Riemann-Roch theorem

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We will evaluate the dimension of $S_k(\Gamma)$ using the Riemann-Roch theorem. It is a general theorem for Riemann surfaces.

Definition

Let X be a topological surface. A Riemann surface structure on X is given by a covering $\mathcal{U} = \{U \subset X\}$ by contractible open subsets together with holomorphic embeddings $\phi_U \colon U \to \mathbb{C}$ satisfying the following compatibility condition: $\phi_U^{-1} \circ \phi_V \colon \phi_U(U \cap V) \to \phi_V(U \cap V)$ is a holomorphic map for every $U, V \in \mathcal{U}$.

We will only deal with a surface X of finite type.

Divisors

Let X be a Riemann surface.

Definition

A divisor in on X is a formal $\mathbb{Z}$-linear combination of points in X. A divisor D is called effective if its coefficients are non-negative. It is convenient to write $D \geq 0$ instead of 'D is effective'.

Let f be a non-zero meromorphic function on X. For a point $P \in X$, let $\mathrm{ord}_P(f)$ be the order of vanishing of f at P; it is negative if f has a pole at P. Define

$$\mathrm{div}(f) = \sum_{P \in X} \mathrm{ord}_P(f) P.$$

Definition

A divisor is principal if it is of the form $\mathrm{div}(f)$.

Meromorphic functions

Let X be a Riemann surface. Let $\mathbb{C}(X)$ be the field of all meromorphic functions on X.
Suppose that $D = \sum_P n_P P$ is a divisor. Then,

$$M(D) := \{f \in \mathbb{C}(X) \colon \mathrm{div}(f) + D \geq 0\}$$

forms a vector space over $\mathbb{C}$. Here we regard $\mathrm{ord}_P(f) = \infty$ when f is constantly zero.

Let $m(D)$ be the dimension of $M(D)$. The Riemann-Roch theorem tells us how to evaluate $m(D)$ in terms of the genus of X and the degree of D:

$$\deg(\sum_P n_P P) := \sum_P n_P.$$

Let g be the genus of X.

Theorem (Riemann-Roch)

If $\deg(D) > 2g - 2$, *then* $m(D) = \deg(D) - g + 1$.

The Riemann-Roch theorem tells us about $m(D)$, but modular forms are not quite meromorphic functions. If k is even and $f(\tau)$ is a modular form of weight k, we have seen that

$$f(\tau)(d\tau)^{\otimes k/2}$$

is invariant. Therefore, a modular form is more like a meromorphic differential.

Definition

A meromorphic differential ω of degree n on a Riemann surface X is determined by the following data: a covering $\mathcal{U}$ of contractible open subsets with local coordinates z_U on $U \in \mathcal{U}$, together with a family of meromorphic functions

$$\omega_U \in \mathbb{C}(U)$$

such that $\omega_U(dz_U)^{\otimes n}$ and $\omega_V(dz_V)^{\otimes n}$ agree on $U \cap V$.

Modular forms and meromorphic differentials

Let $k \geq 2$ be an even integer. Let $M_k^!(\Gamma)$ be the space of weakly holomorphic modular forms of weight k. Here 'weakly' means that we allow poles at cusps. Let $\Omega^{k/2}(\Gamma)$ be the space of meromorphic differentials of degree $k/2$ on X_Γ.

Theorem

We have an isomorphism

$$M_k^!(\Gamma) \xrightarrow{\simeq} \Omega^{k/2}(\Gamma)$$

given by $f \mapsto f(\tau)(d\tau)^{\otimes k/2}$.