{1}{\sqrt{2}}(-i, 1)$

$\rightarrow \frac{1}{\sqrt{2}}(-i\ \ 1) = (0\ \ 1)\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$

$\rightarrow \frac{1}{\sqrt{2}}(-i \;\; 1) = (\alpha_3 \;\; \alpha_4) \qquad \hat{J}_? = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$\rightarrow$ if i choose $|E_i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow |E_f\rangle = \hat{J}_R |E_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\rightarrow \frac{1}{\sqrt{2}}(1 \;\; i) = (1 \;\; 0) \begin{pmatrix} \alpha_1 & \alpha_2 \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = (\alpha_1 \;\; \alpha_2)$

$\rightarrow \boxed{\hat{J}_? = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}} \Longrightarrow \boxed{\hat{J}_? = \hat{J}_R^\dagger}$ → conjugate Transpose.

→ Adjoint operator is also a linear operator in the dual space.

Properties of Adjoint operators :

$(\hat{A}|\psi\rangle^\dagger) \equiv \langle\psi|\hat{A}^\dagger \rightarrow \langle E_i|E_f\rangle = \langle E_i|\hat{J}_M|E_i\rangle \rightarrow z$

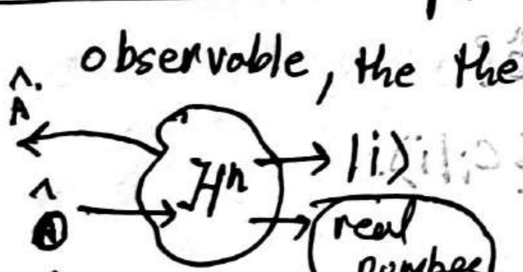
$\langle E_f|E_i\rangle = \langle E_i|\hat{J}_M^\dagger|E_i\rangle \rightarrow z^*$

- Hermetion Operator : i) It's an operator equal to it's Adjoint :
$(\hat{H} = \hat{H}^\dagger)$

$\rightarrow \langle E_i|E_f\rangle = \langle E_i|\hat{H}|E_i\rangle$
$\langle E_f|E_i\rangle = \langle E_i|\hat{H}^\dagger|E_i\rangle$] equal so, $\langle E_i|E_f\rangle = \langle E_f|E_i\rangle$; $z = z^*$

This has to be a Real number.

4th Postulate : All operators for which there is an expected observable, the the operator should be Hermetian.



- It's easier to solve Maxwell's equations with $(\vec{A}, \phi)$ rather than $(\vec{E}, \vec{B})$ inspite of not being Physical observables.

Anti Hermetian operators : $\boxed{\hat{A} = -\hat{A}^\dagger}$

Arbitirary operator : $\hat{O} = \overbrace{\frac{1}{2}(\hat{O} + \hat{O}^\dagger)}^{\text{Hermetian}} + \overbrace{\frac{1}{2}(\hat{O} - \hat{O}^\dagger)}^{\text{Anti-Hermetian.}}$

Unitary operator : $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{\mathbb{I}}$

$$\Rightarrow \boxed{\hat{U}^\dagger = \hat{U}^{-1}}$$

$\rightarrow \hat{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \hat{U}^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$

$\hat{U}|E_i\rangle = |E_i'\rangle$
$\hat{U}|E_f\rangle = |E_f'\rangle$

$\rightarrow \langle E_f | E_i \rangle = \langle E_f | \hat{\mathbb{I}} | E_i \rangle = \langle E_f | \hat{U}^\dagger \hat{U} | E_i \rangle$
$= \langle E_f' | E_i' \rangle$

→ often many distinct operators may not result in different results.

→ many of these operations are equal and will result in same expectation values.

6/08/23

Operators – Hilbert space – Ket vectors.

Postulates : → state vectors → $\mathcal{H}^{(n)}$
→ operators → $\tilde{\mathcal{H}}^n \in \mathcal{H}^{(N)}$
→ observables → Hermetian operators

Rev^n.

① How to determine state vector? (basis)

② Are the basis vectors unique?

③ How to construct orthogonal basis vectors?

④ Can the basis vectors be used to obtain $\hat{O}$?

Framework : $\mathcal{H}^{(n)}$ $\hat{O} = ?$ $|\psi\rangle = \sum c_i |i\rangle$

Eigenvector / Eigenvalue → $|v\rangle \in \mathbb{R}^n$.

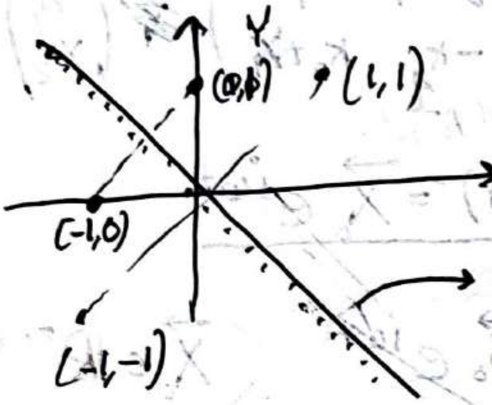
An operator $\hat{A}$ operates on an eigenvector (ket) $|v\rangle$.

$$\boxed{\hat{A}|v\rangle = \lambda|v\rangle}$$

p-Tro

Cartesian space − 2D

$\boxed{P_{2\times 2}\vec{X} = \vec{X}'}$ → reflecting matrix.



$$\begin{pmatrix} -y \\ -x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$-y = ax + by \rightarrow b = -1,\ a = 0$

$-x = cx + dy \rightarrow c = -1,\ d = 0.$

so, $\boxed{P_{2\times 2} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$ ✓.

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \checkmark \neq \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

λ ↓

Thus we are interested in $\hat{P}\,\overline{X} = \lambda\overline{X}$, <u>what are $\overline{X}$ and λ</u>?

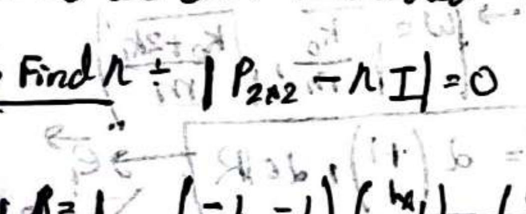
<u>To Find λ</u>: $|P_{2\times 2} - \lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1$

For $\lambda = 1$, $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \alpha \in \mathbb{R} \rightarrow$ The line $x = -y$

For $\lambda = -1$ $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \alpha \in \mathbb{R} \rightarrow$ The line $x = y$.

Q.) <u>Why are Eigenvalues and Eigenvectors relevant</u>?

* Not all vectors are eigen vectors of a matrix. eigen vectors are special.



K_1 : coupling constant.

K_0 : spring constant

$k_1 = 0 \rightarrow$ two independent oscillations

$k_1 > 0 \rightarrow$ coupled oscillations

$$\left.\begin{aligned} m\ddot{x}_1 &= -k_0 x_1 + k_1(x_2 - x_1) \\ m\ddot{x}_2 &= -k_0 x_2 + k_1(x_1 - x_2) \end{aligned}\right] \text{ Coupled Differential Equation.}$$

$$\vec{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \rightarrow m\frac{d^2\vec{X}}{dt} = -\begin{pmatrix} k_0+k_1 & -k_1 \\ -k_1 & k_0+k_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

don't forget.

$$\rightarrow \boxed{m\frac{d^2\vec{X}}{dt} = K_{2\times2}\cdot\vec{X}} \rightarrow \vec{X}(t) = \vec{X}_0 e^{i\omega t}$$

$\rightarrow$ substituting, $-m\vec{X}_0\omega^2 \cancel{e^{i\omega t}} = K_{2\times2}\cdot\vec{X}_0 \cancel{e^{i\omega t}}$ $\quad \vec{X}_0 \in \mathbb{R}^{2\times1}$

$\rightarrow \boxed{(-m\omega^2)\vec{X}_0 = K_{2\times2}\cdot\vec{X}_0} \longrightarrow$ Eigen vector/eigen Value problem.

$$\equiv |K_{2\times2} + m\omega^2 \mathbb{I}| = 0 \rightarrow \left|\begin{pmatrix} -k_0-k_1+m\omega^2 & k_1 \\ k_1 & -k_0+k_1+m\omega_2 \end{pmatrix}\right| = 0$$

$$\rightarrow (m\omega^2-k_0-k_1)^2 - k_1^2 = 0 \rightarrow m\omega^2 - k_0 - k_1 = \pm k_1$$

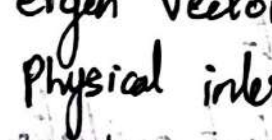
① $\underline{m\omega^2 = k_0}$ ② $\underline{m\omega^2 = k_0+2k_1}$ $\rightarrow \left\{\omega = \sqrt{\frac{k_0}{m}}, \sqrt{\frac{k_0+2k_1}{m}}\right\}$

For $\omega = \sqrt{\frac{k_0}{m}}$, $\begin{pmatrix} -k_1 & k_1 \\ k_1 & -k_1 \end{pmatrix}\vec{X} = 0 \Rightarrow \boxed{\vec{X} = \alpha\begin{pmatrix} +1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}} \rightarrow \vec{e}_1$

For $\omega = \sqrt{\frac{k_0+2k_1}{m}}$, $\begin{pmatrix} k_1 & k_1 \\ k_1 & k_1 \end{pmatrix}\vec{X} = 0 \Rightarrow \boxed{\vec{X} = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}} \rightarrow \vec{e}_2$

eigen vectors of the problem.

Symmetric mode



[Center of Mass]

Antisymmetric mode

[Difference in the co-ordinates]

$\hookrightarrow$ eigen vectors and eigenvalues give us some insight into the Physical interpretation of the problem.

Polarization Example

$J_m|\psi\rangle = |\psi'\rangle$, $\quad |\psi\rangle = \begin{pmatrix} \alpha \\ \beta e^{-i\phi} \end{pmatrix}$, $\underline{\alpha^2+\beta^2=1}$.

$$\left.\begin{matrix} J_x|x\rangle = 1|x\rangle \\ J_y|x\rangle = 0 \neq 0|x\rangle \end{matrix}\right] \quad J_x|R\rangle \neq \lambda|R\rangle$$

- Eigenvectors of an operator provide 100% transfer from the initial to the final states.
- Eigenvalue equation only picks among all ~~states~~ possible states those states whose initial and final are same.

Propertres of Eigen vectors

$|\psi\rangle \in$ eigen ket, $C \in$ complex$/\{0\}$

$\rightarrow \hat{O}|\psi\rangle = \lambda|\psi\rangle \Rightarrow \hat{O}(c|\psi\rangle) = \lambda c|\psi\rangle = \underline{\tilde{\lambda}|\psi\rangle}$

↳ eigen states are represented w.r.t an arbitirary constant and a phase factor.

→ (undetermined) ($\langle\psi|\psi\rangle = 1$)

→ If $|\psi\rangle, |\phi\rangle$ → eigenvectors w.r.t $\hat{O}$, $|\tilde{\Psi}\rangle = c_1|\phi\rangle + c_2|\psi\rangle$

$\rightarrow \hat{O}|\tilde{\Psi}\rangle = \hat{O}(c_1|\phi\rangle + c_2|\psi\rangle) = c_1\lambda_1|\phi\rangle + c_2\lambda_2|\psi\rangle$ = eigenvalue ?

So, if $|\psi\rangle = \sum_i c_i|i\rangle = c_1|1\rangle + c_2|2\rangle + \cdots + c_N|N\rangle$.

* eigen vectors are orthonormal w.r.t each other.

$\Rightarrow \langle j|\psi\rangle = \langle j|(\sum_i c_i|i\rangle) = c_j \cancel{\langle j|j\rangle}^{1} \Rightarrow \boxed{\langle j|\psi\rangle = c_j}$

system $\longleftrightarrow \mathcal{H}^{(N)} \longleftarrow (\hat{O}_1, \hat{O}_2, \cdots\cdots \hat{O}_M) \rightarrow$ eigen vectors will define our Hilbert space.

$\rightarrow \hat{e}_1, \hat{e}_2 \rightarrow$ form a better basis for solving Newton's equation.

- For polarization $|\psi\rangle$ has $\{|x\rangle, |y\rangle\}$, $\{|L\rangle, |R\rangle\}$ > two eigen basis

$\rightarrow |\psi\rangle = \alpha_1|x\rangle + \alpha_2|y\rangle$
$= \tilde{\alpha}_1|L\rangle + \tilde{\alpha}_2|R\rangle$

$|\alpha_1|^2 + |\alpha_2|^2 = 1$
$|\tilde{\alpha}_1|^2 + |\tilde{\alpha}_2|^2 = 1$

However, $\alpha_1 \neq \tilde{\alpha}_1$, $\alpha_2 \neq \tilde{\alpha}_2$

P.T.O

Spectral Decomposition :-

$\boxed{\sum_i |i\rangle\langle i| = \hat{\mathbb{I}}}$ Completeness Relation

This Serves as a Test for "Completeness" i.e, do we have all eigen vectors for said Hilbert Space.

$|\psi\rangle = \sum_i \alpha_i |i\rangle$ and $\langle i|\psi\rangle = \alpha_i$

$\hookrightarrow |\psi\rangle = \sum_i \langle i|\psi\rangle |i\rangle \equiv |\psi\rangle = \underbrace{\sum_i |i\rangle\langle i|}_{\mathbb{I}_{n\times n}}|\psi\rangle$

- For every Hilbert Space $\mathcal{H}^{(N)}$, we are supposed to have N <u>independent</u> eigen vectors.

18/08/23

$\Rightarrow \hat{O}\cdot\hat{\mathbb{1}} = \hat{O}\sum_i |i\rangle\langle i| = \sum_i \hat{O}|i\rangle\langle i|$ only if basis is eigenvectors.

Let, $\hat{O}|i\rangle = \beta_i |i\rangle$ then $\boxed{\hat{O} = \sum \beta_i |i\rangle\langle i|}$

Ex :- $\hat{J} = \beta_1 |x\rangle\langle x| + \beta_2 |y\rangle\langle y| = \tilde{\beta}_1 |L\rangle\langle L| + \tilde{\beta}_2 |R\rangle\langle R|$

$\rightarrow \hat{O}$ is Hermetian, β_i's are real.

$\rightarrow \hat{O}$ is unitary, $\hat{O}^{-1} \xrightarrow[\text{value}]{\text{eigen}} \beta_i^*$

$\rightarrow f(\hat{O}) = \sum_i f(\beta_i)|i\rangle\langle i|$

$\rightarrow \hat{O} \rightarrow$ unitary operator. $|i\rangle_2 = \hat{U}|i_1\rangle_1$, ${}_1\langle i|j\rangle_1 = \delta_{ij}$

${}_2\langle i_2|j\rangle_2 = \delta_{i_2 j_2}$ (You can have a different eigenvector by unitary transformation we moved to another basis)

${}_1\langle i|\hat{\mathbb{1}}|j\rangle_1 = {}_1\langle i|U^\dagger\hat{U}|j\rangle_1 = {}_2\langle i_2|j\rangle_2$.

Let $\hat{O}|i_1\rangle = \lambda_1 |i_1\rangle \longrightarrow U^\dagger\hat{O}\hat{U}|i_1\rangle = \lambda_2 U^\dagger\hat{U}|i_1\rangle$

$\hat{O}|i_2\rangle = \lambda_2 |i_2\rangle$ $\quad \hat{O}_{new}|i_1\rangle = \lambda_2 |i_1\rangle$

$\rightarrow |x\rangle = \hat{U}|R\rangle$

$\boxed{\text{Is } \lambda_x = \lambda_R\ ?} \rightarrow$ Yes.

$\hat{J}_R = \tilde{\beta}_1 |R\rangle\langle R| + \tilde{\beta}_2 |L\rangle\langle L|$ and $\hat{J}_R|R\rangle = \lambda_R|R\rangle$

$\hat{U}^\dagger\hat{J}_R\hat{U}|x\rangle = \lambda_R U^\dagger\hat{U}|x\rangle \longrightarrow \hat{J}_x|x\rangle = \lambda_x|x\rangle$

Sometimes a basis we start with is useless. So we go to a different basis and do all the computations there.

- Unitary Transformations are useful in quantum Mechanics bcoz. we can use them to move from one system to the other. w/o changing the eigen values and hence, the system.

Dynamics and Plank constant.

What Kinematics gave us ? : $|i\rangle \leftarrow \hat{O}$

$|i\rangle \to$ How do they modify.

$\downarrow$

$\mathcal{H}^{(n)}$

Suppose we have a state vector

$|i\rangle_{t=0} \longrightarrow |i\rangle_{t=1s} \longrightarrow |i\rangle_{t=10s}$ } What happens if i leave the system alone and keep measuring it (assume no disturbance) then how does the state vector evolve?

① Pendulum :

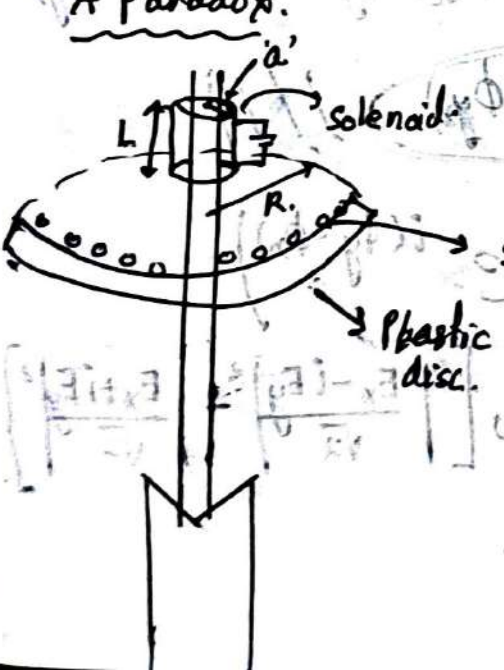
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$\rightarrow$ If i precisely know g, l then i know the Position of the Pendulum at any point of time.

② Gravity : $F = \dfrac{G\, m_1 m_2}{r^2} \Rightarrow \dfrac{d^2 r_1}{dt^2} = \dfrac{G m_2}{r^2}$ $\rightarrow$ we have to determine this to find out how r_1 evolves in time.

* We are interested to know the constant in Q.M

A Paradox. Fey. Vol 2 Sec. 17.4



Small hole in b/w the disc

Small metal spheres at the edge of the plastic disc which small charge q. in such a way that they don't interact with each other or the battery

Battery on $\rightarrow$ Magnetic Field.

$\downarrow$

Suddenly cut : $\vec{B} = 0$

$\downarrow$

Counter clockwise $\vec{E}$ tangential to disc induced

$\downarrow$

Disc starts Spinning

$\downarrow$

Final Angular Momentum

* Final Angular Momentum → Mechanical: where is the initial angular momentum? → Must have been stored in the $\vec{E}$ and $\vec{B}$ in some form.

$$\frac{\partial \vec{\rho}}{\partial t} + \nabla \cdot \vec{j} = 0 , \quad \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{F} = \rho(\vec{E} + \vec{v} \times \vec{B}) = \rho\vec{E} + \vec{j} \times \vec{B}$$

$$= \varepsilon_0 (\nabla \cdot \vec{E})\vec{E} + c^2\left(\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t}\right) \times \vec{B}$$

$$\boxed{\vec{P}_{EM} = \varepsilon_0 \int d^3x \, (\vec{E} \times \vec{B})}$$

$$\vec{f} = \frac{d\vec{p}}{dt}$$ (Refer Jackson)

$$\vec{J}_{EM} = \vec{r} \times \vec{P}_{EM} = \varepsilon_0 \int d^3x \, (\vec{r} \times (\vec{E} \times \vec{B}))$$

$$\vec{E} = \vec{E}_{irr} + \vec{E}_{rot}$$

$$\rightarrow \boxed{\vec{J}_{EM} = J_{EM,\,orbital} + J_{EM,\,spin}} \leftarrow \boxed{\nabla \times \vec{E}_{irr} = 0} \quad \boxed{\nabla \cdot \vec{E}_{rot} = 0}$$

$$\rightarrow \vec{J}_{EM,\,orbital} = \int \frac{d^3x}{4\pi c} \, \vec{r} \times \sum_{j=1}^{3} \vec{E}_{rot,j} (\nabla \cdot \vec{A}_{rot}).$$

$$\vec{J}_{EM,\,spin} = \frac{1}{4\pi c} \int d^3x \, (\vec{E}_{rot} \times \vec{A}_{rot}).$$

<u>Gauge choice</u>

ϕ is constant

Coulomb's Gauge.

$$\boxed{\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \end{aligned}}$$

EM → Z axis

$$A_x = \frac{c}{\omega} E_x^{(0)} \sin(kz - \omega t)$$

$$A_y = \frac{c}{\omega} E_y^{(0)} \sin(kz - \omega t + \phi_y)$$

$$J_{EM}^{(s)} = \frac{V}{4\pi\omega} E_x^{(0)} \cdot E_y^{(0)} \sin(\phi_y - \phi_x) \quad !!!$$

$$J_{EM}^{(s)} = \frac{V}{8\pi(i\omega)} E_x^{(0)} \cdot E_y^{(0)} \left(e^{i(\phi_y - \phi_x)} - e^{-i(\phi_y - \phi_x)}\right)$$

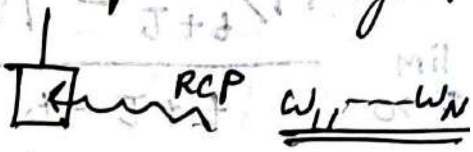
$$= \frac{-iV}{8\pi\omega}\left[E_x^* E_y - E_x E_y^*\right] = \frac{V}{8\pi\omega}\left[\left|\frac{E_x - iE_y}{\sqrt{2}}\right|^2 - \left|\frac{E_x + iE_y}{\sqrt{2}}\right|^2\right]$$

$$= \frac{V}{8\pi\omega}\left[|E_{RCP}|^2 - |E_{LCP}|^2\right]$$

$\vec{J}_{EM} = C_0 \left[|\langle R | \psi\rangle|^2 - |\langle L | \psi \rangle|^2 \right]$

1936 < Beth, holborn] verified the interinsic Angular momentum of EM waves.

[Below a certain Intensity → No torque on the system.



[Depending on the Frequency the torque is different].

Minimum Angular momentum Required to produce torque is,

$$\boxed{\hbar = 1.05 \times 10^{-34}\ Js.}$$

$L_z = n\hbar$ → Discrete.

what Q.M Proved is that $\underline{C_0 = \hbar}$.

* In compton scattering he found out that the change in the momentum of the photon is related to the K.E of the e^-. Thus, $\hbar$ is a universal constant.

→ basis vectors do not change with time. — I will always choose basis vectors such that they do not change with time to make my life easier.

$|\psi\rangle_{t+\tau}$ — describes the state of the system. (state vector)

$|i\rangle$ → same basis vectors at all times.
→ But, $|\psi\rangle_t \neq |\psi\rangle_{t+\tau}$] option I

→ option II [$|i\rangle_t \neq |i\rangle_{t+\tau}$ → Ex: $\hat{x}(t)$, $\hat{y}(t)$
↳ The basis vectors depend on the observer].

t=0 $\quad |\psi_i\rangle_t$ → basis vectors, orthogonal → $\langle \psi_i | \psi_j\rangle_t = \delta_{ij}$

$$\hat{A}_t = \sum \lambda_i^{(t)} |\psi_i\rangle_t\, {}_t\langle \psi_i|$$

t=τ $\quad |\psi_i\rangle_{t+\tau} \longleftrightarrow \langle \psi_i | \psi_j \rangle_{t+\tau}$

$$\hat{A}_{t+\tau} |\psi_i\rangle_{t+\tau} = \lambda_i(t+\tau) |\psi_i\rangle_{t+\tau}$$

P.T.O

How is the state vector / basis vector at time $\underline{t+\tau}$ related to the state/basis vector at time t ?

$$\Rightarrow \lim_{\tau\to 0} \frac{|\psi\rangle_{t+\tau} - |\psi\rangle_t}{t+\tau-t} \doteq \frac{\partial |\psi\rangle_t}{\partial t}$$

no other property of the system is considered except time evolution

$$\hat{A}_t |\psi_i\rangle_t = \lambda_i^{(t)} |\psi_i\rangle_t$$

$$\rightarrow \hat{A}_t \hat{\mathbb{I}} |\psi_i\rangle_t = \lambda_i^{(t)} \hat{\mathbb{I}} |\psi_i\rangle_t \rightarrow \hat{A}\hat{U}^\dagger\hat{U}|\psi_i\rangle_t = \lambda_i^{(t)}\hat{U}^\dagger\hat{U}|\psi_i\rangle_t$$

$$\rightarrow (\hat{U}\hat{A}\hat{U}^\dagger)\hat{U}|\psi_i\rangle_t = \lambda_i^{(t)}\hat{U}|\psi_i\rangle_t$$

$$\rightarrow \boxed{|\psi_i\rangle_{new} = \hat{U}|\psi_i\rangle_t} \; ** \qquad (\hat{U}\hat{A}\hat{U}^\dagger)|\psi_i\rangle_{new} = \lambda_i^{(t)}|\psi_i\rangle_{new}$$

$$\boxed{\hat{A}_{new} = \hat{U}\hat{A}_t\hat{U}^\dagger} \qquad \hat{A}_{new}|\psi_i\rangle_{new} = \lambda_i^{(t)}|\psi_i\rangle_{new}$$

$|\psi_i\rangle_{new}$ → new basis vectors and hence they satisfy the orthogonality and completeness

↳ $${}_{new}\langle\psi_i|\psi_j\rangle_{new} = \delta_{ij} \;/\; \sum_i |\psi_i\rangle_{new}\,{}_{new}\langle\psi_i| = \hat{\mathbb{I}}$$

$$\left\{ |\psi_i\rangle \xrightarrow{\hat{U}} |\psi_i\rangle_{new} \right\}$$

To Fix $\hat{U}$: $|\psi_i\rangle_{new} \underset{??}{\overset{\hat{U}(t+\tau;t)}{\longleftrightarrow}} |\psi_i\rangle_{t+\tau}$

$$\rightarrow |\psi_i\rangle_{t+\tau} = \hat{U}(t+\tau;t)|\psi_i\rangle_t$$

$\Rightarrow$ Unitary time Evolution of States.

Properties of the Unitary operator $\hat{U}$ →

1) $\hat{U}(t+\tau;t)\cdot\hat{U}^\dagger(t+\tau;t) = \hat{\mathbb{I}}$

2) $\lim_{\tau\to 0} U(t+\tau;t) = \hat{\mathbb{I}}$.

3) **Composition Property**:

$$|\psi_i\rangle_{t+\tau} = \hat{U}(t+\tau;t)|\psi_i\rangle_t$$
$$|\psi_i\rangle_{t+2\tau} = \hat{U}(t+2\tau;t+\tau)|\psi_i\rangle_{t+\tau}$$
$$|\psi_i\rangle_{t+2\tau} = \hat{U}(t+2\tau;t).$$

$$\rightarrow \boxed{\hat{U}(t+2\tau;t) = \hat{U}(t+2\tau;t+\tau)\cdot\hat{U}(t+\tau;t)}$$

$t_1, t_2, \ldots t_N$: $\left[\hat{U}(t_N;t_1) = \hat{U}(t_N;t_{N-1})\cdots\hat{U}(t_2;t_1)\right]$

4) **Time Reversal**:

$$\hat{U}^\dagger = \hat{U}^{-1} \;:\; \hat{U}^{-1} = \hat{U}^{-1}(t+\tau;t) = \hat{U}(t,t+\tau) = \hat{U}(t-\tau,t).$$

Construct the unitary operator $\hat{U}(t+\tau;t)$

$\tau \rightarrow dt \rightarrow$ infitesimal time.

$$\hat{U}(t+dt,t) = 1 - i\hat{\Omega}(t)dt. \qquad \hat{U}^\dagger = 1 + i\hat{\Omega}^\dagger(t)dt$$

$$\rightarrow \hat{U}\cdot\hat{U}^\dagger = \hat{\mathbb{I}} - i\hat{\Omega}(t)dt + i\hat{\Omega}^\dagger(t)dt + (dt)^2 \longrightarrow \boxed{\hat{\Omega}^\dagger(t) = \hat{\Omega}(t)}$$

Hermetian

For property 2 to satisfy.

→ Property 2 is satisfied. (check!!)

$$\rightarrow \hat{U} = \hat{\mathbb{I}} - i\hat{\Omega}dt,$$
$$|\psi\rangle_{t+dt} = \hat{U}(t+dt;t)|\psi\rangle_t = (\hat{\mathbb{I}} - i\hat{\Omega}dt)|\psi\rangle_t$$

$$\rightarrow \frac{|\psi\rangle_{t+dt} - |\psi\rangle_t}{dt} = \frac{(\hat{\mathbb{I}} - i\hat{\Omega}dt)|\psi\rangle_t - |\psi\rangle_t}{dt} = \lim_{t\to 0} -i\hat{\Omega}|\psi\rangle_t$$

$$\Rightarrow \frac{\partial|\psi(t)\rangle}{\partial t} = -i\hat{\Omega}|\psi(t)\rangle$$

$$\Rightarrow \frac{\partial\langle\psi(t)|}{\partial t} = +i\langle\psi(t)|\hat{\Omega}$$

What is $\hat{\Omega}$?

The only two possibilities are $\hat{\Omega} = \hbar\hat{K}$ or $\hat{\Omega} = \frac{\hat{O}}{\hbar}$

$[M^{-1}L^{-2}T^{0}]$ — All the unknown Physics into the $\hbar$.

$[M^{1}L^{2}T^{-2}]$ — Classical Mechanics

$$\boxed{\frac{\partial|\Psi\rangle_t}{\partial t} = \frac{-i}{\hbar}\hat{H}(t)\,|\Psi_0\rangle}$$ ✓ (This Equation only describes relastivistic systems.)

$$U(t,t_0) = \exp\left(\frac{-i}{\hbar}\int_{t_0}^{t_1}\hat{H}(t)\,dt\right) = \exp\left[\pm\frac{i}{\hbar}\hat{H}(t_1 - t_0)\right]$$

If $\hat{H}$ independent of t.

$$\Rightarrow \boxed{-i\hbar\frac{\partial\langle\Psi_i(t)|}{\partial t} = \hat{H}\langle\Psi_i(t)|}$$

$$\hat{A}(t) = \sum_{i=1}\lambda_i(t)\,|\Psi_i(t)\rangle\langle\Psi_i(t)| \quad \text{(Spectral Decomposition).}$$

$$\Rightarrow \frac{d\hat{A}}{dt} = \sum_{i=1}^{N}\frac{\partial\lambda_i(t)}{\partial t}|\Psi_i(t)\rangle\langle\Psi_i(t)| + \sum_{i=1}^{N}\lambda_i(t)\left(\frac{\partial|\Psi_i(t)\rangle}{\partial t}\langle\Psi_i(t)| + |\Psi_i(t)\rangle\frac{\partial\langle\Psi_i(t)|}{\partial t}\right).$$

$$\Rightarrow \frac{d\hat{A}}{dt} = \sum_{i=1}\frac{\partial\lambda_i(t)}{\partial t}|\Psi_i(t)\rangle\langle\Psi_i(t)| + \frac{\hat{H}i}{\hbar}\left(|\Psi_i(t)\rangle\cancel{\langle\Psi_i(t)|} - |\Psi_i(t)\rangle\cancel{\langle\Psi_i(t)|}\right)$$

$$\Rightarrow \frac{d\hat{A}}{dt} = \sum_{i=1}^{N}\frac{\partial\lambda_i}{\partial t}|\Psi_i(t)\rangle\langle\Psi_i(t)| + \frac{\hat{H}\hat{A}}{i\hbar} - \frac{\hat{A}\hat{H}}{i\hbar}$$

$$\boxed{\frac{d\hat{A}}{dt} = \frac{\partial\hat{A}(t)}{\partial t} + \frac{1}{i\hbar}[\hat{H},\hat{A}]}$$

Heisenberg's Equation.

[

$\rightarrow \mathcal{H}^{(N)} \rightarrow$ Dual Representation, where we can either represent the state vector evolving in time or the operators evolving in time.

$$\rightarrow |\psi(t)\rangle = \hat{U}\,|\psi(t)\rangle \qquad \underbrace{\langle \hat{A} \rangle}_{\text{number}} = \langle \psi(t)|\hat{A}|\psi(t)\rangle$$

[state vectors evolving in time].

$$\Rightarrow \langle \psi_i | \underbrace{\hat{U}^{\dagger}\hat{A}\,\hat{U}}_{} | \psi_i \rangle = \underline{\langle \psi_i | \hat{A}(t) | \psi_i \rangle}.$$

[operator evolving in time]

$$\rightarrow \underbrace{\langle \psi(t) | \hat{A} | \phi(t) \rangle}_{\text{Schrodinger Representation}} = \langle \psi_i | \hat{U}^{\dagger}\hat{A}\,\hat{U} | \phi_i \rangle = \underbrace{\langle \psi_i | \hat{A}(t) | \phi_i \rangle}_{\text{Heisenburg Representation}}.$$

$\rightarrow$ **Key input** : Hamiltonian : Hence if we know the Hamiltonian well, we don't really have to worry about which representation we use.

$\rightarrow$ Hamiltonian defines the Hilbert space and drives the dynamics of the system. In principle $\hat{H}(t) \equiv$ Hamiltonian can evolve with time but mostly we'll be given a non-time dependent Hamiltonian, i.e, Energy of the system is conserved.

Representation :

$$\underline{|\psi\rangle = c_0|1\rangle + c_1|2\rangle} \rightarrow |c_0|^2 + |c_1|^2 = 1.$$

Two state system

If $c_1 = r_1 e^{i\phi_1}$ and $c_2 = r_2 e^{i\phi_2} \Longrightarrow \boxed{r_1^2 + r_2^2 = 1}$

$$\rightarrow |\psi\rangle = e^{i\phi_1}\left[r_1|1\rangle + r_2 e^{i(\phi_2 - \phi_1)}|2\rangle\right]$$

$| \; | = 1 \rightarrow |\psi\rangle = r_1|1\rangle + r_2 e^{i\phi}|2\rangle \rightarrow \boxed{\phi = \phi_2 - \phi_1}$

$\Rightarrow r_1 = \cos\theta, \; r_2 = \sin\theta \rightarrow |\psi\rangle = \cos\theta|1\rangle + \sin\theta\, e^{i\phi}|2\rangle$

$|\psi\rangle = r_1|1\rangle + r_2 e^{i\phi}|2\rangle = r_1|1\rangle + (x+iy)|2\rangle$

$\underbrace{\langle\psi|\psi\rangle = 1} \Rightarrow r_1^2 + (x+iy)(x-iy) = 1.$

$\Rightarrow r_1^2 + x^2 + y^2 = 1$ (sphere of unit radius).

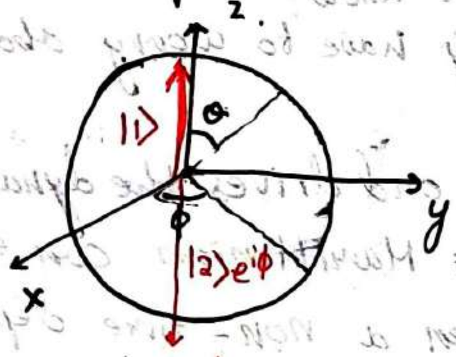
$\underbrace{r_1 \longleftrightarrow z} \rightarrow z = \cos\theta, \quad x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi$

$\hookrightarrow (x+iy) \longleftrightarrow \sin\theta(\cos\phi + i\sin\phi) = \sin\theta e^{i\phi}$

$\hookrightarrow \boxed{|\psi\rangle = \cos\theta|1\rangle + \sin\theta e^{i\phi}|2\rangle}$

$\underline{\theta = 0}: |\psi\rangle = |1\rangle \qquad \underline{\theta = 90}: |\psi\rangle = e^{i\phi}|2\rangle.$

what if i represent it like this? $\Rightarrow |\psi\rangle = \cos\left(\frac{\theta}{2}\right)|1\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|2\rangle$



$\underline{\theta' = 0}: |\psi\rangle = |1\rangle.$

$\theta' = 180: |\psi\rangle = |2\rangle e^{i\phi}$

Bloch Sphere

Two states are represented at the two opposite poles.

Dynamics of 2 State Systems. :

2. State $\longrightarrow$ basis vectors $\longrightarrow$ $\underbrace{|\psi_1\rangle \text{ and } |\psi_2\rangle}_{|1\rangle \text{ and } |2\rangle}$.

$\rightarrow |\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$

Assumption: basis vectors don't change with time

$\langle\psi(t)|\psi(t)\rangle = |c_1(t)|^2 + |c_2(t)|^2 = 1.$

Substitute in the equation

$$\boxed{i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle.}$$

$$i\hbar \frac{\partial}{\partial t}\left(c_1(t)|1\rangle + c_2(t)|2\rangle\right) = \hat{H}\left(c_1(t)|1\rangle + c_2(t)|2\rangle\right)$$

$$\rightarrow i\hbar \frac{\partial c_1(t)}{\partial t}|1\rangle + i\hbar\frac{\partial c_2(t)}{\partial t}|2\rangle = c_1(t)\hat{H}|1\rangle + c_2(t)\hat{H}|2\rangle$$

multiplying by bra $\langle 1|$ and $\langle 2|$ respectively;

$$\rightarrow \left[i\hbar \frac{\partial c_1(t)}{\partial t} = c_1(t)\langle 1|\hat{H}|1\rangle + c_2(t)\langle 1|\hat{H}|2\rangle \right]$$

$$\rightarrow \left[i\hbar \frac{\partial c_2(t)}{\partial t} = c_1(t)\langle 2|\hat{H}|1\rangle + c_2(t)\langle 2|\hat{H}|2\rangle \right]$$

$$\rightarrow i\hbar \frac{\partial c_1(t)}{\partial t} = c_1(t)H_{11} + c_2(t)H_{12}$$

$$\rightarrow i\hbar \frac{\partial c_2(t)}{\partial t} = c_1(t)H_{21} + c_2(t)H_{22}$$

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

*) Hamiltonian by definition is Hermetian : $\boxed{H_{12} = H_{21}^{*}}$

$\Rightarrow$ Scenario. 1 : Hamiltonian Matrix is time independent. $\boxed{H_{11}, H_{22} \in \mathbb{R}}$

$\rightarrow H_{11}, H_{12}, H_{21}, H_{22}$ are independent of t.

$$\rightarrow i\hbar \frac{\partial}{\partial t}\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = e^{-iEt/\hbar}\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} \longrightarrow \text{Assume to be the function.}$$

$$\hookrightarrow i\hbar\left(\frac{-iE}{\hbar}\right)e^{-iEt/\hbar}\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} e^{-iEt/\hbar}\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$$

Solve for eigenvalues

$$\Rightarrow \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} = E\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} \qquad \boxed{|c_1(0)|^2 + |c_2(0)|^2 = 1}$$

$$\rightarrow E_{\pm} = \left(\frac{H_{11} + H_{22}}{2}\right) \pm \left[\left(\frac{H_{22} - H_{11}}{2}\right)^2 + |H_{12}|^2\right]^{1/2}.$$

$$\rightarrow \boxed{\frac{H_{11}+H_{22}}{2} = \overline{E}}\ ,\ \boxed{\frac{H_{22}-H_{11}}{2} = \Delta}\ \boxed{\begin{matrix} H_{12} = V \\ H_{21} = V^* \end{matrix}}$$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} \overline{E}-\Delta & V \\ V^* & \overline{E}+\Delta \end{pmatrix}$$

So, eigenvalues : $E = \overline{E} \pm \left(\Delta^2 + |V|^2\right)^{1/2}$

$$= \overline{E} \pm \Omega$$

↳ And eigen vectors :

① $$\begin{pmatrix} c_1^+ \\ c_2^+ \end{pmatrix} = \frac{1}{\sqrt{V^2+\Omega^2-\Omega\Delta}} \begin{pmatrix} \Omega-\Delta \\ V \end{pmatrix}$$

② $$\begin{pmatrix} c_1^- \\ c_2^- \end{pmatrix} = \frac{1}{\sqrt{V^2+\Omega^2+\Omega\Delta}} \begin{pmatrix} \Omega+\Delta \\ V \end{pmatrix}$$

• We now Assume that V is real

$|c_1(0)|^2 + |c_2(0)|^2 = 1$ and $\Omega^2 = \Delta^2 + V^2$

↳ $\frac{\Delta^2}{\Omega^2} + \frac{V^2}{\Omega^2} = 1 \implies \begin{matrix} \cos 2\theta = \Delta/\Omega \\ \sin 2\theta = -V/\Omega \end{matrix}$

Dynamics of 2-state systems.

$$\{|1\rangle, |2\rangle\}$$

Examples : ① polarization $\rightarrow \{|x\rangle, |y\rangle\}$

② spin of electron $\rightarrow \{|\uparrow\rangle, |\downarrow\rangle\}$ → QIT, → NMR

③ NH_3

④ H_2^+

⑤ 

⑥ K-Meson ⟷ δ-Meson

$$\hat{H} = \begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix} \quad \hat{H}_{ij} = \langle i | H_{ij} | j \rangle$$

$$\boxed{H_{21} = H_{12}^*}$$

$\rightarrow |\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle \rightarrow |c_1|^2 + |c_2|^2 = 1$

normalized for all times: $\langle \psi(t) | \psi(t) \rangle = 1$

$$i\hbar \frac{\partial c_1(t)}{\partial t} = H_{11}(t) \cdot c_1(t) + H_{12}(t) c_2(t)$$

$$i\hbar \frac{\partial c_2(t)}{\partial t} = H_{21}(t) \cdot c_1(t) + H_{22}(t) c_2(t)$$

Assuming $\hat{H}$ independent of time.

$$\rightarrow \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} e^{-iEt/\hbar} \rightarrow \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} = E\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$$

$$\boxed{E_\pm = \frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{22} - H_{11}}{2}\right)^2 + |H_{12}|^2} \; ; \; \begin{pmatrix} c_1^\pm \\ c_2^\pm \end{pmatrix} = \frac{1}{\sqrt{1 + \left(\frac{H_{21}}{E_\pm - H_{22}}\right)^2}} \begin{pmatrix} 1 \\ \frac{H_{21}}{E_\pm - H_{22}} \end{pmatrix}}$$

$$\bar{E} = \frac{H_{11} + H_{22}}{2}, \; \Delta = \frac{H_{22} - H_{11}}{2} \Bigg\} \; \begin{matrix} H_{11} = \bar{E} - \Delta, & H_{12} = V \\ H_{22} = \bar{E} + \Delta, & H_{21} = V^* \end{matrix}$$

So $E_\pm = \bar{E} \pm \sqrt{\Delta^2 + V^2} = \bar{E} \pm \Omega$

So, $\begin{pmatrix} c_1^\pm \\ c_2^\pm \end{pmatrix} = \frac{1}{\sqrt{(\pm\Omega - \Delta)^2 + V^2}} \begin{pmatrix} \pm\Omega - \Delta \\ V \end{pmatrix} \Rightarrow \underbrace{|c_1|^2 + |c_2|^2 = 1}_{\text{Sin and cos}}$

$\Omega^2 = \Delta^2 + V^2$

$$\hookrightarrow V^2 + (\Omega \pm \Delta)^2 = V^2 + \Omega^2 + \Delta^2 \pm 2\Omega\Delta$$

$$= 2\Omega^2\left(1 \pm \frac{\Delta}{\Omega}\right)$$

Now, $1 = \frac{\Delta^2}{\Omega^2} + \frac{V^2}{\Omega^2} \Bigg] \Rightarrow \cos 2\theta = \frac{\Delta}{\Omega}, \; \sin 2\theta = \frac{-V}{\Omega}$

$$\rightarrow V^2 + (\Omega - \Delta)^2 = 4\Omega^2 \sin^2\theta$$

$$V^2 + (\Omega + \Delta)^2 = 4\Omega^2 \cos^2\theta$$

P.T.O

$$\begin{pmatrix} c_1^- \\ c_2^- \end{pmatrix} = \frac{1}{2\Omega\cos\theta}\begin{pmatrix} 2\Omega\cos^2\theta \\ 2\Omega\sin\theta\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

$$|-\rangle \leftarrow \boxed{\begin{pmatrix} c_1^- \\ c_2^- \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} ; \begin{pmatrix} c_1^+ \\ c_2^+ \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}} \rightarrow |+\rangle$$

Now we will replace, $|1\rangle, |2\rangle \longrightarrow |+\rangle, |-\rangle$

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}.$$

✱ $|1\rangle$ and $|2\rangle$: Describe 2 ~~physical properties~~ configurations of a system.

In the new basis, $\boxed{\hat{H} = \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix}}$ Hamiltonian is Diagonalized.

Changing the basis to the eigen basis $|+\rangle$ and $|-\rangle$ we decouple the differential equations solving which are pretty easy.

$$\left[i\hbar \frac{\partial|+\rangle}{\partial t} = E_+|+\rangle, \quad i\hbar\frac{\partial|-\rangle}{\partial t} = E_-|-\rangle \right] \Rightarrow |\psi_\pm(t)\rangle = c_0 e^{-iE_\pm t/\hbar} \, |\psi_\pm(0)\rangle$$

Dynamics :

$$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle \quad (|c_1|^2 + |c_2|^2 = 0)$$

$$\downarrow$$

$$\{|+\rangle, |-\rangle\}.$$

Initial condition at some $t=0$

[Make a prediction]

?? → Physical interpretation.

We have essentially Defined $|+\rangle, |-\rangle$ (eigen basis) using initial $c_1(0)$ and $c_2(0)$

$\underline{t=0}$ $\quad |\psi(t=0)\rangle = c_1(0)|1\rangle + c_2(0)|2\rangle$

Normalization condition fixes both co-ordinates if we fix one.

$i\hbar \dfrac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle$
$|\psi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle$
$i\hbar \dfrac{\partial}{\partial t} c_+(t) = E_+ c_+(t)$
$i\hbar \dfrac{\partial}{\partial t} c_-(t) = E_- c_-(t)$

- $|\psi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle$
- $|\psi(t)\rangle = c_+(0)\, e^{-iE_+t/\hbar}|+\rangle + c_-(t)\, e^{-iE_-t/\hbar}|-\rangle$
- $|\psi(t)\rangle = c_+(0)\, e^{-iE_+t/\hbar}\left(-\sin\theta|1\rangle + \cos\theta|2\rangle\right) + c_-(0)\, e^{-iE_-t/\hbar}\left(\cos\theta|1\rangle + \sin\theta|2\rangle\right)$
- $|\psi(t)\rangle = \left(-c_+(0)\sin\theta\, e^{-iE_+t/\hbar} + c_-(0)\cos\theta\, e^{-iE_-t/\hbar}\right)|1\rangle + \left(c_+(0)\cos\theta\, e^{-iE_+t/\hbar} + c_-(0)\sin\theta\, e^{-iE_-t/\hbar}\right)|2\rangle$

$$|\psi(t)\rangle = \cancel{\left(-c_+(0)\sin\theta\, e^{-iE_+t/\hbar}\right)}\left(\left(1 - \frac{c_-(0)}{c_+(0)}\cot\theta\, e^{-\frac{i(E_--E_+)t}{\hbar}}\right)|1\rangle + \left(-\cot\theta - \frac{c_-(0)}{c_+(0)} e^{-\frac{i(E_--E_+)t}{\hbar}}\right)|2\rangle\right)$$

Arbitirary phase Factor.

$|\psi^p(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$

$\hookrightarrow c_1(t) = 1 - \dfrac{c_-(0)}{c_+(0)}\cot\theta\, e^{+2i\Omega t/\hbar}$

$c_2(t) = -\cot\theta - \dfrac{c_-(0)}{c_+(0)} e^{+2i\Omega t/\hbar}$

Now,

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} \bar{E}-\Delta & V \\ V & \bar{E}+\Delta \end{pmatrix} \quad \text{(Time independent)}$$

$\sin 2\theta = -V/\Omega, \quad \cos 2\theta = \Delta/\Omega, \quad \underline{\Omega^2 = V^2 + \Delta^2}$

Case 1: $\Delta = V = 0 \Rightarrow \Omega = 0$.

$|\psi(t)\rangle$ = does not change with time.

Case 2: $\Delta = 0$ but $V \neq 0$.

$c_1(t) = \{ \quad \} e^{i2i\ldots}$ $c_2(t) = \{ \quad \} e^{-2i\ldots}$ calculate $\lvert c_1\rvert^2$, $\lvert c_2\rvert^2$ and get some answer.

~~$c_1(t)$~~ At $t=0$ $|\psi(0)\rangle = |1\rangle$.

(setting a specific initial condition).

$\rightarrow |1\rangle = -\sin\theta\,|+\rangle + \cos\theta\,|-\rangle$

$\Rightarrow c_+(0) = -\sin\theta, \quad c_-(0) = \cos\theta.$

↳ Now we have initial value of $c_+(0)$ and $c_-(0)$ in terms of θ.

so now, $$|\psi(t)\rangle = \left(\sin^2\theta + \cos^2\theta\, e^{-i2\Omega t/\hbar}\right)|1\rangle + \left(-\sin\theta\cos\theta + \cos\theta\sin\theta\, e^{-i2\Omega t/\hbar}\right)|2\rangle$$

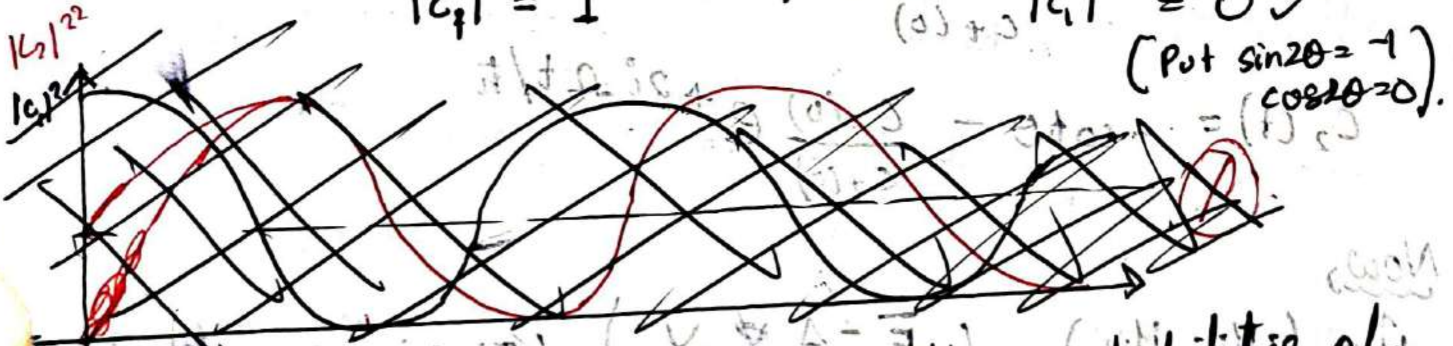
$$|c_2|^2 = \frac{1}{2}\sin^2(2\theta)\left(1 - \cos\left(\frac{2Vt}{\hbar}\right)\right)$$

$$|c_1|^2 = \left(\frac{1}{2}(1+\cos^2 2\theta) + \frac{1}{2}\sin^2 2\theta\cos\left(\frac{2Vt}{\hbar}\right)\right)$$

check. $|c_1|^2 + |c_2|^2 = 1$

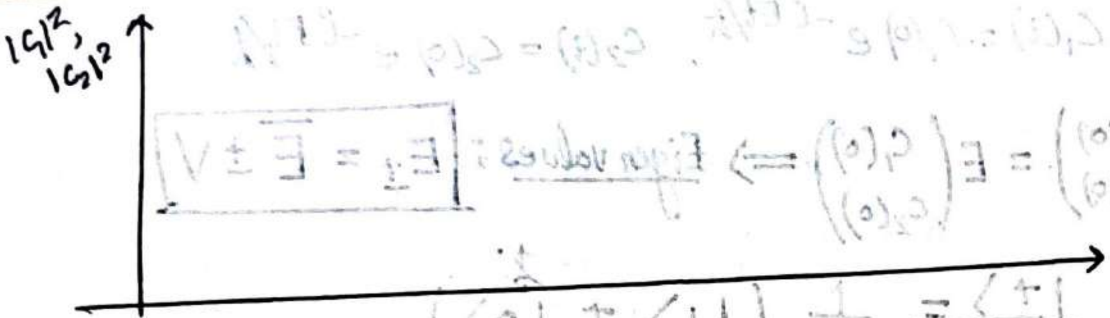
Ex: $\frac{Vt}{\hbar} = \pi$, $|c_2|^2 = 0$, $|c_1|^2 = 1$; $\frac{Vt}{\hbar} = \pi/2$ $|c_2|^2 = 1$ ✓, $|c_1|^2 = 0$ ✓

(Put $\sin 2\theta = 1$, $\cos 2\theta = 0$).



$|c_1|^2$ and $|c_2|^2$ can be interpreted as the probabilities of the particle to be in that state.

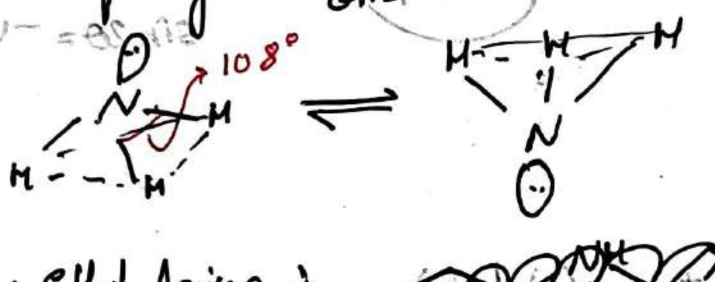
$$\boxed{0 \leq P \leq 1}$$



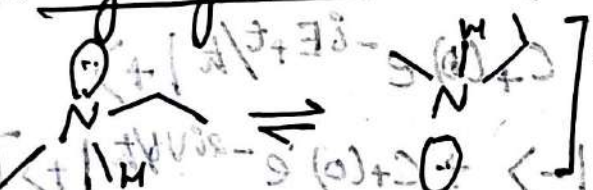
I] Ammonia Molecule (NH_3):

Vol 3 Sec. 8.6

① Flip frequency : 24 GHz ② low temperature : flip frequency is same.



II] Methy Ethyl Amine :



① At room temperature, it flips but at lower frequency : 100 MHz.

② This stops as we cool the molecule.

→ Quantum Mechanics Explains why NH_3 flips while Methyethyl amine does not flip at low temperature.

• Model as a two state system : $|T\rangle, |B\rangle$.

$$\rightarrow \hat{H} = \begin{pmatrix} H_{BB} & H_{BT} \\ H_{TB} & H_{TT} \end{pmatrix}$$

Assume : H_{BT}, H_{TB} is real $\Rightarrow$ $\underline{H_{BT} = H_{TB} = V}$

→ $H_{BB} = H_{TT} = \bar{E}$ since top and bottom is w.r.t us all infact the top and bottom configurations are symmetric w.r.t the plane and there is no difference.

$$\Downarrow$$

$$\hat{H} = \begin{pmatrix} \bar{E} & V \\ V & \bar{E} \end{pmatrix} \Rightarrow \Delta = 0$$

↳ $\bar{E}$, V are different for ammonia and methyl ethyl amine.

$$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle \qquad [|B\rangle \rightarrow |1\rangle, \ |T\rangle \rightarrow |2\rangle]$$

$$\rightarrow i\hbar \frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi(t)\rangle .$$

$$\rightarrow i\hbar \frac{\partial c_1(t)}{\partial t} = c_1(t)\bar{E} + c_2(t)\cdot V , \quad i\hbar \frac{\partial c_2(t)}{\partial t} = c_1(t)\cdot V + c_2(t)\cdot \bar{E}$$

Now Assume, $C_1(t) = C_1(0)\, e^{-iEt/\hbar}$, $C_2(t) = C_2(0)\, e^{-iEt/\hbar}$.

$\rightarrow \begin{pmatrix} \bar{E} & V \\ V & \bar{E} \end{pmatrix} \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} = E \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} \Longrightarrow$ Eigen values: $\boxed{E_\pm = \bar{E} \pm V}$

$\rightarrow$ Eigen vectors: $|\pm\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle \pm |2\rangle \right)$

Step IV] ~~Rep~~ Relation between $|1\rangle, |2\rangle$ and $|+\rangle, |-\rangle$ $(\hat{U})$.

$\hat{U} = \begin{pmatrix} -\sin\theta & +\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$ Here, $\theta = \pi/4$ $\longrightarrow$ Since $\cos 2\theta = \Delta/\Omega$, $\sin 2\theta = -V/\Omega$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

Step VI] $|\psi(0)\rangle = C_-(0)\, |-\rangle + C_+(0)\, |+\rangle$

$|\psi(t)\rangle = C_-(0)\, e^{-iE_- t/\hbar} |-\rangle + C_+(0)\, e^{-iE_+ t/\hbar} |+\rangle$

$= e^{-iE_- t/\hbar} \left[C_-(0)\, |-\rangle + C_+(0)\, e^{-2iVt/\hbar} |+\rangle \right]$

Step VII] $|\psi(t)\rangle = C_1(t)\, |1\rangle + C_2(t)\, |2\rangle$

$\rightarrow |C_1(t)|^2 = \cos^2\left(\frac{2Vt}{\hbar}\right)$, $|C_2(t)|^2 = \sin^2\left(\frac{2Vt}{\hbar}\right)$

$\rightarrow \frac{2VT}{\hbar} = 2\pi \Longrightarrow \boxed{V = \pi \cdot \hbar \cdot \nu} = 4.9 \times 10^{-5}\, eV$ (For Ammonia)

And $E_T = K_B T \sim 2.5 \times 10^{-2}\, eV$ (Thermal energy)

$\Rightarrow \boxed{V_{Ammonia} <<< E_T}$

Classical Two state system: Coin $\rightarrow$ Head, Tail.

$\underleftrightarrow{\text{HTTHTH} \text{ ---}} \longrightarrow 1/2$ Head/Tail

Time independent: $P_T = 1/2 = P_H$.

$\hat{H} = \begin{pmatrix} \bar{E} & V \\ V & \bar{E} \end{pmatrix} = \bar{E} \begin{pmatrix} 1 & V/\bar{E} \\ V/\bar{E} & 1 \end{pmatrix}$, $\bar{E}_\pm = \bar{E}\left(1 \pm \frac{V}{\bar{E}}\right)$

when $\frac{V}{\bar{E}} \ll 1$ then $c_+ \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c_- \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$E_\pm \simeq \bar{E}$

Thus; $\frac{V}{\bar{E}} \lll 1 \longrightarrow$ system stays as in the initial state, no flip.

$\rightarrow V \simeq \bar{E} \rightarrow$ flip / $V > \bar{E} \rightarrow$ flip.

2-state-system :

Diatomic molecule sharing an e^-. (H_2^+)

e^- is jumping b/w the two hydrogen atoms.

$|1\rangle \qquad |2\rangle$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

$$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$$

$$H = \begin{pmatrix} \overline{E} & V \\ V & \overline{E} \end{pmatrix}$$

$\overline{E}$: Energy when e^- is bounded to a single hydrogen atom. (Equilibrium Energy)

We figured out V by experiments. ← V : Potential Energy due to the other H-atom when brought closer to the H-atom to which the e^- is bounded.

[Interatomic Forces acting on Hydrogen atom + e^-]

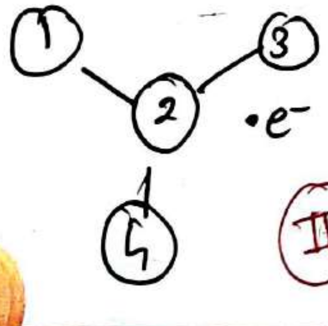
* [State 1 and state 2 are the two lattice points.]

→ Hamiltonian always measures the Energy. The physical significance of the variables inside the Hamitonion changes.

→ various two state systems can be represented by the same mathematics.

→ $|c_1(t)|^2$, $|c_2(t)|^2$ represent the probabilities of the e^- being in either state 1 or state 2.

Tetra Atomic Moleoule :



The Dynamics of this e^- in the tetra atomic molecule

(I) Hilbert space Dimension : 4

(II) $|1\rangle, |2\rangle, |3\rangle, |4\rangle$

$|1\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $|3\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $|4\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

III $|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|3\rangle + c_3(t)|3\rangle + c_4(t)|4\rangle$.

$\boxed{H_{ij} : \text{Interaction Energy of state } i \text{ with state } j\ ?}$

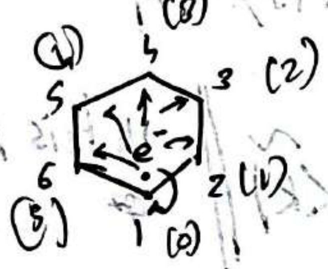
IV $H = \begin{pmatrix} E_0 & V & 0 & 0 \\ V & E_0 & V & V \\ 0 & V & E_0 & 0 \\ 0 & V & 0 & E_0 \end{pmatrix}$

V $i\hbar \dfrac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}\,|\psi(t)\rangle$

$C_i(t) = C_i(0)\, e^{-iEt/\hbar}$

* Solve a 4th degree polynomial for eigenvalues and eigenvectors.

Benzene : [Periodic Boundary condition]



e^- in Benzene and look at dynamics.

Hilbert Space Dimension = 6.

$|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle$

$\rightarrow |\psi(t)\rangle = \sum_{i=0}^{5} c_i(t)|i\rangle$

$\hat{H} = \begin{pmatrix} E_0 & V & 0 & 0 & 0 & V \\ V & E_0 & V & 0 & 0 & 0 \\ 0 & V & E_0 & V & 0 & 0 \\ 0 & 0 & V & E_0 & V & 0 \\ 0 & 0 & 0 & V & E_0 & V \\ V & 0 & 0 & 0 & V & E_0 \end{pmatrix}$

$\hat{H} = \sum_{n=0}^{N-1} \{E_0 |x\rangle\langle x|\} + \{V|x\rangle\langle x+1| + V|x+1\rangle\langle x|\}$

* Hilbert Space and operators must be linear.

$|x\rangle \longrightarrow |\tilde{k}\rangle$

$\sum_{x=1}^{\infty} |x\rangle\langle x| = 1$

$\sum |\tilde{k}\rangle\langle\tilde{k}| = 1.$

$|1\rangle\langle 1|$

$\left.\begin{matrix}|1\rangle\\|2\rangle\end{matrix}\right\} \rightarrow \begin{matrix}|+\rangle\\|-\rangle\end{matrix}$

Check orthogonality of $|\tilde{k}\rangle$ ✓

$$\rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} \sum_{k=0}^{n-1} e^{(2\pi i k \frac{n}{N} |k\rangle)}$$

$$|\tilde{k}\rangle = \sum_{k=0}^{N-1} e^{\frac{-2\pi i k}{N}} \cdot |x\rangle$$

new basis state for the given system (Analogous to $|+\rangle, |-\rangle$)

$|0\rangle = |N\rangle \rightarrow$ Periodicity.

$$\Rightarrow \sum_{k=0}^{N-1} |\tilde{k}\rangle\langle\tilde{k}| \text{ and } \sum_{x=0}^{N-1} |x\rangle\langle x| = \mathbb{1}.$$

The Dual basis is actually a Fourier Transform

[Transformation of basis]

$$\Rightarrow \text{calculate: } \sum_{x=0}^{N-1} |x\rangle\langle x+1| + |x+1\rangle\langle x|$$

$$= \sum_{k=0}^{N-1} 2\cos\left(\frac{2\pi k}{n}\right) |\tilde{k}\rangle\langle\tilde{k}|$$

in this basis we

$$H = \sum_{k=0}^{N-1} \varepsilon_k |\tilde{k}\rangle\langle\tilde{k}|$$

$$\underbrace{\varepsilon_k = E_0 + 2V\cos\left(\frac{2\pi k}{N}\right)}_{\text{Energy eigenvalues.}} \quad \cos\left(\frac{2\pi}{3}\right)$$

→ We removed the need of diagonalizing a n-matrix just by introducing fourier transform.

Calculate the energy eigen value for $k = 1, 5, 0, 2$

$\rightarrow E_0 + V = \varepsilon_1$ || $E_0 + 2V = \varepsilon_0$ || $\varepsilon_3 = E_0 - 2V$

$E_0 + V = \varepsilon_5$ || $E_0 - V = \varepsilon_2$ || $\varepsilon_4 = E_0 - V$

Same energy eigen values

Degenerate states

→ A basis decouples the system.

- A Fourier basis is useful because Hilbert space and operator are linears.

we have Problems in Real space → Mathematics gets simplified in the Fourier space.

we have Problems in Real space → Experiments are done in ~~Fourier~~ Real space.

$|0\rangle,$
$|1\rangle, |2\rangle, |3\rangle, ---|N-1\rangle$ → Move to Fourier Space

$\hat{H}$ (Tough) → $|\tilde{K}\rangle$ (Simplified)

$$|\psi(t)\rangle = \sum_{K=0}^{N-1} C_K(0)\cdot e^{-2E_L t/\hbar}|E_L\rangle$$

find Eigen vectors and values

<u>Linear chain</u> : $N >> L$ ← N atoms

$\boxed{\delta : \text{interatomic space}}$

$\boxed{L = N\delta}$

If L is fixed. If $N(\uparrow)$, $\delta(\downarrow)$. I'm going to consider a limit where $\delta \to 0$.

→ <u>Hilbert Space Dimension</u> : N.
→ Now, I will drop an e^-, how will it spread?
At a given instance of time what is the probability of finding the e^- at a given point in the lattice?

→ $|0\rangle$ ———— $|N-1\rangle$ (State vectors in Real space)

$$\sum_{x=0}^{N-1} |x\rangle\langle x| = \hat{\mathbb{1}} \quad \text{and} \quad \langle i|j\rangle = \delta_{ij}$$

In this basis, how will the hamiltonian look?

P.T.O

Hamiltonian is a tridiagonal matrix.

$$\begin{pmatrix} E_0 & V & & & \\ V & E_0 & V & & \\ & V & E_0 & V & \\ & & \ddots & \ddots & \ddots \\ & & V & E_0 & V \\ & & & V & E_0 \end{pmatrix}$$

And $|\psi(t)\rangle = \sum_{x=0}^{N-1} C_x(t)\,|x\rangle$

13/09/23

Different system $\div$ NH_3, C_6H_6 $\to$ $|1\rangle$, $|2\rangle$

electron of mass 'm' $\longrightarrow$ Lattice $\longrightarrow$ Free e^-

x x x x —— x

atom

$|\psi(t)\rangle = \sum_i C_i(t)\,|i\rangle$ $\to$ Fourier Mode.

$\to$ H_2^+, 4 atom, Benzene (Periodic), Linear chain $\to$ Sin( ) as basis

$\Downarrow$

$\sum e^{ik_\Delta x}$

Boundry condition

$\cos(x) \to e^{ix} + e^{-ix}$

$\sin(x) \to e^{ix} - e^{-ix}$

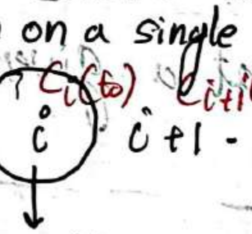
Standinding wave $\to$ Left Moveing + Right Moving

x x x x x ———— x

$N\delta = L$ (finite)
$\uparrow\ \downarrow$

As $N \to \infty$, what happens to $|\psi(t)\rangle$?
$\Downarrow$
$\boxed{\text{Infinite Dimensional Hilbert Space}}$

$t = t_0$ electron $\underline{C_n(t_0)}$ $C_n(t)$



* E_0 and V are different, i.e, they have different Physical interpretations for different physical systems.

$$H = \begin{pmatrix} E_0 & V & & & \\ V & E_0 & V & & \\ & V & E_0 & V & \\ & & \ddots & \ddots & \ddots \\ & & V & E_0 & V \\ & & & V & E_0 \end{pmatrix}$$

?

Let us zone in on a single lattice point

$C_{i-1}(t_0)$ $C_i(t_0)$ $C_{i+1}(t_0)$

.... $i-1$ $\quad i \quad$ $i+1$ ....

What may happen to the e^- at $t = t_0 + \Delta t$.

$t = t_0$
$\downarrow$
$t = t_0 + \Delta t$

If we visualize movement of Energy as Temperature.

Interaction with adjacent atoms in very small time.

1. Stay put at site i
2. hop from $i \to i+1$ or $i-1$ } $\to$ Red.
3. hop from $i+1 \to i$
4. hop from $i-1 \to i$ } $\to$ Black

* The are the possibilities, i.e, possible scenarios that can happen. We are neglecting the possibility of it hopping any more distant than adjacent atoms in such a small time.

- We are treating the e^- as a physical particle whose position we don't know at the given time. Thus, these four possibilities exist.

$\Rightarrow$ At all times, $\sum_{i=1}^{N} |c_i(t)|^2 = 1$ $\qquad |\psi(t)\rangle = \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_N(t) \end{pmatrix}$

Thus, $\sum_{i=1}^{N} |c_i(t_0)|^2 = \sum_{i=1}^{N} |c_i(t_0+\Delta t)|^2 = \underline{1}$ — Eq. (x) $\quad$ complex quantities.

$$\rightarrow c_i(t+\Delta t) = c_i(t) - i\,\omega_{i-1,i}(\Delta t)\, c_{i-1}(t) \longrightarrow (4)$$
$$- i\,\omega_{i+1,i}(\Delta t)\, c_{i+1}(t) \longrightarrow (3)$$
$$- i\,\omega_{i,i}(\Delta t)\, c_i(t) \longrightarrow (1) \text{ and } (2)$$

*) $[\, i\omega_{i-1,i},\ i\omega_{i+1,i},\ i\omega_{i,i}\,] \longrightarrow$ Unknown Parameters $\longrightarrow$ "hopping fraction".

These parameters are in principal arbitrary.

- However, the probability concentration will constrain values of $\omega_{i,j}$ [via Eq. (x)].

$\Rightarrow |c_i(t+\Delta t)|^2 \longrightarrow$ only upto first order in Δt. [Substitute into Eq. (x)].

$$\rightarrow \sum_i |c_i(t)|^2 (\omega_{i,i} - \omega_{i,i}^*) + \sum_i c_i^*(t)\, c_{i+1}(t)\, (\omega_{i+1,i} - \omega_{i,i+1}^*)$$
$$+ \sum_i c_i^*(t)\, c_{i-1}(t)\, (\omega_{i-1,i} - \omega_{i,i-1}^*) = 0.$$

$\Rightarrow \omega_{i,i} = \omega_{i,i}^*$ (Real) $\qquad \left.\begin{matrix} \omega_{i+1,i} = \omega_{i,i+1}^* \\ \omega_{i-1,i} = \omega_{i,i-1}^* \end{matrix}\right\}$

$$W = \begin{pmatrix} \omega_{11} & \omega_{12} & & & \\ \omega_{12}^* & \omega_{22} & \omega_{23} & & \\ & \omega_{32}^* & \omega_{33} & \ddots & \\ & & \ddots & \ddots & \\ & & & & \omega_{NN} \end{pmatrix} \longrightarrow \text{This is a Tridiagonal Matrix}$$

As $N \rightarrow \infty$, the points get closer and closer and thus we cannot use the i representation anymore. $c_i(t)$ will thus go to some distribution function.

$$\rightarrow \underbrace{\sum |c_i(t)|^2 = 1 = \int_{-\infty}^{+\infty} dx\, |\psi(x,t)|^2}= 1.$$

The c_i is so close that we can now represent it like this.

δ δ || I now say that I can represent $|c_i(t)|^2$ as $\delta|\psi_2(x,t)|^2$

$$\Rightarrow \boxed{\frac{c_i(t)}{\delta^{1/2}} \rightarrow \psi(x,t)} \rightarrow$$ For a very small δ we can assume $\psi(x,t)$ to be const.

Different from the state vector would $|\psi(t)\rangle$.

Going back to our probability concentration representation [Eq (x)]

$$\lim_{\Delta t \to 0} \frac{1}{c_i}\frac{\left(c_i(t_0+\Delta t) - c_i(t)\right)}{\Delta t} \rightarrow i\frac{\partial c_i(t)}{\partial t} = \omega_{i-1,i} \cdot \frac{c_{i-1}(t)}{\delta^{1/2}} + \omega_{i+1,i} \cdot \frac{c_{i+1}(t)}{\delta^{1/2}} + \omega_{i,i} \cdot \frac{1}{\delta^{1/2}} c_i(t)$$

$$\Rightarrow i \frac{\partial \psi(x_i,t)}{\partial t} = \omega_{i-1,i}\, \psi(x_{i-1},t) + \omega_{i+1,i}\, \psi(x_{i+1},t) + \omega_{i,i}\, \psi(x_i,t)$$

$$\boxed{\delta = x_{i+1} - x_i}$$

[Taylor expansion] $\leftarrow$ only upto first order in Δt.

$$\rightarrow \psi(x_{i\pm1},t) = \psi(x_i,t) \pm \delta \frac{\partial \psi}{\partial x_i}(x_i,t) + \frac{\delta^2}{2}\frac{\partial^2 \psi(x_i,t)}{\partial x_i^2}$$

$$\rightarrow i\frac{\partial \psi(x,t)}{\partial t} = (\omega_{i+1,i} + \omega_{i-1,i} + \omega_{i,i})[\psi(x_i,t)] + \delta(\omega_{i-1,i} - \omega_{i+1,i})\frac{\partial[\psi(x_i,t)]}{\partial x_i} + \frac{\delta^2}{2}(\omega_{i+1,i} + \omega_{i-1,i})\frac{\partial^2 \psi}{\partial x_i^2}$$

For systems with no velocity dependence Fwd moving is same as Bwd moving

$\Downarrow$

$$\boxed{\omega_{i-1,i} = \omega_{i+1,i}}$$

$$\rightarrow i\frac{\partial \psi(x,t)}{\partial t} = (2\omega_{i+1,i} + \omega_{i,i})\psi(x,t) + \delta^2 \omega_{i,i+1} \frac{\partial^2[\psi(x,t)]}{\partial x^2}$$

for this case: $\omega_{i,i}$ correspond to the E_0 and $\omega_{i+1,i}$, $\omega_{i-1,i}$ correspond to V.

[Heuristic Derivation of S.E].

$\rightarrow \cancel{i\hbar \frac{\partial \psi(x,t)}{\partial t} =}$

our derived equation, looks like a familiar equation.

$$\Rightarrow \boxed{i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)}$$

* ~~E~~ we essentially derived the S.E for a very very specific system, i.e. [Non-Relastivistic particles motion in co-ordinate Space.]

Thus, the Hamiltonian might actually change with the physical situation.

Actual S.E [For all situations] $\rightarrow$ $$\boxed{\frac{i\hbar\, \partial|\psi(t)\rangle}{\partial t} = \hat{H}\,|\psi(t)\rangle}$$

$\rightarrow$ there is no derivation for the schrodinger equation.

* $C_i(t) = \langle i|\psi(t)\rangle$ // $\psi(x,t) = \langle x|\psi(t)\rangle$ kind of a basis.

Our learnt Schrodinger equation is a POSITION SPACE Representation.

∞ Hilbert spaces
- $\psi(x,t) = \langle x|\psi(t)\rangle$ — Position space.
- $\psi(p,t) = \langle p|\psi(t)\rangle$ — Momentum space.

• x, p are continous variables

Finite dimensional Hilbert	Infinite Dimensional Hilbert
$\rightarrow |\psi\rangle = \sum_i c_i|i\rangle$	$\rightarrow \langle\bar{x}|\psi\rangle_t = \psi(\bar{x},t)$
$\rightarrow \hat{H}|i\rangle = \lambda_i|i\rangle$	$\rightarrow \psi(x,t)$ are eigen funtions of $\hat{H}$
$\rightarrow \langle i|j\rangle = \delta_{ij}$	$\rightarrow \langle\bar{x}|\bar{x}'\rangle = \delta(\bar{x}-\bar{x}')$
$\rightarrow \sum|c_i|^2 = 1$.	$\rightarrow \int_{-\infty}^{\infty}|\psi(\bar{x},t)|^2 dx = 1$.
$\rightarrow \hat{\mathbb{I}} = \sum_i |i\rangle\langle i|$	$\rightarrow \hat{\mathbb{I}} = \int_{-\infty}^{\infty} d\bar{x}\,|\bar{x}\rangle\langle\bar{x}|$
$\rightarrow c_i = \langle i|\psi\rangle$.	$\rightarrow P(x,t_0)\,\Delta x = |\psi|^2\Delta x$

~~Active~~ Symmetries of the Q.M system

R shankar ch 11

(Physical)

(I) Classification → a) **Active** ÷ Particle in changing (x, t)

$\boxed{\hat{U} = e^{i\alpha \hat{H}}}$

b) **Passive** ÷ Mathematical constraint which helps simplify the calculation.

$(x_1, x_2) \longrightarrow (X_+, X_-)$

(II) (1) Space-time symmetries.
(2) Discrete Symmetries.

Space - Time Transformation

(i) Space time translation ÷ $x \longrightarrow x' = x + \Delta$, $t \longrightarrow t' = t + \delta$ > System is invariant under these transformations. $\Delta, \delta \ll 1$

(ii) Rotation ÷ $R_z(\theta)$: $x \to x' = x\cos\theta + y\sin\theta$
$y \to y' = -x\sin\theta + y\cos\theta$
$z \to z' = z$.

(iii) Lorentz Transformation ÷ $x \to x' = \gamma(x - \beta t)$, $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$
$t \to t' = \gamma(t - \beta x)$

Discrete Transformations

(i) Space inversion / Parity ÷ $x \longrightarrow x' = -x$ / $y' = -y$.
(ii) Time Reversal ÷ $t \longrightarrow$ ~~...~~ $t' = -t$.

- Spacetime translation, Rotation, Parity can be represented as Unitary transformations $(\hat{U})$, while the time Reversal is an anti-unitary operator.

* Since we are doing non-Relativistic ~~q~~ quantum mechanics, we won't care about the Lorentz. But in any case it can't be represented as an unitary operator.

→ Transformations we can do — (obtain information about the system)

In Quantum,
1) Transformations describing how coordinate variables change. $\boxed{x \to x' = f(x)}$
2) How the state vectors transform.?

$\boxed{|\psi\rangle \to |\psi'\rangle = \hat{U}|\psi\rangle}$ — ~~under~~ $A \longrightarrow \hat{A}'$ also.

we want to Relate symmetries with operators. ÷

$\hat{U} = \exp[i\alpha\hat{G}]$ → What is $\hat{G}$? What physical quantity does it correspond to?

P.T.O

<u>Approach 1</u> : If an operator is invariant under $\hat{U}$, the expection value of the operators are identical.

$$\rightarrow \langle \Psi | \hat{A} | \phi \rangle = \langle \psi' | \hat{A}' | \phi' \rangle$$

- $\hat{A}' = \hat{U} \hat{A} \hat{U}^{\dagger} \longrightarrow \hat{A}' \hat{U} = \hat{U} \hat{A} \longrightarrow$ if the system is invariant $\underline{\hat{A}' = \hat{A}}$

$$\Rightarrow [\hat{A}, \hat{U}] = 0.$$

$\rightarrow$ Thus, if an operator is invariant under under $\hat{U}$, $\hat{A}' = \hat{A} \Rightarrow \boxed{[\hat{A}, \hat{U}] = 0}$

<u>Approach 2</u>: $i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle$ ‖ $|\psi\rangle \rightarrow |\psi'\rangle = \hat{U} |\psi\rangle$, $\hat{H} \rightarrow \hat{H}' = \hat{U} \hat{H} \hat{U}^{\dagger}$

$\rightarrow$ If Hamiltonian is invariant under this transformation. $\hat{H}' = \hat{H}$.

$$\Rightarrow [\hat{H}, \hat{U}] = 0$$

$$\Rightarrow \hat{U} = \exp[i\alpha \hat{G}] \longrightarrow [\hat{H}, \exp[i\alpha \hat{G}]] = 0 \rightarrow \left[\hat{H}, \sum_{p=0}^{\infty} \frac{(i\alpha)^p \hat{G}^p}{p!}\right] = 0$$

$$\rightarrow [\hat{H}, i\alpha \hat{G}] + \left[\hat{H}, \frac{(i\alpha)^2}{2!} \hat{G}^2\right] + \text{------} = 0.$$

$$\rightarrow i\alpha [\hat{H}, \hat{G}] + \frac{(i\alpha)^2}{2!} [\hat{H}, \hat{G}^2] + \text{------} = 0.$$

- only zero if $[\hat{H}, \hat{G}] = 0$.

$$\rightarrow \frac{d \langle \psi | \hat{G} | \phi \rangle}{dt} = \frac{\partial \langle \psi |}{\partial t} \hat{G} | \phi \rangle + \langle \psi | \frac{\partial \hat{G}}{\partial t} | \phi \rangle + \langle \psi | \hat{G} \frac{\partial | \phi \rangle}{\partial t}$$

$$\rightarrow = \frac{i}{\hbar} \langle \psi | \hat{H} \hat{G} | \phi \rangle - \frac{i}{\hbar} \langle \psi | \hat{G} \hat{H} | \phi \rangle + \langle \psi | \frac{\partial \hat{G}}{\partial t} | \phi \rangle$$

$$\dot{\theta} = \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{G}] | \phi \rangle + \langle \psi | \frac{\partial \hat{G}}{\partial t} | \phi \rangle.$$

$$= \langle \psi | \frac{\partial \hat{G}}{\partial t} + \frac{i}{\hbar} [\hat{H}, \hat{G}] | \phi \rangle$$

$\hat{G}$ has partial information about the system ⇓ thus we wist to express $\hat{U}$ as $\hat{G}$.

- Expectation value is invariant under $\hat{U} \Rightarrow \frac{d \langle \psi | \hat{G} | \phi \rangle}{dt} = 0$ if $[\hat{H}, \hat{G}] = 0$.

↳ we need to look at $\hat{G}$ which commute with $\hat{H}$.

$\rightarrow \hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} = \frac{\hat{p}^2}{2m}$. if $[\hat{H}, \hat{P}] = 0 \Rightarrow$ This system is Translationally invariant <u>in x dimension</u>.

$$\Downarrow$$

$$\hat{U} = \exp[i\alpha \hat{P}]$$

↳ By pass the unitary operator. see which ~~terms~~ operators can be written via the Hamiltonian and see if it commutes with $\hat{H}$.

$[\hat{x}, \hat{p}] = i\hbar \hat{I} \longrightarrow$ time independent system $\rightarrow$ All times.
if time dependent $\rightarrow [\hat{x}, \hat{p}]_t \longrightarrow$ measure at every instant.

Theorem: If Hamiltonian of a system is invariant under $\hat{U}$, Generated by an Hermetian operator $\hat{G}$, then the conserved variables associated with $\hat{G}$ satisfy $[\hat{H}, \hat{G}] = 0$.

Symmettery	Conserved Quantity
space - translation	Momentum
time - translation	Energy (Hamiltonian)
Rotation in space	Angular Momentum.
Reflection in space	parity
Gauge Transformation	Charge.

, X - representation $\mathcal{L}(x, \dot{x}, t) \longrightarrow H(x, p, t)$ P representation.

Dirac Delta Function] ÷ $\int_{-\infty}^{\infty} \delta(x - x_0) f(x) = f(x_0) \Rightarrow \int_{-\infty}^{\infty} \delta(x - x_0) dx = 1.$

$\rightarrow \int_{-\infty}^{\infty} dx\, \delta(x - x_0)\, e^{-ipx/\hbar} = e^{-ip(x_0)/\hbar}$ — Fourier Transform

$\hookrightarrow F(P)$

$$F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ipx/\hbar}\, dx. \qquad \text{Fourier}$$

$$f(x) = \frac{1}{\hbar\sqrt{2\pi}} \int_{-\infty}^{\infty} F(p)\, e^{ipx/\hbar}\, dp. \qquad \text{Inverse Fourier.}$$

$\rightarrow$ The better way to write this is in terms of k. $P = \hbar k$, where we loose the constant $\hbar$ and Fourier and inverse Fourier ~~are~~ have same coefficients (Normalized).

$\rightarrow \delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \cdot e^{-ik(x - x_0)}$.

Symmetteries come to the forefront since in infinite hilbert spaces, we can represent $|\phi\rangle$ in coordinates or momentum.

x - representation

$|x\rangle$

$\rightarrow \hat{x}|x\rangle = x|x\rangle$

$\rightarrow \langle x|\psi\rangle = \psi(x)$ $\xleftrightarrow[\text{ke coefficients}]{\text{basis vectors}}$

$\rightarrow \int |x\rangle\langle x| dx = \hat{\mathbb{I}}$

$\rightarrow \langle x|\hat{x} = \langle x|x$

$\rightarrow \langle\psi|\psi\rangle = \langle\psi|\hat{\mathbb{I}}|\psi\rangle = \int \langle\psi|x\rangle\langle x|\psi\rangle dx$

$= \int dx\, \psi^*(x)\cdot\psi(x)$

$\rightarrow \langle x|x_0\rangle = \delta(x-x_0)$

$\rightarrow \langle x|\hat{x}|\psi\rangle = x\,\psi(x)$

$\rightarrow \langle x|k_0\rangle = \frac{1}{\sqrt{2\pi}} e^{ik_0 x}$

$\rightarrow \langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = \langle\psi|\hat{\mathbb{I}}\,\hat{x}\,\hat{\mathbb{I}}|\psi\rangle$

$= \int dx \int dx' \langle\psi|x'\rangle\langle x'|\hat{x}|x\rangle\langle x|\psi\rangle$

$= \int dx \int dx'\, x\, \langle\psi|x'\rangle\,\delta(x-x')\cdot\langle x|\psi\rangle$

$$\boxed{\langle x\rangle = \int dx\, \psi^*(x)\, x\cdot\psi(x).}$$

K (P) representation.

$|k\rangle,\ |p\rangle.$

operator $\rightarrow \hat{p}|p\rangle = p|p\rangle$ ← eingenvalue.

$\langle p|\tilde{\psi}\rangle = \tilde{\psi}(p)$

$\int |k\rangle\langle k| dk = \hat{\mathbb{I}}$

$\langle p|\hat{p} = \langle p|p$

$\langle\tilde{\psi}|\tilde{\psi}\rangle = \int dp\, \tilde{\psi}^*(p)\cdot\tilde{\psi}(p)$

$\langle k|k_0\rangle = \delta(k-k_0)$

$\langle p|\hat{p}|\psi\rangle = p\,\tilde{\psi}(p)$

$\langle k|x_0\rangle = \frac{1}{\sqrt{2\pi}} e^{-ik_0 x_0}$

$\langle p\rangle = \int dp\, \tilde{\psi}^*(p)\cdot p\cdot\tilde{\psi}(p).$

$\Rightarrow \psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\cdot\tilde{\psi}(k)\, e^{+ikx} \rightarrow \frac{\partial\psi(x)}{\partial x} = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\cdot k\,\tilde{\psi}(k)\, e^{ikx}$

$\rightarrow -i\frac{\partial\psi(x)}{\partial x} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\,\tilde{\psi}(k)\, k\, dk \Rightarrow k\,\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\left(-i\frac{\partial\psi(x)}{\partial x}\right) e^{-ikx}$

$\rightarrow \psi^*(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi^*(x')\cdot e^{+ikx'}\, dx'$

$\rightarrow \langle k\rangle = \int_{-\infty}^{\infty} dk\,\tilde{\psi}^*(k)\cdot\tilde{\psi}(k)\cdot k = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} \psi^*(x')\, e^{ikx'} dx' \int_{-\infty}^{\infty} dx\left(-i\frac{\partial\psi(x)}{\partial x}\right) e^{-ikx}$

$\rightarrow$ but $\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik(x-x_0)}\, dk.$

$\rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ik(x-x')} \int_{-\infty}^{\infty} \psi^*(x')\, dx' \int_{-\infty}^{\infty} dx\left(-i\frac{\partial\psi(x)}{\partial x}\right)$

$$\rightarrow \cancel{\delta(x-x')} \int_{-\infty}^{\infty} \psi^*(x')\,dx' \int_{-\infty}^{\infty} -i\,\frac{\partial\psi(x)}{\partial x}\cdot\delta(x-x')\,dx.$$

$$= \int_{-\infty}^{\infty} \psi^*(x')\,dx' \,-i\,\frac{\partial\psi(x')}{\partial x'} \Rightarrow \langle \hat{k}\rangle = \int_{-\infty} dx\,\psi^*(x)\left(-i\frac{\partial}{\partial x}\right)\psi(x)$$

Thus $\langle p\rangle = \langle \psi(x)\,|\, -i\hbar\frac{\partial}{\partial x}\,|\,\psi(x)\rangle$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$.

And thus similarly, $\boxed{\hat{x} = i\hbar\frac{\partial}{\partial p}}$.

$3D \rightarrow (x,y,z) \longrightarrow (P_x, P_y, P_z)$.
[cartesian] => independent coordinates || Phase space $\rightarrow (x, \bar{P}_x, y, P_y, z, P_z)$
Hilbert ~~p~~space $\rightarrow \infty$.

$|\psi\rangle \xrightarrow{x\ rep} (x, y, z, -i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z})$

• Thus, if we have ~~n-representation~~ n-independent co-ordinates then we can have n-independent momentum Hilbert Space coordinate space.

$$\left(x_1, x_2, \text{———}, x_n, -i\hbar\frac{\partial}{\partial x_1}, -i\hbar\frac{\partial}{\partial x_2}, \text{———}, -i\hbar\frac{\partial}{\partial x_n}\right).$$

Thus, $H(\bar{x},\bar{p}) = \frac{p^2}{2m} + V(x)$ and $\boxed{\hat{H}(\hat{x},\hat{p}) = \frac{\hat{p}}{2m} + V(\hat{x})}$.

Thus, $\hat{H}(x,\hat{p}) = \frac{-\hbar^2}{2m}\frac{\partial}{\partial x^2} + V(x)$.

• we generally do not use prepresentation as most functions of x give really really high order differential equations in p ~~as those functions ain't~~ when $\hat{x}$ is substituted in $V(\hat{x})$.

Space Translation:

We are interested in the $\hat{G}$ through which we can represent $\hat{U}$.

① $\hat{U}(\Delta)\,|x\rangle = |x+\Delta\rangle$
② $\hat{U}(\Delta)\cdot\hat{U}(\Delta') = \hat{U}(\Delta+\Delta')$
③ $\hat{U}(\Delta)\cdot\hat{U}(-\Delta) = \hat{\mathbb{I}}$.

Let's say, $\hat{U}(\Delta)\cdot|\psi(x)\rangle = |\psi'(x)\rangle$.

Thus, $|\psi'\rangle = \hat{U}(\Delta)\cdot\hat{\mathbb{I}}\,|\psi\rangle$

$$|\psi'\rangle = \int_{-\infty}^{\infty} \hat{U}(\Delta)\,|x\rangle\langle x|\psi\rangle\,dx = \int_{-\infty}^{\infty} \hat{U}(\Delta)\,|x\rangle\,\psi(x)\,dx,$$

$$\int_{-\infty}^{\infty} dx\,|x+\Delta\rangle\,\psi(x). \quad \text{Let } x' = x+\Delta.$$

$$\langle x|\psi'\rangle = \int_{-\infty}^{\infty} dx'\,\langle x|x'\rangle\,\psi(x'-\Delta) = \int_{-\infty}^{\infty} dx'\,\psi(x'-\Delta)\cdot\delta(x-x')$$
$$= \psi(x-\Delta).$$

$$\Rightarrow \boxed{\psi'(x) = \psi(x-\Delta)}$$

And similarly, we can prove, $\langle \psi | \hat{U}(\Delta) = \langle \psi' |$.

Then, $\boxed{\psi'^{*}(x) = \psi^{*}(x+\Delta)}$ ✓

Now what does $\hat{U}(\Delta)$ do? [what is it's corresponding $\hat{G}$]

$x \rightarrow x'$
$|\psi\rangle \rightarrow |\psi'\rangle$ $\|$ $\underline{\underline{\psi(x) \rightarrow \psi'(x')}}$

Classical Mechanics: $\vec{r} \times \vec{p} = \vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$,

$\rightarrow L_z = x p_y - y p_x \longrightarrow$ Q. Mechanics?

$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$. we Know, $[\hat{x}, \hat{p}_x] = i\hbar$, $[\hat{y}, \hat{p}_y] = i\hbar$

$\rightarrow [\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_x] = [\hat{x}, \hat{y}] = [\hat{p}_x, \hat{p}_y] = 0$.

$\cdot\ \hat{T}_1 = \hat{x}\hat{p}_y$
$\cdot\ \hat{T}_2 = \hat{y}\hat{p}_x$ $\quad \hat{L}_z = \hat{T}_1 + \hat{T}_2$

So,
$$\hat{L}_Y = \hat{Z}\hat{P}_X - \hat{X}\hat{P}_Z$$
$$\hat{L}_Z = \hat{X}\hat{P}_Y - \hat{Y}\hat{P}_X$$
$$\hat{L}_X = \hat{Y}\hat{P}_Z - \hat{Z}\hat{P}_Y$$

• Notes: $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$ will form thier own basis.

We are interested to know the position of $\bar{x}_a$ and $\bar{x}_b$ at all points of time.

$(x, y, z) \longrightarrow (x', y', z')$

$\bar{X}' = RX$ → matric with Euler angles

$|\psi\rangle \rightarrow$ infinite dimensional Hilbert space. $\langle x, y, z|\psi\rangle \longrightarrow \langle x', y', z'|\psi\rangle$.

$[x, \hat{P}_x] \rightarrow \Delta x \Delta P_x \geqslant \hbar/2$.

$* \ |L\rangle = \begin{pmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{pmatrix}$ → $\langle \text{Angular Momentum} | \psi \rangle$.

• We will now ~~assign the~~ define the state vector in terms of the Angular Momentum basis. Since in ~~Ang~~ Rotation, it is much more prudent to define stuff w.r.t. the Angular Basis. → useful

where L_x, L_y, L_z are all infinite dimensional.

$$\boxed{|\psi\rangle = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}}$$

If i write w.r.t the cartesian basis, $\hat{L}_x = y\left(-i\hbar\frac{\partial}{\partial z}\right) + z\left(i\hbar\frac{\partial}{\partial y}\right)$.

If i write w.r.t Momentum representation $\quad \hat{L}_Y = z\left(-i\hbar\frac{\partial}{\partial x}\right) + x\left(i\hbar\frac{\partial}{\partial z}\right)$.

$\hookrightarrow \hat{L}_x = i\hbar\frac{\partial}{\partial p_y} \cdot P_z - i\hbar\frac{\partial}{\partial P_z} P_y$,

$\hat{L}_Y = i\hbar\frac{\partial}{\partial P_z} P_x - i\hbar\frac{\partial}{\partial P_x} P_z$.

Particle in 2D $\psi(x, y)$.

• Rotation about z axis, when will $\psi(x, y)$ be invariant?

In classical Mechanics $(x, y) \rightarrow (x', y')$
$\qquad$ General Representation

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$.

Small Rotation: $\theta \ll 1 \rightarrow$ linear order 'θ'

$$x' \simeq x + y\theta, \quad y' \simeq -x\theta + y.$$

What happens in hilbert space? $\rightarrow \psi(x,y) \longrightarrow \psi'(x',y')$

$$\rightarrow \psi'(x',y') = \psi'(x+y\theta, -x\theta+y)$$

$$= \psi(x,y) + y\theta \cdot \frac{\partial \psi(x,y)}{\partial x} + (-x\theta) \cdot \frac{\partial \psi(x,y)}{\partial y}$$

$$= \psi(x,y) + \theta\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]\psi(x,y).$$ [Taylor Expansion upto first order].

$$+ \theta\left[\frac{y\hat{P}_x - x\hat{P}_y}{-i\hbar}\right]\psi(x,y) = \psi(x,y) + \frac{\theta[-\hat{L}_z]}{-i\hbar}\psi(x,y)$$

$$\rightarrow \psi'(x',y') = \psi(x,y) + \frac{i\theta\hat{L}_z}{\hbar}\psi(x,y) \simeq e^{\frac{i\theta\hat{L}_z}{\hbar}}\psi$$

$$\Rightarrow [\hat{H}, \hat{L}_z] = 0.$$

Consequences of Non-Commuting operators.

- $\hat{H} \rightarrow$ (finite, ∞) $\hat{A}, \hat{B} \rightarrow$ measurement (two different variables). $\hookrightarrow$ Hermetian

$$\rightarrow [\hat{A}, \hat{B}] \neq 0 \longrightarrow \begin{bmatrix} \hat{A}|a_i\rangle = a_i|a_i\rangle \\ \hat{B}|b_j\rangle = b_j|b_j\rangle \end{bmatrix}$$

Def^n:

$$\langle\hat{A}\rangle = \langle\alpha|\hat{A}|\alpha\rangle, \quad \langle\hat{A}^2\rangle = \langle\alpha|\hat{A}^2|\alpha\rangle$$

$$\langle\hat{B}\rangle = \langle\alpha|\hat{B}|\alpha\rangle, \quad \langle\hat{B}^2\rangle = \langle\alpha|\hat{B}^2|\alpha\rangle.$$

$$\rightarrow \sigma_A^2 = \langle\hat{A}^2\rangle - \langle\hat{A}^2\rangle, \quad \sigma_B^2 = \langle\hat{B}^2\rangle - \langle\hat{B}\rangle^2.$$

- $\Delta\hat{A} \equiv \hat{A} - \langle\hat{A}\rangle\hat{\mathbb{I}}, \quad \Delta\hat{B} = \hat{B} - \langle\hat{B}\rangle\hat{\mathbb{I}}$

$$\rightarrow \hat{C} = \sigma_B(\Delta\hat{A}) + i\sigma_A(\Delta\hat{B}) \longrightarrow \text{Non Hermetian}$$

$$\rightarrow \hat{C}^\dagger\hat{C} = (\sigma_B\Delta\hat{A} + i\sigma_A(\Delta\hat{B}))^\dagger(\sigma_B\Delta\hat{A} + i\sigma_A(\Delta\hat{B})).$$

$$= (\sigma_B\Delta\hat{A} - i\sigma_A\Delta\hat{B})(\sigma_B\Delta\hat{A} + i\sigma_A(\Delta\hat{B}))$$

$$= \sigma_B^2(\Delta\hat{A})^2 + \underbrace{i\sigma_A\sigma_B(\Delta\hat{A}\Delta\hat{B}) - i\sigma_A\sigma_B(\Delta\hat{B}\Delta\hat{A})}_{1} + \sigma_A^2(\Delta\hat{B})^2.$$

$$= \text{''} + i\sigma_A\sigma_B(\hat{A}\Delta\hat{B} - \cancel{\langle\hat{A}\rangle\Delta\hat{B}} - \Delta\hat{B}\hat{A} + \cancel{\Delta\hat{B}\langle\hat{A}\rangle}) + \text{''}$$

$$\hat{C}^\dagger\hat{C} = \sigma_B^2(\Delta\hat{A})^2 + \sigma_A^2(\Delta\hat{B})^2 + i\sigma_A\sigma_B[\hat{A},\hat{B}]$$

$$\hat{C}\hat{C}^\dagger = \sigma_B^2(\Delta\hat{A})^2 + \sigma_A^2(\Delta\hat{B})^2 - i\sigma_A\sigma_B[\hat{A},\hat{B}]$$

- $\langle\alpha|\hat{C}\hat{C}^\dagger|\alpha\rangle = \langle\alpha|\hat{C}\,\hat{\mathbb{I}}\,\hat{C}^\dagger|\alpha\rangle$

$$= \int dx\,\langle\alpha|\hat{C}|x\rangle\langle x|\hat{C}^\dagger|\alpha\rangle$$

Let $\langle\alpha|\hat{C}|x\rangle = \beta$

$$\Rightarrow \int dx\,\langle\alpha|\hat{C}|x\rangle\left(\langle\alpha|\hat{C}|x\rangle\right)^* = \int dx\,|\beta|^2 \geq 0.$$

And similarly, $\langle\alpha|\hat{C}^\dagger\hat{C}|\alpha\rangle = \int dx\,|\beta|^2 \geq 0.$

$\rightarrow \langle\alpha|\hat{C}^\dagger\hat{C}|\alpha\rangle \geq 0.$

$\rightarrow \langle\alpha|(\Delta A)^2|\alpha\rangle\sigma_B^2 + \langle\alpha|(\Delta\hat{B})^2|\alpha\rangle\sigma_A^2 + i\sigma_A\sigma_B\langle\alpha|[\hat{A},\hat{B}]|\alpha\rangle \geq 0.$

- $\langle\alpha|(\Delta\hat{A})^2|\alpha\rangle = \langle\alpha|(\hat{A}-\langle\hat{A}\rangle\hat{\mathbb{I}})^2|\alpha\rangle$

$$= \langle\alpha|\hat{A}^2 + \langle\hat{A}\rangle^2 - 2\hat{A}\langle\hat{A}\rangle|\alpha\rangle$$

$$= \langle\alpha|\hat{A}^2|\alpha\rangle - 2\langle\alpha|\hat{A}|\alpha\rangle\cdot\langle\hat{A}\rangle + \langle\hat{A}\rangle^2\langle\alpha|\alpha\rangle.$$

$$= \langle\alpha|\hat{A}^2|\alpha\rangle - 2\langle\hat{A}\rangle^2 + \langle\hat{A}\rangle^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 = \sigma_A^2$$

$\rightarrow \sigma_A^2\sigma_B^2 + \sigma_B^2\sigma_A^2 + i\sigma_A\sigma_B\langle\alpha|[\hat{A},\hat{B}]|\alpha\rangle \geq 0.$

$\rightarrow 2\sigma_A^2\sigma_B^2 + i\sigma_A\sigma_B\langle\alpha|[\hat{A},\hat{B}]|\alpha\rangle \geq 0.$

$\rightarrow 2\sigma_A\sigma_B + i\langle[\hat{A},\hat{B}]\rangle \geq 0$ [since, $\sigma_A.\sigma_B > 0$].

$\rightarrow \boxed{\sigma_A\sigma_B \geq \frac{i}{2}\langle[\hat{A},\hat{B}]\rangle}$ → integrate with a similar expression derived via $\langle\alpha|\hat{C}\hat{C}^\dagger|\alpha\rangle \geq 0$.

$\rightarrow \boxed{\sigma_A\sigma_B \geq \frac{1}{2}\left|i\langle[\hat{A},\hat{B}]\rangle\right|}$ — Most General form of Heisenbury's Uncertainity principle.

- Thus, if Two operators do not commute with each other then one can not precisely measure the corresponding values simultoneously.

- (1) When will the relation vanish : $[\hat{A}\,\hat{B}]=0$ → Measurement in one ~~Disturbance~~ doesn't disturb disturbance in the other.
- (2) When will the uncertainity be minimum?

Ex: $\hat{A}=\hat{x}$, $\hat{B}=\hat{P_x}$ ⟶ $\langle[\hat{x},\hat{P_x}]\rangle = \langle i\hbar\rangle \rightarrow \boxed{\sigma_A\sigma_B \geq \frac{\hbar}{2}}$.

$\langle x|[\hat{x},\hat{P}]|\psi\rangle = \langle x|\hat{x}\hat{P}_x - \hat{P}_x\hat{x}|\psi\rangle$

$$= x\langle x|\hat{p}|\psi\rangle - \langle x|\hat{p}x|\psi\rangle.$$

$$= x(-i\hbar)\langle x|\frac{\partial}{\partial x}|\psi\rangle + i\hbar\langle x|\frac{\partial}{\partial x}|(x|\psi\rangle).$$

$$= \cancel{x(-i\hbar)\langle x|\frac{\partial}{\partial x}|\psi\rangle} + i\hbar\langle x|\frac{\partial}{\partial x}|\psi\rangle + \cancel{i\hbar x\langle x|\frac{\partial}{\partial x}|\psi\rangle}.$$

<u>For minimum Uncertainity</u> $\div$ $|\beta| = 0 \rightarrow \langle\alpha|\hat{C}|\alpha x\rangle = 0$.

$$\rightarrow \langle\alpha|\hat{C}^{\dagger}\hat{C}|\alpha\rangle = 0 \longrightarrow \begin{cases} \hat{C}|\alpha\rangle = 0|\alpha\rangle \\ \hat{C}^{\dagger}|\alpha\rangle = 0|\alpha\rangle \end{cases}$$

Freezing $(\hat{x}, \hat{p}) \rightarrow$ $\hat{C} = \sigma_p(\Delta\hat{x}) + i\sigma_x(\Delta\hat{p})$.

Quantum Harmonic Oscillator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\varphi = E\varphi(x).$$

$\downarrow$ Representation of in invariant manner.

$\downarrow$ S.L equation $\longrightarrow$ solution.

$\rightarrow \hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\cdot\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$, $\hat{a}^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\cdot\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$

$\rightarrow [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$, $[\hat{a}, \hat{a}^\dagger] = 1$, $\underline{\hat{N} = \hat{a}^\dagger\hat{a}}$.

$\rightarrow [\hat{H}, \hat{N}] = 0 \Rightarrow$ eigenstates of these two operators at the same time. $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{\hat{I}}{2}\right) = \hbar\omega\left(\hat{N} + \frac{\hat{I}}{2}\right)$

$\rightarrow \langle x|x'\rangle = \delta_{xx'} \longrightarrow$ orthonormal.

$\rightarrow \hat{H}|E_N\rangle = E_N|E_N\rangle$, $|E_N\rangle \propto |n\rangle$ and thus $\hat{N}|x\rangle = x|x\rangle$.

$\rightarrow \hat{a}|x\rangle = \sqrt{x}\,|x-1\rangle \neq \beta_a|x\rangle$

$\hat{a}^\dagger|x\rangle = \sqrt{x+1}\,|x+1\rangle \neq \beta_a^*|x\rangle$

$\}$ 4 operators: $\hat{H}$, $\hat{N}$ > Same eigen vectors $\not\rightarrow$ eigen vectors of $\hat{a}^\dagger, \hat{a}$

$\rightarrow \boxed{\text{"}n\text{" and } |n\rangle\ ?}$

- $\hat{a}\hat{a}|n\rangle = \hat{a}\sqrt{n}|n-1\rangle = \sqrt{n(n-1)}\,|n-2\rangle$
- $(\hat{a})^m|n\rangle = \sqrt{n(n-1)(n-2)\cdots(n-m+1)}\,|n-m\rangle$
- $(\hat{a}^\dagger)^m|n\rangle = \sqrt{(n+1)(n+2)\cdots(n+m)}\,|n+m\rangle$

$\hat{N}|n\rangle = n|n\rangle$

$\underbrace{\langle n|\hat{N}|n\rangle}_{\geq 0} = n\langle n|n\rangle = n \rightarrow n \geq 0$.

- n can be non-negative. What is the minimum value that n can take?

$\rightarrow \hat{H}|E_n\rangle = E_n|E_n\rangle \rightarrow \langle E_n|\hat{H}|E_n\rangle = E_n \geq 0$.

- $E_n = \hbar\omega\left\langle E_n\left|\hat{a}^\dagger\hat{a} + \frac{\hat{I}}{2}\right|E_n\right\rangle = \hbar\omega\langle E_n|\hat{a}^\dagger\hat{a}|E_n\rangle + \frac{1}{2}\hbar\omega$
- $E_n = \underbrace{\hbar\omega\left|\hat{a}|E_n\rangle\right|^2}_{\geq 0} + \frac{1}{2}\hbar\omega \geq 0$.

Thus $\underline{E_n \geq \frac{1}{2}\hbar\omega}$

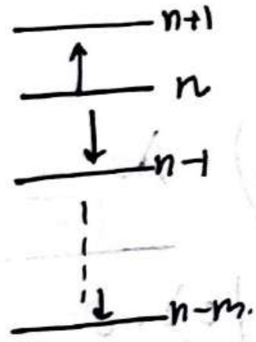
$\hat{N}|n\rangle \rightarrow$ we know $\boxed{n \geq 0}$

$\hat{H}|E_n\rangle \rightarrow$ $E_n \geq \frac{1}{2}\hbar\omega$

$\rightarrow$ what values can n take?

$\rightarrow \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\rightarrow a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$



- In prinincipal, on applying $\hat{a}$ i can go to a State in which $n-m<0$.
  - $\hookrightarrow$ incossistent with $n \geq 0$.
- The only way I can Ensure that $n-m$ never becomes -ve. is that $\boxed{n \in \mathbb{Z}}$.

$|3\rangle \rightarrow |2\rangle \rightarrow |1\rangle \rightarrow \boxed{\hat{a}|0\rangle = 0}$ $\rightarrow$ because lowering stops at 0!

- Thus n is a non-negative integer

and my states are $|0\rangle, |1\rangle, |2\rangle, \ldots, |m\rangle$, $m \in \mathbb{Z}^{+}$

- Also we are solving for non-negative values of E_n. for stable Eq^{m}.

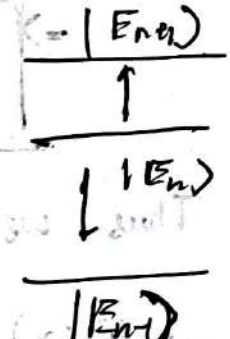
$\rightarrow \hat{N}|n\rangle = n|n\rangle$

$\rightarrow \hat{H}\,\hat{a}|E_n\rangle = \hbar\omega\left(n-\frac{1}{2}\right)\hat{a}|E_n\rangle$

$\hat{H}\,\hat{a}^{\dagger}|E_n\rangle = \hbar\omega\left(n+\frac{3}{2}\right)\hat{a}^{\dagger}|E_n\rangle$

$\rightarrow \hat{H}|E_{n-1}\rangle = \hbar\omega\left(n-\frac{1}{2}\right)|E_{n-1}\rangle = \hbar\omega\left(n+\frac{1}{2}-1\right)|E_{n-1}\rangle$

$\rightarrow \hat{H}|E_{n+1}\rangle = \hbar\omega\left(n+\frac{1}{2}+1\right)|E_{n+1}\rangle$



$\hat{a}|E_n\rangle \rightarrow \hbar\omega(\downarrow)\,|E_n\rangle \longrightarrow$ anhilation operator.

$\hat{a}^{\dagger}|E_n\rangle \longrightarrow \hbar\omega(\uparrow)\,|E_n\rangle \longrightarrow$ Creation operator.

($\hat{N}$: Number operator), ($\hat{H}$: Energy operator)

Movement of ~~Photons~~ Photons b/w states (interaction b/w e^- and photon).

- $\hat{a}$ is an operator which makes one e^- go from one state to another by aquiring energy from an External Agency. [Ex. Photon] It thus doesn't violate Energy condition

Till now, $|0\rangle, |1\rangle, ---, |n\rangle \longrightarrow$ orbitatory Representation.

we take it in x-representation for convinience.

$$\langle x|0\rangle = \varphi_0(x) \qquad \longrightarrow \langle x|\hat{a}|0\rangle = 0$$

$$\langle x|n\rangle = \varphi_n(x) \qquad \hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\lambda} + \frac{i\lambda\hat{P}}{\hbar}\right) \quad \lambda = \left(\frac{\hbar}{m\omega}\right)^{1/2}$$

$$\rightarrow \langle x|\frac{\hat{x}}{\lambda} + \frac{i\lambda}{\hbar}\hat{P}|0\rangle = 0 \longrightarrow \langle x|\frac{\hat{x}}{\lambda} + \lambda\frac{\partial}{\partial x}|0\rangle = 0.$$

$$\rightarrow \frac{x\,\varphi_0(x)}{\lambda} + \lambda\frac{\partial}{\partial x}\varphi_0(x) = 0 \longrightarrow \varphi_0(x) = Ne^{-x^2/2\lambda^2}$$

$$\rightarrow \int_{-\infty}^{\infty}|\varphi_0(x)|^2 dx = 1 \longrightarrow (N : \text{normalization constant})$$

$$\langle x|1\rangle = \varphi_1(x) = \langle x|a^+|0\rangle = \frac{1}{\sqrt{2}}\langle x|\frac{\hat{x}}{\lambda} - \frac{i\lambda}{\hbar}\hat{P}|0\rangle$$

$$= \frac{1}{\sqrt{2}}\langle x|\frac{\hat{x}}{\lambda} - \lambda\frac{\partial}{\partial x}|0\rangle = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \lambda\frac{\partial}{\partial x}\right)(\varphi_0(x))$$

$$\Rightarrow \boxed{\varphi_1(x) = \frac{1}{\sqrt{2}}\left(\frac{x\,\varphi_0(x)}{\lambda} - \lambda\frac{\partial\varphi_0(x)}{\partial x}\right)}.$$

Thus, we can iteratively calulate the wave function of any state

$$\rightarrow \varphi_n(x) \propto H_n(x) \cdot \varphi_0(x) \quad (\text{Hermite polynomial of } x)$$

- solving the differential equation is very lengthy. Now we can figure out any wave function just by knowing the ground state w.f.

$E_0 = \frac{1}{2}\hbar\omega$

Due to symmetric Potential

$E_0 \rightarrow$ Even, symmetric w.r.t $-x \rightarrow +x$.

$E_1 \rightarrow$ odd, antisymmetric w.r.t $-x \rightarrow +x$

$\varphi_n(x) \rightarrow$ even : $n = 0, 2, \ldots$

$\varphi_n(x) \rightarrow$ odd : $n = 1, 3, 5, \ldots$

Probability of finding particle beyond Classical turning point is not zero.

$\langle \hat{x} \rangle = \langle n | \hat{x} | n \rangle$ Let $\hat{x} = c_0 \hat{a} + c_1 \hat{a}^\dagger$
(we'll fix the c_1, c_0 later).

$\langle \hat{x} \rangle = \langle n | c_0 \hat{a} + c_1 \hat{a}^\dagger | n \rangle$

$= \tilde{c}_0 \langle n | \hat{a} | n \rangle + \tilde{c}_1 \langle n | a^\dagger | n \rangle = \underline{0}$.

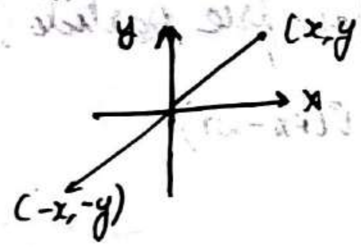
. $(\hat{c}, \hat{c}^\dagger) \rightarrow (\hat{a}, \hat{a}^\dagger)$

$\rightarrow (x, y, p_x, p_y)$
$\downarrow$
$(r, \theta, p_r, p_\theta)$.

: minimum V state . stationary.

. Symmetries in QM → space-time → time trans ($\hat{H}$)
→ space trans ($\hat{P}_x$)
→ Rotation ($\hat{L}_z$)
→ Discrete

Classical Mechanics



: Discrete transformation do not have a corresponding Generation

$x \rightarrow -x$.

$f(x) \rightarrow$ even
$-f(x) \rightarrow$ odd
$f(x) \longrightarrow f(-x)$.

Hilbert Space : parity operator Def^n

$\hat{\Pi} |x\rangle = |-x\rangle$.
$\hat{\Pi} |p\rangle = |-p\rangle$

$\left\{ \begin{array}{l} \psi(x) \rightarrow \psi(-x) \\ \psi(p) \rightarrow \psi(-p) \end{array} \right\}$

Properties : (i) Is it Hermetian? $\rightarrow \langle x' | \hat{\Pi} | x \rangle = \langle x' | -x \rangle = \delta(x' + x)$

$\rightarrow \langle x' | \hat{\Pi} | x \rangle^* = \delta(x' + x)$.

→ Hence, hermetian

(ii) $\hat{\Pi}^2 = \mathbb{I} \rightarrow$ it is it's own inverse. $\rightarrow \hat{\Pi}(\hat{\Pi}|x\rangle) = \hat{\Pi}(|-x\rangle) = |x\rangle$.

(iii) $\hat{\Pi}$ is a unitary operator.

(iv) eigen value of $\hat{\Pi}$? → eigen values +1 or −1 $\rightarrow \hat{\Pi}|+\rangle = 1|+\rangle$
$\hat{\Pi}|-\rangle = -1|-\rangle$

(v) $[\hat{\Pi}, \hat{p}^2]|p\rangle = (\hat{\Pi}\hat{p}^2 - \hat{p}^2\hat{\Pi})|p\rangle = p^2|-p\rangle - p^2|-p\rangle = 0$.

$[\hat{\Pi}, \hat{x}]|x\rangle = (\hat{\Pi}\hat{x} - \hat{x}\hat{\Pi})|x\rangle = x|-x\rangle - (-x)|-x\rangle$
$= 2x|-x\rangle$.

$[\hat{\Pi}, \hat{x}^2]|x\rangle = (\hat{\Pi}\hat{x}^2 - \hat{x}^2\hat{\Pi})|x\rangle = x^2|-x\rangle - x^2|-x\rangle = 0$.

$[\hat{\pi}, \hat{H}] = [\hat{\pi}, V(\hat{x})] \neq 0.$ → consequences?

- **Note**: $[\hat{\pi}; V(\hat{x})] = 0$ if $V(\hat{x})$ is even function.

→ Harmonic oscillator → $[\hat{\pi}, \hat{x}^2] = 0$ but $[\hat{\pi}, \hat{P}] \neq 0$.
Thus the wave functions have a very special parity with half of the w.f's being even ($n=1,3,5$) and half of the wave function being odd ($n=2,4,6$).

→ Free Particle: $[\hat{\pi}, \hat{H}] = 0$
$[\hat{P}, \hat{H}] = 0.$ $\Big\}$? $\left(\hat{H} = \frac{\hat{P}^2}{2m}\right)$.

→ why in 1D, no other system has degeneracy other than free particle?

$\frac{\hat{p}^2}{2m}|\psi\rangle = E|\psi\rangle$, $\psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx-\omega t)}$

→ $E(k) = \frac{\hbar^2 k^2}{2m}$, $p = \hbar k$
$= \frac{p^2}{2m}$

- $e^{i(kx+\omega t)}$ +ve momentum modes, $P>0$.
- $e^{-i(kx+\omega t)}$ −ve momentum modes $P<0$.

$P<0,\ P>0 \longrightarrow E = \frac{p^2}{2m}$

- $|E_1^{(1)}\rangle = |k\rangle_{\sqrt{\frac{2mE}{\hbar}}}$, $|E_1^{(2)}\rangle = |k\rangle_{-\sqrt{\frac{2mE}{\hbar}}}$ but $\underline{[\hat{\pi}, \hat{P}] \neq 0}$

$|E_1^{(1)}\rangle$ and $|E_1^{(2)}\rangle$ are the eigen states of $\hat{H}$ and $\hat{p}$ but they are not the eigen states of $\hat{\pi}$.

→ Construct $|E_1+\rangle = \frac{1}{\sqrt{2}}\left(|E_1^{(1)}\rangle + |E_1^{(2)}\rangle\right)$

$|E_1-\rangle = \frac{1}{\sqrt{2}}\left(|E_1^{(1)}\rangle - |E_1^{(2)}\rangle\right)$

Are these new states eigenstates of the parity operator? ($\hat{\pi}$)

- Thus i'm trying to construct simultaneous eigenstate of $\hat{H}$, $\hat{\pi}$ (Like i constructed simultaneous eigen states of $\hat{H}$ and $\hat{p}$ ÷ both have the same representation).

- $|E_1+\rangle$, $|E_1-\rangle$ will not be stationary states in terms of $\hat{\pi}$, $\hat{P}$ rep.

• $\hat{A}, \hat{B}, \hat{H}$. If $[\hat{A}, \hat{H}] = 0$, $[\hat{B}, \hat{H}] = 0$ but however $[\hat{A}, \hat{B}] \neq 0$ then we will always have a degenerate state.

• Free particle / Harmonic Boscillators are physicists construction.

• common eigen states only possible b/w a set of two operators. ✓.

• Recall: Last week we constructed $[\hat{c}, \hat{c}^\dagger] \rightarrow$ minimum uncertainity

Then for harmonic oscillator $\rightarrow [\hat{a}, \hat{a}^\dagger] \longrightarrow$ Ground state $\equiv$ Gaussian | $\varphi \sim$ Gaussian, $\varphi(x) \sim e^{-\frac{x^2}{4\sigma_x^2}}$.

And $\boxed{\hat{a}|0\rangle = 0} \rightarrow$ minimum uncertainity state.

$\leftarrow$ not min^m uncertainity

$\swarrow$ minimum uncertainity. | $\hat{a} = c_0 \hat{x} + i c_1 \hat{p}$

• We ~~Rel~~ Realize minimum uncertainity states ~~via Electro~~ in the Lab via Electromagnetic pulses (Superposition of Harmonic oscillators).

Stationary States v/s Non-Stationary states $\rightarrow$ Stationary states

by def^n: $\hat{H}|E_n\rangle = E_n|E_n\rangle$ ($|E_n\rangle \rightarrow$ is an eigen state of the hamiltonian)

Let's say $\hat{\varphi}(\hat{x}, \hat{p})$ $\longrightarrow$ TISE $\longleftarrow$ TDSE.

$\rightarrow \langle \hat{\varphi}(\hat{x}, \hat{p}) \rangle = \langle \psi_n, t | \hat{\varphi} | \psi_n, t \rangle$ $\boxed{i\hbar \frac{\partial}{\partial t} \psi_n(x,t) = \hat{H} \psi_n(x,t)}$

• $\hat{H}(\hat{p}, \hat{x}) \longrightarrow$ no time dependence then,

$|\psi_n, t\rangle = |\phi_n, t\rangle |E_n\rangle \rightarrow \boxed{i\hbar \frac{d|\phi_n, t\rangle}{dt} = E_n |\phi_n, t\rangle}$

$\Rightarrow |\phi_n, t\rangle = e^{-iE_n t/\hbar} |\phi_n, 0\rangle$.

t, x $\nearrow$ one pt. at different times.
$\searrow$ one time, multiple operations at different points.

$= \langle \psi_n; t | \hat{\varphi} | \psi_n; t \rangle = e^{iE_n t/\hbar} \langle E_n | \hat{\varphi} | E_n \rangle e^{-iE_n t/\hbar}$

$= \underline{\langle E_n | \hat{\varphi} | E_n \rangle}$.

$\Psi(x,y) \rightarrow V(x,y) = x^2 + y^2 \longrightarrow$ x, y don't talk to each other (no cross term).

$$\left.\begin{array}{l}\rightarrow \Psi_x(x,0) \\ \Psi_y(0,y)\end{array}\right] -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi(x,y)}{\partial x^2} + \frac{\partial^2\Psi(x,y)}{\partial y^2}\right) + (x^2+y^2)\Psi(x,y) = E\,\Psi(x,y)$$

$$\Psi(x,y) = \Psi_x(x)\,\Psi_y(y).$$

18/10/23

1D + one Particle $\rightarrow$ 2D/3D

1D + one Particle $\rightarrow$ Multiparticles

1D - 2particles $\leftrightarrow$ 2D - one particle.

HO

$$H_1 = \frac{P_1^2}{2m} + \frac{1}{2}m\omega_1^2 x_1^2 + \frac{P_2^2}{2m} + \frac{1}{2}m\omega_2^2 x_2^2$$ 1D two uncoupled harmonic oscillator.

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2)$$ [2D(x, y) → Hamiltonian isotrope oscillator]

- $$\hat{H} = \frac{\hat{P}_+^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_+^2 + \frac{\hat{P}_-^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_-^2$$

$\hat{H} = \underline{H_+ \otimes H_-}$

(x_+, x_-, P_+, P_-)

* $[\hat{P}_+, \hat{x}_+] = -i\hbar$, $[\hat{P}_-, \hat{x}_-] = -i\hbar$

* $[\hat{x}_+, \hat{x}_-] = [\hat{P}_+, \hat{P}_-] = 0$

* $[\Delta P_+ \Delta P_- = 0 \;//\; \Delta P_+ \Delta X_- = 0]$.

If changes in one do not affect the other ⇓ Uncoupled

Thus for this case $\boxed{|\Psi\rangle = |\Psi_+\rangle \otimes |\Psi_-\rangle}$

* operations i do on one particle will have no effect on the other particle. $\boxed{|\Psi\rangle = |\Psi_+, \Psi_-\rangle}$

think ideal gas: non-interacting Point Particle

For 1D (2 Harmonic oscillators) → Assumed to be non-interacting! (Realistically bad Assumption).

$-\infty$ ———— x_1 | x_2 ———— $+\infty$

seperated enough to Assume non interaction.

$|\Psi(x_+, x_-)\rangle \neq |\hat{\Psi}(\hat{x}_+, \hat{x}_-)\rangle$

measured in these coordinates

use the physical coordinates in which you measure stuff

* If you swith on interaction, we need to change basis, diagonalize the hamiltonian. However, for the uncoupled case, no change of basis is required.

$(\hat{x}_+, \hat{p}_+) \longleftrightarrow (\hat{a}_+, \hat{a}_+^\dagger)$ and $(\hat{x}_-, \hat{p}_-) \rightleftarrows (\hat{a}_-, \hat{a}_-^\dagger)$

• [Ladder opperators for each Hilbert space].

$$\boxed{\hat{a}_\pm = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x}_\pm + \frac{i\hat{P}_\pm}{m\omega}\right)}, \quad \boxed{\hat{a}_\pm^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x}_\pm - \frac{i\hat{P}_\pm}{m\omega}\right)}$$

$\rightarrow [\hat{a}_+, \hat{a}_+^\dagger] = 1 = [\hat{a}_-, \hat{a}_-^\dagger]$, $\hat{N}_+ = \hat{a}_+^\dagger \hat{a}_+$, $\hat{N}_- = \hat{a}_-^\dagger \hat{a}_-$.

$\rightarrow [\hat{a}_+, \hat{a}_-] = [\hat{a}_+^\dagger, \hat{a}_-^\dagger] = \text{---} = 0.$

$$\rightarrow \boxed{\hat{H} = \left(\hat{a}_+^\dagger a_+ + \frac{1}{2}\right)\hbar\omega + \left(\hat{a}_-^\dagger \hat{a}_- + \frac{1}{2}\right)\hbar\omega}$$

$\Rightarrow \boxed{|\Psi\rangle = |n_+, n_-\rangle.}$

* $\hat{a}_+|0,0\rangle = |0,0\rangle$
$\hat{a}_-|0,0\rangle = |0,0\rangle$
$\hat{a}_+^\dagger|0,0\rangle = |1,0\rangle.$
$\hat{a}_-^\dagger|0,0\rangle = |0,1\rangle$

$\underline{|n_+, n_-\rangle = C_0 (\hat{a}_+^\dagger)^n (\hat{a}_-^\dagger)|0,0\rangle}$

Define: $\hat{J}_+ = \hbar\, \hat{a}_+^\dagger \hat{a}_-$ | $\hat{J} = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$
$\hat{J}_- = \hbar\, a_-^\dagger \hat{a}_+$ | $= \frac{\hbar}{2}(\hat{N}_+ - \hat{N}_-).$

$[\hat{J}, \hat{J}_+] = \hat{J}\hat{J}_+ - \hat{J}_+\hat{J} = \hbar\,\cancel{(\hat{a}_+^\dagger \hat{a}_+ \hat{a}_+^\dagger \hat{a}_- - a_+^\dagger \hat{a}_- \hat{a}_-^\dagger \hat{a}_-)}$
$\cancel{\hbar(a_+^\dagger(\hat{a}_+\hat{a}_+^\dagger - \hat{a}_+\hat{a}_-^\dagger)\hat{a}_-)}$

$= \hbar(\hat{a}_+^\dagger \hat{a}_+ \hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_- \hat{a}_+^\dagger \hat{a}_- - \hat{a}_+^\dagger \hat{a}_- \hat{a}_+^\dagger \hat{a}_+ + \hat{a}_+^\dagger \hat{a}_- \hat{a}_-^\dagger \hat{a}_-)$

$= \hbar\left(\hat{a}_+^\dagger \left(\underbrace{\hat{a}_+\hat{a}_+^\dagger - \hat{a}_-\hat{a}_+^\dagger - \hat{a}_-\hat{a}_+^\dagger + \hat{a}_-\hat{a}_-^\dagger}_{(\hat{a}_+ - \hat{a}_-)a_+^\dagger - \hat{a}_-(\hat{a}_+^\dagger - \hat{a}_-^\dagger)}\right)\hat{a}_-\right).$

Particles in 2 and 3-D

Cartesian $\rightarrow -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y) = E\psi(x,y)$.

$\Rightarrow V(x,y,z) \neq V(x)+V(y)+V(z)$ — Then what?

• Ex: $V(r) = \frac{C_0}{r} = \frac{C_0}{\sqrt{x^2+y^2+z^2}} \neq V(x)+V(y)+V(z)$.

* 2D — Polar ($x = r\cos\theta$, $y = r\sin\theta$).
  3D — spherical polar ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$).
  — cylinderical ($x = r\cos\theta$ $y = r\sin\theta$ $z = z$).

K.E operator : $-\frac{\hbar^2\nabla^2}{2m}$ ‖ 2D :- $-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)$

3D :- spherical polar : $-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}\right)$.

2D — two Examples.

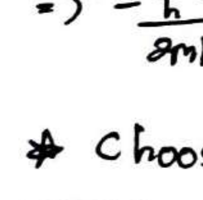
1) Particle on a Ring — Particle of mass 'm' at a fixed radial distance 'R'.

$$-\frac{\hbar^2}{2m}\left(\cancel{\frac{\partial^2}{\partial r^2}}^{\,0}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right)\psi(r,\theta) = E\psi(r,\theta).$$

$$\Rightarrow -\frac{\hbar^2}{2mR^2}\cdot\frac{\partial^2\psi(\theta)}{\partial\theta^2} = E\psi(\theta) \Rightarrow \boxed{\psi(\theta) = c_1 e^{ik\theta}+c_2 e^{-ik\theta}}$$

I can choose one of them. Depending on Boundary conditions.

* Choose : $\psi(\theta) = c_1 e^{ik\theta}$. where $\boxed{k = \sqrt{\frac{2mR^2E}{\hbar^2}}}$ ✓



$\underline{\psi(\theta) = \psi(\theta+2\pi n)}$ $c_1, c_2 \rightarrow$ normalization constants.

$\psi(\theta)$ has to be single valued.

$n = \pm1, \pm2, ---$ so on and so forth.

$e^{i\theta} = e^{ik(\theta+2\pi n)}$
$1 = e^{ik\theta}e^{i2\pi nk}$
$i2\pi nk$ ~~[illegible]~~
$2\pi nk \in$ cond

$E_n = \frac{\hbar^2k^2}{2mR^2}$ • $e^{ik(\theta+2\pi)} = e^{ik(\theta)} \Rightarrow e^{ik2\pi} = 1$

which is only true if $k = 0, \pm1, \pm2,$ —

$\rightarrow \left(E_k = \frac{\hbar^2k^2}{2mR^2}\right)$ ✓ $\rightarrow$ [

Polar $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right)$

E_n & r^2

$V(r) = \begin{cases} 0 & r \le R \\ \infty & r > R \end{cases}$

• Potential has radial dependance
No θ dependance

$\rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta) + V(r)\psi = E\psi$. (Assure that $V(r)$ ↳ no 'θ' dependance)

$\Rightarrow \left(V(r) = \frac{1}{2}m\omega^2 r^2\right)$

General 2D Lagrangian : $\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot\theta^2) - V(r)$

$\begin{cases}(\ddot{r} - r\dot\theta^2) = -\frac{1}{m}\frac{\partial V}{\partial r} \\ (mr^2\dot\theta)^{\cdot} = 0\end{cases}$ $mr^2\dot\theta = \text{const} = l \rightarrow \boxed{\dot\theta = \frac{l}{mr^2}}$ Angular momentum $\downarrow$ z-axis

$(E, l) \rightarrow$ constants of motion $L_z = xp_y - yp_x$

$\rightarrow \hat{H} = -\frac{\hbar^2\nabla^2}{2m} + V(r) = -\frac{\hbar^2}{2m}\cdot\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + V(r)\right)$

$\Rightarrow \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x =$ ~~...~~ $r\cos\theta\left(-i\hbar\left(\cancel{r\cos\theta}\cdot\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\right) - r\sin\theta\left(-i\hbar\left(\sin\theta\frac{\partial}{\partial r}\ldots\right)\right)$

$y = r\sin\theta$ To be continued.

~~$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$~~ $\hat{p}_x = -i\hbar\left(\frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial\cos\theta}{\partial x}\cdot\frac{\partial}{\partial\cos\theta}\right)$

$\Rightarrow \hat{H} \rightarrow$ time independent $\mid \hat{H} = -\frac{\hbar^2}{2m}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] + V(r)$

$\theta \rightarrow \theta' = \theta + \alpha \rightarrow \hat{H}$ is invariant.

↳ corresponding operator that keeps the states invariant

$\hat{R}_\alpha|\psi\rangle_\theta \rightarrow |\psi\rangle_{\theta+\alpha} \Rightarrow \hat{R}_\alpha = \exp\left[\alpha\frac{\partial}{\partial\theta}\right]$ → $\hat{H}$ is invariant thus it must have a unitary operator corresponding to this.

$= \exp\left[\frac{i\alpha\hat{L}_z}{\hbar}\right]$

$\Rightarrow [\hat{H}, \hat{L}] = 0 \rightarrow$ They share the same eigen states.

$\rightarrow \hat{L}|\psi\rangle = l|\psi\rangle$ $\hat{L} \rightarrow$ angular momentum w.r.t the z-axis.

$\rightarrow -i\hbar\frac{\partial}{\partial\theta}\psi(\theta) = l\psi(\theta)$ ↳ $\psi(\theta) = c_0 e^{+il\theta/\hbar}$

Single valued $\varphi(\theta+2\pi) = \varphi(\theta) : e^{il\theta/\hbar} \cdot e^{i2\pi l/\hbar} = e^{il\theta/\hbar}$

$\rightarrow \hat{L}|\psi\rangle = n\hbar|\psi\rangle$ → Angular momentum has to be quantized.

H_0 → Angular momentum.

$$\rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\Psi(r,\theta) + \frac{1}{r^2}\frac{\partial^2\Psi(r,\theta)}{\partial^2\theta}\right] + V(r)\Psi(r,\theta) = E\Psi(r,\theta)$$

$$\boxed{\Psi(r,\theta) = R(r)\,\varphi(\theta)}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d}{dr}\left(r\,dr\,R(r)\right)\right] + \frac{1}{2m}\frac{n^2\hbar^2}{r^2}R(r) + V(r)r = ER(r)$$

$$\rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right] + \left[\frac{n^2\hbar^2}{2mr^2} + V(r)\right]R = ER(r) \quad \boxed{V_{eff}(r)}$$

Region 2] $\varphi = 0$

Region 1] $-\frac{\hbar^2}{2m}\frac{d^2R(r)}{dr^2} - \frac{\hbar^2}{2m}\frac{1}{r}\frac{dR}{dr} + \frac{n^2\hbar^2}{2mr^2}R = ER$ (Bessel Function)

$\varphi(\theta) = c_0 e^{in\theta}$ $\quad n = 0, \pm 1, \pm 2, \ldots$

$\cdot R(\rho) = c_0 J_m(\rho) + c_1 Y_m(\rho)$ $\qquad \rho = kr$; $E = \frac{\hbar^2k^2}{2m}$ $\quad n = 0, 1, \ldots \infty$

. For non trivial soln: $n = 1, 2, \ldots$

$R(\rho) = c_0 J_m(\rho)$ → Plot these in mathematica. → Density Functions

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Schwinger oscillator:

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2x_1^2 + \frac{1}{2}m\omega^2x_2^2 + \lambda x_1 x_2$$

what happens if there is a coupling.

$m\ddot{x}_1 = -k_0x_1 + k_1(x_2 - x_1)$
$m\ddot{x}_2 = -k_0x_2 + k_1(x_1 - x_2)$ $\quad \Big| \quad m\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = K\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ where $K = \begin{pmatrix}k_0+k_1 & -k_1\\ -k_1 & k_0+k_1\end{pmatrix}$

$x(t) = \begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}e^{i\omega t}$ $\quad \Big| \quad \underline{Kx_0 = m\omega^2x_0}$ $\quad \omega_1 = \sqrt{\frac{k_0}{m}}$ $\quad \omega_2 = \sqrt{\frac{k_0+2k_1}{m}}$

$\bar{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\bar{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$.

$$H = \frac{1}{2}[P_1^2 + P_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2] \quad (m=1)$$

$$= \frac{1}{2}\sum_{i=1}^{2} P_i^2 + \frac{1}{2}\sum x_i^T k_{ij} x_j \quad \text{where } [k_{ij}] = \begin{pmatrix} k_0+k_1 & -k_1 \\ -k_1 & k_0+k_1 \end{pmatrix}$$

<u>C Mechanics</u> : $(x_1, x_2, P_1, P_2) \to (x_+, x_-, P_+, P_-)$.

$$x_+ = \frac{x_1 + x_2}{\sqrt{2}}, \quad x_- = \frac{x_1 - x_2}{\sqrt{2}} \longrightarrow \boxed{\begin{aligned} \ddot{x}_+ + \omega_+^2 x_+ &= 0 \\ \ddot{x}_- = \omega_-^2 x_- &= 0 \end{aligned}}$$

• $(x_1, x_2) \to \mathcal{H} \to$ has info about w.f of particle 1 and 2.

$\Psi(x_1, x_2) \longleftarrow$ w.f of particle 1 and 2.

$\hookrightarrow x_+(t), x_2(t) \to \omega$

$\hookrightarrow x_1(t), x_2(t) \to \omega$

$(m=1)$ $\hat{H} = \frac{1}{2}[\hat{P}_1^2 + \hat{P}_2^2 + k_0(\hat{x}_1^2 + \hat{x}_2^2) + k_1(\hat{x}_1 - \hat{x}_0)^2]$

$$\to \left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + k_0(x_1^2 + x_2^2) - k_1(x_1 - x_2)^2\right]\Psi(x_1, x_2)$$

$$= E\,\Psi(x_1, x_2).$$

Assume $\Psi(x_1, x_2)$ to be $\varphi(x_1)\cdot\phi(x_2)$ and see if it leads to any problem on substitution.

$$\to -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2}\varphi(x_1)\cdot\phi(x_2)\right) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2}\phi(x_2)\cdot\varphi(x_1)$$

$$+ k_0(x_1^2 + x_2^2)\varphi(x_1)\cdot\phi(x_2) - k_1(x_1^2 + x_2^2 - 2x_1x_2)\cdot\varphi(x_1)\phi(x_2)$$

This term is problem

$$= E\,\varphi(x_1)\phi(x_2)$$

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + (k_0 - k_1)x_1^2\right]\varphi_1(x_1)\cdot\phi_2(x_2)$$

Does not Allow Seperation of variables

$$+ \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + (k_0 - k_1)x_2^2\right]\varphi(x_1)\cdot\phi(x_2) + \boxed{2k_1 x_1 x_2 \varphi(x_1)\phi(x_2)}$$

$$= E\,\varphi(x_1)\phi(x_2)$$

* We can only solve this eq^n if we differentiate our equation to third order equations (which is a mess) while getting rid of the other terms.

OR CHANGE OF BASIS!! $\to$ eliminates coupling.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \to \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} \quad \left[U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right]$$

$$K_D = UKU^T = \begin{pmatrix} K_0 & 0 \\ 0 & K_0 + 2K_1 \end{pmatrix} \to \boxed{\begin{pmatrix} x_+ \\ x_- \end{pmatrix} = U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \boxed{\begin{pmatrix} p_+ \\ p_- \end{pmatrix} = U \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}$$

- If I can diagonalize $\underline{K}$ then my Hamiltonian is diagonalizable.

{In new coordinates (And NEW FREQUENCIES) : $H = \frac{1}{2}[P_+^2 + P_-^2 + \omega_+^2 x_+^2 + \omega_-^2 x_-^2]$}

- 25/10/23 $H = \frac{1}{2}[p_1^2 + p_2^2 + \underbrace{K_0}_{\text{Spring constant}}(x_1^2 + x_2^2) + \underbrace{k_1(x_1 - x_2)^2}_{\text{Interaction}}]$ $\quad (k_1 > 0)$

$$K_D = U^{-1} K U = \begin{pmatrix} \overset{\omega_+^2}{K_0} & 0 \\ 0 & \underset{\omega_-^2}{K_0 + 2K_1} \end{pmatrix} \quad \Big\| \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; ; \; \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Big/\!\!\Big/ \begin{pmatrix} p_+ \\ p_- \end{pmatrix} = U \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

- $(x_{1,2}, p_{1,2}) \longrightarrow (x_\pm, p_\pm)$ (Canonical Transformation) $\Big|$ $H = \frac{1}{2}[\underbrace{\hat{p}_+^2 + \hat{p}_-^2 + \omega_+^2 x_+^2 + \omega_-^2 x_-^2}_{\text{uncoupled in } (x_\pm;\, p_\pm)}]$.

$$a_\pm = \sqrt{\frac{\omega_\pm}{2\hbar}} \left(\hat{x}_\pm - \frac{i}{\omega_\pm} \hat{p}_\pm \right) \qquad \hat{a}_\pm |0,0\rangle = 0 \; ; \; (a_+^\dagger)^n |0,0\rangle \propto |n_+, 0\rangle$$

$$(a_-^\dagger)^n |0,0\rangle \propto |0, n_-\rangle$$

$$a_\pm^\dagger = \sqrt{\frac{\omega_\pm}{2\hbar}} \left(\hat{x}_\pm + \frac{i}{\omega_\pm} \hat{p}_\pm \right) \qquad \Rightarrow |n_+, n_-\rangle = \left(\frac{(a_+^\dagger)^{n_+}}{(n_+!)^{1/2}} \right) \left(\frac{(a_-^\dagger)^{n_-}}{(n_-!)^{1/2}} \right) |0,0\rangle$$

Now, $|x_+, x_-\rangle = |\overset{H_+}{x_+} \otimes \overset{H_-}{x_-}\rangle$
↳ cross product of Two Hilbert spaces.

$$\to \psi(x_+, x_-) = \left(\frac{\omega_+}{\pi\hbar} \right)^{1/2} \exp\left(-\frac{\omega_+}{2\hbar} x_+^2 \right) \cdot \left(\frac{\omega_-}{\pi\hbar} \right)^{1/2} \exp\left(-\frac{\omega_-}{2\hbar} x_-^2 \right)$$

$$\hookrightarrow \left[\psi(x_1, x_2) = \frac{(\omega_+ \omega_-)^{1/2}}{\pi\hbar} \exp\left(-\frac{\omega_+ + \omega_-}{4\hbar} (x_1 + x_2)^2 \right) \right] \; \text{///}$$

so, if $\underline{k_1 = 0}$ $\psi(x_1, x_2) \propto \exp(-c_0 x_1^2) \exp(-c_0 x_2^2) \propto |x_1\rangle \otimes |x_2\rangle$.

if $\underline{k_1 \neq 0}$ $\psi(x_1 x_2) \not\propto |x_1\rangle \otimes |x_2\rangle$ $\boxed{\text{Entangled States}}$ Doesn't allow seperation
$\exp[-c_1 x_1 x_2]$ of Hilbert spaces.

If i substitute $x_1 + \delta x_1$ in $\underline{\underline{\psi(x_1, x_2)}} \to \psi(x_1 + \delta x_1, x_2)$.

$$\to \left(\frac{\omega_+ \omega_-}{\pi \hbar}\right)^{1/2} \exp\left(-\frac{(\omega_+ + \omega)}{4\hbar}\left(x_1^2 + 2\delta x_1 x_1 + (\delta x_1)^2 + x_2^2 + 2 x_2 \delta x_1 + 2x_1 x_2\right)\right)$$

$$|\psi\rangle \to |1\rangle|2\rangle \not\to |1\rangle|2\rangle \equiv |1\rangle|2\rangle$$

- So, $\psi(x_1 + \delta x_1, x_2) = N \exp(-c_0(x_1 + \delta x_1)^2] \exp[-c_0 x_2^2] \exp[-c_1(x_1 + \delta x_1)x_2]$

$$= N \exp[-c_0 x_1^2] \exp[-c_0 x_2^2] \exp[-2c_0 x_1 \delta x_1 - c_1 x_1 x_2 - c_1 x_2 \delta x_1]$$

$$= N \exp[-c_0(x_1^2 + x_2^2)] \exp[-c_1 x_1 x_2] \exp[-\delta x_1(-2c_0 x_1 + c_1 x_2)]$$

- [For Entangled states, A measurement in x_1 will also affect the State of x_2 !!!]

So by principal $(\delta x_1) \Rightarrow (\delta x_2)$.

Now, $|0,0\rangle = |0\rangle_+ \otimes |0\rangle_-$ (nice representation in $+, -$ basis).

but $|0,0\rangle \overset{?}{=} \sum_{nm} C_{nm} |n\rangle_1 |m\rangle_2$ [How will i represent G.S in 1, 2 basis?]

coefficients we need to determine

State corresponding to oscillator 1

State corresponding to oscillator 2

$(c_0 \hat{a}_+ + c_1 \hat{a}_-)|0,0\rangle = 0 \quad \forall\, c_0, c_1$.

$$\ast\ \hat{a} = \frac{1}{\sqrt{2}}(\hat{a}_+ + \hat{a}_-) = \left(\sqrt{\frac{\omega_+}{2\hbar}}\left(\hat{x}_+ - \frac{i}{\omega_+}\hat{p}_+\right) + \sqrt{\frac{\omega_-}{2\hbar}}\left(x_- - \frac{i}{\omega_-} p_-\right)\right)$$

$$= x_1\left(\frac{1}{2}\sqrt{\frac{\omega_+}{\hbar}} + \frac{1}{2}\sqrt{\frac{\omega_-}{\hbar}}\right) + x_2\left(\frac{1}{2}\sqrt{\frac{\omega_+}{\hbar}} - \frac{1}{2}\sqrt{\frac{\omega_-}{\hbar}}\right)$$

For $\hat{b} = \frac{1}{\sqrt{2}}(\hat{a}_+ - \hat{a}_-)$

$$- i\left[p_1\left(\frac{1}{2\omega_+}\sqrt{\frac{\omega_+}{\hbar}} + \frac{1}{2\omega_-}\sqrt{\frac{\omega_-}{\hbar}}\right) + p_2\left(\frac{1}{2\omega_+}\sqrt{\frac{\omega_+}{\hbar}} - \frac{1}{2\omega_-}\sqrt{\frac{\omega_-}{\hbar}}\right)\right]$$

$\ominus \quad \oplus \quad \ominus \quad \oplus$

- Further simplifying,

$$a = \frac{1}{\sqrt{1 - \xi^2}} \cdot \sqrt{\frac{\omega}{2\hbar}} \cdot \left[\left(x_1 + \frac{i}{\omega} p_1\right) - \xi\left(x_2 - \frac{i}{\omega} p_2\right)\right] \qquad b = \frac{1}{\sqrt{1 - \xi^2}}\sqrt{\frac{\omega}{2\hbar}}\left[\left(x_2 + \frac{i}{\omega} p_2\right) - \xi\left(x_1 \ldots\right.\right.$$

where $\left(\omega = \sqrt{(\omega_+ \omega_-)}\right) \quad \left(\xi = \frac{\sqrt{\omega_-} - \sqrt{\omega_+}}{\sqrt{\omega_-} + \sqrt{\omega_+}}\right)$

Let, $\hat{\alpha} = \sqrt{\frac{\omega}{2\hbar}}\left(\hat{x}_1 + \frac{i}{\omega}\hat{p}_1\right)$, $\hat{\beta} = \sqrt{\frac{\omega}{2\hbar}}\left(\hat{x}_2 + \frac{i}{\omega}\hat{p}\right)$

↳ These do not just correspond to single oscillators as ω contains information about $\underline{\omega_+, \omega_-}$. → (contains both 1, 2 ka information)

$\Rightarrow \hat{a} = \frac{1}{\sqrt{1-\xi^2}}\left(\hat{\alpha} - \xi\hat{\beta}^\dagger\right)$, $\hat{b} = \frac{1}{\sqrt{1-\xi^2}}\left(\hat{\beta} - \xi\hat{\alpha}^\dagger\right)$

$\rightarrow \hat{a}|0,0\rangle = 0, \quad \hat{b}|0,0\rangle = 0$ | $\boxed{\begin{aligned}&\Rightarrow \hat{\alpha}|0,0\rangle = \xi\hat{\beta}^\dagger|0,0\rangle \\ &\Rightarrow \hat{\beta}|0,0\rangle = \xi\hat{\alpha}^\dagger|0,0\rangle\end{aligned}}$

<u>Def^n</u>: $\hat{\alpha}^\dagger|n\rangle_1 = \sqrt{n+1}\,|n+1\rangle$ | $\xi\beta^\dagger|0,0\rangle = \xi\beta^\dagger\sum_{nm}|n\rangle_1|m\rangle_2$.

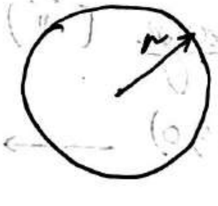
$\beta^\dagger|m\rangle_2 = \sqrt{m+1}\,|m+1\rangle$ | $\xi\alpha^\dagger|0,0\rangle = \xi\alpha^\dagger\sum_{nm}|n\rangle_1|m\rangle_2$.

- $C_{(n+1)m}$

$\Longrightarrow$ $\boxed{\text{To be continued later}}$ → Gallery <u>Oct 28</u>

<u>Charged Particle in Presence of E and B fields.</u>

<u>Classical cyclotron (ω_c)</u>



$\otimes \vec{B}_z$.

$F = -e\left(\vec{E} + \vec{v}\times\vec{B}\right)$

$\boxed{\omega_c = \frac{eB}{m}}$

$\ddot{x} = -\omega_c\dot{y} \rightarrow \dot{x} = -\omega_c(y - Y)$ | $\eta_x = x - X$

$\ddot{y} = \omega_c\dot{x} \qquad \dot{y} = \omega_c(x - X)$ | $\eta_y = y - Y$. (X, Y) ↓ constants of

$\Rightarrow \dot{\eta}_x = -\omega_c\eta_y, \; \dot{\eta}_y = -\omega_c\eta_x \Rightarrow \ddot{\eta}_x = -\omega_c^2\eta_x, \; \ddot{\eta}_y = -\omega_c^2\eta_y$.

$\Rightarrow x(t) = X + r\sin(\omega_c t + \phi) \qquad y(t) = Y + r\cos(\omega_c t + \phi)$.

<u>Quantum</u>:

$p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i} \quad H = \sum_i p_i\dot{q}_i - \mathcal{L} \rightarrow H(p_i, q_i) \longrightarrow \hat{H}$ ← $\mathcal{L}(\dot{q}_i, q_i)$

$\mathcal{L} \rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = 0$ | $\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 + q(A\cdot\dot{\vec{r}} - \phi(\vec{r}))$

$(\dot{x}, x) \qquad (\dot{y}, y)$

$\rightarrow m\ddot{x} = -e\dot{y}(\partial_x A_y - \partial_y A_x) + e\frac{\partial \phi}{\partial x}$

$\rightarrow m\ddot{y} = -e\dot{x}(\partial_x A_y - \partial_y A_x + e\frac{\partial \phi}{\partial y}$.

now $\boxed{\phi = 0}$ ✓

$P_i = \frac{\partial \mathcal{L}}{\partial q_i} \Rightarrow P_x = \frac{\partial \mathcal{L}}{\partial x} = m\dot{x} - eA_x$

$P_y = \frac{\partial \mathcal{L}}{\partial y} = m\dot{y} - eA_y$.

$H = P_x\dot{x} + P_y\dot{y} - \mathcal{L}$

$\stackrel{3}{=} \frac{1}{2m}[(P_x + eA_x)^2 + (P_y + eA_y)^2]$

$\vec{H}(P_x, P_y, x, y) \longrightarrow (A_x, A_y)$

$\vec{B} = B_0\hat{k}$

$\downarrow$

$\vec{A}_1, \vec{A}_2$ ——— $\vec{A}_N$ (i can have multiple A_i that give me the same magnetic field — Gauge invariance of the magnetic field)

$\Rightarrow \vec{B} = \nabla \times \vec{A}$, $\vec{A} + k'A + \nabla\chi$ — Two Gauges

i) London Gauge (ii) Symmetric Gauge

$A_l = -B_0(-y, 0, 0) \longrightarrow B = B_0\hat{k} \longleftarrow A_S = \frac{B_0}{2}(-y, x, 0)$.

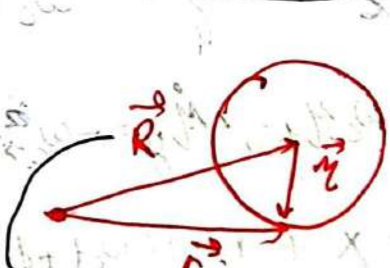
$(x, \dot{x}, y, \dot{y}) \rightarrow$ Gauge invariant || $(P_x, P_y) \rightarrow$ Are not gauge invariant.

$\vec{B} \longrightarrow$

$m\dot{x} = P_x + eA_x = -m\omega_c \eta_y$

$m\dot{y} = P_y + eA_y = +m\omega_c \eta_x$

(Gauge invariant) Sakurai (Gauge invariant)



$\Pi_x = P_x + eA_x$

$\Pi_y = P_y + eA_y$ (Kinetic/Kinematic Momenta).

↳ These momenta are invariant under Gauge Transformations.

$\Rightarrow$ none of these Momenta Physically make sense.

Center of the circle. Arbitrary constant fixed by you as you make the choice of the co-ordinate system.

• The velocities and momentum have very different meanings both in classical as well as Quantum Mechanics — $\boxed{\text{Look up}}$!!!
(independently Gauge dependent)

$$H = \frac{1}{2m}\left[(P_x + eA_x)^2 + (P_y + eA_y)^2\right] = \frac{1}{2m}\left[\Pi_x^2 + \Pi_y^2\right].$$

↳ Gauge independent since Π_x and Π_y are gauge independent.

$$H = \frac{m\omega_c^2}{2}\left[\eta_x^2 + \eta_y^2\right]$$

Classical ⟶ Quantum.

$$(q_i, p_i) \to (\hat{q}_i, \hat{p}_i) \to [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

(Poisson Bracket) ⟶ q & p — ???

↳ Hamiltonian ⟶ $\hat{H}$ ⟶

Thus, $[\hat{x}, \hat{P}_x] = [\hat{y}, \hat{P}_y] = [z, \hat{P}_z] = i\hbar$ ∥ $[\hat{x}, \hat{y}] = [\hat{P}_x, \hat{P}_y] = 0$.

• $\hat{H} = \frac{1}{2m}\left[(\hat{P}_x + eA_x(\hat{x}, \hat{y}))^2 + (\hat{P}_y + eA_y(\hat{x}, \hat{y}))^2\right]$

• $[\hat{\Pi}_x, \hat{\Pi}_y]$ Given: $[f(\hat{A}), \hat{B}] = [\hat{A}, \hat{B}]\frac{\partial f}{\partial A}$

• $[\hat{\eta}_x, \hat{\eta}_y]$. $\Rightarrow [\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ ∥.

• $[P_x + eA_x, P_y + eA_y] = \cancel{[P_x, P_y]}^{0} + [P_x, eA_y] + [eA_x, P_y] + [eA_x, eA_y]$

Calculate: $[\hat{\Pi}_x, \hat{\Pi}_y] = ie\hbar B_0$, $[\hat{\eta}_x, \hat{\eta}_y] = ie\hbar$.

makes the hamiltonian similar to the Quantum Harmonic Oscillator to which this problem can be completely mapped back to.

$\Rightarrow \hat{H} = [\hat{P}^2 + \hat{X}^2]$ and $[\hat{X}, \hat{P}] = i\hbar$ (see Π_x, Π_y here).

THUS
(Creation Annihilation operators)

$$\hat{a}^{\dagger} = \sqrt{\frac{1}{2e\hbar B}}(\Pi_x - i\Pi_y)$$
$$\hat{a} = \sqrt{\frac{1}{2e\hbar B}}(\Pi_x + i\Pi_y)$$

$$\Rightarrow \boxed{\hat{H} = \hbar\omega_c\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)}$$

**
**

Energy Eigenvalues $\rightarrow$ Landau Levels.

1D $\rightarrow H_0 \rightarrow$ (no definite states) || (Degenerate states exist or not)

1D $\rightarrow$ parity $\rightarrow$ free particle.

$[\Pi_x, \Pi_y] \neq 0$. $[\Pi_x, \hat{H}] = [\Pi_y, \hat{H}] = 0 \longrightarrow$ Degeneracy $\rightarrow \infty$.