More homological algebra

The key idea is to manipulate chain complexes 'up to homotopy'.

We defined group cohomology in terms of a bar complex. That's sufficient as a definition. We will review basic notions in homological algebra that will provide us with more tools.

Let (M^{\bullet}, d) be a cochain complex. Recall that

$$H^i(M^{\bullet},d) = Z^i/B^i$$

where $B^{\bullet} = \operatorname{Im}(d)$.

In other words, a cohomology class is a cochain defined up to Im(d). Two cocycles ϕ and ϕ' represent the same cohomology class if

$$\phi' = \phi + d\epsilon$$

for some cochain ϵ . We may view ϵ as a homotopy from ϕ to ϕ' .

A convention

Let M^{\bullet} be a graded module. If $m \in M^r$ is homogeneous of degree r, then write

$$|m|=r$$
.

If $m \in M^{\bullet}$ is a homogeneous element, then let |m| be its degree.

Let M^{\bullet} and N^{\bullet} be two cochain complexes. Graded maps (not necessarily cochain maps) for another graded module

$$\operatorname{Hom}^{\bullet}(M^{\bullet}, N^{\bullet}).$$

There is a natural differential on it; for $f \in \operatorname{Hom}^{\bullet}(M^{\bullet}, N^{\bullet})$

$$(df)(m) = d(fm) + (-1)^{|f|+|d|} f(dm).$$

The sign rule here is an example of the 'Koszul sign'.

An illustration:

$$M^{i-1} \xrightarrow{h} M^{i} \xrightarrow{h} M^{i+1}$$

$$N^{i-1} \xrightarrow{h} N^{i} \xrightarrow{h} N^{i+1}$$

Here, $h \in \operatorname{Hom}^{\bullet}(M^{\bullet}, N^{\bullet})$ is a map of degree -1. Applying the differential,

$$dh = k$$

is a map of degree zero. It acts on cochains as

$$k(m) = (dh)(m) = d(hm) + (-1)^{|d|+|h|}h(dm)$$

or

$$k(m) = (dh)(m) = d(hm) - (-1)^{|h|}h(dm).$$

Proposition

Let M^{\bullet} , N^{\bullet} be cochain complexes. Then, $(\operatorname{Hom}^{\bullet}(M^{\bullet}, N^{\bullet}), d)$ is a cochain complex.

Proof.

Take $h \in \operatorname{Hom}^{\bullet}(M^{\bullet}, N^{\bullet})$ and $m \in M^{\bullet}$.

$$(d(dh))(m) = d((dh)m)) + (-1)^{|dh||d|}(dh)(dm)$$

$$= d((dh)m)) + (-1)^{|h|}(dh)(dm)$$

$$= d \left(d(hm) + (-1)^{|d|+|h|}(h(dm))\right) + (-1)^{|h|}(dh)(dm)$$

$$= (-1)^{|d|+|h|}dhdm + (-1)^{|h|}(dh)(dm)$$

$$= (-1)^{|d|+|h|}dhdm + (-1)^{|h|}\left(dh(dm) + (-1)^{|dh|+|d|}hd(dm)\right)$$

$$= (-1)^{|d|+|h|}dhdm + (-1)^{|h|}dh(dm)$$

$$= 0$$

 M^{\bullet} , N^{\bullet} : cochain complexes

Observation

A cochain map is a 0-cycle in $\operatorname{Hom}^{\bullet}(M^{\bullet}, N^{\bullet})$.

Definition

Let $f, f' \in \text{Hom}^{\bullet}(M^{\bullet}, N^{\bullet})$. They are called homotopic if f' = f + dh for some h.

Proposition

If two cochain maps f and f' are homotopic, then they induce the same map on cohomology.

Proof.

We have df - fd = dh. It suffices to show: for any cocycle m, (dh)(m) is coboundary. Indeed,

$$(dh)(m) = d(hm) + (-1)^{|d|+|h|}h(dm) = d(hm)$$

is a coboundary.

Definition

A quasi-isomorphism from M^{\bullet} to N^{\bullet} is a cochain map $f: M^{\bullet} \to N^{\bullet}$ which induces an isomorphism on cohomology groups.

Remark

A quasi-isomorphism $f: M^{\bullet} \to N^{\bullet}$ may not admit another quasi-isomorphism $f': N^{\bullet} \to M^{\bullet}$ which, on cohomology groups, is inverse to f.

Definition

A cochain complex M^{\bullet} is acyclic if its cohomology groups are zero.

Note that M^{\bullet} is acyclic iff the zero map $0 \to M^{\bullet}$ is a quasi-isomorphism.

Projective resolution

Let R be any A-algebra, possibly non-commutative. Typically, we consider R = A[G]. We work with the category of R-modules.

Definition

A projective resolution of an R-module M is a chain complex consisting of projective R-modules

$$\cdots \to P_2 \to P_1 \to P_0$$

which is quasi-isomorphic to M.

Often, we say that an acyclic complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M$$

is a projective resolution of M. Equivalently, $P_{ullet} o M$ is a quasi-isomorphism.

Let P_{\bullet} , M_{\bullet} be two chain complexes supported in non-negative degrees. Assume that each P_r is projective for $r \geq 1$ and that M_{\bullet} is acyclic.

Lemma (Projective to acyclic lemma)

Any R-linear map $P_0 \to M_0$ extends to a chain map $P_{ullet} \to M_{ullet}$. The extension is unique up to homotopy.

The proof of the lemma is based on an induction argument.

Suppose we have a map $f_0 \colon P_0 \to M_0$. As the induction hypothesis, assume that can extend any such f_0 to f_1 .

$$\cdots P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

$$\downarrow f_1 \qquad \qquad \downarrow f_0$$

$$\cdots M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0$$

From this, one obtains

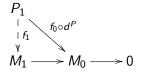
$$\cdots P_2 \longrightarrow B_1^P \longrightarrow 0$$

$$\downarrow^{f_1}$$

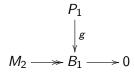
$$\cdots M_2 \longrightarrow B_1^M \longrightarrow 0$$

and the bottom chain complex remains acyclic. Therefore, the hypothesis allows us to extend f_1 to $f_2 \colon P_2 \to M_2$.

We need to initiate the induction process. Suppose we have a map $f_0: P_0 \to M_0$. We get f_1 by applying the lifting property to $f_0 \circ d^P$:



If two maps f_1 , f_1' are obtained in this way, then $g = f_1 - f_1'$ factors through B_1^M by the acyclicity of M_{\bullet} .



The map $M_2 \to B_1$ is surjective by definition. Thus g can be lifted to $h: P_1 \to M_2$. Then, g = dh by construction.

Observe: if $f_0 = dk$, then f_1 can be taken to be $k \circ d^P$. This shows that the any to sequences (f_0, f_1, f_2, \cdots) and $(f_0, f'_1, f'_2, \cdots)$ are homotopic.

Uniqueness of projective resolution

Corollary

If P_{\bullet} , P'_{\bullet} are two projective resolutions of M, then there are quasi-isomorphisms $P_{\bullet} \to P'_{\bullet}$ and $P'_{\bullet} \to P_{\bullet}$ both inducing the identity on homology. Furthermore, these quasi-isomorphisms are unique up to homotopy.