Ext groups and group cohomology

Definition (Homotopy inverse)

A map $\phi\colon M^\bullet \to N^\bullet$ is homotopy inverse to $\psi\colon N^\bullet, \to M^\bullet$ if both $\psi \circ \phi$ and $\phi \circ \psi$ are homotopic to the identity maps. If so, we say ϕ is a homotopy equivalence and $M^\bullet$ is homotopic to $N^\bullet$.

Remark (slogan)

For homological purposes, homotopic objects are regarded identical.

Let A be a commutative ring, R be a central A-algebra. Let M be an R-module. Choose two projective resolutions

$$Q_\bullet, P_\bullet \xrightarrow{\simeq} M.$$

The projective-to-acyclic lemma tells us that two maps

$$\phi \colon Q_\bullet \to P_\bullet$$
$$\psi \colon P_\bullet \to Q_\bullet$$

extending 1_M, the identity on M. Then $\phi \circ \psi$ and $1_{P_\bullet}$ are two extensions of 1_M. Invoking the lemma again, we conclude that they are homotopic. Applying the same argument to $\psi \circ \phi$, we conclude that ϕ is a homotopy inverse to ψ.

Ext groups

Let A be a commutative ring, R be a central A-algebra. Let M be an R-module. Choose a projective resolution

$$P_\bullet \xrightarrow{\simeq} M.$$

Let M' be another R-module. Taking $\mathrm{Hom}(-, M')$, we obtain a cochain complex

$$\mathrm{Hom}_R(P_\bullet, M')$$

of A-modules. We adopt the convention that it is supported in non-negative degrees and $|d| = 1$;

$$\mathrm{Hom}_R(P_\bullet, M') \colon 0 \to \mathrm{Hom}(P_0, M') \to \mathrm{Hom}(P_1, M') \to \cdots$$

Ext groups

Definition

Let M, M' be R-modules and $n \geq 0$ an integer. Define

$$\mathrm{Ext}^n_R(M, M') = H^n(\mathrm{Hom}^\bullet_R(M, M')).$$

The definition involves a choice of $P_\bullet \to M$. The projective-to-acyclic lemma shows that $\mathrm{Ext}^n_R(M, M')$ is well-defined up to unique isomorphism.

Let G be a group and $R = A[G]$. Underlying the homogeneous bar complex is the chain complex, the bar resolution of A,

$$\cdots \to A[G \times G \times G] \to A[G \times G] \to A[G] \to A$$

where the last map is augmentation.

We want to show that this is a resolution of A.

Definition

A complex is contractible if the zero map is homotopic to identity.

Proposition

A contractible complex is acyclic.

Proof.

Take cohomology groups. The zero map is an isomorphism if and only if the module is trivial. $\square$

$$\cdots \to A[G \times G \times G] \to A[G \times G] \to A[G] \to A \qquad (1)$$

Proposition

The complex (1) *is contractible.*

Proof.

The desired homotopy is given by $(\underline{g}) \mapsto (1, \underline{g})$. $\square$

Corollary

The complex (1) *is a free resolution of* A*. In particular, it is a projective resolution.*

Ext and group cohomology

Proposition

Let M be an R-module. We have $\mathrm{Ext}^n_R(A, M) = H^n(G, M)$.

Proof.

Compute $\mathrm{Ext}^n_R(A, M)$ using the bar resolution of A. $\square$

Shapiro's lemma revisited

We come bace to Shapiro's lemma. Let $H \subset G$ be a subgroup of finite index. It says, for an $A[H]$-module V,

$$H^\bullet(G, \mathrm{Ind}_H^G(V)) = H^\bullet(H, V).$$

We will prove it using Ext-description of group cohomology.

Here is a preliminary lemma, called the Frobenius reciprocity. Let V be an $A[H]$-module and W an $A[G]$-module.

Lemma (Frobenius reciprocity)

$$\mathrm{Hom}_{A[G]}(W, \mathrm{Ind}_H^G V) = \mathrm{Hom}_{A[H]}(W, V)$$

Proof.

It is a form of hom-tensor duality. Recall that $\mathrm{Ind}_H^G(V)$ consists of functions $f\colon G \to V$ such that $f(hx) = h(f(x))$ for all $h \in H$. The action is given by $(gf)(x) = f(xg)$. Interpret it as $\mathrm{Ind}_H^G(V) = \mathrm{Hom}_{A[H]}(A[G], V)$. Let $(-)^*$ denote the A-dual. Then,

$$\begin{aligned} \mathrm{Hom}_{A[G]}(W, \mathrm{Ind}_H^G V) &= \mathrm{Hom}_{A[G]}(W, \mathrm{Hom}_{A[H]}(A[G], V)) \\ &= \mathrm{Hom}_{A[G]}(W, V \otimes_{A[H]} A[G]^*) \\ &= \mathrm{Hom}_{A[H]}(W, V). \end{aligned}$$

$\square$

We are ready to prove Shapiro's lemma. Observe that $A[G \times \cdots \times G]$ is a free $A[H]$-module. Then, it is homotopic to the bar resolution $P_\bullet^H \to A$ by the projective-to-acyclic lemma. Let $\eta\colon P_\bullet^H \to P_\bullet^G$ be a homotopy equivalence.

$$\cdots \to A[G \times G] \to A[G] \to A$$

as a resolution $P_\bullet^G \to A$ of A as the trivial $A[H]$-module. This yields a homotopy equivalence:

$$\begin{aligned} C^\bullet(G, \mathrm{Ind}_H^G V) &= \mathrm{Hom}_{A[G]}(P_\bullet^G, \mathrm{Ind}_H^G V) \\ &= \mathrm{Hom}_{A[H]}(P_\bullet^G, V) \\ &\xrightarrow{\eta^*} \mathrm{Hom}_{A[H]}(P_\bullet^H, V) \\ &= C^\bullet(H, V). \end{aligned}$$

Taking cohomology on both sides, we obtain Shapiro's lemma.