# **MATH 7700** Real Analysis 1
# Notes and Selected Exercises

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**Introduction to Real Analysis**, *4 ed.*
By Robert G. Bartle and Donald R. Sherbert

## Contents

# 1 THE REAL NUMBERS

## 1.1 The Absolute Value

[1]

A function that is always happy. It does not allow negative numbers to persist.

**Definition 1.1.** ***Absolute Value*** *of* $a \in \mathbb{R}$*, denoted* $|a|$*, is defined by*

$$|a| := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0 \end{cases}$$

This definition comes with some properties.

**Theorem 1.2.** *1.* $|ab| = |a||b|, \forall a, n \in \mathbb{R}$

*2.* $|a|^2 = a^2, \forall a \in \mathbb{R}$*; it is a special case of 1.*

*3. If* $c \geq 0$*, then* $|a| \leq c$ *iff* $-c \leq a \leq c$

*4.* $-|a| \leq a \leq |a|, \forall a \in \mathbb{R}$*; special case of 3.*

And then the ultimate inequality

[1] This subsection corresponds to section 2.2 in Bartle and Sherbert's Introduction to Real Analysis.

**Theorem 1.3.** ***Triangle Inequality*** *If $a, a \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.*

**NOTE 1.4.** *The triangle inequality can be generalized to $|a_1 + ... + a_n| \leq |a_1| + ...|a_n|$*

From this, we get the following corollaries:

**Corollary 1.5.** *If $a, b \in \mathbb{R}$, then*

1. $||a| - |b|| \leq |a - b|$
2. $|a - b| \leq |a| + |b|$

The absolute value is really a **distance** measure between two points when you consider $|a - b|$. The absolute value function ensures the result is positive. Considering just $|a|$, it is the distance from a to the origin, 0.

While the absolute value function offers a notion of closeness, we need something more precise for how *close* one number is to another. If a number is close to another, the distance between the two points must have some maximum threshold beyond which they are no longer close– a fence between houses, so to speak.

**Definition 1.6.** *Let $a \in \mathbb{R}, \epsilon > 0$, then the $\epsilon$-**neighbourhood** of $a$ is the set $V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}$.*

The interesting part here is that $\epsilon$ can become arbitrarily small, it can even approach 0, i.e., $\epsilon \to 0$. In this case, $|x - a| = 0 \implies x = a$. So, we can write this more formally as:

**Theorem 1.7.** *Let $a \in \mathbb{R}$. If $x \in V_\epsilon(a), \forall \epsilon > 0$, then $|x - a| = 0$, and hence $x = a$.*

## 1.2 Completeness Property of $\mathbb{R}$

[2]

Completeness is a property of the real numbers that, intuitively, implies that there are no *gaps* (in Dedekind's terminology) or *missing points* in the real number line. This contrasts with the rational numbers, whose corresponding number line has a *gap* at each irrational value. This textbook describes completeness by assuming that each nonempty bounded subset of $\mathbb{R}$ has a *supremum* (least upper bound).

**Definition 1.8.** *Let $S \subset \mathbb{R}$ and $S \neq \emptyset$. Then*

1. *$S$ is **bounded above** if $\exists u \in \mathbb{R}$ s.t. $s \leq u$, $\forall s \in S$. Each such $u$ is an **upper bound**.*
2. *$S$ is **bounded below** if $\exists w \in \mathbb{R}$ s.t. $w \leq s$, $\forall s \in S$. Each such $w$ is an **lower bound**.*
3. *If bounded above and below, then **bounded**. If not, **unbounded**.*

**Definition 1.9.** *Let $S \subset \mathbb{R}$ and $S \neq \emptyset$. Then*

1. *$u$ is a **supremum** (denoted sup $S$, for set $S$) if*

[2]This subsection corresponds to section 2.3 in Bartle and Sherbert's Introduction to Real Analysis.

(a) *u is an upper bound of S, and*

(b) *if v is an upper bound of S AND* $u \leq v$*.*

2. *Similar definition for* ***infimum*** *(greatest lower bound), except* $t \leq w$ *where w is the infimum and t is any lower bound. (denoted inf S, for set S)*

We can see that there can be *only* one sup or inf for a subset of $\mathbb{R}$.

*Proof.* Suppose, for contradiction, $u_1$ and $u_2$ are both suprema of S. If $u_1 < u_2$, then $u_2$ cannot be a sup. Similarly, if $u_1 > u_2$, then $u_1$ cannot be a sup. Thus, $u_1 = u_2$ □

The same argument can be made for the uniqueness of the infimum.

Though we have a definition for the supremum, it is beneficial to represent this idea in other ways. One way is to note that a number smaller than sup S $= u$ is not an upper bound, i.e., if $z < u$, z is no upper bound. How can we show this? We must show that an element in S is greater than z but less than u; So $z < s_z < u$, for an element $s_z \in S$. We can write the following lemma:

**Lemma 1.10.** *Upper bound u of* $S \in \mathbb{R}$ *is sup S* $\iff \forall \epsilon > 0, \exists s_\epsilon \in S$ *s.t.* $u - \epsilon < s_\epsilon$

*Proof.* (=>) Suppose u = sup S and $\epsilon > 0$, then $u - \epsilon < u$. So, $u - \epsilon$ is not an upper bound. Thus, $\exists s_\epsilon > (u - \epsilon)$. Thus, $u - \epsilon < s_\epsilon$. (<=) Let u be an upper bound. Consider another upper bound $v < u$. Set $\epsilon = u - v > 0$. By our assumption, $\exists s_\epsilon \in S$ s.t. $v = u - \epsilon < s_\epsilon$. Thus, v is not an upper bound. So, u = sup S. □

**Theorem 1.11 (Completeness Property of $\mathbb{R}$).** *Every nonempty set of* $\mathbb{R}$ *with an upper bound also has a sup in* $\mathbb{R}$*. This property is also called the* ***supremum property*** *of* $\mathbb{R}$*.*

From the property above, we obtain the following theorem:

**Theorem 1.12.** *Every nonempty set of* $\mathbb{R}$ *bounded from below has an infimum.*

*Proof.* Let $S \subseteq \mathbb{R}$ be a nonempty set that is bounded from below. We need to show that $S$ has an infimum, which is the greatest lower bound.

Since $S$ is bounded from below, there exists a lower bound $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$.

Define the set $L$ of all lower bounds of $S$:

$$L = \{l \in \mathbb{R} \mid l \leq s \text{ for all } s \in S\}.$$

Note that $L$ is nonempty because $m \in L$. Also, $L$ is bounded above by any element of $S$. By the completeness property of $\mathbb{R}$, the set $L$ has a supremum, say $\inf S = \sup L$.

We now show that $\inf S = \sup L$ is indeed the infimum of $S$.

1. $\inf S$ is a lower bound: By definition, $\inf S$ is the supremum of the set of lower bounds, so it must be a lower bound of $S$. That is, $\inf S \leq s$ for all $s \in S$.

2. $\inf S$ is the greatest lower bound: Suppose there is a lower bound $l$ of $S$ s.t. $l > \inf S$. This would contradict the fact that $\inf S$ is the supremum of $L$, as $\inf S$ should be the largest element in $L$. Therefore, no such $l$ exists, and $\inf S$ is indeed the greatest lower bound.

Hence, every nonempty set of real numbers that is bounded from below has an infimum. □

## SELECT EXERCISES [3]

**2.3/7 If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S**

*Proof.* Let S contain an upper bound u. Then $\forall s \in S, s \leq u$. Suppose another upper bound v of S, then $v \geq u$. Thus, u = sup S. □

**2.3/8 Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S iff the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$**

*Proof.* Let $S \subseteq \mathbb{R}$ be nonempty. (=>) Let $u \in R$ be an upper bound of S. Let $t \in R and t > u$. Suppose, for contradiction, $t \in S$. Then $t < u$ since u is an upper bound– a contradiction. Thus, $t \notin S$. (<=) Suppose $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$. Suppose for contradiction that $u$ is not an upper bound of $S$. Then, $\exists t \in S$ s.t. $t > u$, a contradiction. Thus, $u$ is an upper bound of S. □

**2.3/9 Let $S \subseteq \mathbb{R}$ be nonempty. Show that if u = sup S, then for every number $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, but $u + 1/n$ is an upper bound**

*Proof.* Let $S \subseteq \mathbb{R}$ be nonempty and u = sup S.

1. Consider $v = u - 1/n$. Set $\epsilon = 1/n > 0$, since $n \in \mathbb{N}$. Since u = sup S, $\exists s_\epsilon$ s.t. $u - \epsilon = u - 1/n < s_\epsilon$. Thus, $u - 1/n$ is not an upper bound.

2. Consider $w = u + 1/n$, then $w > u$ since $1/n > 0$. Thus, $\forall s \in S < u < w$, showing that w is an upper bound.

□

**2.3/10 Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $sup(A \cup B) = sup\{supA, supB\}$**

[3] Page 39 of the textbook, section 2.3

*Proof.*

1. Let A and B be bounded sets. Then $\exists N \in \mathbb{Z}+$ s.t. $-N < a < N$, $\forall a \in A$. Similarly, $\exists M \in \mathbb{Z}+$ s.t. $-M < b < M$, $\forall b \in B$. Define $P = max(M, N)$. It is sufficient to show that $-P < c < P$, $\forall c \in A \cup B$ to prove $A \cup B$ is bounded.

   Suppose, for contradiction, there exists $c \in A \cup B$, s.t. $|c| \geq P$. Then $c \geq max(N, M)$. If $c \in A$, then $N > c \geq P = max(N, M)$– a contradiction. A similar argument can be applied to B. Thus, there does not exist $c \in A \cup B$, s.t. $|c| \geq P$ implying $-P < c < P$, $\forall c \in A \cup B$ showing bounded-ness.

2. (a) Suppose $x \in A$, then $x \leq supA$. Similarly, for $x \in B$, then $x \leq supB$. So, for every $x \in A \cup B$, we have $x \leq max(supA, supB) = sup\{supA, supB\}$. Thus, $sup\{supA, supB\}$ is an upper bound on $A \cup B$; it must also be greater than or equal to the least upper bound, so $sup(A \cup B) \leq sup\{supA, supB\}$

   (b) We know that $sup(A \cup B) \geq x$, $\forall x \in (A \cup B)$. So, $sup(A \cup B) \geq a$, $\forall a \in A$ and $sup(A \cup B) \geq b$, $\forall b \in B$. So, $sup(A \cup B) \geq supA$ and $sup(A \cup B) \geq supB$ implying $sup(A \cup B) \geq sup\{supA, supB\}$.

   From (a) and (b), we get the desired equality.

□

**2.3/11 Let S be a bounded set in $\mathbb{R}$ and let $S_0$ be a nonempty subset of S. Show that $infS \leq infS_0 \leq supS_0 \leq supS$**

*Proof.* Let $S_0 \subset S$ and $S_0 \neq \emptyset$. Since $S_0$ is a subset of $S$, every element of $S_0$ is also an element of $S$. Therefore, any lower bound for $S$ is also a lower bound for $S_0$.

Let $m = \inf S$. By the definition of the infimum, $m$ is a lower bound for $S$, which implies $m \leq x$ for all $x \in S$. Since $S_0$ is a subset of $S$, we have $m \leq x$ for all $x \in S_0$. This means that $m$ is also a lower bound for $S_0$, leading to the conclusion $\inf S \leq \inf S_0$. Furthermore, since $S_0$ is nonempty, there exists at least one element $x_0 \in S_0$.

Thus, $\inf S_0$ is less than or equal to any element in $S_0$, including $x_0$, and since $\sup S_0$ is an upper bound for all elements of $S_0$, we have $\inf S_0 \leq x_0 \leq \sup S_0$. Hence, it follows that $\inf S_0 \leq \sup S_0$.

Finally, since $S_0 \subset S$, every element of $S_0$ is also an element of $S$. Let $M = \sup S$. By the definition of the supremum, $M$ is an upper bound for $S$, so $x \leq M$ for all $x \in S$. Consequently, since $S_0$ is a subset of $S$, we have $\sup S_0 \leq M$, or equivalently, $\sup S_0 \leq \sup S$. Thus, we conclude that $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$.

□

## 1.3 Applications of Supremum Property

[4]

In the previous section, we saw the existence of sup and inf on sets. However, this concept can extend to **functions**. So, for instance, a function is **bounded above** if $f(D) = \{f(x) : x \in D\}$ is bounded above in $\mathbb{R}$. The concept is similar for when **bounded below**. If bounded above and below, we say f is **bounded**, i.e., $\exists B \in \mathbb{R}$ s.t. $|f(x)| \leq B \ \forall x \in D$.

Here, we can say a few things about the relationships between any two functions, $f$ and $g$.

1. If $f(x) \leq g(x) \ \forall x \in D$, then $sup\ f(D) \leq sup\ g(D)$. This can be shown by $f(x) \leq g(x) \leq sup\ g(D) \implies f(x) \leq sup\ g(D)$, showing that $sup\ g(D)$ is an upper bound on $f(x) \implies sup\ f(D) \leq sup\ g(D)$.

2. Note that the previous relationship does NOT comment on the infima of both functions. That is not known.

3. However, if $f(x) \leq g(y) \ \forall x, y \in D$ (note that it is not g(x), but g(y)), then we can do one better than just comparing infima of both: we can say $sup\ f(D) \leq inf\ g(D)$.

**ARCHIMEDEAN PROPERTY**

Though, we intuitively know that $\mathbb{N}$ is unbounded, how can we prove this? We need to use the completeness property of $\mathbb{R}$ and the inductive property of $\mathbb{N}$ (i.e., if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$). Since $\mathbb{N}$ knows no bounds, if you have any $x \in \mathbb{R}$, there has to be $n \in \mathbb{N}$ s.t. $x < n$.

**Theorem 1.13.** ***Archimedean Property*** *If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ s.t. $x \leq n_x$.*

*Proof.* Suppose not. Then $n \leq x \ \forall n \in \mathbb{N}$; thus, x is an upper bound of $\mathbb{N}$. So, by completeness property, the nonempty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Since $sup\mathbb{N} = u \implies u - 1$ is not an upper bound. So, $m \in \mathbb{N}$ s.t. $u - 1 < m$. Then, adding 1 gives $u < m + 1 \in \mathbb{N}$, contradicting u is an upper bound of $\mathbb{N}$. □

Another important aspect of the **supremum property** is that it can guarantee the existence of numbers $\in \mathbb{R}$ under certain hypotheses. For instance, it can give us the existence of $x \in \mathbb{R}$ s.t. $x^2 = 2$. A version of the proof appears in one of the exercises below. Assume we can guarantee the existence of $\sqrt{2}$.

**DENSITY of RATIONALS in $\mathbb{R}$**

We know that rational numbers are **countable** (there exists a bijection of $\mathbb{N}$ onto the set of rationals). Irrational numbers, however, are not, i.e., **uncountable** (no existence of a bijection of $\mathbb{N}$ onto the set). Despite this countable-ness, we can show that there exists a rational number between any two irrational numbers (in fact, there are infinite such rationals), and this property is called the **density of rationals in** $\mathbb{R}$.

**Theorem 1.14.** ***Density Theorem*** *If $x, y \in \mathbb{R}$ with $x < y, \exists r \in \mathbb{Q}$ s.t. $x < r < y$.*

[4] This subsection corresponds to section 2.4 in Bartle and Sherbert's Introduction to Real Analysis.

*Proof.* We can assume without loss of generality that $x > 0$. The reasoning behind this assumption is that if $x < 0$, we can consider the shifted interval $[x + y, y + y]$ where both endpoints are positive, and any rational number $r$ found in this new interval will correspond to a rational number in the original interval. The theorem holds for the positive case, which implies it holds for all $x$.

Since $y - x > 0$, we apply a result of the Archimedean property (If $t > 0$, there exists $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$) to find a natural number $n \in \mathbb{N}$ such that:

$$0 < \frac{1}{n} < y - x$$

Multiplying through by $n$, we have:

$$nx + 1 < ny$$

Next, since $nx > 0$, there exists $m \in \mathbb{N}$ (Archimedean property to the rescue again) such that:

$$m - 1 \leq nx < m$$

Thus, combining these inequalities, we have:

$$nx < m < ny$$

This shows that the rational number $r = \frac{m}{n}$ satisfies $x < r < y$, which completes the proof. □

We can corollarize this for the density of irrational numbers as well.

**Corollary 1.15.** *If* $x, y \in \mathbb{R}$ *with* $x < y, \exists z \in \mathbb{R} - \mathbb{Q}$ *s.t.* $x < z < y$.

**SELECT EXERCISES** [5]

**2.4/1 Show that** $sup\{1 - 1/n : n \in \mathbb{N}\} = 1$

*Proof.* We need to show the following two properties:

1. u = 1 is an upper bound: Since $n > 0 \implies 1/n > 0 \implies -1/n < 0 \implies 1 - 1/n < 1$, showing that 1 is an upper bound on the set.

2. For any other upper bound v, $v \geq u$: For contradiction, suppose an upper bound $v < u$. Then $v = u - 1/\mu$, for some $\mu > 0, \mu \in \mathbb{R}$. By the Archimedean property, we know that for $\mu \in \mathbb{R}, \exists n_\epsilon \in \mathbb{N}$ s.t. $\mu \leq n_\epsilon \implies 1/n_\epsilon \leq 1/\mu \implies 1 - 1/n_\epsilon \geq 1 - 1/\mu = u - 1/\mu = v$, showing that there is an element greater than v but less than u. Thus, for any upper bound $v$, $u \leq v$.

□

[5] Page 44 of the textbook, section 2.4

**2.4/2 If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find inf S and sup S.**

*Proof.* We see that as $m \to \infty, 1/n - 1/m = 1/n$.Thus, the largest value S can have is the largest value of 1/n = 1. Similarly, the smallest value of S is the smallest value of -1/m = -1. So sup S and inf S are 1 and -1 respectively. □

**2.4/3 Let $S \subseteq R$ be nonempty. Prove that if a number $u \in \mathbb{R}$ has the properties: (i) for every $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, and (ii) for every number $n \in \mathbb{N}$ the number $u + 1/n$ is an upper bound of S, then $u = supS$. (This is the converse of Exercise 2.3.9.)**

*Proof.* Let properties (i) and (ii) be true. Then to show u = supS, we need to show the following two properties:

1. u is an upper bound on S: From property (i), we see that there is no such $n \in \mathbb{N}$ s.t. $u - 1/n$ is an upper bound, i.e., $\exists s_\epsilon \in S$, where $u - 1/n < s_\epsilon \leq u$. Thus, $\forall s \in S, s \leq u$.

2. Show that for any other upper bound $v$, $u \leq v$: Suppose, for contradiction, an upper bound v s.t. $v < u$. Then $v = u - \epsilon$, for some $\epsilon > 0 \in \mathbb{R}$. Choose $\epsilon = 1/n \implies v = u - 1/n$, which from property (i) is not an upper bound. Thus, for any upper bound $v$, $u \leq v$.

□

**2.4/7 Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that sup(A+B) = supA + supB and inf(A+B) = infA + infB.**

*Proof.*

1. sup(A+B) = supA + supB:
   (a) $sup(A + B) \leq supA + supB$: By definition of $A + B$, we have $a + b \leq sup(A+B), a \in A, b \in B$. Then we can get $a = a+b-b \leq sup(A+B)-b$. If we fix $b \in B$, then $sup(A+B)-b$ is an upper bound for $A+B-B = A$. By definition of $supA, \forall b \in B, supA \leq suo(A + B) - b \implies b \leq sup(A+B)-supA, \forall b \in B$. Thus, $sup(A+B)-supA$ in an upper bound for B. So, $supB \leq sup(A+B) - supA \implies supB+supA \leq sup(A+B)$.
   (b) $sup(A+B) \geq supA+supB$: Since supA is an upper bound for A, $a \in A$, and same for b. So $a + b \leq supA + supB, \forall x \in A$ and $y \in B$. So, supA + supB is an upper bound for A + B. Thus, $sup(A+B) \geq supA+supB$.
2. inf(A+B) = infA + infB: SAME AS ABOVE?

□

**2.4/16 Modify the argument in Theorem 2.4.7 to show that if $a > 0$, then there exists a positive real number $z$ such that $z^2 = a$.**

*Proof.* Define a set $S := \{s \in \mathbb{R} : 0 \leq s, s^2 < a\}$. Since $0 \in S$, the set is nonempty. The set is bounded by a, because if $t > a$, then $a^2 > a$, so $t \notin S$. Since the set is bounded, it has a supremum, call it x.

FINISH

□

## 1.4 Intervals

[6]

The start of this section is all terminology:

- If $a, b \in \mathbb{R}$ satisfy $a < b$, then **open interval** is the set $(a, b) := x \in \mathbb{R} : a < x < b$.
- In the above set, a and b are **endpoints** of the interval.
- Similar to an open interval, we define a **closed interval** on a and b when the endpoints are in the set: $[a, b] := x \in \mathbb{R} : a \leq x \leq b$.
- If one side (can be left or right) is open while the other is closed, the interval is **half-open** (or **half-closed**).
- Each of the four scenarios ($(a, b), [a, b], [a, b), (a, b]$) are bounded and the **length** is given by $b - a$.
- The opposite of bounded intervals and unbounded intervals (**infinite open/closed intervals**) that involve $\pm\infty$.

### CHARACTERIZATION of INTERVALS & NESTED INTERVALS

If two points x,y with $x < y$ are in an interval $I$, then any point in between is also in the interval, i.e., if $x, y \in I$ and $x < t < y$, then $t \in I$. We can formalize this as the following theorem.

**Theorem 1.16.** ***Characterization Theorem*** *If $S$ is a subset of $\mathbb{R}$ that contains at least two points and has the property: if $x, y \in S$ and $x < y$, then $[x, y] \subseteq S$; then $S$ is an interval.*

If a sequence of intervals $I_n, n \in \mathbb{N}$ follows $... \subseteq I_{n+1} \subseteq I_n ... \subseteq I_2 \subseteq I_1$, then the interval sequence is **nested**. Now, you can have many nested intervals, and, intuitively, they keep getting smaller the more you nest. So, eventually, when you reach the end of your nests, you will have at least one element; if you did not, that particular nest must be an empty set. This one element that is present in the most-nested interval is then a part of all the previous intervals as well. We formalize this as follows.

**Theorem 1.17.** ***Nested Intervals Property*** *If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed, bounded intervals, then $\exists \xi \in I_n \ \forall n \in \mathbb{N}$.*

[6]This subsection corresponds to section 2.5 in Bartle and Sherbert's Introduction to Real Analysis.

*Proof.* **SKETCH ONLY** Since the intervals are nested, the n-th interval $\subseteq$ the first interval, i.e., $I_n \subseteq I_1$. Then the infima $(a_n)$ of $I_n$ is less than or equal to $sup\ I_1 = b_1$. So, it is clear that $I_n$ is bounded above, so it must have a supremum, call is $\xi$. Since $\xi = sup\ I_n \implies a_n \leq \xi\ \forall n \in \mathbb{N}$.

Next, we must show that $\xi \leq b_n$ (then putting together the first half with the second, we get $a_n < \xi < b_n \implies \xi \in [a_n, b_n]$). Do this by showing that for any $n$, $b_n$ is an upper bound for the set $\{a_k : k \in \mathbb{N}\}$. Consider two cases (i) $k \geq n$, and (ii) $k < n$. $\square$

Now that we have the notion of nested intervals and common entries in intervals, let us dive slightly deeper. Construct a set of all the lengths $\{b_n - a_n : n \in \mathbb{N}\}$. It may be obvious that the supremum of the set would be $b_1 - a_1$, seeing that it would be the largest interval. However, the infimum of this set of lengths can give us information about the number of unique points in all the sets. Note that the infimum of the set is the smallest interval $(b_n - a_n)$. If $inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then the last interval contains one element (since $x - x = 0\ \forall x \in \mathbb{R}$). This must mean that there is ONLY one element $\xi$ common to all intervals, and it is unique.

**Theorem 1.18.** *If $I_n := [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed, bounded intervals s.t. the lengths $b_n - a_n$ of $I_n$ satisfy $inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then $\xi \in I_n\ \forall n \in \mathbb{N}$ is unique.*

With deeper notions of intervals and their properties, we can demonstrate the following theorem.

**Theorem 1.19.** *The set $\mathbb{R}$ of real numbers is not countable.*

*Proof.* Suppose, for contradiction, that $\mathbb{R}$ is countable. Then subsets of it would also be countable. Let $I = [0, 1]$ which can be enumerated as $I = \{x_1, ..., x_n\}$ for $n \in \mathbb{N}$. Select a closed subinterval $I_1$ s.t. $x_1 \notin I_1$ but $x_1 \in I$. Keep going, i.e., then create $I_2$ s.t. $x_2 \notin I_2$. In this way, we construct $... \subseteq I_n ... \subseteq I_2 \subseteq I_1$ s.t. $I_n \subseteq I$ and $x_n \notin I_n\ \forall n$. The nested interval property described earlier implies the existence of $\xi \in I$ s.t. $\xi \in I_n\ \forall n$. Thus, $\xi \neq x_n\ \forall n \in \mathbb{N}$. So the earlier enumeration of I is not complete, and thus, is uncountable. $\square$

Since $\mathbb{Q}$ is countable and $\mathbb{R}$ is not, we get $\mathbb{R} \backslash \mathbb{Q}$ to be uncountable; the irrational are uncountable.

## SELECT EXERCISES [7]

**2.5/3 If $S \subseteq \mathbb{R}$ is a nonempty bounded set, and $I_s := [infS, supS]$, show that $S \subseteq I_s$. Moreover, if J is any closed bounded interval containing S, show that $I_s \subseteq J$.**

*Proof.* Let $S \subseteq \mathbb{R}, S \neq \emptyset$, and $I_s := [infS, supS]$. Suppose an element $s \in S$, then $\forall s \geq infS$, by the definition of infimum. Similarly, $\forall s \leq supS$. So, we have $infS \leq s \leq supS,\ \forall s \in S$. Thus, $s \in [infS, supS] \implies s \in I_s$. Since s was arbitrary, $S \subseteq I_s$.

Let $S \subseteq J$, where J is a closed bounded interval [a,b]. Then $a \leq s \leq b,\ \forall s \in S$. So, $a \leq infS$ and $b \geq supS$ since $a \leq s$ and $b \geq s\ \forall s \in S \implies a \leq infS \leq supS \leq b \implies [infS, supS] \subseteq [a, b] \implies I_s \subseteq J$. $\square$

[7] Section 2.5 exercises, page 52

**2.5/6 If ... $\subseteq I_n ... \subseteq I_2 \subseteq I_1$ is a nested sequence of intervals and if $I_n = [a_n, b_n]$, show that $a_1 \leq ... \leq a_n ...$ and $b_1 \geq ... \geq b_n ...$.**

*Proof.* Let $I_n = [a_n, b_n]$ be the intervals in the nested sequence such that:

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \ldots$$

Since $I_{n+1} \subseteq I_n$ for each $n$, every element in $I_{n+1}$ must also be in $I_n$. In particular, the left endpoint $a_{n+1}$ of $I_{n+1}$ must satisfy:

$$a_n \leq a_{n+1},$$

because $a_{n+1}$ lies within $I_n = [a_n, b_n]$. This means the sequence $\{a_n\}$ is non-decreasing, i.e.,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \ldots$$

Similarly, since $I_{n+1} \subseteq I_n$, the right endpoint $b_{n+1}$ of $I_{n+1}$ must satisfy:

$$b_{n+1} \leq b_n,$$

because $b_{n+1}$ also lies within $I_n = [a_n, b_n]$. This implies that the sequence $\{b_n\}$ is non-increasing, i.e.,

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq \ldots$$

Hence, proved. □

**2.5/7 Let $I_n := [0, 1/n], n \in \mathbb{N}$. Prove that $\cap_{\infty}^{n=1} I_n = \{0\}$.**

*Proof.* Let $I_n$ be as stated in the problem. We will prove the claim in two parts:

1. 0 is a value that satisfies $\cap_{\infty}^{n=1} I_n$: We see that as n increases, the length of the interval decreases. Note that the length of the interval is given by 1/n. So, for any $x \in \cap_{\infty}^{n=1} I_n,\ x \in \mathbb{R}$, we must have $0 \leq x \leq 1/n,\ \forall n \in \mathbb{N}$. As n increases, 1/n starts to converge to 0. So, 0 satisfies $\cap_{\infty}^{n=1} I_n$.

2. 0 is the ONLY value that satisfies $\cap_{\infty}^{n=1} I_n$: For contradiction, assume that intersection contains b. So, $b < 1/n\ \forall n \in \mathbb{N}$. Since $b > 0$, the Archimedean property states that $\exists n \in \mathbb{N}$ s.t. $1/n < b$, which contradicts our assumption.

□

**2.5/9 Let $K_n := (n, \infty), n \in \mathbb{N}$. Prove that $\cap_{\infty}^{n=1} K_n = \emptyset$.**

*Proof.* Assume $K_n$ as in the problem statement, so $K_n$ is the interval $(n, \infty)$. Thus, for $k \in \mathbb{R}$ to be in $K_n$, i.e., for $k \in K_n$, we need k s.t. $n < k < \infty$. Suppose, for contradiction, that $\cap_{\infty}^{n=1} K_n \neq \emptyset$, then $\exists x$ s.t. $x > n,\ \forall n \in \mathbb{N}$. However, by the Archimedean principle, $\exists n_x$ s.t. $n_x > x,\ n_x \in \mathbb{N}$, a contradiction. Thus, $\cap_{\infty}^{n=1} K_n = \emptyset$. □

# 2 SEQUENCES & SERIES

## 2.1 Sequences and their Limits

[8]

**Definition 2.1.** *A* ***sequence of real numbers*** *is a function defined on the set* $\mathbb{N} = \{1, 2, ...\}$ *of natural numbers whose range is contained in the set* $\mathbb{R}$ *of real numbers.*

A sequence is denoted by $X,\ (x_n),\ (x_n : n \in \mathbb{N})$. Each value of $x_n$ is a **term** or **element**. You may have also noticed that sequences are often constructed using a formula for the $n$th term. When the first few $k$ terms are given and a formula for the $x_{n+k-1}$ is given, we say that the sequence is defined **inductively** or **recursively**. There are other types of sequences as well.

The most trivial of them is the **constant** sequence, I will not bother defining this sequence. Then we have an **exponential** sequence, $B := (b^n)$ where b is a representative variable. We also have the famous **Fibonacci** sequence defined as $F := (f_n)$, where $f_1 := 1,\ f_2 := 1,$ and $f_{n+1} := f_{n-1} + f_n$, for $n \geq 2$.

**LIMITS of a SEQUENCE**

While there are many limit-based concepts or definitions within real analysis, the limit of a sequence is one of the most basic.

**Definition 2.2.** *A sequence* $X = (x_n)$ *is* $\mathbb{R}$ *is said to* ***converge*** *to* $x \in \mathbb{R}$*, or* $x$ *is said to be* ***limit*** *of* $(x_n)$*, if* $\forall \epsilon > 0\ \exists K(\epsilon) \in \mathbb{N}$ *s.t.* $\forall n \geq K(\epsilon)$*, the terms* $x_n$ *satisfy* $|x_n - x| < \epsilon$*.*

If a sequence has a limit, it is **convergent**, and if not, it is **divergent**.

**NOTE 2.3.** *The notation* $K(\epsilon)$ *emphasizes the choice of* $K$ *depends on the value of* $\epsilon$*. Shorthanded, we write just* $K$*. On the topic of notation,* $x_n \to x$ *indicates the intuitive idea that the values of* $x_n$ *approach the number* $x$ *as* $n \to \infty$*.*

Earlier, we established that a set can have only one supremum, or least upper bound. Similarly, a sequence can have only one limit.

**Theorem 2.4.** *A sequence in* $\mathbb{R}$ *can have at most one limit.*

*Proof.* Suppose, for contradiction, there are two limits x' and x" for a sequence $x_n$. For each $\epsilon > 0$, $\exists K'$ s.t. $|x_n - x'| < \epsilon/2$, $\forall n \geq K'$, and similarly, $\exists K''$ s.t. $|x_n - x''| < \epsilon/2$, $\forall n \geq K''$. We let $K$ be the larger of $K'$ and $K''$. Then for $n \geq K$, use the triangle inequality to get $|x' - x''| = |x' - x_n + x_n - x''| \leq |x' - x_n| + |x_n - x''| < \epsilon/2 + \epsilon/2 = \epsilon$. Since $\epsilon > 0$ is arbitrary, $x' - x'' = 0$. □

We can also describe convergence of $x_n$ to x in the following ways.

**Theorem 2.5.** *Let* $X = (x_n)$ *be a sequence of real numbers, and let* $x \in \mathbb{R}$*. The following statements are equivalent.*

[8] This subsection corresponds to section 3.1 in Bartle and Sherbert's Introduction to Real Analysis.

1. *$X$ converges to $x$.*

2. $\forall \epsilon > 0,\ \exists K$ *s.t.* $\forall n \geq K,\ x_n$ *satisfies* $|x_n - x| < \epsilon$.

3. $\forall \epsilon > 0,\ \exists K$ *s.t.* $\forall n \geq K,\ x_n$ *satisfies* $x - \epsilon < x_n < x + \epsilon$.

4. *For every $\epsilon-$neighbourhood $V_\epsilon(x)$ of $x$, there exists $K \in \mathbb{N}$ s.t. $\forall n \geq K$, the terms $x_n$ belong to $V_\epsilon(x)$.*

With the language of neighbourhoods, we can say that, in a sequence, all but a finite no. of terms are in $V_\epsilon(x)$. The terms that are all less than $K \in \mathbb{N}$.

**TAILS of SEQUENCES**

**Definition 2.6.** *If $X = (x_1, x_2, ..., x_n, ...)$ is a sequence of real numbers and if $m$ is a given natural number, then the $m-$**tail** is the sequence $X_m := (x_{m+n} : n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, ...)$.*

So, if you are asked about $X_3$, you give them the sequence $(x_4, x_5, ...)$.
The idea of a tail is about cutting out the beginning bullshit to see if the end of a sequence really converges. If the original sequence converges and I remove the first $m$ terms, the remaining $n - m$ terms are bound to converge. If you ask me why this concept is important enough to note in a book, it is to filter the beginning useless terms. It can also suffice to say that lim $X_m$ = lim X.

**SELECT EXERCISES** [9]

**3.1/1 The sequence $(x_n)$ is defined by the following formulas for the $n$th term. Write the first five terms in each case:**

1. $x_n := 1 + (-1)^n$: $1 + (-1) = 0$, $1 + (-1)^2 = 2$, $1 + (-1)^3 = 0$, $1 + (-1)^4 = 2$, $1 + (-1)^5 = 0$

2. $x_n := (-1)^n/n$: $-1^1/1 = -1$, $-1^2/2 = 1/2$, $-1^3/3 = -1/3$, $-1^4/4 = 1/4$, $-1^5/5 = -1/5$

3. $x_n := 1/(n(n{+}1))$: $1/(1(1{+}1)) = 1/2$, $1/(2(2{+}1)) = 1/6$, $1/(3(3{+}1)) = 1/12$, $1/(4(4+1)) = 1/20$, $1/(5(5+1)) = 1/30$

4. $x_n := 1/(n^2 + 2)$: $1/(1^2 + 2) = 1/3$, $1/(2^2 + 2) = 1/6$, $1/(3^2 + 2) = 1/11$, $1/(4^2 + 2) = 1/18$, $1/(5^2 + 2) = 1/27$

**3.1/2 The first few terms of a sequence $(x_n)$ are given below. Assume that the *natural pattern* indicated by these terms persists, give a formula for the $n$th**

[9] Page 61 of the textbook, section 3.1

**term $x_n$.**

1. 5, 7, 9, 11,... : $a_1 = 5,\ d = 2 \implies a_n = 5 + 2(n-1)$
2. 1/2, -1/4, 1/8, -1/16,... : $a_n = (-1)^{n-1}/2^n$
3. 1/2, 2/3, 3/4, 4/5,... : $a_n = n/(n+1)$
4. 1, 4, 9, 16,... : $a_n = n^2$

**3.1/4 For any $b \in \mathbb{R}$, prove that $lim(b/n) = 0$.**

*Proof.* Let $\epsilon > 0$ be given. We need to show that there exists a natural number $N$ such that for all $n > N$,

$$\left| \frac{b}{n} - 0 \right| < \epsilon.$$

Since

$$\left| \frac{b}{n} - 0 \right| = \left| \frac{b}{n} \right| = \frac{|b|}{n},$$

we want to find $N$ such that

$$\frac{|b|}{n} < \epsilon.$$

To achieve this, we solve

$$n > \frac{|b|}{\epsilon}.$$

Let $N = \left\lceil \frac{|b|}{\epsilon} \right\rceil$. Then for any $n > N$, we have

$$n \geq N > \frac{|b|}{\epsilon} \implies \frac{|b|}{n} < \frac{|b|}{N} \leq \epsilon \implies \left| \frac{b}{n} \right| < \epsilon$$

for all $n > N$, which shows that

$$\lim_{n \to \infty} \frac{b}{n} = 0.$$

$\square$

**3.1/5 Use the definition of the limit of a sequence to establish the following limits.**

- (b) $lim(2n/(n+1)) = 2$:

  Let $\epsilon > 0$. We wish to show

  $$\left|\frac{2n}{n+1} - 2\right| < \epsilon$$

  Simplifying LHS gives $|\frac{2n-2(n+1)}{n+1}| = |\frac{-1}{n+1}| < 1/n$. If $1/K < \epsilon$, then for any $n \geq K$, we also have $1/n < \epsilon$. Therefore, the limit is 2.

- (c) $lim((3n+1)/(2n+5)) = 3/2$:

  Let $\epsilon > 0$. We wish to show

  $$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \epsilon$$

  Simplifying LHS gives $|\frac{2(3n+1)-3(2n+5)}{2(2n+5)}| = |\frac{-13}{4n+10}| = |\frac{13}{4n+10}| < 13/4n$. If $13/4K < \epsilon$, then for any $n \geq K$, we also have $13/4n < \epsilon$. Therefore, the limit is 3/2.

**3.1/8 Prove that $lim(x_n) = 0 \iff lim(|x_n|) = 0$. Give an example to show that the convergence $(|x_n|)$ need not imply the convergence of $(x_n)$.**
*Proof.* We know that $lim(x_n) = 0$, so $\exists k \in \mathbb{N}$ s.t. $|x_n - 0| < \epsilon,\ \forall n \geq k \iff ||x_n| - 0| < \epsilon \iff lim(|x_n|) = 0$. □

**3.1/9 Show that if $x_n \geq 0,\ \forall n \in \mathbb{N}$ and $lim(x_n) = 0$, then $lim(\sqrt{x_n}) = 0$.**
*Proof.* Since $lim(x_n) = 0$, we can write $|x_n - 0| < \epsilon^2 \implies x^n = \epsilon^2$ (since $x_n > 0,\ \forall n \in \mathbb{N}) \implies \sqrt{x^n} = \epsilon \implies |\sqrt{x_n} - 0| < \epsilon \implies lim(\sqrt{x_n}) = 0$. □

**3.1/11 Show that $lim\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$.**
*Proof.* $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{N(N+1)} = \frac{1}{N} - \frac{1}{N+1},\ \forall n \geq N$. Choose $\epsilon$ s.t. $N > 1/\epsilon \implies 1/n - 1/(n+1) < \epsilon,\ \forall n \geq N$. □

**3.1/12 Show that $lim(\sqrt{n^2+1} - n) = 0$.**
*Proof.* $\sqrt{n^2+1} - n = (\sqrt{n^2+1} - n) \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} \leq \frac{1}{n} \implies 0 \leq lim(\frac{1}{\sqrt{n^2+1}+n}) \leq lim(\frac{1}{n}) \implies lim(\sqrt{n^2+1} - n) = 0$ □

**3.1/13 Show that $lim(1/3^n) = 0$.**
*Proof.* Similar to the question above, $1/3^n < 1/n,\ \forall n \in \mathbb{N} \implies 0 \leq lim(1/3^n) < lim(1/n) \implies lim(1/3^n) = 0$. □

**3.1/14 Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $lim(nb^n) = 0$. (Hint: Use Binomial theorem)**
*Proof.* Using the ratio test, we know that $|nx^n|$ converges to 0, when $0 < x < 1$. So, the limit must go to 0 as well. □

## 2.2 Limit Theorems

[10]

**Definition 2.7.** *A sequence $X = (x_n)$ of real numbers is said to be **bounded** if there exists a real number $M > 0$ s.t. $|x_n| \leq M, \ \forall n \in \mathbb{N}$.*

So, the sequence is bounded iff the set $\{x_n : n \in \mathbb{N}\}$ of its values is a bounded subset of $\mathbb{R}$.

**Theorem 2.8.** *A convergent sequence of real numbers is bounded.*

*Proof.* Suppose the limit of the sequence is $x$ and let $\epsilon := 1$. Then $\exists K \in \mathbb{N}$ s.t. $|x_n - x| < 1, \ \forall \geq K$. Then applying the triangle inequality with $n \geq K$, we get: $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$. If we set $M := sup\{|x_1|, |x_2|, ..., |x_{K-1}|, 1 + |x|\}$, then it follows that $|x_n| \leq M, \ \forall n \in \mathbb{N}$. □

We can also prove the theorem above using neighborhood language. If a neighborhood is given for the limit x, then all but a finite no. of terms of the sequence belong to the neighborhood. Thus, the neighborhood is bounded, and finite sets are bounded, so the sequence is bounded.

Limits of sequences follow the operations of addition (**sum**: $X + Y = (x_n + y_n)$), subtraction (**difference**: $X - Y = (x_n - y_n)$), multiplication (**product**: $XY = (x_n y_n)$), and division (**quotient**: $X/Y = (x_n/y_n), \ \forall y_n \neq 0$). With these operations, we develop a deeper understanding of convergent sequences.

**Theorem 2.9.**

1. *If $X$ $(x_n)$ and $Y$ $(y_n)$ are two convergent sequences, converging to $x$ and $y$ respectively. Then any of the operations mentioned above (in addition to scalar multiplication, $c \in \mathbb{R}$; and excluding division) also converge to x+y,x-y,xy, and cx.*

2. *Division: If $X$ converges to $x$ and $Z$ is a sequence of nonzero real numbers that converge to $z$ and if $z \neq 0$, then $X/Z$ converges to $x/z$.*

**Theorem 2.10.** *If $X$ is a convergent sequence of real numbers and if $x_n \geq 0, \ \forall n \in \mathbb{N}$, then $x = lim(x_n) \geq 0$.*

A more robust version of the theorem above is given below.

**Theorem 2.11.** *If $X$ and $Y$ are convergent sequences of real numbers and if $x_n \leq y_n, \ \forall n \in \mathbb{N}$, then $lim(x_n) \leq lim(y_n)$.*

This also has the equivalence in bounds for the terms for sequence elements as given below.

**Theorem 2.12.** *If $X$ is a convergent sequence and if $a \leq x_n \leq b, \ \forall n \in \mathbb{N}$, then $a \leq lim(x_n) \leq b$.*

You may recognize the above theorem is similar to the squeeze theorem, so let us formally write that down too!

[10] This subsection corresponds to section 3.2 in Bartle and Sherbert's Introduction to Real Analysis.

**Theorem 2.13.** ***Squeeze Theorem*** *Suppose X, Y, and Z are sequences s.t.* $x_n \leq y_n \leq z_n$, $\forall n \in \mathbb{N}$, *and that* $lim(x_n) = lim(z_n)$. *Then* $Y = (y_n)$ *is convergent and* $lim(x_n) = lim(y_n) = lim(z_n)$.

In addition to all these theorems, there are two more (1) $(|x_n|) \to |x|$, and (2) $(\sqrt{x_n}) \to \sqrt{x}$, where $\to$ indicates *converges to*; and it is assumed that $(x_n) \to x$.

**SELECT EXERCISES** [11]

**3.2/1 For $x_n$ given by the following formulas, establish either convergence or divergence of sequence X.**

1. $x_n := n/(n+1)$
   Since $n < (n+1)$, X converges. We have $lim(x_n) = 1$.
2. $x_n := (-1)^n n/(n+1)$
   Sequence diverges due to $(-1)^n$ term.
3. $x_n := n^2/(n+1)$
   Using L'hopital's rule, the limit of the sequence is equivalent to $2n/1$, which diverges.
4. $x_n := (2n^2 + 3/(n^2+1)$
   If we take $n^2$ common from the numerator and denominator and cancel them, we are left with $(2 + 3/n^2)/(1 + 1/n^2)$. The limit converges to 2.

**3.2/4 Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.**

[11] Page 69 of the textbook, section 3.2

*Proof.* We know that:

- $\lim_{n\to\infty} x_n = x$ where $x \neq 0$.
- $\lim_{n\to\infty}(x_n y_n) = L$, i.e., the sequence $XY$ converges.

We need to show that the sequence $Y = \{y_n\}$ converges to some limit $y$.
**Step 1: Express the limit of $y_n$.**
Since $x_n \to x$ and $x \neq 0$, for sufficiently large $n$, $x_n$ will be close to $x$ and, in particular, $x_n \neq 0$. Thus, we can write:

$$y_n = \frac{x_n y_n}{x_n}$$

**Step 2: Use the limit properties.**
Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_n y_n}{x_n}$$

Using the limit laws:

$$\lim_{n\to\infty} y_n = \frac{\lim_{n\to\infty}(x_n y_n)}{\lim_{n\to\infty} x_n} = \frac{L}{x}$$

Since $L$ and $x$ are both real numbers and $x \neq 0$, the right-hand side is a real number. Therefore, $Y = \{y_n\}$ converges to $y = \frac{L}{x}$.

□

**3.2/5 Show that the following sequences are not convergent.**
**(a)** $(2^n)$
The sequence $\{2^n\}$ is not convergent because it tends to infinity as $n$ increases. Specifically, for any real number $M$, there exists some $N \in \mathbb{N}$ such that for all $n > N$, $2^n > M$. Since the sequence grows without bound, it cannot converge to a finite limit.
**(b)** $((-1)^n n^2)$
The sequence $\{(-1)^n n^2\}$ is not convergent because it oscillates between positive and negative values while increasing in magnitude. Specifically, the sequence alternates between $n^2$ and $-n^2$, and as $n$ increases, the magnitude $|(-1)^n n^2| = n^2$ increases without bound. Therefore, the sequence does not approach a single finite value and cannot converge.

**3.2/7 If $b_n$ is bounded and $lim(a_n) = 0$, show that $lim(a_n b_n) = 0$. Explain why theorem 3.2.3. (theorem 2.9 in this notes' text) cannot be used.**

*Proof.* Since $\lim_{n\to\infty} a_n = 0$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n| < \frac{\epsilon}{M}$, where $M$ is an upper bound for the sequence $\{b_n\}$. That is, $|b_n| \leq M$ for all $n \in \mathbb{N}$.
Now, consider the sequence $\{a_n b_n\}$:

$$|a_n b_n| = |a_n| \cdot |b_n| < \frac{\epsilon}{M} \cdot M = \epsilon$$

Thus, for all $n > N$, $|a_n b_n| < \epsilon$, which implies $\lim_{n\to\infty}(a_n b_n) = 0$.

□

Theorem 3.2.3. (2.9 in this text ) cannot be used directly in our proof because $\{b_n\}$ is only given as a bounded sequence, not necessarily a convergent sequence. The theorem requires both sequences to be convergent, which is not guaranteed for $\{b_n\}$.

**3.2/9 Let $y_n := \sqrt{n+1} - \sqrt{n}$. $\forall n \in \mathbb{N}$. Show that $(\sqrt{n} y_n)$ converges. Find the limit.**
*Proof.* We can simplify $y$ by multiplying it with its conjugate

$$y_n := \sqrt{n+1} - \sqrt{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \implies \sqrt{n} y_n = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1 + \frac{1}{n}} + \frac{\sqrt{n}}{\sqrt{n}}} = \frac{1}{2}$$

Thus, $\sqrt{n} y_n$ converges, and it converges to 1/2. □

**3.2/10 Determine the limits.**
**(a)** $(\sqrt{4n^2 + n} - 2n)$
Multiply with conjugate to get $\frac{n}{\sqrt{4n^2+n}+2n}$. Then divide numerator and denominator by n to get 1/4

**(b)** $(\sqrt{n^2 + 5n} - n)$
Use similar process as above.

**3.2/17**
**(a)** Give an example of a convergent sequence $(x_n)$ of positive numbers with $lim(x_{n+1}/x_n) = 1$.
Consider the sequence $x_n = 1/n$. Then $lim\frac{x_{n+1}}{x_n} = \frac{n}{n+1} = \frac{n/n}{n/1+1/n} = 1$.

**(b)** Give an example of a divergent sequence with this property.
Simply consider $x_n = n$. Then we are in a similar situation as above, so the limit is 1.

**3.2/20 Let $(x_n)$ be a sequence of positive real numbers such that $lim(x_n^{1/n}) = L < 1$. Show that there exists a number $r$ with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $lim(x_n) = 0$.**

*Proof.* We have $lim(x_n^{1/n}) = L < 1$. So we can find another $\epsilon \in \mathbb{R}$ s.t. $L + \epsilon < 1$. For instance $\epsilon = (1 - L)/2 \implies L + \epsilon = (L + 1)/2 < 1$. So, $L + \epsilon < 1$, let this quantity be $r$, then $0 < r < 1$. We get $0 < L < r < 1 \implies 0 < lim(x_n^{1/n}) < r < 1$. Since $x_n^{1/n} \to L$ as $n \to \infty$, $\exists n \in \mathbb{N}$ s.t. $n \geq N$ for which $x_n^{1/n} = L < r \implies x_n^{1/n} < r \implies x_n < r^n$. We also know that for any $r \in \mathbb{R}$ s.t. $0 < r < 1$, $lim(r^n) = 0$. So, we have $0 < lim\ x_n < lim\ r^n = 0$. By the squeeze theorem, $lim\ x_n = 0$. □

**3.2/22 Suppose that $(x_n)$ is a convergent sequence and $(y_n)$ is such that for any $\epsilon > 0$ there exists M such that $|x_n - y_n| < \epsilon$ for all $n \geq M$. Does it follow that $(y_n)$ is convergent?**

*Proof.* Let $(x_n) \to L$ (We want to show $(y_n) \to L$). So, $\exists M_1$ s.t. for $n \geq M_1$, $|x_n - L| < \epsilon/2$. By the question, we get $\exists M_2$ s.t. for $n \geq M_2$, $|x_n - y_n| < \epsilon/2$. Consider $|y_n - L| \geq |x_n - L| + |-x_n + y_n| < \epsilon/2 + \epsilon/2 = \epsilon$, showing $y_n \to L$. □

## 2.3 Monotone Sequences

[12]

**Definition 2.14.** *Let $X = (x_n)$ be a sequence of real numbers. We say that $X$ is **increasing** if it satisfies the inequalities*

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots$$

*We say that $X$ is **decreasing** if*

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \ldots$$

*We say $X$ is **monotone** if it either increasing or decreasing.*

**Theorem 2.15** (Monotone Convergence Theorem)**.** *A monotone sequence of real numbers is convergent iff it is bounded. Further:*

1. *If $X = (x_n)$ is a bounded increasing squence, then $lim(x_n) = sup\{x_n : n \in \mathbb{N}\}$*
2. *If $Y = (Y_n)$ is a bounded decreasing squence, then $lim(y_n) = inf\{y_n : n \in \mathbb{N}\}$*

The Monotone Convergence Theorem establishes the existence of the limit of a bounded monotone sequence. It also gives us a way of calculating the limit of the sequence provided we can evaluate the supremum in case (a), or the infimum in case (b). Sometimes it is difficult to evaluate this supremum (or infimum), but once we know that it exists, it is often possible to evaluate the limit by other methods.

**EULER'S NUMBER**

Let $e_n := (1 + 1/n)^n\ n \in \mathbb{N}$. We will now show that the sequence $E = (e_n)$ is bounded and increasing; hence it is convergent. The limit of this sequence is the famous **Euler number** $e$, whose approximate value is 2.718281828459045..., which is taken as the base of the **natural logarithm**.

[12] This subsection corresponds to section 3.3 in Bartle and Sherbert's Introduction to Real Analysis.

**SELECT EXERCISES** [13]

**3.3/1 Let $x_1 := 8$ and $x_{n+1} := x_n/2 + 2,\ \forall n \in \mathbb{N}$. Show that $(x_n)$ is bounded and monotone. Find the limit.**

Let the sequence converge to $L$. So if $x_n \to L$, then $x_{n+1} \to L$, for some $n \geq N \in \mathbb{N}$. So, $x_{n+1} := x_n/2 + 2 \implies L = L/2 + 2 \implies L = 4$. Now we must show that 4 is indeed the limit, and that the sequence is monotone. We use induction to prove the 4 is the lower bound.

Base case: $n = 1$. We see that $x_1 = 8 > 4$, thus, the base case holds true.

Ind. Hyp.: Suppose the claim holds for any $n = k > 1$.

Now consider, $k + 1$: Then $x_{n+1} = \frac{x_n}{2} + 2$. We know that $x_n > 4 \implies \frac{x_n}{2} > 2 \implies \frac{x_n}{2} + 2 > 4 \implies x_{n+1} > 4$. Thus, 4 is the lower bound for $x_{n+1} > 4$ when $x_n > 4$.

Now, to show it is monotone, consider:

$$x_{n+1} - x_n = \frac{x_n}{2} + 2 - x_n = 2 - \frac{x_n}{2} < 2 - 4/2 = 0,$$

thus, $x_{n+1} < x_n$. Since the sequence is decreasing, it is bounded above by the first element $x_1 = 8$.

**3.3/2 Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n,\ \forall n \in \mathbb{N}$. Show that $(x_n)$ is bounded and monotone. Find the limit.**

Let the sequence converge to $L$. So if $x_n \to L$, then $x_{n+1} \to L$, for some $n \geq N \in \mathbb{N}$. So, $x_{n+1} := 2 - 1/x_n \implies L = 2 - 1/L \implies L^2 - 2L + 1 = 0 \implies L = 1$. To show it converges, we must show it is bounded and monotone; we show bounded-ness:

We have $x_1 > 1 \implies 1/x_1 < 1 \implies 2 - 1/x_1 > 1 \implies x_2 > 1$. Then, by induction, $x_n > 1,\ \forall n \in \mathbb{N}$. Thus, the sequence is bounded below by 1.

To show monotonicity: Consider $x_{n+1}x_n = 2 - 1/x_n - x_n = -(x_n - 1)^2/x_n < 0$. Thus, the sequence is decreasing.

**3.3/3 Let $a > 0$ and  and let $z_1 > 1$. Define $z_{n+1} := \sqrt{a + z_n}\ \forall n \in \mathbb{N}$. Show that $(z_n)$ converges and find the limit.**

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[13] Page 77 of the textbook, section 3.3

We can divide this problem into two cases:

1. $z_n \geq z_{n+1}$:
   Then $z_n \geq z_{n+1} = \sqrt{a + z_n} \iff z_n^2 \geq z_n + a \iff (z_n - 1/2)^2 \geq a + 1/4 \iff z_n \geq \sqrt{a + 1/4} + 1/2$ or $z_n \leq -\sqrt{a + 1/4} + 1/2 < 0$. So if, $z_n \geq z_{n+1} \implies z_{n+1} \geq z_{n+2}$. So, by induction, $(z_n)$ is decreasing. For bounded-ness, consider base case $z_n > \sqrt{a} \implies z_{n+1} = \sqrt{z_n + a} \geq \sqrt{a}$. By induction, $z_n \geq \sqrt{a}$, $\forall n \in \mathbb{N}$. Thus, the limit exists.

2. $z_n \leq z_{n+1}$:
   Then $(z_1 - 1/2)^2 \leq a + 1/4 \iff 1/2 - \sqrt{a + 1/4} \leq z_1 \leq 1/2 + \sqrt{a + 1/4}$. If $0 \leq z_1 \leq 1/2 + \sqrt{a + 1/4}$, then $z_1 \leq z_2$. By induction, we know that $(z_n)$ is increasing. For bounded-ness, consider base case $z_1 \leq 1/2 + \sqrt{a + 1/4}$. Assume the claim is true for some $n = k$. Then consider $z_{k+1} = \sqrt{a + z_k} \leq \sqrt{1/2 + \sqrt{a + 1/4} + a} = \sqrt{1/4 + \sqrt{a + 1/4} + a + 1/4} = \sqrt{(\sqrt{a + 1/4} + 1/2)^2} = 1/2 + \sqrt{a + 1/4}$. So the limit exists.

The limit is given by

$$L = \sqrt{L + a} \iff L^2 = a + L \iff L^2 - L - a = 0 \iff L = \frac{1 + \sqrt{1 + 4a}}{2}$$

**3.3/9 Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := sup\ A$. Show there exists an increasing sequence $(x_n)$ with $x_n \in A\ \forall n \in \mathbb{N}$ s.t. $u = lim(x_n)$.**
*Proof.* If $u \in A$, then suppose an $x_u = u$. If $u \notin A$, then by the definition of supremum,

1. Pick $x_1$, for $1 > 0$, $\exists x_1 \in A$ s.t. $u \geq x_1 > u - 1$, i.e., $0 < u - x_1 < 1$.

2. Pick $x_2$, for $1/2 > 0$, $\exists x_2 \in A$ s.t. $u \geq x_1 > max\{u - 1/2, x_1\}$, i.e., $0 < u - x_2 < 1/2$.

If we picked an $x_k$, then pick $x_{k+1}$ to be $u \geq x_{k+1} > max\{u - 1/(k+1), x_k\}$. Then we have a sequence $(x_n)$ s.t. $0 < u - x_n < 1/n$ and $x_{n+1} > x_n \implies$ monotonicity. Thus $lim(x_n) = u$. $\square$

## 2.4 Subsequences and the Bolzano-Weierstrass Theorem

[14]

In this section we will introduce the notion of a subsequence of a sequence of real numbers. Informally, a subsequence of a sequence is a selection of terms from the given sequence such that the selected terms form a new sequence. Usually the selection is made for a definite purpose. For example, subsequences are often useful in establishing the convergence or the divergence of the sequence. We will also prove the important existence theorem known as

[14] This subsection corresponds to section 3.4 in Bartle and Sherbert's Introduction to Real Analysis.

the Bolzano-Weierstrass Theorem, which will be used to establish a number of significant results.

**Definition 2.16.** *Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \ldots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ is given by*

$$(x_{n_1}, x_{n_2}, ..., x_{n_k}, ...)$$

*is called a* ***subsequence****.*

A tail of a sequence is a special type of subsequence. In fact, the $m$-tail corresponds to the sequence of indices $n_1 = m+1, n_2 = m+2, ...$

**Theorem 2.17.** *If a sequence $X = (x_n)$ of real numbers converges to a real number $x$, then any subsequence $X' = (x_{n_k})$ of $X$ also converges to $x$.*

*Proof.* Let $\epsilon > 0$ be given and let $K(\epsilon)$ s.t. $n \geq K(\epsilon) \implies |x_n - x| < \epsilon$. Since $n_1 < n_2 < \cdots < n_k < \ldots$ is an increasing sequence of natural numbers, it is easily proved that $n_k \geq k$. Hence, if $k \geq K(\epsilon)$, we also have $n_k \geq k \geq K(\epsilon)$ so that $|x_{n_k} - x| < \epsilon$. Therefore, the subsequence also converges to $x$. □

**Theorem 2.18.** *Let $X = (x_n)$ be a sequence of real numbers, then the following are equivalent:*

1. *The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$.*
2. *$\exists \epsilon_0 > 0$ s.t. $\forall k \in \mathbb{N} \implies n_k \in \mathbb{N}$ s.t. $n_k \geq k$ and $|x_{n_k} - x| \geq \epsilon_0$*
3. *$\exists \epsilon_0 > 0$ and $X' = (x_{n_k})$ s.t. $|x_{n_k} - x| \geq \epsilon_0 \ \forall k \in \mathbb{N}$.*

**Theorem 2.19** (Divergence Criteria)**.** *If a sequence $X = (x_n)$ of real numbers has either of the following properties, then $X$ is divergent.*

1. *$X$ has two convergent subsequences whose limits are not equal.*
2. *$X$ is unbounded.*

**EXISTENCE of MONOTONE SUBSEQUENCES**

While not every sequence is a monotone sequence, we will now show that every sequence has a monotone subsequence.

**Theorem 2.20** (Monotone Subsequence Theorem)**.** *If $X$ is a sequence of real numbers, then there is a subsequence of $X$ that is monotone.*

**BOLZANO-WEIESTRASS THEOREM**

We will now use the Monotone Subsequence Theorem to prove the Bolzano-Weierstrass Theorem. Because of the importance of this theorem we will also give a second proof of it based on the Nested Interval Property.

**Theorem 2.21** (Bolzano-Weierstrass Theorem)**.** *A bounded sequence of real numbers has a convergent subsequence.*

*Proof.* It follows from the Monotone Subsequence Theorem that if Xis a bounded sequence, then it has a subsequence X' that is monotone. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem that the subsequence is convergent. □

An alternate proof:

*Proof.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Since the sequence is bounded, there exist real numbers $a$ and $b$ such that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$. Therefore, the sequence is contained within the interval $I_1 := [a, b]$.

We will use the nested interval property to find a convergent subsequence. Begin with $I_1 = [a, b]$ and denote it as $I_1 = [a_1, b_1]$.

Bisect $I_1$ into two equal subintervals $I_{1,1}$ and $I_{1,2}$, where $I_{1,1} = [a_1, \frac{a_1+b_1}{2}]$ and $I_{1,2} = [\frac{a_1+b_1}{2}, b_1]$. Divide the set of indices $\{n \in \mathbb{N} : n > 1\}$ into two subsets:

$$A_1 := \{n \in \mathbb{N} : n > 1 \text{ and } x_n \in I_{1,1}\}$$

$$B_1 := \{n \in \mathbb{N} : n > 1 \text{ and } x_n \in I_{1,2}\}$$

If $A_1$ is infinite, take $I_2 = I_{1,1}$ and let $n_2$ be the smallest natural number in $A_1$. If $A_1$ is finite, then $B_1$ must be infinite. In this case, take $I_2 = I_{1,2}$ and let $n_2$ be the smallest natural number in $B_1$.

Bisect $I_2$ into two equal subintervals $I_{2,1}$ and $I_{2,2}$. Divide the set $\{n \in \mathbb{N} : n > n_2\}$ into two subsets:

$$A_2 := \{n \in \mathbb{N} : n > n_2 \text{ and } x_n \in I_{2,1}\}$$
$$B_2 := \{n \in \mathbb{N} : n > n_2 \text{ and } x_n \in I_{2,2}\}$$

If $A_2$ is infinite, take $I_3 = I_{2,1}$ and let $n_3$ be the smallest natural number in $A_2$. If $A_2$ is finite, then $B_2$ must be infinite. In this case, take $I_3 = I_{2,2}$ and let $n_3$ be the smallest natural number in $B_2$.

Continue this process to construct a sequence of nested intervals $I_k$, where each $I_k$ is contained in $I_{k-1}$ and has length $\frac{b-a}{2^k}$. For each $k$, let $n_k$ be the smallest index in the appropriate set determined by whether $A_k$ or $B_k$ is infinite.

Since the length of $I_k$ is $\frac{b-a}{2^k}$, which approaches 0 as $k$ increases, the nested intervals $I_k$ have a common point $c$ in $\bigcap_{k=1}^{\infty} I_k$ by the nested interval property.

Moreover, since $x_{n_k}$ is in $I_k$ for all $k$, and $I_k \to c$ as $k \to \infty$, it follows that $x_{n_k} \to c$. Thus, the subsequence $\{x_{n_k}\}$ converges to $c$.

Therefore, every bounded sequence has a convergent subsequence, proving the Bolzano-Weierstrass theorem. □

**Theorem 2.22.** *Let $X$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of $X$ converges to $x$. Then the sequence $X$ converges to $x$.*

## SELECT EXERCISES [15]

**3.4/2 Use the method of example 3.4.3(b) to show that if $0 < c < 1$, then $lim(c^{1/n}) = 1$.**

*Proof.* Let $z_n = c^{1/n}$, where $0 < c < 1$. We want to show that $\lim_{n\to\infty} z_n = 1$.
First, observe that $z_n = c^{1/n}$. Since $0 < c < 1$, $c$ raised to any positive power will be less than 1. Therefore, $z_n < 1$ for all $n \in \mathbb{N}$.
Next, let's analyze the behavior of $z_n$ as $n \to \infty$. Consider the sequence $\{z_n\}$:

$$z_n = c^{1/n}.$$

To use the Monotone Convergence Theorem, we need to show that $\{z_n\}$ is monotone. We will show that $z_n$ is increasing.
For $n \in \mathbb{N}$, we have:

$$z_{n+1} = c^{1/(n+1)}.$$

We need to compare $z_n$ and $z_{n+1}$. Note that:

$$z_{n+1} = c^{1/(n+1)} = \left(c^{1/n}\right)^{n/(n+1)} = z_n^{n/(n+1)}.$$

Since $0 < c < 1$, $z_n < 1$, and therefore $z_n^{n/(n+1)} > z_n$. Thus:

$$z_{n+1} > z_n.$$

So, $\{z_n\}$ is an increasing sequence.
To apply the Monotone Convergence Theorem, we also need to check that $\{z_n\}$ is bounded above. We see that $z_n < 1$ for all $n$, so $\{z_n\}$ is bounded above by 1.
Since $\{z_n\}$ is increasing and bounded above, it converges by the Monotone Convergence Theorem. Let $L = \lim_{n\to\infty} z_n$.
To find $L$, consider the limit of $z_n$ as $n \to \infty$:

$$L = \lim_{n\to\infty} c^{1/n}.$$

Taking the natural logarithm of both sides, we get:

$$\ln L = \lim_{n\to\infty} \ln(c^{1/n}) = \lim_{n\to\infty} \frac{\ln c}{n}.$$

Since $\ln c < 0$ and $\frac{\ln c}{n} \to 0$ as $n \to \infty$, we have:

$$\ln L = 0 \implies L = e^0 = 1.$$

Thus:

$$\lim_{n\to\infty} c^{1/n} = 1.$$

□

[15] Page 84 of the textbook, section 3.4

**3.4/4 Show the following sequences are divergent.**

1. $(1 - (-1)^n + 1/n)$ Consider two subsequences, $n \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$. We see that when $n$ is even, the subsequence converges to 0. When odd, subsequences converges to 2. Thus, the sequence is divergent.

2. $sin(n\pi/4)$ Similar to above, consider two subsequences: $n \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 8$. Then we see that the former subsequence converges to 1 and the latter converges to 0. Thus, the sequence is divergent.

**3.4/9 Suppose every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $lim\ X = 0$.**
*Proof.* If the limit exists, $lim\ X = 0$. If it does not exist, then it diverges, i.e., sequence is either unbounded or has different convergent subsequences.

1. Different convergent subsequences:
subsequence s.t. lim of subsequence = a $\neq 0$. Since any subsequence of this subsequence has limit $a \neq 0$, we find a contradiction.

2. Unbounded:
Assume unbounded from above. Then for

$$1 \in \mathbb{N},\ \exists x_{n_1} \text{ s.t. } x_1 > 1$$

$$2 \in \mathbb{N},\ \exists x_2 \text{ s.t. } x_{n_2} > max\{x_1, 2\}$$

$$\vdots$$

$$k+1 \in \mathbb{N},\ \exists x_{k+1} \text{ s.t. } x_{n_{k+1}} > max\{x_{n_k}, k+1\}$$

Then we find a subsequence s.t. $x_{n_k} > 0$. Since $|x_{n_k} - 0| > k$, no subsequence of $(x_{n_k})$ converges to 0, a contradiction.

Since we found a contradiction in both cases. □

**3.4/11 Suppose that $x_n \geq 0\ \forall n \in \mathbb{N}$ and $lim((-1)^n x_n$ exists. Show that $(x_n)$ converges.**
*Proof.* We have given

$$lim_{n\to\infty}(-1)^n x_n = L \implies lim_{n\to\infty}|(-1)^n x_n| = |L|$$

Then, we can write

$$||(-1)^n x_n| - |L|| < \epsilon$$

Using triangular inequality

$$||(-1)^n x_n| - |L|| = ||x_n| - |L|| = |x_n - L| < \epsilon$$

We get $|x_n| = x_n$ since $x_n \geq 0$. So, $x_n$ converges. □

**3.4/12 Show that if $(x_n)$ is unbounded, then there exists a subsequence $(x_{n_k})$ s.t.** $lim(1/x_{n_k}) = 0$

*Proof.* Use the sequence found in problem 3.4/9. Since $(x_{n_k} > k \implies 1/(x_{n_k} < 1/k \implies lim(1/(x_{n_k}) = 0$. □

## 2.5 The Cauchy Criterion

[16]

The Monotone Convergence Theorem is extraordinarily useful and important, but it has the significant drawback of applying only to monotone sequences. We need to have a condition implying the convergence of a sequence that does not require us to know the value of the limit in advance and is not restricted to monotone sequences. The Cauchy Criterion, which will be established in this section, is such a condition.

**Definition 2.23.** *A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\epsilon > 0$ there exists a natural number $H(\epsilon)$ s.t. $\forall n, m \geq H(\epsilon)$, the terms $x_n, x_m$ satisfy $|x_n - x_m| < \epsilon$.*

The significance of the concept of Cauchy sequence lies in the main theorem of this section, which asserts that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. This will give us a method of proving a sequence converges without knowing the limit of the sequence.

**Lemma 2.24.** *If $X = (x_n)$ is a convergent sequence of real numbers, then $X$ is a Cauchy sequence.*

*Proof.* If $x := lim\ X$, then given $\epsilon > 0, \exists K(\epsilon/2) \in \mathbb{N}$ s.t. $n \geq K(\epsilon)$ then $|x_n - x| < \epsilon/2$. Thus, if $H(\epsilon) := K(\epsilon/2)$ and if $n, n \geq H(\epsilon)$, then we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x_m - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

Since $\epsilon > 0$ is arbitrary, it follows that $(x_n)$ is Cauchy sequence. □

**Lemma 2.25.** *A Cauchy sequence of real numbers is bounded.*

*Proof.* Let $X := (x_n)$ be a Cauchy sequence and let $\epsilon := 1$. If $H := H(1)$ and $n \geq H \implies |x_n - x_H| < 1$. Hence, by the triangle inequality, we have $|x_n| \leq |x_H| + 1\ \forall n \geq H$. If we set

$$M := sup\{|x_1|, |x_2|, \dots, |x_{H-1}|, |x_H| + 1\},$$

then it follows that $|x_n| \leq M\ \forall n \in \mathbb{N}$. □

**Theorem 2.26** (Cauchy Convergence Criterion). *A sequence of real numbers is convergent if and only if it is a Cauchy sequence*

[16]This subsection corresponds to section 3.5 in Bartle and Sherbert's Introduction to Real Analysis.

**Definition 2.27.** *We say that a sequence $X := (x_n)$ of real numbers is* ***contractive*** *if there exists a constant $C$, $0 < C < 1$ s.t.*

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

*for all $n \in \mathbb{N}$. The number $C$ is called the* ***constant*** *of the contractive sequence.*

**Theorem 2.28.** *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

**Corollary 2.29.** *If $X := (x_n)$ is a contraction sequence with constant $C$, $0 < C < 1$, and if $x^* := lim\ X$, then*

1. $|x^* - x_n| \leq \frac{C^{n-1}}{1-C}|x_2 - x_1|$
2. $|x^* - x_n| \leq \frac{C}{1-C}|x_n - x_{n-1}|$

**SELECT EXERCISES** [17]

**3.5/2 Show directly from the definition that the following are Cauchy sequences.**

1. $\frac{n+1}{n}$

   *Proof.* Consider the sequence $x_n = \frac{n+1}{n}$. We must show that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_n - x_m| < \epsilon$. Observe that:

   $$x_n = 1 + \frac{1}{n}.$$

   Thus, for $n, m \geq N$, we have:

   $$|x_n - x_m| = \left|1 + \frac{1}{n} - \left(1 + \frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{nm}\right|.$$

   As $n, m \to \infty$, this difference tends to 0. Given any $\epsilon > 0$, we can find $N$ such that for all $n, m \geq N$, $|x_n - x_m| < \epsilon$. Therefore, the sequence is Cauchy. □

2. $(1 + \frac{1}{2!} + \cdots + \frac{1}{n!})$

   *Proof.* Let $x_n = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$. To show that this sequence is Cauchy, for any $\epsilon > 0$, we need to find $N$ such that for all $m, n \geq N$, $|x_n - x_m| < \epsilon$. Since the terms in the sequence become smaller as $n$ increases (due to the factorial in the denominator), we have:

   $$|x_n - x_m| = \left|\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!}\right|.$$

   As $n \to \infty$, the sum tends to 0. Thus, for any $\epsilon > 0$, we can find $N$ such that for all $m, n \geq N$, $|x_n - x_m| < \epsilon$. Therefore, the sequence is Cauchy. □

[17] Page 91 of the textbook, section 3.5

**3.5/3 Show directly from the definition that the following are not Cauchy sequences.**

1. $((-1)^n)$

   *Proof.* The sequence $x_n = (-1)^n$ oscillates between 1 and -1. For any $n$ and $m$, $|x_n - x_m|$ does not tend to 0 as $n, m \to \infty$. For example, if $n$ is odd and $m$ is even, $|x_n - x_m| = 2$, which is a constant and does not become arbitrarily small. Thus, the sequence is not Cauchy. □

2. $(n + \frac{(-1)^n}{n})$

   *Proof.* Consider the sequence $x_n = n + \frac{(-1)^n}{n}$. As $n \to \infty$, the term $n$ grows without bound, while $\frac{(-1)^n}{n}$ oscillates between positive and negative values. For large $n$ and $m$, the difference $|x_n - x_m|$ is approximately $|n - m|$, which does not tend to 0. Therefore, the sequence is not Cauchy. □

3. $(\ln n)$

   *Proof.* The sequence $x_n = \ln n$ grows without bound as $n \to \infty$. For large $n$ and $m$, $|x_n - x_m| = |\ln n - \ln m| = |\ln(n/m)|$. Since $\ln n \to \infty$, the difference between terms does not tend to 0. Thus, the sequence is not Cauchy. □

**3.5/5 If $x_n = \sqrt{n}$, show that $(x_n)$ satisfies $\lim |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.**
*Proof.* Let $x_n = \sqrt{n}$. We first show that $\lim |x_{n+1} - x_n| = 0$. We have:

$$|x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}|.$$

Using the difference of squares, we can rewrite this as:

$$|\sqrt{n+1} - \sqrt{n}| = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

As $n \to \infty$, this expression tends to 0, so $\lim |x_{n+1} - x_n| = 0$. However, the sequence is not Cauchy because for large $n$ and $m$, $|x_n - x_m| \approx |n - m|^{1/2}$, which does not tend to 0. Hence, the sequence is not Cauchy. □

**3.5/9 If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that it is a Cauchy sequence.**

*Proof.* Let $(x_n)$ be a sequence such that $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, where $0 < r < 1$. To show that $(x_n)$ is Cauchy, for any $\epsilon > 0$, we need to find $N$ such that for all $m, n \geq N$, $|x_n - x_m| < \epsilon$. For $m > n$, we have:

$$|x_n - x_m| \leq \sum_{k=n}^{m-1} |x_{k+1} - x_k| < \sum_{k=n}^{m-1} r^k = r^n \frac{1 - r^{m-n}}{1 - r}.$$

As $n \to \infty$, this sum tends to 0. Therefore, for any $\epsilon > 0$, we can find $N$ such that for all $m, n \geq N$, $|x_n - x_m| < \epsilon$. Thus, the sequence is Cauchy. □

## 2.6 Introducton to Infinite Series

[18]

**Definition 2.30.** *If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series generated by** $X$ is the sequence $S := (s_k)$ defined by $s_k := x_1 + \cdots + x_k$.*

The $x$ are **terms** of the series and $s_k$ is the **partial sum**. If $lim\ S$ exists, the series is **convergent** and is called the **sum** or **value**. If limit does not exists, the limit is **divergent**.

**Definition 2.31.** *Consider the sequence $X := (r^n)_{n=0}^{\infty}$, $r \in \mathbb{R}$, which generates the **geometric series**:*

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \ldots$$

*The sum of the series is given by*

$$S_n = \frac{a}{(1 - r)}$$

*where $a$ is the first term and $r$ is the common ratio.*

**Theorem 2.32** ($n$th Term Test). *If the series $\sum x_n$ converges, then $lim(x_n) = 0$.*

**Theorem 2.33** (Cauchy Criterion for Series). *If the series $\sum x_n$ converges iff $\forall \epsilon > 0, \exists M(\epsilon) \in \mathbb{N}$ s.t. $m > n \geq M(\epsilon) \implies$*

$$|s_m - s_n| = |x_{n+1} + \ldots x_m| < \epsilon$$

**Theorem 2.34.** *Let $(x_n)$ be a sequence of nonnegative real numbers. Then the series $\sum x_n$ converges $\iff$ the sequence $S = (s_k)$ of partial sums is bounded. In this case*

$$\sum_{n=1}^{\infty} x^n = lim(s_k) = sup\{s_k : k \in \mathbb{N}\}$$

*Proof.* Since $x_n \geq 0$, the sequence $S$ of partial sums is monotone increasing. By the Monotone Convergence Theorem, $S$ converges iff it is bounded, in which case its limit equals $sup\{s_k\}$. □

[18] This subsection corresponds to section 3.7 in Bartle and Sherbert's Introduction to Real Analysis.

## COMPARISON TESTS

**Theorem 2.35** (**Comparison Test**). *Let $X$ and $Y$ be real sequences and suppose that for some $K \in \mathbb{N}$, we have*

$$0 \leq x_n \leq y_n \ for \ n \geq K$$

1. *Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.*
2. *The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.*

*Proof.* Suppose $\sum y_n$ converges and $\epsilon > 0$, let $M(\epsilon) \in \mathbb{N}$ s.t. $m > n \geq M(\epsilon)$, then

$$y + n + 1 + \cdots + y_m < \epsilon$$

If $m > sup\{K, M(\epsilon)\}$, then it follows that

$$0 \leq x_{n+1} + \cdots + x_m \leq y + n + 1 + \cdots + y_m < \epsilon,$$

showing that $\sum x_n$ converges. □

**Theorem 2.36** (**Limit Comparison Test**). *Let $X$ and $Y$ are strictly positive sequences and suppose that the following limit exists in $\mathbb{R}$:*

$$r := lim(\frac{x_n}{y_n})$$

1. *If $r \neq 0 \implies \sum x_n$ is convergent $\iff \sum y_n$ is convergent.*
2. *If $r = 0$ and $\sum y_n$ is convergent $\implies \sum x_n$ is convergent.*

## SELECT EXERCISES [19]

**3.7/3 By using partial fractions, show that $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$**
We can decompose the function as follows

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

Writing out the first few terms, we see that we get

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \dots$$

We observe that all the terms following the first term cancel out. Thus, we must be left with the limit of the second term in the decomposition:

$$1 - lim(\frac{1}{n+2}) = 1 - 0 = 1$$

showing the sum to be 1.

[19]Page 100 of the textbook, section 3.7

**3.7/5 Can you give an example of a convergent series $\sum x_n$ and a divergent series $\sum y_n$ s.t. $\sum(x_n + y_n)$ is convergent? Explain.**
It is not possible. Since $\sum y_n$ is divergent, adding a convergent sequence to it does not bound it any manner.

**3.7/6 Below**

1. Calculate the value of $\sum_{n=2}^{\infty}(2/7)^n$
   We see that $a = \frac{4}{49}$ and $r = \frac{2}{7}$. Using formula, we find the sum is given by
   $$\frac{\frac{4}{49}}{1 - \frac{2}{7}} = \frac{4}{35}$$
2. Calculate the value of $\sum_{n=1}^{\infty}(1/3)^{2n}$
   We see that $a = \frac{1}{9}$ and $r = \frac{1}{9}$. Using formula, we find the sum is given by
   $$\frac{\frac{1}{9}}{1 - \frac{1}{9}} = \frac{1}{8}$$

**3.7/7 Find a formula for the series $\sum_{n=1}^{\infty} r^{2n}$ when $|r| < 1$.**
Using the formula, we see that the first term and common ratio are both $r^2$, so we get formula $\frac{r^2}{1-r^2}$

**3.7/11 If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum a_n^2$ always convergent? Either prove or offer a counterexample.**
*Proof.* Since $a_n > 0,\ n \geq 1 \implies a_n \to 0,\ n \geq K \in \mathbb{N} \implies a_n^2 < a_n$. Thus, $\sum a_n^2$ converges. $\square$

# 3 LIMITS

## 3.1 Limits of Functions

[20]

In order for the idea of the limit of a function f at a point c to be meaningful, it is necessary that f be defined at points near c. It need not be defined at the point c, but it should be defined at enough points close to c to make the study interesting. This is the reason for the following definition.

**Definition 3.1.** *Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of $A$ if $\forall \delta > 0 \implies \exists x \in A,\ x \neq c$ s.t. $|x - c| < \delta$.*

Alternatively, we can say that a point $c$ is a cluster point of the set A if every $\delta$-neighborhood $V_\delta(c) = (c - \delta, c + \delta)$ of $c$ contains at least one point of $A$ distinct from $c$.

[20] This subsection corresponds to section 4.1 in Bartle and Sherbert's Introduction to Real Analysis.

**Theorem 3.2.** *A number $c \in \mathbb{R}$ is a cluster point of a subset $A$ of $\mathbb{R} \iff \exists (a_n) \in A$ s.t. $lim(a_n) = c$ and $a_c \neq c\ \forall n \in \mathbb{N}$.*

## DEFINITION of the LIMIT

We now state the precise definition of the limit of a function $f$ at a point $c$. It is important to note that in this definition, whether $f$ is defined at $c$ or not is material. In any case, we exclude $c$ from consideration when determining the limit.

**Definition 3.3.** *Let $A \subseteq \mathbb{R}$, and $c$ is a cluster point of $A$. For a function $f : A \to \mathbb{R}$, $L \in \mathbb{R}$ is said to be a* ***limit of*** $f$ ***at*** $c$ *if, $\epsilon > 0 \implies \exists \delta > 0$ s.t. $x \in A$ and $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$.*

If $L$ is a limit of $f$ at $c$, $\implies$ $f$ **converges to** $L$ **at** $c$. We denote this as

$$L = \lim_{x \to c} f(x) = \lim_{x \to c} f$$

If not, it **diverges**.

**Theorem 3.4.** *If $f : A \to \mathbb{R}$ and if $c$ is a cluster point of $A$, then $f$ can have only one limit at $c$.*

**Theorem 3.5.** *If $f : A \to \mathbb{R}$ and if $c$ is a cluster point of $A$. Then the following statements are equivalent:*

1. $\lim_{x \to c} f = L$
2. *Given any $\epsilon$-neighborhood $V_\epsilon(L)$ of $L$, there exists a $\delta$-neighborhood $V_\delta(c)$ of $c$ such that if $x \neq c$ is any point in $V_\delta(c) \cup A \implies f(x) \in V\epsilon(L)$.*

## CRITERIA for LIMITS

**Theorem 3.6** (**Sequential Criterion**)**.** *If $f : A \to \mathbb{R}$ and if $c$ is a cluster point of $A$. Then the following statements are equivalent:*

1. $\lim_{x \to c} f = L$
2. $\forall (x_n) \in A \to c$ *s.t.* $x_n not = c$, $\forall n \in \mathbb{N}$, *the sequence* $(f(x)) \to L$.

**Theorem 3.7** (**Divergence Criterion**)**.** *Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A$.*

(a) *If $L \in \mathbb{R}$, then $f$ does not have limit $L$ at $c$ if and only if there exists a sequence $(x_n)$ in $A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence $(x_n)$ converges to $c$ but the sequence $(f(x_n))$ does not converge to $L$.*

(b) *The function $f$ does not have a limit at $c$ if and only if there exists a sequence $(x_n)$ in $A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence $(x_n)$ converges to $c$ but the sequence $(f(x_n))$ does not converge in $\mathbb{R}$.*

**Definition 3.8.** *Let the **signum function*** sgn *be defined by:*

$$\operatorname{sgn}(x) := \begin{cases} +1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

**SELECT EXERCISES** [21]

**4.1/6 Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ and $c \in I$. Suppose $\exists K, L$ s.t. $|f(x) - L| \le K|x - c|,\ x \in I$. Show $\lim_{x\to c} f(x) = L$.**
To prove that $\lim_{x\to c} f(x) = L$, we need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.
We can use neighborhood language: $\lim_{x\to c} f(x) = L \iff \forall\epsilon > 0, \exists\delta > 0$ s.t. $\forall x \in V_\delta(c),\ f(x) \in V_\epsilon(L)$
Then for $\epsilon > 0$, take $\delta = \frac{\epsilon}{k}$.
Then we have:

$$|f(x) - L| < K|x - c| < K\delta = K\left(\frac{\epsilon}{K}\right) = \epsilon.$$

This shows that for every $\epsilon > 0$, we can find a $\delta > 0$ (specifically $\delta = \frac{\epsilon}{K}$) such that whenever $0 < |x - c| < \delta$, it follows that $|f(x) - L| < \epsilon$.

**4.1/9 Establish the following limits**

1. $\lim_{x\to 2} \frac{1}{1-x} = -1$

   *Proof.* Given $\epsilon > 0$, we want $|\frac{1}{1-x} + 1| < \epsilon$ for $|x - 2| < \delta$.

   $$|\frac{1}{1-x} + 1| = |\frac{1 + (1 - x)}{1 - x}| = |\frac{2 - x}{1 - x}|.$$

   We bound $x$ using $\delta = \frac{1}{2}$. Then $|x - 2| < 1/2 \implies \frac{1}{2} < x - 1 < \frac{3}{2}$. This gives us

   $$\frac{x-2}{x-1} < \epsilon \implies |x - 2| < \epsilon|x - 1| < \frac{\epsilon}{2}$$

   Therefore, we choose $\delta = \{\frac{\epsilon}{2}, \frac{1}{2}\}$, then $|\frac{1}{1-x} + 1| < \epsilon$. □

2. $\lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$

   *Proof.* Given $\epsilon > 0$, we want $|\frac{1}{1+x} - \frac{1}{2}| < \epsilon$ for $|x - 1| < \delta$. Now,

   $$|\frac{1}{1+x} - \frac{1}{2}| = |\frac{2x - 1 - x}{2(1 + x)}| = |\frac{x - 1}{2(1 + x)}|.$$

[21] Page 110 of the textbook, section 4.1

We bound $x$ using $\delta = \frac{1}{4}$. Then $|x-1| < 1/4 \implies \frac{7}{4} < x+1 < \frac{9}{4}$. This gives us

$$\frac{x-1}{2(x+1)} < \epsilon \implies |x-1| < 2\epsilon|1+x| < \frac{11\epsilon}{2}$$

Therefore, we choose $\delta = min\{\frac{11\epsilon}{2}, \frac{1}{4}\}$, then $|\frac{1}{1+x} - 1\frac{1}{2}| < \epsilon$. □

3. $\lim_{x\to 0} \frac{x^2}{|x|} = 0$

   *Proof.* Note that $\frac{x^2}{|x|} = |x|$, a short proof:

   (a) If $x > 0$: $x = |x| \implies \frac{x^2}{|x|} = \frac{x^2}{x} = x = |x|$.

   (b) If $x < 0$: $x = -|x| \implies \frac{x^2}{|x|} = \frac{x^2}{-x} = -x = |x|$.

   For $\epsilon > 0$, take $\delta = \epsilon$, then if $|x-0| < \delta \implies ||x|-0| = |x| < \delta = \epsilon$. This gives us $\lim_{x\to 0}|x| = 0$.

   □

4. $\lim_{x\to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$

   *Proof:* Similar to above, we calculate $\frac{x^2-x+1}{x+1} - \frac{1}{2} = \frac{(2x-1)(x+1)}{(2x-1)}$. So, choose $|x-1| < \delta = 1/2 \implies -1/2 < x-1 < 1/2 \implies 1/2 < x < 3/2$. Then $\frac{(2x-1)(x+1)}{(2x-1)} < \epsilon \implies |x-1| < \frac{|x+1|}{|2x-1|}\epsilon$. Choose the numerator $x = 1/2 \implies |x+1| = 3/2$ and the denominator $x = 3/2 \implies |2x-1| = 2$. Putting together, we get $|x-1| < \frac{|x+1|}{|2x-1|}\epsilon < \frac{3}{4}\epsilon$. So, $\delta = \{1/2, 3\epsilon/4\}$

   □

## 4.1/10 Establish the following limits

1. $\lim_{x\to 2} x^2 + 4x = 12$

   *Proof.* We have $|x^2+4x-12| \implies |x+6|\cdot|x-2| < \epsilon \implies |x-2| < \frac{\epsilon}{|x+6|}$. Let $|x-2| < 1 \implies 1 < x < 3 \implies 7 < x+6 < 9$. So we get $|x-2| < \frac{\epsilon}{|x+6|} = \frac{\epsilon}{9}$. So, $\delta = min\{1, \frac{\epsilon}{9}\}$ □

2. $\lim_{x\to -1} \frac{x+5}{2x+3} = 4$

   *Proof.* Let initial bound be 1. Then we get $\delta = min\{1, \frac{9\epsilon}{7}\}$. □

**4.1/15 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if rational, and $f(x) = 0$ if irrational.**

1. **Show that $f$ has a limit at $x = 0$**

   *Proof.* Let $\epsilon > 0$. Put $\delta = \epsilon$. If $0 < |x - 0| < \delta$, then

   $$|f(x) - 0| = |x| < \epsilon \text{ (rational case)}$$

   $$|f(x) - 0| = 0 < \epsilon \text{ (irrational case)}$$

   Since $f(x) - 0| < \epsilon$ whenever $0 < |x - 0| < \delta$, the limit tends to 0. □

2. **Use a sequential argument to show that if $c \neq 0$, then $f$ does not have a limit at $c$.**

   *Proof.* By density theorem, we can pick $(x_n), (y_n)$ s.t. $\forall n \in \mathbb{N}$:

   (a) $x_n \in \mathbb{Q}, y_r \in \mathbb{R} \backslash \mathbb{Q}$,

   (b) $x_n, y_n \in (c, c + \frac{1}{n})$

   Note that (b) tells us that the two sequences are not equal to $c$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$. Now, (a) shows that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} f(y_n) = 0$. Therefore, $\lim_{n\to\infty} f(x)$ does not exist when $c \neq 0$. □

## 3.2 Limits Theorems

[22]

**Definition 3.9.** *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in \mathbb{R}$ be a cluster point of $A$. We say that $f$* ***is bounded on a neighborhood of c*** *if there exists a $\delta$-neighborhood $V_\delta(c)$ of $c$ and a constant $M > 0$ such that we have $|f(x)| \leq M$, $\forall x \in A \cap V_\delta(c)$.*

**Theorem 3.10.** *If $A \subseteq \mathbb{R}$, and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then $f$ is bounded on some neighborhood of $c$.*

**Definition 3.11.** *Let $A \subseteq \mathbb{R}$ and let $f$ and $g$ be functions defined on $A \to \mathbb{R}$. We define the* ***sum*** *$f + g$,* ***difference*** *$f - g$,* ***product*** *$fg$ on $A \to \mathbb{R}$ to be the functions given by*

$$(f + g)(x) := f(x) + g(x),$$

$$(f - g)(x) := f(x) - g(x),$$

$$(fg)(x) := f(x)g(x)$$

[22]This subsection corresponds to section 4.2 in Bartle and Sherbert's Introduction to Real Analysis.

*for all $x \in A$. Further, if $b \in \mathbb{R}$, we define the* ***multiple*** *$bf$ to be the function given by*

$$(bf)(x) := bf(x) \ \forall x \in A.$$

*Finally*

**Theorem 3.12.** *Let $A \subseteq \mathbb{R}$, let $f$ and $g$ be functions on $A$ to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A$. Further, let $b \in \mathbb{R}$.*

1. *If $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then:*
   (a) $\lim_{x\to c}(f+g) = L+M$
   (b) $\lim_{x\to c}(f-g) = L-M$
   (c) $\lim_{x\to c}(fg) = LM$
   (d) $\lim_{x\to c}(bf) = bL$
2. *If $h : A \to \mathbb{R}$, if $h(x) \neq 0 \ \forall x \in A$ and if $\lim_{x\to c} h = H \neq 0$, then*

$$\lim_{x\to c} \frac{f}{h} = \frac{L}{H}$$

**SELECT EXERCISES** [23]

**4.2/3 Find**

$$\lim_{x\to 0} \frac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2}, \ x > 0$$

We cannot use the theorem that allows us to evaluate the quotient since the denominator tends to 0. Thus, if $x \neq 0$, we get

$$\frac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2} \cdot \frac{\sqrt{1+2x} + \sqrt{1+3x}}{\sqrt{1+2x} + \sqrt{1+3x}} = \frac{1+2x-1+3x}{(x+2x^2)(\sqrt{1+2x} + \sqrt{1+3x})}$$

This becomes

$$\frac{-1}{(1+2x)(\sqrt{1+2x} + \sqrt{1+3x})},$$

which can be evaluated. Thus, the $\lim_{x\to 0} \frac{-1}{(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} = -1/2$

**4.2/4 Prove that $\lim_{x\to 0} \cos(1/x)$ does not exist but that $\lim_{x\to 0} x\cos(1/x) = 0$.**

[23] Page 116 of the textbook, section 4.2

**(a)** Showing $\lim_{x \to 0} \cos(1/x)$ does not exist:

*Proof:* We employ the divergence criteria: we need to find a sequence $(x_n)$ such that $x_n \to 0$ but $\cos(1/x_n) \not\to L$, for any L. Take $x_n := \frac{1}{\pi n}$. It is clear that $x \neq 0$ and $\lim(x_n) = 0$. Then consider $\cos(1/x_n) = \cos(\pi n)$. However, $\cos(\pi n)$ is not convergent as we can find two sequences (odd and even values of n that converge to -1 and 1 respectively) that converge to different quantities. Thus, the limit does not exist.

□

**(b)** Showing $\lim_{x \to 0} x \cos(1/x) = 0$:

*Proof:* Observe that $-1 \leq \cos(1/x) \leq 1 \implies -x \leq x\cos(1/x) \leq x \implies \lim_{x \to 0} -x \leq \lim_{x \to 0} x\cos(1/x) \leq \lim_{x \to 0} x \implies 0 \leq \lim_{x \to 0} x\cos(1/x) \leq 0$. Using the squeeze theorem, we find that the limit is 0.

□

**4.2/11 Determine whether the following limits exist in $\mathbb{R}$.**

**(a)** $\lim_{x \to 0} \sin(1/x^2),\ x \neq 0$

Similar to **4.2/4(a)**, we choose a sequence $(x_n) := \sqrt{\frac{2}{n\pi}}$. Observe that $x_n \to 0$ and $x_n \neq 0$. Then $\sin(1/x_n^2) = \sin(n\pi/2)$, which is not a convergent sequence. We can find two subsequences that converge to two different values.

**(b)** $\lim_{x \to 0} x \sin(1/x^2),\ x \neq 0$

Using the squeeze theorem identical to **4.2/4(b)**, the limit tends to 0.

**(c)** $\lim_{x \to 0} sgn \sin(1/x),\ x \neq 0$

Note that when $x < 0 \implies sgn(\sin(-1/x)) = sgn(-sin(1/x)) = -1$ since $-sin(1/x) < 0$. Similarly, if $x > 0 \implies sgn(\sin(1/x)) = 1$. So we see that The function can converge to two different values depending on whether sin is positive or negative. So we can find two sequences that do the job. Thus, the limit does not exist.

**(b)** $\lim_{x \to 0} \sqrt{x} \sin(1/x^2),\ x > 0$

Use the squeeze theorem once again, similar to **4.2/4(b)**. However, acknowledge the square root– note that the limit remains unchanged, by a result **4.2/15**. The limit tends to 0.

## 3.3 Some Extensions of the Limit Concept

[24]

[24] This subsection corresponds to section 4.3 in Bartle and Sherbert's Introduction to Real Analysis.

In this section, we shall present three types of extensions of the notion of a limit of a function that often occur. Since all the ideas here are closely parallel to ones we have already encountered, this section can be read easily.

### ONE-SIDED LIMITES

There are times when a function $f$ may not possess a limit at a point $c$, yet a limit does exist when the function is restricted to an interval on one side of the cluster point $c$.

For example, the signum function considered in Example 4.1.10(b) has no limit at $c = 0$. However, if we restrict the signum function to the interval $(0, \infty)$, the resulting function has a limit of 1 at $c = 0$. Similarly, if we restrict the signum function to the interval $(-\infty, 0)$, the resulting function has a limit of $-1$ at $c = 0$. These are elementary examples of right-hand and left-hand limits at $c = 0$.

**Definition 3.13.** *Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$.*

*(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that $L \in \mathbb{R}$ is a right-hand limit of $f$ at $c$ and we write*

$$\lim_{x \to c^+} f(x) = L$$

*if given any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \epsilon$.*

*(ii) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, c) = \{x \in A : x < c\}$, then we say that $L \in \mathbb{R}$ is a left-hand limit of $f$ at $c$ and we write*

$$\lim_{x \to c^-} f(x) = L$$

*if given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \epsilon$.*

**NOTE 3.14.** *1. The limits $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ are called one-sided limits of $f$ at $c$. It is possible that neither one-sided limit may exist. Also, one of them may exist without the other existing.*

*2. If $A$ is an interval with left endpoint $c$, then it is readily seen that $f : A \to \mathbb{R}$ has a limit at $c$ if and only if it has a right-hand limit at $c$. Moreover, in this case, the limit $\lim_{x \to c} f(x)$ and the right-hand limit $\lim_{x \to c^+} f(x)$ are equal.*

**Theorem 3.15.** *Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:*

*(i) $\lim_{x \to c} f(x) = L$.*

*(ii) For every sequence $(x_n)$ that converges to $c$ such that $x_n \in A$ and $x_n > c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to $L$.*

## INFINITE LIMITS

The function $f(x) = \frac{1}{x^2}$ for $x \to 0$ is not bounded on a neighborhood of 0, so it cannot have a limit in the sense of Definition 4.1.4. While the symbols $\infty$ and $-\infty$ do not represent real numbers, it is sometimes useful to be able to say that "$f(x) = \frac{1}{x^2}$ tends to $\infty$ as $x \to 0$."

**Definition 3.16.** *Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A$.*

*(i) We say that $f$ tends to $\infty$ as $x \to c$, and write*

$$\lim_{x \to c} f(x) = \infty,$$

*if for every $a \in \mathbb{R}$ there exists $\delta = \delta(a) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > a$.*

*(ii) We say that $f$ tends to $-\infty$ as $x \to c$, and write*

$$\lim_{x \to c} f(x) = -\infty,$$

*if for every $a \in \mathbb{R}$ there exists $\delta = \delta(a) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < a$.*

### Example

(a) $\lim_{x \to 0}(1/x^2) = \infty$. For, if $a > 0$ is given, let $\delta = 1/\sqrt{a}$. It follows that if $0 < |x| < \delta$, then $x^2 < 1/a$ so that $1/x^2 > a$.

(b) Let $g(x) = 1/x$ for $x \to 0$. The function $g$ does not tend to either $\infty$ or $-\infty$ as $x \to 0$. For, if $a > 0$, then $g(x) < a$ for all $x < 0$, so that $g$ does not tend to $\infty$ as $x \to 0$.

While many of the results in Sections 4.1 and 4.2 have extensions to this limiting notion, not all of them do since $\infty$ and $-\infty$ are not real numbers.

**Theorem 3.17.** *Let $A \subset \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A$. Suppose that $f(x) < g(x)$ for all $x \in A, x \neq c$.*

*(a) If $\lim_{x \to c} f(x) = \infty$, then $\lim_{x \to c} g(x) = \infty$.*

*(b) If $\lim_{x \to c} g(x) = -\infty$, then $\lim_{x \to c} f(x) = -\infty$.*

## LIMITS at INFINITY

Limits at infinity examine the behavior of functions as the input approaches positive or negative infinity.

**Definition 3.18.** *Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that:*

*(i) $L \in \mathbb{R}$ is the limit of $f(x)$ as $x \to \infty$ if*

$$\lim_{x \to \infty} f(x) = L$$

*if for every $\epsilon > 0$, there exists $M = M(\epsilon) > 0$ such that for all $x \in A$ with $x > M$, we have $|f(x) - L| < \epsilon$.*

*(ii)* $L \in \mathbb{R}$ *is the limit of* $f(x)$ *as* $x \to -\infty$ *if*

$$\lim_{x \to -\infty} f(x) = L$$

*if for every* $\epsilon > 0$*, there exists* $M = M(\epsilon) < 0$ *such that for all* $x \in A$ *with* $x < M$*, we have* $|f(x) - L| < \epsilon$*.*

**Examples**

(a) For the function $f(x) = \frac{1}{x}$, we find:

$$\lim_{x \to \infty} f(x) = 0.$$

(b) For $g(x) = 2x^2 + 3x - 1$, it follows that:

$$\lim_{x \to \infty} g(x) = \infty.$$

Similar to finite limits, the Squeeze Theorem applies for limits at infinity.

**Theorem 3.19** (**Squeeze Theorem**). *Let* $A \subset \mathbb{R}$*, and let* $f, g, h : A \to \mathbb{R}$ *such that* $f(x) \leq g(x) \leq h(x)$ *for all* $x \in A$*. If*

$$\lim_{x \to \infty} f(x) = L \quad \textit{and} \quad \lim_{x \to \infty} h(x) = L,$$

*then*

$$\lim_{x \to \infty} g(x) = L.$$

**SELECT EXERCISES** [25]

**4.3/3 Let** $f(x) := |x|^{-1/2},\ x \neq 0$**. Show that left-hand = right-hand limit =** $+\infty$

*Proof:* Note that when $x < 0 \implies f(x) := \frac{1}{\sqrt{x}}$ and $x > 0 \implies f(x) := \frac{1}{\sqrt{x}}$. This means that $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{1}{\sqrt{x}}$. Also note that as $x \to 0$, for $\sqrt{x} \geq x \implies \frac{1}{x} \leq \frac{1}{\sqrt{x}}$, $x \leq 1$. Since we know that $\lim_{x \to 0} 1/x = \infty \implies \lim_{x \to 0} 1/\sqrt{x} = \infty$, by a theorem in this section.

□

**4.3/4 Let** $c \in \mathbb{R}$ **and** $f$ **be defined for** $x \in (c, \infty)$ **and** $f(x) > 0,\ \forall x \in (c, \infty)$**. Show that** $\lim_{x \to c} f = \infty \iff \lim_{x \to c} 1/f = 0$.

[25] Page 123 of the textbook, section 4.3

*Proof:* ( $\Longrightarrow$ ) Assume $\lim_{x\to c} f = \infty$. Then for any constant $M > 0, \exists\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) > M$. We can rewrite $f(x) > M \implies f(x) - 0 > M \implies \frac{1}{f(x)} < \frac{1}{M} \implies |\frac{1}{f(x)} - 0| < \frac{1}{M}$ since $f(x) > 0$. Choose $\epsilon = 1/M$. Then by $\epsilon - \delta$ definition of a limit, this direction is shown.

( $\Longleftarrow$ ) Assume $\lim_{x\to c} 1/f = 0$. So, for any $\epsilon > 0, \exists\delta > 0$ such that if $|x - c| < \delta$, we have $|\frac{1}{f} - 0| < \epsilon$. Then we have that $\frac{1}{f} < \epsilon$, since $f(x) > 0 \implies 1/f > 0 \implies |1/f - 0| = 1/f$. We rewrite at $f > 1/\epsilon$. So, choose $M = 1/\epsilon > 0$. Then, for any $M > 0$, we have $f > M \implies \lim_{x\to c} f = \infty$.

$\square$

**4.3/11 Suppose that** $\lim_{x\to c} f(x) = L,\ L > 0$, **and** $\lim_{x\to c} g(x) = \infty$. **Show that** $\lim_{x\to c} f(x)g(x) = \infty$. **If** $L = 0$, **show by example that this conclusion may fail.**

**(a)** Show that $\lim_{x\to c} f(x)g(x) = \infty$:

*Proof:* Since $f(x) \to L \implies \forall\epsilon, \exists\delta_1 > 0$ s.t. $|x - c| < \delta_1 \implies |f(x) - L| < \epsilon$. We can write this fact as $L - \epsilon < f(x) < L + \epsilon$, for any $x$ where $f$ is defined; we write $f = f(x)$ and $g = g(x)$ for shorthand.

Since $g \to \infty$ as $x \to c$, by definition, we must have for every $M > 0, \exists\delta_2$ s.t. $|x - c| < \delta_2 > 0 \implies g > M$. Since $0 < L - \epsilon < f$, we can multiply with M to get $\implies 0 < M(L - \epsilon) < Mf < fg$, since $M < g$. So choose a new constant $N = M(L - \epsilon)$. So for any $N > 0, \exists\delta = \min\{\delta_1, \delta_2\}$ s.t. $|x - c| < \delta \implies fg > N$.

Thus, the limit goes to infinity.

$\square$

**(b)** $L = 0$:

*Proof:* Take $c = 0$, and two functions $f(x) := x$ and $g(x) := 1/x$. Then the product of the two functions is $fg = 1 \implies$ a constant, which is clearly not $\infty$.

$\square$

# 4 CONTINUOUS FUNCTIONS

## 4.1 Continuous Functions

[26]

[26] This subsection corresponds to section 5.1 in Bartle and Sherbert's Introduction to Real Analysis.

**Definition 4.1.** *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and $c \in A$. We say $f$ is* ***continuous at*** $c$ ***if, given any*** $\epsilon > 0$, $\exists \delta > 0$ ***s.t.*** $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

If not continuous, $f$ is **discontinuous at** $c$.

**Theorem 4.2.** *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A \iff$ given any $\epsilon$-neighborhood $V_\epsilon(f(c))$ of $f(c)$, $\exists \delta$-neighborhood $V_\delta(c)$ of $c$ such that if $x$ is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\epsilon(f(c))$, that is,*

$$f(A \cap V_\delta(c)) \subseteq V_\epsilon(c))$$

**Theorem 4.3 (Sequential Criterion for Continuity).** *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A \iff \forall (x_n) \in A$ and $(x_n) \to c$, the sequence $(f(x_n)) \to f(c)$.*

**Theorem 4.4 (Discontinuity Criterion).** *A function $f : A \to \mathbb{R}$ is discontinuous at a point $c \in A \iff \exists (x_n) \in A$ and $(x_n) \to c$, but the sequence $(f(x_n)) \not\to f(c)$.*

**Definition 4.5.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. If $B \subset A$ and $f$ is continuous at every point of $B \implies f$ is* ***continuous on the set*** $B$.

**SELECT EXERCISES** [27]

**5.1/7 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $c$ and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of $c$ such that if $x \in V_\delta(c) \implies f(x) > 0$.**
Since $f$ is continuous at $c$ and $f(c) > 0$, we use the $\varepsilon$-$\delta$ definition of continuity. By the continuity of $f$ at $c$, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

Now choose $\varepsilon = f(c)$. Since $f(c) > 0$, this choice of $\varepsilon$ is valid and positive. By the definition of continuity, there exists a $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < f(c).$$

This implies

$$-f(c) < f(x) - f(c) < f(c),$$

or equivalently,

$$0 < f(x) < 2f(c),$$

Thus, $f(x) > 0$ for all $x \in V_\delta(c) = (c - \delta, c + \delta)$.

**5.1/10 Show that the absolute value function $f(x) := |x|$ is continuous at every point $c \in \mathbb{R}$.**

[27] Page 129 of the textbook, section 5.1

*Proof:* To prove that the absolute value function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$, we need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$, it follows that $|f(x) - f(c)| < \epsilon$. That is, we need to show:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |x - c| < \delta \implies ||x| - |c|| < \epsilon.$$

Let $\epsilon > 0$. We will consider two cases for $c$.
**Case 1:** $c = 0$
In this case, $f(x) = |x|$ and $f(0) = |0| = 0$.
Clearly, we can choose $\delta = \epsilon$, and if $|x| < \delta$, then $|x| = ||x| - 0| < \epsilon$. Thus, the function is continuous at $c = 0$.
**Case 2:** $c \neq 0$
We now need to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $||x| - |c|| < \epsilon$.
By the triangle inequality, we know that:

$$||x| - |c|| \leq |x - c|.$$

Thus, if we choose $\delta = \epsilon$, then whenever $|x - c| < \delta$, we have:

$$||x| - |c|| \leq |x - c| < \epsilon.$$

This shows that $f(x) = |x|$ is continuous at $c \neq 0$.
Since the function is continuous at $c = 0$ and $c \neq 0$, we conclude that the absolute value function $f(x) := |x|$ is continuous at every point $c \in \mathbb{R}$.

□

**5.1/11 Let $K > 0$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$, $\forall x, y \in \mathbb{R}$. Show that $f$ is continuous at every point $c \in \mathbb{R}$.**

*Proof:* We are given that the function $f$ satisfies the Lipschitz condition with constant $K$:

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in \mathbb{R}.$$

To prove that $f$ is continuous at any point $c \in \mathbb{R}$, we must show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x-c| < \delta$ implies $|f(x)-f(c)| < \epsilon$.

Let $\epsilon > 0$. By the given Lipschitz condition, we have:

$$|f(x) - f(c)| \leq K|x - c|.$$

To ensure that $|f(x)-f(c)| < \epsilon$, it suffices to choose $\delta$ such that $K|x-c| < \epsilon$. This can be achieved by setting:

$$\delta = \frac{\epsilon}{K}.$$

Thus, whenever $|x - c| < \delta$, we have:

$$|f(x) - f(c)| \leq K|x - c| < K\delta = \epsilon.$$

Therefore, $f$ is continuous at $c$.

Since $c \in \mathbb{R}$ was arbitrary, this shows that $f$ is continuous at every point in $\mathbb{R}$.

$\square$

## 4.2 Combinations of Continuous Functions

[28]

**Theorem 4.6.** *Let $A \subseteq \mathbb{R}$, let $f$ and $g$ be functions on $A \to \mathbb{R}$, and let $b \in \mathbb{R}$. Suppose that $c \in A$ and that $f$ and $g$ are continuous at $c$.*

*(a) Then $f + g, f - g, fg, bf$ are continuous at $c$.*

*(b) If $h : A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0,\ \forall x \in A$, the quotient $f/h$ is continuous at $c$.*

We can apply this result to all points $c \in A$. Then we can modify the theorem to state that $f$ and $g$ are continuous on all of $A$.

**Theorem 4.7.** *Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$, and let $|f|$ be defined by $|f|(x) := |f(x)|,\ x \in A$.*

*(a) If $f$ is continuous at a point $c \in A$, then $|f|$ is continuous at $c$.*

*(b) If $f$ is continuous on $A$, then $|f|$ is continuous on $A$.*

**NOTE 4.8.** *We can say modify the above theorem to show that $\sqrt{f}$ is continuous as well.*

[28] This subsection corresponds to section 5.2 in Bartle and Sherbert's Introduction to Real Analysis.

## COMPOSITION OF CONTINUOUS FUNCTIONS

**Theorem 4.9.** *Let $A, B \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}, g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq f(B)$. If $f$ is continuous at a point $c \in A$ and $g$ is continuous at $b = f(c) \in B$, then the composition $g \circ f : A \to \mathbb{R}$ is continuous at $c$.*

We can modify this result to demonstrate that the composition is continuous on all of $A$.

## SELECT EXERCISES [29]

**5.2/3 Give an example of functions $f$ and $g$ that are both discontinuous at a point $c \in R$ such that (a) the sum $f + g$ is continuous at $c$, (b) the product $fg$ is continuous at $c$.**
Let $f(x) = (x)$ and $g(x) = -(x)$, where $(x)$ is the signum function, defined as:

$$(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Both $f$ and $g$ are discontinuous at $c = 0$, as the signum function is not continuous at $x = 0$.
**(a)** The sum $f(x) + g(x) = (x) + (-(x)) = 0$ for all $x \in \mathbb{R}$. Therefore, $f + g$ is the zero function, which is continuous at every point, including $c = 0$.
**(b)** The product $f(x)g(x) = (x)(-(x)) = -1$ for $x \neq 0$, and $f(0)g(0) = 0 \cdot 0 = 0$. Thus, $fg$ is continuous at $c = 0$, as there is no jump or discontinuity in the product function.

**5.2/7 Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.**
Let $f(x)$ be the function defined as:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ -1, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]. \end{cases}$$

This function is discontinuous at every point in $[0, 1]$, because both rationals and irrationals are dense in $[0, 1]$, and $f$ takes different values for rationals and irrationals.
However, $|f(x)| = 1$ for all $x \in [0, 1]$, which is a constant function and hence continuous on $[0, 1]$.

**5.2/8 Let $f, g$ be continuous $\mathbb{R} \to \mathbb{R}$ and suppose that $f(r) = g(r),\ \forall r \in \mathbb{Q}$. Is it true that $f(x) = g(x),\ \forall x \in \mathbb{R}$?**

[29] Page 133 of the textbook, section 5.2

Yes, it is true that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

*Proof:* Suppose $f(x_0) \neq g(x_0)$ for some $x_0 \in \mathbb{R}$. Then $|f(x_0) - g(x_0)| > 0$ and thus we can take $\epsilon = |f(x_0) - g(x_0)|/2$ in the definition of continuity. Since $f$ is continuous, $\exists \delta_1 > 0$ s.t.

$$x \in (x_0 - \delta_1, x_0 + \delta_1) \quad \Rightarrow \quad |f(x) - f(x_0)| < \frac{|f(x_0) - g(x_0)|}{2} \tag{1}$$

Since $g$ is continuous, $\exists \delta_2 > 0$ s.t.

$$x \in (x_0 - \delta_2, x_0 + \delta_2) \quad \Rightarrow \quad |g(x) - g(x_0)| < \frac{|f(x_0) - g(x_0)|}{2} \tag{2}$$

Take a rational number $r \in (x_0 - \delta, x_0 + \delta)$, where $\delta = \min\{\delta_1, \delta_2\}$. It follows from (1) and (2) that

$$\begin{aligned}
|f(x_0) - g(x_0)| &\leq |f(x_0) - f(r)| + |f(r) - g(r)| + |g(r) - g(x_0)| \\
&< \frac{|f(x_0) - g(x_0)|}{2} + 0 + \frac{|f(x_0) - g(x_0)|}{2} \\
&= |f(x_0) - g(x_0)|,
\end{aligned}$$

which is a contradiction.

□

## 4.3 Continuous Functions on Intervals

[30]

**Definition 4.10.** *A function $f : A \to \mathbb{R}$ is said to be **bounded on** $A$ if there exists a constant $M > 0$ such that $|f(x)| \leq M, \ \forall x \in A$.*

In other words, a function is bounded on a set if its range is abounded set in $\mathbb{R}$. A function $f$ is not bounded on the set $A$ if given any $M > 0$, $\exists x_M \in A$ s.t. $|f(x_M)| > M$. Then, $f$ is **unbounded**.

**Theorem 4.11 (Boundedness Theorem).** *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ is bounded on $I$.*

### MAX-MIN THEOREM

**Definition 4.12.** *Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$. We say that $f$ has an **absolute maximum** on $A$ if $\exists x^* \in A$ s.t.*

$$f(x^*) \geq f(x), \ \forall x \in A.$$

*We say that $f$ has an absolute minimum on $A$ if $\exists x_* \in A$ s.t.*

$$f(x_*) \geq f(x), \ \forall x \in A.$$

[30] This subsection corresponds to section 5.3 in Bartle and Sherbert's Introduction to Real Analysis.

**NOTE 4.13.** *A continuous function on a set $A$ does not necessarily have an absolute maximum or an absolute minimum on the set.*

**Theorem 4.14 (Maximum-Mininum Theorem).** *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then $f$ has an absolute maximum and an absolute minimum.*

## BOLZANO'S THEOREM

**Theorem 4.15.** *Let $I$ be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then $\exists c \in I$ between $a$ and $b$ such that $f(c) = k$.*

**Corollary 4.16.** *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfies*

$$inf\ f(I) \leq k \leq sup\ f(I),$$

*then there exists a number $c \in R$ s.t. $f(c) = k$.*

**Theorem 4.17.** *Let $I$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) := \{f(x) : x \in I\}$ is a closed bounded interval.*

**Theorem 4.18 (Preservations of Intervals Theorem).** *Let $I$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.*

## SELECT EXERCISES [31]

**5.3/1 Let $I := [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function such that $f > 0,\ \forall x \in I$. Prove $\exists \alpha > 0$ s.t. $f(x) \geq \alpha,\ \forall x \in I$.**

*Proof:* Using the Maximum-Minimum theorem from this section, we know that there exists an absolute minimum $x_*$, such that $f(x_*) \geq f(x)$. Since $f(x) > 0,\ \forall x \in I \implies f(x_*) > 0$. Choose $\alpha = f(x_*) \implies 0 < \alpha \leq f(x),\ \forall x \in I$.

$\square$

**5.3/3 Let $I := [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function on $I$ such that $\forall x \in I,\ \exists y \in I$ s.t. $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove $\exists c \in I$ s.t. $f(c) = 0$.**

*Proof:* Let $(x_n)$ be a sequence. If $x_1 \in I$, then $\exists x_2$ s.t. $f(x_2) \leq \frac{1}{2}f(x_1)$. Continuing, we see that for $n$-terms, we get $f(x_n) \leq \frac{1}{2^{n-1}}f(x_1)$. Since $\frac{1}{\lim 2^{n-1}} \to 0 \implies |f(x_n)| \to 0$. Then using Bolzano Weierstrass theorem, we can find a subsequence for which $c$ is the limit. Then using $f$'s continuity, we have that $f(c) = \lim f(x_{n_k}) = 0$. Hence, proved.

$\square$

[31] Page 140 of the textbook, section 5.3

**5.3/13 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $\lim\ f = 0$ as $n \to \pm\infty$. Prove that $f$ is bounded on $\mathbb{R}$ and attain either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and minimum need not be attained.**

*Proof:* From $\lim_{n\to\infty} f = 0$, we have

$$\forall \epsilon > 0,\ \exists \beta > 0 \text{ s.t. } \text{ if } x > \beta \implies |f(x) - 0| < \epsilon$$

Similarly, from $\lim_{n\to-\infty} f = 0$, we have

$$\forall \epsilon > 0,\ \exists \alpha < 0 \text{ s.t. } \text{ if } x < \alpha \implies |f(x) - 0| < \epsilon$$

We end up with if $x \in (-\infty, a) \cup (b, \infty)$, then $|f(x)| < \epsilon$. If $f(x) = 0$, we are done, the function is bounded by $\epsilon$. However, if $f(x) \neq 0 \implies \exists c$ s.t. $f(c) \neq 0$. Choose $M = f(c) = \epsilon$. Then $|f(x)| < M$, showing that $f$ is bounded.

We end up with an interval $I := [a, b]$ for which we are unsure what is going on. However, since $f$ is continuous on $\mathbb{R}$, $f$ is continuous on $I$. Thus, by the Min-Max Theorem, we know there exists an absolute maximum or minimum on $I$.

□

For a counter-example, think of the normal distribution with mean 0 and standard deviation 1. Then as $n \to \pm\infty$, we see the distribution tends to 0. However, we only have an absolute maximum at mean = 0.

**5.3/14 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $\beta \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) < \beta$, then there exists a $\delta$-neighborhood $U$ of $x_0$ such that $f(x) < \beta,\ \forall x \in U$.**

*Proof:* Define a new function $F(x) = f(x) - \beta$. Then $\lim_{x\to x_0} F(x) < 0$. Using a result from the previous chapter, we know that $\exists U(x_0)$ s.t. $F(x) < 0 \implies f(x) < \beta$.

□

## 4.4 Uniform Continuity

[32]

**Definition 4.19.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. We say that $f$ is **uniformly continuous** on $A$ if $\forall \epsilon > 0\ \exists \delta(\epsilon) > 0$ s.t. if $x, u \in A$ satisfy $|x - u| < \delta(\epsilon) \implies |f(x) - f(u)| < \epsilon$.*

**Theorem 4.20 (Nonuniform Continuity Criteria).** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Then the following are equivalent:*

[32] This subsection corresponds to section 5.4 in Bartle and Sherbert's Introduction to Real Analysis.

*(i)* $f$ *is not uniformly continuous on* $A$.

*(ii)* $\exists \epsilon_0$ *s.t.* $\forall \delta > 0$ *there are points* $x_\delta, u_\delta \in A$ *s.t.* $|x_\delta - u_\delta| < \delta$ *and* $|f(x_\delta) - f(u_\delta)| \geq \epsilon_0$.

*(iii)* $\exists \epsilon_0$ *and* $(x_n), (u_n) \in A$ *s.t.* $\lim(x_n - u_n) = 0$ *and* $|f(x_n) - f(u_n)| \geq \epsilon_0$ $\forall n \in \mathbb{N}$

**Theorem 4.21.** *Let* $I$ *be a closed bounded interval and let* $f : I \to \mathbb{R}$ *be continuous on* $I$. *Then* $f$ *is uniformly continuous on* $I$.

### LIPSCHITZ FUNCTIONS

If a uniformly continuous function is given on a set that is not a closed-bounded interval, then it is sometimes difficult to establish its uniform continuity. However, there is a condition that frequently occurs that is sufficient to guarantee uniform continuity.

**Definition 4.22.** *Let* $A \subseteq \mathbb{R}$ *and* $f : A \to \mathbb{R}$. *If there exists* $K > 0$ *such that*

$$|f(x) - f(u)| \leq K|x - u|$$

*for all* $x, u \in A$, *then* $f$ *is said to be a* ***Lipschitz function*** *(or to satisfy a* ***Lipschitz condition****) on* $A$.

**Theorem 4.23.** *If* $f : A \to \mathbb{R}$ *is a Lipschitz function, then* $f$ *is uniformly continuous on* $A$.

### SELECT EXERCISES [33]

**5.4/2 Show that the function $f(x) := 1/x^2$ is uniformly continuous on $A := [1, \infty)$, but that it is not uniformly continuous on $B := (0, \infty)$.**

- Show uniform continuity on A:
  We have

  $$|f(x) - f(u)| = |1/x^2 - 1/u^2| = \frac{|u + x|}{u^2x^2}|u - x| = (\frac{1}{ux^2} + \frac{1}{u^2x})|x - u| \leq 2|x - u|,$$

  So, $f$ is Lipschitz. Observe that since $A := [1, \infty) \implies 1 \leq 1/y,\ \forall y \in A$.

- Show discontinuity on B:
  Using discontinuity criteria from this section, we can find two sequences $x_n = 1/\sqrt{n}$ and $u_n = 1/\sqrt{n+1} \implies \lim(x_n - u_n) = 0$. However, $|f(x_n) - f(u_n)| = |n - n - 1| = 1$. If we choose $\epsilon_0 = 1/2$, we see that $f$ is not continuous on $B$.

**5.4/3 Use the Nonuniform Continuity Criterion 5.4.2 to show that the following functions are not uniformly continuous on the given sets.**

[33] Page 148 of the textbook, section 5.4

(a) $f(x) := x^2,\ A := [0, \infty)$
Choose two sequences: $x_n := n+1/n$ and $u_n := n$. Follow the same procedure as the second part of the previous problem.

(a) $g(x) := sin(1/x),\ B := (0, \infty)$
We must find two sequences again. Observe that we have a reciprocal within the $sin$ function, thus, our sequences should be reciprocals– we end up with no reciprocals in the $sin$ function. Choose $x_n := 1/2n\pi$ and $u_n := 1/(2n\pi+\pi/2)$.

**5.4/6 Show that of $f$ and $g$ are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are both bounded on $A$, then their product $fg$ is uniformly continuous on $A$.**

*Proof:* Suppose $f$ and $g$ are uniformly continuous and bounded on $A$. Let $M_1$ be the maximum $|f|$ attains and $M_2$ be the maximum that $|g|$ attains. Then define $M := \sup\{M_1, M_2\}$. So, when $|x - u| < \delta$, we have

$$|f(x) - f(u)| < \epsilon/2M \text{ and}$$

$$|g(x) - g(u)| < \epsilon/2M \text{ and}$$

So if we have $|x - u| < \delta$, consider

$$|f(x)g(x) - f(u)g(u)| = |f(x)g(x) - f(x)g(u) + f(x)g(u) - f(u)g(u)|$$

$$\leq |f(x)||g(x) - g(u)| + |g(u)||f(x) - f(u)|$$

$$< \frac{\epsilon}{2}(\frac{f}{M} + \frac{g}{M}) \leq \frac{\epsilon}{2}(\frac{M}{M} + \frac{M}{M}) = \frac{\epsilon}{2}(2) = \epsilon$$

□

**5.4/7 If $f(x) := x$ and $g(x) := \sin x$, show that both $f$ and $g$ are uniformly continuous on $\mathbb{R}$, but that their product $fg$ is not uniformly continuous on $\mathbb{R}$.**

*Proof:*

- Continuity of $f$:
  Choose $\delta = \epsilon$. Then $|x-u| = |f(x) - f(u)| < \delta = \epsilon$

- Continuity of $g$:
  We get $|f(x) - f(u)| = |\sin x - \sin u| = |2\cos\frac{x+u}{2}\sin\frac{x-u}{2}| \leq 2|\sin\frac{x-u}{2}|$. We also know that $|\sin x| \leq |x|$. So, $\forall \epsilon > 0$, we need a $\delta$ such that
  $$|x-u| < \delta \implies 2|\sin\frac{x-u}{2}| < \epsilon$$
  Now,
  $$2|\sin\frac{x-u}{2}| \leq 2|\frac{x-u}{2}| < 2\delta$$
  Choose $\delta = \epsilon/2$.

- Discontinuity of $fg$:
  We have $fg = x\sin x$. We choose two sequences $x_n := \pi n + 1/n$ and $u_n := \pi n$. We see that $\lim_{n\to\infty} x_n - u_n = \lim_{n\to\infty} \pi n + 1/n - \pi n = \lim_{n\to\infty} 1/n = 0$. However, $\lim_{n\to\infty} f(x_n) - f(u_n) = \lim_{n\to\infty}(\pi n + 1/n)\sin(\pi n + 1/n) - (\pi n)\sin(\pi n) = \lim_{n\to\infty} \pi(-1)^n \neq 0$.

□

**5.4/9 If $f$ is uniformly continuous on $A \subseteq \mathbb{R}$, and $|f(x)| \geq k > 0,\ \forall x \in A$, show that $1/f$ is uniformly continuous on A.**

*Proof:* Since $|f(x)| \geq k > 0 \implies 0 < \frac{1}{|f|} \leq \frac{1}{k}$. It is clear that $1/f$ is bounded by $1/k$. Now, define a new function $g := 1/f$. For $|x-u| < \delta$, we have $|f(x) - f(u)| < \epsilon k^2$. Consider

$$|g(x) - g(u)| = |\frac{1}{f(x)} - \frac{1}{f(u)}| = |\frac{f(u) - f(x)}{f(x)f(u)}| \leq |\frac{f(u) - f(x)}{k^2}| < \frac{\epsilon k^2}{k^2} = \epsilon$$

So, $1/f$ is continuous.

□

# 5 DIFFERENTIATION

## 5.1 The Derivative

[34]

**Definition 5.1.** *Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and $x \in I$. We say that a real number $L$ is the* ***derivative of** $f$ **at*** *$c$ if given any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ s.t. if $x \in I$*

[34] This subsection corresponds to section 6.1 in Bartle and Sherbert's Introduction to Real Analysis.

*satisfies* $0 < |x - c| < \delta(\epsilon)$*, then*

$$\left|\frac{f(x) - f(c)}{x - c} - L\right| < \epsilon$$

*in this case we say that* $f$ *is* ***differentiable*** *at* $c$*. and we write* $f'(c)$ *for* $L$*. It is also given by the limit*

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

*provided the limit exists.*

**Theorem 5.2.** *If* $f : I \to \mathbb{R}$ *has a derivative at* $c \in I$*, then* $f$ *is continuous at* $c$*.*

**NOTE 5.3.** *The continuity of* $f$ *at a point does not assure the existence of the derivative at that point.*

**Theorem 5.4.** *Let* $I \subseteq \mathbb{R}$ *be an interval, let* $c \in I$*, and let* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *be functions that are differentiable at* $c$*. Then:*

*(a) If* $a \in \mathbb{R}$*, then the function* $af$ *is differentiable at* $c$*, and*

$$(af)'(c) = af'(c).$$

*(b) The function* $f + g$ *is differentiable at* $c$*, and*

$$(f + g)'(c) = f'(c) + g'(c).$$

*(c) (Product Rule) The function* $fg$ *is differentiable at* $c$*, and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

*(d) (Quotient Rule) If* $g(c) \neq 0$*, then the function* $\frac{f}{g}$ *is differentiable at* $c$*, and*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

**CHAIN RULE**

**Theorem 5.5 (Carathéodory's Theorem).** *Let* $f$ *be defined on an interval* $I$ *containing the point* $c$*. Then* $f$ *is differentiable at* $c \iff$ *there exists a function* $\varphi$ *on* $I$ *that is continuous at* $c$ *and satisfies*

$$f(x) - f(c) = \varphi(x)(x - c),\ x \in I$$

*In this case, we have* $\varphi(c) = f'(c)$*.*

**Theorem 5.6 (Chain Rule).** *Let* $I, J$ *be intervals in* $\mathbb{R}$*, let* $g : I \to \mathbb{R}$ *and* $f : J \to \mathbb{R}$ *be functions such that* $f(J) \subseteq I$*, and let* $c \in J$*. If* $f$ *is differentiable at* $c$ *and* $g$ *is differentiable at* $f(c)$*, then the composite function* $g \circ f$ *is differentiable at* $c$ *and*

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

**INVERSE FUNCTIONS**

**Theorem 5.7.** *Let $I$ be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on $I$. Let $J := f(I)$, and let $g : J \to \mathbb{R}$ be the strictly monotone and continuous function inverse to $f$. If $f$ is differentiable at $c \in I$ and $f'(c) \neq 0$, then $g$ is differentiable at $d := f(c)$, and*

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}.$$

**Theorem 5.8.** *Let $I$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone on $I$. Let $J := f(I)$, and let $g : J \to \mathbb{R}$ be the function inverse to $f$. If $f$ is differentiable on $I$ and $f'(x) \neq 0$ for $x \in I$, then $g$ is differentiable on $J$, and*

$$g' = \frac{1}{f' \circ g}.$$

**SELECT EXERCISES** [35]

**6.1/1 Use the definition to find the derivative of each of the following functions:**

(a) $f(x) := x^3,\ x \in \mathbb{R}$
$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2$

(b) $g(x) := 1/x,\ x \in \mathbb{R}, x \neq 0$
$\lim_{h\to 0} \frac{1/(x+h) - 1/x}{h} = \lim_{h\to 0} \frac{-h}{h(x)(x+h)} = -1/x^2$

(c) $h(x) := \sqrt{x},\ x > 0$
$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$

(d) $k(x) := 1/\sqrt{x},\ x > 0$
$\lim_{h\to 0} \frac{1/\sqrt{(x+h)} - 1/\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{2x\sqrt{x}}$

**6.1/2 Show that $f(x) := x^{1/3}, x \in \mathbb{R}$ is not differentiable at $x = 0$.**

*Proof:* Suppose, for contradiction, that $f$ is differentiable at $x = 0$. Then by Carathèodory's theorem, $\exists \varphi$ s.t.

$$f(x) - f(0) = \varphi(x)(x - 0) \implies \varphi(x) = x^{-2/3}$$

Then,

$$\varphi(0) = \lim_{x\to 0} \frac{1}{x^{2/3}},$$

which does not exist– a contradiction. Thus, $f(x)$ is not differentiable at $x = 0$.

$\square$

[35] Page 170 of the textbook, section 6.1

**6.1/4 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for x rational, $f(x) := 0$ for x irrational. Show that $f$ is differentiable at $x = 0$, and find $f'(0)$.**
*Proof:* We $f$ defined as above. Then we must find

$$f'(o) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

If $h \in \mathbb{Q} \implies \frac{f(h)}{h} = h$. If $h \in \mathbb{R} - \mathbb{Q} \implies \frac{f(h)}{h} = 0$. So it is true that $0 < \frac{f(h)}{h} \leq h$. So, we can rewrite

$$0 < \lim_{h\to 0} \frac{f(h)}{h} \leq \lim_{h\to 0} h = 0.$$

By the squeeze theorem, we have that $f'(0) = 0$.

□

**6.1/10 Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) := 0$. Show that $g$ is differentiable $\forall x \in \mathbb{R}$. Also, show that the derivative $g'$ is not bounded on the interval $[-1, 1]$.**

- Show $g$ is differentiable:

  *Proof:* When $x \neq 0 \implies g'(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2)$, so $g$ is differentiable. Consider $x = 0$:

$$\lim_{h\to 0} \frac{h^2 \sin(1/h^2) - 0}{h} = \lim_{h\to 0} h\sin(1/h^2) - 0 = 0$$

□

- Unboundedness:

  *Proof:* We have $g'(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2)$. Consider the sequence $x_n := \frac{1}{\sqrt{2n\pi}}$. Then

$$g'(x_n) = 2x_n \sin(1/x_n^2) - (2/x_n)\cos(1/x_n^2) = \frac{2}{\sqrt{2n\pi}}\sin(2n\pi) - (2\sqrt{2n\pi})\cos(2n\pi)$$

$$= \frac{2}{\sqrt{2n\pi}} - (2\sqrt{2n\pi})$$

  Note that as $n \to \infty \implies g' \to -\infty$, thus, it is not bounded on the interval $[-1, 1]$.

□

**6.1/14 Given $h(x) := x^3 + 2x + 1$, $\forall x \in \mathbb{R}$ has an inverse $h^{-1}$ on $\mathbb{R}$, find the value of $(h^{-1})'(y)$ at points $x = 0, 1, -1$.**

We have $h(x)$ given, so we can find $h'(x) := 3x^2 + 2 \neq 0, \ \forall x \in \mathbb{R}$. Also note that $h$ is a strictly increasing and monotone function due it being the sum of two strictly increase functions. So, by a theorem in this section, we know the inverse is differentiable at each point and that it is given by $(h^{-1})'(x) = 1/h'(x)$.

At $x = 0 \implies (h^{-1})'(0) = 1/h'(0) = 1/2$.
At $x = 1 \implies (h^{-1})'(1) = 1/h'(1) = 1/5$.
At $x = -1 \implies (h^{-1})'(-1) = 1/h'(-1) = 1/5$.

**6.1/15 Given that the restriction of the cosine function cos to $I := [0, \pi]$ is strictly decreasing and $\cos 0 = 1$ and $\cos \pi = -1$, let $J := [-1, 1]$, and let $\arccos : J \to \mathbb{R}$ be the function inverse to the restriction of $\cos$ to $I$. Show that $\arccos$ is differentiable on $(-1, 1)$ and $D \arccos y = (-1)/(1 - y^2)^{1/2}, \ y \in (-1, 1)$. Show that $\arccos$ is not differentiable at $-1$ and $1$.**

*Proof:* First note that $\arccos y = x \implies y = \cos x$. We know that the inverse exists and that $D \cos x = -\sin x \neq 0$, for $\cos x \in (-1, 1)$. Thus, we can find, using the same theorem from the previous problem, that

$$D \arccos y = \frac{1}{D \cos x} = \frac{1}{-\sin x} = \frac{-1}{\sqrt{s - \cos^2 x}} = \frac{-1}{\sqrt{1 - y^2}}$$

However, note that when $y = \pm 1 \implies \frac{-1}{sin(0 \text{ or } \pi)}$ which is undefined, thus arccos is not differentiable at $y = \pm 1$.

□

**6.1/14 Given that the restriction of $\tan$ to $I := [-\pi/2, \pi/2]$ is strictly increasing and that $\tan(I) = \mathbb{R}$, let $\arctan : \mathbb{R} \to \mathbb{R}$ be the function inverse to the restriction of $\tan$ to $I$. Show that $\arctan$ is differentiable on $\mathbb{R}$ and $D \arctan y = (1)/(1+y^2), \ y \in \mathbb{R}$.**

*Proof:* First note that $\arctan y = x \implies y = \tan x$. Also, $D \tan x = \sec^2 x \neq 0, \ x \in I$. Thus, we can find using the same theorem that

$$D \arctan y = \frac{1}{D \tan x} = \frac{1}{\sec^2 x} = \frac{1}{1 + tan^2 x} = \frac{-1}{1 + y^2}$$

Hence, proved.

□

## 5.2 Mean Value Theorem

[36]

The Mean Value Theorem, which relates the values of a function to the values of its derivative, is one of the most useful results in real analysis.

Recall the notion of a **relative maximum/minimum**. If a function $f$ has either, then

[36]This subsection corresponds to section 6.2 in Bartle and Sherbert's Introduction to Real Analysis.

we say that $f$ has a **relative extremum**. We find these points by examining the zeros of the derivative.

**NOTE 5.9.** *Examining the zeros to find relative extrema only applies to **interior** points of an interval. The endpoints provide a unique circumstance– though they may be the local minimum or maximum, their derivative need not be zero.*

**Theorem 5.10** (**Interior Extremum Theorem**). *Let $c \in I$ be an interior point at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of $f$ at $c$ exists, then $f'(c) = 0$.*

**Corollary 5.11.** *Let $f : I \to \mathbb{R}$ be continuous on $I$ and suppose that $f$ has a relative extremum at an interior point $c$. Then either the derivative of $f$ at $c$ does not exist, or it is equal to 0.*

**Theorem 5.12** (**Rolle's Theorem**). *Suppose that $f$ is continuous on $I := [a, b]$, that f' exists at every point in $(a, b)$, and that $f(a) = f(b) = 0$. Then there exists at least one point $c \in (a, b)$ s.t. $f'(c) = 0$.*

From the theorem above, we can obtain the famous Mean Value Theorem.

**Theorem 5.13.** *Suppose that $f$ is continuous on a closed interval $I$, and that $f$ has a derivative in the open interval $(a, b)$. Then there exists at least one point $c \in (a, b)$ s.t.*

$$f(b) - f(a) = f'(c)(b - a).$$

The geometric view of the MVT is that there is some point on the curve $y = f(x)$ at which the tangent line is parallel to the line segment through the points $(a, f(a)), (b, f(b))$. Thus it is easy to remember the statement of the MVT by drawing appropriate diagrams. Naturally, we have many consequence results:

**Theorem 5.14.** *Suppose that $f$ is continuous on the closed interval $I$, that $f$ is differentiable on the open interval, and that $f' = 0,\ x \in (a, b)$. Then $f$ is constant on $I$.*

**Corollary 5.15.** *Suppose that $f$ and $g$ are continuous on $I$, that they are differentiable on the open interval, and that $f' = g',\ \forall x \in (a, b)$. Then there exists a constant $C$ such that $f = g + C$ on $I$.*

**Theorem 5.16.** *Let $f$ be differentiable on $I$. Then:*

*(a) $f$ is increasing on $I \iff f'(x) \geq 0\ \forall x \in I$.*

*(b) $f$ is decreasing on $I \iff f'(x) \leq 0\ \forall x \in I$.*

**Theorem 5.17.** *Let $f$ be differentiable on $I$ and $c$ be an interior point of $I$. Assume that $f$ is differentiable on $(a, c)$ and $(c, b)$ Then:*

*(a) If there is a neighborhood $(c-\delta, c+\delta) \subseteq I$ s.t. $f'(x) \geq 0$ for $c-\delta < x < c$ and $f'(x) \leq 0$ for $c < x < c + \delta$, then $f$ has a relative maximum at $d$.*

*(b) If there is a neighborhood $(c-\delta, c+\delta) \subseteq I$ s.t. $f'(x) \leq 0$ for $c-\delta < x < c$ and $f'(x) \geq 0$ for $c < x < c + \delta$, then $f$ has a relative minimum at $d$.*

**INEQUALITIES**

One very important use of the Mean Value Theorem is to obtain certain inequalities. Whenever information concerning the range of the derivative of a function is available, this information can be used to deduce certain properties of the function itself.

**SELECT EXERCISES** [37]

**6.2/1 For each of the following functions on the reals, find the points of relative extrema, the intervals on which the function is increasing, and those on which it is decreasing.**

(c) $h(x) := x^3 - 3x - 4$

We find the derivative $h'(x) = 3x^2 - 3$. Set $h'$ equal to 0 to find the critical points $\pm 1$. Using the second derivative test ($h''(x) = 6x$) to obtain at $x = 1 \implies h''(1) = 6 > 0 \implies$ relative minimum. Similarly, $x = -1$ is found to be a relative maximum. After knowing what kind of extrema points $x = \pm 1$, we can deduce that $x = -1 \implies$ the left is increasing and the right is decreasing $\implies (-\infty, -1)$ is increasing and $(-1, 1)$ is decreasing. Similarly, we find that $(1, \infty)$ is increasing.

(d) $k(x) := x^4 + 2x^2 - 4$

Doing what we did above, we see that $k'(x) = 0 \implies 4x(x^2 + 1) = 0$. Since $k$ is defined on $\mathbb{R}$. The only possible critical point is $x = 0$. Using the second derivative test, $k''(0) = 4 > 0 \implies$ relative minimum. This leads us to $(-\infty, 0)$ is decreasing and $(0, \infty)$ is increasing.

**6.2/4 Let $a_1, \ldots, a_n \in \mathbb{R}$ and let $f$ be defined on $\mathbb{R}$ by**

$$f(x) := \sum_{i=1}^{n} (a_i - x)^2, \ \ x \in \mathbb{R}$$

**Find the unique point of relative minimum for $f$.**

First, we find the derivative:

$$f'(x) = \sum_{i=1}^{n} 2(a_i - x)(-1) = \sum_{i=1}^{n} -2(a_i - x) = 2nx - 2\sum_{i=1}^{n} a_i$$

Setting the derivative equal to 0, we see that

$$f'(x) = 0 \implies 2nx = 2\sum_{i=1}^{n} a_i \implies x = \frac{1}{n}\sum_{i=1}^{n} a_i$$

which appears to be the average of $a_1, \ldots, a_n$.

**6.2/6 Use the MVT to prove that $|\sin x - \sin y| \le |x - y|, \ \forall x, y \in \mathbb{R}$.**

[37] Page 179 of the textbook, section 6.2

*Proof:* By the MVT, $\exists c \in (x, y)$ s.t.

$$\frac{\sin x - \sin y}{x - y} = \cos c$$

Since $|\cos c| \leq 1 \implies$

$$\left|\frac{\sin x - \sin y}{x - y}\right| \leq 1 \implies |\sin x - \sin y| \leq |x - y|$$

□

**6.2/7 Use the MVT to prove that $(x-1)/x < \ln x < x - 1,\ x > 1$.**
*Proof:* Similar to the question above, using the MVT, we have

$$\frac{\ln x - \ln 1}{x - 1} = \frac{1}{c},\ c \in (1, x)$$

for $f$ on $[1, x]$. Since $f'(x) = 1/x$, we have

$$\frac{1}{x} < \frac{\ln x - 0}{x - 1} < 1 \implies \frac{x-1}{x} < \ln x < x - 1$$

□

**6.2/9 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := 2x^4 + x^4 \sin(1/x), x \neq 0$ and $f(0) := 0$. Show that $f$ has an absolute minimum at $x = 0$, but that its derivative has both positive and negative values in every neighborhood of 0.**
*Proof:* We find $f'(x) = 8x^3 + 4x^3 \sin(1/x) - x^2 \cos(1/x)$. We see that $f'(0) = 0 \implies x = 0$ is a relative extremum. Also, note that $f = x^4(2+\sin(1/x))$. Observe that $\sin(1/x) \in [-1, 1] \implies (2 + \sin(1/x)) \in [1, 3]$. Thus, the behaviour of $f$ is determined by $x^4$. So, as $x \to 0 \implies f \to 0$. Thus, $x = 0$ is an absolute minimum.
Observe that in the derivative, we have $\sin$ and $\cos$ terms that oscillate in sign. Thus, $f'$ has both positive and negative values in every neighborhood.

□

**6.2/10 Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + 2x^2 \sin(1/x), x \neq 0$ and $g(0) := 0$. Show that $g'(0) = 1$, but in every neighborhood of the derivative $g'(x)$ takes on both positive and negative values. Thus $g$ is not monotonic in any neighborhood of 0.**

*Proof:* If $x = 0$, then $g'(x) = lim_{x \to 0}(g(x) - g(0))/x = lim_{x \to 0}(1 + 2x \sin(1/x)) = 1$. If $x \neq 0$, we can find its derivative $g'(x) := 1 + 4x \sin(1/x) - 2\cos(1/x)$. We can observe that $\lim_{x \to 0} 1 + 4x \sin(1/x) = 1$. We now analyze the sign of $g'(x)$ near $x = 0$. Notice that:
- When $x = \frac{1}{2n\pi}$, where $n \in \mathbb{N}$, we have $\sin(1/x) = \sin(2n\pi) = 0$, and $\cos(1/x) = \cos(2n\pi) = 1$. Therefore:

$$g'\left(\frac{1}{2n\pi}\right) = 1 - 2 = -1.$$

So, $g'(x) < 0$ for $x = \frac{1}{2n\pi}$.
- When $x = \frac{1}{(4n+1)\pi}$, we have $\sin(1/x) = \sin((4n+1)\pi) = 1$, and $\cos(1/x) = \cos((4n+1)\pi) = -1$. Therefore:

$$g'\left(\frac{1}{(4n+1)\pi}\right) = 1 + 4\left(\frac{1}{(4n+1)\pi}\right) - 2(-1) > 0.$$

So, $g'(x) > 0$ for $x = \frac{1}{(4n+1)\pi}$.
Thus, in every neighborhood of $x = 0$, the derivative $g'(x)$ takes on both positive and negative values. Therefore, $g(x)$ is not monotonic in any neighborhood of $x = 0$.

□

**6.2/15 Let $I$ be an interval. Prove that if $f$ is differentiable on $I$ and if the derivative $f'$ is bounded on $I$, then $f$ satisfies a Lipschitz condition on $I$.**

*Proof:* Let $I := [a, b]$. Since $f$ is differentiable, we can find $f'(c)$ for $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Since the derivative $f'$ is bounded, we have that $|f'| \leq M$ for $M > 0$. Then

$$f(b) - f(a) \leq M(b - a)$$

showing that we satisfy Lipschitz condition. We know that this demonstrates continuity.

□

**6.2/20 Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, and that $f(0) = 0, f(1) = 1, f(2) = 1$.**

(a) Show that there exists $c_1 \in (0,1)$ s.t. $f'(c_1) = 1$.
Since $f$ is differentiable and continuous, we can apply the MVT to show that $\exists c_1 \in (0,1)$ such that

$$f(1) - f(0) = f'(c_1)(1-0) \implies f'(c_1) = 1$$

(b) Show that there exists $c_2 \in (1,2)$ s.t. $f'(c_2) = 0$.
Using a similar argument as above, we can find $c_2 \in (1,2)$.

(c) Show that there exists $c \in (0,2)$ s.t. $f'(c) = 1/3$.
Since $f$ is differentiable on $I$ and $f'(c_2) < 1/3 < f'(c_1)$ for $c_1, c_2 \in (a,b)$, we can apply Darboux's theorem to show that $\exists c \in (c_2, c_1) \subseteq (a,b)$ s.t. $f(c) = 1/3$.

## 5.3 L'Hôpital's Rules

[38]

**Theorem 5.18.** *Let $f$ and $g$ be defined on [a,b], let $f(a) = g(a) = 0$, and $g(x) \neq 0$ for $a < x < b$. If $f$ and $g$ are differentiable at $a$ and if $g'(a) \neq 0$, then the limit of $f/g$ at $a$ exists and is equal to $f'(a)/g'(a)$. Thus,*

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

**Theorem 5.19** (**Cauchy's Mean Value Theorem**). *Let $f$ and $g$ be defined on [a,b] and differentiable on (a,b), and assume that $g(x) \neq 0, \forall x \in (a,b)$. Then there exists $c \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

**Theorem 5.20** (**L'Hôspital's Rule, I**). *Let $-\infty \leq a < b \leq \infty$ and let $f, g$ be differentiable on $(a,b)$ such that $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that*

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x).$$

*(a) If*

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R},$$

*then*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

*(b) If*

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in (-\infty, \infty),$$

[38] This subsection corresponds to section 6.3 in Bartle and Sherbert's Introduction to Real Analysis.

*then*

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L.$$

**NOTE 5.21.** *You can also use the above theorem when the denominator function* $g(x) \to \pm\infty$ *as* $x \to +a$

**SELECT EXERCISES** [39]

**6.3/5 Let** $f(x) := x^2\sin(1/x), x \neq 0, f(0) := 0$ **and** $g(x) := \sin x, x \in \mathbb{R}$**. Show that** $\lim_{x\to 0} f/g = 0$**, but that lim f'/g' does not exist.**
We see that

$$\lim_{x\to 0}\frac{x^2\sin(1/x)}{\sin(x)} = \frac{0}{\sin 0} = \frac{0}{0}.$$

Now, observe that

$$\lim_{x\to 0}\frac{f'}{g'} = \lim_{x\to 0}\frac{2x\sin(1/x)-\cos(1/x)}{\cos x}$$

Using limit properties, we can find

$$\frac{\lim_{x\to 0} 2x\sin(1/x)-\cos(1/x)}{\lim_{x\to 0}\cos x} = \frac{\lim_{x\to 0} 2x\sin(1/x)-\cos(1/x)}{1} = \lim_{x\to 0} 2x\sin(1/x)-\cos(1/x)$$

Choose a sequence $x_n := \frac{1}{n\pi}$. we see that $x_n \to 0$, but $\cos(1/x_n) = \cos(n\pi)$ oscillates between $\{1,-1\}$. So, the limit does not exist.

**6.3/6 Evaluate the following limits.**

(a) $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}$
When we evaluate at $x = 0$, we get $0/0$. So, we use this section's namesake rule to obtain

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$$

and then we do it again

$$\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x} = 2/1 = 2$$

(b) $\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}$
Same process as above. Evaluate limit after every derivative to ensure the viability of L'Hôspital's rule.

## 5.4 Taylor's Theorem

[40]

[39] Page 187 of the textbook, section 6.3
[40] This subsection corresponds to section 6.4 in Bartle and Sherbert's Introduction to Real Analysis.

WIP

# 6 RIEMANN INTEGRAL

## 6.1 Riemann Integral

[41]

### PARTITIONS

**Definition 6.1.** *If $I := [a, b]$ is a real closed bounded interval, then a* ***partition*** *of $I$ is a finite, ordered set $P := (x_0, \ldots, x_n)$ of points in $I$ such that*

$$a = x_0 < \cdots < x_n = b$$

We can denote this partition alternatively by $P = \{[x_{i-1}, x_i]\}_{i=1}^n$.

**Definition 6.2.** *The* ***norm*** *or* ***mesh*** *of $P$ is defined as*

$$||P|| := \max\{x_1 - x_0, \ldots, x_n - x_{n-1}\}.$$

Furthermore, if a point $t_i$ has been selected from each subinterval $I_i = [x_{i-1}, x_i]$ for $i \in \mathbb{N}$ then the points are called **tags** of the subintervals. The set of ordered pairs of subintervals and corresponding tags is called a **tagged partition** of $I$, and is denoted by $\dot{\mathrm{P}}$ (note the dot on top).

### RIEMANN INTEGRAL

The next definition is akin to the limit definitions we are familiar with.

**Definition 6.3.** *A function $f : [a, b] \to \mathbb{R}$ is said to be* ***Riemann integrable*** *on [a, b] if $\exists L \in \mathbb{R}$ s.t. $\forall \epsilon > 0 \exists \delta_\epsilon > 0$ s.t. if $\dot{P}$ is any tagged partition of [a, b] with its norm less than $\delta_\epsilon$, then*

$$|S(f, \dot{P}) - L| < \epsilon$$

*We often denote this by*

$$L = \int_a^b f \text{ or } \int_a^b f(x)dx.$$

The set of all Riemann integrable functions on an interval $[a, b]$ is denoted by $R[a, b]$.

**Theorem 6.4.** *If $f \in R[a, b]$, then the value of the integral is uniquely determined.*

**Theorem 6.5.** *If $g \in R[a, b]$ and $f(x) = g(x)$ except for a finite number of points, then $f$ is integrable and $\int_a^b f = \int_a^b g$.*

**Theorem 6.6.** *Integration is distributive on addition, you can pull out constants, and if you have two functions with one less than each other, their integrals will reflect that property.*

**Theorem 6.7.** *If $f \in R[a, b]$, then $f$ is bounded on [a, b].*

[41] This subsection corresponds to section 7.1 in Bartle and Sherbert's Introduction to Real Analysis.

**SELECT EXERCISES** [42]

**7.1/1 If $I := [0, 4]$, calculate the norms of the following partitions:**

(a) $P_1 := (0, 1, 2, 4)$ The norm is defined as the maximum of all the consecutive differences. we have $||P_1|| = 4 - 2 = 2$.

(c) $P_3 := (0, 1, 1.5, 2, 3.4, 4)$ Similar to above, we have $3.4 - 2 = 1.4$.

**7.1/2 If $f(x) = x^2, \quad x \in [0, 4]$, calculate the Reimann sums, where $\dot{\mathbf{P}}_i$ has the same partition points as above. The tags are as indicated.**

(a) $\dot{\mathrm{P}}_1$ with tags at left end points.
We have the sum $S(x^2, \dot{P}_1, x_i) = 0^2 + 1^2 + 2^2(2) = 9$

(b) $\dot{\mathrm{P}}_1$ with tags at right end points.
$S(x^2, \dot{P}_1, x_{i+1}) = 1^2 + 2^2 + 4^2(2) = 37$

**7.1/6 Solve**

(a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in R[0, 2]$ and evaluate its integral.

*Proof:* $\int_0^2 f = \int_0^1 f + \int_1^2 f = \int_0^2 2 + \int_0^2 1 = 2 + 1 = 3$

□

(b) Similar to above.

**7.1/8 If $f \in R[a, b]$ and $|f| \le M \quad \forall x \in [a, b]$, show that $|\int_a^b f| \le M(b - a)$.**

*Proof:* Let P be an arbitrary partition. Then the Reimann sum is given by $|S(f; P)| = |\sum f(t_i)(x_i - x_{i-1})| \le \sum |f(t_i)||(x_i - x_{i-1})| \le \sum M|(x_i - x_{i-1})| = M \sum |(x_i - x_{i-1})| = M(b - a)$.

□

**7.1/9 If $f$ is a bounded function on $[a, b]$, and $(\mathcal{P}_n)$ is any sequence of tagged partitions of $[a, b]$ such that $\|\mathcal{P}_n\| \to 0$, prove that**

$$\int_a^b f = \lim_{n \to \infty} S(f; \mathcal{P}_n).$$

[42] Page 206 of the textbook, section 7.1

*Proof:* Since $f \in R[a,b]$, by definition, $\exists L \in \mathbb{R}$ s.t. $\forall \epsilon > 0\ \exists \delta_\epsilon > 0 : ||\dot{P}|| < \delta_\epsilon \implies$

$$|S(f;\dot{P}) - L| < \epsilon$$

and further, we can write

$$|S(f;\dot{P}) - \int_a^b f| < \epsilon$$

We can choose $\epsilon = 1/n$ by the Archimedean principle, then

$$0 \leq |S(f;\dot{P}) - \int_a^b f| < 1/n$$

Note that a sequence pf partitions whose norm tends to 0 means that $||\dot{P}_n|| = \max\{x_1 - x_0, \ldots, x_i - x_{i-1}\} \to 0 \implies x_i = x_{i-1}$, for a larger enough i; the partitions get finer and finer. The limit of both sides are zero. Thus, using the Squeeze theorem

$$0 \leq |S(f;\dot{P}_n) - \int_a^b f| < 1/n \implies \lim S(f;\dot{P}_n) - \int_a^b f = 0 \implies \lim S(f;\dot{P}_n) = \int_a^b f$$

□

**7.1/10 Let $g(x) = 0$ if $x \in [0,1] \cap \mathbb{Q}$ and $g(x) = \frac{1}{x}$ if $x \in [0,1] \cap \mathbb{R} \setminus \mathbb{Q}$. Explain why $g \notin R[0,1]$. However, show there exists a sequence $(\mathcal{P}_n)$ of tagged partitions of $[0,1]$ such that $\|\mathcal{P}_n\| \to 0$ and $\lim_{n\to\infty} S(g;\mathcal{P}_n)$ exists.**

*Proof:*
- We can pick two partitions, one rational and the other irrational. Then the Riemann integral would be 0 in one and greater than 0 in the other. Since the value is not unique, $g \notin R[0,1]$.
- Pick any sequence with partitions such that $||P_n|| \to 0$ and the tags are irrational. Then the sum is greater than 0 and exists.

□

**7.1/11 Suppose $f$ is bounded on $[a,b]$, and there exist two sequences of tagged partitions of $[a,b]$, $(\mathcal{P}_n)$ and $(\mathcal{Q}_n)$, such that $\|\mathcal{P}_n\| \to 0$ and $\|\mathcal{Q}_n\| \to 0$, but**

$$\lim_{n\to\infty} S(f;\mathcal{P}_n) \neq \lim_{n\to\infty} S(f;\mathcal{Q}_n).$$

**Show that $f \notin R[a,b]$.**

*Proof:* By the definition of Riemann integrability, the Riemann sums for all sequences of partitions whose norms tend to zero should converge to the same limit (the integral). This uniqueness is violated in the question, thus, we cannot find the function to be integrable.

□

**7.1/12 Consider the Dirichlet function defined by**

$$f(x) = \begin{cases} 1 & \textbf{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \textbf{if } x \in [0,1] \cap \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

**Use the preceding exercise to show that $f$ is not Riemann integrable on $[0,1]$.**

*Proof:* Consider two tagged partition sequences with rational $\dot{Q}_n$ and irrational tags $\dot{I}_n$ such that their norms tend to 0. We can find such sequences due to the interval's density in the rationals and the irrationals.

Then, we have

$$S(f;\dot{Q}_n) = \sum_i^n f(t_i)(x_i - x_{i-1}) = \sum_i^n 1\cdot(x_i - x_{i-1}) = 1\cdot\sum_i^n (x_i - x_{i-1}) = 1\cdot 1 = 1$$

Similarly, for the irrationals, the Riemann integral would be 0. Since they are unequal, we can apply the result of the previous question to obtain the desired result.

□

## 6.2 Riemann Integrable Functions

[43]

**Theorem 6.8** (**Cauchy Criterion**). *$f : [a,b] \to \mathbb{R} \in R[a,b] \iff \forall \epsilon > 0\ \exists \eta_\epsilon > 0$ s.t. if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of $[a,b]$ with their norms less than $\eta_\epsilon$, then*

$$|S(f;\dot{P}) - S(f:\dot{Q})| < \epsilon$$

The following theorem is one of our favorites, but now for integrals.

**Theorem 6.9** (**Squeeze Theorem**). *Let $f : [a,b] \to \mathbb{R}$. Then $f \in R[a,b] \iff \forall \epsilon > 0\ \exists$ functions $\alpha_\epsilon$ and $\omega_\epsilon \in R[a,b]$ with*

$$\alpha_\epsilon(x) \le f(x) \le \omega_\epsilon(x) \quad \forall x \in [a,b],$$

*and such that*

$$\int_a^b (\omega_\epsilon - \alpha_\epsilon) < \epsilon$$

**Theorem 6.10.** *If $\varphi : [a,b] \to \mathbb{R}$ is a step function, then $\varphi \in R[a,b]$.*

**Theorem 6.11.** *If $f : [a,b] \to \mathbb{R}$ is continuous, then $f \in R[a,b]$.*

**Theorem 6.12.** *If $f : [a,b] \to \mathbb{R}$ is monotone, then $f \in R[a,b]$.*

[43] This subsection corresponds to section 7.2 in Bartle and Sherbert's Introduction to Real Analysis.

**ADDITIVITY**

**Theorem 6.13.** *Let $F : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then $f \in R[a,b] \iff$ its restrictions to [a, c] and [c, b] are both Riemann integrable. In this case*

$$\int_a^b f = \int_a^c f + \int_a^b f$$

We can also restate the theorem above to be such that the integral from p to q is the same as adding the integral from p to c and c to q, for any points $p, q, c \in [a,b]$.

**Definition 6.14.** *If $f \in R[a,b]$ and $\alpha, \beta \in [a,b]$ and $\alpha < \beta$, we define*

$$\int_\beta^\alpha f := -\int_\alpha^\beta f \text{ and } \int_\alpha^\alpha f := 0.$$

**SELECT EXERCISES** [44]

**7.2/1 Let $f : [a,b] \to \mathbb{R}$. Show that $f \not\in R[a,b] \iff \exists \epsilon_o > 0$ s.t. $\forall n \in \mathbb{N}\ \exists \dot{P}_n$ and $\dot{Q}_n$ with $||\dot{P}_n|| < 1/n$ and $||\dot{Q}_n|| < 1/n$ such that $|S(f;\dot{P}_n) - S(f;\dot{Q}_n)| \geq \epsilon_0$.**

*Proof:* ( $\implies$ ) Let $f \notin \in R[a,b]$. Then by the definition, we have must have two partitions whose norms tend to 0, but their Riemann sums do not:

$$|S(f;\dot{P}_n) - S(f;\dot{Q}_n)| > 0 \implies S(f;\dot{P}_n) - S(f;\dot{Q}_n) \geq \epsilon_0$$

for some $\epsilon_0 > 0$.

( $\impliedby$ ) Since we have $|S(f;\dot{P}_n) - S(f;\dot{Q}_n)| \geq \epsilon_0$, showing that the Riemann sums are unequal. Thus, $f \notin \in R[a,b]$.

□

**7.2/2 Consider $h(x) := x+1,\ x \in [0,1] \cap \mathbb{Q}$ and $h(x) := 0,\ x \in [0,1] \cap \mathbb{R} \backslash \mathbb{Q}$. Show that $h \notin R[0,1]$.**

*Proof:* Find two partition sequences such that they tend to 0, but one is rational and the other is irrational. Then we can show that the Riemann sum of the irrational is 0, and the sum for the rational partition sequence is bounded below by 1 and above by 2. Thus, they are not equal. So, $h \notin R[0,1]$.

□

**7.2/4 If $p(x) := -x$ and $q(x) := x$ and $p \leq f \leq q \quad \forall x \in [0,1]$, does it follow from the Squeeze Theorem that f is integrable?**

[44] Page 215 of the textbook, section 7.2

Using the squeeze theorem, we must have, given $\epsilon > 0$:

$$\int_0^1 2x dx = x^2|_0^1 = 1 - 0 = 1.$$

If we choose $\epsilon = 0.5$, this does not work. So, it does not necessary follow.

**7.2/6 If $y : [a,b] \to \mathbb{R}$ takes on only a finite number of distinct values, is $y$ a step function?**

Since y takes only a finite number of distinct values and can be expressed as a constant function on a fir
y is a step function. Therefore, we conclude that:

$$\boxed{y \text{ is a step function.}}$$

**7.2/7 If $S(f; \dot{P})$ is any Riemann sum of $f : [a,b] \to \mathbb{R}$, show that there exists a step function $\phi : [a,b] \to \mathbb{R}$ s.t. $\int_a^b \phi = S(f; \dot{P})$.**

*Proof:* Define the step function $\phi$ by setting $\phi(x) = f(t_i)$ for $x \in [x_{i-1}, x_i)$, where $t_i$ is the tag corresponding to the subinterval $[x_{i-1}, x_i]$. Since $\phi$ is constant on each subinterval of the partition $\mathcal{P}$, we can compute its integral as follows:

$$\int_a^b \phi(x)\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(t_i)\,dx = \sum_{i=1}^n f(t_i) \int_{x_{i-1}}^{x_i} 1\,dx = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}),$$

which is exactly the definition of the Riemann sum $S(f; \mathcal{P})$. Therefore, $\int_a^b \phi(x)\,dx = S(f; \mathcal{P})$.

□

**7.2/8 Suppose $f$ is continuous on $[a,b]$, $f \geq 0$, $\forall x \in [a,b]$ and $\int_a^b f = 0$. Prove that $f = 0$, $\forall x \in [a,b]$.**

*Proof:* To prove that $f(x) = 0$ for all $x \in [a, b]$, assume, for contradiction, that $f(x)$ is not identically zero. Then there exists some $c \in (a, b)$ such that $f(c) > 0$. By continuity, there exists $\delta > 0$ such that

$$f(x) > \frac{1}{2}f(c) > 0 \quad \text{for all } x \in (c - \delta, c + \delta) \cap [a, b].$$

Now consider the integral:

$$\int_{c-\delta}^{c+\delta} f(x)\, dx > \int_{c-\delta}^{c+\delta} \frac{1}{2}f(c)\, dx = f(c) \cdot \delta > 0.$$

This implies

$$\int_{a}^{b} f(x)\, dx \geq \int_{c-\delta}^{c+\delta} f(x)\, dx > 0,$$

contradicting the assumption $\int_a^b f(x)\, dx = 0$. Therefore, $f(x) = 0$ for all $x \in [a, b]$.

□

**7.2/10 If $f$ and $g$ are continuous on $[a, b]$ and if $\int f = \int g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.**

*Proof:* Define $h(x) = f(x) - g(x)$. Since $f$ and $g$ are continuous, $h$ is also continuous on $[a, b]$. Given that

$$\int_a^b h(x)\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = 0,$$

it follows that the integral of $h$ over $[a, b]$ is zero.
By the properties of continuous functions, if $h(x)$ did not take on the value 0 anywhere on $[a, b]$, $h(x)$ would be either strictly positive or strictly negative, which would make $\int_a^b h(x) \neq 0$, a contradiction. Therefore, there must exist a point $c \in [a, b]$ where $h(c) = 0$, meaning $f(c) = g(c)$, as required by Bolzano's Theorem.

□

**7.2/11 If $f$ is bounded by $M$ on $[a, b]$ and if the restriction of $f$ to every interval $[c, b]$ where $c \in (a, b)$ is Riemann integrable, show that $f \in R[a, b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$.**

*Proof:* We will show that $f$ is integrable on $[a,b]$ by constructing partitions satisfying the definition of Riemann integration.

Let $\epsilon > 0$ and $M \geq |f|$. Now choose $\delta > 0$ such that $M\delta < \frac{\epsilon}{4}$. Fix a partition $P'$ of $[a+\delta, b]$ such that the difference between the upper Riemann sum and lower Riemann sum of $f$ on $P'$ is at most $\frac{\epsilon}{2}$, which we know exists as $f$ is Riemann integrable on $[a+\delta, b]$.

We construct a partition $P$ for the interval $[a,b]$ by adding the single point $a$ to the partition $P'$ of $[a+\delta, b]$. The difference between the upper and lower Riemann sums for $P$ is at most the difference on $[a, a+\delta]$ added to the difference on $[a+\delta, b]$. Therefore, we have

$$2 \cdot \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

Thus, for any $\epsilon > 0$, we have explicitly constructed a partition satisfying the requirements for Riemann integrability on $[a,b]$.
For the second part: now that we know $f$ is integrable on $[a,b]$, we can write

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Since

$$\left| \int_a^c f(x)\, dx \right| \leq M|c-a| \to 0 \quad \text{as } c \to a,$$

we have that $\int_c^b f \to \int_a^b f$ as $c \to a^+$.

$\square$

**7.2/12 Show that $g(x) := \sin(1/x),\ x \in (0,1]$ and $g(0) := 0$ belongs to $R[0,1]$.**
*Proof:* Since $|g(x)| \leq 1$ for all $x \neq 0$ and $g(x)$ is continuous on every interval $[c,1]$ where $0 < c < 1$, we can consider $g(x)$ as piecewise continuous over $[0,1]$ by defining $g(0) = 0$. This definition ensures that $g(x)$ satisfies the conditions of the preceding exercise, establishing that $g \in R[0,1]$.

$\square$

**7.2/16 If $f$ is continuous on $[a,b], a < b$, show that there exists $c \in [a,b]$ such that we have $\int_a^b f = f(c)(b-a)$. This result is sometimes called the *Mean Value Theorem for Integrals.***

*Proof:* Let $m := \inf_{x\in[a,b]} f(x)$ and $M := \sup_{x\in[a,b]} f(x)$. Then, we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Since $f$ is continuous on $[a,b]$, the Intermediate Value Theorem (Bolzano's Theorem) guarantees there exists a point $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Rearranging, we obtain

$$\int_a^b f(x)\,dx = f(c)(b-a),$$

which completes the proof of the Mean Value Theorem for Integrals.

□

## 6.3 The Fundamental Theorem

[45]

**FIRST FORM**

This part is the theoretical basis for the method of calculating an integral. If f has an **antiderivative**, then we can calcuate its integral.

**Theorem 6.15** (**Fundamental Theorem of Calculus, First Form**). *Suppose there is a* ***finite*** *set $E \in [a,b]$ and functions $f, F := [a,b] \to \mathbb{R}$ such that:*

*(a) F is continuous on [a,b]*

*(b) $F'(x) = f(x) \forall x \in [a,b] \backslash E$ (It is okay to have a finite number of points such that F'(c) does not exist in $\mathbb{R}$ or does not equal $f(c)$.)*

*(c) $f \in R[a,b]$. Then*

$$\int_a^b f = F(b) - F(a)$$

**SECOND FORM**

**Definition 6.16.** *If f$\in R[a,b]$, then the function defined by*

$$F(z) := \int_a^z f \quad z \in [a,b]$$

[45]This subsection corresponds to section 7.3 in Bartle and Sherbert's Introduction to Real Analysis.

*is called the* ***indefinite integral of*** $f$ *with* ***basepoint*** $a$.

**Theorem 6.17.** *The indefinite integral* $F$ *defined by the previous definition is continuous on* $[a, b]$. *In fact, if* $|f(x)| \leq M \ \forall x \in [a, b]$, *then* $|F(z) - F(w)| \leq M|z - w| \ \forall z, w \in [a, b]$.

**Theorem 6.18** (**Fundamental Theorem of Calculus, Second Form**). *Let* $f \in R[a, b]$ *and let* $f$ *be continuous at a point* $c \in [a, b]$. *Then the indefinite integral defined above is differentiable at* $c$ *and* $F'(c) = f(c)$.

**Theorem 6.19.** *IF* $f$ *is continuous on* $[a, b]$, *then the indefinite integral* $F$ *defined above is differentiable on* $[a, b]$ *and* $F'(x) = f(x) \ \forall x \in [a, b]$.

## OTHER THEOREMS

The following theorem lets us change variables during integration.

**Theorem 6.20** (**Substitution Theorem**). *Let* $J := [a, b]$ *and let* $\varphi : J \to \mathbb{R}$ *have a continuous derivative on* $J$. *If* $f : I \to \mathbb{R}$ *is continuous on an interval* $I$ *containing* $\varphi(J)$, *then*

$$\int_a^b f(\varphi(t)) \cdot \varphi'(t)dt = \int_{\varphi(a)}^{\varphi(b)} f(x)dx.$$

**Theorem 6.21** (**Lebesgue's Integrability Criterion**). *A bounded function* $f : [a, b] \to \mathbb{R}$ *is Riemann integrable* $\iff$ *it is continuous almost everywhere on* $[a, b]$.

**Theorem 6.22** (**Composition Theorem**). *Let* $f \in R[a, b]$ *with* $f([a, b]) \subseteq [c, d]$ *and* $\varphi : [c, d] \to \mathbb{R}$ *be continuous. Then the composition* $\varphi \circ f \in R[a, b]$.

There is another theorem similar to the above one. If two functions are integrable, then their product is as well.

**Theorem 6.23** (**Integration by Parts**). *Let* $F, G$ *be differentiable on* $[a, b]$ *and let* $f := F'$ *and* $g := G' \in R[a, b]$. *Then*

$$\int_a^b fG = FG|_a^b - \int_a^b Fg.$$

## SELECT EXERCISES [46]

**7.3/3 If** $g(x) := x, |x| \geq 1$ **and** $g(x) := -x, |x| < 1$ **and** $G(x) := 1/2|x^2 - 1|$**, show that** $\int_{-2}^3 g(x)dx = G(3) - G(-2) = 5/2$**.**

[46] Page 223 of the textbook, section 7.3

*Proof:* Using additivity, we can split the integral into

$$\int_{-2}^{3} g(x) = \int_{-2}^{-1} g(x) + \int_{-1}^{1} g(x) + \int_{1}^{3} g(x)$$

Then, we can use the fundamental theorem of calculus, which gives

$$\int_{-2}^{3} g(x) = G(-1) - G(-2) + G(1) - G(-1) + G(3) - G(1) = G(3) - G(-2) = 5/2$$

$\square$

**7.3/5 Let $f : [a, b] \to \mathbb{R}$ and let $C \in \mathbb{R}$.**

(a) If $\Phi : [a, b] \to \mathbb{R}$ is an antiderivative of $f$ on $[a, b]$, show that $\Phi_C(x) = \Phi(x)+C$ is also an antiderivative.

*Proof:* Since $\Phi(x)$ is an antiderivative of $f$, we have $\Phi'(x) = f(x)$ for all $x \in [a, b]$. Now, consider $\Phi_C(x) = \Phi(x) + C$, where $C$ is a constant. Then, by the linearity of differentiation:

$$\Phi_C'(x) = \frac{d}{dx}(\Phi(x) + C) = \Phi'(x) + \frac{d}{dx}(C).$$

Since $C$ is constant, $\frac{d}{dx}(C) = 0$, so we have

$$\Phi_C'(x) = \Phi'(x) = f(x).$$

Thus, $\Phi_C(x)$ is also an antiderivative of $f$ on $[a, b]$.

□

(b) If $\Phi_1$ and $\Phi_2$ are antiderivatives, show that $\Phi_1 - \Phi_2$ is a constant function on $[a, b]$.

*Proof:* Let $\Phi_1$ and $\Phi_2$ be antiderivatives of $f$ on $[a, b]$. This means that $\Phi_1'(x) = f(x)$ and $\Phi_2'(x) = f(x)$ for all $x \in [a, b]$.
Consider the function $\Phi_1 - \Phi_2$ on $[a, b]$. To show that it is constant, we can calculate its derivative:

$$\frac{d}{dx}(\Phi_1(x) - \Phi_2(x)) = \Phi_1'(x) - \Phi_2'(x).$$

Since $\Phi_1'(x) = f(x)$ and $\Phi_2'(x) = f(x)$, we have

$$\Phi_1'(x) - \Phi_2'(x) = f(x) - f(x) = 0.$$

Thus, $\frac{d}{dx}(\Phi_1(x) - \Phi_2(x)) = 0$, which implies that $\Phi_1 - \Phi_2$ is constant on $[a, b]$.
Therefore, $\Phi_1(x) - \Phi_2(x) = C$ for some constant $C$ on $[a, b]$.

□

**7.3/10 Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $v : [c, d] \to \mathbb{R}$ be differentiable on $[c, d]$ with $v([c, d]) \subseteq [a, b]$. If we define $G(x) := \int_a^{v(x)} f$, show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c, d]$.**

*Proof:* Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and let $v : [c,d] \to \mathbb{R}$ be differentiable on $[c,d]$ with $v([c,d]) \subseteq [a,b]$. Define the function

$$G(x) := \int_a^{v(x)} f(t)\,dt.$$

We want to show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c,d]$.
To compute $G'(x)$, we use the Fundamental Theorem of Calculus and the Chain Rule. First, observe that $G(x)$ is an integral with an upper limit depending on $x$. By the Fundamental Theorem of Calculus, we have:

$$\frac{d}{dx}\int_a^{v(x)} f(t)\,dt = f(v(x)) \cdot \frac{d}{dx}v(x).$$

Since $v(x)$ is differentiable on $[c,d]$, we have $v'(x) = \frac{d}{dx}v(x)$. Therefore,

$$G'(x) = f(v(x)) \cdot v'(x),$$

which is what we wanted to show.

$\square$

**7.3/11 Find $F'(x)$ when $F$ is defined on $[0,1]$ by:**

(a) $F(x) := \int_0^{x^2}(1+t^3)^{-1}dt$

To differentiate $F(x) := \int_0^{x^2} \frac{1}{1+t^3}\,dt$, we apply the Fundamental Theorem of Calculus along with the Chain Rule. We have

$$F'(x) = \frac{d}{dx}\int_0^{x^2} \frac{1}{1+t^3}\,dt = \frac{1}{1+(x^2)^3} \cdot \frac{d}{dx}(x^2) = \frac{2x}{1+x^6}.$$

(b) $F(x) := \int_{x^2}^{x} \sqrt{1+t^2}dt$

To differentiate $F(x) := \int_{x^2}^{x} \sqrt{1+t^2}\,dt$, we again use the Fundamental Theorem of Calculus, noting that both the upper and lower limits depend on $x$. We have

$$F'(x) = \sqrt{1+x^2} \cdot \frac{d}{dx}(x) - \sqrt{1+(x^2)^2} \cdot \frac{d}{dx}(x^2).$$

Simplifying, this gives

$$F'(x) = \sqrt{1+x^2} - 2x\sqrt{1+x^4}.$$

**7.3/13 The function $g$ is defined on $[0,3]$ by $g(x) := -1, 0 \le x < 2$ and $g(x) := 1, 2 \le x < 3$. Find the indefinite integral $G(x) = \int_0^x g(x)$ for $0 \le x < 3$, and sketch the graphs of $g$ and $G$. Does $G'(x) = g(x),\ \forall x \in [0,3]$?**

*Proof:* To compute $G(x) = \int_0^x g(t)\,dt$ for $0 \le x < 3$, we consider the two cases based on the definition of $g(x)$.
For $0 \le x < 2$, we have $g(x) = -1$, so

$$G(x) = \int_0^x g(t)\,dt = \int_0^x -1\,dt = -x.$$

For $2 \le x < 3$, we can split the integral as follows:

$$G(x) = \int_0^x g(t)\,dt = \int_0^2 g(t)\,dt + \int_2^x g(t)\,dt.$$

The first part evaluates to $\int_0^2 g(t)\,dt = \int_0^2 -1\,dt = -2$. For the second part, we have $g(x) = 1$ on $[2, x)$, giving

$$G(x) = -2 + \int_2^x 1\,dt = -2 + (x-2) = x - 4.$$

Thus, the function $G(x)$ is defined as

$$G(x) = \begin{cases} -x, & 0 \le x < 2, \\ x - 4, & 2 \le x < 3. \end{cases}$$

Now, we check if $G'(x) = g(x)$ for all $x \in [0, 3]$. For $0 \le x < 2$, $G(x) = -x$, so $G'(x) = -1 = g(x)$. For $2 \le x < 3$, $G(x) = x - 4$, so $G'(x) = 1 = g(x)$. However, $G(x)$ is not differentiable at $x = 2$, as there is a discontinuity in $g(x)$ at this point. Therefore, $G'(x) = g(x)$ holds for $x \in [0, 3] \setminus \{2\}$.

□

## 6.4 The Darboux Integral

**Definition 6.24.** *Let $f : I \to \mathbb{R}$ be a bounded function on $I = [a, b]$ and let $P = (x_0, \ldots, x_n$ be a partition of I. For k=1,2,...,n we let*

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k\} \text{ and } M_k := \sup\{f(x) : x \in [x_{k-1}, x_k\}$$

*Then the* ***lower sum*** *is*

$$L(f; P) := \sum_{k=1}^{n} m_k(x_k - x_{k-1}),$$

*and the* ***upper sum*** *is given by*

$$U(f; P) := \sum_{k=1}^{n} M_k(x_k - x_{k-1}),$$

**Lemma 6.25.** *If $f := I \to \mathbb{R}$ is bonded and $P$ is any partition of $I$, then $L(f; P) \le U(f; P)$.*

We can different partitions for the same sum. If P and Q are two partitions, Q is a **refinement** of P if each partition point in P is contained in Q ($P \subseteq Q$). So a refinement is more accurate, thus, naturally the difference between the upper/lower sum and the true value lessens; i.e.,

**Lemma 6.26.** *If Q is a refinement of P, then $L(f;P) \leq U(f;Q)$ and $U(f;Q) \leq U(f;P)$*

We can start to compare different partitions and their upper/lower sums. For any two partitions, the lower sum is always then the upper sum, regardless of the partition you choose.

We progress into integrals. In the following definition $\mathcal{P}(I)$ denotes the collection of all partitions of the interval I.

**Definition 6.27.** *Let $I := [a,b]$ and $f : I \to \mathbb{R}$ be a bounded function. The **lower integral of f on I** is the number*

$$L(f) := \sup\{L(f;P) : P \in \mathcal{P}(I)\}$$

*Similarly, the **upper integral** is*

$$U(f) := \inf\{U(f;P) : P \in \mathcal{P}(I)\}$$

**Theorem 6.28.** *Let $I := [a,b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then the lower and upper integral exists on I. Moreover,*

$$L(f) \leq U(f).$$

What if the lower sums and the upper sums are the same? We reach this section's highlight:

**Definition 6.29.** *Let $I := [a,b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then f is **Darboux integrable on I** of $L(f) = U(f)$. The Darboux integral's value is the sum (L(f) or U(f)).*

To check whether something is Darboux integrable, we can introduce an epsilon definition that we are already familiar with:

**Theorem 6.30** (Darboux Integrability Criterion)**.** *Let $I := [a,b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then f is Darboux integrable on I $\iff \forall \epsilon > 0, _\exists P_\epsilon$ s.t.*

$$U(f;P_\epsilon) - L(f;P_\epsilon) < \epsilon$$

**Corollary 6.31.** *Let $I := [a,b]$ and $f : I \to \mathbb{R}$ be a bounded function. If $\{P_n : n \in \mathbb{N}\}$ is a sequence of partitions of I such that*

$$\lim_n (U(f;P_n) - L(f;P_n)) = 0$$

*then f is integrable and* $\lim_n(U(f;P_n)) = \int_a^b f = \lim_n(L(f;P_n))$

Finally, some implications

**Theorem 6.32.** *If the function f on the interval $I = [a,b]$ is either continuous or monotone on I, then f is Darboux integrable on I.*

Since this section was slightly different than the Riemann integral we learned about in the past three sections, let's relate them.

**Theorem 6.33.** *A function f is Darboux integral $\iff$ f is Riemann integrable.*

## SELECT EXERCISES [47]

**7.4/1 Let $f(x) := |x|$ for $-1 \leq x \leq 2$. Calculate $L(f;P)$ and $U(f;P)$ for the following partitions:**

(a) $P_1 := (-1, 0, 1, 2)$
Using the definition, $L(f;P_1) = \sum_k^n m_k(x_k - x_{k-1}) = 0(1) + 0(1) + 1(1) = 1$.
Similarly, $U(f;P_1) = \sum_k^n M_k(x_k - x_{k-1}) = 1(1) + 1(1) + 2(1) = 4$.

(a) $P_2 := (-1, -1/2, 0, 1/2, 1, 3/2, 2)$
$L(f;P_2) = \sum_k^n m_k(x_k - x_{k-1}) = 1/2(1/2 + 0 + 0 + 1/2 + 1 + 3/2) = 7/4$, and
$U(f;P_2) = \sum_k^n M_k(x_k - x_{k-1}) = 1/2(1 + 1/2 + 1/2 + 1 + 3/2 + 2) = 13/4$

**7.4/2 Prove if $f(x) := c\ x \in [a,b]$, then its Darboux integral is equal to $c(a-b)$.**

*Proof:* Since $\forall x \in [a,b]$, $f(x) = c$, then $m_k = M_k = c$. So using the definitions of lower and upper sums, we get $L(f) = \sum m_k(x_k - x_{k-1}) = \sum c(x_k - x_{k-1}) = c\sum(x_k - x_{k-1}) = c(b-a)$. The same can be shown for the upper sum, since $M_k = c$. Thus, $L(f) = c(b-a) = U(f)$, showing the Darboux integral exists and is $c(b-a)$.

□

**7.4/5 Let $f, g, h$ be bounded on $I := [a,b]$ such that $f \leq g \leq h$, $\forall x \in I$. Show that if $f$ and $h$ are Darboux integrable and if $\int_a^b f = \int_a^b h$, then g is also Darboux integrable with $\int_a^b g = \int_a^b f$.**

*Proof:* Since $f$ and $g$ are Darboux integrable, we have $L(f) = U(f)$ and $L(g) = U(g)$. Since we know $f \leq g \leq h \implies L(f) \leq L(g) \leq L(h)$, and the same for upper sums. If we subtract all the lower sums from their corresponding upper sums, we get $U(f) - L(f) \leq U(g) - L(g) \leq U(h) - L(h) \implies 0 \leq U(g) - L(g) \leq 0$. Thus, $L(g) = U(g)$, showing $g$ is Darboux integrable.

Then we can write $\int_a^b f \leq U(g) \leq \int_a^b h = \int_a^b f \implies U(g) = \int_a^b f \implies \int_a^b g = \int_a^b f$.

□

**7.4/6 Let $f$ be defined on $[0,2]$ by $f(x) := 1, x \neq 1$ and $f(1) := 0$. Show that the Darboux integral exists and find its value.**

[47] Page 233 of the textbook, section 7.4

*Proof:* Consider the partition $P_\epsilon = \{0, 1-\epsilon/2, 1+\epsilon/2, 2\}$ for some $\epsilon > 0$, which divides $[0,2]$ into three subintervals: $[0, 1-\epsilon/2]$, $[1-\epsilon/2, 1+\epsilon/2]$, and $[1+\epsilon/2, 2]$.
Lower Sum $L(f; P_\epsilon)$

- On $[0, 1-\epsilon/2]$, $\inf f = 1$ and the interval length is $1-\epsilon/2$, so contribution is $(1)(1-\epsilon/2) = 1-\epsilon/2$.
- On $[1-\epsilon/2, 1+\epsilon/2]$, $\inf f = 0$ and the interval length is $\epsilon$, so contribution is $0 \cdot \epsilon = 0$.
- On $[1+\epsilon/2, 2]$, $\inf f = 1$ and the interval length is $1-\epsilon/2$, so contribution is $(1)(1-\epsilon/2) = 1-\epsilon/2$.

Thus, $L(f; P_\epsilon) = (1-\epsilon/2) + 0 + (1-\epsilon/2) = 2-\epsilon$.
Upper Sum $U(f; P_\epsilon)$

- On $[0, 1-\epsilon/2]$, $\sup f = 1$ and the interval length is $1-\epsilon/2$, so contribution is $(1)(1-\epsilon/2) = 1-\epsilon/2$.
- On $[1-\epsilon/2, 1+\epsilon/2]$, $\sup f = 1$ and the interval length is $\epsilon$, so contribution is $(1)(\epsilon) = \epsilon$.
- On $[1+\epsilon/2, 2]$, $\sup f = 1$ and the interval length is $1-\epsilon/2$, so contribution is $(1)(1-\epsilon/2) = 1-\epsilon/2$.

Thus, $U(f; P_\epsilon) = (1-\epsilon/2) + \epsilon + (1-\epsilon/2) = 2$. Since $U(f; P_\epsilon) - L(f; P_\epsilon) = 2 - (2-\epsilon) = \epsilon$, we see that $f$ is Darboux integrable, with

$$\int_0^2 f(x)\,dx = 2.$$

□

**7.4/8 Let $f$ be continuous on $I := [a,b]$ and assume $f \geq 0,\ \forall x \in I$. Prove if $L(f) = 0$, then $f(x) = 0,\ \forall x \in I$.**

*Proof:* Assume, for contradiction, that $f \neq 0$. Then there must be some $x_o \in I$ s.t. $f(x_0) > 0$. Then $\exists (x_0 - \delta, x_0 + \delta) \subset I$ s.t. the interval is positive. Specifically, by continuity, $\exists \epsilon > 0$ and $\delta > 0$ s.t.

$$f(x) > f(x_0)/2 > 0 \ \forall x \in (x_0 - \delta, x_0 + \delta)$$

Then we can construct a partition such that the lower Darboux sum includes this infimum of this positive valued interval, which is at least $f(x_0)/2$. Thus,

$$L(f; P) = f(x_0)/2 \cdot 2\delta = f(x_0)\delta > 0$$

This contradicts our assumption that $L = 0 \implies f = 0$.

□

**7.4/9 Let $f_1$ and $f_2$ be bounded on $[a, b]$. Show that $L(f_1) + L(f_2) \leq L(f_1 + f_2)$.**

*Proof:* We know that $L(f; P) = \sum (x_k - x_{k-1}) m_k$, where $m_k = \inf f(x), x \in [x_{k-1}, x_k]$. By the properties of infimum, we know that $\inf f_1 + \inf f_2 \leq \inf f_1 + f_2$, then for any partition,

$$L(f_1; P) + L(f_2; P) \leq L(f_1 + f_1; P)$$

. If we take the supremum over all partitions, we get

$$L(f_1) + L(f_1 2) \leq L(f_1 + f_2)$$

□

**7.4/11 If $f$ is bounded function on $[a, b]$ such that $f(x) = 0$ except for x in $\{c_1, \ldots, c_n\} \in [a, b]$, show that $U(f) = L(f) = 0$.**

*Proof:* Since $f$ is bounded, $|f| \leq M$. Given $\epsilon > 0$, we construct intervals containing each of the finite points in the question such that the length of the intervals does not exceed $\epsilon/nM$. If the interval does not contain any of the exceptional points, the inf and sup are both 0 since f would be 0. Then

$$U(f) - L(f) < \sum_{1}^{n} M(\epsilon/nM) - 0 = nM\epsilon/nM - 0 = \epsilon$$

Showing that $U(f) = L(f)$. Since $L(f) = 0$ for all intervals, $U(f) = L(f) = 0$.

□

**7.4/15 Let $f$ be defined on $I := [a, b]$ and assume $f$ satisfies the Lipschitz condition $|f(x) - f(y)| \leq K|x - y| \ \forall x, y \in I$. If $P_n$ is the partition of $I$ into $n$ equal parts, show that $0 \leq U(f; P_n) - \int_a^b f \leq K(b - a)^2/n$.**

*Proof:* Consider the partition $P_n = \{x_0, x_1, \ldots, x_n\}$ of $I$ into $n$ equal parts, where $\Delta x = \frac{b-a}{n}$. Let $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$ and $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$. The Lipschitz condition implies that $M_i - m_i \le K\Delta x$ on each subinterval, so

$$U(f;P_n) - L(f;P_n) = \sum_{i=1}^{n}(M_i - m_i)\Delta x \le \sum_{i=1}^{n} K\Delta x \cdot \Delta x = K\frac{(b-a)^2}{n}.$$

Since $f$ is uniformly continuous, it is Riemann integrable on $[a,b]$, and we have

$$0 \le U(f;P_n) - \int_a^b f = U(f;P_n) - L(f;P_n) \le \frac{K(b-a)^2}{n}.$$

Thus,

$$0 \le U(f;P_n) - \int_a^b f \le \frac{K(b-a)^2}{n}.$$

□

# 7 SEQUENCES OF FUNCTIONS

In previous chapters, we have often made use of sequences of real numbers. In this chapter, we shall consider sequences whose terms are functions rather than real numbers. Sequences of functions arise naturally in real analysis and are especially useful in obtaining approximations to a given function and defining new functions from known ones.

## 7.1 Pointwise and Uniform Convergence

Let $A \subseteq \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$ there is a function $f_n : A \to \mathbb{R}$; we shall say that $(f_n)$ is a **sequence of functions** on $A$ to $\mathbb{R}$. Clearly, for each $x \in A$, such a sequence gives rise to a sequence of real numbers, name the sequence

$$(f_n(x))$$

obtained by evaluating each of the functions at the point $x$.

**Definition 7.1.** *Let $(f_n)$ be a sequence of functions on $A \subseteq \mathbb{R} \to \mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_o \to \mathbb{R}$. We say that the **sequence** $(f_n)$ **converges on** $A_0$ **to** $f$ if, $\forall x \in A_0$, the sequence $(f_n(x))$ converges to $f(x) \in \mathbb{R}$. In this case, we call $f$ the **limit on** $A_0$ **of the sequence** $(f_n)$. When such a function $f$ exists, we say that the sequence $(f_n)$ is **convergent on** $A_0$, or that $(f_n)$ **converges pointwise on** $A_0$.*

Some notation:

$$f = \lim(f_n) \text{ on } A_0 \quad \text{or} \quad f_n \to f \text{ on } A_0$$

**Lemma 7.2.** *A sequence $(f_n)$ of functions on $A \subseteq \mathbb{R} \to \mathbb{R}$ converges to a function $f : A_0 \to \mathbb{R}$ on $A_0 \iff \forall \epsilon > 0$ and $\forall x \in A_0$ there is a natural $K(\epsilon, x)$ such that if $n \ge K(\epsilon, x)$, then*

$$|f_n(x) - f(x)| < \epsilon$$

**UNIFORM CONVERGENCE**

**Definition 7.3.** *$(f_n)$ of functions on $A \subseteq \mathbb{R} \to \mathbb{R}$* ***converges uniformly on*** *$A_0 \subseteq A$ to $f : A_0 \to \mathbb{R}$ if $\forall \epsilon > 0$ $\exists K(\epsilon)$ s.t. if $n \geq K(\epsilon)$, then*

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in A_0.$$

**Lemma 7.4.** *$(f_n)$ on $A \subseteq \mathbb{R} \to \mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R} \iff \exists \epsilon_0 > 0$ $\exists (f_{n_k})$ of $(f_n)$ and a sequence $(x_k) \in A_0$ such that*

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0 \quad \forall k \in \mathbb{N}.$$

**UNIFORM NORM**

**Definition 7.5.** *$\varphi : A \to \mathbb{R}$ is a function, $\varphi$ is* ***bounded on*** *$A$ if the set $\varphi(A)$ is a bounded subset of $\mathbb{R}$. If $\varphi$ is bounded, we define the* ***uniform norm of*** *$\varphi$* ***on*** *$A$ by*

$$||\varphi||_A := \sup\{|\varphi(x)| : x \in A\}$$

**Lemma 7.6.** *$(f_n)$ is bounded on $A \subseteq \mathbb{R}$ converges uniformly on $A$ to $f \iff ||f_n - f||_A \to 0$*

**Theorem 7.7** (**Cauchy Criterion for Uniform Convergence**)**.** *$(f_n)$ is a sequence of bounded functions on $A \subseteq \mathbb{R}$. This sequence converges uniformly on $A$ to a bounded function $f \iff \forall \epsilon > 0$ $\exists H(\epsilon) \in \mathbb{N}$ s.t. $\forall m, n \geq H(\epsilon)$, then $||f_m - f_n||_A \leq \epsilon$.*

**SELECT EXERCISES**

**8.1/2 Show that** $\lim(nx/(1 + x^2n^2)) = 0$ **for all** $x \in \mathbb{R}$**.**

## 7.2 Interchange of Limits

## 7.3 Exponential and Logarithmic Functions

# 8 GLIMPSE INTO TOPOLOGY

In this chapter, we hope to generalize the notion of an interval. So far, we define functions on intervals, so now, we go beyond that.

## 8.1 Open and Closed Sets in $\mathbb{R}$

First, let us begin with extending the notion of point neighborhoods.

**Definition 8.1.** *A* ***neighborhood*** *of $x \in \mathbb{R}$ is any set $V$ that contains an $\epsilon$-neighborhood $V_\epsilon(x) := (x - \epsilon, x + \epsilon)$ of $x$ for some $\epsilon > 0$.*

With this newly generalized definition of a neighborhood, we introduce the following.

**Definition 8.2.** *Open and closed sets:*

- *A subset $G$ of $\mathbb{R}$ is* ***open*** *in $\mathbb{R}$ if for each $x \in G$ there exists a neighborhood $V$ of $x$ such that $V \subseteq G$.*
- *A subset $F$ of $\mathbb{R}$ is* ***closed*** *in $\mathbb{R}$ if the complement $C(F) := \mathbb{R} \backslash F$ is open in $\mathbb{R}$.*

Here are some properties that we observe for open and closed sets:

**Theorem 8.3 (Open/Closed Set Properties).** *(a) The union of an arbitrary collection of open/closed subsets in $\mathbb{R}$ is open/closed.*

*(b) The intersection of any finite collection of open/closed sets in $\mathbb{R}$ is open/closed.*

We can offer some characterization of closets in terms of sequences as well:

**Theorem 8.4.** *$F \subseteq \mathbb{R}$, then the following are equivalent*

1. *$F$ is a closed subset of $\mathbb{R}$.*
2. *If $X = (x_n)$ is any convergent sequence of elements in $F$, then lim $X$ belongs to $F$.*

Naturally, we can also think about cluster points. Recall that a point $x$ is a cluster point of a set F if every $\epsilon$-neighborhood of $x$ contains a point of F different from F.

**Theorem 8.5.** *A subset of $\mathbb{R}$ is closed if and only if it contains all of its cluster points.*

Let us offer a similar treatment of open sets.

**Theorem 8.6.** *A subset of $\mathbb{R}$ is open if and only if it is the union of countably many disjoint open intervals in $\mathbb{R}$.*

A special set is the **Cantor set**. This set enabled us to learn more deeply about the nature of sets.

**Definition 8.7.** *The Cantor set $\mathbb{F}$ is the intersection of the sets $F_n$, $n \in \mathbb{N}$, obtained by successive removal of the open middle thirds, starting with $[0, 1]$.*

## SELECT EXERCISES [48]

[48] Page 332 of the textbook, section 11.1

**11.1/2 Show that the intervals $(a, \infty)$ and $(-\infty, a)$ are open sets, and that the intervals $[b, \infty)$ and $(-\infty, b]$ are closed sets.**

- $(a, \infty)$
  Take $x \in (a, \infty)$. Then $x > a$. Choose $\epsilon = (x-a)/2 > 0$. Then $(x-\epsilon, x+\epsilon) = (x - (x - a)/2, x + (x - a)/2) \subseteq (a, \infty)$. So, open.

- $(-\infty, a)$
  Similar to previous one.

- $[b, \infty)$
  Observe that the complement of this interval is $(-\infty, b)$, which is open. So, closed. You can also make an alternate argument, choose $b$; observe that any neighborhood $(b - \delta, b + \delta) \nsubseteq [b, \infty)$, since $b - \delta < b$.

- $(-\infty, b]$
  Similar to previous part.

**11.1/6 Show that $A = \{1/n : n \in \mathbb{N}\}$ is not a closed set, but that $A \cup \{0\}$ is a closed set.**

*Proof:* **Show that $A = \{1/n : n \in \mathbb{N}\}$ is not a closed set.**
The elements of $A$ are $1, 1/2, 1/3, \ldots$, and the sequence $\{1/n\}$ converges to 0. Thus, 0 is a limit point of $A$. However, $0 \notin A$, so $A$ does not contain all its limit points and is not closed.
**Show that $A \cup \{0\}$ is a closed set.**
Consider the set $B := A \cup \{0\} = \{1, 1/2, 1/3, \ldots\} \cup \{0\}$. The only limit point of $B$ is 0, and since $0 \in B$, the set contains all its limit points. Therefore, $B$ is closed.

□

**11.1/7 Show that the set $\mathbb{Q}$ of rational numbers is neither open nor closed.**

*Proof:* **Showing that $\mathbb{Q}$ is not open:**
A set $S$ in $\mathbb{R}$ is open if for every point $x \in S$, there exists an $\epsilon > 0$ such that the interval $(x-\epsilon, x+\epsilon) \subseteq S$. Consider any rational point $r \in \mathbb{Q}$. For any $\epsilon > 0$, the interval $(r - \epsilon, r + \epsilon)$ contains irrational numbers since the irrational numbers are dense in $\mathbb{R}$. Therefore, no neighborhood around any rational point can be entirely contained in $\mathbb{Q}$.
**Showing that $\mathbb{Q}$ is not closed:**
A set $S$ in $\mathbb{R}$ is closed if it contains all its limit points. Consider a sequence $\{r_n\}$ of rational numbers that converges to an irrational number $\alpha$. Since irrational numbers are limit points of $\mathbb{Q}$, and $\alpha \notin \mathbb{Q}$, the set $\mathbb{Q}$ does not contain all of its limit points.

□

**11.1/8 Show that if G is an open set and F is a closed set, then $G \backslash F$ is an open set and $F \backslash G$ is a closed set.**

*Proof:* **Show that $G \setminus F$ is open:**

Let $G$ be an open set and $F$ be a closed set. Consider any point $x \in G \setminus F$. Since $x \in G$ and $x \notin F$, and $G$ is open, there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq G$. Also, since $x \notin F$ and $F$ is closed, $(x-\epsilon, x+\epsilon) \cap F = \emptyset$. Thus, $(x-\epsilon, x+\epsilon) \subseteq G \setminus F$, so $G \setminus F$ is open.

**2. Show that $F \setminus G$ is closed:**

To show that $F \setminus G$ is closed, we consider its complement: The complement of $F \setminus G$ is $\mathbb{R} \setminus (F \setminus G) = (\mathbb{R} \setminus F) \cup G$. Since $F$ is closed, $\mathbb{R} \setminus F$ is open, and since $G$ is open, the union $(\mathbb{R} \setminus F) \cup G$ is open.

Since the complement of $F \setminus G$ is open, $F \setminus G$ is closed.

□

**11.1/9 A point $x \in \mathbb{R}$ is an interior point of $A \subseteq \mathbb{R}$ in case there is a neighborhood V of x such that $V \subseteq A$. Show that a set $A \subseteq \mathbb{R}$ is open $\iff$ $\forall x \in A$, x is an interior point of A.**

*Proof:* **(1) If $A$ is open, then $\forall x \in A$, $x$ is an interior point of $A$:**

Assume $A$ is open. By definition, for every $x \in A$, there exists a neighborhood $V$ of $x$ such that $V \subseteq A$. Since $V$ is a neighborhood of $x$, there exists some $\epsilon > 0$ such that the interval $(x-\epsilon, x+\epsilon) \subseteq A$. Thus, $x$ is an interior point of $A$.

**(2) If $\forall x \in A$, $x$ is an interior point of $A$, then $A$ is open:**

Assume that for all $x \in A$, $x$ is an interior point of $A$. By the definition of an interior point, for each $x \in A$, there exists a neighborhood $V_x = (x-\epsilon_x, x+\epsilon_x) \subseteq A$. Therefore, we can express $A$ as the union of all such neighborhoods:

$$A = \bigcup_{x \in A} V_x.$$

Since the union of open sets is open, $A$ is open.

□

**11.1/10 A point $x \in \mathbb{R}$ is an boundary point of $A \subseteq \mathbb{R}$ in case every neighborhood V of x contains points in A and points in $A^c$. Show that a set A and its complement $A^c$ have exactly the same boundary points.**

*Proof:* **1. If $x$ is a boundary point of $A$, then $x$ is a boundary point of $A^c$:**

Assume $x$ is a boundary point of $A$. By definition, for every neighborhood $V$ of $x$, we have:

$$V \cap A \neq \emptyset \quad \text{and} \quad V \cap A^c \neq \emptyset.$$

Since $V \cap A^c \neq \emptyset$ and $V$ is arbitrary, it follows that $x$ is also a boundary point of $A^c$.

**2. If $x$ is a boundary point of $A^c$, then $x$ is a boundary point of $A$:**

Assume $x$ is a boundary point of $A^c$. By definition, for every neighborhood $V$ of $x$, we have:

$$V \cap A^c \neq \emptyset \quad \text{and} \quad V \cap A \neq \emptyset.$$

Thus, $x$ is a boundary point of $A$.

□

**11.1/11 Show that a set $G \subseteq \mathbb{R}$ is open $\iff$ it does not contain any of its boundary points.**

*Proof:* ( $\Longrightarrow$ ) Assume $G$ is open. Let $x \in \partial G$ be a boundary point of $G$. By the definition of a boundary point, every neighborhood of $x$ intersects both $G$ and $G^c$.

Since $G$ is open, there must be a neighborhood $V$ of $x$ such that $V \subseteq G$. This contradicts the assumption that $x \in \partial G$, because $V$ would not intersect $G^c$. Therefore, $x \notin G$. Thus, $G$ does not contain any of its boundary points.

( $\Longleftarrow$ ) Assume $G$ does not contain any of its boundary points. Let $x \in G$. Since $x \notin \partial G$, there exists a neighborhood $V$ of $x$ such that $V \subseteq G$. Thus, $x$ is an interior point of $G$, and $G$ is open.

□

DO

**11.1/14 If $A \subseteq \mathbb{R}$, let $A^\circ$ be the union of all open sets that are contained in $A$; the set $A^\circ$ is called the interior of A. Show that**

1. **$A^\circ \subseteq A$,**

2. **$(A^\circ)^\circ = A^\circ$,**

3. **$(A \cap B)^\circ = A^\circ \cap B^\circ$,**

4. **$A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$,**

5. **An example showing that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ may be strict.**

**Solution:**
**1.**
**2.**
**3.** $(A \cap B)^\circ = A^\circ \cap B^\circ$: If $U$ is an open set contained in $A \cap B$, then $U \subseteq A^\circ$ and $U \subseteq B^\circ$, so $U \subseteq A^\circ \cap B^\circ$. Conversely, if $U \subseteq A^\circ \cap B^\circ$, then $U \subseteq A \cap B$, so $U \subseteq (A \cap B)^\circ$. Thus, $(A \cap B)^\circ = A^\circ \cap B^\circ$.
**4.** $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$: If $x \in A^\circ \cup B^\circ$, then $x \in A^\circ$ or $x \in B^\circ$. In both cases, there exists an open neighborhood of $x$ contained in $A \cup B$, so $x \in (A \cup B)^\circ$. Thus, $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.
**5.** Example: Let $A = [0, 2]$ and $B = [1, 3]$.

- $A^\circ = (0, 2)$,
- $B^\circ = (1, 3)$,
- $A^\circ \cup B^\circ = (0, 2) \cup (1, 3) = (0, 3)$,
- $(A \cup B)^\circ = (0, 3)$.

In this example, $A^\circ \cup B^\circ = (A \cup B)^\circ$, but for some other sets, the inclusion $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ may be strict.

**11.1/15 If $A \subseteq \mathbb{R}$, let $A^-$ be the intersection of all closed sets that are containing $A$; the set $A^-$ is called the closure of A. Show that $A^-$ is a closed set, the smallest closed set containing A, and that a point $\omega$ belongs to $A^- \iff \omega$ is either an interior point or a boundary point of A.**

*Proof:*

DO

$\square$

**11.1/18 Show that if $F \subseteq \mathbb{R}$ is a closed nonempty set that is bounded above, then $\sup F \in F$.**

*Proof:* Let $F \subseteq \mathbb{R}$ be a closed, non-empty set that is bounded above. Since $F$ is bounded above, there exists some real number $M$ such that for all $x \in F$, $x \leq M$. Therefore, $F$ has an upper bound.

Since $F$ is non-empty and bounded above, the least upper bound (supremum) $\sup F$ exists by the completeness property of $\mathbb{R}$.

By the definition of the supremum, for all $x \in F$, we have $x \leq \sup F$. Additionally, for any $\epsilon > 0$, there exists some $x \in F$ such that $\sup F - \epsilon < x \leq \sup F$, meaning that the elements of $F$ can be arbitrarily close to $\sup F$.

Since $\sup F$ is the least upper bound, it is also a limit point of $F$, because elements of $F$ can be found arbitrarily close to $\sup F$.

Given that $F$ is closed, it contains all of its limit points. Therefore, since $\sup F$ is a limit point of $F$, we conclude that $\sup F \in F$.

$\square$

## 8.2 Compact Sets

Let us start with the notion of an open cover before moving on to understanding compactness.

**Definition 8.8.** *Let $A \subset \mathbb{R}$. An **open cover** of $A$ is a collection $G = \{G_\alpha\}$ of open sets in $\mathbb{R}$ whose union contains $A$; that is,*

$$A \subseteq \bigcup_\alpha G_\alpha$$

If $G' \subset G$ and $A \subseteq G'$, then $G'$ is a **subcover** of G, and if $G'$ contains finitely many sets, then it is a **finite subcover** of G.

**Definition 8.9.** *A subset $K$ of $\mathbb{R}$ is said to be **compact** if every open cover of $K$ has a finite subcover.*

**Theorem 8.10.** *If $K$ is a compact subset of $\mathbb{R}$, then $K$ is closed and bounded.*

Compactness has some stronger theorems:

**Theorem 8.11** (**Heine-Borel Theorem**). *A subset $K$ of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

**Theorem 8.12.** *A subset $K$ of $\mathbb{R}$ is compact if and only if every sequence in $K$ has a subsequence that converges to a point in $K$.*

### SELECT EXERCISES [49]

**11.2/1 Exhibit an open cover of the interval $(1, 2]$ that has no finite subcover.** Let $G_n := (0, 2 - 1/n)$, $n \in \mathbb{N}$. Then clearly $(1, 2] \subseteq \cup_{n=1}^{\infty} G_n$, which is an open cover. However, if $\{G_{n_1}, \ldots G_{n_k}\}$ is a finite subcollection and if we let $m := \sup\{n_1, \ldots, n_k,\}$ then

$$G_{n_1} \cup \cdots \cup G_{n_k} = G_m = (0, 2 - 1/m)$$

We see that $2 \notin G_m$. Thus, there is no finite subcover $\implies$ $(1, 2]$ is not compact.

**11.2/3 Exhibit an open cover of the set $\{1/n : n \in \mathbb{N}\}$ that has no finite subcover.** Let $G_n := (1/2n, 2)$, $n \in \mathbb{N}$. Since $1/2n < 1/n$, we have that $1/n \in G_n$. Consider a finite subcollection, and similar to above, let $m$ be the supremum of all the indices. Then $G_m = (1/2m, 2)$. Since $0 < 1/2m$, by the Archimedean principle, we can find a number such that $1 < \epsilon < 1/2m$. This $\epsilon = 1/a$, $a \in \mathbb{N}$. Thus, $1/a \notin G_m$.

**11.2/4 Prove, using definition 11.2.2 (7.9 in this document), that if $F$ is a closed subset of a compact set $K$ in $\mathbb{R}$, then $F$ is compact.**

[49] Page 336 of the textbook, section 11.2

*Proof:* Since K is compact, every open cover $G_n$ of K has a finite subcover $G_m,\ m < \infty$. We can write this as $K \subseteq G_m \subseteq G_n$. Since F is a closed subset of K, we have $F \subseteq K \implies F \subseteq K \subseteq G_m \subseteq G_n \implies F \subseteq G_m \subseteq G_n$. Thus, every open cover of K is an open cover of F such that F has a finite subcover, the same one that covers K.

□

**11.2/8 Prove that the intersection of an arbitrary collection of compact sets in $\mathbb{R}$ is compact.**

*Proof:* Let $\{K_i\}_{i\in I}$ be an arbitrary collection of compact sets in $\mathbb{R}$, where $I$ is an indexing set (possibly infinite). We need to prove that the intersection

$$K = \bigcap_{i\in I} K_i$$

is compact. We will show that $K$ is both closed and bounded.

**1.** $K$ **is closed:** Each $K_i$ is compact, and therefore closed in $\mathbb{R}$. Consider a sequence $\{x_n\} \subseteq K$ converging to $x$. Since $x_n \in K$, we have $x_n \in K_i$ for all $i \in I$. Since each $K_i$ is closed, $x \in K_i$ for all $i$, which implies $x \in K$. Thus, $K$ is closed.

**2.** $K$ **is bounded:** Each $K_i$ is compact, so it is bounded. Hence, for each $i \in I$, there exists $M_i$ such that $K_i \subseteq [-M_i, M_i]$. Define $M = \sup_{i\in I} M_i$, which is finite since the $M_i$'s are bounded. Then, $K \subseteq [-M, M]$, and $K$ is bounded.

Since $K$ is both closed and bounded, it is compact in $\mathbb{R}$.

□

**11.2/10 Let $K \neq \emptyset$ be a compact set in $\mathbb{R}$. Show that $\inf K$ and $\sup K$ exist and belong to $K$.**

*Proof:* Since $K$ is compact, it is closed and bounded. Therefore, $\inf K$ and $\sup K$ exist in $\mathbb{R}$ by the Least Upper Bound Property of the real numbers.

**1. Proving** $\inf K \in K$**:**

Consider the set

$$K_n := \{k \in K : k < \inf K + \frac{1}{n}\}.$$

Each $K_n$ is compact and non-empty, so by the previous exercise, $K_n$ is non-empty and contains points arbitrarily close to $\inf K$. Thus, there exists a sequence $\{x_n\} \subseteq K_n$ such that $x_n \to \inf K$. By the compactness of $K$, this sequence has a subsequence converging to $\inf K$, and hence $\inf K \in K$.

**2. Proving** $\sup K \in K$**:**

Similarly, consider the set

$$K'_n := \{k \in K : k > \sup K - \frac{1}{n}\}.$$

Each $K'_n$ is compact and non-empty, and by the same reasoning, there exists a sequence $\{y_n\} \subseteq K'_n$ such that $y_n \to \sup K$. Again, by compactness, this sequence has a subsequence converging to $\sup K$, so $\sup K \in K$.

Thus, $\inf K \in K$ and $\sup K \in K$.

□

## 8.3 Continuous Functions

**Lemma 8.13.** *A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A \iff \forall U(f(c))$, $\exists V(c)$ s.t. if $x \in V \cap A \implies f(x) \in U$. $U$ and $V$ are neighborhoods.*

**Theorem 8.14 (Global Continuity Theorem).** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function with domain $A$. Then the following are equivalent:*

*(a) $f$ is continuous at every point of $A$.*

*(b) For every open set $G$ in $\mathbb{R}$, there exists an open set $H \in \mathbb{R}$ such that $H \cap A = f^{-1}(G)$.*

**Corollary 8.15.** *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous $\iff$ $f^{-1}(G)$ is open in $\mathbb{R}$ whenever $G$ is open.*

**Theorem 8.16 (Preservation of Compactness).** *If $K$ is a compact subset of $\mathbb{R}$ and if $f : K \to \mathbb{R}$ is continuous on $K$, then $f(K)$ is compact.*

**Theorem 8.17.** *If $K$ is a compact subset of $R$ and $f : K \to \mathbb{R}$ is injective and continuous, then $f^{-1}$ is continuous on $f(K)$.*

## SELECT EXERCISES [50]

**11.3/1 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for $x \in \mathbb{R}$.**

[50] Page 340 of the textbook, section 11.3

(a) **Show that the inverse image $f^{-1}(I)$ of an open interval $I := (a, b)$ is either an open interval, the union of two open intervals, or empty, depending on $a$ and $b$.**

(b) **Show that if I is an open interval containing 0, then the direct image $f(I)$ is not open.**

**Part (a):** The function $f(x) = x^2$ is a parabola opening upwards with vertex at $(0, 0)$. It is not one-to-one over all of $\mathbb{R}$, but is increasing for $x > 0$ and decreasing for $x < 0$. Thus, for a given $y = f(x)$, there are typically two values of $x$ corresponding to it: one positive and one negative, unless $y = 0$, where the only solution is $x = 0$. We need to find the set $f^{-1}(I) = \{x \in \mathbb{R} \mid f(x) \in (a, b)\}$, i.e., the set of $x$ values such that $f(x) = x^2 \in (a, b)$.

- If $a \geq 0$ and $b > a$, then $x^2 \in (a, b)$ implies that:

  $$\sqrt{a} < |x| < \sqrt{b}.$$

  This gives two intervals for $x$:

  $$-\sqrt{b} < x < -\sqrt{a} \quad \text{or} \quad \sqrt{a} < x < \sqrt{b}.$$

  So, in this case, $f^{-1}(I) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$, which is the union of two open intervals.

- If $a < 0$ and $b > 0$, then $x^2 \in (a, b)$ implies that $x^2 \in (0, b)$, i.e., $|x| < \sqrt{b}$, so the inverse image is the open interval:

  $$-\sqrt{b} < x < \sqrt{b}.$$

  Thus, $f^{-1}(I) = (-\sqrt{b}, \sqrt{b})$.

- If $a = 0$ and $b > 0$, then $x^2 \in (0, b)$, which is the same as the previous case, so $f^{-1}(I) = (-\sqrt{b}, \sqrt{b})$.

- If $a < 0$ and $b \leq 0$, then the set $f^{-1}(I)$ is empty because $f(x) = x^2$ is always non-negative, and there are no $x$ values such that $x^2 \in (a, b)$ when $a < 0$ and $b \leq 0$.

**Part (b):** Let $I$ be an open interval containing 0, say $I = (-\epsilon, \epsilon)$ for some small $\epsilon > 0$. The image of $I$ under $f(x) = x^2$ is:

$$f(I) = \{x^2 \mid x \in (-\epsilon, \epsilon)\} = [0, \epsilon^2).$$

This is a closed interval $[0, \epsilon^2)$, which is not open because it includes the point 0, but no points to the left of 0 (there's no neighborhood around 0 entirely contained within $f(I)$). If $I$ is an open interval containing 0, then the direct image $f(I) = [0, \epsilon^2)$ is not open because it includes 0 but has no open neighborhood around 0 within it.

**11.3/2 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/(1 + x^2)$ for $x \in \mathbb{R}$.**

(a) **Find an open interval $(a, b)$ whose direct image under $f$ is not open.**
Consider the open interval $I = (0, 1)$. We want to find the preimage $f^{-1}(I)$, which is the set of all $x \in \mathbb{R}$ such that $f(x) \in (0, 1)$. This implies:

$$0 < \frac{1}{1 + x^2} < 1 \quad \text{for} \quad x \neq 0.$$

Thus, the preimage is $f^{-1}((0, 1)) = \mathbb{R} \setminus \{0\}$, which is not open because 0 is a limit point but not included in the set.

(b) **Show that the direct image of the closed interval $(0, \infty]$ is not closed.**
The function $f(x) = \frac{1}{1+x^2}$ takes values in the interval $(0, 1]$, with the maximum value $f(0) = 1$ and the function approaching 0 as $|x| \to \infty$. Thus, the direct image of $(0, \infty]$ under $f$ is:

$$f(\mathbb{R}) = (0, 1]\,.$$

Since the interval $(0, 1]$ does not include the point 0, though it is a limit point, it is not closed in $\mathbb{R}$.

**11.3/4 Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) := 1, 0 \leq x \leq 1$ and $h(x) := 0$ otherwise. Find an open set $G$ such that $h^{-1}(G)$ is not open, and a closed set $F$ such that $h^{-1}(F)$ is not closed.**

*Proof:* Consider the open set $G = (1/2, 3/2)$. We compute the preimage of $G$ under $h$. For $x \in [0, 1]$, we have $h(x) = 1$, which is in the interval $(1/2, 3/2)$. For $x \notin [0, 1]$, we have $h(x) = 0$, which is not in $(1/2, 3/2)$. Therefore, the preimage $h^{-1}(G)$ is the interval $[0, 1]$, which is not open in $\mathbb{R}$ since it contains the boundary points 0 and 1 but is not an open set.
Thus, the open set $G = (1/2, 3/2)$ has the property that $h^{-1}(G) = [0, 1]$, which is not open.

Next, consider the closed set $F = [-1/2, 1/2]$. We compute the preimage of $F$ under $h$. For $x \in [0, 1]$, $h(x) = 1$, and for $x \notin [0, 1]$, $h(x) = 0$. The set $F = [-1/2, 1/2]$ contains 0 but does not contain 1, so the preimage $h^{-1}(F)$ will include all points where $h(x) = 0$, which are outside the interval $[0, 1]$, but it will not include the boundary points where $h(x) = 1$. This leads to $h^{-1}(F) = (-\infty, 0) \cup (1, \infty)$, which is not closed in $\mathbb{R}$ since it does not contain the limit points 0 and 1.
Thus, the closed set $F = [-1/2, 1/2]$ has the property that $h^{-1}(F) = (-\infty, 0) \cup (1, \infty)$, which is not closed.

□

**11.3/8 Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that the set $\{x \in \mathbb{R} : f(x) = 1\}$ is neither open nor closed in $\mathbb{R}$.**

*Proof:* Consider the Dirichlet discontinuous function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

The set $A = \{x \in \mathbb{R} : f(x) = 1\}$ consists of all rational numbers $\mathbb{Q}$. We now show that this set is neither open nor closed in $\mathbb{R}$:

$A$ is not open: For any $x \in A$, i.e., for any $x \in \mathbb{Q}$, every open interval around $x$ contains both rational and irrational numbers. Since $f(x) = 1$ for rational $x$, but $f(x) = 0$ for irrational $x$, any open interval around a rational point contains points where $f(x) = 0$. Thus, $A$ does not contain an open neighborhood around any of its points, meaning $A$ is not open.

$A$ is not closed: The set $A$ (the set of rational numbers $\mathbb{Q}$) is not closed because it is not equal to its closure. The closure of $A$ is $\mathbb{R}$, since every irrational number is a limit point of rational numbers. Hence, $A$ does not contain all of its limit points, so it is not closed.

$\square$

**11.3/9 Prove that $f : \mathbb{R} \to \mathbb{R}$ is continuous $\iff$ for each closed set $F \in \mathbb{R}$, the inverse image $f^{-1}(F)$ is closed.**

*Proof:* ( $\implies$ ) To prove that $f^{-1}(F)$ is closed, consider a sequence $\{x_n\}$ in $f^{-1}(F)$ that converges to some point $x \in \mathbb{R}$. Since $x_n \in f^{-1}(F)$, we know that $f(x_n) \in F$ for all $n$. Since $F$ is closed and $f(x_n) \to f(x)$ (by the continuity of $f$), it follows that $f(x) \in F$. Thus, $x \in f^{-1}(F)$, which shows that $f^{-1}(F)$ contains all its limit points and is therefore closed.

( $\impliedby$ ) By definition, $f$ is continuous if for every open set $U \subseteq \mathbb{R}$, the preimage $f^{-1}(U)$ is open. To prove this, recall that every open set $U$ in $\mathbb{R}$ can be written as the complement of a closed set, i.e., $U = \mathbb{R} \setminus F$, where $F = \mathbb{R} \setminus U$ is closed. Since we assume that the preimage of every closed set is closed, we know that $f^{-1}(F)$ is closed. Therefore, the preimage of the open set $U$ is:

$$f^{-1}(U) = f^{-1}(\mathbb{R} \setminus F) = \mathbb{R} \setminus f^{-1}(F).$$

Since $f^{-1}(F)$ is closed, its complement $f^{-1}(U)$ is open. Thus, $f^{-1}(U)$ is open for every open set $U$, which proves that $f$ is continuous.

$\square$