Eisenstein series

Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$. It is convenient to introduce the weight-k operator:

$$(f|_k\gamma)(\tau) := f(\gamma\tau)(c\tau + d)^{-k}.$$

It acts on the right of functions defined on $\mathfrak{H}$.

Note that a modular form for Γ of weight k is invariant under $|_k\gamma$ for all $\gamma \in \Gamma$.

A typical way to construct such a function is to take an average.

Let $k \in \mathbb{Z}$. Consider the following formal series

$$G_k(\tau) = \sum_{c,d}{}' \frac{1}{(c\tau + d)^k}$$

where

$$\sum_{c,d}{}'$$

denotes a series indexed by $(c, d) \in \mathbb{Z}^2 - \{0\}$.

Let $\gamma = \left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$. Then,

$$\begin{aligned} G_k(\gamma\tau) &= \sum_{c,d}{}' \frac{1}{\left(c\,\frac{a'\tau+b'}{c'\tau+d'} + d\right)^k} \\ &= \left(c'\tau + d'\right)^{-k} \times \sum_{c,d}{}' \frac{1}{\left(c\left(a'\tau + b'\right) + d\left(c'\tau + d'\right)\right)^k} \\ &= \left(c'\tau + d'\right)^{-k} \times \sum_{c,d}{}' \frac{1}{\left((ca' + dc')\tau + cb' + dd'\right)^k} \\ &= \left(c'\tau + d'\right)^{-k} \times \sum_{C,D}{}' \frac{1}{\left(C\tau + D\right)^k} \\ &= \left(c'\tau + d'\right)^{-k} G_k(\tau). \end{aligned}$$

If we ignore convergence issues, then $G_k(\tau)$ is invariant under the weight-k operator by construction.

Proposition

Assume that $k > 2$ is even. Then, the series $G_k(\tau)$ converges absolutely and uniformly on any compact subset of $\mathfrak{H}$. In particular, it defines a holomorphic function on $\mathfrak{H}$.

Proof.

Exercise. $\square$

Proposition

Assume that $k > 2$ is even. As $\tau \to i\infty$, $G_k(\tau)$ is bounded. That is to say, $G_k(\tau)$ is holomorphic at infinity.

Proof.

Exercise. $\square$

q-expansion of $G_k(\tau)$

Put $q = e^{2\pi i\tau}$. Let

$$\sigma_s(m) = \sum_{0<d|m} d^{s-1}, \quad m \in \mathbb{Z}_+$$

be the divisor function.

Theorem

The Fourier expansion of $G_k(\tau)$ at infinity is given by

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sigma_{k-1}(m)q^m.$$

sketch of proof

Consider the partial sum

$$\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k}.$$

Let us try to apply the Poisson summation formula:

$$\sum_{d\in\mathbb{Z}} h(x+d) = \sum_{m\in\mathbb{Z}} \hat{h}(m)e^{-2\pi imx}.$$

Here y is a fixed constant and $h(x) = 1/(x+yi)^k$ is a $\mathbb{C}$-valued function of $x \in \mathbb{R}$.

Assuming that $h(x) = 1/(x + yi)^k$ satisfied all the growth conditions necessary for the Poisson summation, it remains to evaluate

$$\hat{h}(x) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i t x} dt.$$

To do this, use the residue formula at the pole of τ^{-k} at $\tau = 0$, or $x = -yi$.

sketch of proof, continueed

To finish the proof, use the Fourier expansion of

$$\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k}.$$

to obtain that for

$$\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^k}$$

and hence for

$$G_k(\tau) = {\sum_{c,d}}' \frac{1}{(c\tau+d)^k}.$$

We define $\Delta(\tau)$ and $j(\tau)$:

Definitions

1. $g_2(\tau) = 60\,G_4(\tau)$
2. $g_3(\tau) = 140\,G_6(\tau)$
3. $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = q + \cdots$
4. $j(\tau) = 1728g_2(\tau)^3/\Delta(\tau) = 1/q + \cdots$