Cohomology of $\mathrm{SL}_2(\mathbb{Z})$ and modular forms

Let $N \geq 1$ be a fixed positive integer.

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon c \equiv 0\,(N) \right\}$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon c \equiv a - 1 \equiv d - 1 \equiv 0\,(N) \right\}$$

These are called congruence subgroups of level N.

Proposition

$\Gamma_1(N) \subset \Gamma_0(N)$ *is a normal subgroup.*

It is convenient to interprete congruence subgroups in terms of lattices. Consider two dimensional free abelian group

$$\Lambda = \mathbb{Z} \oplus \mathbb{Z}$$

viewed as a lattice in $\mathbb{R}^2$, and a sublattice

$$\Lambda' \subset \Lambda$$

such that

$$\Lambda/\Lambda' \simeq \mathbb{Z}/N\mathbb{Z}.$$

Regard elements in $\mathbb{R}^2$ as column vectors. Let $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{R}^2$ on the left. Then, $\mathrm{SL}_2(\mathbb{Z})$ acts on the set of all such sublattices as well. Take a standard one $\Lambda_N = \{(x, y) \in \mathbb{Z}^2 \colon y \equiv 0\,(N)\}$.

Lemma

$\Gamma_0(N)$ *is the stabilizer of* Λ_N.

Since $\Gamma_0(N)$ fixes Λ_N, we have an action

$$\Gamma_0(N) \times \Lambda/\Lambda_N \to \Lambda/\Lambda_N$$

or a homomorphism

$$\chi \colon \Gamma_0(N) \to \mathrm{Aut}\,(\Lambda/\Lambda_N) = (\mathbb{Z}/N\mathbb{Z})^{\times}.$$

Lemma

$\Gamma_1(N)$ is the kernel of χ.

Lemma

χ is surjective.

Proof.

Let $k \in \mathbb{Z}$ with $(N, k) = 1$. Suffices to solve $ad - bc = 1$ with constraints $d \equiv k\,(N)$ and $c \equiv 0\,(N)$. Reduce it to solving $xN + bk = 1$, which is possible whenever $(N, k) = 1$. $\square$

A digression with $\mathrm{PSL}_2(\mathbb{Z})$

Recall that

$$\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}.$$

Lemma

If $N \geq 3$, $\Gamma_1(N) \to \mathrm{PSL}_2(\mathbb{Z})$ is injective.

fractional linear transformation

Let $\mathfrak{H}$ be the upper half plane:

$$\mathfrak{H} = \{\tau \in \mathbb{C} \colon \tau = x + iy, y > 0\}$$

Then, $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathfrak{H}$ as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

This is also known as Möbius transformation.

The action can be extended to $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

hyperbolic, parabolic, elliptic elements elements

Nontrivial elements in $\mathrm{PSL}_2(\mathbb{R})$ fall into three types; hyperbolic, parabolic, and elliptic.

Definition

Let $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ and $t = \mathrm{Tr}(\gamma)$. Then γ is

1. hyperbolic if $t > 2$
2. parabolic if $t = 2$
3. elliptic if $t < 2$.

Proposition

An element $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ is

1. *hyperbolic iff it has two fixed points in $\mathbb{P}^1(\mathbb{R})$*
2. *parabolic iff it has exactly one fixed point in $\mathbb{P}^1(\mathbb{R})$*
3. *elliptic iff it has exactly one fixed point in $\mathfrak{H}$.*

Definition

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a discrete subgroup. $x \in \mathbb{P}^1(\mathbb{R})$ is a cusp if x is fixed by a parabolic element of Γ.

Proposition

Cusps of $\mathrm{PSL}_2(\mathbb{Z})$ *are* $\mathbb{P}^1(\mathbb{Q})$*.*

Proposition

A parabolic element in $\mathrm{PSL}_2(\mathbb{R})$ is conjugate to a unipotent matrix.

Proposition

Let $\Gamma' \subset \Gamma$ be a subgroup of finite index. Then, Γ and Γ' have the same set of cusps.

Corollary

For all $N \geq 1$ and $i = 0, 1$, the cusps of $\Gamma_i(N)$ is $\mathbb{P}^1(\mathbb{Q})$.

Let $f(\tau)\colon \mathfrak{H} \to \mathbb{C}$ be a holomorphic function. Further assume that it is periodic; $f(\tau) = f(\tau + 1)$. Turn in into a holomorphic function $g(q)$ on the punctured unit disc

$$\{q \in \mathbb{C}\colon 0 < |q| < 1\}$$

by putting $q = e^{2\pi i\tau}$.

Definition

$f(\tau)$ is holomorphic at infinity if $g(q)$ can be extended to the whole unit disc. One can similarly define holomorphicity at any given point in $\mathbb{P}^1(\mathbb{R})$.

Definition

Let $\Gamma = \Gamma_1(N)$. Let $k \in Z$. A holomorphic function $f(\tau)\colon \mathfrak{H} \to \mathbb{C}$ is a modular form of weight k if

1. $f(\tau)$ is holomorphic at all cusps
2. $f(\gamma\tau)(c\tau + d)^{-k} = f(\tau)$ for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$.

If, in addition, $f(\tau)$ vanishes at all cusps, then $f(\tau)$ is called cuspidal, or a cusp form.

k is called the weight of $f(\tau)$.

Proposition

If the weight is even, say $2k$, then $f(\tau)(d\tau)^{\otimes k}$ is invariant.

Proof.

Use $d(\gamma\tau) = (d\tau)(c\tau + d)^{-2}$. $\square$

modular forms and cohomology of $\mathrm{SL}_2(\mathbb{Z})$

We would like to connect modular forms and cohomology of $\mathrm{SL}_2(\mathbb{Z})$. Recall the Shapiro lemma:

$$H^\bullet(\mathrm{SL}_2(\mathbb{Z}), \mathrm{Ind}_{\Gamma_1 N}^{\mathrm{SL}_2(\mathbb{Z})} M) = H^\bullet(\Gamma_1(N), M).$$

In other words, cohomology groups of $\mathrm{SL}_2(\mathbb{Z})$ subsumes those of $\Gamma_1(N)$ for all N.

Let $S_k(\Gamma_1(N))$ be the space of all holomorphic cusp forms of weight k for the group $\Gamma_1(N)$. It is a vector space over $\mathbb{C}$.

Consider the special case by taking $k = 2$. Define the period map

$$S_2(\Gamma_1(N)) \to H^1(\Gamma_1(N), \mathbb{C})$$

by the formula

$$f(\tau) \mapsto \left(P_f : \gamma \mapsto \int_{\infty}^{\gamma\infty} f(\tau) d\tau\right)$$

Proposition

P_f is a homomorphism.

Choosing a different cusp

The choice of the cusp $\infty \in \mathbb{P}^1(\mathbb{Q})$ seems arbitrary. Any other cusp $x \in \mathbb{P}^1(\mathbb{Q})$ can be used to define a cocycle

$$P_f^x(\gamma) := \int_x^{\gamma x} f(\tau) d\tau.$$

Proposition

$P_f = P_f^x$

Proof.

Take $\gamma \in \Gamma_1(N)$.

$$\begin{aligned}
\int_\infty^{\gamma\infty} f(\tau)d\tau - \int_x^{\gamma x} f(\tau)d\tau &= \int_\infty^{x} f(\tau)d\tau - \int_{\gamma\infty}^{\gamma x} f(\tau)d\tau \\
&= \int_\infty^{x} f(\tau)d\tau - \int_{\infty}^{x} f(\gamma\tau)d(\gamma\tau) \\
&= \int_\infty^{x} f(\tau)d\tau - \int_{\infty}^{x} f(\tau)d(\tau) \\
&= 0
\end{aligned}$$

Define

$$H^1_{\mathrm{cusp}}(\Gamma_1(N), \mathbb{C}) = \ker \left(H^1(\Gamma_1(N), \mathbb{C}) \to \bigoplus_{x \in \mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_x, \mathbb{C}) \right)$$

where the sum is taken over all cusps x with stabilizer Γ_x.

Proposition

The period map takes values in $H^1_{\mathrm{cusp}}(\Gamma_1(N), \mathbb{C})$.

Proof.

Let $\gamma \in \Gamma_x$. Then,

$$P_f^x(\gamma) = \int_x^{\gamma x} f(\tau) d\tau = \int_x^x f(\tau) d\tau = 0$$

$\square$

Proposition

$H^1_{\mathrm{cusp}}(\Gamma_1(N), \mathbb{C}) = \bigoplus_{x \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_x, \mathbb{C})$.

Proof.

Let $\gamma \in \Gamma_1(N)$. Let $x \in \mathbb{P}^1(\mathbb{Q})$ be a fixed cusp and let $y = \gamma x$. Then, $\Gamma_x \xrightarrow{\sim} \Gamma_y$ under $g \mapsto \gamma g \gamma^{-1}$. It induces an isomorphism $\gamma^* \colon H^\bullet(\Gamma_y, \mathbb{C}) \to H^\bullet(\Gamma_x, \mathbb{C})$. Let $r_z \colon H^\bullet(\Gamma, \mathbb{C}) \to H^\bullet(\Gamma_z, \mathbb{C})$ be the restriction map for a cusp z. We observe

$$r_x = \gamma^* \circ r_y$$

from the interpretation of H^1 as homomorphisms. The proposition follows from the above. □

Without the cuspidality condition

It is customary (for good reasons) to choose ∞ as the base point for period integrals. However, any point $\tau \in \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ works. If $\tau \in \mathfrak{H}$, then we do not need to worry about convergence and the period map can be extended to a larger class of forms.

Let $M_k(\Gamma_1(N))$ be the space of holomorphic modular forms of weight k for the group $\Gamma_1(N)$. Here we do not require cuspidality. In this case, one can choose $\tau \in \mathfrak{H}$ and define the period map

$$M_2(\Gamma_1(N) \to H^1(\Gamma_1(N), \mathbb{C})$$

as before. It is independent of τ. On the other hand, there is no reason for this map to take values i $H^1_{\mathrm{cusp}}(\Gamma_1(N), \mathbb{C})$.