Moduli of Complex Elliptic Curves

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We follow Diamond and Shurman's book 'A First Course on Modular Forms', Springer GTM 228.

0 Introduction

Elliptic curves are important geometric/arithmetic objects that gathered much interests of geometers and number theorists for a long time. There are various ways of studying elliptic curves and their structures.

A particular way that we are to focus on is to study the *moduli space* of elliptic curves(with some enhanced structures). In this note we study the complex analytic theory, which is equivalent to studying the theory over $\mathbb{C}$.

We will clarify various concepts of moduli spaces and work through the theory of elliptic curves over arbitrary rings and more on later weeks.

1 Complex Elliptic Curves

Definition 1.1. A *complex elliptic curve* is a pointed compact Riemann surface of genus 1.

It is well-known that an elliptic curve is isomorphic to a complex torus $\mathbb{C}/\Lambda$ where Λ is a lattice of rank 2. The distingushed point corresponds to 0.

Theorem 1.2. *Any elliptic curve is isomorphic to a complex torus of the form*

$$\mathbb{C}/\Lambda_\tau \ \ (\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau, \ \ \mathrm{Im}\tau > 0).$$

Moreover, this τ is unique up to the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane $\mathfrak{H} = \{\mathrm{Im}\tau > 0\}$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Hence the isomorphism classes of elliptic curves are in one-to-one correspondence with the points of the quotient space

$$\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z}) \overset{j}{\simeq} \mathbb{C},$$

where j is a weight 0 modular function known as the j-invariant of the elliptic curve $\mathbb{C}/\Lambda_\tau$. Here, if we write

$$G_{2k}(\tau) = \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \frac{1}{\omega^{2k}}, \ \ g_2(\tau) = 60G_4(\tau), \ \ g_3(\tau) = 140G_6(\tau),$$

then

$$j = \frac{1728 g_2^3}{g_2^3 - 27g_3^2}.$$

We call the space $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z}) \simeq \mathbb{C}$ the *moduli space* of elliptic curves for this reason.

2 Compactifying the Moduli Space

One of the issues of the moduli space $\mathbb{C}$ is that it is not *compact*. We will compactify $\mathbb{C}$ by adding cusps.

Definition 2.1. The extended upper half-plane $\mathfrak{H}^*$ is a topological space defined as follows: as a set, it is $\mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$. We then have an obvious extension of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathfrak{H}^*$. The basic open sets for the topology are the open sets of $\mathfrak{H}$, and the sets

$$\gamma \cdot \{\mathrm{Im}\tau > \delta\}, \ \ \gamma \in \mathrm{SL}(2,\mathbb{Z}), \ \ \delta > 0.$$

The topology defined reflects some topological properties of actions near cusps. One motivation of adding the rational and the infinity points to the upper half-plane is to think them of the limit points of the action. We would like to explain the significance of the following theorem.

Theorem 2.2. *The quotient space $\mathfrak{H}^*/\mathrm{SL}(2,\mathbb{Z})$ has a structure of a compact Riemann surface. It is isomorphic to $\mathbb{CP}^1$ and contain $\mathbb{C} = \mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$ as an open submanifold, of which the complement is the point ∞.*

The stabilizer of points of $\mathfrak{H}$ are generally $\{\pm I\}$, but orbits of i and the third root of unity ρ have stabilizers of order 4 and 6 respectively, and ∞ has a stabilizer group which is an extension of $\mathbb{Z}$ by $\pm I$. The finite stabilizers mentioned above is not an issue by following well-known lemma from Riemann surface theory(take $G = \mathrm{PSL}(2,\mathbb{Z})$):

Lemma 2.3. *Let X be a Riemann surface and G be an abstract group. If G acts faithfully and holomorphically and properly discontinuously on X, the orbit space X/G has a (unique) Riemann surface structure so the projection map $X \to X/G$ is holomorphic.*

What is interesting is that despite the infinite cyclic stabilizer of ∞ in $\mathrm{PSL}(2,\mathbb{Z})$ we can give chart near ∞ so that $\mathfrak{H}^*/\mathrm{SL}(2,\mathbb{Z})$ is a compactification of $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$. This happens because some open sets of $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$ pulled back to $\mathfrak{H}$ 'straightens' near ∞.

Definition 2.4. Let X be a Riemann surface. A *hole chart* is a chart $\phi : U \to V$ from an open subset of X's to $\mathbb{C}$'s such that there exists a closed subset C of X contained in U that is mapped to a punctured disk of $\mathbb{C}$ contained in V.

Lemma 2.5. *Maintain the notation of the above definition. Then there exists a Riemann surface X' which is set-theoretically X with a point added. The charts of X' are the charts of X and the chart obtained by the inverting extended ϕ^{-1}, where the extended ϕ^{-1}'s domain is the union of V with the punctured point of the disk.*

So the picture is clear by now: the topology on $\mathfrak{H}^*$ described above is defined so the quotient space $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$ will have a hole chart 'near ∞'.

The straightening can be observed by considering the behaviour of the j-invariant, which established the isomorphism $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z}) \simeq \mathbb{C}$. In fact, the j-invariant's Fourier expansion near ∞ is:

$$j = \frac{1}{q} + 744 + 196884q + \cdots \ \ (q = e^{2\pi i\tau}).$$

Hence the open set $\{\mathrm{Im}\tau > \delta\}$ is mapped to $j(\{|q| < e^{-2\pi\delta}\})$, giving the hole chart punctured at ∞ for sufficient large δ ('disk punctured at ∞' is just $\{|z| > R\}$ for some $R > 0$), because j is injective modulo 1 near ∞ by the fact

$$\left(\frac{d}{dq}\right)_{q=0} \frac{1}{j} = \left(\frac{d}{dq}\right)_{q=0} (q - 744q^2 + \cdots) = 1 \neq 0.$$

3 Enhanced Structures on Elliptic Curves

One of the other issues about the moduli space $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$ is that it is not a **fine moduli space**. Existence of nontrivial automorphisms of $\mathbb{C}/(\Lambda_\tau)$ when $\tau = i$ or ρ causes this failure. Such can be overcome when we enhance the elliptic curves. Other reasons to enhance the elliptic curves are to keep track of torsion data.

Definition 3.1. Let E be an elliptic curve, and N a natural number. We denote by $E[N]$ the group of N-torsion points of E. Fixing as isomorphism $E \simeq \mathbb{C}/\Lambda_\tau$, if P, $Q \in E[N]$, then we associate an N^{th} root of unity $e(P,Q)$ defined by

$$e(P,Q) := e^{\frac{2\pi i(ad-bc)}{N}} \text{ where } P = \frac{a+b\tau}{N}, Q = \frac{c+d\tau}{N}.$$

This is a well-defined bilinear pairing on $E[N]$ called the *Weil pairing.*

The Weil pairing is intrinsic to E, which is not a priori clear.

Definition 3.2. Let N be a natural number. We define following level N structures on elliptic curves.

$$\begin{cases} \Gamma_0(N)\text{-structure on } E: & \text{A cyclic subgroup of order } N. \\ \Gamma_1(N)\text{-structure on } E: & \text{A point of order } N. \\ \Gamma(N)\text{-structure on } E: & \text{A pair of generators of } E[N] \text{ with Weil pairing } e^{\frac{2\pi i}{N}}. \end{cases}$$

hence a Γ-enhanced elliptic curve is a tuple of an elliptic curve with additional data, where Γ is one of the above groups.

The gamma groups above are exactly those from the class. The results of the previous sections hold almost word in word.

Theorem 3.3. *Let N be a natural number and Γ be one of the groups $\Gamma_0(N)$, $\Gamma_1(N)$ or $\Gamma(N)$. Then any Γ-enhanced elliptic curve is isomorphic to*

$$\begin{cases} (\mathbb{C}/\Lambda_\tau, < \frac{1}{N} >), & \text{if } \Gamma = \Gamma_0(N). \\ (\mathbb{C}/\Lambda_\tau, \frac{1}{N}), & \text{if } \Gamma = \Gamma_1(N). \\ (\mathbb{C}/\Lambda_\tau, \frac{1}{N}, \frac{\tau}{N}), & \text{if } \Gamma = \Gamma(N). \end{cases}$$

Furthermore, this τ is unique up to Γ-action. Therefore, the moduli space of Γ-enhanced elliptic curves is the quotient

$$\mathfrak{H}/\Gamma$$

which has a structure of a noncompact Riemann surface. Such moduli spaces can be compactified by adding the cusps $\mathbb{Q} \cup \{\infty\}$, giving the compact Riemann surface

$$\mathfrak{H}^*/\Gamma$$

where the added points are Γ-orbits of $\mathbb{Q} \cup \{\infty\}$, which is a finite set since $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma] < \infty$.

Additionally, moduli spaces associated to $\Gamma_1(N)$ and $\Gamma(N)$ above are fine moduli spaces when $N > 4$.