Dimension formula

Recall the Poincare(-Lefschetz) duality: let M be a compat oriented n-manifold with boundary ∂M. Let L be a coefficient system.

Theorem (Poincare duality)

Let L be a $\pi_1(M)$-module. Then,

$$H^{n-p}(M, \partial M; L) = H_p(M; L)$$

for all p.

On the other hand, we have the long exact sequence for a pair (X, Z).

$$0 \to H^0(X; Z) \to H^0(X) \to H^0(Z) \to H^1(X; Z) \to \cdots$$

Let us apply this to the pair $X = X_1(N)$ and $Z = \{\text{cusps}\}$. Also possible is $X = X_1^{\mathrm{BS}}(N)$ and $Z = \partial X$. In either case, suppressing the coefficient system from notation, we have

$$\begin{array}{ccccccc} 0 & \to & H^0(X;Z) & \to & H^0(X) & \to & H^0(Z) \\ & \to & H^1(X;Z) & \to & H^1(X) & \to & H^1(Z) \\ & \to & H^2(X;Z) & \to & H^2(X) & \to & H^2(Z) = 0 \end{array}$$

Here are a few observations.

- Exactness implies that dimensions add up to zero.
- $(X_1(N), \{\text{cusps}\}) \simeq (X_1^{\mathrm{BS}}(N), \partial Z)$ by excision.
- The middle column is given by the Euler characteristic of X.
- The right column is about cusps.
- $H^0(X;Z) = 0$.
- $H^1(X;Z)$ surjects onto $H^1_{\mathrm{cusp}}(\Gamma)$.
- $H^2(X;Z)$ can be computed using Poincare duality.

$$\begin{array}{ccccccc} 0 & \to & H^0(X;Z) & \to & H^0(X) & \to & H^0(Z) \\ & \to & H^1(X;Z) & \to & H^1(X) & \to & H^1(Z) \\ & \to & H^2(X;Z) & \to & H^2(X) & \to & H^2(Z) = 0 \end{array}$$

Proposition

Let $g = g_1(N)$ be the genus of $X_1(N)$. Then,

$$\sum_{i=0}^{2} (-1)^i \dim_F H^i(X_1(N), L_{k-2}(F)) = (2g-2)(k-1).$$

Proof.

A topological oriented surface of genus g has Euler characteristic $2g-2$. On the other hand, L_{k-2} has dimension $k-1$. $\square$

We have similar formulas:

Proposition

Let $g = g_1(N)$ be the genus of $X_1(N)$. Then,

$$\sum_{i=0}^{2}(-1)^i \dim_F H^i(X_1^{\mathrm{BS}}(N), L_{k-2}(F))$$
$$= (2g-2)(k-1) - \#\{\mathit{cusps}\} \times (k-1)$$

$$\begin{array}{rcccccc} 0 & \to & H^0(X;Z) & \to & H^0(X) & \to & H^0(Z) \\ & \to & H^1(X;Z) & \to & H^1(X) & \to & H^1(Z) \\ & \to & H^2(X;Z) & \to & H^2(X) & \to & H^2(Z) = 0 \end{array}$$

We can reduce relative H^2 to H_0:

$$\begin{aligned} & H^2(X_1(N), \{\text{cusps}\}; L_{k-2}) \\ \simeq\, & H^2(X_1^{\mathrm{BS}}(N), \partial X_1^{\mathrm{BS}}(N); L_{k-2}) \\ \simeq\, & H_0(X_1^{\mathrm{BS}}(N), L_{k-2}). \end{aligned}$$

Here we used the excision and the Poincare duality.

Let us compute $H_0(X_1^{\mathrm{BS}}(N), L_{k-2}(F))$. Since $Y_1(N) \to X_1^{\mathrm{BS}}(N)$ is a homotopy equivalence, it suffices to compute $H_0(Y_1(N), L_{k-1}) = H_0(\Gamma_1(N), L_{k-1})$.

Lemma

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. For all $k \geq 2$, $L_{k-2}(F)$ is irreducible as Γ-module.

We postpone its proof for the moment. As a corollary:

$$\dim_F H^2(X_1(N), \{\text{cusps}\}; L_{k-2}(F)) = \begin{cases} 1 & \text{if } k = 2 \\ 0 & \text{if } k > 2. \end{cases}$$

Here, $H_0(\Gamma, L_{k-2})$ is the largest quotient of L_{k-2} on which Γ acts trivially.

Lemma

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. For all $k \geq 2$, $L_{k-2}(F)$ is irreducible as Γ-module.

Proof.

Recall that the action of $\mathrm{SL}_2(\mathbb{Z})$ on L_{k-2} is algebraic. Recall that $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective for all N. Fix a prime p. The surjectivity implies that

$$\Gamma \to \lim_{\leftarrow} \mathrm{SL}_2(\mathbb{Z}/p^n) = \lim_{\leftarrow} \mathrm{SL}_2(\mathbb{Z}_p)$$

has a Zariski dense image. Thus, it suffices to show that the action of $\mathrm{SL}_2(\mathbb{Z}_p)$ on $L_{k-2}(F)$ is irreducible for some algebraically closed field $F \supset \mathbb{Z}_p$. Note that $\mathbb{Z}_p^\times$ is infinite.

Let $V \subset L_{k-2}(F)$ be a non-trivial Γ-stable subspace. Pick a polynomial with maximal Y-degree, say n. Use $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ to show that $n = k-2$. Use $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ with $a \in \mathbb{Z}_p^\times$ to show $Y^n \in V$. Apply $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ to Y^n to show $X^j Y^{k-2-j} \in V$ for all $j < k-2$. $\square$

We return to the long exact sequence:

$$\begin{array}{ccccccc} 0 & \to & H^0(X;Z) & \to & H^0(X) & \to & H^0(Z) \\ & \to & H^1(X;Z) & \to & H^1(X) & \to & H^1(Z) \\ & \to & H^2(X;Z) & \to & H^2(X) & \to & H^2(Z) = 0 \end{array}$$

Proposition

$H^0(X, Z; L_{k-2}(F)) = 0$

Proof.

Identify H^0 with invariant subspaces of $L_{k-2}(F)$. It shows that

$$H^0(X_1^{\mathrm{BS}}(N), L_{k-2}(F)) \to H^0(\partial X_1^{\mathrm{BS}}(N), L_{k-2}(F))$$

is injective. $\square$

Alternatively, one can prove the proposition by triangulating $X_1(N)$ with vertices supported on cusps.

Take $X = X_1^{\mathrm{BS}}(N)$. Let us consider this part:

$$\begin{array}{ccccccc} & & & & H^0(X) & \to & H^0(Z) \\ \to & H^1(X;Z) & \to & H^1(X) & \to & H^1(Z) \end{array}$$

Then $H^0(X) = 0$. We have

$$H^1_{\mathrm{cusp}}(\Gamma_1(N), \mathbb{Q}) = \ker\left(H^1(X) \to H^1(Z)\right).$$

The exactness shows

$$\begin{aligned} &\dim_F H^1_{\mathrm{cusp}}(\Gamma_1(N), L_{k-2}(F)) \\ = &\dim_F H^1(X, Z; L_{k-2}(F)) - \dim_F H^0(Z, L_{k-2}(F)). \end{aligned}$$

Let $Z = \partial X_1^{\mathrm{BS}}(N)$. We have

$$H^1(Z, L_{k-2}(F)) = H_0(Z, L_{k-2}(F)).$$

On each connected component of Z, which is a circle, pick a generator γ for the fundamental group. Then,

$$H_0(Z, L_{k-2}(F)) = (k-1) - \dim_F \operatorname{Im}(\gamma - 1\colon L_{k-2} \to L_{k-2}).$$

The image of the multiplication map has varying dimension depending on cusps.

Let α be a cusp of $\Gamma = \Gamma_1(N)$. Let γ be a generator of the stabilizer subgroup.

Definition

A cusp α is regular if γ is conjugate to $\left[\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right]$. Otherwise, it is conjugate to $\left[\begin{smallmatrix} -1 & h \\ 0 & -1 \end{smallmatrix}\right]$, and α is called irregular.

Let $u = \dim_F \operatorname{Im}(\gamma - 1\colon L_{k-2} \to L_{k-2})$.

Proposition

If α is irregular and n is odd, then $u = k - 1$. Otherwise, $u = k - 2$.

Proof.

To be added later. $\square$

Let s be the number of cusps, and t be that of irregular cusps. Let $\delta_k = 0$ if k is even and 1 otherwise.

Theorem (Dimension formula)

We have

$$\begin{aligned} &\dim_K H^1_{\mathrm{cusp}}(\Gamma_1(N), L_k(F)) \\ &= \begin{cases} (2g-2)(k-1) + (k-2)s + \delta_k t & \text{if } k > 2 \\ 2g & \text{if } k = 2. \end{cases} \end{aligned}$$