MŰEGYETEM 1782

Budapest University of Technology and Economics
Faculty of Electrical Engineering and Informatics
Department of Measurement and Information Systems

Configurable Stochastic Analysis Framework for Asynchronous Systems

Scientific Students' Associations Report

Authors:

Attila Klenik
Kristóf Marussy

Supervisors:

dr. Miklós Telek
Vince Molnár
András Vörös

2015.

Contents

Összefoglaló A kritikus rendszerek – biztonságkritikus, elosztott és felhőalkalmazások – helyességének biztosításához szükséges a funkcionális és nemfunkcionális követelmények matematikai igényességű ellenőrzése. Számos, szolgáltatásbiztonsággal és teljesítményvizsgálattal kapcsolatos tipikus kérdés általában sztochasztikus analízis segítségével válaszolható meg.

A kritikus rendszerek elosztott és aszinkron tulajdonságai az *állapottér robbanás* jelenségéhez vezetnek. Emiatt méretük és komplexitásuk gyakran megakadályozza a sikeres sztochasztikus analízist, melynek számításigénye nagyban függ a lehetséges viselkedések számától. A modellek komponenseinek jellegzetes időbeli viselkedése a számításigény további jelentős növekedését okozhatja.

A szolgáltatásbiztonsági és teljesítményjellemzők kiszámítása markovi modellek állandósult állapotbeli és tranziens megoldását igényli. Számos eljárás ismert ezen problémák kezelésére, melyek eltérő reprezentációkat és numerikus algoritmusokat alkalmaznak ; ám a modellek változatos tulajdonságai miatt nem választható ki olyan eljárás, mely minden esetben hatékony lenne.

A markovi analízishez szükséges a modell lehetséges viselkedéseinek, azaz állapoterének felderítése, illetve tárolása, mely szimbolikus módszerekkel hatékonyan végezhető el. Ezzel szemben a sztochasztikus algoritmusokban használt vektor- és indexműveletek szimbolikus megvalósítása nehézkes. Munkánk célja egy olyan, integrált keretrendszer fejlesztése, mely lehetővé teszi a komplex sztochasztikus rendszerek kezelését a szimbolikus módszerek, hatékony mátrix reprezentációk és numerikus algoritmusok előnyeinek ötvözésével.

Egy teljesen szimbolikus algoritmust javasolunk a sztochasztikus viselkedéseket leíró mátrix-dekompozíciók előállítására a szimbolikus formában adott állapottérből kiindulva. Ez az eljárás lehetővé teszi a temporális logikai kifejezéseken alapuló szimbolikus technikák használatát.

A keretrendszerben megvalósítottuk a konfigurálható sztochasztikus analízist : megközelítésünk lehetővé teszi a különböző mátrix reprezentációk és numerikus algoritmusok kombinált használatát. Az implementált algoritmusokkal állandósult állapotbeli költség- és érzékenység analízis, tranziens költséganalízis és első hiba várható bekövetkezési idő analízis végezhető el sztochasztikus Petri-háló (SPN) markovi költségmodelleken. Az elkészített eszközt integráltuk a PetriDotNet modellező szoftverrel. Módszerünk gyakorlati alkalmazhatóságát szintetikus és ipari modelleken végzett mérésekkel igazoljuk.

Abstract Ensuring the correctness of critical systems – such as safety-critical, distributed and cloud applications – requires the rigorous analysis of the functional and extra-functional properties of the system. A large class of typical quantitative questions regarding dependability and performability are usually addressed by stochastic analysis.

Recent critical systems are often distributed/asynchronous, leading to the well-known phenomenon of *state space explosion*. The size and complexity of such systems often prevents the success of the analysis due to the high sensitivity to the number of possible behaviors. In addition, temporal characteristics of the components can easily lead to huge computational overhead.

Calculation of dependability and performability measures can be reduced to steady-state and transient solutions of Markovian models. Various approaches are known in the literature for these problems differing in the representation of the stochastic behavior of the models or in the applied numerical algorithms. The efficiency of these approaches are influenced by various characteristics of the models, therefore no single best approach is known.

The prerequisite of Markovian analysis is the exploration of the state space, i.e. the possible behaviors of the system. Symbolic approaches provide an efficient state space exploration and storage technique, however their application to support the vector operations and index manipulations extensively used by stochastic algorithms is cumbersome. The goal of our work is to introduce a framework that facilitates the analysis of complex, stochastic systems by combining together the advantages of symbolic algorithms, compact matrix representations and various numerical algorithms.

We propose a fully symbolic method to explore and describe the stochastic behaviors. A new algorithm is introduced to transform the symbolic state space representation into a decomposed linear algebraic representation. This approach allows leveraging existing symbolic techniques, such as the specification of properties with *Computational Tree Logic* (CTL) expressions.

The framework provides configurable stochastic analysis: an approach is introduced to combine the different matrix representations with numerical solution algorithms. Various algorithms are implemented for steady-state reward and sensitivity analysis, transient reward analysis and mean-time-to-first-failure analysis of stochastic models in the *Stochastic Petri Net* (SPN) Markov reward model formalism. The analysis tool is integrated into the PetriDotNet modeling application. Benchmarks and industrial case studies are used to evaluate the applicability of our approach.

Chapter 1

Introduction

Árvíztűrő tükörfúrógép

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Chapter 2

Background

2.1 Petri nets

Petri nets are a widely used graphical and mathematical modeling tool for systems which are concurrent, asynchronous, distributed, parallel or nondeterministic.

Definition 2.1 A *Petri net* is a 5-tuple $PN = (P, T, F, W, M_0)$, where

- $P = \{p_0, p_1, \ldots, p_{n-1}\}$ is a finite set of places;
- $T = \{t_0, t_1, \ldots, t_{m-1}\}$ is a finite set of transitions;
- $F \subseteq (P \times T) \cup (P \times T)$ is a set of arcs, also called the flow relation;
- $W : F \to \mathbb{N}^+$ is an arc weight function;
- $M_0 : P \to \mathbb{N}$ is the initial marking;
- $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$ [32].

Arcs from P to T are called *input arcs*. The input places of a transition t are denoted by ${}^\bullet t = \{p : (p,t) \in F\}$. In contrast, arcs of the form (t,p) are called *output arcs* and the output places of t are denoted by $t^\bullet = \{p : (t,p) \in F\}$.

A *marking* $M : P \to \mathbb{N}$ assigns a number of *tokens* to each place. The transition t is *enabled* in the marking M (written as $M\,[t\rangle$) when $M(p) \geq W(p,t)$ for all $p \in {}^\bullet t$.

Petri nets are graphically represented as edge weighted directed bipartite graphs. Places are drawn as circles, while transitions are drawn as bars or rectangles. Edge weights of 1 are ususally omitted from presentation. Dots on places correspond to tokens in the current marking.

If $M\,[t\rangle$ the transition t can be *fired* to get a new marking M' (written as $M\,[t\rangle\,M'$) by decreasing the token counts for each place $p \in {}^\bullet t$ by $W(p,t)$ and increasing the token counts for each place $p \in t^\bullet$ by $W(t,p)$. Note that in general, ${}^\bullet t$ and $t^\bullet$ need not

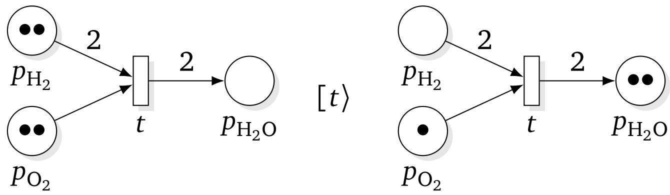


Figure 2.1 A Petri net model of the reaction of hydrogen and oxygen.

be disjoint. Thus, the firing rule can be written as

$$M'(p) = M(p) - W(p,t) + W(t,p), \tag{2.1}$$

where we take $W(x,y) = 0$ if $(x,y) \notin F$ for brevity.

A marking M' is *reachable* from the marking M (written as $M \rightsquigarrow M'$) if there exists a sequence of markings and transitions for some finite k such that

$$M = M_1 \, [t_{i_1}\rangle \, M_2 \, [t_{i_2}\rangle \, M_3 \, [t_{i_3}\rangle \cdots [t_{i_{k-1}}\rangle \, M_{k-1} \, [t_{i_k}\rangle \, M_k = M'.$$

A marking M is in the reachable *state space* of the net if $M_0 \rightsquigarrow M$. The set of all markings reachable from M_0 is denoted by

$$RS = \{M : M_0 \rightsquigarrow M\}.$$

Definition 2.2 The Petri net PN is *k-bounded* if $M(p) \le k$ for all $M \in RS$ and $p \in P$. PN is *bounded* if it is k-bounded for some (finite) k.

The reachable state space RS is finite if and only if the Peti net is bounded.

Example 2.1 The Petri net in Figure 2.1 models the chemical reaction

$$2\,H_2 + O_2 \rightarrow 2\,H_2O.$$

In the initial marking (left) there are two hydrogen and two oxygen molecules, represented by tokens on the places p_{H_2} and p_{O_2}, therefore the transition t is enabled. Firing t yields the marking on the right where the two tokens on p_{H_2O} are the reaction products. Now t is no longer enabled.

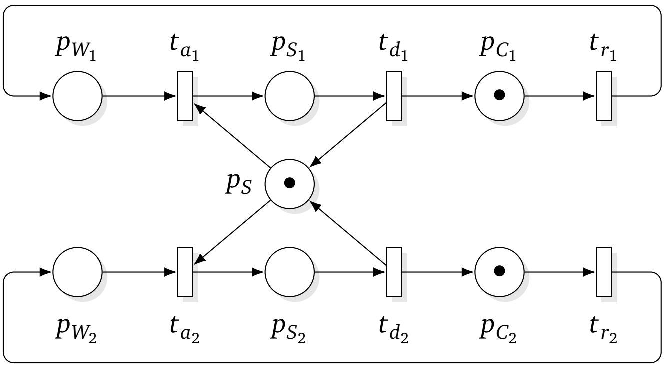


Figure 2.2 The *SharedResource* Petri net model.

Running example 2.2 In Figure 2.2 we introduce the *SharedResource* model which will serve as a running example throughout this report.

The model consists of a single shared resource S and two consumers. Each consumer can be in one of the C_i (calculating locally), W_i (waiting for resource) and S_i (using shared resource) states. The transitions r_i (request resource), a_i (acquire resource) and d_i (done) correspond to behaviors of the consumers. The net is 1-bounded, therefore it has finite RS.

The Petri net model allows the verification of safety properties, e.g. we can show that there is mutual exclusion – $M(S_1) + M(S_2) \leq 1$ for all reachable markings – or that deadlocks cannot occur. In contrast, we cannot compute dependability or performability measures (e.g. the utilization of the shared resource or number of calculations completed per unit time) because the model does not describe the temporal behavior of the system.

2.1.1 Petri nets extended with inhibitor arcs

One of the most frequently used extensions of Petri nets is the addition of inhibitor arcs, which constrains the rule for transition enablement. This modification gives Petri nets expressive power equivalent to Turing machines [10].

Definition 2.3 A *Petri net with inhibitor arcs* is a 3-tuple $PN_I = (PN, I, W_I)$, where

- $PN = (P, T, F, W, M_0)$ is a Petri net;
- $I \subseteq P \times T$ is the set of inhibitor arcs;
- $W_I : I \to \mathbb{N}^+$ is the inhibitor arc weight function.

Let $^{\circ}t = \{p : (p,t) \in I\}$ denote the set of inhibitor places of the transition t. The enablement rule for Petri nets with inhibitor arcs can be formalized as

$$M\,[t\rangle \iff M(p) \geq W(p,t) \text{ for all } p \in {}^{\bullet}t \text{ and } M(p) < W_I(p,t) \text{ for all } p \in {}^{\circ}t.$$

The firing rule (2.1) remains unchanged.

2.2 Continuous-time Markov chains

Continuous-time Markov chains are mathematical tools for describing the behavior of systems in countinous time where the random behavior of the system only depends on its current state.

Definition 2.4 A *Continuous-time Markov Chain* (CTMC) $X(t) \in S, t \geq 0$ over a finite or countable infinite state space $S = \{0, 1, \dots, n-1\}$ is a continuous-time random process with the *Markovian* or memoryless property

$$\begin{aligned}\mathbb{P}(X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}, X(t_{k-2}) = x_{k-2}, \dots, X(t_0) = x_0)\\ = \mathbb{P}(X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}),\end{aligned}$$

where $t_0 \leq t_1 \leq \cdots \leq t_k$. A CTMC is said to be *time-homogenous* if it also satisfies

$$\mathbb{P}(X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}) = \mathbb{P}(X(t_k - t_{k-1}) = x_k \mid X(0) = x_{k-1}),$$

i.e. it is invariant to time shifting.

In this report we will restrict our attention to time-homogenous CTMCs over finite state spaces. The state probabilities of these stochastic processes at time t form a finite-dimensional vector $\boldsymbol{\pi}(t) \in \mathbb{R}$,

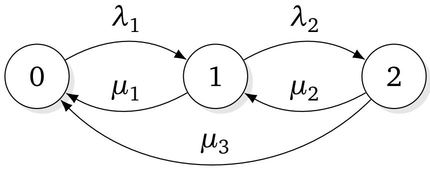
$$\pi(t)[x] = \mathbb{P}(X(t) = x)$$

that satisfies the differential equation

$$\frac{\mathrm{d}\boldsymbol{\pi}(t)}{\mathrm{d}t} = \boldsymbol{\pi}(t)Q \tag{2.2}$$

for some square matrix Q. The matrix Q is called the *infinitesimal generator matrix* of the CTMC and can be interpreted as follows:

- The diagonal elements $q[x,x] < 0$ describe the holding times of the CTMC. If $X(t) = x$, the *holding time* $h_x = \inf\{h > 0 : X(t+h) \neq x\}$ spent in state x is exponentially distributed with rate $\lambda_x = -q[x,x]$. If $q[x,x] = 0$, then no transitions are possible from state x and it is said to be *absorbing*.



$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ \mu_1 & -\lambda_2-\mu_1 & \lambda_2 \\ \mu_3 & \mu_2 & -\mu_2-\mu_3 \end{pmatrix} \end{array}$$

Figure 2.3 Example CTMC with 3 states and its generator matrix.

- The off-diagonal elements $q[x,y] \geq 0$ describe the state transitions. In state x the CTMC will jump to state y at the next state transition with probability $-q[x,y]/q[x,x]$. Equivalently, there is expontentially distributed countdown in the state x for each $y : q[x,y] > 0$ with *transition rate* $\lambda_{xy} = q[x,y]$. The first countdown to finish will trigger a state change to the corresponding state y. Thus, the CTMC is a transition system with exponentially distributed timed transitions.
- Elements in each row of Q sum to 0, hence it satisfies $Q\mathbf{1}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$.

For more algebraic properties of infinitesimal generator matrices, we refer to Plemmons and Berman [34] and Stewart [47].

A state y is said to be *reachable* from the state x ($x \rightsquigarrow y$) if there exists a sequence of states

$$x = z_1, z_2, z_3, \dots, z_{k-1}, z_k = y$$

such that $q[z_i, z_{i+1}] > 0$ for all $i = 1, 2, \dots, k-1$. If y is reachable from x for all $x, y \in S$ y, the Markov chain is said to be *irreducible*.

The *steady-state probability distribution* $\boldsymbol{\pi} = \lim_{t\to\infty} \boldsymbol{\pi}(t)$ exists and is independent from the *initial distribution* $\boldsymbol{\pi}(0) = \boldsymbol{\pi}_0$ if and only if the finite CTMC is irreducible. The steady-state distribution is a stationary solution of eq. (2.2), therefore it satisfies the linear equation

$$\frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}t} = \boldsymbol{\pi} Q = \mathbf{0}, \quad \boldsymbol{\pi}\mathbf{1}^{\mathrm{T}} = 1. \tag{2.3}$$

Example 2.3 Figure 2.3 shows a CTMC with 3 states. The transitions from state 0 to 1 and from 1 to 2 are associated with exponentially distributed countdowns with rates λ_1 and λ_2 respectively, while transitions in the reverse direction have rates μ_1 and μ_2. The transition form state 2 to 0 is also possible with rate μ_3.

The rows (corresponding to source states) and columns (destination states) of the infinitesimal generator matrix Q are labeled with the state numbers. The diagonal element $q[1,1]$ is $-\lambda_2 - \mu_1$, hence the holding time in state 1 is exponentially distributed with rate $\lambda_2 + \mu_1$. The transition to 0 is taken with probability

$-q[1,0]/q[1,1] = \mu_1/(\lambda_2+\mu_1)$, while the transition to 2 is taken with probability $\lambda_2/(\lambda_2+\mu_1)$.

The CTMC is irreducible, because every state is reachable from every other state. Therefore, there is a unique steady-state distribution $\boldsymbol{\pi}$ independent from the initial distribution $\boldsymbol{\pi}_0$.

2.2.1 Markov reward models

Continuous-time Markov chains may be employed in the estimation of performance measures of models by defining *rewards* that associate *reward rates* with the states of a CTMC. The momentary reward rate random variable $R(t)$ can describe performance measures defined at a single point of time, such as resource utilization or probability of failure, while the *accumulated reward* random variable $Y(t)$ may correspond to performance measures associated with intervals of time, such as total downtime.

Definition 2.5 A *Continuous-time Markov Reward Process* over a finite state space $S = \{0, 1, \dots, n-1\}$ is a pair $(X(t), \mathbf{r})$, where $X(t)$ is a CTMC over S and $\mathbf{r} \in \mathbb{R}^n$ is a *reward rate vector*.

The element $r[x]$ of the reward vector is a momentary reward rate in state x, therefore the reward rate random variable can be written as $R(t) = r[X(t)]$. The accumulated reward until time t is defined by

$$Y(t) = \int_0^t R(\tau)\,\mathrm{d}\tau.$$

The computation of the distribution function of Y(t) is a computationally intensive task (a summary is available at [36, Table 1]), while its mean, $\mathbb{E}\,Y(t)$, can be computed efficiently as discussed below.

Given the initial probability distribution vector $\boldsymbol{\pi}(0) = \boldsymbol{\pi}_0$ the expected value of the reward rate at time t can be calculated as

$$\mathbb{E}\,R(t) = \sum_{i=0}^{n-1} \pi(t)[i]r[i] = \boldsymbol{\pi}(t)\,\mathbf{r}^{\mathrm{T}}, \tag{2.4}$$

which requires the solution of the initial value problem [20, 40]

$$\frac{\mathrm{d}\boldsymbol{\pi}(t)}{\mathrm{d}t} = \boldsymbol{\pi}(t)Q, \quad \boldsymbol{\pi}(0) = \boldsymbol{\pi}_0$$

to form the inner product $\mathbb{E}\,R(t) = \boldsymbol{\pi}(t)\,\mathbf{r}^{\mathrm{T}}$. To obtain the expected steady-state reward rate (if it exists) the linear equation (2.3) should be solved instead for the steady-state probability vector $\boldsymbol{\pi}$.

The expected value of the accumulated reward is

$$\begin{aligned}
\mathbb{E}\,Y(t) &= \mathbb{E}\left[\int_0^t R(\tau)\,\mathrm{d}\tau\right] = \int_0^t \mathbb{E}[R(\tau)]\,\mathrm{d}\tau \\
&= \int_0^t \sum_{i=0}^{n-1} \pi(\tau)[i] r[i]\,\mathrm{d}\tau = \sum_{i=0}^{n-1} \int_0^t \pi(\tau)[i]\,\mathrm{d}\tau\, r[i] \\
&= \int_0^t \boldsymbol{\pi}(t)\,\mathrm{d}\tau\,\mathbf{r}^{\mathrm{T}} = \mathbf{L}(t)\,\mathbf{r}^{\mathrm{T}},
\end{aligned}$$

where $\mathbf{L}(t) = \int_0^t \boldsymbol{\pi}(t)\,\mathrm{d}\tau$ is the accumulated probability vector, which is the solution of the initial value problem [40]

$$\frac{\mathrm{d}\mathbf{L}(t)}{\mathrm{d}t} = \boldsymbol{\pi}(t), \quad \frac{\mathrm{d}\boldsymbol{\pi}(t)}{\mathrm{d}t} = \boldsymbol{\pi}(t)Q, \quad \mathbf{L}(0) = \mathbf{0}, \quad \boldsymbol{\pi}(0) = \boldsymbol{\pi}_0. \tag{2.5}$$

Example 2.4 Let c_0, c_1 and c_2 denote operating costs per unit time associated with the states of the CTMC in Figure 2.3. Consider the Markov reward process $(X(t), \mathbf{r})$ with reward rate vector

$$\mathbf{r} = \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix}.$$

The random variable $R(t)$ describes the momentary operating cost, while $Y(t)$ is the total operating expenditure until time t. The steady-state expectation of R is the average maintenance cost per unit time of the long-running system.

2.2.2 Sensitivity

Consider a reward process $(X(t), \mathbf{r})$ where both the infinitesimal generator matrix $Q(\boldsymbol{\theta})$ and the reward rate vector $\mathbf{r}(\boldsymbol{\theta})$ may depend on some *parameters* $\boldsymbol{\theta} \in \mathbb{R}^m$. The *sensitivity* analysis of the rewards $R(t)$ may reveal performance or reliability bottlenecks of the modeled system and aid designers in achieving desired performance measures.

Definition 2.6 The *sensitivity* of the expected reward rate $\mathbb{E}R(t)$ to the parameter $\theta[i]$ is the partial derivative

$$\frac{\partial\, \mathbb{E}R(t)}{\partial \theta[i]}.$$

The model reacts to the change of parameters with high absolute sensitivity more prominently, therefore they can be promising avenues of system optimization.

To calculate the sensivity of $\mathbb{E}R(t)$, the partial derivative of both sides of eq. (2.4) is taken, yielding

$$\frac{\partial\, \mathbb{E}R(t)}{\partial \theta[i]} = \frac{\partial \boldsymbol{\pi}(t)}{\partial \theta[i]}\mathbf{r}^{\mathrm{T}} + \boldsymbol{\pi}(t)\left(\frac{\partial \mathbf{r}}{\partial \theta[i]}\right)^{\mathrm{T}} = \mathbf{s}_i(t)\,\mathbf{r}^{\mathrm{T}} + \boldsymbol{\pi}(t)\left(\frac{\partial \mathbf{r}}{\partial \theta[i]}\right)^{\mathrm{T}},$$

where $\mathbf{s}_i$ is the sensitivity of $\boldsymbol{\pi}$ to the parameter $\theta[i]$.

In transient analysis, the sensitivity vector $\mathbf{s}_i$ is the solution of the initial value problem

$$\frac{\mathrm{d}\mathbf{s}_i(t)}{\mathrm{d}t} = \mathbf{s}_i(t)Q + \pi(t)V_i, \quad \frac{\mathrm{d}\boldsymbol{\pi}(t)}{\mathrm{d}t} = \boldsymbol{\pi}_i(t)Q, \quad \mathbf{s}_i(0) = \mathbf{0}, \quad \boldsymbol{\pi}(0) = \boldsymbol{\pi}_0,$$

where $V_i = \partial Q(\boldsymbol{\theta})/\partial \theta[i]$ is the partial derivative of the generator matrix [38]. A similar initial value problem can be derived for the sensitivity of $\mathbf{L}(t)$ and $Y(t)$.

To obtain the sensitivity $\mathbf{s}_i$ of the steady-state probability vector $\boldsymbol{\pi}$, the system of linear equations

$$\mathbf{s}_i Q = -\boldsymbol{\pi} V_i, \quad \mathbf{s}_i \mathbf{1}^T = 0 \tag{2.6}$$

is solved [4].

Another type of sensitivity analysis considers *unstructured* small perturbations of the infinitesimal generator matrix Q instead of dependecies on parameters [18, 23]. This latter, unstructured analysis may be used to study the numerical stability and conditioning of the solutions of the Markov chain.

2.2.3 Time to first failure

Let $D \subsetneq S$ be a set of *failure states* of the CTMC $X(t)$ and $U = S \setminus D$ be a set of operating states. We will assume without loss of generality that $U = \{0, 1, \dots, n_U - 1\}$ and $D = \{n_U, n_U + 1, \dots, n - 1\}$.

The matrix

$$Q_{UD} = \begin{pmatrix} Q_{UU} & \mathbf{q}_{UD}^{\mathrm{T}} \\ \mathbf{0} & 0 \end{pmatrix}$$

is the infinitesimal generator of a CTMC $X_{UD}(t)$ in which all the failures states D were merged into a single state n_U and all outgoing transitions from D were removed. The matrix Q_{UU} is the $n_U \times n_U$ upper left submatrix of Q, while the vector $\mathbf{q}_{UD} \in \mathbb{R}^{n_U}$ is defined as

$$q_{UD}[x] = \sum_{y \in D} q[x, y].$$

If the initial distribution $\boldsymbol{\pi}_0$ is 0 for all failure states (i.e. $\pi_0[x] = 0$ for all $x \in D$), the *Time to First Failure*

$$\mathit{TFF} = \inf\{t \geq 0 : X(t) \in D\} = \inf\{t \geq 0 : X_{UD}(t) = n_U\}$$

is *phase-type distributed* with parameters $(\boldsymbol{\pi}_U, Q_{UU})$ [33], where $\boldsymbol{\pi}_U$ is the vector containing the first n_U elements of $\boldsymbol{\pi}_0$. In particular, the *Mean Time to First Failure* is

$$MTFF = \mathbb{E}[TFF] = -\boldsymbol{\pi}_U Q_{UU}^{-1} \mathbf{1}^{\mathrm{T}}. \tag{2.7}$$

The probability of a D'-mode failure ($D' \in D$) is

$$\mathbb{P}(X(TFF_{+0}) = y) = -\boldsymbol{\pi}_D U Q_{UU}^{-1} \mathbf{q}_{UD'}^{\mathrm{T}}, \tag{2.8}$$

where $\mathbf{q}_{UD'} \in \mathbb{R}^{n_U}$, $q_{UD'}[x] = \sum_{y \in D'} q[x, y]$ is the vector of transition rates from operational states to failure states D'.

2.3 Stochastic Petri nets

While reward processes based on continuous-time Markov chains allow the study of dependability or reliability measurements, the explicit specification of stochastic processes and rewards is often cumbersome. More expressive formalisms include queueing networks, stochastic process algebras such as PEPA [14, 19], Stochastic Automata Networks [15] and Stochastic Petri Nets (SPN).

Stochastic Petri Nets extend Petri nets by assigning random exponentially distributed random delays to transitions [27]. After the delay associated with an enabled transition is elapsed the transition fires *atomically* are transitions delays are reset.

Definition 2.7 A Stochastic Petri Net is a pair $SPN = (PN, \Lambda)$, where PN is a Petri net (P, T, F, W, M_0) and $\Lambda : T \to \mathbb{R}^+$ is a transition rate function.

Likewise, a stochastic Petri net with inhibitor arcs is a pair $SPN_I = (PN_I, \Lambda)$, where PN_I is a Petri net with inhibitor arcs.

A finite CTMC can be associated with a bounded stochastic Petri net (with inhibitor arcs) as follows:

1. The reachable state space of the Petri net is explored. We associate a consecutive natural numbers with the states such that the state space is

$$RS = \{M_0, M_1, M_2, \ldots, M_{n-1}\},$$

where M_0 is the initial marking. From now on, we will use markings $M_x \in RS$ and natural numbers $x \in \{0, 1, \ldots, n-1\}$ to refer to states of the model interchangably.

2. We define a CTMC $X(t)$ over the finite state space

$$S = \{0, 1, 2, \ldots, n-1\}.$$

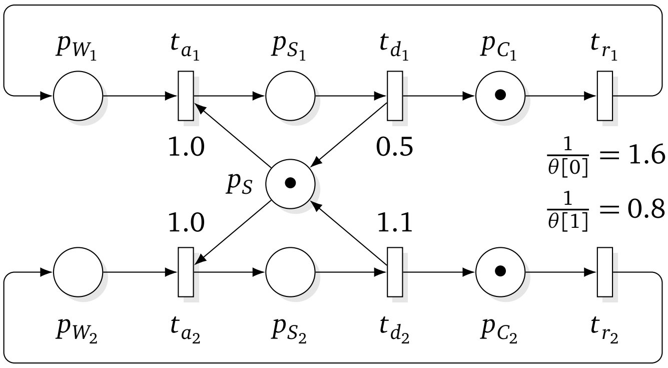


Figure 2.4 Example stochastic Petri net for the *SharedResource* model.

The initial distribution vector will be set to

$$\boldsymbol{\pi}(0) = \boldsymbol{\pi}_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

in the analysis steps ($\pi_0[x] = \delta_{0,x}$).

3. The generator matrix $Q \in \mathbb{R}^{n\times n}$ encodes the possible state transitions of the Petri net and the associated transition rate $\Lambda(\cdot)$ as

$$\begin{gathered} q_O[x,y] = \sum_{\substack{t\in T \\ M_x[t\rangle M_y}} \Lambda(t) \quad \text{if } x \neq y, \\ q_O[x,x] = 0, \\ Q = Q_O - \operatorname{diag}\{Q_O \mathbf{1}^{\mathrm{T}}\}, \end{gathered}$$

where the summation is done over all transition from the marking M_x to M_y, while Q_O and $Q_D = -\operatorname{diag}\{Q_O\mathbf{1}^{\mathrm{T}}\}$ are the off-diagonal and diagonal parts of Q, respectively.

Running example 2.5 Figure 2.4 shows the SPN model for *SharedResouce*, which is the Petri net from Figure 2.2 on page 5 extended with exponential transition rates.

The transitions a_1, d_1, a_2 and d_2 have rates 1.0, 0.5, 1.0 and 1.1, respectively. The parameter vector $\boldsymbol{\theta} = (0.625, 1.25) \in \mathbb{R}^2$ is introduced such that the transitions r_1 and r_2 have rates $1/\theta[0]$ and $1/\theta[1]$.

The reachable state space (Table 2.1) contains 8 markings which are mapped to the integers $S = \{0, 1, \ldots, 7\}$. The state space graph along with the transition rates of

$RS =$

P:	S	C_1	W_1	S_1	C_2	W_2	S_2	
M_0	1	1	0	0	1	0	0	initial
M_1	1	0	1	0	1	0	0	client 1 waiting
M_2	1	1	0	0	0	1	0	client 2 waiting
M_3	1	0	1	0	0	1	0	1 waiting, 2 waiting
M_4	0	0	0	1	1	0	0	client 1 shared working
M_5	0	0	0	1	0	1	0	1 shared working, 2 waiting
M_6	0	1	0	0	0	0	1	client 2 shared working
M_7	0	0	1	0	0	0	1	1 waiting, 2 shared working

Table 2.1 Reachable state space of the *SharedResource* model.

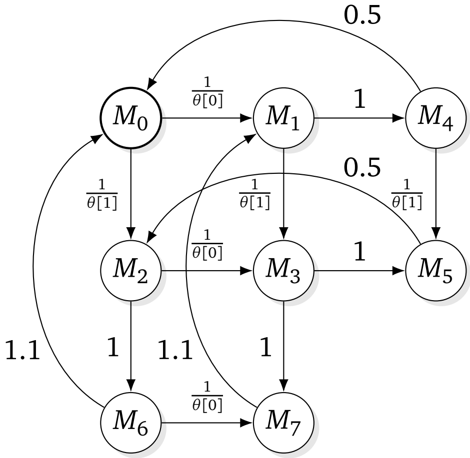


Figure 2.5 The CTMC associated with the *SharedResource* SPN model.

the CTMC is shown in Figure 2.5. The generator matrix is

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} * & \frac{1}{\theta[0]} & \frac{1}{\theta[1]} & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & \frac{1}{\theta[1]} & 1 & 0 & 0 & 0 \\ 0 & 0 & * & \frac{1}{\theta[0]} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & 0 & 1 & 0 & 1 \\ 0.5 & 0 & 0 & 0 & * & \frac{1}{\theta[1]} & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & * & 0 & 0 \\ 1.1 & 0 & 0 & 0 & 0 & 0 & * & \frac{1}{\theta[0]} \\ 0 & 1.1 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \end{array},$$

where in each row the diagonal element is the negative of the sum of the other elemens so that $Q\mathbf{1}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$. The CTMC is irreducible, therefore it has a well-defined steady-state distribution.

Extensions of stochastic Petri nets include transitions with general or phase-type delay distributions [26, 28], Generalized Stochastic Petri Nets (GSPN) with immediate transitions [29, 48] and Deterministic Stochastic Petri Nets (DSPN) with deterministic firing delays [41]. Among these, only phase-type distributed delays and GSPNs can be handled with purely Markovian analysis. Stochastic Well-formed Nets (SWN) are a class of colored Petri nets especially amenable to stochastic analysis [9]. Stochastic Activity Networks (SAN) also allow colored places, moreover, they introduce input and output gates for more flexible modeling [22].

2.3.1 Stochastic reward nets

Definition 2.8 A *Stochastic Reward Net* is a triple $SRN = (SPN, rr, ir)$, where SPN is a stochastic Petri net, $rr : \mathbb{N}^P \to \mathbb{R}$ is a *rate reward function* and $ir : T \times \mathbb{N}^P \to \mathbb{R}$ is an *impulse reward* function. A stochastic Reward net with inhibitor arcs is a triple $SRN_I = (SPN_I, rr, ir)$, where SPN_I is a stochastic Petri net with inhibitor arcs.

The rate reward $rr(M)$ is the reward gained per unit time in marking M, while $ir(t, M)$ is the reward gained when the transition t fires in marking M.

If $ir(t, M) \equiv 0$, the SRN is equivalent to the Markov reward process $(X(t), \mathbf{r})$, where $X(t)$ is the CTMC associated with the stochastic Petri net and

$$\mathbf{r} \in \mathbb{R}^n, \quad r[x] = rr(M_x).$$

If there are impulse rewards, exact calculation of the expected reward rate $\mathbb{E}R(t)$ and expected accumulated reward $\mathbb{E}\,Y(t)$ can be performed on reward process $(X, \mathbf{r})$,

$$r[x] = rr(M_x) + \sum_{t \in T, M_x[t\rangle} \Lambda(t)\, ir(t, M_x),$$

where the summation is taken over all enabled transitions [11]. In general, the distribution of $Y(t)$ cannot be derived by this method [37].

Running example 2.6 The SRN model

$$rr_1(M) = M(P_{S_1}) + M(P_{S_2}), \quad ir_1(t, M) \equiv 0$$

describes the utilization of the shared resouce in the *SharedResouce* SPN (Figure 2.4 on page 12). $R_1(t) = 1$ if the resource is allocated, hence $\mathbb{E}R_1(t)$ is the probability that the resource is in use at time t, while $Y(t)$ is the total usage time until t.

Another reward structure

$$rr_2(M) \equiv 0, \quad ir_2(t, M) = \begin{cases} 1 & \text{if } t \in \{t_{r_1}, t_{r_2}\}, \\ 0 & \text{otherwise} \end{cases}$$

counts the completed calculations, which are modeled by tokens leaving the places C_1 and C_2. The exprected steady-state reward rate $\lim_{t\to\infty} \mathbb{E}R(t)$ equals the number of calculations per unit time in a long-running system, while $Y(t)$ is the number of calculations performed until time t.

The reward vectors associated with these SRNs are

$$\begin{array}{rcccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{r}_1 = \big(& 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\big), \\ \mathbf{r}_2 = \big(& \frac{1}{\theta[0]} + \frac{1}{\theta[1]} & \frac{1}{\theta[1]} & \frac{1}{\theta[0]} & 0 & \frac{1}{\theta[1]} & 0 & \frac{1}{\theta[0]} & 0\big). \end{array}$$

2.3.2 Superposed stochastic Petri nets

Definition 2.9 A *Superposed Stochastic Petri Net* (SSPN) is a pair $SSPN = (SPN, \mathcal{P})$, where $\mathcal{P} = \{P^{(0)}, P^{(1)}, \dots, P^{(J-1)}\}$ is partition of the set of places $P = P^{(0)} \cup P^{(1)} \cup \cdots \cup P^{(J-1)}$ [13]. Superposed stochastic Petri nets with inhibitor arcs $SSPN_I = (SPN_I, \mathcal{P})$ are defined analogously.

The jth *local net* $LN^{(j)} = ((P^{(j)}, T^{(j)} = T_L^{(j)} \cup T_S^{(j)}, F^{(j)}, W^{(j)}, M_0^{(j)}), \Lambda^{(j)})$ can be constructed as follows:

- $P^{(j)}$ is the corresponding set from the partition of the original net.
- $T^{(j)}$ contains the local transition $T_L^{(j)}$ and synchronizing transitions $T_S^{(j)}$. A transition is *local* to $LN^{(j)}$ if it only affects places in $P^{(j)}$, that is,

$$T_L^{(j)} = \{t \in T : {}^\bullet t \cup t^\bullet \subseteq P^{(j)}\}. \tag{2.9}$$

  No transition may be local to more than one local net.

  A transition *synchronizes* with $LN^{(j)}$ if it affects some places in $P^{(j)}$ but it is not local to $LN^{(j)}$,

$$T_S^{(j)} = \{t \in T : ({}^\bullet t \cup t^\bullet) \cap P^{(j)} \neq \emptyset\} \setminus T_L^{(j)}. \tag{2.10}$$

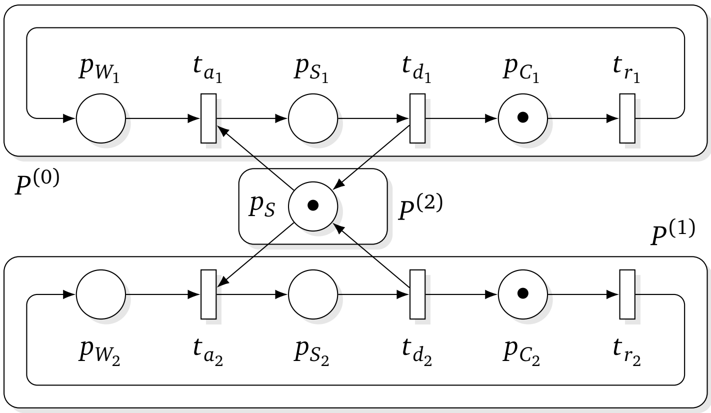


Figure 2.6 A partitioning of the *SharedResource* Petri net.

- The relation $F^{(j)}$ and the functions $W^{(j)}$, $M_0^{(j)}$, $\Lambda^{(j)}$ are the appropriate restrictions of the original structures, $F^{(j)} = F \cap ((P^{(j)} \times T^{(j)}) \cup (T^{(j)} \times J^{(j)}))$, $W^{(j)} = W|_{F^{(j)}}$, $M_0^{(j)} = M_0|_{P^{(j)}}$, $\Lambda^{(j)} = M_0|_{T^{(j)}}$.

If there are inhibitor arcs in $SSPN_I$, inhibitor arcs must be considered when local net $LN_I^{(j)}$ is constructed. The set ${}^\bullet t \cup t^\bullet$ is replaced with ${}^\bullet t \cup t^\bullet \cup {}^\circ t$ in eqs. (2.9) and (2.10) so that the enablement of local transitions only depends on the marking of places in $P^{(j)}$ and only places in $P^{(j)}$ may be affected upon firing. In addition, the inhibitor arc relation and weight function are restricted as $I^{(j)} = I \cap (P^{(j)} \cap T^{(j)})$, $W_I^{(j)} = W_I|_{I^{(j)}}$.

> **Running example 2.7** Figure 2.6 shows a possible partitioning of the *Shared-Resource* SPN into a SSPN. The components $P^{(0)}$ and $P^{(1)}$ model the two consumers, while $P^{(2)}$ contains the unallocated resource S.
>
> The transitions r_1 and r_2 are local to $LN^{(0)}$ and $LN^{(1)}$, respectively, while a_1, d_1, a_2 and d_2 synchronize with $LN^{(2)}$ and the local net associated with their consumers.

The *local reachable state space* $RS^{(j)}$ of $LN^{(j)}$ is the the set of markings beloning to the state space RS of the original net restricted to the places $P^{(j)}$ (duplicates removed),

$$RS^{(j)} = \{M^{(j)} : M \in RS, M^{(j)} = M|_{P^{(j)}}\}.$$

This is a *subset* of the reachable state space of $LN^{(j)}$, in particular, $RS^{(j)}$ is always finite if RS is finite, even if $LN^{(j)}$ is not bounded. Analysis techniques for generating local state spaces include *partial P-invariants* [7] and explicit projection of global reachable markings [5].

$$RS^{(0)} = \left\{ \begin{array}{cccc} \hline P: & C_1 & W_1 & S_1 \\ \hline M_0^{(0)} & 1 & 0 & 0 \\ M_1^{(0)} & 0 & 1 & 0 \\ M_2^{(0)} & 0 & 0 & 1 \\ \hline \end{array} \right\},$$

$$RS^{(1)} = \left\{ \begin{array}{cccc} \hline P: & C_2 & W_2 & S_2 \\ \hline M_0^{(1)} & 1 & 0 & 0 \\ M_1^{(1)} & 0 & 1 & 0 \\ M_2^{(1)} & 0 & 0 & 1 \\ \hline \end{array} \right\}, \quad RS^{(2)} = \left\{ \begin{array}{cc} \hline P: & S \\ \hline M_0^{(2)} & 1 \\ M_1^{(2)} & 1 \\ \hline \end{array} \right\}$$

Table 2.2 Local reachable markings of the *SharedResouce* SSPN from Figure 2.6.

The *potential state space PS* of an SSPN is the Descares product of the local reachable state spaces of its components

$$PS = RS^{(0)} \times RS^{(1)} \times \cdots \times RS^{(J-1)},$$

which is a (possibly not proper) superset of the global reachable state space RS.

We will associate the natural numbers $S^{(j)} = \{0, 1, \ldots, n_j-1\}$ with the local reachable markings $RS^{(j)} = \{M_0, M_1, \ldots, M_{n_j-1}\}$ to aid the construction of Markov chains and use them interchangably. The notation

$$M = \mathbf{x} = (x^{(0)}, x^{(1)}, \ldots, x^{(J-1)})$$

refers to the global state $\mathbf{x}$ composed from the local markings $x^{(j)}$, i.e. the marking

$$M(p) = M^{(j)}_{x^{(j)}}(p), \quad \text{if } p \in P^{(j)},$$

which is the union of the local markings $M^{(0)}_{x^{(0)}}, M^{(1)}_{x^{(1)}}, \ldots, M^{(J-1)}_{x^{(J-1)}}$.

Running example 2.8 The local reachable markings of the *SharedResource* SSPN are enumerated in Table 2.2.

The transitions d_1 and d_2 are always enabled in $LN^{(2)}$ because all their input places are located in other components, thus $LN^{(2)}$ is an unbounded Petri net. Despite this, $RS^{(2)}$ is finite, because it only contains the local markings which are reachable in the original net.

The potential state space PS contains $3 \cdot 3 \cdot 2 = 18$ potential markings, although only 8 are reachable (Table 2.1 on page 13). For example, the marking $(2, 2, 0)$ is not reachable, as it would violate mutual exclusion.

2.4 Kronecker algebra

Definition 2.10 The *Kronecker product* of matrices $A \in \mathbb{R}^{n_1 \times m_1}$ and $B \in \mathbb{R}^{n_2 \times m_2}$ is the matrix $C = A \otimes B \in \mathbb{R}^{n_1 n_2 \times m_1 m_3}$, where

$$c[i_1 n_1 + i_2, j_1 m_1 + j_2] = a[i_1, j_1] b[i_2, j_2].$$

Some properties of the Kroncker product are

1. Associativity:
$$A \otimes (B \otimes C) = (A \otimes B) \otimes C,$$
which makes J-way Kronecker products $A^{(0)} \otimes A^{(1)} \otimes \cdots \otimes A^{(J-1)}$ well-defined.
2. Distributivity over matrix addition:
$$(A + B) \otimes (C + D) = A \otimes C + B \otimes C + A \otimes D + B \otimes D,$$
3. Compatibility with ordinary matrix multiplication:
$$(AB) \otimes (CD) = (A \otimes C)(B \otimes D),$$
in particular,
$$A \otimes B = (A \otimes I_2)(I_1 \otimes B)$$
for appropriately-sized identity matrices I_1 and I_2.

We will occasionally employ multi-index notation to refer to elements of Kronecker product matrices. For example, we will write

$$b[\mathbf{x}, \mathbf{y}] = b[(x^{(0)}, x^{(1)}, \ldots, x^{(J-1)}), (y^{(0)}, y^{(1)}, \ldots, y^{(J-1)})] = \\ a^{(0)}[x^{(0)}, y^{(0)}] a^{(1)}[x^{(1)}, y^{(1)}] \cdots a^{(J-1)}[x^{(J-1)}, y^{(J-1)}],$$

where $\mathbf{x} = (x^{(0)}, x^{(1)}, \ldots, x^{(J-1)})$, $\mathbf{y} = (y^{(0)}, y^{(1)}, \ldots, y^{(J-1)})$ and B is the J-way Kronecker product $A^{(0)} \otimes A^{(1)} \otimes \cdots \otimes A^{(J-1)}$.

Definition 2.11 The *Kronecker sum* of matrices $A \in \mathbb{R}^{n_1 \times m_1}$ and $B \in \mathbb{R}^{n_2 \times m_2}$ is the matrix $C = A \oplus B \in \mathbb{R}^{n_1 n_2 \times m_1 m_3}$, where

$$C = A \otimes I_2 + I_1 \otimes B,$$

where $I_1 \in \mathbb{R}^{n_1 \times m_1}$ and $I_2 \in \mathbb{R}^{n_2 \times m_2}$ are identity matrices.

Example 2.9 Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Their Kronecker product is

$$A \otimes B = \begin{pmatrix} 1\cdot 0 & 1\cdot 1 & 2\cdot 0 & 2\cdot 1 \\ 1\cdot 2 & 1\cdot 0 & 2\cdot 2 & 2\cdot 0 \\ 3\cdot 0 & 3\cdot 1 & 4\cdot 0 & 4\cdot 1 \\ 3\cdot 2 & 3\cdot 0 & 4\cdot 2 & 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \\ 6 & 0 & 8 & 0 \end{pmatrix},$$

while their Kronecker sum is

$$A \oplus B = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 2 & 4 \end{pmatrix}.$$

Chapter 3

Overview of the approach

3.1 General workflow

3.2 Problems

3.3 Out workflow

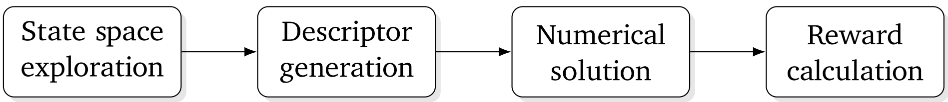


Figure 3.1 The general stochastic analysis workflow.

Chapter 4

Efficient generation and storage of continuous-time Markov chains

4.1 State-space exploration

4.1.1 Explicit state-space exploration

4.1.2 Symbolic methods

Multivalued decision diagrams

Edge-labeled decision diagrams

4.2 Storage of generator matrices

4.2.1 Explicit matrix storage

Dense matrices

Sparse matrices

Column major versus row major storage

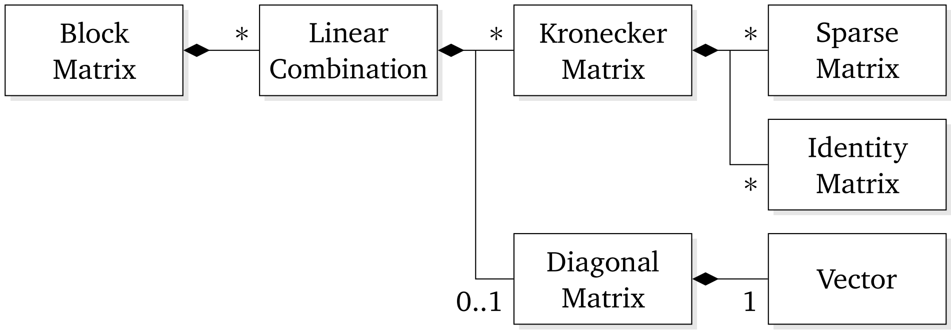


Figure 4.1 Data structure for block Kronecker matrices.

4.2.2 Kronecker decomposition

4.2.3 Block Kronecker decomposition

4.3 Matrix composition

4.3.1 Generating sparse matrices from symbolic state spaces

4.3.2 Explicit block Kronecker decomposition

4.3.3 Symbolic block Kronecker decomposition

Chapter 5

Algorithms for stochastic analysis

Steady state, transient, accumulated and sensitivity analysis problems pose several numerical challanges, especially when the state space of the CTMC and the vectors and matrices involved in the computation are extemely large.

In steady-state and sensitivty analysis, linear equations of the form $\mathbf{x}A = \mathbf{b}$ are solved, such as eqs. (2.3) and (2.6) on page 7 and on page 10. The steady-state probability vector is the solution of the linear system

$$\frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}t} = \boldsymbol{\pi} Q = \mathbf{0}, \quad \boldsymbol{\pi}\mathbf{1}^{\mathrm{T}} = 1, \tag{2.3 revisited}$$

where the infinitesimal generator Q is a rank-deficient matrix. Therefore, steady-state solution methods must handle various generator matrix decompositions and homogenous linear equation with rank deficient matrices. Convergence and compuation times of linear equations solvers depend on the numerical properties of the Q matrices, thus different solvers may be preferred for different models.

In transient analysis, initial value problems with first-order linear differetial equations such as eqs. (2.2) and (2.5) on page 6 and on page 9 are considered. The decomposed generator matrix Q must handled efficiently. Another difficulty is caused by the *stiffness* of differential equations arising from some models, which may significantly increase computation times.

To facilitate configurable stochastic analysis, we implemented several linear equation solvers and transient analysis methods. Where it is reasonable, the implementation is independent of the form of the generator matrix Q. Genericity is achieved by defining an interface between the algorithms and the data structures with operations including

- multiplication of a matrix with a vector from left or right,
- scalar product of vectors with other vectors and columns of matrices,
- specialized operations like accessing the diagonal or off-diagonal parts of a matrix and replacing columns of matrices.

The implementation of these low-level operations is also decoupled from the data structure. This strategy enables further configurability by replacing the operations at runtime, for example, switching between sequential and parallel execution for different parts of the analysis workflow.

While high level configurability allows the modeler to select analysis algorithms appropriate for the model and performance measures under study, low leven configurability of the operations enables additional customization of algorithm execution for the structure of the model as well as the hardware in use. Benchmark results for the workflow are discussed in Section 6.5 on page 47.

In this chapter, we describe the algorithms implemented in our stochastic analysis framework. The pseudocode of the algorithms is annotated with the low level operations performed on the configurable data structure by the high level algorithms.

5.1 Linear equation solvers

5.1.1 Explicit solution by LU decomposition

LU decomposition is a direct method for solving linear equations with forward and backward substitution, i.e. it does not require iteration to reach a given precision.

The decomposition computes the lower triangular matrix L and upper triangular matrix U such that

$$A = LU.$$

To solve the equation

$$\mathbf{x}A = \mathbf{x}LU = \mathbf{b}$$

forward substitution is applied first to find $\mathbf{z}$ in

$$\mathbf{z}U = \mathbf{b},$$

then $\mathbf{x}$ is computed by back substitution from

$$\mathbf{x}L = \mathbf{b}.$$

We used Crout's LU decomposition [35, Section 2.3.1], presented in Algorithm 5.1), which ensures

$$u[i,i] = 1 \text{ for all } i = 0, 1, \ldots, n-1,$$

i.e. the diagonal of the U matrix is uniformly 1. The matrix is filled in during the decomposition even if it was initially sparse, therefore it should first be copied to a dense array storage for efficiency. This considerably limits the size of Markov chains that can be analysed by direct solution due to memory requirements. Our data structure allows access to upper and lower diagonal parts to matrices and linear combinations, therefore no additional storage is needed other than A itself.

Algorithm 5.1 Crout's LU decomposition without pivoting.

Input: the matrix $A \in \mathbb{R}^{n \times n}$ operated on in-place
Output: $L, U \in \mathbb{R}^{n \times n}$ such that $A = LU$, $u[i,i] = 1$ for all $i = 0, 1, \dots, n-1$

for $i \leftarrow 0$ **to** $n-1$ **do**
    for $j \leftarrow 0$ **to** i **do** $a[i,j] \leftarrow a[i,j] - \sum_{k=0}^{j-1} a[i,k]a[k,j]$
    for $j \leftarrow i+1$ **to** $n-1$ **do** $a[i,j] \leftarrow \big(a[i,j] - \sum_{k=0}^{i-1} a[i,k]a[i,j]\big)\big/a[i,i]$
Let A_L, A_D and A_U refer to the strictly lower triangular, diagonal and strictly upper triangular parts of A, respectively.
$L \leftarrow A_L + A_D$
$U \leftarrow A_U + I$
return L, U

Algorithm 5.2 Forward and back substitution.

Input: $U, L \in \mathbb{R}^{n \times n}$, right vector $\mathbf{b} \in \mathbb{R}^n$
Output: solution of $\mathbf{x}LU = \mathbf{b}$

1 **allocate** $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$
2 **if** $\mathbf{b} = \mathbf{0}$ **then** $\mathbf{z} \leftarrow \mathbf{0}$ // Skip forward substitution for homogenous equations
3 **else for** $j \leftarrow 0$ **to** $n-1$ **do** $z[j] \leftarrow b[j] \cdot \sum_{i=0}^{j-1} u[i,j]$
4 **if** $l[n-1,n-1] \approx 0$ **then**
5   **if** $z[n-1] \approx 0$ **then** $x[n-1] \leftarrow 0$ // Set the free parameter to 1
6   **else error** "inconsistent linear equation system"
7 **else** $x[n-1] \leftarrow z[n-1]/l[n-1,n-1]$
8 **for** $j \leftarrow n-2$ **downto** 0 **do**
9   **if** $l[j,j] \approx 0$ **then error** "more than one free parameter"
10     $x[j] \leftarrow \big(z[i] - \sum_{i=j+1}^{n-1} x[i]l[i,j]\big)\big/l[j,j]$
11 **return x**

The forward and back substitution process is shown in Algorithm 5.2. If multiple equations are solver with the same matrix, its LU decomposition may be cached.

Matrices of less than full rank

If the matrix Q is of rank $n-1$, the element $l[n-1,n-1]$ in Crout's LU decomposition will be 0. In this case, $x[n-1]$ is a free parameter and will be set to 1 to yield a nonzero solution vector when $z[n-1] = 0$. If $z[n-1] \neq 0$, the equation $\mathbf{x}L = \mathbf{z}$ does not have a solution and the error condition in line 6 is triggered. A matrix of rank less than $n-1$ triggers the error condition in line 9.

In practice, the algorithm can be used to solve homogenous equations in Markovian

Algorithm 5.3 Basic iterative scheme for solving linear equations.

Input: matrix $A \in \mathbb{R}^{n \times n}$, right vector $\mathbf{b} \in \mathbb{R}^n$, initial guess $\mathbf{x} \in \mathbb{R}^n$, tolerance $\tau > 0$
Output: approximate solution of $\mathbf{x}A = \mathbf{b}$ and its residual norm

allocate $\mathbf{x}' \in \mathbb{R}^n$ // Previous iterate for convergence test
repeat
  $\mathbf{x}' \leftarrow \mathbf{x}$ // Save the previous vector
  $\mathbf{x} \leftarrow f(\mathbf{x}')$
until $\|\mathbf{x}' - \mathbf{x}\| \leq \tau$
return $\mathbf{x}$ and $\|\mathbf{x}Q - \mathbf{b}\|$

analysis, because the infinitesimal generator matrix Q of an irreducible CTMC is always of rank $n-1$. The solution vector $\mathbf{x}$ is not a probability vector in general, so it must be normalized as $\boldsymbol{\pi} = \mathbf{x}/\mathbf{x}\mathbf{1}^{\mathrm{T}}$ to get a stationary probability distribution vector.

5.1.2 Iterative methods

Iterative methods express the solution of the linear equation $\mathbf{x}A = \mathbf{b}$ as a recurrence

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}),$$

where $\mathbf{x}_0$ is an initial guess vector. The iteration converges to a solution vector when $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$ exists and $\mathbf{x}$ equals the true solution vector $\mathbf{x}^*$. The iteration is illustrated in Algorithm 5.3.

The process is assumed to have converged if subsequent iterates are sufficiently close, i.e. the stopping criterion at the kth iteration is

$$\|\mathbf{x}_k - \mathbf{x}_{k-1}\| \leq \tau \tag{5.1}$$

for some prescribed tolerance τ. In our implementation, we selected the L^1-norm

$$\|\mathbf{x}_k - \mathbf{x}_{k-1}\| = \sum_i \left| x_k[i] - x_{k-1}[i] \right|$$

as the vector norm used for detecting convergence.

Premature termination may be avoided if iterates spaced $m > 1$ iterations apart are used for convergence test ($\|\mathbf{x}_k - \mathbf{x}_{k-m}\| \leq \tau$), but only at the expense of additional memory required for storing m previous iterates. In order to handle large Markov chains with reasonable memory consumption, we only used the convergence test with a single previous iterate.

Correctness of the solution can be checked by observing the norm of the residual $\mathbf{x}_k A - \mathbf{b}$, since the error vector $\mathbf{x}_k - \mathbf{x}^*$ is generally not available. Because the additional matrix multiplication may make the latter check costly, it is performed only after

Algorithm 5.4 Power iteration.

$\alpha^{-1} \leftarrow 1/\max_i |a[i,i]|$
repeat
  $\mathbf{x}' \leftarrow \mathbf{x}A$ // Vector-matrix product
  $\mathbf{x}' \leftarrow \mathbf{x}' + (-1) \cdot \mathbf{x}$ // In-place scaled vector addition
  $\epsilon \leftarrow \alpha^{-1} \|\mathbf{x}'\|$ // Vector norm calculation
  $\mathbf{x} \leftarrow \mathbf{x} + \alpha^{-1}\mathbf{x}'$ // In-place scaled vector addition
until $\epsilon \le \tau$

detecting convergence by eq. (5.1) on page 28. Unfortunately, the residual norm may not be representative of the error norm if the problem is ill-conditioned.

For a detailed discussion stopping criterions and iterate normalization in steady-state CTMC analysis, we refer to [46, Section 10.3.5].

Power iteration

Power iteration [46, Section 10.3.1] is the one of the simplest iterative methods for Markovian analysis. Its iteration function has the form

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}) = \mathbf{x}_{k-1} + \frac{1}{\alpha}(\mathbf{x}_{k-1}A - \mathbf{b}).$$

The iteration converges if the diagonal elements $a[i,i]$ of A are strictly negative, the off-diagonal elements $a[i,j]$ are nonnegative and $\alpha \ge \max_i |a[i,i]|$. The matrix A satisfies these properties if it is an inifinitesimal generator matrix of an irreducible CTMC. The fastest convergence is achieved when $\alpha = \min_i |a[i,i]|$.

Power iteration can be realized by replacing lines 2–5 in Algorithm 5.3 on page 28 with the loop in Algorithm 5.4.

This realization uses memory efficiently, because it only requires the allocation of a single vector $\mathbf{x}'$ in addition to the initial guess $\mathbf{x}$.

Observation 5.1 If $\mathbf{b} = 0$ and A is an inifitesimal generator matrix, then

$$\begin{aligned}
\mathbf{x}_k \mathbf{1}^{\mathrm{T}} &= \left[\mathbf{x}_{k-1} + \frac{1}{\alpha}(\mathbf{x}_{k-1}A - \mathbf{b})\right]\mathbf{1}^{\mathrm{T}} \\
&= \mathbf{x}_{k-1}\mathbf{1}^{\mathrm{T}} + \frac{1}{\alpha}\mathbf{x}_{k-1}A\mathbf{1}^{T} - \mathbf{b}\mathbf{1}^{\mathrm{T}} \\
&= \mathbf{x}_{k-1}\mathbf{1}^{\mathrm{T}} + \frac{1}{\alpha}\mathbf{x}_{k-1}\mathbf{0}^{\mathrm{T}} - \mathbf{0}\mathbf{1}^{\mathrm{T}} = \mathbf{x}_{k-1}\mathbf{1}^{\mathrm{T}}.
\end{aligned}$$

This means the sum of the elements of the result vector $\mathbf{x}$ and the initial guess vector $\mathbf{x}_0$ are equal, because the iteration leaves the sum unchanged.

To solve an equation of the form

$$\mathbf{x}Q = \mathbf{0}, \quad \mathbf{x}\mathbf{1}^{\mathrm{T}} = 1 \tag{5.2}$$

where Q is an infinitesimal generator matrix, the initial guess $\mathbf{x}_0$ is selected such that $\mathbf{x}_0\mathbf{1}^{\mathrm{T}} = 1$. If the CTMC described by Q is irreducible, we may select

$$x_0[i] \equiv \frac{1}{n}, \tag{5.3}$$

where n is the dimensionality of $\mathbf{x}$. After the initial guess is selected, the equation $\mathbf{x}\mathbf{1}^{\mathrm{T}}$ may be ignored to solve $\mathbf{x}Q = \mathbf{0}$ with the power method. This process yields the solution of the original problem (5.2).

Jacobi and Gauss–Seidel iteration

Jordan and Gauss–Seidel iterative methods [46, Section 10.3.2–3] repeatedly solve a system of simultaneous equations of a specific form.

In Jordan iteration, the system

$$\left.\begin{aligned}
b[0] &= x_k[0]a[0,0] + x_{k-1}[1]a[1,0] + \cdots + x_{k-1}[n-1]a[n-1,0], \\
b[1] &= x_{k-1}[0]a[0,1] + x_k[1]a[1,1] + \cdots + x_{k-1}[n-1]a[n-1,1], \\
&\vdots \\
b[n-1] &= x_{k-1}[0]a[0,n-1] + x_{k-1}[1]a[1,n-1] + \cdots + x_k[n-1]a[n-1,n-1],
\end{aligned}\right\}$$

is solved for $\mathbf{x}_k$ at each iteration, i.e. there is a single unknown in each row and the rest of the variables are taken from the previous iterate. In vector form, the iteration can be expressed as

$$\mathbf{x}_k = A_D^{-1}(\mathbf{b} - A_O\mathbf{x}_{k-1}),$$

where A_D and A_O are the diagonal (all off-diagonal elements are zero) and off-diagonal (all diagonal elements are zero) parts of $A = A_D + A_O$.

In Gauss–Seidel iteration, the linear system

$$\left.\begin{aligned}
b[0] &= x_k[0]a[0,0] + x_{k-1}[1]a[1,0] + \cdots + x_{k-1}[n-1]a[n-1,0], \\
b[1] &= x_k[0]a[0,1] + x_k[1]a[1,1] + \cdots + x_{k-1}[n-1]a[n-1,1], \\
&\vdots \\
b[n-1] &= x_k[0]a[0,n-1] + x_k[1]a[1,n-1] + \cdots + x_k[n-1]a[n-1,n-1],
\end{aligned}\right\}$$

is considered, i.e. the ith equation contains the first i elements of $\mathbf{x}_k$ as unknowns. The equations are solved for successive elements of $\mathbf{x}_k$ from top to bottom.

Jacobi over-relaxation, a generalized form of Jacobi iteraion, is realized in Algorithm 5.5. The value 1 of the over-relaxation paramter ω corresponds to ordinary

Algorithm 5.5 Jacobi over-relaxation.

Input: matrix $A \in \mathbb{R}^{n \times n}$, right vector $\mathbf{b} \in \mathbb{R}^n$, initial guess $\mathbf{x} \in \mathbb{R}^n$, tolerance $\tau > 0$, over-relaxation parameter $\omega > 0$
Output: approximate solution of $\mathbf{x}A = \mathbf{b}$

1 **allocate** $\mathbf{x}' \in \mathbb{R}^n$
2 Let A_O refer to the off-diagonal part of A.
3 **repeat**
4     $\mathbf{x}' \leftarrow \mathbf{x}A_O$ // Matrix-vector product
5     $\mathbf{x}' \leftarrow \mathbf{x}' + (-1) \cdot \mathbf{b}$ // In-place scaled vector addition
6     $\epsilon \leftarrow 0$
7   **for** $i \leftarrow 0$ **to** $n - 1$ **do**
8         $y \leftarrow (1 - \omega)x[i] - \omega x'[i]/a[i,i]$
9         $\epsilon \leftarrow \epsilon + |y - x[i]|$
10         $x[i] \leftarrow y$
11 **until** $\epsilon \leq \tau$
12 **return** $\mathbf{x}$

Jacobi iteration. Values $\omega > 1$ may accelerate convergence, while $0 < \omega < 1$ may help diverging Jacobi iteration converge.

Jacobi over-relaxation has many parallelization opportunities. The matrix multiplication in line 4 and the vector addition in line 5 can be parallelized, as well as the for loop in line 7. Our implementation takes advantage of the configurable linear algebra operations framework to execute lines 4 and 5 with possible paralellization considering the structures of both the vectors $\mathbf{x}, \mathbf{x}'$ and the matrix A. However, the inner loop is left sequential to reduce implementation complexity, as it represents only a small fraction of execution time compared to the matrix-vector product.

Algorithm 5.6 shows an implementation of successive over-relaxation for Gauss–Seidel iteration, where the notation $\mathbf{a}_O[\cdot, i]$ refers to the ith column of A_O.

Gauss–Seidel iteration cannot easily be parallelized, because calculation of successive elements $x[0], x[1], \ldots$ depend on all of the prior elements. However, in contrast with Jacobi iteration, no memory is required in addition to the vectors $\mathbf{x}$, $\mathbf{b}$ and the matrix X, which makes the algorithm suitable for very large vectors and memory-constrained situations. In addition, convergence is often significantly faster.

The sum of elements $\mathbf{x}\mathbf{1}^\mathrm{T}$ does not stay constant during Jacobi or Gauss–Seidel iteration. Thus, when solving equations of the form $\mathbf{x}Q = \mathbf{0}, \mathbf{x}\mathbf{1}^\mathrm{T} = 1$, normalization cannot be entierly handled by the initial guess. We instead transform the equation into

Algorithm 5.6 Gauss–Seidel successive over-relaxatation.

Input: matrix $A \in \mathbb{R}^{n \times n}$, right vector $\mathbf{b} \in \mathbb{R}^n$, initial guess $\mathbf{x} \in \mathbb{R}^n$, tolerance $\tau > 0$, over-relaxation parameter $\omega > 0$
Output: approximate solution of $\mathbf{x}A = \mathbf{b}$

allocate $\mathbf{x}' \in \mathbb{R}^n$
Let A_O refer to the off-diagonal part of A.
repeat
  $\epsilon \leftarrow 0$
  for $i \leftarrow 0$ **to** $n-1$ **do**
    $\mathit{scalarProduct} \leftarrow \mathbf{x} \cdot \mathbf{a}_O[\cdot, i]$ // Scalar product with column of matrix
    $y \leftarrow \omega(b[i] - \mathit{scalarProduct})/a[i,i] + (1-\omega) \cdot x[i]$
    $\epsilon \leftarrow \epsilon + |y - x[i]|$
    $x[i] \leftarrow y$
until $\epsilon \leq \tau$
return $\mathbf{x}$

the form

$$\mathbf{x} \begin{pmatrix} q[0,0] & q[0,1] & \cdots & q[0,n-2] & 1 \\ q[1,0] & q[1,1] & \cdots & q[1,n-2] & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q[n-2,0] & q[n-2,1] & \cdots & q[n-2,n-2] & 1 \\ q[n-1,0] & q[n-1,1] & \cdots & q[n-2,n-1] & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{5.4}$$

where we take advantage of the fact that the infinitesimal generator matrix is not of full rank, therefore one of the columns is redundant and can be replaced with the condition $\mathbf{x}\mathbf{1}^{\mathrm{T}} = 1$. While this transformation may affect the convergence behavior of the algorithm, it allows uniform handling of homogenous and non-homogenous linear equations.

Group iterative methods

Group or *block* iterative methods Stewart [46, Section 10.4] assume the block structure for the vectors $\mathbf{x}$, $\mathbf{b}$ and the matrix A

$$\mathbf{x}[i] \in \mathbb{R}^{n_i}, \mathbf{b}[j] \in \mathbb{R}^{n_j}, A[i,j] \in \mathbb{R}^{n_i \times n_j} \text{ for all } i,j \in \{0,1,\ldots,N-1\},$$

Infinitesimal generator matrices in the block Kronecker decomposition along with appropriately partitioned vectors match this structure **TODO ref az elozo fejezetre**. Each block of $\mathbf{x}$ corresponds to a group a variables that are simultaneously solved for.

Algorithm 5.7 Group Jacobi over-relaxation.

Input: block matrix A, block right vector $\mathbf{b}$, block initial guess $\mathbf{n}$, tolerance $\tau > 0$, over-relaxation parameter $\omega > 0$
Output: approximate solution of $\mathbf{x}A = \mathbf{b}$ and its residual norm

```
allocate x′ and c with the same block structure as x and b
Let A_OB represent the off-diagonal part of the block matrix A with the blocks
along the diagonal set to zero.
repeat
    x′ ← x, c ← b
    c ← c + (−1) · x′A_OB            // Scaled accumulation of vector-matrix product
    parallel for i ← 0 to N − 1 do                  // Loop over all blocks
        Solve x[i]A[i,i] = c[i] for x[i]
    ε ← 0
    for k ← 0 to n − 1 do                           // Loop over all elements
        y ← ωx[k] + (1 − ω)x′[k]
        ε ← ε + |y − x′[k]|
        x[k] ← y
until ε ≤ τ
```

Group Jacobi iteration solves the linear system

$$\left.\begin{aligned}
\mathbf{b}[0] &= \mathbf{x}_k[0]A[0,0] + \mathbf{x}_{k-1}[1]A[1,0] + \cdots + \mathbf{x}_{k-1}[n-1]A[n-1,0],\\
\mathbf{b}[1] &= \mathbf{x}_{k-1}[0]A[0,1] + \mathbf{x}_k[1]A[1,1] + \cdots + \mathbf{x}_{k-1}[n-1]A[n-1,1],\\
&\vdots\\
\mathbf{b}[n-1] &= \mathbf{x}_{k-1}[0]A[0,n-1] + \mathbf{x}_{k-1}[1]A[1,n-1] + \cdots + \mathbf{x}_k[n-1]A[n-1,n-1],
\end{aligned}\right\}$$

while group Gauss–Seidel considers

$$\left.\begin{aligned}
\mathbf{b}[0] &= \mathbf{x}_k[0]A[0,0] + \mathbf{x}_{k-1}[1]A[1,0] + \cdots + \mathbf{x}_{k-1}[n-1]A[n-1,0],\\
\mathbf{b}[1] &= \mathbf{x}_k[0]A[0,1] + \mathbf{x}_k[1]A[1,1] + \cdots + \mathbf{x}_{k-1}[n-1]A[n-1,1],\\
&\vdots\\
\mathbf{b}[n-1] &= \mathbf{x}_k[0]A[0,n-1] + \mathbf{x}_k[1]A[1,n-1] + \cdots + \mathbf{x}_k[n-1]A[n-1,n-1].
\end{aligned}\right\}$$

Implementations of group Jacobi over-relaxation and group Gauss–Seidel successive over-relaxation are shown in Algorithms 5.7 and 5.8 on this page and. The inner linear equations of the form $\mathbf{x}[i]A[i,i] = \mathbf{c}$ may be solved by any algorithm, for example, LU decomposition, iterative methods, or even block-iterative methods if A has a two-level block structure. The choice of the inner algorithm may significantly affect performance and care must be taken to avoid diverging inner solutions in an iterative solver is used.

Algorithm 5.8 Group Gauss–Seidel successive over-relaxation.

Input: block matrix A, block right vector $\mathbf{b}$, block initial guess $\mathbf{n}$, tolerance $\tau > 0$, over-relaxation parameter $\omega > 0$
Output: approximate solution of $\mathbf{x}A = \mathbf{b}$ and its residual norm

allocate $\mathbf{x}'$ and $\mathbf{c}$ large enough to store a single block of $\mathbf{x}$ and $\mathbf{b}$.
repeat
  $\epsilon \leftarrow 0$
  for $i \leftarrow 0$ **to** $N-1$ **do** // Loop over all blocks
    $\mathbf{x}' \leftarrow \mathbf{x}[i], \mathbf{c} \leftarrow \mathbf{b}[i]$
    for $j \leftarrow 0$ **to** $N-1$ **do**
      if $i \neq j$ **then** // Scaled accumulation of vector-matrix product
        $\mathbf{c} \leftarrow \mathbf{c} + (-1) \cdot \mathbf{x}[j]A[i,j]$
    Solve $\mathbf{x}[i]A[i,i] = \mathbf{c}$ for $\mathbf{x}[i]$
    for $k \leftarrow 0$ **to** $n_i - 1$ **do**
      $y \leftarrow \omega x[i][k] + (1-\omega)x'[k]$
      $\epsilon \leftarrow \epsilon + |y - x'[k]|$
      $x[i][k] \leftarrow y$
until $\epsilon \leq \tau$

In Jacobi over-relaxation, paralellization of both the matrix multiplication and the inner loop is possible. However, two vectors of the same size as $\mathbf{x}$ are required for temporary storage.

Gauss–Seidel successive over-relaxation cannot be parallelized easily, but it requires only two temporary vectors of size equal to the largest block of $\mathbf{x}$, much less than Jacobi over-relaxation. Moreover, it often requires fewer steps to converge, making it preferable over Jacobi iteration.

Because the inner solver may be selected by the user and thus its convergence behaviour varies widely, we do not perform the transformation for homogenous equations (5.4). Instead, the normalization $\boldsymbol{\pi} = \mathbf{x}/\mathbf{x}\mathbf{1}^{\mathrm{T}}$ is performed only after finding any nonzero solution of $\mathbf{x}Q = \mathbf{0}$.

For a detailed analysis of the convergence behaviour of group iterative methods, we refer to Greenbaum [21, Chapter 14] and

BiConjugate Gradient Stabilized (BiCGSTAB)

BiConjugate Gradient Stabilized (BiCGSTAB) [43, Section 7.4.2; 49] is an iterative algorithm belonging to the class of Krylov subspace methods, which includes other algorithms such as the Generalized Minimum Residual (GMRES) [42], Conjugate Gradient Squared (CGS) [44] and IDR(s) [45].

We selected BiCGSTAB as the Krylov subspace solver in our framework because of its good convergence behaviour and low memory requirements. BiCGSTAB only requires the storage of 7 vectors, which makes it suitable even for large state spaces with large states vectors, unlike e.g. GMRES, which allocates an additional vector every iteration.

Algorithm 5.9 on page 41 shows the pseudocode for BiCGSTAB. Our implementation is based on the MATLAB code[1] by Barrett et al. [2].

Solving preconditioned equations in the form $\mathbf{x}AM^{-1} = \mathbf{b}M^{-1}$ could improve convergence, but was omitted from our current implementation. As the choice is appropriate preconditioner matrices M is not trivial [25], implementation and sudy of preconditioners for Markov chains, especially with block Kronecker decomposition, is in the scope of our future work.

Because six vectors are allocated in addition to $\mathbf{x}$ and $\mathbf{b}$, the amount of available memory may be a significant bottleneck.

Similar to Observation 5.1 on page 29, it can be seen that the sum $\mathbf{x}\mathbf{1}^{\mathrm{T}}$ stays constant throughout BiCGSTAB iteration. Thus, we can find probability vectors satisfying homogenous equations by the initialization in eq. (5.3) on page 30.

5.2 Transient analysis

5.2.1 Uniformization

The *uniformization* or *randomization* method solves the initial value problem

$$\frac{\mathrm{d}\boldsymbol{\pi}(t)}{\mathrm{d}t} = \boldsymbol{\pi}(t)Q, \quad \boldsymbol{\pi}(t) = \boldsymbol{\pi}0 \tag{2.2 revisited}$$

by computing

$$\boldsymbol{\pi}(t) = \sum_{k=0}^{\infty} \boldsymbol{\pi}_0 P^k e^{-\alpha t} \frac{(\alpha t)^k}{k!}, \tag{5.5}$$

where $P = \alpha^{-1}Q + I$, $\alpha \ge \max_i |a[i,i]|$ and $e^{-\alpha t}\frac{(\alpha t)^k}{k!}$ is the value of the Poisson probabilty function with rate αt at k.

Integrating both sides of eq. (5.5) to compute $\mathbf{L}(t)$ yields [40]

$$\begin{aligned}\int_0^t \boldsymbol{\pi}(u)\,\mathrm{d}u = \mathbf{L}(t) &= \sum_{k=0}^{\infty} \boldsymbol{\pi}_0 P^k \int_0^t e^{-\alpha u}\frac{(\alpha u)^k}{k!}\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \boldsymbol{\pi}_0 P^k \frac{1}{\alpha} \sum_{l=k+1}^{\infty} e^{-\alpha t}\frac{(\alpha t)^l}{l!}\end{aligned}$$

[1] `http://www.netlib.org/templates/matlab//bicgstab.m`

$$= \frac{1}{\alpha} \sum_{k=0}^{\infty} \boldsymbol{\pi}_0 P^k \left(1 - \sum_{l=0}^{k} e^{-\alpha t} \frac{(\alpha t)^l}{l!} \right). \tag{5.6}$$

Both eqs. (5.5) and (5.6) on page 35 and on the current page can be realized as

$$\mathbf{x} = \frac{1}{W} \left(\sum_{k=0}^{k_{\text{left}}-1} w_{\text{left}} \boldsymbol{\pi}_0 P^k + \sum_{k=k_{\text{left}}}^{k_{\text{right}}} w[k - k_{\text{left}}] \boldsymbol{\pi}_0 P^k \right), \tag{5.7}$$

where $\mathbf{x}$ is either $\boldsymbol{\pi}(t)$ or $\mathbf{L}(t)$, k_{left} and k_{right} are *trimming constants* selected based on the required precision, $\mathbf{w}$ is a vector of (possibly accumulated) Poisson weights and W is a scaling factor. The weight before the left cutoff w_{left} is 1 if the accumulated probability vector $\mathbf{L}(t)$ is calculated, 0 otherwise.

Eq. (5.7) is implemented by Algorithm 5.10 on page 42. The algorithm performs *steady-state* detection in line 9 to avoid unneccessary work once the iteration vector $\mathbf{p}$ reaches the steady-state distribution $\boldsymbol{\pi}(\infty)$, i.e. $\mathbf{p} \approx \mathbf{p}P$. If the initial distribution $\boldsymbol{\pi}_0$ is not further needed or can be generated efficiently (as it is the case with a single initial state), the result vector $\mathbf{x}$ may share the same storing, resulting in a memory overhead of only two vectors $\mathbf{p}$ and $\mathbf{q}$.

The weights and trimming constants may be calculated by the famous algorithm of Fox and Glynn [17]. However, their algorithm is extremely complicated due to the limitations of single-precision floating-point arithmetic [24]. We implemented Burak's significantly simpler algorithm [8] in double precision instead (Algorithm 5.11 on page 43), which avoids underflow by a scaling factor $W \gg 1$.

5.2.2 TR-BDF2

A weakness of the uniformization algorithm is the poor tolerance of *stiff* Markov chains. The CTMC is called stiff if the $|\lambda_{\text{min}}| \ll |\lambda_{\text{max}}|$, where λ_{min} and λ_{max} are the nonzero eigenvalues of the infinitesimal generator matrix Q of minimum and maximum absolute value [39]. In other words, stiff Markov chains have behaviors on drastically different timescales, for example, clients are served frequently while failures happen infrequently.

Stiffness leads to very large small rates α in line of Algorithm 5.10 on page 42, thus a large right cutoff k_{right} is required for computing the transient solution with sufficient accuracy. Moreover, the slow stabilization results in taking many iterations before steady-state detection in line 9.

Some methods that can handle stiff CTMCs efficiently are stochastic complementation [30], which decouples the slow and fast behaviors of the system, and adaptive uniformization [31], which varies the uniformization rate α. Alternatively, an L-stable differential equation solver may be used to solve eq. (2.2) on page 6, such as TR-BDF2 [1, 39].

TR-BDF2 is an implicit integrator with alternating trapezoid rule (TR) steps

$$\boldsymbol{\pi}_{k+\gamma}(2I + \gamma h_k Q) = 2\boldsymbol{\pi}_k + \gamma h_k \boldsymbol{\pi}_k Q$$

and second order backward difference steps

$$\boldsymbol{\pi}_{k+1}[(2-\gamma)I - (1-\gamma)h_k Q] = \frac{1}{\gamma}\boldsymbol{\pi}_{k+\gamma} - \frac{(1-\gamma)^2}{\gamma}\boldsymbol{\pi}_k,$$

which advance the time together by a step of size h_k. The constant $0 < \gamma < 1$ sets the breakpoint between the two steps. We set it to $\gamma = 2 - \sqrt{2} \approx 0.59$ following the recommendation of Bank et al. [1].

As a guess for the initial step size h_0, we chose the uniformization rate of Q. The kth step size $h_k > 0$, including the 0th one, is selected such that the local error estimate

$$LTE_{k+1} = \left\| 2\frac{-3\gamma^4 + 4\gamma - 2}{24 - 12\gamma} h_k \left[-\frac{1}{\gamma}\boldsymbol{\pi}_k + \frac{1}{\gamma(1-\gamma)}\boldsymbol{\pi}_{k+\gamma} - \frac{1}{1-\gamma}\boldsymbol{\pi}_{k+1} \right] \right\| \tag{5.8}$$

is bounded by the local error tolerance

$$LTE_{k+1} \leq \left(\frac{\tau - \sum_{i=0}^{k} LTE_i}{t - \sum_{i=0}^{k} k_i} \right) h_{k+1}.$$

This Local Error per Unit Step (LEPUS) error control "produces excellent results for many problems", but is usually costly [39]. Moreover, the accumulated error at the end of integration may be larger than the prescribed tolerance τ, since eq. (5.8) is only an approximation of the true error.

An implementation of TR-BDF2 based on the pseudocode of A. L. Reibman and Trivedi [39] is shown in Algorithm 5.12 on page 44.

In lines 12 and 16 any linear equation solver from Section 5.1 on page 26 may be used except power iteration, since the matrices, in general, do not have strictly negative diagonals. Due to the way the matrices, which are linear combinations of I and Q, are passed to the inner solvers, our TR-BDF2 integrator is currently limited to Q matrices which are not in block form.

The vectors $\boldsymbol{\pi}_0, \boldsymbol{\pi}_k$ and $\boldsymbol{\pi}_{k+\gamma}, \mathbf{d}_{k+1}$ may share storage, respectively, therefore only 4 state-space sized vectors are required in addition to the initial distribution $\boldsymbol{\pi}_0$.

The most computationally intensive part is the solution of two linear equation per every attempted step, which may make TR-BDF2 extremely slow. However, its performance does *not* depend on the stiffness of the Markov chain, which may make it better suited to stiff CTMCs than uniformization [39].

5.3 Mean time to first failure

In MTFF calculation (Section 2.2.3 on page 10), quantities of the forms

$$MTFF = -\underbrace{\boldsymbol{\pi}_U Q_{UU}^{-1}}_{\gamma} \mathbf{1}^{\mathrm{T}}, \quad \mathbb{P}(X(TFF_{+0}) = y) = -\underbrace{\boldsymbol{\pi}_U Q_{UU}^{-1}}_{\gamma} \mathbf{q}_{UD'}^{\mathrm{T}} \qquad \text{(2.7, 2.8 revisited)}$$

are computed, where U, D, D' are the set of operations states, failure states and a specific failure mode $D' \subsetneq D$, respectively.

The vector $\boldsymbol{\gamma} \in \mathbb{R}^{|U|}$ is the solution of the linear equation

$$\boldsymbol{\gamma} Q_{UU} = \boldsymbol{\pi}_U \tag{5.9}$$

and may be obtained by any linear equation solver.

The sets $U, D = D_1 \cup D_2 \cup \cdots$ are constructed by the evaluation of CTL expressions. If the failure mode D_i is described by φ_i, then the sets D and U are described by CTL formulas $\varphi_D = \neg \mathsf{AX}\,\mathsf{true} \vee \varphi_1 \vee \varphi_2 \vee \cdots$ and $\varphi_U = \neg \varphi_D$, where the deadlock condition $\neg \mathsf{AX}\,\mathsf{true}$ is added to make 5.9 irreducible.

After the set U is generated symbolically, the matrix Q_{UU} may be decomposed in the same way as the whole state space S. Thus, the vector-matrix operations required for solving (5.9) can be executed as in steady-state analysis.

5.4 Efficient vector-matrix products

Iterative linear equation and transient distribution solvers require several vector-matrix products per iteration. Therefore, efficient vector-matrix multiplication algorithms are required for the various matrix storage methods (i.e. dense, sparse and block Kronecker matrices) to support configurable stochastic analysis.

Our data structure supports run-time reconfiguration of operations, for example, to switch between parallel and sequential matrix multiplication implementations for different parts of an algorithm, depending on the characteristics of the model and the hardware which runs the analysis.

Implemented matrix multiplication for the data structure (see Figure 4.1 on page 24) routines are

- Multiplication of vectors with dense and sparse matrices. Sparse matrix multiplication may be parallelized by splitting the columns of the matrix into chunck and submitting each chunk to the executor thread pool.

  Operations with vectors and sparse matrices are implemented in an `unsafe`[2] context. The elements of the data structures are not under the influence of the

[2] `https://msdn.microsoft.com/en-us/library/chfa2zb8.aspx`

Garbage Collector runtime, but stored in natively allocated memory. This allows the handling of large matrices without adversely impacting the performance of other parts of the program, albeit the cost of allocations in increased.

- Multiplication with block matrices by delegation to the constituent blocks of the matrix (Algorithm 5.13 on page 45). The input and output vectors are converted to block vectors before multiplication. If parallel execution is required, each block of the output vector can be computed in a different task, since it is independent from the others.

- Multiplication by a linear combination of matrices is delegated to the constituent matrices (Algorithm 5.14 on page 45). An in-place scaled addition of vector-matrix product to a vector operation is required for this delegation. To facilitate this, each vector-matrix multiplication algorithm is implemented also as an in-place addition and in-place scaled addition of vector-matrix product, and the appropriate implementation is selected based on the function call aruments.

- Multiplications $\mathbf{b} \cdot \operatorname{diag}\{\mathbf{a}\}$ by diagonal matrices are executed as elementwise product $\mathbf{b} \odot \mathbf{a}$. The special case of multiplication by an identity matrix is equivalent to a vector copy.

- Multiplications by Kronecker products is performed by the Shuffle algorithm [3, 6] as shown in Algorithm 5.15 on page 45.

  The algorithm requires access to slices of a vector, denoted as $\mathbf{x}[i_0{:}s{:}l]$, which refers to the elements $x[i], x[i+s], x[i+2s], \ldots, x[i+(l-1)s]$. Thus, slices were integrated into the operations framework as first-class elements, and multiplication algorithms are implemented with support for vector slice indexing.

  Shuffle rewrites the Kronecker products as

$$\bigotimes_{h=0}^{k-1} A^{(h)} = \prod_{h=0}^{k-1} I_{\prod_{f=0}^{h-1} n_f \times \prod_{f=0}^{h-1} n_f} \otimes A^{(h)} \otimes I_{\prod_{f=h+1}^{k-1} m_f \times \prod_{f=h+1}^{k-1} m_f},$$

  where $I_{a \times a}$ denotes an $a \times a$ identity matrix. Multiplications by terms of the form $I_{N \times N} \otimes A^{(h)} \otimes I_{M \times M}$ are carried out in the loop at line 8 of Algorithm 5.15.

  The temporary vectors $\mathbf{x}, \mathbf{x}'$ are large enough store the results of the successive matrix multiplications. They are cached for every worker thread to avoid repeated allocations.

  Other algorithms for vector-Kronecker product multiplication are the Slice [16] and Split [12] algorithms, which are more amenable to parallel execution than Shuffle. Their implementation is in the scope of our future work.

5.5 Processing results

5.5.1 Calculation of rewards

Symbolic storage of reward functions

5.5.2 Calculation of sensitivity

Sensitivity of state probabilities

Sensitivity of rewards

Algorithm 5.9 BiCGSTAB iteration without preconditioning.

Input: matrix $A \in \mathbb{R}^{n\times n}$, right vector $\mathbf{b} \in \mathbb{R}^n$, initial guess $\mathbf{x} \in \mathbb{R}^n$, tolerance $\tau > 0$
Output: approximate solution of $\mathbf{x}A = \mathbf{b}$

allocate $\mathbf{r}, \mathbf{r}_0, \mathbf{v}, \mathbf{p}, \mathbf{s}, \mathbf{t} \in \mathbb{R}^n$
$\mathbf{r} \leftarrow \mathbf{b}$
$\mathbf{r} \leftarrow \mathbf{r} + (-1) \cdot \mathbf{x}A$ // Scaled accumulation of vector-matrix product
if $\|\mathbf{r}\| \le \tau$ **then**
    message "initial guess is correct, skipping iteration"
    return $\mathbf{x}$
$\mathbf{r}_0 \leftarrow \mathbf{r}, \mathbf{v} \leftarrow \mathbf{0}, \mathbf{p} \leftarrow \mathbf{0}, \rho' \leftarrow 1, \alpha \leftarrow 1, \omega \leftarrow 1$
while true **do**
    $\rho \leftarrow \mathbf{r_0} \cdot \mathbf{r}$ // Scalar product
    if $\rho \approx 0$ **then error** "breakdown: $\mathbf{r} \perp \mathbf{r}_0$"
    $\beta \leftarrow \rho/\rho' \cdot \alpha/\omega$
    $\mathbf{p} \leftarrow \mathbf{r} + \beta \cdot \mathbf{p}$ // Scaled vector addition
    $\mathbf{p} \leftarrow \mathbf{p} + (-\beta\omega) \cdot \mathbf{v}$ // In-place scaled vector addition
    $\alpha \leftarrow \rho/(\mathbf{r}_0 \cdot \mathbf{v})$ // Scalar product
    $\mathbf{r} \leftarrow \mathbf{s} + (-\alpha) \cdot \mathbf{s}$ // Scaled vector addition
    if $\|\mathbf{s}\| < \tau$ **then**
        $\mathbf{x} \leftarrow \mathbf{x} + \alpha \cdot \mathbf{p}$ // In-place scaled vector addition
        message "early return with vanishing **s**"
        return $\mathbf{x}$
    $\mathbf{t} \leftarrow \mathbf{s}A$ // Vector-matrix multiplication
    $tLengthSquared \leftarrow \mathbf{t} \cdot \mathbf{t}$ // Scalar product
    if $tLengthSquared \approx 0$ **then error** "breakdown: $\mathbf{t} \approx \mathbf{0}$"
    $\omega \leftarrow (\mathbf{t} \cdot \mathbf{s})/tLengthSquared$ // Scalar product
    if $\omega \approx 0$ **then error** "breakdown: $\omega \approx 0$"
    $\epsilon \leftarrow 0$
    for $i \leftarrow 0$ **to** $n-1$ **do**
        $change \leftarrow \alpha p[i] + \omega s[i]$
        $\epsilon \leftarrow \epsilon + |change|$
        $x[i] \leftarrow x[i] + change$
    if $\epsilon \le \tau$ **then return** $\mathbf{x}$
    $\mathbf{s} \leftarrow \mathbf{t} + (-\omega) \cdot \mathbf{r}$ // Scaled vector addition
    $\rho' \leftarrow \rho$

Algorithm 5.10 Uniformization.

Input: infinitesimal generator $Q \in \mathbb{R}^{n \times n}$, initial probability vector $\boldsymbol{\pi}_0 \in \mathbb{R}^n$, truncation parameters $k_{\text{left}}, k_{\text{right}} \in \mathbb{N}$, weights $w_{\text{left}} \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^{k_{\text{right}} - k_{\text{left}}}$, scaling constant $W \in \mathbb{R}$, tolerance $\tau > 0$

Output: instantaneous or accumulated probability vector $\mathbf{x} \in \mathbb{R}^n$

allocate $\mathbf{x}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^n$
$\alpha^{-1} \leftarrow 1/\max_i |a[i,i]|$
$\mathbf{p} \leftarrow \boldsymbol{\pi}_0$
if $w_{\text{left}} = 0$ **then** $\mathbf{x} \leftarrow \mathbf{0}$ **else** $\mathbf{x} \leftarrow w_{\text{left}} \cdot \mathbf{p}$ // Vector scaling
for $k \leftarrow 1$ **to** k_{right} **do**
    $\mathbf{q} \leftarrow \mathbf{p}Q$ // Vector-matrix product
    $\mathbf{q} \leftarrow \alpha^{-1} \cdot \mathbf{q}$ // In-place vector scaling
    $\mathbf{q} \leftarrow \mathbf{q} + \mathbf{q}$ // In-place vector addition
    if $\|\mathbf{q} - \mathbf{p}\| \leq \tau$ **then**
        $\mathbf{x} \leftarrow \mathbf{x} + \left(\sum_{l=k}^{k_{\text{right}}} w[l - k_{\text{left}}]\right) \cdot \mathbf{q}$ // In-place scaled vector addition
        break
    if $k < k_{\text{left}} \wedge w_{\text{left}} \neq 0$ **then** $\mathbf{x} \leftarrow \mathbf{x} + w_{\text{left}} \cdot \mathbf{q}$ // In-place scaled vector addition
    else if $k \geq k_{\text{left}}$ **then** $\mathbf{x} \leftarrow \mathbf{x} + w[k - k_{\text{left}}] \cdot \mathbf{q}$ // In-place scaled vector addition
    Swap the references to $\mathbf{p}$ and $\mathbf{q}$
$\mathbf{x} \leftarrow W^{-1} \cdot \mathbf{x}$ // In-place vector scaling
return $\mathbf{x}$

Algorithm 5.11 Burak's algorithm for calculating the Poisson weights.

Input: Poisson rate $\lambda = \alpha t$, tolerance $\tau > 10^{-50}$
Output: truncation parameters $k_{\text{left}}, k_{\text{right}} \in \mathbb{N}$, weights $\mathbf{w} \in \mathbb{R}^{k_{\text{right}}-k_{\text{left}}}$, scaling constant $W \in \mathbb{R}$

$M_w \leftarrow 30, M_a \leftarrow 44, M_s \leftarrow 21$
$m \leftarrow \lfloor \lambda \rfloor, \mathit{tSize} \leftarrow \lfloor M_w \sqrt{\lambda} + M_a \rfloor, \mathit{tStart} \leftarrow \max\{m + M_s - \lfloor \mathit{tSize}/2 \rfloor, 0\}$
allocate tWeights $\in \mathbb{R}^{\mathit{tSize}}$
$\mathit{tWeights}[m - \mathit{tStart}] \leftarrow 2^{176}$
for $j \leftarrow m - \mathit{tStart}$ **downto** 1 **do**
  $\mathit{tWeights}[j-1] = (j + \mathit{tStart})\,\mathit{tWeights}[j]/\lambda$
for $j \leftarrow m - \mathit{tStart} + 1$ **to** tSize **do**
  $\mathit{tWeights}[j+1] = \lambda\,\mathit{tWeights}[j]/(j + \mathit{tStart})$
$W \leftarrow 0$
for $j \leftarrow 0$ **to** $m - \mathit{tStart} - 1$ **do**
  $W \leftarrow W + \mathit{tWeights}[j]$
$\mathit{sum1} \leftarrow 0$ // Avoid adding small numbers to larger numbers
for $j \leftarrow \mathit{tSize} - 1$ **downto** $m - \mathit{tStart}$ **do**
  $\mathit{sum1} \leftarrow \mathit{sum1} + \mathit{tWeights}[j]$
$W \leftarrow W + \mathit{sum1}, \mathit{threshold} \leftarrow W\tau/2, \mathit{cdf} \leftarrow 0, i \leftarrow 0$
while $\mathit{cdf} < \mathit{threshold}$ **do**
  $\mathit{cdf} \leftarrow \mathit{cdf} + \mathit{tWeights}[i]$
  $i \leftarrow i + 1$
$k_{\text{left}} \leftarrow \mathit{tStart} + i, \mathit{cdf} \leftarrow 0, i \leftarrow \mathit{tSize} - 1$
while $\mathit{cdf} < \mathit{threshold}$ **do**
  $\mathit{cdf} \leftarrow \mathit{cdf} + \mathit{tWeights}[i]$
  $i \leftarrow i - 1$
$k_{\text{right}} \leftarrow \mathit{tStart} + i$
allocate $\mathbf{w} \in \mathbb{R}^{k_{\text{right}}-k_{\text{left}}}$
for $j \leftarrow k_{\text{left}}$ **to** k_{right} **do**
  $w[j - k_{\text{left}}] \leftarrow \mathit{tWeights}[j - \mathit{tStart}]$
return $k_{\text{left}}, k_{\text{right}}, \mathbf{w}, W$

Algorithm 5.12 TR-BDF2 for transient analysis.

Input: infinitesimal generator $Q \in \mathbb{R}^{n\times n}$, initial distribution $\boldsymbol{\pi}_0$, mission time $t > 0$, tolerance $\tau > 0$

Output: transient distribution $\boldsymbol{\pi}(t)$

allocate $\boldsymbol{\pi}_k, \boldsymbol{\pi}_{k+\gamma}, \boldsymbol{\pi}_{k+1}, \mathbf{d}_k, \mathbf{d}_{k+1}, \mathbf{y} \in \mathbb{R}^n$
$\mathit{maxIncrease} \leftarrow 10, \mathit{leastDecrease} \leftarrow 0.9$
$\mathit{timeLeft} \leftarrow t, h \leftarrow 1/\max_i |a[i,i]|, \gamma \leftarrow 2-\sqrt{2}, C \leftarrow \left|\frac{-3\gamma^4+4\gamma-2}{24-12\gamma}\right|, \mathit{errorSum} \leftarrow 0$
$\boldsymbol{\pi}_k \leftarrow \boldsymbol{\pi}_0$
$\mathbf{d}_k \leftarrow \boldsymbol{\pi}_k Q$ // Vector-matrix product
while $\mathit{timeLeft} > 0$ **do**
  $\mathit{stepFailed} \leftarrow$ false, $h \leftarrow \min\{h, \mathit{timeLeft}\}$
  while true **do**
    /* TR step */
    $\mathbf{y} \leftarrow 2 \cdot \boldsymbol{\pi}_k$ // Vector scaling
    $\mathbf{y} \leftarrow \mathbf{y} + \gamma h \cdot \mathbf{d}_k$ // In-place vector addition
    Solve $\boldsymbol{\pi}_{k+\gamma}(2I + -\gamma hQ) = \mathbf{y}$ for $\boldsymbol{\pi}_{k+\gamma}$ with initial guess $\boldsymbol{\pi}_k$
    /* BDF2 step */
    $\mathbf{y} \leftarrow -\frac{(1-\gamma)^2}{\gamma} \cdot \boldsymbol{\pi}_k$ // Vector scaling
    $\mathbf{y} \leftarrow \frac{1}{\gamma} \cdot \boldsymbol{\pi}_{k+\gamma}$ // In-place scaled vector addition
    Solve $\boldsymbol{\pi}_{k+1}((2-\gamma)I + (\gamma-1)hQ) = \mathbf{y}$ for $\boldsymbol{\pi}_{k+1}$ with initial guess $\boldsymbol{\pi}_{k+\gamma}$
    /* Error control and step size estimation */
    $\mathbf{y} \leftarrow -\frac{1}{\gamma}\mathbf{d}_k$ // Vector scaling
    $\mathbf{y} \leftarrow \mathbf{y} + \frac{1}{\gamma(1-\gamma)}\boldsymbol{\pi}_{k+\gamma} Q$ // In-place scaled addition of vector-matrix product
    $\mathbf{d}_{k+1} \leftarrow \boldsymbol{\pi}_{k+1} Q$ // Vector-matrix product
    $\mathbf{y} \leftarrow \mathbf{y} + \left(-\frac{1}{1-\gamma}\right)\mathbf{d}_{k+1}$ // In-place scaled vector addition
    $\mathit{LTE} \leftarrow 2Ch\|\mathbf{y}\|, \mathit{localTol} \leftarrow (\tau - \mathit{errorSum})/\mathit{timeLeft} \cdot h$
    if $\mathit{LTE} < \mathit{localTol}$ **then** // Successful step
      $\mathit{timeLeft} \leftarrow \mathit{timeLeft} - h, \mathit{errorSum} \leftarrow \mathit{errorSum} + \mathit{LTE}$
      // Do not try to increase h after a failed step
      if $\neg\mathit{stepFailed}$ **then** $h \leftarrow h \cdot \min\{\mathit{maxIncrease}, \sqrt[3]{\mathit{localTol}/\mathit{LTE}}\}$
      break
    $\mathit{stepFailed} \leftarrow$ true, $h \leftarrow h \cdot \min\{\mathit{leastDecrease}, \sqrt[3]{\mathit{localTol}/\mathit{LTE}}\}$
  Swap the references to $\boldsymbol{\pi}_k, \boldsymbol{\pi}_{k+1}$ and $\mathbf{d}_k, \mathbf{d}_{\mathbf{k+1}}$
return $\boldsymbol{\pi}_k$

Algorithm 5.13 Parallel block vector-matrix product.

Input: block vector $\mathbf{b} \in \mathbb{R}^{n_0+n_1+\cdots+n_{k-1}}$,
block matrix $A \in \mathbb{R}^{(n_0+n_1+\cdots+n_{k-1})\times(m_0+m_1+\cdots+m_{l-1})}$
Output: $\mathbf{c} = \mathbf{b}A \in \mathbb{R}^{m_0+m_1+\cdots+m_{l-1}}$
allocate $\mathbf{c} \in \mathbb{R}^{m_0+m_1+\cdots+m_{l-1}}$
parallel for $j \leftarrow 0$ **to** $l-1$ **do**
  $\mathbf{c}[j] \leftarrow \mathbf{0}$
  for $i \leftarrow 0$ **to** $k-1$ **do**
    $\mathbf{c}[j] \leftarrow \mathbf{c}[j] + \mathbf{b}[i]A[i,j]$ // Scaled addition of vector-matrix product

Algorithm 5.14 Product of a vector with a linear combination matrix.

Input: $\mathbf{b} \in \mathbb{R}^n$, $A = \nu_0 A_0 + \nu_1 A_1 + \cdots + \nu_{k-1}A_{k-1}$, where $A_h \in \mathbb{R}^{n\times m}$
Output: $\mathbf{c} = \mathbf{b}A \in \mathbb{R}^m$
allocate $\mathbf{c} \in \mathbb{R}^m$ if no target buffer is provided
$\mathbf{c} \leftarrow \mathbf{0}$
for $h \leftarrow 0$ **to** $k-1$ **do**
  $\mathbf{c} \leftarrow \nu_h \cdot \mathbf{b}A_h$ // In-place scaled addition of vector-matrix product
return c

Algorithm 5.15 The SHUFFLE algorithm for vector-matrix multiplication.

Input: $\mathbf{b} \in \mathbb{R}^{n_0 n_1 \cdots n_{k-1}}$, $A = A^{(0)} \otimes A^{(1)} \otimes \cdots \otimes A^{(k-1)}$, where $A^{(h)} \in \mathbb{R}^{n_h \times m_h}$
Output: $\mathbf{c} = \mathbf{b}A \in \mathbb{R}^{m_0 m_1 \cdots m_{k-1}}$
$n \leftarrow n_0 n_1 \cdots n_{k-1}$, $m \leftarrow m_0 m_1 \cdots m_{k-1}$
$\mathit{tempLength} \leftarrow \max_{h=-1,0,1,\ldots,k-1} \prod_{f=0}^{h} m_f \prod_{f=h+1}^{k-1} n_f$
allocate $\mathbf{x}, \mathbf{x}'$ with at least *tempLength* elements
$\mathbf{x}[0{:}1{:}n] \leftarrow \mathbf{b}$, $i_{\text{left}} \leftarrow 1$, $i_{\text{right}} \leftarrow \prod_{h=1}^{k-1} n_h$
for $h \leftarrow 0$ **to** $k-1$ **do**
  if $A^{(h)}$ is not an identity matrix **then**
    $i_{\text{base}} \leftarrow 0$, $j_{\text{base}} \leftarrow 0$
    for $il \leftarrow 0$ **to** $i_{\text{left}} - 1$ **do**
      for $ir \leftarrow 0$ **to** $i_{\text{right}} - 1$ **do**
        $\mathbf{x}'[j_{\text{base}}{:}m_h{:}i_{\text{right}}] \leftarrow \mathbf{x}[i_{\text{base}}{:}n_h{:}i_{\text{right}}]A^{(h)}$
        $i_{\text{base}} \leftarrow i_{\text{base}} + n_h i_{\text{right}}$, $j_{\text{base}} \leftarrow j_{\text{base}} + m_h i_{\text{right}}$
    Swap the references to $\mathbf{x}$ and $\mathbf{x}'$
  $i_{\text{left}} \leftarrow i_{\text{left}} \cdot m_h$
  if $h \neq k-1$ **then** $i_{\text{right}} \leftarrow i_{\text{right}} / n_{h+1}$
return $\mathbf{c} = \mathbf{x}[0{:}1{:}m]$

Chapter 6

Evaluation

6.1 Combinatorial testing

6.2 Software redundancy

6.3 Benchmark models

6.3.1 Synthetic models

Resource sharing

Kanban

Dining philosophers

6.3.2 Case studies

Performability of clouds

6.4 Baselines

6.4.1 PRISM

6.4.2 SMART

6.5 Results

Chapter 7

Conclusion

7.1 Future work

References

[1] Randolph E. Bank, William M. Coughran Jr., Wolfgang Fichtner, Eric Grosse, Donald J. Rose, and R. Kent Smith. “Transient Simulation of Silicon Devices and Circuits”. In: *IEEE Trans. on CAD of Integrated Circuits and Systems* 4.4 (1985), pp. 436–451. DOI: 10.1109/TCAD.1985.1270142.

[2] Richard Barrett, Michael W Berry, Tony F Chan, James Demmel, June Donato, Jack Dongarra, Victor Eijkhout, Roldan Pozo, Charles Romine, and Henk Van der Vorst. *Templates for the solution of linear systems: building blocks for iterative methods*. Vol. 43. Siam, 1994.

[3] Anne Benoit, Brigitte Plateau, and William J Stewart. “Memory efficient iterative methods for stochastic automata networks”. In: (2001).

[4] James T. Blake, Andrew L. Reibman, and Kishor S. Trivedi. “Sensitivity Analysis of Reliability and Performability Measures for Multiprocessor Systems”. In: *SIGMETRICS*. 1988, pp. 177–186. DOI: 10.1145/55595.55616.

[5] Peter Buchholz. “Hierarchical Structuring of Superposed GSPNs”. In: *IEEE Trans. Software Eng.* 25.2 (1999), pp. 166–181. DOI: 10.1109/32.761443.

[6] Peter Buchholz, Gianfranco Ciardo, Susanna Donatelli, and Peter Kemper. “Complexity of Memory-Efficient Kronecker Operations with Applications to the Solution of Markov Models”. In: *INFORMS Journal on Computing* 12.3 (2000), pp. 203–222. DOI: 10.1287/ijoc.12.3.203.12634.

[7] Peter Buchholz and Peter Kemper. “On generating a hierarchy for GSPN analysis”. In: *SIGMETRICS Performance Evaluation Review* 26.2 (1998), pp. 5–14. DOI: 10.1145/288197.288202.

[8] Maciej Burak. “Multi-step Uniformization with Steady-State Detection in Nonstationary M/M/s Queuing Systems”. In: *CoRR* abs/1410.0804 (2014). URL: http://arxiv.org/abs/1410.0804.

[9] Giovanni Chiola, Claude Dutheillet, Giuliana Franceschinis, and Serge Haddad. “Stochastic Well-Formed Colored Nets and Symmetric Modeling Applications”. In: *IEEE Trans. Computers* 42.11 (1993), pp. 1343–1360. DOI: 10.1109/12.247838.

[10] Piotr Chrzastowski-Wachtel. "Testing Undecidability of the Reachability in Petri Nets with the Help of 10th Hilbert Problem". In: *Application and Theory of Petri Nets 1999, 20th International Conference, ICATPN '99, Williamsburg, Virginia, USA, June 21-25, 1999, Proceedings*. Vol. 1639. Lecture Notes in Computer Science. Springer, 1999, pp. 268–281. DOI: 10.1007/3-540-48745-X_16.

[11] Gianfranco Ciardo, Jogesh K. Muppala, and Kishor S. Trivedi. "On the Solution of GSPN Reward Models". In: *Perform. Eval.* 12.4 (1991), pp. 237–253. DOI: 10.1016/0166-5316(91)90003-L.

[12] Ricardo M. Czekster, César A. F. De Rose, Paulo Henrique Lemelle Fernandes, Antonio M. de Lima, and Thais Webber. "Kronecker descriptor partitioning for parallel algorithms". In: *Proceedings of the 2010 Spring Simulation Multiconference, SpringSim 2010, Orlando, Florida, USA, April 11-15, 2010*. SCS/ACM, 2010, p. 242. ISBN: 978-1-4503-0069-8. URL: http://dl.acm.org/citation.cfm?id=1878537.1878789.

[13] Susanna Donatelli. "Superposed Generalized Stochastic Petri Nets: Definition and Efficient Solution". In: *Application and Theory of Petri Nets 1994, 15th International Conference, Zaragoza, Spain, June 20-24, 1994, Proceedings*. Vol. 815. Lecture Notes in Computer Science. Springer, 1994, pp. 258–277. DOI: 10.1007/3-540-58152-9_15.

[14] Susanna Donatelli. "Superposed stochastic automata: a class of stochastic Petri nets with parallel solution and distributed state space". In: *Performance Evaluation* 18.1 (1993), pp. 21–36.

[15] Paulo Fernandes, Brigitte Plateau, and William J. Stewart. "Numerical Evaluation of Stochastic Automata Networks". In: *MASCOTS '95, Proceedings of the Third International Workshop on Modeling, Analysis, and Simulation On Computer and Telecommunication Systems, January 10-18, 1995, Durham, North Carolina, USA*. IEEE Computer Society, 1995, pp. 179–183. DOI: 10.1109/MASCOT.1995.378690.

[16] Paulo Fernandes, Ricardo Presotto, Afonso Sales, and Thais Webber. "An Alternative Algorithm to Multiply a Vector by a Kronecker Represented Descriptor". In: *21st UK Performance Engineering Workshop*. 2005, pp. 57–67.

[17] Bennett L. Fox and Peter W. Glynn. "Computing Poisson Probabilities". In: *Commun. ACM* 31.4 (1988), pp. 440–445. DOI: 10.1145/42404.42409.

[18] Robert E Funderlic and Carl Dean Meyer. "Sensitivity of the stationary distribution vector for an ergodic Markov chain". In: *Linear Algebra and its Applications* 76 (1986), pp. 1–17.

[19] Stephen Gilmore and Jane Hillston. “The PEPA Workbench: A Tool to Support a Process Algebra-based Approach to Performance Modelling”. In: *Computer Performance Evaluation, Modeling Techniques and Tools, 7th International Conference, Vienna, Austria, May 3-6, 1994, Proceedings*. Vol. 794. Lecture Notes in Computer Science. Springer, 1994, pp. 353–368. DOI: 10.1007/3-540-58021-2_20.

[20] Winfried K. Grassmann. “Transient solutions in markovian queueing systems”. In: *Computers & OR* 4.1 (1977), pp. 47–53. DOI: 10.1016/0305-0548(77)90007-7.

[21] A. Greenbaum. *Iterative Methods for Solving Linear Systems*. Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 1997. ISBN: 9781611970937. URL: https://books.google.hu/books?id=IX9rrFe1YLQC.

[22] *International Workshop on Timed Petri Nets, Torino, Italy, July 1-3, 1985*. IEEE Computer Society, 1985. ISBN: 0-8186-0674-6.

[23] Ilse CF Ipsen and Carl D Meyer. “Uniform stability of Markov chains”. In: *SIAM Journal on Matrix Analysis and Applications* 15.4 (1994), pp. 1061–1074.

[24] David N Jansen. “Understanding Fox and Glynn’s “Computing Poisson probabilities”. In: (2011).

[25] Amy Nicole Langville and William J. Stewart. “Testing the Nearest Kronecker Product Preconditioner on Markov Chains and Stochastic Automata Networks”. In: *INFORMS Journal on Computing* 16.3 (2004), pp. 300–315. DOI: 10.1287/ijoc.1030.0041.

[26] Francesco Longo and Marco Scarpa. “Two-layer Symbolic Representation for Stochastic Models with Phase-type Distributed Events”. In: *Intern. J. Syst. Sci.* 46.9 (2015), pp. 1540–1571. DOI: 10.1080/00207721.2013.822940.

[27] Marco Ajmone Marsan. “Stochastic Petri nets: an elementary introduction”. In: *Advances in Petri Nets 1989, covers the 9th European Workshop on Applications and Theory in Petri Nets, held in Venice, Italy in June 1988, selected papers*. Vol. 424. Lecture Notes in Computer Science. Springer, 1988, pp. 1–29. DOI: 10.1007/3-540-52494-0_23.

[28] Marco Ajmone Marsan, Gianfranco Balbo, Andrea Bobbio, Giovanni Chiola, Gianni Conte, and Aldo Cumani. “The Effect of Execution Policies on the Semantics and Analysis of Stochastic Petri Nets”. In: *IEEE Trans. Software Eng.* 15.7 (1989), pp. 832–846. DOI: 10.1109/32.29483.

[29] Marco Ajmone Marsan, Gianni Conte, and Gianfranco Balbo. “A Class of Generalized Stochastic Petri Nets for the Performance Evaluation of Multiprocessor Systems”. In: *ACM Trans. Comput. Syst.* 2.2 (1984), pp. 93–122. DOI: 10.1145/190.191.

[30] Carl D Meyer. "Stochastic complementation, uncoupling Markov chains, and the theory of nearly reducible systems". In: *SIAM review* 31.2 (1989), pp. 240–272.

[31] Aad PA van Moorsel and William H Sanders. "Adaptive uniformization". In: *Stochastic Models* 10.3 (1994), pp. 619–647.

[32] Tadao Murata. "Petri nets: Properties, analysis and applications". In: *Proceedings of the IEEE* 77.4 (1989), pp. 541–580.

[33] M. Neuts. "Probability distributions of phase type". In: *Liber Amicorum Prof. Emeritus H. Florin*. University of Louvain, 1975, pp. 173–206.

[34] RJ Plemmons and A Berman. *Nonnegative matrices in the mathematical sciences*. Academic Press, New York, 1979.

[35] William H Press. *Numerical recipes 3rd edition: The art of scientific computing*. Cambridge university press, 2007.

[36] S. Rácz, Á. Tari, and M. Telek. "MRMSolve: Distribution estimation of Large Markov reward models". In: *Tools 2002*. Springer, LNCS 2324, 2002, pp. 72–81.

[37] S. Rácz and M. Telek. "Performability Analysis of Markov Reward Models with Rate and Impulse Reward". In: *Int. Conf. on Numerical solution of Markov chains*. 1999, pp. 169–187.

[38] A. V. Ramesh and Kishor S. Trivedi. "On the Sensitivity of Transient Solutions of Markov Models". In: *SIGMETRICS*. 1993, pp. 122–134. DOI: 10.1145/166955.166998.

[39] Andrew L. Reibman and Kishor S. Trivedi. "Numerical transient analysis of markov models". In: *Computers & OR* 15.1 (1988), pp. 19–36. DOI: 10.1016/0305-0548(88)90026-3.

[40] Andrew Reibman, Roger Smith, and Kishor Trivedi. "Markov and Markov reward model transient analysis: An overview of numerical approaches". In: *European Journal of Operational Research* 40.2 (1989), pp. 257–267.

[41] *Advances in Petri Nets 1987, covers the 7th European Workshop on Applications and Theory of Petri Nets, Oxford, UK, June 1986*. Vol. 266. Lecture Notes in Computer Science. Springer, 1987. ISBN: 3-540-18086-9.

[42] Youcef Saad and Martin H Schultz. "GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems". In: *SIAM Journal on scientific and statistical computing* 7.3 (1986), pp. 856–869.

[43] Yousef Saad. *Iterative methods for sparse linear systems*. Siam, 2003.

[44] Peter Sonneveld. "CGS, a fast Lanczos-type solver for nonsymmetric linear systems". In: *SIAM journal on scientific and statistical computing* 10.1 (1989), pp. 36–52.

[45] Peter Sonneveld and Martin B van Gijzen. “IDR (s): A family of simple and fast algorithms for solving large nonsymmetric systems of linear equations”. In: *SIAM Journal on Scientific Computing* 31.2 (2008), pp. 1035–1062.

[46] William J Stewart. *Probability, Markov chains, queues, and simulation: the mathematical basis of performance modeling*. Princeton University Press, 2009.

[47] Williams J Stewart. *Introduction to the numerical solutions of Markov chains*. Princeton Univ. Press, 1994.

[48] Enrique Teruel, Giuliana Franceschinis, and Massimiliano De Pierro. “Well-Defined Generalized Stochastic Petri Nets: A Net-Level Method to Specify Priorities”. In: *IEEE Trans. Software Eng.* 29.11 (2003), pp. 962–973. DOI: 10.1109/TSE.2003.1245298.

[49] Henk A Van der Vorst. “Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems”. In: *SIAM Journal on scientific and Statistical Computing* 13.2 (1992), pp. 631–644.