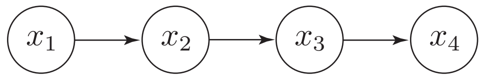
ECE 239AS.3, Spring 2024 **Project 2: Diffusion models**
Neural Networks & Deep Learning 2 Prof. J. Kao
UCLA TAs: T. Monsoor, Y. Liu, S. Rajesh, K. Pang

Due Monday, May 6, 2024, by 11:59 pm to Gradescope.
60 points total.

1. (10 points) Consider the following directed graph:



   (a) (2 points) Write the joint distribution $P(x_1, x_2, x_3, x_4)$ in terms of a product of conditional distributions, one for each node in the graph.

   (b) (4 points) A possible application of this directed graph is to model a stimulus that changes over time. Let

$$x_1 \sim \mathcal{N}\left(0,\, \sigma^2\right)$$
$$x_t | x_{t-1} \sim \mathcal{N}\left(x_{t-1},\, \sigma^2\right),$$

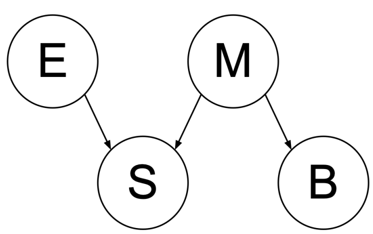
   where $t = 2, 3, 4$. This is known as a *random walk model*, since x_t is obtained by adding a random increment (in this case, a Gaussian) to x_{t-1}. What is the 4×4 covariance matrix of the vector $[x_1\ x_2\ x_3\ x_4]^{\mathrm{T}}$ in terms of σ^2?

   (c) (2 points) Find the inverse of the covariance matrix found in part (b) in terms of σ^2. This is known as the *precision matrix*.

   (d) (2 points) Relate where zeros appear in the precision matrix to the graph structure.

$P(E)$	
$+e$	0.4
$-e$	0.6

$P(S\|E, M)$			
$+e$	$+m$	$+s$	1.0
$+e$	$+m$	$-s$	0.0
$+e$	$-m$	$+s$	0.8
$+e$	$-m$	$-s$	0.2
$-e$	$+m$	$+s$	0.3
$-e$	$+m$	$-s$	0.7
$-e$	$-m$	$+s$	0.1
$-e$	$-m$	$-s$	0.9



$P(M)$	
$+m$	0.1
$-m$	0.9

$P(B\|M)$		
$+m$	$+b$	1.0
$+m$	$-b$	0.0
$-m$	$+b$	0.1
$-m$	$-b$	0.9

Figure 1: A directed acyclic graph representing the relationship between the firing of neurons B, E, M, S along with the conditional probability tables (CPT). In the CPT, the firing of neurons is represented by Bernoulli random variables where a $'+'$ indicates that the neuron has fired and a $'-'$ indicates that the neuron is quiescent.

2. (20 points) The Neuromodulation lab at UCLA is studying an alien brain consisting of 4 neurons only. During the course of the study, they have found some relationships between the firing of these neurons. They have decided to express the relationships using a probabilistic graphical model shown in Fig 1.

   In this question, you do not have to evaluate the final value of an expression you find (i.e., no calculator is needed). For example, an answer like:

   $$\frac{(0.6)(0.7)}{0.1^2}$$

   is acceptable.

   (a) (2 points) What is the probability of neurons E and S being quiescent, given that neurons M and B are quiescent?

   (b) (2 points) What is the probability of neuron M firing, given that neurons B, E, and S have fired?

   (c) (16 points) In a directed acyclic graph $\mathcal{G}$, the Markov blanket of a node X is the smallest subset of nodes in $\mathcal{G}$ such that X is conditionally independent of every other node in $\mathcal{G}$ given it's Markov blanket. If we denote the Markov blanket of X by $\mathcal{M}(X)$ and the subset of nodes in $\mathcal{G}$ excluding the Markov blanket by $\{\mathcal{G} - \mathcal{M}(X)\}$, then $\mathcal{M}(X)$ satisfies the following condition

   $$X \perp\!\!\!\perp \{\mathcal{G} - \mathcal{M}(X)\} | \mathcal{M}(X)$$

   i. (6 points) Show that B is independent of $\{E, S\}$ given M. Use the result along with the definition of Markov blanket to compute $\mathcal{M}(B)$. You must show this mathematically rigorously, not with an intuition statement.

   ii. (6 points) Show that E is independent of B given $\{M, S\}$. You must show this mathematically rigorously, not with an intuition statement.

   iii. (2 points) Assuming $M \not\perp\!\!\!\perp E|S$ (we have shown this result in class for a collider graph example) and using the result from (c)(ii) along with the definition of Markov blanket compute $\mathcal{M}(E)$.

   iv. (2 points) Based on the Markov blanket computations, does the Markov blanket of a node only contain its parents and children?

3. (30 points) **Implement Conditional DDPM.** Complete **ConditionalDDPM.ipynb**, **ResUNet.py** and **DDPM.py**. Print out **all three files along with your solutions for Questions 1 and 2**, and submit them as one PDF to Gradescope.