Compactifications of Y_Γ

Observe that $Y_{\mathrm{SL}_2(\mathbb{Z})}$ is not compact. It as a cusp towards the infinity.

It follows that $Y_1(N)$ is never compact, either.

Compactifying $Y_{\mathrm{SL}_2(\mathbb{Z})}$

Around infinity, $Y_{\mathrm{SL}_2(\mathbb{Z})}$ looks like a punctured disc. Filling in with a point gives us a compact space $Y_{\mathrm{SL}_2(\mathbb{Z})}$.

The additional point at infinity can be identified with $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{P}^1(\mathbb{Q})$, the $\mathrm{SL}_2(\mathbb{Z})$-orbit of cusps.

Compactifying $Y_1(N)$

To compactify $Y_1(N)$, one can add a point to each $\Gamma_1(N)$-orbit of $\mathbb{P}^1(\mathbb{Q})$.

$X_1(N)$ denotes the compactified surface. It is still a Riemann surface.

Little changes if we consider $\Gamma_0(N)$ instead of $\Gamma_1(N)$ as long as it has no fixed points. Even if there are fixed points, one can use inflation restriction sequence and just consider $\Gamma_1(N)$.

Borel-Serre compactification

There is another compactification of $Y_1(N)$. A compactification of this type in full generality is due to Borel and Serre.

In the case of $Y_1(N)$, it is simply given by adding discs around cusps.

Let $X_1^{\mathrm{BS}}(N)$ be the resulting compact topological surface. Note that it cannot be given a structure of a Riemann surface.

One advantage of considering the Borel-Serre compactification:-

$$Y_1(N) \hookrightarrow X_1^{\mathrm{BS}}(N)$$

is a homotopy equivalence.

We need compactifications of Y_Γ because we will rely on duality theorems to prove the dimension formula.

Here is a general form: let M be a compat oriented n-manifold with boundary ∂M. Let L be a coefficient system.

Theorem (Poincare duality)

Let L be a $\pi_1(M)$-module. Then,

$$H^{n-p}(M, \partial M; L) = H_p(M; L)$$

for all p.

On the other hand, we have the long exact sequence for a pair (X, Z).

$$0 \to H^0(X;Z) \to H^0(X) \to H^0(Z) \to H^1(X;Z) \to \cdots$$

Let us apply this to the pair $X = X_1(N)$ and $Z = \{\text{cusps}\}$. Also possible is $X = X_1^{\mathrm{BS}}(N)$ and $Z = \partial X$. In either case, suppressing the coefficient system from notation, we have

$$\begin{array}{ccccccc} 0 & \to & H^0(X;Z) & \to & H^0(X) & \to & H^0(Z) \\ & \to & H^1(X;Z) & \to & H^1(X) & \to & H^1(Z) \\ & \to & H^2(X;Z) & \to & H^2(X) & \to & H^2(Z) = 0 \end{array}$$

Here are a few observations.

- Exactness implies that dimensions add up to zero.
- $(X_1(N), \{\text{cusps}\}) \simeq (X_1^{\mathrm{BS}}(N), \partial Z)$ by excision.
- The middle column is given by the Euler characteristic of X.
- The right column is about cusps.
- $H^0(X;Z) = 0$.
- $H^1(X;Z)$ surjects onto $H^1_{\mathrm{cusp}}(\Gamma)$.
- $H^2(X;Z)$ can be computed using Poincare duality.

Let $L = L_{k-2}(F)$, where $F \subset \mathbb{C}$ is a subfield. Note that

$$\dim_F(L_{k-2}(F)) = k - 1.$$

Let $g_1(N)$ be the genus of $X_1(N)$. Let $Z_1(N)$ be the set of $\Gamma_1(N)$-orbits of cusps. For each $\alpha \in Z_1(N)$ let γ_α be a generator of its stabilizer subgroup.

Theorem (Dimension formula)

We have

$$\begin{aligned} &\dim_K H^1_{\mathrm{cusp}}(\Gamma_1(N), L_k(F)) \\ &= (2g-2)(k-1) + \sum_{\alpha \in Z_1(N)} \dim_F(\gamma_\alpha - 1)L_{k-2}(F). \end{aligned}$$