# Functoriality of the bar construction

Consider the category with:

1. objects: pairs $(G, M)$, $G$ is a group and $M$ is an $A[G]$-module,
2. morphisms: $(\phi, \alpha)\colon (G_1, M_1) \to (G_2, M_2)$ such that $\alpha$ is $\phi$-equivariant

and the category of cochain complexes suppored in non-negative degrees:

1. objects: cochain complexes in non-negative degrees,
2. morphisms: cochain maps($=A$-linear maps commuting with differentials).

The bar construction

$$(G, M) \longmapsto C^\bullet(G, M)$$

is functorial.

# The case of restriction

Here is a special case. Let $G$ be a group and $H \subset G$ be any subgroup. Let $M$ be a $A[G]$-module. $M$ has an $H$-action induced by $H \subset G$. There is a map

$$C^\bullet(G, M) \to C^\bullet(H, M)$$

given by 'restriction'.

It is easy to check that it is a cochain map.

$\Rightarrow$ Induces a map $H^\bullet(G, M) \to H^\bullet(H, M)$, also called 'restriction'.

# Restriction on $H^0(G, M)$

The restriction map $H^0(G, M) \to H^0(H, M)$, or

$$M^G \to M^H$$

is the natural inclusion.

# Restriction on $H^1(G, A)$

Let $G$ act trivially on $A$.

The restriction map $H^1(G, A) \to H^1(H, A)$ is given by restricting a homomorphism $G \to A$ to $H$.

# Induction

If $G$ acts on $M$, then any subgroup $H \subset G$ acts on $M$. A construction dual to it is called induction.

Let $H$ act on the left of $M$. Consider

$$\mathrm{Ind}_H^G(M) := \{f\colon G \to M\colon f(hx) = h(f(x))\}.$$

This is equivalent to considering the right action

$$(f \cdot h)(x) = h^{-1} \cdot (f(hx))$$

of $H$ on all functions and taking the right $H$-invariant part.

# Inductrion, continued

## Proposition

$\mathrm{Ind}_H^G(M)$ *is a left* $G$*-module under* $(g \cdot f)(x) = f(xg)$.

## Proof.

Let $f \in \mathrm{Ind}_H^G(M)$ and $g \in G$. We want to show $g \cdot f$ is right $H$-invariant.

$$\begin{aligned}
((g \cdot f) \cdot h)(x) &= h^{-1}((g \cdot f)(hx)) \\
&= h^{-1}(f(hxg)) \\
&= h^{-1}(f(h(xg))) \\
&= (f \cdot h)(xg) \\
&= f(xg) \\
&= (g \cdot f)(x)
\end{aligned}$$

$\Rightarrow g \cdot f$ is also right $H$-invariant. $\square$

# Shapiro's lemma

This is a fundamental tool for computation.

## Lemma

*Let $H \subset G$ be a subgroup and $M$ be any $A[H]$-module. The map*

$$\begin{aligned} e_1 \colon \mathrm{Ind}_H^G M &\to M \\ f &\mapsto f(1) \end{aligned}$$

*induces a morphism of pairs $(H, M) \to (G, \mathrm{Ind}_H^G(M))$.*

## Proof.

Let $h \in H$. We want to show $h \cdot e_1(f) = e_1(h \cdot f)$. The left-hand-side is $h \cdot f(1)$. The right-hand-side is $f(h)$. They are equal because $f \cdot h = f$. $\square$

## Lemma (Shapiro)

*We have a natural isomorphism*

$$H^\bullet(G, \mathrm{Ind}_H^G M) \simeq H^\bullet(H, M).$$

# Shapiro's lemma, proof

We have a map (by abuse of notation)

$$e_1\colon H^\bullet(G, \mathrm{Ind}_H^G M) \longrightarrow H^\bullet(H, M).$$

induced by $e_1$.

Our goal is to show that $e_1$ is an isomorphism on cohomology. The idea is to use two different ways to compute the cohomology of the group $H$.

Details of the proof to be added..

# The corestriction map

Let $H \subset G$ be a subgroup of finite index. We have a map

$$\mathrm{Ind}_H^G(M) \to M$$

given by

$$f \mapsto \sum_{g \in G/H} g(f(g^{-1}).)$$

The well-definedness follows from the $H$-equivariance of $f \in \mathrm{Ind}_H^G(M)$.

# Corestriction, continued

## Definition

Let $H \subset G$ be a subgroup of finite index. The corestriction map

$$H^\bullet(H, M) \to H^\bullet(G, M)$$

is given by the composition of

$$H^\bullet(H, M) \simeq H^\bullet(G, \mathrm{Ind}_H^G M)$$

and

$$H^\bullet(G, \mathrm{Ind}_H^G M) \to H^\bullet(G, M).$$

# An application

**Proposition**

*Let $G$ be a group with subgroup $H_1, H_2$ of finite index. An isomorphism $\phi\colon H_1 \xrightarrow{\sim} H_2$ induces an endomorphism on $H^\bullet(G, M)$.*

**Proof.**

Compose the restriction, $\phi^*$ and corestriction maps. $\square$