Petersson inner product

Let $k \geq 1$ be any integer, $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. For modular forms

$$f(\tau) \in S_k(\Gamma)$$
$$g(\tau) \in M_k(\Gamma)$$

we would like to define the Petersson inner product

$$\langle f, g \rangle_\Gamma \in \mathbb{C}$$

by the formula

$$\langle f, g \rangle_\Gamma = \int_{Y_\Gamma} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k \frac{dxdy}{y^2}.$$

$$\langle f, g\rangle_\Gamma = \int_{Y_\Gamma} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k \frac{dxdy}{y^2}$$

We need to address two issues:

- whether the integrand is well-defined on $Y_\Gamma = \Gamma\backslash\mathfrak{H}$.
- whether the integral over Y_Γ is convergent.

Once two issues are resolved, $\langle -, -\rangle_\Gamma$ is clearly Hermitian-symmetric and and positive definite on $S_k(\Gamma)$.

$$\langle f, g\rangle_\Gamma = \int_{Y_\Gamma} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k \frac{dxdy}{y^2}$$

Let us check whether the integrand is Γ-invariant.

Proposition

$dxdy/y^2$ is $\mathrm{PSL}_2(\mathbb{R})$-invariant.

Proof.

Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$. Note that

$$4dxdy/y^2 = d\tau d\bar{\tau}\mathrm{Im}(\tau)^{-2}.$$

We have seen

$$\gamma^* d\tau = (c\tau + d)^{-2} d\tau$$
$$\gamma^* d\bar{\tau} = (c\bar{\tau} + d)^{-2} d\bar{\tau}$$
$$\mathrm{Im}(\gamma\tau) = \mathrm{Im}(\tau)|c\tau + d|^{-2}$$

from which the proposition follows. $\square$

$$\langle f, g\rangle_\Gamma = \int_{Y_\Gamma} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k \frac{dxdy}{y^2}$$

Let us handle the convergence issue. Recall that the standard fundemandtal domain D of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$ satisfies

$$\overline{D} = \left\{\tau \in \mathfrak{H} \colon |\tau| > 1, -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2}\right\}.$$

Using this, a fundamental domain for Γ can be taken so that its closure is the union of $\gamma\overline{D}$'s where γ's are taken from a set of representatives for $\Gamma\backslash\mathrm{PSL}_2(\mathbb{Z})$.

It suffices to check that

$$\int_{\gamma\overline{D}} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k \frac{dxdy}{y^2}$$

is convergent for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

The convergence of

$$\int_{\gamma\overline{D}} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k \frac{dxdy}{y^2}$$

is equivalent to that of

$$\int_{\overline{D}} (f|_k\gamma)(\tau)\overline{(g|_k\gamma)(\tau)}\mathrm{Im}(\gamma\tau)^k \frac{dxdy}{y^2}.$$

The limit

$$\lim_{t\to\infty} \int_{x+iy\in\overline{D},\, y<t} (f|_k\gamma)(\tau)\overline{(g|_k\gamma)(\tau)}\mathrm{Im}(\gamma\tau)^k \frac{dxdy}{y^2}.$$

is convergent since

$$(f|_k\gamma)(\tau) = O(e^{-Cy})$$

as $y \to \infty$ for some positive C, while other terms have at most polynomial growth as $y \to \infty$.

Here is the conclusion.

Theorem

Let $k \geq 1$ be any integer and Γ be a congruence subgroup. The Petersson inner product $\langle -, - \rangle$ can be defined on $S_k(\Gamma) \times M_k(\Gamma)$ and induces a Hermitian-symmetric and positive definite form on $S_k(\Gamma)$.

Towards injectivity

We would like to use Petersson inner product to show the injectivity of

$$\mathrm{ES}\colon S_k(\Gamma) \to H^1_{\mathrm{cusp}}(\Gamma, L_{k-2}(\mathbb{R})).$$

For this, we need the counterpart of $\langle -, - \rangle$ on $H^1_{\mathrm{cusp}}(\Gamma, L_{k-2}(\mathbb{R}))$. It arises from a $(-1)^k$-symmetric pairing on $L_{k-2}(\mathbb{R})$.

Recall that $L_1(\mathbb{R})$ is the standard representation on $\mathrm{SL}_2(\mathbb{R})$; action of $\mathrm{SL}_2(\mathbb{R})$ on the vector space V of column vectors of size two. Then, $L_{k-2}(\mathbb{R}) = \mathrm{Sym}^{k-2} V$.

An important fact we use here is $V \xrightarrow{\sim} V^*$ as $\mathrm{SL}_2(\mathbb{R})$-representations. The isomorphism is given by the determinant, which is skew-symmetric.

From the skew-symmetric pairing on V and the identification $L_{k-2}(\mathbb{R}) = \mathrm{Sym}^{k-2}V$, we obtain a $(-1)^k$-symmetric pairing on $L_{k-2}(\mathbb{R})$ for all $k \geq 2$.

As a result, we get a pairing on

$$H^1_{\mathrm{cusp}}(\Gamma, L_{k-2}(\mathbb{R})) \times H^1(\Gamma, L_{k-2}(\mathbb{R})) \to \mathbb{R}$$

via the cup product and $H^2(X_\Gamma^{\mathrm{BS}}, \partial X_\Gamma^{\mathrm{BS}}; \mathbb{R}) \xrightarrow{\simeq} H_0(X_\Gamma^{\mathrm{BS}}, \mathbb{R}) = \mathbb{R}$.

In terms of cocycles, the cup product is given by

$$(\phi \cup \psi(g, h)) = -\phi(g) \otimes g\psi(h).$$

Let us denote by $(-,-)$ the induced pairing on $H^1_{\mathrm{cusp}}(\Gamma, \mathbb{R})$.

$$\mathrm{ES}\colon S_k(\Gamma) \to H^1_{\mathrm{cusp}}(\Gamma, L_{k-2}(\mathbb{R}))$$

Note that the source of ES is a vector space over $\mathbb{C}$, while its target is over $\mathbb{R}$.

Theorem

We have

$$(\mathrm{ES}(f), \mathrm{ES}(g)) = -(2i)^{k-3}\left(\langle f, g\rangle + (-1)^{k+1}\langle g, f\rangle\right)$$
$$(\mathrm{ES}(f), \mathrm{ES}(i^{k+1}g)) = 2^{k-2} Re(\langle f, g\rangle)$$
$$(\mathrm{ES}(f), \mathrm{ES}(i^{k+2}g)) = -2^{k-2} i Im(\langle f, g\rangle)$$

for $f, g \in S_k(\Gamma)$

Sketch of proof.

Based on the explicit calculation of the pairing on $L_{k-2}(\mathbb{R})$. A useful observation is $\langle (X - \tau Y)^{k-2}, (X - \bar{\tau} Y)^{k-2}\rangle = (\bar{\tau} - \tau)^{k-2}$. □

Let us use

$$(\mathrm{ES}(f), \mathrm{ES}(i^{k+1}g)) = 2^{k-2}\mathrm{Re}\,(\langle f, g\rangle)$$
$$(\mathrm{ES}(f), \mathrm{ES}(i^{k+2}g)) = -2^{k-2}i\mathrm{Im}\,(\langle f, g\rangle)$$

to establish the injectivity of ES.
If $\mathrm{ES}(f) = 0$, then, $\mathrm{Re}\,(\langle f, g\rangle) = \mathrm{Im}\,(\langle f, g\rangle) = 0$. By the non-degeneracy of the Petersson inner product, we conclude that $\mathrm{ES}(f) = 0$.

Variants

Let $S_k(\Gamma)^c$ be the space of anti-holomorphic cusp forms. We have

$$S_k(\Gamma)^c = \{\overline{f(\tau)} \colon f \in S_k(\Gamma)\}$$

so $S_k(\Gamma)^c \cong S_k(\Gamma)$.

Theorem

We have an isomorphism $S_k(\Gamma) \oplus S_k(\Gamma)^c \xrightarrow{\sim} H^1_{\mathrm{cusp}}(\Gamma, L_{k-2}(\mathbb{C}))$.

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We have an isomorphism $S_k(\Gamma) \oplus S_k(\Gamma)^c \oplus M_k(\Gamma) \xrightarrow{\sim} H^1(\Gamma, L_{k-2}(\mathbb{C}))$.

This is reminiscent of Hodge decomposition. If $k = 2$, it is the Hodge decomposition for the de Rham cohomology of Y_Γ. For $k > 2$, one considers the sheaf of differential forms with values on $L_{k-2}(\mathbb{C})$ and apply the Hodge theory.