# A resolution from modular curve

Let $\Gamma = \Gamma_1(N)$. We have seen that any projective resolution of $A$ as a trivial $A[\Gamma_1(N)]$-module produces a formula for cohomology groups of $\Gamma_1(N)$.

A bar resolution is such a resolution. Is there anything else? In general, if a group $G$ acts freely on contractible space $X$, then such a resolution can be obtained in terms of singular chains in $X$.

The group $\Gamma_1(N)$ naturally acts on $\mathfrak{H}$, which is contractible.

### Proposition

*If $N \geq 4$, the group $\Gamma_1(N)$ is torsion-free.*

### Proof.

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and $\gamma^m = 1$ with $m > 2$. Then, $|\mathrm{tr}(\gamma|) < 2$. It implies that $|\mathrm{tr}(\gamma)| = 0, 1$. On the other hand, if $\gamma \in \Gamma_1(N)$, then $\mathrm{tr}(\gamma) \equiv 2$ modulo $N$. Both cannot hold true simultaneously if $N \geq 4$. □

### Proposition (1)

*Suppose $N \geq 4$. The action of $\Gamma_1(N)$ on $\mathfrak{H}$ is free.*

### Proposition (2)

*Any $\tau \in \mathfrak{H}$ has a finite cyclic stabilizer subgroup.*

### Proposition (3)

*If $\tau \in \mathfrak{H}$ is fixed by a non-trivial element $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, then $\tau^3 = 1$ or $\tau^4 = 1$.*

Note that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). There is an elementary proof for (3). For example, Diamona-Shurman Lemma 2.3.2.

### Proposition

*If $\tau \in \mathfrak{H}$ is fixed by a non-trivial element $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, then $\tau^3 = 1$ or $\tau^4 = 1$.*

Here is a sketch another proof of the above proposition based on a moduli interpretation of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$. This interpretation will play important roles later.

### Proof.

An elliptic curve is $\mathbb{C}/\Lambda$ where $\Lambda$ is a lattice; a discrete subgroup isomorphic to $\mathbb{Z}^2$. Choosing a positively oriented bases, we get $\Lambda \xrightarrow{\sim} \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Positivity ensure $\tau = \omega_2/\omega_1$ has positive imaginary part. Since there is a $\mathrm{SL}_2(\mathbb{Z})$ action on the choice $\Lambda \xrightarrow{\sim} \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, we get the action on the upper half plane parametrizing $\tau$. The stabilizer of $\tau$ corresponds to automorphisms of $\mathbb{C}/\Lambda$.

It turns out that $\mathbb{C}/\Lambda$ can have a non-trivial automorphism if $\Lambda = \mathcal{O}$, where $\mathcal{O} \subset \mathbb{C}$ is a subring. We know that $\mathcal{O}^\times/\{\pm 1\}$ can have order 2 or 3 from algebraic number theory. $\square$

Alternatively, we can prove:-

## Proposition

*Any $\tau \in \mathfrak{H}$ has a finite cyclic stabilizer subgroup.*

## Proof.

Identify $\mathfrak{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. This comes from the decomposition

$$A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} R_\theta$$

where $R_\theta$ is the rotation by $\theta \in \mathbb{R}$. Here $\tau = x + y\sqrt{-1}$. A stabilizer subgroup $\Gamma_\tau \subset \mathrm{SL}_2(\mathbb{Z})$ is isomorphic to a discrete subgroup of $\mathrm{SO}_2(\mathbb{R})$. Since $\mathrm{SO}_2(\mathbb{R})$ is a circle, its discrete subgroup is necessarily finite and cyclic. □

# Topology on $Y_\Gamma = \Gamma\backslash\mathfrak{H}$.

## Proposition

*Let $\gamma \in \Gamma$, $\tau \in \mathfrak{H}$, and $\tau' = \gamma\tau$. Suppose $\tau \neq \tau'$. Then, there exists neighborhoods $\tau \in U$ and $\tau' \in U'$ such that $U \cap U' = \emptyset$.*

## Proof.

It follows from the existence of the $j$-function. □

## Corollary

*$Y_\Gamma$ is a topological surface.*

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