# Moduli of Complex Elliptic Curves

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We follow Diamond and Shurman's book 'A First Course on Modular Forms', Springer GTM 228.

## 1 Complex Elliptic Curves

**Definition 1.1.** *A **complex elliptic curve** is a pointed compact Riemann surface of genus 1.*

It is well-known that an elliptic curve is isomorphic to a complex torus $\mathbb{C}/\Lambda$ where $\Lambda$ is a lattice of rank 2. The distingushed point corresponds to 0.

**Theorem 1.2.** *Any elliptic curve is isomorphic to a complex torus of the form*

$$\mathbb{C}/\Lambda_\tau \ \ (\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau, \ \ \mathrm{Im}\tau > 0).$$

*Moreover, this $\tau$ is unique up to the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane $\mathfrak{H} = \{\mathrm{Im}\tau > 0\}$ given by*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.\tau = \frac{a\tau + b}{c\tau + d}.$$

*Hence the isomorphism classes of elliptic curves are in one-to-one correspondence with the points of the quotient space*

$$\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z}) \overset{j}{\simeq} \mathbb{C},$$

*where $j$ is a weight 0 modular function known as the $j$-invariant of the elliptic curve $\mathbb{C}/\Lambda_\tau$. Here, if we write*

$$G_{2k}(\tau) = \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \frac{1}{\omega^{2k}}, \ \ g_2(\tau) = 60G_4(\tau), \ \ g_3(\tau) = 140G_6(\tau),$$

*then*

$$j = \frac{1728 g_2^3}{g_2^3 - 27g_3^2}.$$

We call the space $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z}) \simeq \mathbb{C}$ the **moduli space** of elliptic curves for this reason.

## 2 Compactifying the Moduli Space

One of the issues of the moduli space $\mathbb{C}$ is that it is not **compact**. We will compactify $\mathbb{C}$ by adding cusps.

**Definition 2.1.** *The extended upper half-plane $\mathfrak{H}^*$ is a topological space defined as follows: as a set, it is $\mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$. We then have an obvious extension of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathfrak{H}^*$. The basic open sets for the topology are the open sets of $\mathfrak{H}$, and the sets*

$$\gamma.\{\mathrm{Im}\tau > \delta\}, \ \ \gamma \in \mathrm{SL}(2,\mathbb{Z}), \ \ \delta > 0.$$

The topology defined reflects some topological properties of actions near cusps. One motivation of adding the rational and the infinity points to the upper half-plane is to think them of the limit points of the action. We would like to explain the significance of the following theorem.

**Theorem 2.2.** *The quotient space $\mathfrak{H}^*/\mathrm{SL}(2,\mathbb{Z})$ has a structure of a compact Riemann surface. It is isomorphic to $\mathbb{CP}^1$ and contain $\mathbb{C} = \mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$ as an open submanifold, of which the complement is the point $\infty$.*

The stabilizer of points of $\mathfrak{H}$ are generally $\{\pm I\}$, but orbits of $i$ and the third root of unity $\rho$ have stabilizers of order 4 and 6 respectively, and $\infty$ has a stabilizer group which is an extension of $\mathbb{Z}$ by $\pm I$. The finite stabilizers mentioned above is not an issue by following well-known lemma from Riemann surface theory(take $G = \mathrm{PSL}(2, \mathbb{Z})$):

**Lemma 2.3.** *Let $X$ be a Riemann surface and $G$ be an abstract group. If $G$ acts faithfully and holomorphically and properly discontinuously on $X$, the orbit space $X/G$ has a (unique) Riemann surface structure so the projection map $X \to X/G$ is holomorphic.*

What is interesting is that despite the infinite cyclic stabilizer of $\infty$ in $\mathrm{PSL}(2, \mathbb{Z})$ we can give chart near $\infty$ so that $\mathfrak{H}^*/\mathrm{SL}(2, \mathbb{Z})$ is a compactification of $\mathfrak{H}/\mathrm{SL}(2, \mathbb{Z})$. This happens because some open sets of $\mathfrak{H}/\mathrm{SL}(2, \mathbb{Z})$ pulled back to $\mathfrak{H}$ 'straightens' near $\infty$.

**Definition 2.4.** *Let $X$ be a Riemann surface. A **hole chart** is a chart $\phi : U \to V$ from an open subset of $X$'s to $\mathbb{C}$'s such that there exists a closed subset $C$ of $X$ contained in $U$ that is mapped to a punctured disk of $\mathbb{C}$ contained in $V$.*

**Lemma 2.5.** *Maintain the notation of the above definition. Then there exists a Riemann surface $X'$ which is set-theoretically $X$ with a point added. The charts of $X'$ are the charts of $X$ and the chart obtained by the inverting extended $\phi^{-1}$, where the extended $\phi^{-1}$'s domain is the union of $V$ with the punctured point of the disk.*

So the picture is clear by now: the topology on $\mathfrak{H}^*$ described above is defined so the quotient space $\mathfrak{H}/\mathrm{SL}(2, \mathbb{Z})$ will have a hole chart 'near $\infty$'.

The straightening can be observed by considering the behaviour of the $j$-invariant, which established the isomorphism $\mathfrak{H}/\mathrm{SL}(2, \mathbb{Z}) \simeq \mathbb{C}$. In fact, the $j$-invariant's Fourier expansion near $\infty$ is:

$$j = \frac{1}{q} + 744 + 196884q + \cdots \quad (q = e^{2\pi i\tau}).$$

Hence the open set $\{\mathrm{Im}\tau > \delta\}$ is mapped to $j(\{|q| < e^{-2\pi\delta}\})$, giving the hole chart punctured at $\infty$ for sufficient large $\delta$ ('disk punctured at $\infty$' is just $\{|z| > R\}$ for some $R > 0$), because $j$ is injective modulo 1 near $\infty$ by the fact

$$\left(\frac{d}{dq}\right)_{q=0} \frac{1}{j} = \left(\frac{d}{dq}\right)_{q=0} (q - 744q^2 + \cdots) = 1 \neq 0.$$

# 3 Enhanced Structures on Elliptic Curves

One of the other issues about the moduli space $\mathfrak{H}/\mathrm{SL}(2, \mathbb{Z})$ is that it is not a **fine moduli space**. Existence of nontrivial automorphisms of $\mathbb{C}/(\Lambda_\tau)$ when $\tau = i$ or $\rho$ causes this failure. Such can be overcome when we enhance the elliptic curves.

**Definition 3.1.** *Let $E$ be an elliptic curve, and $N$ a natural number. We denote by $E[N]$ the group of $N$-torsion points of $E$. Fixing as isomorphism $E \simeq \mathbb{C}/\Lambda_\tau$, if $P$, $Q \in E[N]$, then we associate an $N^{th}$ root of unity $e(P, Q)$ defined by*

$$e(P, Q) := e^{\frac{2\pi i(ad-bc)}{N}} \text{ where } P = \frac{a + b\tau}{N}, Q = \frac{c + d\tau}{N}.$$

*This is a well-defined bilinear pairing on $E[N]$ called the **Weil pairing**.*

The Weil pairing is intrinsic to $E$, which is not a priori clear.

**Definition 3.2.** *Let $N$ be a natural number. We define following level $N$ structures on elliptic curves.*

$$\begin{cases} \Gamma_0(N)\text{-structure on } E: & \text{A cyclic subgroup of order } N. \\ \Gamma_1(N)\text{-structure on } E: & \text{A point of order } N. \\ \Gamma(N)\text{-structure on } E: & \text{A pair of generators of } E[N] \text{ with Weil pairing } e^{\frac{2\pi i}{N}}. \end{cases}$$

*hence a $\Gamma$-enhanced elliptic curve is a tuple of an elliptic curve with additional data, where $\Gamma$ is one of the above groups.*

The gamma groups above are exactly those from the class. The results of the previous sections hold almost word in word.

**Theorem 3.3.** *Let $N$ be a natural number and $\Gamma$ be one of the groups $\Gamma_0(N)$, $\Gamma_1(N)$ or $\Gamma(N)$. Then any $\Gamma$-enhanced elliptic curve is isomorphic to*

$$\begin{cases} (\mathbb{C}/\Lambda_\tau, < \frac{1}{N} >), & \textit{if } \Gamma = \Gamma_0(N). \\ (\mathbb{C}/\Lambda_\tau, \frac{1}{N}), & \textit{if } \Gamma = \Gamma_1(N). \\ (\mathbb{C}/\Lambda_\tau, \frac{1}{N}, \frac{\tau}{N}), & \textit{if } \Gamma = \Gamma(N). \end{cases}$$

*Furthermore, this $\tau$ is unique up to $\Gamma$-action. Therefore, the moduli space of $\Gamma$-enhanced elliptic curves is the quotient*

$$\mathfrak{H}/\Gamma$$

*which has a structure of a noncompact Riemann surface. These are fine moduli spaces when $N > 1$. Such moduli spaces can be compactified by adding the cusps $\mathbb{Q} \cup \{\infty\}$, giving the compact Riemann surface*

$$\mathfrak{H}^*/\Gamma$$

*where the added points are $\Gamma$-orbits of $\mathbb{Q} \cup \{\infty\}$, which is a finite set since $[\mathrm{SL}(2, \mathbb{Z}) : \Gamma] < \infty$.*