Functoriality of the bar construction

Consider the category with:

1. objects: pairs (G, M), G is a group and M is an $A[G]$-module,
2. morphisms: $(\phi, \alpha)\colon (G_1, M_1) \to (G_2, M_2)$ such that α is ϕ-equivariant

and the category of cochain complexes suppored in non-negative degrees:

1. objects: cochain complexes in non-negative degrees,
2. morphisms: cochain maps(=A-linear maps commuting with differentials).

The bar construction

$$(G, M) \longmapsto C^{\bullet}(G, M)$$

is functorial.

The case of restriction

Here is a special case. Let G be a group and $H \subset G$ be any subgroup. Let M be a $A[G]$-module. M has an H-action induced by $H \subset G$. There is a map

$$C^{\bullet}(G, M) \to C^{\bullet}(H, M)$$

given by 'restriction'.

It is easy to check that it is a cochain map.

$\Rightarrow$ Induces a map $H^{\bullet}(G, M) \to H^{\bullet}(H, M)$, also called 'restriction'.

Restriction on $H^0(G, M)$

The restriction map $H^0(G, M) \to H^0(H, M)$, or

$$M^G \to M^H$$

is the natural inclusion.

Restriction on $H^1(G, A)$

Let G act trivially on A.

The restriction map $H^1(G, A) \to H^1(H, A)$ is given by restricting a homomorphism $G \to A$ to H.

Induction

If G acts on M, then any subgroup $H \subset G$ acts on M. A construction dual to it is called induction.

Let H act on the left of M. Consider

$$\mathrm{Ind}_H^G(M) := \{f\colon G \to M\colon f(hx) = h(f(x))\}.$$

This is equivalent to considering the right action

$$(f \cdot h)(x) = h^{-1} \cdot (f(hx))$$

of H on all functions and taking the right H-invariant part.

Inductrion, continued

Proposition

$\mathrm{Ind}_H^G(M)$ *is a left* G*-module under* $(g \cdot f)(x) = f(xg^{-1})$.

Proof.

Let $f \in \mathrm{Ind}_H^G(M)$ and $g \in G$. We want to show $g \cdot f$ is right H-invariant.

$$\begin{aligned} ((g \cdot f) \cdot h)(x) &= h^{-1}\left((g \cdot f)(hx)\right) \\ &= h^{-1}\left(f(hxg^{-1})\right) \\ &= h^{-1}\left(f\left(h\left(xg^{-1}\right)\right)\right) \\ &= (f \cdot h)\left(xg^{-1}\right) \\ &= f(xg^{-1}) \\ &= (g \cdot f)(x) \end{aligned}$$

$\Rightarrow g \cdot f$ is also right H-invariant. $\square$

Shapiro's lemma

This is a fundamental tool for computation.

Lemma

Let $H \subset G$ be a subgroup and M be any $A[H]$-module. The map

$$\begin{aligned} e_1 \colon \mathrm{Ind}_H^G M &\to M \\ f &\mapsto f(1) \end{aligned}$$

induces a morphism of pairs $(H, M) \to (G, \mathrm{Ind}_H^G(M))$.

Proof.

Let $h \in H$. We want to show $h \cdot e_1(f) = e_1(h \cdot f)$. The left-hand-side is $h \cdot f(1)$. The right-hand-side is $f(h^{-1})$. They are equal because $f \cdot h = f$. $\square$

Lemma (Shapiro)

We have a natural isomorphism

$$H^\bullet(G, \mathrm{Ind}_H^G M) \simeq H^\bullet(H, M).$$

Shapiro's lemma, proof

We have a map (by abuse of notation)

$$e_1\colon H^\bullet(G, \mathrm{Ind}_H^G M) \longrightarrow H^\bullet(H, M).$$

induced by e_1.

Our goal is to show that e_1 is an isomorphism on cohomology. The idea is to use two different ways to compute the cohomology of the group H.

Details of the proof to be added..

The corestriction map

Let $H \subset G$ be a subgroup of finite index. We have a map

$$\mathrm{Ind}_H^G(M) \to M$$

given by

$$f \mapsto \sum_{g \in G/H} g(f(g^{-1}).)$$

The well-definedness follows from the H-equivariance of $f \in \mathrm{Ind}_H^G(M)$.

Corestriction, continued

Definition

Let $H \subset G$ be a subgroup of finite index. The corestriction map

$$H^\bullet(H, M) \to H^\bullet(G, M)$$

is given by the composition of

$$H^\bullet(H, M) \simeq H^\bullet(G, \mathrm{Ind}_H^G M)$$

and

$$H^\bullet(G, \mathrm{Ind}_H^G M) \to H^\bullet(G, M).$$