ELLIPTIC FUNCTIONS AND ELLIPTIC INTEGRALS

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Abstract. In this article, we briefly browse the history of elliptic functions and addition theorem.

1. Addition Theorem

Definition 1.1. An addtion theorem for a function f is a formula expressing $f(u+v)$ in terms of $f(u)$ and $f(v)$.

Here f is an one variable function.

Example 1.2. For $F(x)=\int_0^x \frac{1}{\sqrt{1-u^2}}\,du$,

$$\int_0^{\sin(x)} \frac{1}{\sqrt{1-u^2}}\,du + \int_0^{\sin(y)} \frac{1}{\sqrt{1-u^2}}\,du = \int_0^{\sin(x+y)} \frac{1}{\sqrt{1-u^2}}\,du.$$

Example 1.3 (Euler's addition theorem)**.** For $F(x)=\int_0^x \frac{1}{\sqrt{1-u^4}}\,du$,

$$\int_0^{x} \frac{1}{\sqrt{1-u^4}}\,du + \int_0^{y} \frac{1}{\sqrt{1-u^4}}\,du = \int_0^{\frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2}} \frac{1}{\sqrt{1-u^4}}\,du.$$

If we set $f(s)=x, f(t)=y$, then

$$\int_0^{f(s)} \frac{1}{\sqrt{1-u^4}}\,du + \int_0^{f(t)} \frac{1}{\sqrt{1-u^4}}\,du = \int_0^{f(s+t)} \frac{1}{\sqrt{1-u^4}}\,du.$$

Here f is the inverse of F. In short, an addition theorem is related to the inverse functions of the antiderivative of a function.

Note that the integral is an elliptic integral. Elliptic integral is $\int R(x,\sqrt{P(x)})\,dx$ where $R(x,y)$ is a rational function of two variables x,y and $P(x)$ is a polynomial of degree 3 or 4. That such integrals cannot be evaluated in terms of the elementary functions was finally proved by Liouville.

Let

$$P(x) = a_0x^{2n-1} + a_1x^{2n-2} + ... + a_{2n-1}$$

set $x = c+1/y$, where c is not a root of $P(x)$. Then

$$P(x) = P(c) + P'(c)\frac{1}{y} + ... + \frac{P^{2n-1}(c)}{(2n-1)!}\frac{1}{y^{2n-1}} = \frac{P_1(y)}{y^{2n}}.$$

Then $P_1(x)$ has degree 2n. Thus $\int R(x,\sqrt{P(x)})\,dx=\int R_1(x,\sqrt{P_1(x)})\,dx$.

Legendre showed that integration of the elliptic integral of fourth degree, can be reduced to the integration of the three integrals,

$$\int \frac{1}{\sqrt{1-u^2}\sqrt{1-l^2u^2}}\,du, \int \frac{u^2}{\sqrt{1-u^2}\sqrt{1-l^2u^2}}\,du, \int \frac{1}{(u-a)\sqrt{1-u^2}\sqrt{1-l^2u^2}}\,du.$$

Next, Weierstrass showed that

$$u(p) = \int_0^p \frac{1}{\sqrt{4t^3 - g_2t - g_3}}\,dt,$$

where g_2 and g_3 are constants such that $g_2^3 - 27g_3^2 \neq 0$ is the fundamental elliptic function. the fundamentality means every elliptic function could be expressed in terms of $p(u)$ and $p'(u)$. It also related to the inverse function of the function defined by antiderivative.

Abel generalized an addition theorem.

Theorem 1.4. *Let $\psi x = \int R(x, \sqrt{P(x)})\,dx$. Let n be positive integers, and $h_1, .., h_n$ rational numbers. Then there exist g that*

$$h_1\psi x_1 + .. + h_n\psi x_n = v + \psi x_1' + .. + \psi x_g'$$

where v is an elementary function of $x_1, .., x_n$ and $x_1', .., x_g'$ are algebraic function of $x_1, .., x_n$ and g is minimal : given algebraic functions $x_1', .., x_{g-1}'$ of $x_1, .., x_g$, for any constant v, there exists ψx such that

$$\psi x_1 + .. + \psi x_g \neq \psi x_1' + .. + \psi x_{g-1}' + v$$

Here ψ is another bounded integral made by $\sqrt{P(x)}$.

Next time We shall discuss what g means.

References

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