More homological algebra

We defined group cohomology in terms of a bar complex. That's sufficient as a definition. We will review basic notions in homological algebra that will provide us with more tools.

The key idea is to manipulate chain complexes 'up to homotopy'.

Let $(M^\bullet, d)$ be a cochain complex. Recall that

$$H^i(M^\bullet, d) = Z^i/B^i$$

where $B^\bullet = \mathrm{Im}(d)$.

In other words, a cohomology class is a cochain defined up to $\mathrm{Im}(d)$. Two cocycles ϕ and ϕ' represent the same cohomology class if

$$\phi' = \phi + d\epsilon$$

for some cochain ϵ. We may view ϵ as a homotopy from ϕ to ϕ'.

A convention

Let $M^\bullet$ be a graded module. If $m \in M^r$ is homogeneous of degree r, then write

$$|m| = r.$$

If $m \in M^\bullet$ is a homogeneous element, then let $|m|$ be its degree.

Let $M^\bullet$ and $N^\bullet$ be two cochain complexes. Graded maps (not necessarily cochain maps) for another graded module

$$\mathrm{Hom}^\bullet(M^\bullet, N^\bullet).$$

There is a natural differential on it; for $f \in \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$

$$(df)(m) = d(fm) + (-1)^{|f|+|d|} f(dm).$$

The sign rule here is an example of the 'Koszul sign'.

An illustration:

$$\begin{tikzcd}
M^{i-1} \arrow[r] & M^i \arrow[r] \arrow[dl, "h"'] \arrow[d, "k"'] & M^{i+1} \arrow[dl, "h"] \\
N^{i-1} \arrow[r] & N^i \arrow[r] & N^{i+1}
\end{tikzcd}$$

Here, $h \in \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ is a map of degree -1. Applying the differential,

$$dh = k$$

is a map of degree zero. It acts on cochains as

$$k(m) = (dh)(m) = d(hm) + (-1)^{|d|+|h|}h(dm)$$

or

$$k(m) = (dh)(m) = d(hm) - (-1)^{|h|}h(dm).$$

Proposition

Let $M^\bullet, N^\bullet$ be cochain complexes. Then, $(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet), d)$ is a cochain complex.

Proof.

Take $h \in \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ and $m \in M^\bullet$.

$$\begin{aligned}
(d(dh))(m) &= d((dh)m)) + (-1)^{|dh||d|}(dh)(dm) \\
&= d((dh)m)) + (-1)^{|h|}(dh)(dm) \\
&= d\left(d(hm) + (-1)^{|d|+|h|}(h(dm))\right) + (-1)^{|h|}(dh)(dm) \\
&= (-1)^{|d|+|h|}dhdm + (-1)^{|h|}(dh)(dm) \\
&= (-1)^{|d|+|h|}dhdm + (-1)^{|h|}\left(dh(dm) + (-1)^{|dh|+|d|}hd(dm)\right) \\
&= (-1)^{|d|+|h|}dhdm + (-1)^{|h|}dh(dm) \\
&= 0
\end{aligned}$$

$\square$

$M^\bullet$, $N^\bullet$: cochain complexes

Observation

A cochain map is a 0-cycle in $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$.

Definition

Let $f, f' \in \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$. They are called homotopic if $f' = f + dh$ for some h.

Proposition

If two cochain maps f and f' are homotopic, then they induce the same map on cohomology.

Proof.

We have $df - fd = dh$. It suffices to show: for any cocycle m, $(dh)(m)$ is coboundary. Indeed,

$$(dh)(m) = d(hm) + (-1)^{|d|+|h|} h(dm) = d(hm)$$

is a coboundary. $\square$

Definition

A quasi-isomorphism from $M^\bullet$ to $N^\bullet$ is a cochain map $f\colon M^\bullet \to N^\bullet$ which induces an isomorphism on cohomology groups.

Remark

A quasi-isomorphism $f\colon M^\bullet \to N^\bullet$ may not admit another quasi-isomorphism $f'\colon N^\bullet \to M^\bullet$ which, on cohomology groups, is inverse to f.

Definition

A cochain complex $M^\bullet$ is acyclic if its cohomology groups are zero.

Note that $M^\bullet$ is acyclic iff the zero map $0 \to M^\bullet$ is a quasi-isomorphism.

Projective resolution

Let R be any A-algebra, possibly non-commutative. Typically, we consider $R = A[G]$. We work with the category of R-modules.

Definition

A projective resolution of an R-module M is a chain complex consisting of projective R-modules

$$\cdots \to P_2 \to P_1 \to P_0$$

which is quasi-isomorphic to M.

Often, we say that an acyclic complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M$$

is a projective resolution of M. Equivalently, $P_\bullet \to M$ is a quasi-isomorphism.

Let $P_\bullet$, $M_\bullet$ be two chain complexes supported in non-negative degrees. Assume that each P_r is projective for $r \geq 1$ and that $M_\bullet$ is acyclic.

Lemma (Projective to acyclic lemma)

Any R-linear map $P_0 \to M_0$ extends to a chain map $P_\bullet \to M_\bullet$. The extension is unique up to homotopy.

The proof of the lemma is based on an induction argument.

Suppose we have a map $f_0\colon P_0 \to M_0$. As the induction hypothesis, assume that can extend any such f_0 to f_1.

$$
\begin{tikzcd}
\cdots P_2 \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
\cdots M_2 \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r] & 0
\end{tikzcd}
$$

From this, one obtains

$$
\begin{tikzcd}
\cdots P_2 \arrow[r] & B_1^P \arrow[r] \arrow[d, "f_1"] & 0 \\
\cdots M_2 \arrow[r] & B_1^M \arrow[r] & 0
\end{tikzcd}
$$

and the bottom chain complex remains acyclic. Therefore, the hypothesis allows us to extend f_1 to $f_2\colon P_2 \to M_2$.

We need to initiate the induction process. Suppose we have a map $f_0 \colon P_0 \to M_0$. We get f_1 by applying the lifting property to $f_0 \circ d^P$:

$$\begin{tikzcd}
P_1 \arrow[d, dashed, "f_1"] \arrow[dr, "f_0 \circ d^P"] & & \\
M_1 \arrow[r] & M_0 \arrow[r] & 0
\end{tikzcd}$$

If two maps f_1, f_1' are obtained in this way, then $g = f_1 - f_1'$ factors through B_1^M by the acyclicity of $M_\bullet$.

$$\begin{tikzcd}
& P_1 \arrow[d, "g"] & \\
M_2 \arrow[r, two heads] & B_1 \arrow[r] & 0
\end{tikzcd}$$

The map $M_2 \to B_1$ is surjective by definition. Thus g can be lifted to $h \colon P_1 \to M_2$. Then, $g = dh$ by construction.

Observe: if $f_0 = dk$, then f_1 can be taken to be $k \circ d^P$. This shows that the any to sequences $(f_0, f_1, f_2, \cdots)$ and $(f_0, f_1', f_2', \cdots)$ are homotopic.

Uniqueness of projective resolution

Corollary

If $P_\bullet$, $P'_\bullet$ are two projective resolutions of M, then there are quasi-isomorphisms $P_\bullet \to P'_\bullet$ and $P'_\bullet \to P_\bullet$ both inducing the identity on homology. Furthermore, these quasi-isomorphisms are unique up to homotopy.