Riemann-Roch theorem

Riemann-Roch theorem

We will evaluate the dimension of $S_k(\Gamma)$ using the Riemann-Roch theorem. It is a general theorem for Riemann surfaces.

Definition

Let X be a topological surface. A Riemann surface structure on X is given by a covering $\mathcal{U} = \{U \subset X\}$ by contractible open subsets together with holomorphic embeddings $\phi_U \colon U \to \mathbb{C}$ satisfying the following compatibility condition: $\phi_U^{-1} \circ \phi_V \colon \phi_U(U \cap V) \to \phi_V(U \cap V)$ is a holomorphic map for every $U, V \in \mathcal{U}$.

We will only deal with a surface X of finite type.

Divisors

Let X be a Riemann surface.

Definition

A divisor in on X is a formal $\mathbb{Z}$-linear combination of points in X. A divisor D is called effective if its coefficients are non-negative. It is convenient to write $D \geq 0$ instead of 'D is effective'.

Let f be a non-zero meromorphic function on X. For a point $P \in X$, let $\mathrm{ord}_P(f)$ be the order of vanishing of f at P; it is negative if f has a pole at P. Define

$$\mathrm{div}(f) = \sum_{P \in X} \mathrm{ord}_P(f) P.$$

Definition

A divisor is principal if it is of the form $\mathrm{div}(f)$.

Meromorphic functions

Let X be a Riemann surface. Let $\mathbb{C}(X)$ be the field of all meromorphic functions on X.
Suppose that $D = \sum_P n_P P$ is a divisor. Then,

$$M(D) := \{f \in \mathbb{C}(X) \colon \operatorname{div}(f) + D \geq 0\}$$

forms a vector space over $\mathbb{C}$. Here we regard $\operatorname{ord}_P(f) = \infty$ when f is constantly zero.

Let $m(D)$ be the dimension of $M(D)$. The Riemann-Roch theorem tells us how to evaluate $m(D)$ in terms of the genus of X and the degree of D:

$$\deg(\sum_P n_P P) := \sum_P n_P.$$

Let g be the genus of X.

Theorem (Riemann-Roch)

If $\deg(D) > 2g - 2$, *then* $m(D) = \deg(D) - g + 1$.

The Riemann-Roch theorem tells us about $m(D)$, but modular forms are not quite meromorphic functions. If k is even and $f(\tau)$ is a modular form of weight k, we have seen that

$$f(\tau)(d\tau)^{\otimes k/2}$$

is invariant. Therefore, a modular form is more like a meromorphic differential.

Definition

A meromorphic differential ω of degree n on a Riemann surface X is determined by the following data: a covering $\mathcal{U}$ of contractible open subsets with local coordinates z_U on $U \in \mathcal{U}$, together with a family of meromorphic functions

$$\omega_U \in \mathbb{C}(U)$$

such that $\omega_U(dz_U)^{\otimes n}$ and $\omega_V(dz_V)^{\otimes n}$ agree on $U \cap V$.

Divisor associated to a meromorphic modular form

Let $f(\tau)$ be a meromorphic modular form for some $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. This means, by definition, that $f(\tau)$ can have poles in X_Γ. Let $\alpha \in X_\Gamma - Y_\Gamma$ be a cusp, and q_α be a uniformizer at α. Then,

$$f(\tau) = \sum_{m=r}^{\infty} c(f, \alpha) q_\alpha^m$$

with $c(f, \alpha) \neq 0$. Define

$$\mathrm{ord}_\alpha(f) = r$$

and

$$\mathrm{div}(f) = \sum_P \mathrm{ord}_P(f).$$

Divisor associated to a meromorphic differential

Suppose $\omega \in \Omega^{k/2}(\Gamma)$ is a meromorphic differential. For each $P \in X_\Gamma$, define

$$\mathrm{ord}_P(\omega) := \mathrm{ord}_P(\omega_U)$$

for a small open set U containing P, so that

$$\omega|_U = \omega_U (dz_U)^{\otimes k/2}.$$

Note that $\mathrm{ord}_P(\omega_U)$ does not depend on the choice of U.

Define

$$\mathrm{div}(\omega) = \sum_P \mathrm{ord}_P(\omega).$$

Modular forms and meromorphic differentials

Let $k \geq 2$ be an even integer. Let $A_k(\Gamma)$ be the space of meromorphic modular forms of weight k. Let $\Omega^{k/2}(\Gamma)$ be the space of meromorphic differentials of degree $k/2$ on X_Γ.

Theorem

We have an isomorphism

$$A_k(\Gamma) \xrightarrow{\sim} \Omega^{k/2}(\Gamma)$$

given by $f \mapsto f(\tau)(d\tau)^{\otimes k/2}$.

The isomorphism

$$A_k(\Gamma) \xrightarrow{\sim} \Omega^{k/2}(\Gamma)$$

is not compatible with ord$(-)$. Indeed, if $f \in A_k(\Gamma)$ correspondes to f, then

$$\mathrm{div}(\omega) = \mathrm{div}(f) + \frac{k}{2}\mathrm{div}(d\tau).$$

This is because

$$dq = d(e^{2\pi i\tau}) = 2\pi i q d\tau$$

or

$$d\tau = 2\pi i \frac{dq}{q}.$$

Proposition

For any even $k \geq 2$, there exists a non-zero form $f_k \in A_k(\Gamma)$.

Proof.

Recall that $j(\tau)$ is meromorphic function. We have $dj(\tau) \in A_2(\Gamma)$ and $(d(j\tau))^{k/2} \in A_k(\Gamma)$. $\square$

Then, we can identify

$$A_k(\Gamma) = \mathbb{C}(X) \cdot f_k(\tau)$$

once we fix a non-zero $f_k(\tau) \in A_k(\Gamma)$.

Theorem (Riemann-Roch)

If $\omega \in \Omega^1(\Gamma)$, then $deg(\omega) = 2g - 2$.

Corollary

$div(f_k(\tau)) = k(g-1)$.

Let $D_\infty \subset X_\Gamma$ be the sum of all cusps. Combining previous results, one obtains: for even $k \geq 2$

$$\begin{aligned} S_k(\Gamma) &= \{f \in A_k(\Gamma)\colon \operatorname{div}(f) - D_\infty \geq 0\} \\ &= \{f \in \mathbb{C}(X)\colon \operatorname{div}(f) + \operatorname{div}(f_k) - D_\infty \geq 0\} \\ &= M(\operatorname{div}(f_k) - D_\infty). \end{aligned}$$

To apply Riemann-Roch, need to check

$$\deg(\operatorname{div}(f_k) - D_\infty) > 2g - 2.$$

If $k \geq 4$, this follows from the genus formula; $g = 1 + d/12 - s/2$.

Theorem

If $k \geq 4$ *and even, then* $\dim_{\mathbb{C}} S_k(\Gamma) = (k-1)(g-1) - s$

Proof.

Collecting previous results, we get

$$\dim_{\mathbb{C}} S_k(\Gamma) = m(\mathrm{div}(f_k) - D_\infty) = \deg(\mathrm{div}(f_k) - D_\infty) - g + 1$$

and

$$\deg(\mathrm{div}(f_k) - D_\infty) - g + 1 = k(g-1) - s.$$

$\square$

Corollary

If $k \geq 4$ *and even, then*

$$\dim_{\mathbb{Q}} H^1_{\mathrm{cusp}}(\Gamma, L_{k-2}(\mathbb{Q})) = k(g-1) - s$$

For $k = 2$, need to use a more general version of Riemann-Roch:-

Theorem (Riemann-Roch)

Let X be a compact Riemann surface. There exists a non-zero $\omega_X \in \Omega^1(X)$. Furthermore, for any D,

$$m(D) = \deg(D) - g + 1 + m(div(\omega_X) - D.$$

For odd $k \geq 3$, use $M_k(\Gamma) \to M_{2k}(\Gamma)$ sending $f(\tau) \mapsto f(\tau)^2$, and apply the same strategy.