L-function and the theorem on arithmetic progressions

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March 22, 2019

Abstract. We define the Dirichlet L-function and use its properties to prove that there exist infinitely many prime numbers p such that $p \equiv a$(mod m) where a and m are relatively prime integers ≥ 1.

Let (λ_n) be an increasing sequence of real numbers tending to infinity. A Dirichlet series with exponents (λ_n) is a series of the form

$$f(z) = \sum a_n e^{-\lambda_n z} \quad (a_n, z \in \mathbb{C})$$

These are the properties of Dirichlet series from complex analysis.

Proposition 1. If f converges for $z = z_0$, it converges for $Re(z) > Re(z_0)$ and it is holomorphic in that domain.

Proposition 2. Let a_n are real ≥ 0. Suppose that f converges for $Re(z) > \rho$ and that f can be extended analytically to a function holomorphic in a neighborhood of the point $z = \rho$. Then there exists $\epsilon > 0$ such that f converges for $Re(z) > \rho - \epsilon$.

When $\lambda_n = \log n$, we get $f(s) = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, which is a form of the zeta function and L-function. The notation s being traditional for the variable.

Recall the properties of the zeta function $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}}$, which equalities holds for $Re(s) > 1$.

Proposition 3. (a) $\zeta(s)$ is holomorphic and nonzero for $Re(s) > 1$.
(b) $\zeta(s) = \frac{1}{s-1} + \phi(s)$, where $\phi(s)$ is holomorphic for $Re(s) > 0$. Thus $\zeta(s)$ extends analytically for $Re(s) > 0$ and has a simple pole at $s = 1$.

Let G be a finite abelian group. A character of G is a homomorphism of G into the multiplicative group $\mathbb{C}^*$ of complex numbers. The characters of G form a group $Hom(G, \mathbb{C}^*)$ which we denote by $\hat{G}$ and call the *dual* of G. Note that the group $\hat{G}$ is also a finite abelian group of the same order as G. For $\chi \in \hat{G}$ and $x \in G$, we have $|\chi(x)| = 1$ because $\chi(x)^n = \chi(x^n) = \chi(1) = 1$ where n is the order of x.

Proposition 4. Let n be the order of G and let $\chi \in \hat{G}$. Then

$$\sum_{x \in G} \chi(x) = \begin{cases} n, & if\ \chi = 1 \\ 0, & if\ \chi \neq 1. \end{cases}$$

Proof) The first formula is obvious. To prove the second, choose $y \in G$ such that $\chi(y) \neq 1$. Then $\chi(y) \sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(xy) = \sum_{x \in G} \chi(x)$, hence $(\chi(y) - 1) \sum_{x \in G} \chi(x) = 0$. Since $\chi(y) \neq 1$, this implies $\sum_{x \in G} \chi(x) = 0$.

Proposition 5. Let $x \in G$. Then

$$\sum_{\chi \in \hat{G}} \chi(x) = \begin{cases} n, & if\ x = 1 \\ 0, & if\ x \neq 1. \end{cases}$$

This follows from Proposition 4 applied to the dual group $\hat{G}$.

Let $m \geq 1$ be a fixed integer. We let $(\mathbb{Z}/m\mathbb{Z})^*$ the multiplicative group of invertible elements of the ring $\mathbb{Z}/m\mathbb{Z}$ and let χ be a character of $(\mathbb{Z}/m\mathbb{Z})^*$. We extend the domain of χ to whole $\mathbb{Z}$ by putting $\chi(a) = 0$ if a is not prime to m.
The corresponding L-function is defined by the Dirichlet series

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$$

Proposition 6. For $\chi = 1$, we have $L(s,1) = F(s)\zeta(s)$ with $F(s) = \prod_{p|m}(1-p^{-s})$.
In particular $L(s,1)$ extends analytically for $Re(s) > 0$ and has a simple pole at $s = 1$.

Proposition 7. For $\chi \neq 1$, the series $L(s,\chi)$ converges absolutely in $Re(s) > 1$; one has

$$L(s,\chi) = \prod_{p\ prime} \frac{1}{1-\chi(p)p^{-s}} \quad for\ Re(s) > 1$$

Proof) Since $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ converges for $\alpha > 1$, $\alpha \in \mathbb{R}$, and $\chi(n)$ are bounded, we see that $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ converges absolutely for $Re(s) > 1$.
Since $\chi(ab) = \chi(a)\chi(b)$ for every $a, b \in \mathbb{Z}/m\mathbb{Z}$, we get

$$\sum_{n=1}^{\infty} \chi(n)/n^s = \prod_{p\ prime} \left(\sum_{m=1}^{\infty} \chi(p^m)/p^{-ms} \right) = \prod_{p\ prime} \frac{1}{1-\chi(p)p^{-s}}$$

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The key point of Dirichlet's proof is to show that $L(1,\chi) \neq 0$ for all $\chi \neq 1$.
We continue on the next paper.

References

[1] J.-P.Serre, *A Course in Arithmetic*, Springer, 1973