The Tate Curve

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We discuss the Tate curve, and how it is utilized to compute the cusps. The 0th section is taken from the article of Katz in 'Modular Functions of One Variable III', Springer LNM 350, while the others are from the book of Katz and Mazur.

This week's notes is going to be a preparation for the next week's one, where we will calculate the cusps for one of our moduli problems.

0 The Tate Curve

Consider a holomorphic function f defined on the complex upper half-plane $\mathfrak{H}$. If f satisfies the usual transformation behaviour

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$$

without the holomorphic/meromorphic assumption at ∞, f defines a holomorphic function on the punctured disk

$$\tilde{f}(q) = f(\tau), \quad q = e^{2\pi i\tau} \in \mathbb{D} \setminus \{0\}.$$

Therefore, $\tilde{f}$ admits a Laurent series expansion at 0, with possibly infinite tale(=negative degree part). To say f is meromorphic at 0 is to say the tail is finite, and holomorphic is to say it has no tale.

Now define for a pair of elliptic curve and an invariant differential (E, ω)

$$F(E,\omega) = (\omega_2)^{-k} f(\frac{\omega_1}{\omega_2})$$

where $\mathrm{Im}\frac{\omega_1}{\omega_2} > 0$ and ω_1, ω_2 generate the periods $\{\int_\gamma \omega : \gamma \in H_1(E;\mathbb{Z})\}$. Then we are asking about whether the function

$$F(\mathbb{C}^\times/q^{\mathbb{Z}}, 2\pi i z dz)$$

is contained in $\mathbb{C}((q))$ or $\mathbb{C}[[q]]$. Now, using the Weierstrass $\mathscr{P}$-functions and an appropriate affine coordinate change, the term appearing can easily be seen to be embedded in $\mathbb{P}^2$ as

$$y^2 + xy = x^3 + B(q)x + C(q), \omega = \frac{dx}{x+2y}, \text{ where } B(q), C(q) \in q\mathbb{Z}[[q]].$$

This last equation defines an elliptic curve over $\mathbb{Z}((q))$. We denote this by $Tate(q)$ and call it the *Tate Curve* and denote by ω_{can} the differential form above.

Over this curve we have the Eisenstein series E_4, E_6 and the discriminant Δ, and also the j-invariant

$$j = \frac{1}{q} + 744 + \cdots, \ E_4 = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n, \ E_6 = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n, \ \Delta = q\prod_{n=1}^\infty (1-q^n)^{24}.$$

1 Coarse Moduli Scheme near ∞ via Tate Curve

The formula from the last section shows that $A[[q]] = A[[1/j]]$. The Tate curve from the last section looks like they might be useful for cusp analysis from the motivative definition above. Indeed,

Proposition 1.1. *Let A be a ring, $\mathscr{P}$ be a representable moduli problem on Ell/A, $\mathfrak{M}(\mathscr{P})$ the base scheme of the representing elliptic curve. Then there exists a morphism*

$$\mathscr{P}_{Tate(q)/A((q))} \to \mathfrak{M}(\mathscr{P})_{A((q))}$$

of $A((q))$ scheme which is a finite Galois covering, hence giving as isomorphism

$$\mathscr{P}_{Tate(q)/A((q))}/\pm 1 \simeq \mathfrak{M}(\mathscr{P})_{A((q))}$$

where the ± 1 denotes the automorphism of the Tate curve $x \mapsto x$, $y \mapsto -x-y$.

Recall the notations from the last notes: A is an excellent regular noetherian ring, $\mathscr{P}$ a finite relatively representable moduli problem on Ell/A, normal near infinity. Then we obtain:

Theorem 1.2. *In the above settings, if some prime number l is invertible in A or $\mathscr{P}$ is representable near infinity. Then the finite $A[[q]]$-scheme $\widehat{Cusps}(\mathscr{P})$ is the normalization of $A[[q]]$ in the finite normal $A((q))$ scheme*

$$\mathscr{P}_{Tate(q)/A((q))}/\pm 1.$$

Therefore, to study the cusps one can study the Tate curve. Moreover, if we introduce the commutative group scheme $T[N]$ over $\mathbb{Z}[q, q^{-1}]$ defined by

$$T[N](A) = \{ \text{ collection of pairs } (X, i) \text{ where } X \in A, \ \ 0 \le i < N, \ \ X^N = q^i \}$$

with obvious multiplication $(X, i) * (Y, j) = (XY, i+j)$, then we can furthermore reduce the calculation to $T[N]$ for our 'level N' moduli problems.

This seems plausible, because the groups $T[N]$ algother look like they are made to count torsion points on the Tate curve, and hence for 'level N problems', there must play a crucial role. $T[N]$also possesses an alternating pairing $e_N : T[N] \times T[N] \to \mu_N$ defined by

$$e_N(X, Y) = X^j / Y^i$$

where μ_N is the group scheme

$$\mu_N(A) = \{\text{primitive } N^{\text{th}} \text{ root of unity of } A\}.$$

Also from transcendental theory, one can recover the x and y of the Tate curve of N-torsion points as as elements of $\mathbb{Z}((q)) \otimes \mathbb{Z}[\zeta_N, 1/N]$ where ζ_N is a primitive N^{th} root of unity. Therefore, our level N moduli problems should actually be considered at least over the ring $\mathbb{Z}[\zeta_N]$.

We will do that next week. I will finish the notes for the course by stating theorems of the book concerning the cusps, and do some calculations for special case in the next week.