A fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$

Fundamental domain

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a discrete subgroup.

Definition

A subset $D \subset \mathfrak{H}$ is a fundamental domain for Γ if

1. D is open,
2. $\gamma D \cap D = \emptyset$ for all $\gamma \in \Gamma - \{1\}$,
3. $\mathfrak{H} = \bigcup_{\gamma \in \Gamma} \gamma \overline{D}$, where $\overline{D}$ is the closure of D inside $\mathfrak{H}$.

Theorem

Let $D = \{\tau \in \mathfrak{H} \colon |\tau| > 1, -1/2 < \mathrm{Re}(\tau) < 1/2\}$. *Then* D *is a fundamental domain for* $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

Lemma

$\bigcup_{\gamma \in \Gamma} \gamma \overline{D} = \mathfrak{H}$.

Proof.

Let $X = \bigcup_{\gamma \in \Gamma} \gamma \overline{D}$. We will show that it is both open and closed.

Openness: only non-trivial to show an open neighborhood around τ with non-trivial stabilizer subgroup.

Closedness: use local finiteness. $\square$

Lemma

Let $\tau, \tau' \in D$ with $\mathrm{Im}(\tau) \leq \mathrm{Im}(\tau')$. If $\gamma\tau = \tau'$ and $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, then $\gamma = 1$.

Proof.

Let $\tau = x + yi$ and $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Note that $\mathrm{Im}(\gamma\tau) = \mathrm{Im}(\tau)|c\tau + d|^{-2}$. Since we assumed $\mathrm{Im}(\tau) \leq \mathrm{Im}(\tau')$, it implies that

$$|c\tau + d|^2 = (cx + d)^2 + cy^2 \leq 1.$$

Because $y \geq 1/2$, one can deduce $|c| \leq 1$.

If $c = 0$, then γ is unipotent. One can rule this out.

If $c = 1$, then $|\tau + d| \leq 1$. One can rule this out, too.

If $c = -1$, then $|-\tau + d| \leq 1$. Essentially the same as the previous case. $\square$

Theorem

Suppose $\Gamma = \Gamma_1(N)$ *with* $N \geq 4$. *Then,* $\Gamma\backslash\mathfrak{H}$ *is a topological surface of finite type.*

It is customary to denote $\Gamma\backslash\mathfrak{H}$ by Y_Γ or $Y(\Gamma)$. Also common is $Y_0(N)$ or $Y_1(N)$.

Corollary

For each $\Gamma = \Gamma_1(N)$, $N \geq 4$, *one can talk about genus of* $Y(\Gamma)$.