L-function and the Class Number Formula

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Abstract. We study how we can calculate the class number of a number ring by using the Dedekind zeta function. We establish the specific class number formula by using the properties of L-functions under the assumption that underlied number field is an abelian extension of $\mathbb{Q}$.

1 Dedekind zeta function

Denote $\mathbb{A}$ the set of algebraic integers in $\mathbb{C}$. Let K be a number field of degree n over $\mathbb{Q}$, and let $R = \mathbb{A} \cap K$ be the ring of algebraic integers in K. The *class number of* R is the number of ideal classes of R. For each real number $t \geq 0$, let $i(t)$ the number of ideals I of R with index $|R/I| \leq t$.

We give a result about the number of ideals of a number ring from algebraic number theory that we need in our discussion.

Lemma 1. There exists a real number κ, depending on R such that $i(t) = h\kappa t + \epsilon(t)$ where h is the number of ideal classes in R and $\epsilon(t)$ is $O(t^{1-1/n})$. Furthermore, κ can be calculated from the properties of R.

We will deal with the ordinary Dirichlet series, so we use its property from complex analysis.

Lemma 2. Suppose $\sum_{n\leq t} a_n$ is $O(t^r)$ for some real $r \geq 0$. Then $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $\Re(s) > r$, and it is analytic on that domain.

The *Dedekind zeta function* ζ_K of a number field K is defined for $\Re(s) > 1$ by

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{j_n}{n^s}$$

where j_n denotes the number of ideals I of $R = \mathbb{A} \cap K$ with $|R/I| = n$. Lemma 1 shows that $\sum_{n\leq t} j_n$ is $O(t)$, so $\zeta_K(s)$ converges and is analytic on the half-plane $Re(s) > 1$.

Recall that the Riemann zeta function $\zeta(s)$ (which is $\zeta_{\mathbb{Q}}$) extends analytically in $\Re(s) > 0$ except for a simple pole at $s = 1$. We can extend $\zeta_K(s)$ as

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{j_n - k\kappa}{n^s} + h\kappa\zeta(s)$$

for $\Re(s) > 1$, where h is the number of ideal classes in $R = \mathbb{A} \cap K$ and κ is the number occurring in lemma 1. By lemma 1, 2, the ordinary Dirichlet series with coefficients $j_n - h\kappa$ converges to an analytic function on the half-plane $Re(s) > 1 - 1/[K : \mathbb{Q}]$. Thus we obtain a meromorphic extension of $\zeta_K(s)$ on the half-plane $Re(s) > 1 - 1/[K : \mathbb{Q}]$, analytic everywhere except for a simple pole at $s = 1$. Now, from the extended formula for $\zeta_K(s)$, we get $h = \rho/\kappa$, where

$$\rho = \lim_{s \to 1} \frac{\zeta_K(s)}{\zeta(s)}.$$

The value of κ can be computed from the properties of R, hence if we can calculate ρ (without first knowing h), we can get the value of h. In other words, we can use the Dedekind zeta function to calculate the number of ideal classes in a number ring.

In the next paper, we prove explicit class number formula by using L-functions under the assumption that K is an abelian extension of $\mathbb{Q}$.

References

[1] Daniel A. Marcus, *Number Fields*, 2nd ed., Springer, 2018