Review3 : Introduction to Elliptic Curves and Modular Forms

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Abstract

Last assignment, we defined g_2, g_3, G_4, G_6 as coefficients of elliptic functions. However, these functions can be examples of modular forms for Γ. In this assignment we are going to calculate some examples of modular forms.

Recall the definition and properties of modular forms and cusp forms.

Definition 1. *Let $f(z)$ be a holomorphic function on $\mathbb{H}$ and $k \in \mathbb{Z}$. The function $f(z)$ is a modular form of weight k if $f(\tau)$ is holomorphic at every cusps and $f(\gamma\tau)(c\tau+d)^{-k} = f(\tau)$ for all $\gamma \in \Gamma$. In addition, if $f(\tau)$ vanishes at all cusps, the function $f(z)$ is called a cusp form. Also we denote the set of modular forms of weight k as $M_k(\Gamma)$, denote the set of cusp forms of weight k as $S_k(\Gamma)$*

Proposition 1. *If weight is even$(= 2k)$, then $f(\tau)(d\tau)^{\bigotimes k}$ is invariant.*

Proof. This is from the fact that $\frac{d\gamma\tau}{d\tau} = \frac{d}{d\tau}(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{-2}$. Thus, we have $f(\tau) = (c\tau+d)^{-2k} f(\gamma\tau) = (\frac{d\gamma\tau}{d\tau})^k f(\gamma\tau)$.

□

Remark that since $\overline{\Gamma}$ is generated by $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, the definition of modular forms can be represented as $f(S\tau) = f(\tau)$ and $f(T\tau) = (-\tau)^k f(\tau)$. It is equivalent to $f(-\frac{1}{\tau}) = (\tau)^k f(\tau)$ and $f(\tau+1) = f(\tau)$. We can understand the Eisenstein series as an example of modular forms.

Definition 2. *Let k be an even integer greater than 2, define the Eisenstein series as*

$$G_k(\tau) = \sum_{m,n}{}' \frac{1}{(m\tau+n)^k}$$

, where $\sum_{m,n}{}'$ implies the summation for every integers m, n except $(0,0)$.

It is the same as the definition of $G_k(L) = \sum_{m,n}{}' \frac{1}{(m\omega_1 + n\omega_2)^k}$ where L is the lattice generated by 1, τ.

Proposition 2. $G_k \in M_k(\Gamma)$

Proof. First, $k > 2$ implies $G_k(\tau)$ converges absolutely and uniformly on compact sets. Thus $G_k(\tau)$ defines holomorphic functions on $\mathbb{H}$. We can easily check that $G_k(\tau+1) = G_k(\tau)$. Also we have

$$\lim_{\tau\to i\infty} \sum_{m,n}{}' (m\tau+n)^{-k} = \sum_{n\neq 0} n^{-k} = 2\zeta(k) < \infty.$$

Thus, $f(\tau)$ is holomorphic at ∞. Also, $\tau^{-k}G_k(-\frac{1}{\tau}) = \sum_{m,n}{}'(-m+n\tau)^{-k} = G_k(\tau)$. Thus, $G_k(\tau) \in M_k(\Gamma)$.

□

Since $f(\tau)$ is meromorphic at infinity, the Fourier expansion of $f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n$ has at most finite non-zero a_n's for negative n's, where $q = e^{2\pi i\tau}$. This expansion is called "q-expansion" of modular form.

Proposition 3. *$G_k(\tau)$ has q-expansion*

$$G_k(\tau) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

where B_k implies Bernoulli numbers, i.e. $\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$.

Proof. We have $\pi\cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty}\left(\frac{1}{a+n} + \frac{1}{a-n}\right) = \frac{1}{a} + \sum_{n\in\mathbb{N}} \frac{2a}{a^2-n^2}$. By differentiate both sides, we get $\sum_{n\in\mathbb{Z}} \frac{1}{(a+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{n\in\mathbb{N}} n^{k-1}e^{2\pi i n a}$.

Also, we have $a\pi\cot(\pi a) = a\pi i\left(\frac{e^{\pi i a}+e^{-\pi i a}}{e^{\pi i a}-e^{-\pi i a}}\right) = a\pi i\left(1 + \frac{2}{e^{2\pi i a}-1}\right)$. By replacing $2\pi i a$ by x, we get $a\pi\cot(\pi a) = \frac{x}{2} + \frac{x}{e^x-1} = \frac{x}{2} + \sum_{k=0}^{\infty} B_k\frac{x^k}{k!}$. Compare with the formula

$$a\pi\cot(\pi a) = 1 - \frac{2}{n^2}\sum_{n\in\mathbb{N}} \frac{a^2}{1-(\frac{a^2}{n^2})} = 1 - 2\sum_{n\geq 1}\sum_{k\geq 1}\frac{a^{2k}}{n^{2k}}$$

, we get $\zeta(k) = -\frac{(2\pi i)^k}{2}\frac{B_k}{k!}$ for positive even k.

Finally, replacing a by $m\tau$, we get

$$\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{n\in\mathbb{N}} n^{k-1}e^{2\pi i n m\tau} = -\frac{2k}{B_k}\zeta(k)\sum_{n\in\mathbb{N}} n^{k-1}q^{nm} = -\frac{2k}{B_k}\zeta(k)\sum_{n\in\mathbb{N}}\sum_{d|n} d^{k-1}q^n$$

$= -\frac{2k}{B_k}\zeta(k)\sum_{n\in\mathbb{N}}\sigma_{k-1}(n)q^n$, where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. □

Thus, it is natural to define "normalized Eisenstein series" as $E_k(\tau) = \frac{1}{2\zeta(k)}G_k(\tau)$.

Proposition 4. *The normalized Eisenstein series, $E_k(\tau)$ can be expressed as*

$$E_k(\tau) = \frac{1}{2} \sum_{(m,n)=1} \frac{1}{(m\tau + n)^k}.$$

Proof. By absolute convergence of the series, we can change the order of summation freely. Thus,

$$\zeta(k) \sum_{(m,n)=1} \frac{1}{(m\tau+n)^k} = \sum_{d\in\mathbb{N}} \frac{1}{d^k} \sum_{(m,n)=1} \frac{1}{(m\tau+n)^k} = \sum_{d\in\mathbb{N}} \sum_{(m,n)=1} \frac{1}{(dm\tau+dn)^k}$$

$$= \sum_{m,n}{}' \frac{1}{(m\tau+n)^k} = G_k(\tau).$$

□

Recall that we have defined $g_2(\tau)$ and $g_3(\tau)$ from differential equation of Weierstrass $\mathfrak{P}$-function, $\mathfrak{P}'(z)^2 = f(\mathfrak{P}(z))$ where $f(x) = 4x^3 - g_2x - g_3$.

Proposition 5. *Of course, $g_2(\tau)$ and $g_3(\tau)$ are modular forms, and $g_2(\tau) = 60G_4(\tau) = \frac{4}{3}\pi^4 E_4(\tau)$, $g_3(\tau) = 140G_6(\tau) = \frac{8}{27}\pi^6 E_6(\tau)$.*

Since for nontrivial lattices, the equation $f(x) = 4x^3 - g_2x - g_3$ should have distinct solutions, which means the discriminant is nonvanishing for nontrivial τ. Thus we can define the discriminant modular form $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$. Then we can check it is not a trivial form and the constant term is zero. Thus $\Delta(\tau)$ is a cusp form.

Proposition 6. $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = \frac{(2\pi)^{12}}{1728}(E_4(\tau)^3 - E_6(\tau)^2)$ *is a cusp form of weight 12.*

In addition we can check easily from $E_4(\tau) = 1+240\sum_{n\in\mathbb{N}} \sigma_3(n)q^n$ and $E_6(\tau) = 1-504\sum_{n\in\mathbb{N}} \sigma_5(n)q^n$, $(2\pi)^{-12}\Delta(\tau)$ has rational q-expansion with (the coefficient of q) = 1.

In next assignment, we are going to find $M_k(\Gamma)$ and $S_k(\Gamma)$ for the case $\Gamma = SL_2(\mathbb{Z})$ and check that the discriminant modular form is a cusp form of the smallest weight.