RIEMANN SURFACES

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1. Riemann surfaces

In view of historical origin and current development in theoretical physics, Riemann surfaces are very important topics. A Riemann surface is a one dimensional complex manifold. But why we call the manifolds Riemann surfaces? Through 5 essay, we shall see the reason and connections with elliptic curves.

Definition 1.1. Let X be a two-dimensional manifold. *A complex chart* on X is a homeomorphism $\phi : U \to V$, where $U \subset X$ is an open set in X, and $V \subset \mathbb{C}$ is an open set in the complex plane. Let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be two complex charts on X. We say that ϕ_1 and ϕ_2 are compatible if either $U_1 \cap U_2 = \varnothing$, or

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is holomorphic.

Definition 1.2. A *complex atlas* on X is a system $\mathcal{U} = \{\phi_i : U_i \to V_i, i \in I\}$ of charts which are compatible and cover X.

Definition 1.3. Two complex atalses $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if every chart of one is compatible with every chart of the other.

Definition 1.4. A *complex structure* on X is a maximal complex atlas on X, or equivalently, and equivalence class of complex atlases on X

Definition 1.5. A $Riemann\ surface$ is a connected two-dimensional manifold X with a complex structure.

Example 1.6. *The Riemann sphere* $\mathbb{P}^1$. $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ endowed with one point compactification of $\mathbb{C}$ and complex structure. One can identify with the unit sphere in $\mathbb{R}^3$ using the stereographic projection.

Example 1.7. $Tori$. Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. Define

$$\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}.$$

The Quotient topology on $\mathbb{C}/\Gamma$ endowed with a complex structure is a Tori. For detail, See [1].

Definition 1.8. Suppose X and Y are Riemann surfaces. A continuous mapping $f : X \to Y$ is called *holomorphic*, if for every pair of charts $\psi_1 : U_1 \to V_1$ on X and $\psi_2 : U_2 \to V_2$ on Y with $f(U_1) \subset U_2$, the mapping

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2$$

is holomorphic in the usual sense.

A mapping $f : X \to Y$ is called *biholomorphic* if it is bijective and both f, f^{-1} are holmorphic. Two Riemann surface X and Y are called *isomorphic* if there exists a biholomorphic mapping $f : X \to Y$.

Proposition 1.9. *Let* $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *and* $\Gamma' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ *be two lattices in* $\mathbb{C}$. *Then* $\Gamma = \Gamma'$ *if and only if there exists a matrix* $A \in SL(2,\mathbb{Z}) := \{A \in GL(2,\mathbb{Z}) : detA = 1\}$ *such that*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Proof. It suffices to show 'only if' part. If we let A $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$ since $A \in SL(2,\mathbb{Z})$. Then $c\omega_1' - a\omega_2' = (cb - ad)\omega_2 = -\omega_2$. Then $\omega_2 \in \Gamma'$. Since $\omega_1' = a\omega_1 + b\omega_2$ and a, b are integers, also $\omega_1 \in \Gamma'$. Hence $\Gamma \subset \Gamma'$. As $A^{-1} \in SL(2,\mathbb{Z})$, $\Gamma' \subset \Gamma$ by same argument.

Next we want to propose more concrete criterion for classifying complex torus.

Proposition 1.10. *Every torus* $X = \mathbb{C}/\Gamma$ *is isomorphic to a torus of the form*

$$X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

where $\tau \in \mathbb{C}$ *satisfies* $Im(\tau) > 0$.

Proof. Let ω_1, ω_2 be generator of Γ. Then we can choose α such that $\alpha\omega_1$ is real in complex plane and $|\alpha|{=}(|\omega_1|)^{-1}$. If $\arg(\alpha\omega_2) > \pi$, replace by $-\alpha$. If we set $\tau{=}\alpha\omega_2$, multiplying α on $\mathbb{C}$ induces a holomorphic map from $\mathbb{C}/\Gamma$ to $X(\tau)$ since $\alpha\Gamma \subset \Gamma'$. But the two lattice is identified by multiplying α. So the map is biholomorphic. Hence the two tori is isomorphic. Note that the volume of the lattice of $X(\tau)$ is differ from that of Γ.

Corollary 1.11. *Suppose* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ *and* $Im(\tau) > 0$. *Let*

$$\tau' := \frac{a\tau + b}{c\tau + d}.$$

Then the tori $X(\tau)$ *and* $X(\tau')$ *are isomorphic.*

Proof. Set $\binom{\omega_1}{\omega_2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{\tau}{1}$. Let Γ',Γ be each lattice generated by $(\tau', 1)$ and $(\tau, 1)$. Let $\alpha{=}(c\tau + d)^{-1}$. Then $\Gamma'{=}\alpha\Gamma$. Then by proposition (1.9),(1.10), $X(\tau)$ and $X(\tau')$ are isomorphic.

In general, Given a map from $\mathbb{C}/\Gamma$ to $\mathbb{C}/\Gamma'$, then there exists α, β with $\alpha\Gamma \subset \Gamma'$ such that the image of map is $\alpha\Gamma + \beta {+} \Gamma'$.

References

[1] O.Forster, *Lectures on Riemann Surfaces.* Springer-Verlag, New York, 1999.