Modular forms of higher weights

Higher weight case

We have seen the period map $S_2(\Gamma_1(N)) \to H^1_{\mathrm{cusp}}(\Gamma_1(N), \mathbb{C})$.

What happens to a general integer $k \geq 2$?

Answer: there is a coefficient system L_{k-2} and an analogous map

$$S_k(\Gamma_1(N)) \to H^1(\Gamma_1(N), L_{k-2}).$$

Let $\mathrm{PSL}_2(\mathbb{Z})$ act on polynomials in $\mathbb{Z}[X, Y]$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot X = aX + bY$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot Y = cX + dY$$

Note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * X = dX - bY$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * Y = -cX + aY$$

also defines an action. Indeed, $\gamma \mapsto (\gamma^{-1})^t$ is a homomorphism.

Forms with values in $\mathbb{C}[X, Y]$

Let $f(\tau)$ be a modular form of weight $k \geq 2$ for the group $\Gamma_1(N)$. Associate to $f(\tau)$ a form

$$f(\tau)(X - \tau Y)^{k-2} d\tau$$

with 'values in polynomials'. It is a map $\mathfrak{H} \to \mathbb{C}[X, Y]$.

Note that $\Gamma_1(N)$ acts on both the source and the target of the map.

Proposition

$f(\tau)(X - \tau Y)^{k-2} d\tau$ is invariant under $\Gamma_1(N)$, if it acts through $(aX + bY, cX + dY)$.

Proof.
Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

$$
\begin{aligned}
& f\left(\frac{a\tau+b}{c\tau+d}\right)\left((aX+bY)-\frac{a\tau+b}{c\tau+d}(cX+dY)\right)^{k-2} d\left(\frac{a\tau+b}{c\tau+d}\right) \\
&= f(\tau)\left((aX+bY)(c\tau+d)-(a\tau+b)(cX+dY)\right)^{k-2} d\tau \\
&= f(\tau)(X-\tau Y)^{k-2} d\tau
\end{aligned}
$$

$\square$

Let $f(\tau)$ be a modular form of weight $k \geq 2$ for the group $\Gamma_1(N)$. Let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ be two cusps. Then, the integral

$$\int_\alpha^\beta f(\tau)(X - \tau Y)^{k-2} d\tau$$

can be defined and takes its value in $\mathbb{C}[X, Y]$. Let

$$L_{k-2}(\mathbb{C}) \subset \mathbb{C}[X, Y]$$

be the span of monimials of degree $k - 2$. Define:

$$\begin{aligned} P_f \colon \Gamma_1(N) &\longrightarrow L_{k-2}(\mathbb{C}) \\ \gamma &\longmapsto \int_\infty^{\gamma\infty} f(\tau)(X - \tau Y)^{k-2} d\tau \end{aligned}$$

Proposition

P_f is a cocycle, where $\Gamma_1(N)$ act through $(dX - bY, -cX + aY)$.

Proof.

We need to show that $P_f(\gamma_1\gamma_2) = \gamma_1 * P_f(\gamma_2) + P_f(\gamma_1)$

$$\int_{\infty}^{\gamma_1\gamma_2\infty} f(\tau)(X-\tau Y)^{k-2}d\tau$$
$$= \int_{\infty}^{\gamma_1\infty} f(\tau)(X-\tau Y)^{k-2}d\tau + \int_{\gamma_1\infty}^{\gamma_1\gamma_2\infty} f(\tau)(X-\tau Y)^{k-2}d\tau.$$

On the other hand, the second term above can be rewritten as:

$$\int_{\infty}^{\gamma_2\infty} f(\gamma_1\tau)(X-(\gamma_1\tau)\,Y)^{k-2}d\,(\gamma_1\tau)$$
$$=\gamma_1\gamma_1^{-1} * \int_{\infty}^{\gamma_2\infty} f(\gamma_1\tau)(X-(\gamma_1\tau)\,Y)^{k-2}d\,(\gamma_1\tau)$$
$$=\gamma_1 * \int_{\infty}^{\gamma_2\infty} f(\tau)\cdot(X-\tau Y)^{k-2}d\tau.$$

$\square$

Choosing a different cusp

Let $\alpha \in \mathbb{P}^1(\mathbb{Q})$ be another cusp. Define

$$P_f^{\alpha}(\gamma) := \int_{\alpha}^{\gamma\alpha} f(\tau)(X - \tau Y)^{k-2} d\tau.$$

Then, $P_f \neq P_f^{\alpha}$ in general. However, the difference

$$P_f - P_f^{\alpha}$$

is a coboundary; $d\epsilon$ for some $\epsilon \in C^0(\Gamma_1(N), L_{k-2}(\mathbb{C})) = L_{k-2}(\mathbb{C})$.

For the moment, put $\omega(f) = f(\tau)(X - \tau Y)^{k-2} d\tau$. Then, for $\gamma \in \Gamma_1(N)$, we have

$$\begin{aligned}
P_f(\gamma) - P_f^{\alpha}(\gamma) &= \int_{\infty}^{\alpha} \omega(f) - \int_{\gamma\infty}^{\gamma\alpha} \omega(f) \\
&= \int_{\infty}^{\alpha} \omega(f) - \gamma * \int_{\infty}^{\alpha} \omega(f) \\
&= (1 - \gamma) * \int_{\infty}^{\alpha} \omega(f).
\end{aligned}$$

This means $P_f - P_f^{\alpha} = d\epsilon$ with

$$\epsilon = \int_{\infty}^{\alpha} \omega(f) \in L_{k-2}(\mathbb{C}).$$

Proposition

Let $k \geq 2$ be an integer. We have a map

$$S_k(\Gamma_1(N)) \to H^1_{\mathrm{cusp}}(\Gamma_1(N), L_{k-2}(\mathbb{C}))$$

given by $f \mapsto P_f$.

Proof.

Let α be a cusp with stabilizer Γ_α. To show P_f is a coboundary when restricted to Γ_α, take P_f^α as a representative for the same cohomology class. $\square$

Similarly, one can show the existence of a map

$$M_k(\Gamma_1(N)) \to H^1(\Gamma_1(N), L_{k-2}(\mathbb{C})).$$

Taking the real part

One can take the real part of the period map to define a map

$$\mathrm{ES}\colon S_k(\Gamma_1(N)) \to H^1_{\mathrm{cusp}}(\Gamma_1(N), \mathbb{R}).$$

Theorem (Eichler-Shimura)

The map ES *is an isomorphism between real vector spaces.*