Exam Problems October

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1 Problem 1

You have been in Bilka and bought 17 distinct items. When you come home you look at the receipt and wonder how you can apply the tools from DM551 to the information on the receipt. You soon realize that you may be able to find an application of the pigeon-hole principle to the info on the receipt. Prove that that, no matter how the 17 things listed on the receipt there will always be 5 items on the receipt (top to bottom) so that their price is either increasing (not smaller than the previous) or decreasing (not larger than the previous) in the order from top to bottom.

Answer
Associate two numbers with each item (I_i, D_i), which corresponds to the tuple (longest increasing sequence before this item, and the longest decreasing sequence before this item). [When we meet a new distinct price, it may increment I or D.] If one of these are 5, then we have proved, that there are either 5 decreasing or increasing sequences of prices. So now we try to prove the opposite, the worst case that would still hold would be where there are 4 items of increasing price and 4 items of decreasing price.

You want to argue that $(I_i, D_i) \neq (I_j, D_j)$ for $i \neq j$. This is not a sufficient argument :

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proof by contradiction:
Assume $D_i, I_i \leq 4$ for $i = 1, 2...17$.
We have $4 \cdot 4 < 17$, since we have $4 \cdot 4$ possibilities of the sequences, to meet the requirements. Now since there are 17 distinct items/prices, there must be one of these sequences which is at least 5, otherwise, we would only have 16 items. This is using the pigeonhole principle.

2 Problem 2

At a party there are 12 men and 8 women.

2.0.1 (a) How many different pairs (m, w), where m is a man and w is a woman, can one make?

Answer
For each of the 12 men we can choose 8 women, meaning we have

$$\binom{12}{1} \cdot \binom{8}{1} = 12 \cdot 8 = 96$$

Possibilities

2.0.2 (b) In how many ways can we form 8 pairs $(m_1, w_1), ..., (m_6, w_6)$ where m_i is a man, $m_i \neq m_j \ for \ i \neq j, w_i$ is a woman, $w_i = w_j \ for \ i \neq j$ and the order of these pairs is not important (so that all permutations of the same 8 pairs count as one solution)? Hint: how many solutions are there for a fixed set of 8 distinct men?

Answer
We can choose 8 men from 12 men in $\binom{12}{8}$ ways. For the first man in each possible choosing of the 8 men, there are 8 possibilities, the second man only has 7 options for which women to pick, since one has already been taken by the first man. This continues until there is only 1 woman left.

$$\binom{12}{8} \cdot 8! = 19958400$$

2.0.3 (c) In how many ways can we arrange the 8 women in a circle if we consider two arrangements identical when each woman has the same two women next to her (so for example $w_1w_2w_3w_4w_5w_6w_7w_8w_1$ is the same as $w_1w_8w_7w_6w_5w_4w_3w_2w_1$)?

Answer
We have to choose 8 out of the 8 women, meaning we have $\binom{8}{8}$ possibilities to do this. For each of these choices, which is only 1 in this case, we can order them in 8!-ways. Now, we have to remove the once that have we over counted, as we consider some arrangements identical. To do this we have 8 rotation options around the table, meaning for each of the permutations there are 8 rotations which are the same, due to the identical arrangement just being rotated. Now, for each of these rotations, we also have to multiply with two to ensure we remove all vertical/horizontal flips, which will end up being excluded due to the arrangements we see as identical. This means we can calculate it by:

$$\binom{8}{8} \cdot \frac{8!}{8 \cdot 2} = \frac{7!}{2} = 2520$$

2.0.4 **(d) Now consider a fixed cyclic ordering $w_{i1}w_{i2}w_{i3}w_{i4}w_{i5}w_{i6}w_{i7}w_{i8}w_{i1}$ of the eight women. We want to place the men into the circle in such a way that no two women stand next to each other. In how many ways can this be done if we do not distinguish between the men? Hint: Compare with Exercise 6.5.48.**

Answer

As we cannot distinguish the men from each other, we use the **stars and bars** theorem. We represent the women as the bars, using them as indicators, given the inability to distinguish between the men. Effectively, we have to sit 8 women, to separate them from each other. Since the women sit in a fixed cyclic order, we will need 8 men to separate them from each other. This can be seen as 8 bins, where we have 12 identical objects to distribute between them. This can be seen as:

$$x_1 + x_2 + x_3 ... x_8 = 12$$

The objects cannot be distinguished from each other, and all the bins need to have at least one object inside. We can visualize this by applying a constrain on all of the x's, where $x'_i \geq 1$ where $i \in \{1, 2, 3...8\}$. Since each of these has to be at least one, we end up with the equation: more Precision

$$x'_1 + x'_2 + x'_3 ... x'_8 = 12 - 8 = 4$$

Using the formula for choosing with duplication's: $\binom{n+k-1}{n}$, where n is the objects left over after distributing one object in each of the bins, in this case, $12 - 8 = 4$. k is the number of bins. This will result in:

$$\binom{4+8-1}{4} = \binom{11}{4} = 330$$

(✓)

3 Problem 3

3.0.1 (a) Suppose we choose a random letter x from the string 'RECURRENCE' and a random letter y from the string 'RELATION'. What is the probability that $x = y$?

Answer

We only need to take into account the characters that are in both strings. Since there is no way to find a certain character in both string if the character isn't present in both. The probability of picking 'R' in the string 'RECCURENCE', when chosen at random is $\frac{k}{n}$, where k is the number of times the specific character is present in the string, and n is the total length of the string. We use the product rule, to find the probability that we hit a specific character in the first string and in the second string, because these are independent events.

$$P(R) = \frac{3}{10} \cdot \frac{1}{8}$$
$$P(E) = \frac{3}{10} \cdot \frac{1}{8}$$
$$P(N) = \frac{1}{10} \cdot \frac{1}{8}$$

We use the sum rule to add these together since the event of picking these characters is disjoint, and then we will have the total probability that a randomly chosen character in each of the strings is the same character.

✓

$$P(R) + P(E) + P(N) = \frac{3}{10} \cdot \frac{1}{8} + \frac{3}{10} \cdot \frac{1}{8} + \frac{1}{10} \cdot \frac{1}{8} = \frac{7}{80}$$

3.0.2 (b) How many different permutations are there of the string 'RECURRENCE'?

Answer

First we find the number of occurrences of each unique letter.
There are the following counts of the letters:

- R: 3
- E: 3
- C: 2
- U: 1
- N: 1

The number of ways to permute the string with no duplications, i.e assuming we can distinguish the letters from each other, even though there are duplicates is 10!. But we have to remove those that we cannot distinguish, for example REC_1URENC_2E and REC_2URENC_1E will be considered the same string. To get rid of these countings where we cannot distinguish them from each other, we have to remove 2! as these are the number of ways to permute the 2 C's. Now, for each of these 2! permutations, there are for example

3! permutations of the letter R as well, so we have to remove this as well. Following the logic, we get the calculation:

$$\binom{10}{3} \cdot \binom{7}{3} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = \frac{10!}{3! \cdot 3! \cdot 2! \cdot 1! \cdot 1!} = 50400$$

4 Problem 4

Prove that for all non-negative integers n we have $\sum_{k=0}^{n}\binom{n}{k}17^k 3^{n-k} = \sum_{k=0}^{n}\binom{n}{k}10^n$

Answer
Using the Binomial Theorem from chapter 6.4 Rosen, we can prove the following.

$$\sum_{k=0}^{n}\binom{n}{k}\cdot 17^k \cdot 3^{n-k} = (17+3)^n = (10+10)^n$$

$$(10+10)^n = \sum_{k=0}^{n}\binom{n}{k}10^k 10^{n-k} = \sum_{k=0}^{n}\binom{n}{k}10^{k+(n-k)} = \sum_{k=0}^{n}\binom{n}{k}10^n$$

5 Problem 5

Consider an experiment where we roll two dice once. Let X denote the minimum value of the two dice (so if we roll 3 and then 2, we have X= 2).

5.0.1 (a) What are the different values that X can take?

Answer
Since we only care about the lowest die, we have

- 1, if we hit (1,1), (1,2)...(1,6) and vice versa, if the first die is the increasing one, and the second die is the fixed
- 2, if we hit (2,2), (2,3)...(2,6) and vice versa
- 3, if we hit (3,3), (3,4)...(3,6) and vice versa.
- 4, if we hit (4,4), (4,5)...(4,6) and vice versa.
- 5, if we hit (5,5), (5,6) and vice versa.
- 6, if (6,6) and vice versa. What would that be?

Meaning we the possibilities of X=1, X=2, X=3, X=4, X=5 and X=6.

5.0.2 (b) What is the probability of X taking the different values? That is, find for all possible values r the quantity $p(X = r)$.

Answer

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

The different probability of X can be calculated by $\frac{(number\ of\ ways\ we\ get\ i)}{all\ permutations\ of\ two\ dice\ rolls}$ for all $i \in \{1, 2, 3, 4, 5, 6\}$

- $P(X = 1) = \frac{11}{36}$
- $P(X = 2) = \frac{9}{36}$
- $P(X = 3) = \frac{7}{36}$
- $P(X = 4) = \frac{5}{36}$
- $P(X = 5) = \frac{3}{36}$
- $P(X = 6) = \frac{1}{36}$

5.0.3 (c) Determine $E(X)$

Answer
The formula for calculating the expectation of a given random variable is:
Let the set S be the set of all possible values that X can take

$$E(X) = \sum_{i \in S} p(X = i) \cdot i$$

$$\frac{11}{36} \cdot 1 + \frac{9}{36} \cdot 2 + \frac{7}{36} \cdot 3 + \frac{5}{36} \cdot 4 + \frac{3}{36} \cdot 5 + \frac{1}{36} \cdot 6 = \frac{91}{36} \approx 2.52$$

6 Problem 6

6.0.1 (a) Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 15$

Answer

Here we once again can use Stars and Bars, we recognize that we have 3 bins, which are the 3 variables, $x_1,\ x_2\ and\ x_3$. To distinguish these bins from each other, we will only need 2 bars. We have 15 stars since we can choose from 15 balls. This means the solution is

$$\binom{3-1+15}{15}$$

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6.0.2 (b) Solve the problem above with the extra condition that $x_1 \geq 4, x_3 \geq 5$.

Answer

We have the same scenario that we have 3 bins, and we only need 2 bars to distinguish them from each other. We use the stars and bars principle again. The scenario is now that, since we have to meet the conditions, we introduce $x_1' = 4, x_3 = 5'$ giving us the equation:] X_1' and X_3' are variables.

$$x_1' + x_2 + x_3' = 15 - 4 - 5 = 6$$

Since 4 of the points are already put into x_1' and 5 into x_3'. We then only have to distribute 6 points into the bins, which are distinguished by 2 bars, meaning the result is

$$\binom{3-1+6}{6} = 28$$

6.0.3 (c) In how many ways can one distribute 15 identical balls into 3 distinct boxes such that box 1 contains at most 5 balls, box 2 at most 8 balls, and box 3 at most 9 balls? Hint: use inclusion-exclusion.

Answer

First, we can calculate the total number of ways to distribute the 15 balls into the 3 distinct boxes This is done by $\binom{3-1+15}{15} = \binom{17}{15} = 136$, now we have a substantial overcount. This can be fixed by excluding all the ways that the conditions are not met, these are all the combinations where there are more than 5 balls in box 1, where there are more than 8 balls in box 2, and where there are more than 9 balls in box 3. These conditions all need to be accounted for. But since when we subtract the cases where one of the conditions are not met, we have removed too many. This can be seen when looking up the explanation of inclusion-exclusion, and becomes more clear when one looks at for example. figure 3 chapter 8.5 in Rosen. The following notation will be used:

- *box1*: where box1 condition is satisfied. Contains at most 5 balls.
- $box_2 \cup box_1$: where box2 and box1 condition is satisfied. Box 2 contains at most 8 balls and box 1 at most 5 balls.

We end up with the calculation:

$$|box_1 \cup box_2 \cup box_3| = |box_1| + |box_2| + |box_3| - |box_1 \cap box_2| - |box_1 \cap box_3| - |box_2 \cap box_3| + |box_1 \cap box_2 \cap box_3|$$

1 Condition is met: Here we take the complement of the conditions, i.e where box1 has more than the 5 balls in it (it has at least 6 balls inside, meaning we have to distribute $15-6$ balls), box 2 has more than 8 (it has at least 9 balls inside), etc. Because then we know how many combinations exist where each of the conditions isn't met. These results will be subtracted from the total amount of combinations.

- box1 :
$$\binom{3-1+(15-6)}{15-6} = \binom{11}{9} = 55$$
- box2:
$$\binom{3-1+(15-9)}{15-9} = \binom{8}{6} = 28$$
- box3:
$$\binom{3-1+(15-10)}{(15-10)} = \binom{7}{5} = 21$$

By the principle of inclusion-exclusion, we know that we have removed too many cases, and we have to add back the cases where 2 conditions are not met again, as some of these cases are part of the cases where 3 conditions are not met. Next up we add the cases where 2 conditions are not met:

- box1 & box2:
$$\binom{3-1+(15-6-9)}{15-6-9} = \binom{2}{0} = 1$$
- box1 & box3:
$$\binom{3-1+(15-6-10)}{15-6-10} = \binom{1}{-1} = 0$$
- box2 & box3:
$$\binom{3-1+(15-9-10)}{15-9-10} = \binom{-2}{-4} = 0$$

The intersection between the cases, where all of the conditions are not met, must be zero as we would need at least $6+9+10=25$ balls.
We then calculate the result by:

Margins! ☺

$$|box_1 \cup box_2 \cup box_3| = \binom{17}{15} - (\binom{11}{9} + \binom{8}{6} + \binom{7}{5}) + (\binom{2}{0} + \binom{1}{-1} + \binom{-2}{-4}) = 136-55-28-21+1+0+0 = 33$$

6.0.4 (d) Explain why the following does not lead to the correct answer: The sum of the upper bounds is 22 = 5 + 8 + 9 so 7 more than 15. Find the number of ways to distribute 7 balls in three boxes and return this as the answer. Here the distribution of the 7 balls would tell us how much to lower each upper bound so that we lower them by 7 in total.

Answer

The reason this approach doesn't work is because it will lead to overcount. As this approach would result in the solution

$$\binom{9}{7} = 36$$

Therefore we will have over count by $36 - 33 = 3$, since we also take into account the instances where we would break the boundaries of the boxes. Like the instances where we put 7 or 6 balls into box1, which would break the limit of 5. This means that in order for this approach to work, we would have to raise the upper bounds of box1 to at least 7, such that all boxes have boundaries of at least 7. As this would remove any overcount.

7 Problem 7

7.0.1 (a) In how many ways can we assign persons to the n skills if we just want to cover each skill by either a man or a woman?

Answer
Since we have n skills, it doesn't matter whether the person possessing the skill is a woman or a man. i.e. there are two choices for each of the skills. Using this information, it can be seen that for each of our skills, we have a choice between a man or a woman. Therefore we have 2^n ways of assigning the n skills to 2 choices of genders. unclear

7.0.2 (b) Prove using probabilistic method

Answer
As a committee has to have at least 1 woman and 1 man to be a valid committee, we see that if we have to form a committee with k people (which implies k skills). This means that picking the first person out of the **k** people can never cause the committee to not meet the condition. However, when we have picked the first person, either a woman or a man, then the next person we can pick has a $\frac{1}{2}$ chance of making the committee not satisfy the condition. This holds for the leftover $k-1$ people we have to choose. The probability of a committee being satisfied is then $1-\left(\frac{1}{2}\right)^{k-1}$

We now use random indicator variables, where the random indicator variable $X_i = 1$ iff the committee i satisfies the condition, else it takes the value 0. This means that $E(X_i) = 1 \cdot (1-(\frac{1}{2})^{k-1})$, since the probability of the i*'th* committee is satisfied is $1-(\frac{1}{2})^{k-1}$, and the random indicator variable is equal to 1 when its satisfied, else its 0. We then do this for all of the m committees, i.e have random indicator variables for all $1 \le i \le m$, and use **linearity of expectation** which results in the calculation

$$E(X) = E(\sum_{i=1}^{m} X_i) = \sum_{i=1}^{m} 1-(\frac{1}{2})^{k-1} = m \cdot (1-(\frac{1}{2})^{k-1})$$

We can rewrite:

$$1-(\frac{1}{2})^{k-1}$$

To its alternate form:

$$2^{-k-1}(2^{k-1}-1) = \frac{2^{k-1}-1}{2^{k-1}}$$

We now have, that the expected number of committees that satisfy the condition is

$$m(\frac{2^{k-1}-1}{2^{k-1}})$$

then we expect to satisfy the condition in less than 2^{k-1} of the committees. This means that there must exist a choosing from the m created committees, where we choose less than 2^{k-1} of the committees which then satisfies, that at least 1 woman and at least 1 man is in the committee. This is because the number of satisfied committees must be an integer, meaning that we have

$$\lceil n(\frac{2^{k-1}-1}{2^{k-1}})\rceil = n$$

whenever

$$0 \leq n < 2^{k-1}$$

meaning we satisfy n committees, whenever it is in the given range, which is less than 2^{k-1}.

8 Problem 8

8.0.1 (a) Argue that A is always correct if it returns 'true'

Answer
If it finds the majority pair, then it will return *true*, and in the case it does not find the majority pair then it will simply start a new round. If it has not found the majority pair in the allowed m rounds, then it will return *false*.

you should argue that A cannot return true unless it finds a majority pair.

X

8.0.2 (b) Argue that S can have at most one majority pair

Answer
We see that $\lfloor\frac{n}{3}\rfloor + 1 > \frac{n}{3}$ since the condition holds for $\lfloor\frac{n}{3}\rfloor \leq \frac{n}{3} < \lfloor\frac{n}{3}\rfloor + 1$. This means that, since $\frac{n}{3} < \lfloor\frac{n}{3}\rfloor + 1$, then $3K > n$, i.e there can only be two numbers which appears K times, else $3K$ would not be bigger than n.

✓

8.0.3 (c) Prove that when there is a majority pair in S, then the probability of the loop finding the pair in any execution of its loop is at least 1/3

Answer
At first, we see that picking one of the majority numbers in the majority pair is independent in each step, as picking either of these in the first step is acceptable, and since we then remove it again, we are left to pick the last majority number. This means it does not matter if we pick one or the other majority number in the first trial.

If we define two events (as in the hint)

- E_1 the event that we pick $x_r \in \{x, y\}$ where $\{x, y\}$ is the majority pair.
- E_2 The event that we pick $x_t \in \{x, y\} - \{x_r\}$.

The probability of satisfying event E_1 can be calculated by taking how many times each of the two majority pair numbers appears divided by the total amount of numbers in the set, which is the size of the sample space. In conclusion, the probability of picking one of the majority numbers in step 1 is then

$$P(E_1) = \frac{2K}{n} > \frac{2}{3}$$

since the total amount of times the two numbers appear is more than $\frac{2}{3}$ of the total size of the set.

$E_2 | E_1$ you calculate this

The probability of satisfying event $\cancel{E_2}$ can be calculated by first removing all the copies of x_r from S, this leaves us with a new set S' where x_r doesn't exist. We know that x_r occurred at least K times, therefore we can assume that there must be $n - K$ elements left in S'. Therefore the probability of picking x_t, which are the remaining elements of the majority pair is $\frac{K}{n-K}$. We have a known inequality from problem 8.b, namely $3K > n$, this implies that $2K > n - K$ also holds. Therefore

$$P(E_2) = \frac{K}{n - K} > \frac{1}{2}$$

must hold true as well. We see this because

$$\frac{K}{n-K} > \frac{K}{2K}$$

Since the events ~~are independent~~ this allows us to calculate their intersection like this:

almost right

(X)

$$P(E_1 \cap E_2) = P(E_1) \cdot \underset{P(E_2|E_1)}{\cancel{P(E_2)}} > \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

To conclude, if there is a majority pair in S, then the probability that Algorithm A finds the pair in any execution of its loop is at least $\frac{1}{3}$.

8.0.4 (d) What should the value of m be if we wish to ensure that the probability of S not having a majority pair is at least 99% if the loop returns *false*?

Answer

The probability of finding the majority pair, when it is in the set is at least $\frac{1}{3}$. So the probability of not finding it, when it is in the set is $1 - \frac{1}{3} = \frac{2}{3}$. Since the trials of running the algorithm are independent of each other, after running the algorithm m times, and not having returned the majority pair yet is then $(\frac{2}{3})^m$. So the probability of the complement, i.e. not having found it in m rounds is then $1 - (\frac{2}{3})^m$. Setting this equal to $0.99 = 1 - (\frac{2}{3})^m$ and solving, we get

$$m = \lceil 11.35774717 \rceil = 12$$

rounds.

8.0.5 (e) What is the expected number of times we need to repeat the loop?

Answer

We can rewrite the expected value for a random variable with infinitely many outcomes according to Rosen section 7.4.5 as:

$$E(X) = \sum_{j=1}^{\infty} j \cdot p(X = j) = \sum_{j=1}^{\infty} j(1-p)^{j-1}p = p\sum_{j=1}^{\infty} j(1-p)^{j-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

In the prior problem, we showed that the probability of success is greater than 1/3. We can now insert this into the formula for the expected and we get the following:

$$E(X) < \frac{1}{\frac{1}{3}} = 3$$

Since this number is between 2 and 3, since the probability of success is less than $\frac{1}{2}$ but more than $\frac{1}{3}$. We then have to round this up, as we cannot take fractions of a round. In conclusion, the expected number of times we need to repeat the loop is 3.