MATH 8440 Advanced ODEs
Notes and Selected Exercises

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Contents

1 BASIC CONCEPTS

1.1 Definitions

By an *ODE* of order n, we mean a relation of form $F(x, y, y', y'', ...y^{(n)}) = 0$, involving function F of $n+1$ variables; x is the *independent variable*, $y = y(x)$ is an unknown function, and $y', ...$ are the first n derivatives of y w.r.t. x.

A function $y = \phi(x)$ is a *solution* of the DE if the substitution

$$y = \phi(x), y' = \phi'(x), ..., y^{(n)} = \phi^{(n)}(x)$$

reduces the DE to an identity. Unless stated otherwise, the *quantities* $x, y, y', y'', ...y^{(n)}$ *take only (finite) real values* and all *functions are single-valued* (each input gives a single output).

Often times, many functions may satisfy a DE, i.e., non-unique solutions. However, to obtain a specific solution for a problem, we require a supplemental condition, one that specifies a specific value for the independent variable called the *initial condition*.

The process of finding solutions of a DE is called the *integration* of the equation.

1.2 Geometric Interpretation

Consider the DE

$$y' = f(x, y), \tag{1.2.1}$$

where the function $f(x, y)$ is defined on some xy-plane. Suppose, for every point (x, y) of the domain, we draw a short line segment with slope $f(x, y)$ (the two directions of a line segment are not distinguished, so no arrows on the slope field). This set of directions (or slopes) in the domain is called its *direction field*, in this case, it is field for equation (1.2.1).

Now, the problem of solving a DE has changed: we must find all graphs $y = \phi(x), a < x < b$ in the domain whose tangents have directions belonging to the direction field of (1.2.1). In other words, we are looking for all $y = \phi(x)$ that are consistent with the direction field– each point on the graph y, the tangest to the curve should align with the direction indicated by the direction field at that point.

Need for Robustness

From a geometric POV, we have two problems:

1. Since the slope of (x, y) is given by $f(x, y)$, we naturally exclude all directions parallel to the y-axis. Since the slop is undefined. So this method in this rudimentary form is not robust– we need a way to include points that have direction fields parallel to the y-axis.

2. By considering only graphs with equations of the form $y = \phi(x)$, we exclude curves which are intersected more than once by perpendiculars to the x-axis. This happens since each function is single-valued.

To deal with the first problem listed above, we must allow $f(x, y) \to \infty$. So, we can introduce the following

$$\frac{dx}{dy} = f_1(x, y), \tag{1.2.2}$$

where

$$f_1(x, y) = \frac{1}{f(x, y)}.$$

At points where f and f_1 are defined, we can choose either. However, where $f \to \infty$, we use equation (1.2.2). Note that, all times at least one of the functions is defined; when one of them $\to \infty$, the other $\to 0$.

Then, we allow general curves in parametric form instead of graphs. This leads to the following generalization of the problem: find all curves $x = \lambda(t), y = \mu(t), \alpha < t < \beta$ whose tangents have directions specified by (1.2.1) and (1.2.2). Such curves are called *integral curves* of (1.2.1) and (1.2.2). Note that a graph can be an integral curve, but not vice versa. By considering parametric curves, we generalize the problem to account for all possible curves whose tangents align with the direction fields. This approach is useful in scenarios where a single-valued function might not be sufficient.

Now, we have something more robust.

Example: Consider

$$\frac{dy}{dx} = \frac{y}{x} \tag{1.2.3}$$

If we set a constant $k = y/x \implies y = kx$. So the solution is given by curves defined by

$$y = kx. \tag{1.2.4}$$

However, the set of all curves is given by

$$ax + by = 0, \tag{1.2.5}$$

where a and b are arbitrary constants and not both zero. Graphically, the solution looks like beams of rays shooting out the origin since k can be any real value slope.

Example: Consider

$$\frac{dy}{dx} = \frac{-x}{y}, \tag{1.2.6}$$

the reciprocal of the previous example.

From basic slope stuff, from the previous example, each point's perpendicular tangent is the solution to this example.

2 SIMPLE EQUATIONS

2.1 $y' = f(x)$

We consider the equation

$$\frac{dy}{dy} = f(x) \tag{2.1.7}$$

which can be distinguished into two cases:

Case 1: *The function is continuous on* (a, b)
Then one solution to (2.1.7) is given by

$$y(x) = \int_{x_0}^{x} f(s)ds + C \tag{2.1.8}$$

where $x, x_0 \in (a, b)$ and all other solutions differ by the additive constant C. If the given IC has to pass through a point $(x_0, y_0) \implies C = y_0$, where $a < x < b$. The IC would be unique.

Case 2: *The function is continuous on* (a, b) *except at point* c *(*$a < c < b$*) where* $F(x) \to \infty$ *as* $x \to c$.
We use the same method used to obtain equation (1.2.2) to determine the direction field at $x = c$:

$$\frac{dx}{dy} = \frac{1}{f(x)} \tag{2.1.9}$$

Note that the direction field becomes **steeper** and steeper as $x \to c$. We can divide the interval (a, b) into two sections (a, c) and (c, b). We can replicate case 1 in each subinterval; however, combining the two with a discontinuous point c is where case 2 differs from case 1. if $(x_0, y_0) \in (a, c) \implies \exists!\ IC$ s.t.

$$y = y_0 + \int_{x_0}^{x} f(s)ds \tag{2.1.10}$$

(a) Consider

$$\int_{x_0}^{x} f(s)ds \quad a < x_0 < c \tag{2.1.11}$$

Suppose (2.1.11) converges as $x \to c - 0$. Then, (2.1.10) IC approaches a definite point on $x = c$. Suppose (2.1.11) diverges as $x \to c - 0$. We opt to study the subinterval (c, b) in a similar manner.

(b) Let $f \to +\infty$ as $x \to c \pm 0$, and suppose (2.1.11) converges as $x \to c + 0$ (note that $c < x_0 < b$ rather than $a < x_0 < c$) if and only if the integral also converges as $x \to c - 0$. Then we obtain indefinitely many ICs passing through any given (x_0, y_0). This is true because all converge at $x = c$, resulting in multiple entry and exit points.

(c) If we suppose the previous situation except that (2.1.11) diverges. Then we obtain unique solutions as $x = c$ is virtually untouched. We obtain **asymptotic regimes**.

(d) Finally, consider a final situation where $f \to -\infty$ from the left and $f \to +\infty$ from the right and the integrals corresponding to each subinterval converge. Since we obtain a vertical line at $x = c$ with each IC converging onto it, we lose uniqueness.

We observe that the convergence of ICs most likely leads to a loss in uniqueness of solutions. Divergence gives us what we need for uniqueness.

2.2 $y' = f(y)$

The difference between

$$\frac{dy}{dx} = f(y) \tag{2.2.12}$$

and (2.1.7) from the previous subsection is that the roles of x and y are reversed. If $f(y)$ is continuous and nonzero for $y \in (a, b)$, we write

$$\frac{dx}{dy} = \frac{1}{f(y)} \tag{2.2.13}$$

So, we get a unique IC

$$x = x_0 + \int_{y_0}^{y} \frac{ds}{f(s)} + C$$

passing through every $(x_0, y_0) \in (a, b)$. Like before, all shifts parallel to the x-axis generate alternate solutions, represented by the additive constant C.

Performing a similar analysis as in the previous section: suppose $f(y)$ is continuous on $(a, b) - \{c\}$ where $f(y) \to 0$ as $y \to c$. We find ourselves with the following cases:

1. If

$$\int_{y_0}^{y} \frac{ds}{f(s)} \tag{2.2.14}$$

diverges as $y \to c \pm 0, \exists! IC$ passing through each $y \in (a, b)$, and the line $y = c$ is the asymptote of all ICs.

2. If (2.2.14) converges as $y \to c \pm 0$ and does not change sign as y passed through $y = c$, there are indefinitely many ICs passing through– no uniqueness.

3. If we are in a situation similar to above, but (2.2.14) changes sign– then we remain unique in $y \in (a, c)$ and $y \in (c, b)$, but not at $y = c$.

2.3 Equations with Separated Variables

We refer to equations of the form

$$\frac{dy}{dx} = f_1(x) f_2(y) \tag{2.3.15}$$

THEOREM. *Suppose $f_1(x)$ is continuous for $x \in (a, b)$ and $f_2(y)$ is continuous and nonzero for $y \in (c, d)$. Then there exists a unique IC for (2.3.15) that passes through the rectangle $(a, b) \times (c, d)$.*

Proof. We first prove uniqueness. Suppose $\varphi(x)$ is a solution and $\varphi(x_0) = y_0$. We have the identity:

$$\frac{d\varphi(x)}{dx} = f_1(x) f_2(\varphi(x)) \implies \frac{d\varphi(x)}{f_2[\varphi(x)]} = f_1(x) dx,$$

Integrating both sides between x_0 and x gives

$$F_2(\varphi(x)) - F_2(y_0) = F_1(x) - F_1(x_0),$$

where $F_2(y)$ is a primitive of $1/f_2(y)$ and $F_1(x)$ is a primitive of $f_1(x)$. Since F_2 is strictly monotonic, we can solve uniquely for $\varphi(x)$ as:

$$\varphi(x) = F_2^{-1} \left(F_2(y_0) + F_1(x) - F_1(x_0) \right).$$

This shows that the solution is unique. For existence, note that the function $\varphi(x)$ defined above satisfies the differential equation in a neighborhood of x_0 and meets the initial condition. Differentiating shows that $\varphi(x)$ satisfies the equation:

$$\frac{1}{f_2(\varphi(x))} \varphi'(x) = f_1(x),$$

and since $\varphi(x_0) = y_0$, the initial condition is satisfied. This completes the proof.

$\blacksquare$

NOTE. If $f(y)$ vanishes at $y = y_1$, then uniqueness is dependent on the convergence of (2.2.14) as $y \to y_1$. If convergent, no uniqueness; else, uniqueness. Vanish $\implies$ $f_2(y_1) = 0$.

2.4 Homogeneous Equations

We refer to equations of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{2.4.16}$$

Suppose $f(u)$ is defined in (a, b)/ Then $f(y/x)$ will be defined in the pair of angles consisting of the points (x, y) such that $a < y/x < b$. The domain with these angles is G.

THEOREM. *Suppose $f(u) \in C(a, b)$ and $f(u) \neq u$. Then $\exists! IC$ for (2.3.15) passing through $(x_0, y_0) \in G$.*

Proof. Rewrite $y = xu \implies y' = u + xu' = f(u) \implies u' = \frac{f(u)-u}{x}$, which is separable (we can use uniqueness result from previous subsection). Since the numerator is never zero as $f(u) \neq u$, $\forall u \in (a, b)$, we have a unique solution $\phi(u) = \int \frac{du}{f(u)-u}$. We integrate to get

$$\ln|x| = \phi(y/x) + C \tag{2.4.17}$$

where ϕ is the primitive of $1/(f(u) - u)$. We find the solutions remain invariant as $x \mapsto cx$ and $y \mapsto cy$. Thus, solution is unique.

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NOTE. If $f(u) = u$ at separate points, then several ICs pass through $(x_0, y_0) \in G$. Moreover if ϕ converges as $u \to c$, then we obtain a line $y = cx$ (similar to x=c in subsection 2.1) where ICs converge. We lose uniqueness.

2.5 Linear Equations

We refer to equations of the form

$$\frac{dy}{dx} = a(x)y + b(x) \tag{2.5.18}$$

THEOREM. *If $a(x), b(x) \in C(\alpha, \beta)$, then there exists a unique $IC(x_0, y_0)$ for all $x_0 \in (\alpha, \beta)$, $y_0 \in \mathbb{R}$*

Proof. Let's first consider the **homogeneous case** where $b = 0$. The differential equation becomes:

$$y' = a(x)y.$$

This is a *separable differential equation*, meaning we can rewrite it as:

$$\frac{dy}{y} = a(x)dx.$$

Integrating both sides, we get:

$$\ln |y| = \int a(x)dx.$$

Exponentiating both sides, we have the solution:

$$y = y_0 \exp\left(\int_{x_0}^{x} a(\zeta)d\zeta\right),$$

where y_0 is the value of the solution at $x = x_0$. This confirms that there is a unique solution passing through (x_0, y_0).

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Example: Consider the differential equation:

$$y' = a(x)y, \quad a(x+T) = a(x) \text{ for all } x \in (-\infty, \infty).$$

This is a **periodic differential equation** with period T. The solution to the equation $\phi' = a(x)\phi$ defines a **monodromy map**:

$$\Lambda : \mathbb{R} \to \mathbb{R}, \quad \Lambda(\phi(0)) = \phi(T).$$

There exists a constant λ such that $\phi(T) = \lambda\phi(0)$. Specifically, we can compute λ as:

$$\lambda = \exp\left(\int_{x_0}^{x_0+T} a(\zeta)d\zeta\right).$$

This represents the **amplification factor** after one period T.

- If $\lambda = 1$, the solution is **periodic** with period T.
- If $\lambda < 1$, the solution decays to 0 as $x \to \infty$, meaning the solution is stable.
- If $\lambda > 1$, the solution grows unbounded as $x \to \infty$, meaning the solution is **unstable**.

Example: Now, consider the **non-homogeneous equation**:

$$y' = a(x)y + b(x),$$

with initial condition $y(0) = y_0$.
To solve this, we set $y = zv$, where z is to be determined, and solve for v using:

$$v' = a(x)v, \quad v(0) = 1.$$

The solution for v is:

$$v(x) = \exp\left(\int_{x_0}^{x} a(\zeta)d\zeta\right).$$

Substituting into the original equation, we get:

$$z' = \frac{b(x)}{v(x)}.$$

Solving for z, we obtain:

$$z(x) = y_0 + \int_{x_0}^{x} b(s)\exp\left(-\int_{x_0}^{s} a(\zeta)d\zeta\right) ds.$$

Thus, the solution to the original equation is:

$$y(x) = v(x)\left(y_0 + \int_{x_0}^{x} b(s)\exp\left(-\int_{x_0}^{s} a(\zeta)d\zeta\right) ds\right).$$

Using the monodromy map, we can rewrite the solution in terms of λ as:

$$y = \lambda y_0 + C,$$

where $\lambda = \exp\left(\int_{\cdot}^{T} a(\zeta)d\zeta\right)$.

THEOREM. *If $\lambda \neq 1$, there exists a unique initial condition y_0^* such that the solution is periodic. The initial condition y_0^* satisfies:*

$$y_0^* = \frac{C}{1-\lambda}.$$

- *If $\lambda < 1$, the solution is **stable**. The periodic solution persists and decays as $x \to \infty$.*
- *If $\lambda > 1$, the solution is **unstable** and grows unbounded as $x \to \infty$.*

Proof. Consider the sequence $\phi_n = \lambda\phi_{n-1} + C$. We have:

$$|\phi_n - \phi_{n-1}| = \lambda|\phi_{n-1} - \phi_{n-2}| = \cdots = \lambda^n|\phi_1 - \phi_0| \to 0 \text{ as } n \to \infty.$$

Thus, $\{\phi_n\}$ is a **Cauchy sequence** and converges to a fixed point $\phi^* = y_0^*$. Therefore, the solution remains bounded as $x \to \infty$, indicating stability.
If $\lambda > 1$, the solution grows without bound, indicating **instability**.

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We start with a first-order linear ODE with periodic coefficients:

$$y' = a(x)y + b(x), \qquad a(x+T) = a(x), \qquad b(x+T) = b(x). \tag{2.5.19}$$

The **monodromy map** $P(y_0)$ is introduced, which maps the initial value y_0 to the value of the solution after one period T, i.e.,

$$P(y_0) = y(x_0 + T). \tag{2.5.20}$$

In the non-homogeneous case, the map takes the form

$$P(y_0) = \lambda y_0 + C, \quad \text{where } \lambda = \exp\left(\int_{x_0}^{x_0+T} a(s)ds\right). \tag{2.5.21}$$

If $\lambda \neq 1$, there is a unique periodic solution given by

$$y_0^* = \lambda y_0^* + C. \tag{2.5.22}$$

STABILITY DISCUSSION The solution behavior stabilizes as you repeatedly apply the monodromy map, so the solution $P^n(y_0)$ approaches the periodic solution.

Example: Consider the equation $y' = -y + \sin(x)$. For this equation, the coefficients are periodic with period 2π. Since $\lambda < 1$, there exists a unique periodic solution. By solving for the periodic solution, we find that

$$y(x) = \frac{1}{2}(\sin(x) - \cos(x)) + ce^{-x}.$$

As $x \to \infty$, the non-periodic term decays, and the solution converges to the periodic solution $\frac{1}{2}(\sin(x) - \cos(x))$.

LOGISTIC GROWTH MODEL w/ PERIODIC HARVESTING The logistic ODE with a periodic harvesting term is introduced:

$$y' = (1 - y)y + h(x), \tag{2.5.23}$$

where $h(x)$ is periodic with period $T = 1$. If the coefficients are not periodic, the monodromy map is instead called a **Poincaré map**.

A **Poincaré map** is a mathematical tool used in the study of dynamical systems, particularly for analyzing periodic orbits. It transforms the continuous dynamics of a system into a discrete framework by mapping a point in the phase space to another point after one complete cycle (period) of the system's evolution. This approach is particularly useful for systems with periodic or quasi-periodic behavior, as it helps to determine the stability of orbits and the existence of fixed points. Fixed points of the Poincaré map correspond to periodic orbits of the original system, and their stability can provide valuable insights into the long-term behavior of the system.

THEOREM. *A solution $y(x, y_0)$ is periodic if and only if the Poincaré map has a fixed point.*

THEOREM. *The logistic equation with a harvesting term has at most two periodic solutions.*

Proof. We will assume that the initial value problem depends smoothly on the initial data $y(0) = y_0$. Thus, we can define $\phi(x, y_0) = \frac{\partial}{\partial y_0} y(x, y_0)$. Let $\phi' := \frac{\partial}{\partial x}\phi = \frac{\partial}{\partial y_0}\frac{\partial}{\partial x} y(x, y_0)$. Here, we assume that the partial derivatives commute. Then,

$$\begin{aligned} \phi' &= \frac{\partial}{\partial y_0}\left((1-y)y + h(x)\right) \\ &= (\phi - 2y\phi) \\ &= (1 - 2y(x, y_0))\,\phi(x). \end{aligned}$$

This is a linear ODE in ϕ if we assume y_0 is fixed. Then

$$\phi(x) = \phi(0)\exp\left(\int_0^x (1 - 2y(s, y_0))ds\right).$$

Observe that $\phi(0) = 1$. Therefore,

$$\phi(x) = \exp\left(\int_0^x (1 - 2y(s, y_0))ds\right).$$

This is periodic with period 1. Moreover,

$$\begin{aligned} \phi(1) &= \exp\left(\int_0^1 (1 - 2y(s, y_0))ds\right) \\ &= \frac{\partial}{\partial y_0} y(1, y_0) \qquad\qquad = \frac{\partial}{\partial y_0} P(y_0) = P'(y_0). \end{aligned}$$

Since $\exp(\cdot) > 0$, then $P'(y_0) > 0$. Let's compute the second derivative:

$$\begin{aligned} P''(y_0) &= \frac{\partial}{\partial y_0}\exp\left(\int_0^1 1 - 2y(s, y_0)ds\right) \\ &= \phi(1)\frac{\partial}{\partial y_0}\int_0^1 1 - 2y(s, y_0)ds \\ &= \phi(1)(-2)\int_0^1 \frac{\partial}{\partial y_0} y(s, y_0)ds \\ &= \phi(1)(-2)\int_0^1 \phi(s)ds, \end{aligned}$$

which is negative. Hence, the Poincaré map is concave down for all y_0.
There are exactly three scenarios: 0, 1, or 2 fixed points y_0 such that $P(y_0) = y_0$.

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Example: Harvest Quota Equation This is a variation of the logistic equation:

$$y' = (1-y)y - c. \tag{2.5.24}$$

Here, $h(x) = -c$, and this is periodic with period $T = 1$. The roots of the equation $y^2 - y + c = 0$ are

$$y = \frac{1}{2}(1 \pm \sqrt{1-4c}).$$

If $c < 1/4$, there will be two periodic solutions; if $c > 1/4$, none; and if $c = 1/4$, one solution. Looking at the phase diagram, we see that $c = 1$ produces an attractor and $c = 1/4$ produces a repeller.

$$\begin{aligned}
\tilde{y} &:= y - \frac{1}{2} \\
\dot{\tilde{y}} &= \dot{y} = -\tilde{y}^2 \\
\tilde{y}^2 &= (y - \frac{1}{2})^2 = y^2 - y + \frac{1}{4},
\end{aligned}$$

so $\dot{\tilde{y}} = -\tilde{y}^2$. Then

$$\begin{aligned}
-\frac{\dot{\tilde{y}}}{y^2} &= c \\
\frac{1}{\tilde{y}} &= x + c \\
\tilde{y} &= \frac{1}{x+c} \\
\tilde{y}(0) &= \tilde{y}_0.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{y} &= \frac{1}{x + \frac{1}{\tilde{y}}} = \frac{1}{x + \frac{1}{y_0 - \frac{1}{2}}} \\
\Rightarrow y - \frac{1}{2} &= \frac{1}{x + \frac{1}{y_0 - \frac{1}{2}}} \\
\Rightarrow y(x) &= \frac{1}{2} + \frac{1}{x + \frac{1}{y_0 - \frac{1}{2}}} = \frac{y_0 - \frac{1}{2}}{y_0 + \frac{1}{2}} + \frac{1}{2}.
\end{aligned}$$

The Poincaré map at $T = 1$ is

$$P(y_0) = \frac{1}{1 + \frac{1}{y_0 - \frac{1}{2}}} + \frac{1}{2}.$$

Notice that $\frac{1}{2}$ is a fixed point, so $y_0 = 0$ is not periodic in the case of harvesting.

2.6 Exact Equations and Integrating Factors

We discuss equations of the form

$$M(x,y)dx + N(x,y)dy = 0$$

If the LHS of the equation above is an *exact* differential of some function of x and y, then the equation is called an **exact equation**. If partial w.r.t. y and x for M and N existed are continuous, we need the following condition to hold

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

on some simply connected domain G.

A domain is **simply connected** if whenever G contains a closed polygonal line L with no self-intersections, it also contains the interior of L.

THEOREM. *Let M,N, and their partials be continuous on a rectangle $Q : a < x < b, c < y < d$, and suppose N is nonvanishing in Q and the exact condition (stated above) holds everywhere in Q. Then there exists a unique IC passing through every point of Q.*

Proof. Suppose a function z(x,y) satisfying the LHS of exact equation above. Since N is nonvanishing, we can write

$$M + Ny' = 0 \implies \frac{dz[x,y(x)]}{dx} = 0$$

since $M \equiv \partial z/\partial x$ and $N \equiv \partial z/\partial y$. Then y(x) is a solution $\iff$ $z[x,y(x)] = C$, which is some constant. So, for some $C = z(x_0, y_0)$, we have a solution. Then by the Implicit Function Theorem, we have a unique curve passing through the point; we obtain our desired solution.

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INTEGRATING FACTORS Even when the exact condition is not met, we can transform the differential equation into an exact equation with the use of an **integrating factor**. It is a function $\mu(x,y)$ such that if M,N, and μ have continuous partial derivatives, then the following is the new exact condition

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \implies M\frac{\partial \mu}{\partial y} - M\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Then, to transform the LHS of the exact equation, we need only a particular solution of what is above– a complicated endeavour in its own right.

3 GENERAL THEORY

3.1 Contraction Mappings (Banach Fixed Point Theorem)

The following theorem is the principal of contraction mappings, a.k.a., the Banach Fixed Point Theorem.

Visualize a dot moving toward a target point, covering half the remaining distance with each step. Eventually, the dot settles at this target, known as a "fixed point," since further applications of the mapping do not change its position. The **contraction** property ensures that regardless of the starting point, repeated applications of the mapping will lead to this unique fixed point.

THEOREM. *Let X be a banach space and suppose $K : D \to D \subset X$, for some closed set D. Assume that $\forall x \in D \quad ||Kx - Ky|| \leq \theta||x - y||$, for some $\theta \in (0, 1)$. Then $\exists! x_* \in D$ s.t. $Kx_* = x_*$. Moreover, we can estimate*

$$||x_* - K^n x_0|| \leq \frac{\theta^n}{1-\theta}||Kx_0 - x_0||$$

and

$$\sum_{n=1}^{\infty} \theta_n < \infty.$$

In this context, we are dealing with a Banach space X and a closed set D within it. The mapping $K : D \to D$ is a contraction, meaning it brings points closer together by a factor of θ (where $0 < \theta < 1$). This contraction property ensures that for any two points x and y in D, the distance between their images under K is less than or equal to θ times the distance between x and y, represented by the LHS.
The first part of the statement guarantees that there exists a unique fixed point x_* in D such that $Kx_* = x_*$. The estimates provided give us a way to measure how close the iterates $K^n x_0$ (the result of repeatedly applying K starting from some initial point x_0) are to this fixed point x_*. Specifically, as n increases, the distance between $K^n x_0$ and x_* decreases exponentially due to the factor θ^n.
The sum condition shows that the series of distances converges:

$$\sum_{n=1}^{\infty} \theta^n < \infty,$$

implying that the iterates will settle down to the fixed point x_* as n goes to infinity. Intuitively, this can be thought of as a "pulling" effect where each application of K brings us closer to a stable point, much like a ball rolling down a bowl will eventually come to rest at the bottom.

The following is one of the most fundamental theorems in analysis, the implicit function theorem.

THEOREM. *Let $f(x,y)$ be defined on the strip $a \leq x \leq b, -\infty < y\infty$, and suppose $f(x,y)$ is a continuous in x and differentiable in y, where*

$$0 < m \leq f_y(x,y) \leq M < \infty$$

everywhere in the strip. Then the equation $f(x,y) = 0$ has one and only one continuous solution $y(x)$ on $a \leq x \leq b$.

The statement describes a function $f(x,y)$ defined on a vertical strip where $a \leq x \leq b$ and $-\infty < y < \infty$. This function is continuous in x and differentiable in y, with the partial derivative $f_y(x,y)$ bounded between two positive constants m and M. Intuitively, this means that as y varies, $f(x,y)$ increases at a consistent rate, ensuring stability. The equation $f(x,y) = 0$ represents a level curve where the function equals zero. For any fixed x, the positivity of f_y guarantees that there is exactly one y that makes $f(x,y) = 0$ because as y increases, $f(x,y)$ will transition from negative to positive (or vice versa) without ever oscillating back down to zero. Therefore, the combination of continuity, differentiability, and the positive lower bound on the derivative ensures the existence of a unique continuous solution $y(x)$ for each x in the interval $[a,b]$.

3.2 Euler Lines

Suppose we have a differential equation of the form $y' = f(x,y)$, where $f(x,y)$ is defined on a domain G. The equation defines a direction field which has ICs.

Choosing any point (x_0, y_0) in G, we draw a line segment through it with slope $f(x_0, y_0)$. Then on this line segment choose another point $(x_1, y_1) \in G$, and draw a line segment with slope $f(x_1, y_1)$. Keep repeating this construction such that $x_0 < x_1 < x_2 < \dots$. The resulting polygonal line is a **Euler line**.

Every such line passing points resemble an IC, provided that such Euler lines are sufficiently short and the ICs exist.

Euler lines approximate the IC almost; like infinite sums approximate and ultimately define an integral.

The Euler line converges to the IC provided that f is continuous. Though, we do not have uniqueness– we can have more than one IC through a point $(x_0, y_0) \in G$. So, we need more than continuity to demonstrate uniqueness.

3.3 Arzelà's Theorem

The main theorem is as follows

THEOREM. *Let $\{f(x)\}$ be an infinite family of uniformly bounded, equicontinuous functions defined on a finite interval (a,b). Then $\{f(x)\}$ contains a uniformly convergent infinite subsequence.*

Equicontinuous refers to uniform continuity.
The uniform boundedness ensures that all functions in the family stay within a fixed range, preventing wild oscillations, while equicontinuity guarantees that the functions do not vary too abruptly. Together, these properties imply the existence of a uniformly convergent subsequence via the compactness criteria: since the interval (a, b) allows for uniform convergence, we can extract a subsequence that converges uniformly.

The proof of this theorem is referred to while proving Peano's Existence Theorem– the next subsection.

3.4 Peano's Existence Theorem

The main theorem is as follows
THEOREM. *Let f be a bounded and continuous on a domain G. Then at least one IC of the differential equation $dy/dx = f(x, y)$ passes through each point $x_0, y_0)$ of G.*

Note that this theorem says *at least one* implying that uniqueness is yet to be guaranteed. We achieve this in the next subsection.

3.5 Osgood's Uniqueness Theorem

The main theorem is as follows.
THEOREM. *Suppose the function $f(x, y)$ satisfies the condition*

$$|f(x, y_2) - f(x, y_1)| \leq \varphi(|y_2 - y_1|)$$

for every pair of points $(x, y_1), (x, y_2)$ in a domain G, where $\varphi(u) > 0$ is a continuous function on $0 < u \leq a$ and

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{a} \frac{du}{\varphi(u)} = \infty$$

Then there is no more than one solution of the equation $y' = f(x, y)$ passing through each point (x_0, y_0) of G.

NOTE. The Lipschitz condition is such a φ

This theorem ensures the uniqueness of solutions to the differential equation $y' = f(x, y)$ under the given conditions. The function $f(x, y)$ satisfies a form of the Lipschitz condition, where the difference in function values is bounded by $\varphi(|y_2 - y_1|)$. The condition on $\varphi(u)$ ensures that the function does not allow multiple solutions through the same point, thus guaranteeing that there is at most one solution through each initial condition.

3.6 Cauchy's Theorem

The main theorem is given below.

THEOREM. *If $f(x, y)$ is analytic in both x and y in a neighborhood of the point (x_0, y_0), then the differential equation*

$$\frac{dy}{dx} = f(x, y)$$

has a unique solution $y(x)$ satisfying the initial condition

$$y(x_o) = y_0,$$

and this solution is analytic in a neighborhood of x_0.

An **analytic function** is one that can be locally represented by a convergent power series around any point in its domain. So, in other words, f is analytic if it has a **Taylor series expansion** around that point that converges to the function in some neighborhood of the point.

Proof. Without loss of generality, assume $x_0 = y_0 = 0$. Let the function $f(x, y)$ be represented as a power series:

$$f(x, y) = \sum_{k,\ell \geq 0}^{\infty} a_{k\ell} x^k y^\ell.$$

Uniqueness: Suppose $y = c_0 + c_1 x + c_2 x^2 + \ldots$ is an analytic solution to the initial value problem (IVP). Then, at $x = 0$, we have $y(0) = c_0 = 0$. Since $y'(x) = f(x, y(x))$, we can evaluate the first derivative at $x = 0$: $y'(0) = f(0, 0) = a_{00}$. For the second derivative, we compute: $2c_2 = y''(0) = f_x(0, 0) + f_y(0, 0)y'(0) = a_{01} + a_{10}c_1$. For the third derivative:

$$6c_3 = y'''(0) = f_{xx}(0, 0) + f_{xy}(0, 0)y'(0) + f_{y,x}(0, 0)y'(0) + f_{yy}(0, 0)y'^2(0) + f_y(0, 0)y''(0),$$

which is some well-defined function of the coefficients in the power series of f and the previously determined coefficients c_k. This process uniquely determines the identity of $y(x)$.

Existence: Next, consider the equation

$$(*) \quad \begin{cases} \dfrac{dz}{dx} = F(x, z), \\ z(0) = 0, \end{cases}$$

where $F(x, y) = \sum_{k,\ell} A_{k\ell} x^k y^\ell$ and $|a_{k\ell}| \leq |A_{k\ell}|$ for all k, ℓ. We claim that there exists a unique solution to this equation, which we denote as $z = c_0^* + c_1^* x + c_2^* x^2 + \ldots$. Furthermore, we claim that if $|c_k| \leq c_k^*$, then $y = y(c_0, c_1, \ldots)$ is also analytic.

First, at $x = 0$, we have: $c_0 = c_0^* = 0, \quad c_1 = f(0, 0) = a_{00}, \quad |c_1| \leq |a_{00}| \leq A_{00} = c_1^*$. Repeating the argument from the uniqueness section, we get: $c_1 = f_x(0, 0) + f_y(0, 0)c_1$, so that

$$|c_1| \leq \frac{1}{2}|f_x(0, 0)| + |f_y(0, 0)||c_1| \leq \frac{1}{2}\left(F_x(0, 0) + F_y(0, 0)c_1^*\right) = c_2^*.$$

By repeating this argument, we can bound each c_k in terms of the corresponding coefficients $A_{k\ell}$, and thus $y(x)$ remains analytic.

Now, assume that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has a radius of convergence $R > 0$. For any fixed r such that $0 < r < R$, we have

$$\left| \sum_{k=0}^{\infty} a_k x^k \right| \leq \sum_{k=0}^{\infty} |a_k| r^k < \infty \quad \text{for all} \quad |x| < r.$$

Thus, there exists $M > 0$ such that

$$|a_k| r^k \leq M, \quad |a_k| \leq \frac{M}{r^k} \quad \text{for all} \quad k.$$

Define

$$F(x) = \sum_{k=1}^{\infty} A_k x^k.$$

We claim that there exists an $r > 0$ such that

$$F(x, y) = \frac{M}{(1 - \frac{x}{r})(1 - \frac{y}{r})}$$

will majorize $f(x, y)$. Now, consider the equation

$$\frac{dz}{dx} = \frac{M}{1 - \frac{x}{r}} \frac{1}{1 - \frac{z}{r}},$$

which is separable. The solution is

$$z - \frac{z^2}{2r} = -rM \ln \left(1 - \frac{x}{r} \right).$$

Solving this quadratic equation for z, we obtain

$$z = r \left(1 \pm \sqrt{1 - 4M \ln \left(1 - \frac{x}{r} \right)} \right).$$

We choose the negative branch of the square root to ensure that $z(0) = 0$. Thus, we claim that

$$|x| < r \left(1 - e^{-\frac{1}{2M}} \right).$$

This shows that z is analytic with a radius of analyticity less than the above expression.

Finally, we rewrite the problem as

$$\frac{dy}{dx} = a(x) y + b(x),$$

where $a(x)$ and $b(x)$ are analytic near zero, and $f = a(x)y + b(x)$. Then

$$|a(x)|, |b(x)| \leq \frac{M}{1 - \frac{x}{r}}.$$

Moreover, the equation

$$\frac{dz}{dx} = F(x, z) = \frac{M}{1 - \frac{x}{r}}(z + 1)$$

is separable, and its integral curve is

$$\ln(1 + z) = Mr \ln\left(1 - \frac{x}{r}\right).$$

Thus, z is analytic on a disk of radius less than r, and the solution $y(x)$ is also analytic.

$\square$

RUNDOWN. This proof shows that if the function $f(x, y)$ is analytic (i.e., it can be expressed as a power series) near a point (x_0, y_0), then the differential equation $\frac{dy}{dx} = f(x, y)$ has a unique solution that is also analytic in a neighborhood around x_0.

- **Uniqueness of the Solution:**
  - We assume there is a solution $y(x)$ to the equation, expressed as a power series.
  - Using the power series for $f(x, y)$, we compute the derivatives of the solution at $x = 0$ (or another point, but we choose $x_0 = 0$).
  - Each coefficient in the power series of $y(x)$ is uniquely determined by the coefficients of $f(x, y)$.
  - Thus, the solution $y(x)$ is unique.
- **Existence of the Solution:**
  - We solve a simpler equation using the power series representation of $f(x, y)$.
  - We show that the solution to this simpler equation is analytic, i.e., it can be expressed as a convergent power series.
  - Since the solution is analytic, it also solves the original equation $\frac{dy}{dx} = f(x, y)$.

Thus, the solution to the differential equation is both unique and analytic in a neighborhood around x_0.

3.7 Smoothness of Solutions

The main theorem is given below.

THEOREM. *If $f(x, y)$ has continuous derivatives with respect to x and y up to order $p \geq 0$, then every solution of the differential equation*

$$y' = f(x, y)$$

has continuous derivatives with respect to x and y up to order $p + 1$.

When a solution is said to be **smooth**, it means that the function is *infinitely differentiable* and its derivatives are *continuous* for all orders. In geometric terms, a function $y(x)$ is smooth if its graph can be drawn without any sharp corners, cusps, or breaks.

3.8 Stability (Dependence on Initial Data)

Often times, theoretical differential equations are idealized. So it is necessary that nearby solutions to theoretical ones are somewhat stable, i.e., as initial data changes, the outputs change proportionally (as opposed to drastically).

The following theorem asserts that under certain conditions, the solution of a differential equation depends continuously on the equation itself and the initial conditions.

THEOREM. *Let $f(x, y)$ be continuous and bounded on G, and suppose there is a unique solution to*

$$y' = f(x, y)$$

passing through (x_0, y_0) of G. Then the solution to the above equation depends continuously on f and (x_0, y_0) in the following sense:

- *Suppose the solution $y(x_0)$ passing through (x_0, y_0) is continued onto some interval $[a, b]$ where $a < x_0 < b$.*
- *Let $(x_0^*, y_0^*) \in G$ and f^* be any continuous function on G.*
- *Then given $\epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ s.t.*

$$|x_0^* - x_0| < \delta, \quad |x_0^* - x_0| < \delta, , \quad \sup_{(x,y)\in G} |f^*(x, y) - f(x, y)| < \delta$$

imply that the equation

$$y' = f^*(x, y)$$

has a solution $y_0^*(x)$ passing through (x_0^*, y_0^*) and defined on $[a, b]$ which satisfy

$$\sup_{a\le x\le b} |y_0^*(x) - y_0(x)| < \epsilon$$

This theorem gives a result about the *continuity* of solutions to the initial value problem (IVP) for the differential equation

$$y' = f(x, y),$$

where $f(x, y)$ is continuous and bounded on a domain G, and there exists a unique solution to this equation passing through a given point (x_0, y_0).
The core idea of the theorem is that the solution $y(x)$ depends *continuously* on the function $f(x, y)$ and the initial condition (x_0, y_0). This means that small changes in either the function or the initial condition will lead to small changes in the solution, as long as the changes are small enough.
Key Interpretation: If you perturb the initial condition (x_0, y_0) or the function $f(x, y)$, the new solution $y_0^*(x)$ will be *close* to the original solution $y(x)$. The difference between the solutions can be made arbitrarily small by making the perturbations small enough.
Applications: This theorem is useful in numerical methods, control theory, and mathematical modeling, where it is important to know that small changes in the problem setup (such as the initial condition or the equation itself) will result in small, predictable changes in the solution.

3.9 Hadamard's Lemma

LEMMA. Let G be a domain in $(x_1, \ldots, x_n, z_1, \ldots, z_m)$ space which is convex in $x_1, \ldots, x_n$ and suppose $F(x_1, \ldots, x_n, z_1, \ldots, z_m)$ has continuous derivatives with respect to $x_1, \ldots, x_n$ up to order $p > 0$ (inclusive) on G.

Then there are n functions

$$\Phi_i(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m) \quad (i = 1, \ldots, n)$$

with continuous derivatives with respect to $x_1, \ldots, x_n, y_1, \ldots, y_n$ up to order $p - 1$ on G such that

$$F(y_1, \ldots, y_n, z_1, \ldots, z_m) - F(x_1, \ldots, x_n, z_1, \ldots, z_m)$$

$$= \sum_{i=1}^{n} \Phi_i(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m)(y_1 - x_i)$$

This theorem provides a result about the relationship between a function $F(x_1, \ldots, x_n, z_1, \ldots, z_m)$ and its partial derivatives with respect to the $x_1, \ldots, x_n$-coordinates. It tells us that under certain smoothness conditions, the change in the function F can be expressed as a sum involving certain functions Φ_i, which capture how the change in the x_i-coordinates affects the function.
Key Points: - G is a convex domain in the space of variables $(x_1, \ldots, x_n, z_1, \ldots, z_m)$.
- The function $F(x_1, \ldots, x_n, z_1, \ldots, z_m)$ has continuous derivatives up to order p with respect to $x_1, \ldots, x_n$. - There exist functions $\Phi_i(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m)$ that have continuous derivatives up to order $p-1$ with respect to $x_1, \ldots, x_n, y_1, \ldots, y_n$. This equation allows us to understand how the changes in the x_i-coordinates lead to a change in the function F, expressed through the functions Φ_i.
Applications: - This result is useful for linearizing nonlinear functions and in perturbation theory, where small changes in the input variables are analyzed. - It is also relevant in optimization and numerical methods, providing a way to approximate the effects of small changes in the system.

A **convex** domain (or simply a convex set) in mathematical terms refers to a subset of a vector space (or more generally, a topological space) with a specific geometric property: for any two points within the set, the line segment connecting them is also entirely contained within the set.