Cusps for congruence subgroups

Let $U = \left\{ \pm \left[\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix} \right] \right\}$ be the stabilizer subgroup of $\infty \in \mathbb{P}^1(\mathbb{Q})$. Let $\frac{a}{c} \in \mathbb{Q} \subset \mathbb{P}^1(\mathbb{Q})$, with $\gcd(a, c) = 1$. Send it to

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

for some $c, d \in \mathbb{Z}$, so that $\gamma\infty = \frac{a}{c}$.

Proposition

It is an $\mathrm{SL}_2(\mathbb{Z})$*-equivariant bijection* $\mathbb{P}^1(\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{Z})/U$.

Proof.

The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ is transitive. $\square$

Corollary

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ *be a subgroup. Then, there is a bijection* $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/U \simeq \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$*.*

Corollary

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ *be a subgroup of finite index. Then, the cusps of* Y_Γ *are in bijection with double cosets* $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/U$*.*

Consider the principal congruence subgroup $\Gamma(N)$. We let U act on $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ via the map $U \to \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Proposition

The cusps of $\Gamma(N)$ *are in bijection with* $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/U$.

Proof.

The cusps of $\Gamma(N)$ are in bijection with the double cosets $\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})/U$. One can show that

$$\Gamma(N)\gamma U \mapsto \bar{\gamma} U$$

induces a bijection $\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})/U \simeq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/U$. $\square$

The cosets

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/U$$

can be described explicitly. Let

$$N = \prod_p N_p$$

be the factorization of N into prime powers. Observe that

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_p \mathrm{SL}_2(\mathbb{Z}/N_p\mathbb{Z})$$

by Chinese remainder theorem. Also, we have

$$\mathrm{Im}\left(U \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right) = \prod_p \mathrm{Im}\left(U \to \mathrm{SL}_2(\mathbb{Z}/N_p\mathbb{Z})\right).$$

So we may work 'prime-by-prime'.

Lemma

For all $r \geq 0$, we have

$$\#\mathrm{GL}_2(\mathbb{Z}/p^{r+1}\mathbb{Z}) = (p^2-1)(p^2-p)p^{4r}.$$

Proof.

If $r = 0$, use linear algebra to count all bases of $(\mathbb{Z}/p\mathbb{Z})^2$. To handle the case $r \geq 1$, use the subnormal series

$$\mathrm{GL}_2(\mathbb{Z}/p^{r+1}\mathbb{Z}) \supset 1 + \mathrm{M}_2(p\mathbb{Z}) \supset 1 + \mathrm{M}_2(p^2\mathbb{Z}) \supset \cdots$$

with successive quotients isomorphic to $(\mathbb{Z}/p\mathbb{Z})^4$. □

Corollary

$\#\mathrm{SL}_2(\mathbb{Z}/p^{r+1}\mathbb{Z}) = (p^3-p)p^{3r}$

Proof.

Use $\#(\mathbb{Z}/p^{r+1}\mathbb{Z}) = (p-1)p^r$. □

Consider

$$\overline{U} = \mathsf{Im}\left(U \to \mathrm{SL}_2(\mathbb{Z}/p^{r+1}\mathbb{Z})\right).$$

Proposition

We have

$$\#\overline{U} = \begin{cases} 2 & \text{if } p^{r+1} = 2 \\ 2 \times p^{r+1} & \text{otherwise.} \end{cases}$$

Proof.

Look at the image of $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. □

Corollary

The cardinality of

$$\Gamma(p^{r+1})\backslash\mathbb{P}^1(\mathbb{Q})$$

is given by

$$\#\left(\mathrm{SL}_2(\mathbb{Z}/p^{r+1}\mathbb{Z})/U\right) = \begin{cases} 3 & \text{if } p^{r+1} = 2 \\ \frac{1}{2}(p^2-1)p^{2r} & \text{otherwise.} \end{cases}$$

Proof.

Combine the previous formulas. $\square$

An orbit in

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/U$$

has two representatives

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \gamma' \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix},$$

then, we have

$$\begin{bmatrix} a \\ c \end{bmatrix} = \pm \begin{bmatrix} a' \\ c' \end{bmatrix}.$$

Conversely, $\gamma U = \gamma' U$ if $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] = \pm \left[\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right]$.

$\gcd(a, c) = \gcd(a', c') = 1.$

$$\alpha = \begin{bmatrix} a \\ c \end{bmatrix} \quad \alpha' = \begin{bmatrix} a' \\ c' \end{bmatrix}$$

Proposition

$\Gamma(N)\alpha = \Gamma(N)\alpha'$ *if and only if*

$$\begin{bmatrix} a \\ c \end{bmatrix} \equiv \pm \begin{bmatrix} a' \\ c' \end{bmatrix} \pmod{N}.$$

Description of $\Gamma_1(N)$-orbits. Let U_+ be the subgroup of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ generated by $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Then,

$$U_+ \backslash \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/U$$

is the set of cusps for $\Gamma_1(N)$. Recall that

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/U$$

is classified by

$$\{\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \in (\mathbb{Z}/N\mathbb{Z})^2 \colon \gcd(a, c, N) = 1\}/\{\pm 1\}.$$

We let U_+ act on it by multiplication on the left.

$\gcd(a, c) = \gcd(a', c') = 1.$

$$\alpha = \begin{bmatrix} a \\ c \end{bmatrix} \quad \alpha' = \begin{bmatrix} a' \\ c' \end{bmatrix}$$

Proposition

$\Gamma_1(N)\alpha = \Gamma_1(N)\alpha'$ if and only if

$$\begin{bmatrix} a \\ c \end{bmatrix} \equiv \pm \begin{bmatrix} a' + jc' \\ c' \end{bmatrix} \quad \textit{(mod } N\textit{).}$$

Keep the assumptions: $\gcd(a, c) = \gcd(a', c') = 1$ and $\alpha = \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right], \alpha' = \left[\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right]$

Description of $\Gamma_0(N)$-orbits. Note that $\Gamma_0(N) \subset \Gamma_1(N)$ is normal with quotient isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$. An element $x \in (\mathbb{Z}/N\mathbb{Z})^\times$ acts on $\Gamma_1(N)$-orbits by

$$x \cdot \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} xa \\ x^{-1}c \end{bmatrix}$$

where x^{-1} denotes the multiplicative inverse modulo N.

Proposition

$\Gamma_0(N)\alpha = \Gamma_0(N)\alpha'$ *if and only if*

$$\begin{bmatrix} xa \\ x^{-1}c \end{bmatrix} \equiv \begin{bmatrix} a' + jc' \\ c' \end{bmatrix} \pmod{N}.$$

A sample quiz problem

Let $N = 10$. The genus of $X_1(N)$ is zero.

1. List representatives for the orbits of $\Gamma_1(N)$ acting on $\mathbb{P}^1(\mathbb{Q})$.
2. Among the representatives listed above, determine which one is equivalent to $\frac{5}{12}$.
3. Compute the dimensions of $H^1_{\mathrm{cusp}}(\Gamma_1(N), L_{k-2})$ for $k = 2, 4$.

Example $N = 12$

	$a = 0$	1	2	3	4	5	6	7	8	9	10	11
$c = 0$	$\cdot$	$\otimes$	$\cdot$	$\cdot$	$\cdot$	$\otimes$	$\cdot$	$\otimes$	$\cdot$	$\cdot$	$\cdot$	$\otimes$
1	$\otimes$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
2	$\cdot$	$\otimes$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$
3	$\cdot$	$\otimes$	$\otimes$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$
4	$\cdot$	$\otimes$	$\cdot$	$\otimes$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$
5	$\otimes$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$c = 6$	$\cdot$	$\otimes$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\otimes$
7	$\otimes$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
8	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\otimes$	$\cdot$	$\otimes$
9	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\cdot$	$\otimes$	$\otimes$
10	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\otimes$
11	$\otimes$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Figure 3.2. The cusps of $\Gamma(12)$ and of $\Gamma_1(12)$

We will use the dimension formula:

$$\begin{aligned}&\dim_F H^1_{\mathrm{cusp}}(\Gamma_1(N), L_{k-2}(F))\\ = &\begin{cases}(2g-2)(k-1)+(k-2)s+\delta_k t & \text{if } k>2\\ 2g & \text{if } k=2.\end{cases}\end{aligned}$$

If k is even, $\delta_k t = 0$. Therefore,

$$\dim_{\mathbb{Q}} H^1_{\mathrm{cusp}}(\Gamma_1(10)), L_0(\mathbb{Q})) = 2 \times 0 = 0$$

and

$$\begin{aligned}&\dim_{\mathbb{Q}} H^1_{\mathrm{cusp}}(\Gamma_1(10)), L_2(\mathbb{Q}))\\ =&(-2)\times(4-1)+(4-2)\times 10 = -6+20 = 14.\end{aligned}$$

You my use the lmfdb website to check the answers. Warning: they display dimensions over $\mathbb{C}$, so our dimensions should be twice theirs.

Genus formula

In fact, one can compute the genus of $X_1(N)$, too.

1. $d = [\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma_1(N)]$
2. s: the number of cusps in $X_1(N)$.
3. $s^{(2)}$: the number of elliptic points of order two in $X_1(N)$
4. $s^{(3)}$: the number of elliptic points of order three in $X_1(N)$

Theorem

$$g_1(N) = 1 + \frac{d}{12} - \frac{s^{(2)}}{4} - \frac{s^{(3)}}{3} - \frac{s}{2}$$