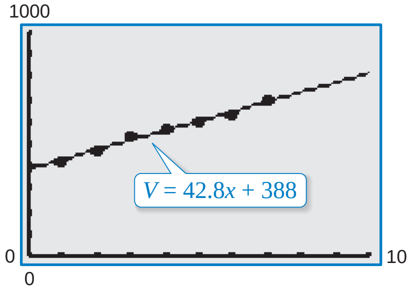
1 Functions and Their Graphs



Section 1.7, Example 4
Alternative-Fueled Vehicles

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Introduction to Library of Parent Functions

In Chapter 1, you will be introduced to the concept of a *function*. As you proceed through the text, you will see that functions play a primary role in modeling real-life situations.

There are three basic types of functions that have proven to be the most important in modeling real-life situations. These functions are algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. These three types of functions are referred to as the *elementary functions*, though they are often placed in the two categories of *algebraic functions* and *transcendental functions*. Each time a new type of function is studied in detail in this text, it will be highlighted in a box similar to those shown below. The graphs of these functions are shown on the inside covers of this text.

ALGEBRAIC FUNCTIONS

These functions are formed by applying algebraic operations to the linear function $f(x) = x$.

Name	*Function*	*Location*
Linear	$f(x) = x$	Section 1.1
Quadratic	$f(x) = x^2$	Section 2.1
Cubic	$f(x) = x^3$	Section 2.2
Rational	$f(x) = \dfrac{1}{x}$	Section 2.7
Square root	$f(x) = \sqrt{x}$	Section 1.2

TRANSCENDENTAL FUNCTIONS

These functions cannot be formed from the linear function by using algebraic operations.

Name	*Function*	*Location*
Exponential	$f(x) = a^x, a > 0, a \neq 1$	Section 3.1
Logarithmic	$f(x) = \log_a x, x > 0, a > 0, a \neq 1$	Section 3.2
Trigonometric	$f(x) = \sin x$	Section 4.5
	$f(x) = \cos x$	Section 4.5
	$f(x) = \tan x$	Section 4.6
	$f(x) = \csc x$	Section 4.6
	$f(x) = \sec x$	Section 4.6
	$f(x) = \cot x$	Section 4.6
Inverse trigonometric	$f(x) = \arcsin x$	Section 4.7
	$f(x) = \arccos x$	Section 4.7
	$f(x) = \arctan x$	Section 4.7

NONELEMENTARY FUNCTIONS

Some useful nonelementary functions include the following.

Name	*Function*	*Location*
Absolute value	$f(x) = \lvert x \rvert$	Section 1.2
Greatest integer	$f(x) = [\![x]\!]$	Section 1.3

1.1 Lines in the Plane

What you should learn

- Find the slopes of lines.
- Write linear equations given points on lines and their slopes.
- Use slope-intercept forms of linear equations to sketch lines.
- Use slope to identify parallel and perpendicular lines.

Why you should learn it

The slope of a line can be used to solve real-life problems. For instance, in Exercise 97 on page 14, you will use a linear equation to model student enrollment at Penn State University.

The Slope of a Line

In this section, you will study lines and their equations. The **slope** of a nonvertical line represents the number of units the line rises or falls vertically for each unit of horizontal change from left to right. For instance, consider the two points

$$(x_1, y_1) \qquad \text{and} \qquad (x_2, y_2)$$

on the line shown in Figure 1.1.

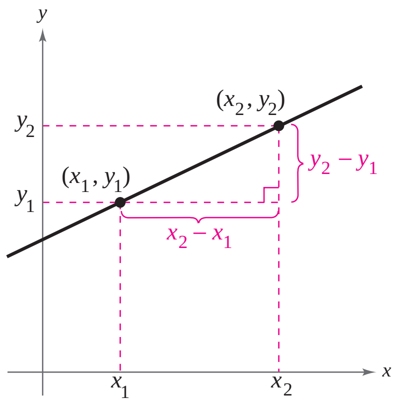

Figure 1.1

As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction. That is,

$$y_2 - y_1 = \text{the change in } y$$

and

$$x_2 - x_1 = \text{the change in } x.$$

The slope of the line is given by the ratio of these two changes.

Definition of the Slope of a Line

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

where $x_1 \neq x_2$.

When this formula for slope is used, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). Once you have done this, however, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{\cancel{m = \frac{y_2 - y_1}{x_1 - x_2}}}_{\text{Incorrect}}$$

Throughout this text, the term *line* always means a *straight* line.

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Example 1 Finding the Slope of a Line

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$

b. $(-1, 2)$ and $(2, 2)$

c. $(0, 4)$ and $(1, -1)$

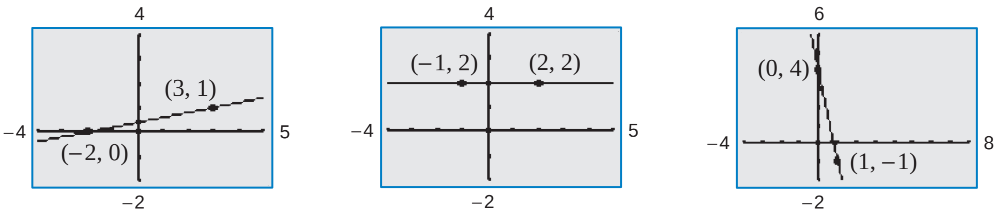
Solution

$$\textbf{a.}\quad m = \frac{\overbrace{y_2 - y_1}^{\text{Difference in } y\text{-values}}}{\underbrace{x_2 - x_1}_{\text{Difference in } x\text{-values}}} = \frac{1 - 0}{3 - (-2)} = \frac{1}{3 + 2} = \frac{1}{5}$$

$$\textbf{b.}\quad m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0$$

$$\textbf{c.}\quad m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5$$

The graphs of the three lines are shown in Figure 1.2. Note that the *square setting* gives the correct "steepness" of the lines.



(a) (b) (c)

Figure 1.2

✓CHECKPOINT Now try Exercise 15.

The definition of slope does not apply to vertical lines. For instance, consider the points $(3, 4)$ and $(3, 1)$ on the vertical line shown in Figure 1.3. Applying the formula for slope, you obtain

$$m = \frac{4 - 1}{3 - 3} = \cancel{\frac{3}{0}}. \qquad \text{Undefined}$$

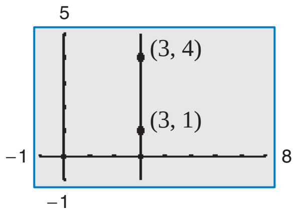


Figure 1.3

Because division by zero is undefined, the slope of a vertical line is undefined.

From the slopes of the lines shown in Figures 1.2 and 1.3, you can make the following generalizations about the slope of a line.

The Slope of a Line

1. A line with positive slope $(m > 0)$ *rises* from left to right.
2. A line with negative slope $(m < 0)$ *falls* from left to right.
3. A line with zero slope $(m = 0)$ is *horizontal*.
4. A line with undefined slope is *vertical*.

Explore the Concept

Use a graphing utility to compare the slopes of the lines $y = 0.5x$, $y = x$, $y = 2x$, and $y = 4x$. What do you observe about these lines? Compare the slopes of the lines $y = -0.5x$, $y = -x$, $y = -2x$, and $y = -4x$. What do you observe about these lines? (*Hint:* Use a *square setting* to obtain a true geometric perspective.)

The Point-Slope Form of the Equation of a Line

When you know the slope of a line *and* you also know the coordinates of one point on the line, you can find an equation of the line. For instance, in Figure 1.4, let (x_1, y_1) be a point on the line whose slope is m. When (x, y) is any *other* point on the line, it follows that

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the **point-slope form** of the equation of a line.

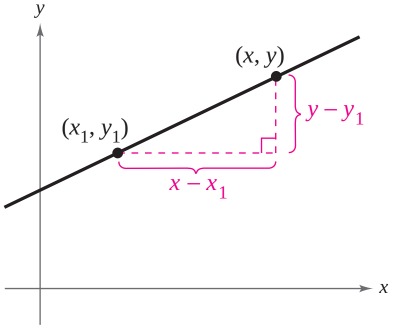


Figure 1.4

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

Example 2 The Point-Slope Form of the Equation of a Line

Find an equation of the line that passes through the point

$$(1, -2)$$

and has a slope of 3.

Solution

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-2) = 3(x - 1) \quad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y + 2 = 3x - 3 \quad \text{Simplify.}$$
$$y = 3x - 5 \quad \text{Solve for } y.$$

The line is shown in Figure 1.5.

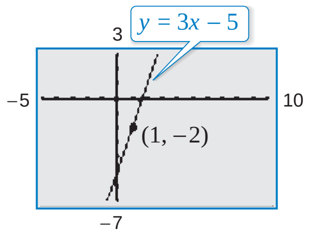


Figure 1.5

CHECKPOINT Now try Exercise 25.

The point-slope form can be used to find an equation of a nonvertical line passing through two points

$$(x_1, y_1) \quad \text{and} \quad (x_2, y_2).$$

First, find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

Then use the point-slope form to obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is sometimes called the **two-point form** of the equation of a line.

Study Tip

When you find an equation of the line that passes through two given points, you need to substitute the coordinates of only one of the points into the point-slope form. It does not matter which point you choose because both points will yield the same result.

Example 3 A Linear Model for Profits Prediction

During 2006, Research In Motion's net profits were \$631.6 million, and in 2007 net profits were \$1293.9 million. Write a linear equation giving the net profits y in terms of the year x. Then use the equation to predict the net profits for 2008. (Source: Research In Motion Limited)

Solution

Let $x = 0$ represent 2000. In Figure 1.6, let $(6, 631.6)$ and $(7, 1293.9)$ be two points on the line representing the net profits. The slope of this line is

$$m = \frac{1293.9 - 631.6}{7 - 6} = 662.3.$$

By the point-slope form, the equation of the line is as follows.

$$y - 631.6 = 662.3(x - 6)$$

$$y = 662.3x - 3342.2$$

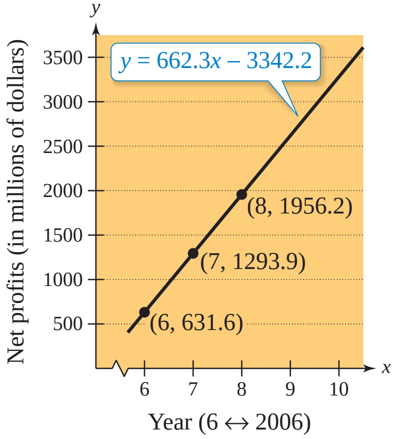


Figure 1.6

Now, using this equation, you can predict the 2008 net profits ($x = 8$) to be

$$y = 662.3(8) - 3342.2 = 5298.4 - 3342.2 = \$1956.2 \text{ million}.$$

(In this case, the prediction is quite good—the actual net profits in 2008 were \$1968.8 million.)

CHECKPOINT Now try Exercise 33.

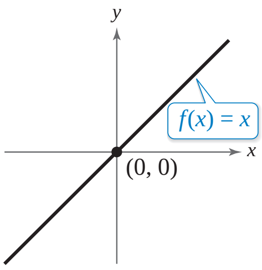
Library of Parent Functions: Linear Function

In the next section, you will be introduced to the precise meaning of the term *function*. The simplest type of function is the *parent linear function*

$$f(x) = x.$$

As its name implies, the graph of the parent linear function is a line. The basic characteristics of the parent linear function are summarized below and on the inside cover of this text. (Note that some of the terms below will be defined later in the text.)

Graph of $f(x) = x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing



The function $f(x) = x$ is also referred to as the *identity function*. Later in this text, you will learn that the graph of the linear function $f(x) = mx + b$ is a line with slope m and y-intercept $(0, b)$. When $m = 0$, $f(x) = b$ is called a *constant function* and its graph is a horizontal line.

Sketching Graphs of Lines

Many problems in coordinate geometry can be classified as follows.

1. Given a graph (or parts of it), find its equation.

2. Given an equation, sketch its graph.

For lines, the first problem is solved easily by using the point-slope form. This formula, however, is not particularly useful for solving the second type of problem. The form that is better suited to graphing linear equations is the **slope-intercept form** of the equation of a line, $y = mx + b$.

Slope-Intercept Form of the Equation of a Line

The graph of the equation

$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Example 4 Using the Slope-Intercept Form

Determine the slope and y-intercept of each linear equation. Then describe its graph.

a. $x + y = 2$

b. $y = 2$

Algebraic Solution

a. Begin by writing the equation in slope-intercept form.

$x + y = 2$ Write original equation.

$y = 2 - x$ Subtract x from each side.

$y = -x + 2$ Write in slope-intercept form.

From the slope-intercept form of the equation, the slope is -1 and the y-intercept is

$(0, 2)$.

Because the slope is negative, you know that the graph of the equation is a line that falls one unit for every unit it moves to the right.

b. By writing the equation $y = 2$ in slope-intercept form

$y = (0)x + 2$

you can see that the slope is 0 and the y-intercept is

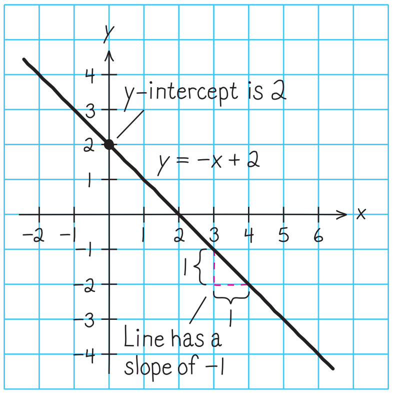
$(0, 2)$.

A zero slope implies that the line is horizontal.

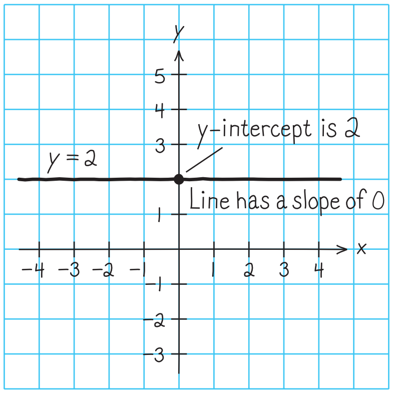
CHECKPOINT Now try Exercise 35.

Graphical Solution

a.



b.



From the slope-intercept form of the equation of a line, you can see that a horizontal line $(m = 0)$ has an equation of the form

$$y = b. \qquad \text{Horizontal line}$$

This is consistent with the fact that each point on a horizontal line through $(0, b)$ has a y-coordinate of b. Similarly, each point on a vertical line through $(a, 0)$ has an x-coordinate of a. So, a vertical line has an equation of the form

$$x = a. \qquad \text{Vertical line}$$

This equation cannot be written in slope-intercept form because the slope of a vertical line is undefined. However, every line has an equation that can be written in the general form

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

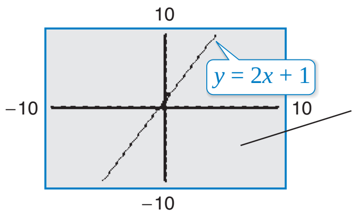
where A and B are not *both* zero.

Summary of Equations of Lines

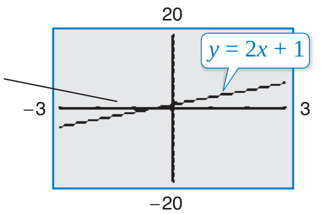
1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

Example 5 Different Viewing Windows

When a graphing utility is used to graph a line, it is important to realize that the graph of the line may not visually appear to have the slope indicated by its equation. This occurs because of the viewing window used for the graph. For instance, Figure 1.7 shows graphs of $y = 2x + 1$ produced on a graphing utility using three different viewing windows. Notice that the slopes in Figures 1.7(a) and (b) do not visually appear to be equal to 2. When you use a *square setting*, as in Figure 1.7(c), the slope visually appears to be 2.

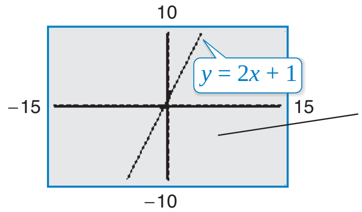


Using a *nonsquare setting*, you do *not* obtain a graph with a true geometric perspective. So, the slope does *not* visually appear to be 2.



(a) ***Nonsquare setting***

(b) ***Nonsquare setting***



Using a *square setting*, you can obtain a graph with a true geometric perspective. So, the slope visually appears to be 2.

(c) ***Square setting***

Figure 1.7

CHECKPOINT Now try Exercise 61.

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular.

> **Parallel Lines**
>
> Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is,
>
> $m_1 = m_2$.

Explore the Concept

Graph the lines $y_1 = \frac{1}{2}x + 1$ and $y_2 = -2x + 1$ in the same viewing window. What do you observe?

Graph the lines $y_1 = 2x + 1$, $y_2 = 2x$, and $y_3 = 2x - 1$ in the same viewing window. What do you observe?

Example 6 Equations of Parallel Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is parallel to the line

$$2x - 3y = 5.$$

Solution

Begin by writing the equation of the given line in slope-intercept form.

$2x - 3y = 5$	Write original equation.
$-2x + 3y = -5$	Multiply by -1.
$3y = 2x - 5$	Add $2x$ to each side.
$y = \frac{2}{3}x - \frac{5}{3}$	Write in slope-intercept form.

Therefore, the given line has a slope of

$$m = \tfrac{2}{3}.$$

Any line parallel to the given line must also have a slope of $\frac{2}{3}$. So, the line through $(2, -1)$ has the following equation.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-1) = \frac{2}{3}(x - 2)$	Substitute for y_1, m, and x_1.
$y + 1 = \frac{2}{3}x - \frac{4}{3}$	Simplify.
$y = \frac{2}{3}x - \frac{7}{3}$	Write in slope-intercept form.

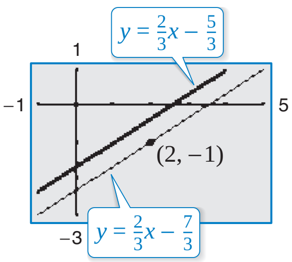


Figure 1.8

Notice the similarity between the slope-intercept form of the original equation and the slope-intercept form of the parallel equation. The graphs of both equations are shown in Figure 1.8.

CHECKPOINT Now try Exercise 67(a).

> **Perpendicular Lines**
>
> Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,
>
> $m_1 = -\dfrac{1}{m_2}$.

Example 7 Equations of Perpendicular Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is perpendicular to the line

$$2x - 3y = 5.$$

Solution

From Example 6, you know that the equation can be written in the slope-intercept form

$$y = \frac{2}{3}x - \frac{5}{3}.$$

You can see that the line has a slope of $\frac{2}{3}$. So, any line perpendicular to this line must have a slope of $-\frac{3}{2}$ (because $-\frac{3}{2}$ is the negative reciprocal of $\frac{2}{3}$). So, the line through the point $(2, -1)$ has the following equation.

$$y - (-1) = -\frac{3}{2}(x - 2) \qquad \text{Write in point-slope form.}$$

$$y + 1 = -\frac{3}{2}x + 3 \qquad \text{Simplify.}$$

$$y = -\frac{3}{2}x + 2 \qquad \text{Write in slope-intercept form.}$$

The graphs of both equations are shown in Figure 1.9.

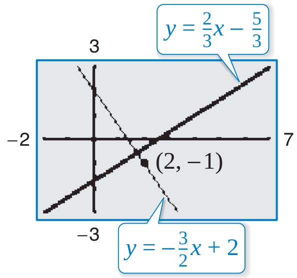


Figure 1.9

CHECKPOINT Now try Exercise 67(b).

Example 8 Graphs of Perpendicular Lines

Use a graphing utility to graph the lines $y = x + 1$ and $y = -x + 3$ in the same viewing window. The lines are supposed to be perpendicular (they have slopes of $m_1 = 1$ and $m_2 = -1$). Do they appear to be perpendicular on the display?

Solution

When the viewing window is nonsquare, as in Figure 1.10, the two lines will not appear perpendicular. When, however, the viewing window is square, as in Figure 1.11, the lines will appear perpendicular.

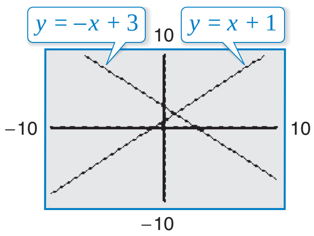


Figure 1.10

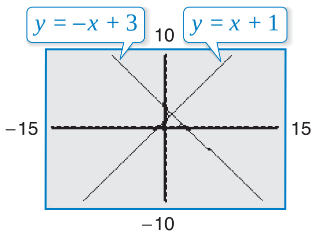
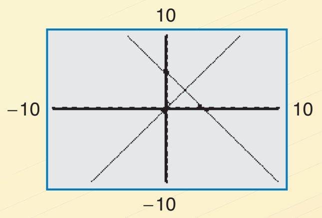


Figure 1.11

CHECKPOINT Now try Exercise 81.

What's Wrong?

You use a graphing utility to graph $y_1 = 1.5x$ and $y_2 = -1.5x + 5$, as shown in the figure. You use the graph to conclude that the lines are perpendicular. What's wrong?



1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match each equation with its form.

(a) $Ax + By + C = 0$	(i) vertical line
(b) $x = a$	(ii) slope-intercept form
(c) $y = b$	(iii) general form
(d) $y = mx + b$	(iv) point-slope form
(e) $y - y_1 = m(x - x_1)$	(v) horizontal line

In Exercises 2 and 3, fill in the blank.

2. For a line, the ratio of the change in y to the change in x is called the _______ of the line.

3. Two lines are _______ if and only if their slopes are equal.

4. What is the relationship between two lines whose slopes are -3 and $\frac{1}{3}$?

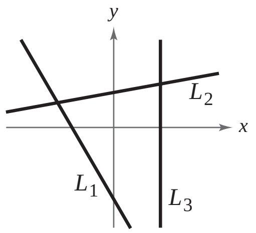
5. What is the slope of a line that is perpendicular to the line represented by $x = 3$?

6. Give the coordinates of a point on the line whose equation in point-slope form is $y - (-2) = \frac{1}{2}(x - 5)$.

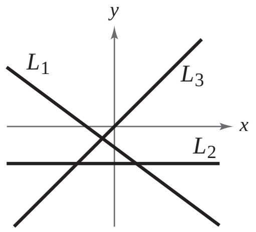
Procedures and Problem Solving

Using Slope **In Exercises 7 and 8, identify the line that has the indicated slope.**

7. (a) $m = \frac{2}{3}$ (b) m is undefined. (c) $m = -2$

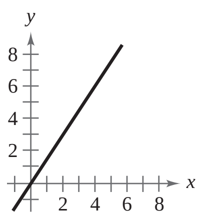


8. (a) $m = 0$ (b) $m = -\frac{3}{4}$ (c) $m = 1$

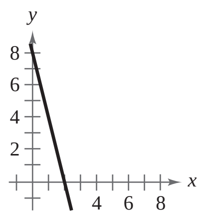


Estimating Slope **In Exercises 9 and 10, estimate the slope of the line.**

9.



10.



Sketching Lines **In Exercises 11 and 12, sketch the lines through the point with the indicated slopes on the same set of coordinate axes.**

	Point	*Slopes*			
11.	(2, 3)	(a) 0	(b) 1	(c) 2	(d) -3
12.	$(-4, 1)$	(a) 3	(b) -3	(c) $\frac{1}{2}$	(d) Undefined

Finding the Slope of a Line **In Exercises 13–16, find the slope of the line passing through the pair of points. Then use a graphing utility to plot the points and use the *draw* feature to graph the line segment connecting the two points. (Use a *square setting*.)**

13. $(0, -10), (-4, 0)$ **14.** $(2, 4), (4, -4)$

✓ **15.** $(-6, -1), (-6, 4)$ **16.** $(-3, -2), (1, 6)$

Using Slope **In Exercises 17–24, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)**

	Point	*Slope*
17.	(2, 1)	$m = 0$
18.	$(3, -2)$	$m = 0$
19.	(1, 5)	m is undefined.
20.	$(-4, 1)$	m is undefined.
21.	$(0, -9)$	$m = -2$
22.	$(-5, 4)$	$m = 2$
23.	$(7, -2)$	$m = \frac{1}{2}$
24.	$(-1, -6)$	$m = -\frac{1}{2}$

The Point-Slope Form of the Equation of a Line **In Exercises 25–32, find an equation of the line that passes through the given point and has the indicated slope. Sketch the line by hand. Use a graphing utility to verify your sketch, if possible.**

25. $(0, -2), \quad m = 3$ **26.** $(-3, 6), \quad m = -2$

27. $(2, -3), \quad m = -\frac{1}{2}$ **28.** $(-2, -5), \quad m = \frac{3}{4}$

29. $(6, -1), \quad m$ is undefined

30. $(-10, 4), \quad m$ is undefined

31. $\left(-\frac{1}{2}, \frac{3}{2}\right), \quad m = 0$ **32.** $(2.3, -8.5), \quad m = 0$

33. Finance The median player salary for the New York Yankees was \$1.6 million in 2001 and \$5.2 million in 2009. Write a linear equation giving the median salary y in terms of the year x. Then use the equation to predict the median salary in 2017.

34. Finance The median player salary for the Dallas Cowboys was \$441,300 in 2000 and \$1,326,720 in 2008. Write a linear equation giving the median salary y in terms of the year x. Then use the equation to predict the median salary in 2016.

Using the Slope-Intercept Form **In Exercises 35–42, determine the slope and y-intercept (if possible) of the linear equation. Then describe its graph.**

35. $x - 2y = 4$ **36.** $3x + 4y = 1$

37. $2x - 5y + 10 = 0$ **38.** $4x - 3y - 9 = 0$

39. $x = -6$ **40.** $y = 12$

41. $3y + 2 = 0$ **42.** $2x - 5 = 0$

Using the Slope-Intercept Form **In Exercises 43–48, (a) find the slope and y-intercept (if possible) of the equation of the line algebraically, and (b) sketch the line by hand. Use a graphing utility to verify your answers to parts (a) and (b).**

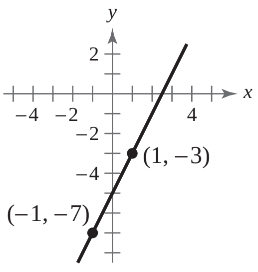
43. $5x - y + 3 = 0$ **44.** $2x + 3y - 9 = 0$

45. $5x - 2 = 0$ **46.** $3x + 7 = 0$

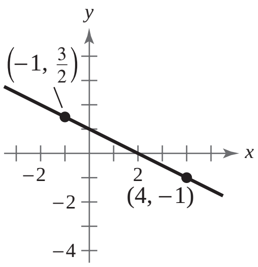
47. $3y + 5 = 0$ **48.** $-11 - 8y = 0$

Finding the Slope-Intercept Form **In Exercises 49 and 50, find the slope-intercept form of the equation of the line shown.**

49.



50.



Finding the Slope-Intercept Form **In Exercises 51–60, write an equation of the line that passes through the points. Use the slope-intercept form (if possible). If not possible, explain why and use the general form. Use a graphing utility to graph the line (if possible).**

51. $(5, -1), (-5, 5)$ **52.** $(4, 3), (-4, -4)$

53. $(-8, 1), (-8, 7)$ **54.** $(-1, 4), (6, 4)$

55. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$ **56.** $(1, 1), \left(6, -\frac{2}{3}\right)$

57. $\left(-\frac{1}{10}, -\frac{3}{5}\right), \left(\frac{9}{10}, -\frac{9}{5}\right)$ **58.** $\left(\frac{3}{4}, \frac{3}{2}\right), \left(-\frac{4}{3}, \frac{7}{4}\right)$

59. $(1, 0.6), (-2, -0.6)$ **60.** $(-8, 0.6), (2, -2.4)$

Different Viewing Windows **In Exercises 61 and 62, use a graphing utility to graph the equation using each viewing window. Describe the differences in the graphs.**

61. $y = 0.5x - 3$

Xmin = -5	Xmin = -2	Xmin = -5
Xmax = 10	Xmax = 10	Xmax = 10
Xscl = 1	Xscl = 1	Xscl = 1
Ymin = -1	Ymin = -4	Ymin = -7
Ymax = 10	Ymax = 1	Ymax = 3
Yscl = 1	Yscl = 1	Yscl = 1

62. $y = -8x + 5$

Xmin = -5	Xmin = -5	Xmin = -5
Xmax = 5	Xmax = 10	Xmax = 13
Xscl = 1	Xscl = 1	Xscl = 1
Ymin = -10	Ymin = -80	Ymin = -2
Ymax = 10	Ymax = 80	Ymax = 10
Yscl = 1	Yscl = 20	Yscl = 1

Parallel and Perpendicular Lines **In Exercises 63–66, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.**

63. L_1: $(0, -1), (5, 9)$
L_2: $(0, 3), (4, 1)$

64. L_1: $(-2, -1), (1, 5)$
L_2: $(1, 3), (5, -5)$

65. L_1: $(3, 6), (-6, 0)$
L_2: $(0, -1), \left(5, \frac{7}{3}\right)$

66. L_1: $(4, 8), (-4, 2)$
L_2: $(3, -5), \left(-1, \frac{1}{3}\right)$

Equations of Parallel and Perpendicular Lines **In Exercises 67–76, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line.**

67. $(2, 1), \quad 4x - 2y = 3$ **68.** $(-3, 2), \quad x + y = 7$

69. $\left(-\frac{2}{3}, \frac{7}{8}\right), \quad 3x + 4y = 7$ **70.** $\left(\frac{2}{5}, -1\right), \quad 3x - 2y = 6$

71. $(-3.9, -1.4), \quad 6x + 2y = 9$

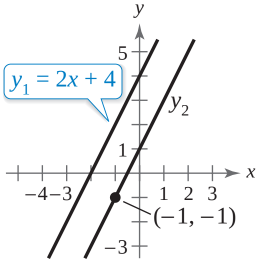
72. $(-1.2, 2.4), \quad 5x + 4y = 1$

73. $(3, -2), \quad x - 4 = 0$ **74.** $(3, -1), \quad y - 2 = 0$

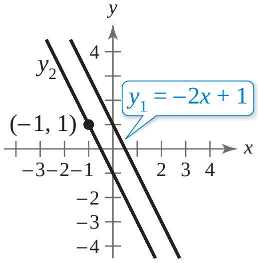
75. $(-4, 1), \quad y + 2 = 0$ **76.** $(-2, 4), \quad x + 5 = 0$

Equations of Parallel Lines **In Exercises 77 and 78, the lines are parallel. Find the slope-intercept form of the equation of line y_2.**

77.

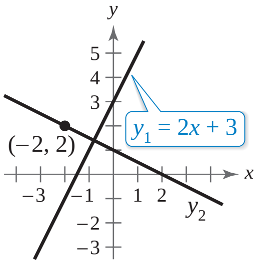


78.

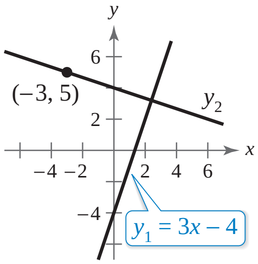


Equations of Perpendicular Lines **In Exercises 79 and 80, the lines are perpendicular. Find the slope-intercept form of the equation of line y_2.**

79.



80.



Graphs of Parallel and Perpendicular Lines **In Exercises 81–84, identify any relationships that exist among the lines, and then use a graphing utility to graph the three equations in the same viewing window. Adjust the viewing window so that each slope appears visually correct. Use the slopes of the lines to verify your results.**

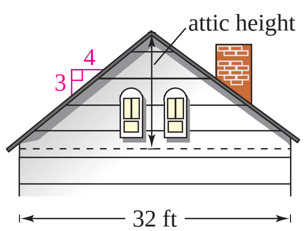
81. (a) $y = 2x$ (b) $y = -2x$ (c) $y = \frac{1}{2}x$

82. (a) $y = \frac{2}{3}x$ (b) $y = -\frac{3}{2}x$ (c) $y = \frac{2}{3}x + 2$

83. (a) $y = -\frac{1}{2}x$ (b) $y = -\frac{1}{2}x + 3$ (c) $y = 2x - 4$

84. (a) $y = x - 8$ (b) $y = x + 1$ (c) $y = -x + 3$

85. Architectural Design The "rise to run" ratio of the roof of a house determines the steepness of the roof. The rise to run ratio of the roof in the figure is 3 to 4. Determine the maximum height in the attic of the house if the house is 32 feet wide.



86. Highway Engineering When driving down a mountain road, you notice warning signs indicating that it is a "12% grade." This means that the slope of the road is $-\frac{12}{100}$. Approximate the amount of horizontal change in your position if you note from elevation markers that you have descended 2000 feet vertically.

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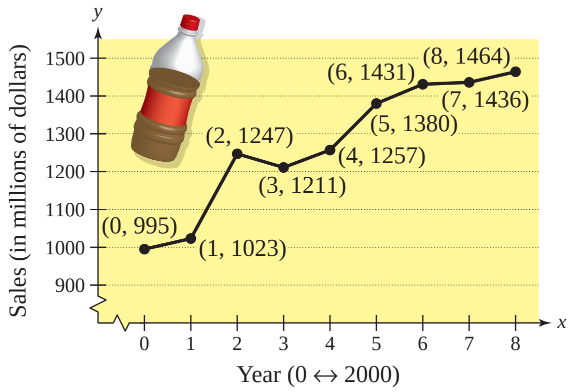
87. MODELING DATA

The graph shows the sales y (in millions of dollars) of the Coca-Cola Bottling Company each year x from 2000 through 2008, where $x = 0$ represents 2000. (Source: Coca-Cola Bottling Company)



(a) Use the slopes to determine the years in which the sales showed the greatest increase and greatest decrease.

(b) Find the equation of the line between the years 2000 and 2008.

(c) Interpret the meaning of the slope of the line from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the sales of the Coca-Cola Bottling Company in 2010. Do you think this is an accurate estimate? Explain.

88. MODELING DATA

The table shows the profits y (in millions of dollars) for Buffalo Wild Wings for each year x from 2002 through 2008, where $x = 2$ represents 2002. (Source: Buffalo Wild Wings Inc.)

Year, x	Profits, y
2	3.1
3	3.9
4	7.2
5	8.9
6	16.3
7	19.7
8	24.4

(a) Sketch a graph of the data.

(b) Use the slopes to determine the years in which the profits showed the greatest and least increases.

(c) Find the equation of the line between the years 2002 and 2008.

(d) Interpret the meaning of the slope of the line from part (c) in the context of the problem.

(e) Use the equation from part (c) to estimate the profit for Buffalo Wild Wings in 2010. Do you think this is an accurate estimate? Explain.

Using a Rate of Change to Write an Equation **In Exercises 89–92, you are given the dollar value of a product in 2009 and the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 9$ represent 2009.)**

	2009 Value	*Rate*
89.	\$2540	\$125 increase per year
90.	\$156	\$4.50 increase per year
91.	\$20,400	\$2000 decrease per year
92.	\$245,000	\$5600 decrease per year

93. Accounting A school district purchases a high-volume printer, copier, and scanner for \$25,000. After 10 years, the equipment will have to be replaced. Its value at that time is expected to be \$2000.

(a) Write a linear equation giving the value V of the equipment for each year t during its 10 years of use.

(b) Use a graphing utility to graph the linear equation representing the depreciation of the equipment, and use the value or trace feature to complete the table. Verify your answers algebraically by using the equation you found in part (a).

t	0	1	2	3	4	5	6	7	8	9	10
V											

94. Meterology Recall that water freezes at 0°C (32°F) and boils at 100°C (212°F).

(a) Find an equation of the line that shows the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F.

(b) Use the result of part (a) to complete the table.

C		$-10°$	$10°$			$177°$
F	$0°$			$68°$	$90°$	

95. Business A contractor purchases a bulldozer for \$36,500. The bulldozer requires an average expenditure of \$9.25 per hour for fuel and maintenance, and the operator is paid \$18.50 per hour.

(a) Write a linear equation giving the total cost C of operating the bulldozer for t hours. (Include the purchase cost of the bulldozer.)

(b) Assuming that customers are charged \$65 per hour of bulldozer use, write an equation for the revenue R derived from t hours of use.

(c) Use the profit formula $(P = R - C)$ to write an equation for the profit gained from t hours of use.

(d) Use the result of part (c) to find the break-even point (the number of hours the bulldozer must be used to gain a profit of 0 dollars).

96. Real Estate A real estate office handles an apartment complex with 50 units. When the rent per unit is \$580 per month, all 50 units are occupied. However, when the rent is \$625 per month, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear.

(a) Write the equation of the line giving the demand x in terms of the rent p.

(b) Use a graphing utility to graph the demand equation and use the *trace* feature to estimate the number of units occupied when the rent is \$655. Verify your answer algebraically.

(c) Use the demand equation to predict the number of units occupied when the rent is lowered to \$595. Verify your answer graphically.

97. *Why you should learn it* *(p. 3)* In 1990, Penn State University had an enrollment of 75,365 students. By 2009, the enrollment had increased to 87,163. (Source: Penn State Fact Book)

(a) What was the average annual change in enrollment from 1990 to 2009?

(b) Use the average annual change in enrollment to estimate the enrollments in 1995, 2000, and 2005.

(c) Write the equation of a line that represents the given data. What is its slope? Interpret the slope in the context of the problem.

98. Writing Using the results of Exercise 97, write a short paragraph discussing the concepts of *slope* and *average rate of change*.

Conclusions

True or False? **In Exercises 99 and 100, determine whether the statement is true or false. Justify your answer.**

99. The line through $(-8, 2)$ and $(-1, 4)$ and the line through $(0, -4)$ and $(-7, 7)$ are parallel.

100. If the points $(10, -3)$ and $(2, -9)$ lie on the same line, then the point $\left(-12, -\frac{37}{2}\right)$ also lies on that line.

Exploration **In Exercises 101–104, use a graphing utility to graph the equation of the line in the form**

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0.$$

Use the graphs to make a conjecture about what a and b represent. Verify your conjecture.

101. $\dfrac{x}{5} + \dfrac{y}{-3} = 1$

102. $\dfrac{x}{-6} + \dfrac{y}{2} = 1$

103. $\dfrac{x}{4} + \dfrac{y}{-\frac{2}{3}} = 1$

104. $\dfrac{x}{\frac{1}{2}} + \dfrac{y}{5} = 1$

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Using Intercepts **In Exercises 105–108, use the results of Exercises 101–104 to write an equation of the line that passes through the points.**

105. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

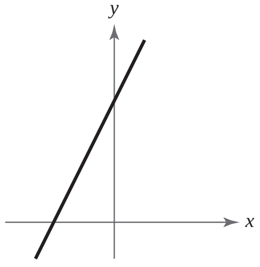
106. x-intercept: $(-5, 0)$
y-intercept: $(0, -4)$

107. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

108. x-intercept: $\left(\frac{3}{4}, 0\right)$
y-intercept: $\left(0, \frac{4}{5}\right)$

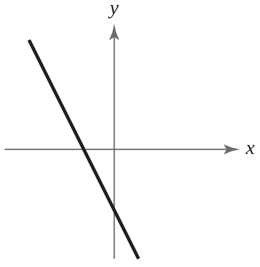
Think About It **In Exercises 109 and 110, determine which equation(s) may be represented by the graphs shown. (There may be more than one correct answer.)**

109.



(a) $2x - y = -10$
(b) $2x + y = 10$
(c) $x - 2y = 10$
(d) $x + 2y = 10$

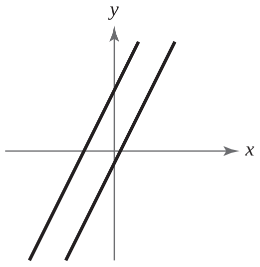
110.



(a) $2x + y = 5$
(b) $2x + y = -5$
(c) $x - 2y = 5$
(d) $x - 2y = -5$

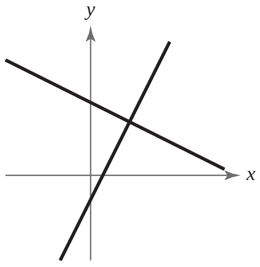
Think About It **In Exercises 111 and 112, determine which pair of equations may be represented by the graphs shown.**

111.



(a) $2x - y = 5$
$2x - y = 1$
(b) $2x + y = -5$
$2x + y = 1$
(c) $2x - y = -5$
$2x - y = 1$
(d) $x - 2y = -5$
$x - 2y = -1$

112.



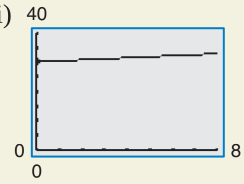
(a) $2x - y = 2$
$x + 2y = 12$
(b) $x - y = 1$
$x + y = 6$
(c) $2x + y = 2$
$x - 2y = 12$
(d) $x - 2y = 2$
$x + 2y = 12$

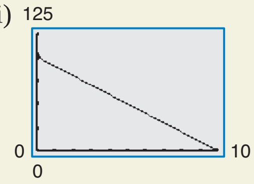
113. **Think About It** Does every line have both an x-intercept and a y-intercept? Explain.

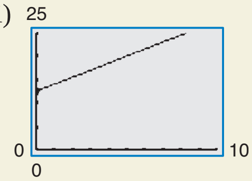
114. **Think About It** Can every line be written in slope-intercept form? Explain.

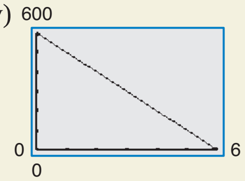
115. **Think About It** Does every line have an infinite number of lines that are parallel to it? Explain.

116. **CAPSTONE** Match the description with its graph. Determine the slope of each graph and how it is interpreted in the given context. [The graphs are labeled (i), (ii), (iii), and (iv).]

(i)


(ii)


(iii)


(iv)


(a) You are paying \$10 per week to repay a \$100 loan.
(b) An employee is paid \$12.50 per hour plus \$1.50 for each unit produced per hour.
(c) A sales representative receives \$30 per day for food plus \$0.35 for each mile traveled.
(d) A computer that was purchased for \$600 depreciates \$100 per year.

Cumulative Mixed Review

Identifying Polynomials **In Exercises 117–122, determine whether the expression is a polynomial. If it is, write the polynomial in standard form.**

117. $x + 20$

118. $3x - 10x^2 + 1$

119. $4x^2 + x^{-1} - 3$

120. $2x^2 - 2x^4 - x^3 + 2$

121. $\dfrac{x^2 + 3x + 4}{x^2 - 9}$

122. $\sqrt{x^2 + 7x + 6}$

Factoring Trinomials **In Exercises 123–126, factor the trinomial.**

123. $x^2 - 6x - 27$

124. $x^2 - 11x + 28$

125. $2x^2 + 11x - 40$

126. $3x^2 - 16x + 5$

127. ***Make a Decision*** To work an extended application analyzing the numbers of bachelor's degrees earned by women in the United States from 1985 through 2007, visit this textbook's *Companion Website*. (Source: National Center for Education Statistics)

The *Make a Decision* exercise indicates a multipart exercise using large data sets. Go to this textbook's *Companion Website* to view these exercises.

1.2 Functions

Introduction to Functions

Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation.** Here are two examples.

1. The simple interest I earned on an investment of \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.

2. The area A of a circle is related to its radius r by the formula $A = \pi r^2$.

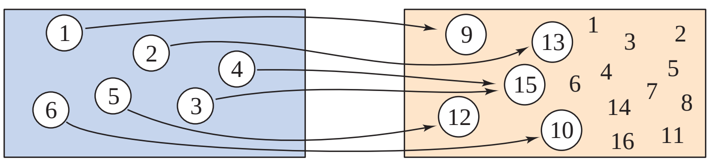
Not all relations have simple mathematical formulas. For instance, people commonly match up NFL starting quarterbacks with touchdown passes, and time of day with temperature. In each of these cases, there is some relation that matches each item from one set with exactly one item from a different set. Such a relation is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

To help understand this definition, look at the function that relates the time of day to the temperature in Figure 1.12.

Time of day (P.M.) Temperature (in degrees C)



Set A is the domain.
Inputs: 1, 2, 3, 4, 5, 6

Set B contains the range.
Outputs: 9, 10, 12, 13, 15

Figure 1.12

This function can be represented by the ordered pairs

$$\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 10°)\}.$$

In each ordered pair, the first coordinate (x-value) is the **input** and the second coordinate (y-value) is the **output.**

Characteristics of a Function from Set A to Set B

1. Each element of A must be matched with an element of B.

2. Some elements of B may not be matched with any element of A.

3. Two or more elements of A may be matched with the same element of B.

4. An element of A (the domain) cannot be matched with two different elements of B.

What you should learn

- Decide whether a relation between two variables represents a function.
- Use function notation and evaluate functions.
- Find the domains of functions.
- Use functions to model and solve real-life problems.
- Evaluate difference quotients.

Why you should learn it

Many natural phenomena can be modeled by functions, such as the force of water against the face of a dam, explored in Exercise 78 on page 27.

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Study Tip

Be sure you see that the *range* of a function is not the same as the use of *range* relating to the viewing window of a graphing utility.

To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. When any input value is matched with two or more output values, the relation is not a function.

Example 1 Testing for Functions

Decide whether the relation represents y as a function of x.

a.

Input, x	2	2	3	4	5
Output, y	11	10	8	5	1

b.

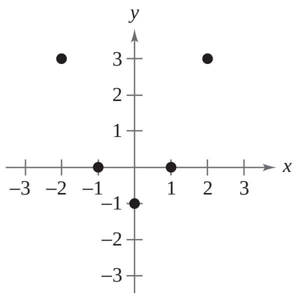


Figure 1.13

Solution

a. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

b. The graph in Figure 1.13 *does* describe y as a function of x. Each input value is matched with exactly one output value.

CHECKPOINT Now try Exercise 11.

In algebra, it is common to represent functions by equations or formulas involving two variables. For instance, the equation $y = x^2$ represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Example 2 Testing for Functions Represented Algebraically

Determine whether the equation represents y as a function of x.

a. $x^2 + y = 1$ **b.** $-x + y^2 = 1$

Solution

To determine whether y is a function of x, try to solve for y in terms of x.

a. $x^2 + y = 1$ Write original equation.

$y = 1 - x^2$ Solve for y.

Each value of x corresponds to exactly one value of y. So, y is a function of x.

b. $-x + y^2 = 1$ Write original equation.

$y^2 = 1 + x$ Add x to each side.

$y = \pm\sqrt{1 + x}$ Solve for y.

The $\pm$ indicates that for a given value of x there correspond two values of y. For instance, when $x = 3$, $y = 2$ or $y = -2$. So, y is not a function of x.

CHECKPOINT Now try Exercise 23.

Explore the Concept

Use a graphing utility to graph $x^2 + y = 1$. Then use the graph to write a convincing argument that each x-value corresponds to at most one y-value.

Use a graphing utility to graph $-x + y^2 = 1$. (*Hint:* You will need to use two equations.) Does the graph represent y as a function of x? Explain.

Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as the *value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *output value* of the function at the *input value* x. In function notation, the *input* is the independent variable and the *output* is the dependent variable. For instance, the function $f(x) = 3 - 2x$ has *function values* denoted by $f(-1)$, $f(0)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder." Consequently, the function could be written as

$$f(\square) = (\square)^2 - 4(\square) + 7.$$

Example 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find each value of the function.

a. $g(2)$

b. $g(t)$

c. $g(x + 2)$

Solution

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$\begin{aligned} g(x+2) &= -(x+2)^2 + 4(x+2) + 1 && \text{Substitute } x + 2 \text{ for } x. \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 && \text{Multiply.} \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 && \text{Distributive Property} \\ &= -x^2 + 5 && \text{Simplify.} \end{aligned}$$

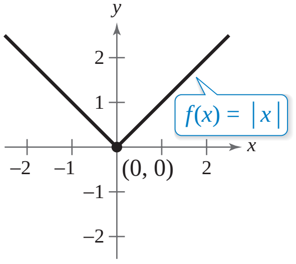
CHECKPOINT Now try Exercise 31.

In Example 3, note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

Library of Parent Functions: Absolute Value Function

The *parent absolute value function* given by $f(x) = |x|$ can be written as a piecewise-defined function. The basic characteristics of the parent absolute value function are summarized below and on the inside cover of this text.

Graph of $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$

A function defined by two or more equations over a specified domain is called a **piecewise-defined function.**

Example 4 A Piecewise-Defined Function

Evaluate the function when $x = -1$ and $x = 0$.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution

Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 \qquad \text{Substitute } -1 \text{ for } x.$$
$$= 2. \qquad \text{Simplify.}$$

For $x = 0$, use $f(x) = x - 1$ to obtain

$$f(0) = 0 - 1 \qquad \text{Substitute 0 for } x.$$
$$= -1. \qquad \text{Simplify.}$$

The graph of f is shown in Figure 1.14.

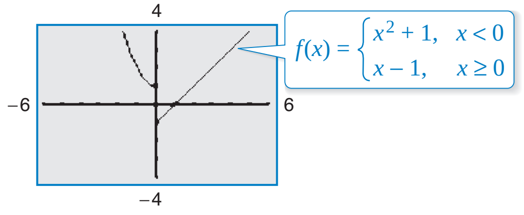


Figure 1.14

CHECKPOINT Now try Exercise 39.

Technology Tip

Most graphing utilities can graph piecewise-defined functions. For instructions on how to enter a piecewise-defined function into your graphing utility, consult your user's manual. You may find it helpful to set your graphing utility to *dot mode* before graphing such functions.

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The Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4}$$ Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real numbers x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x}$$ Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

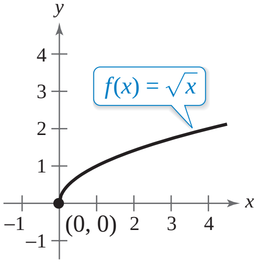
Explore the Concept

Use a graphing utility to graph $y = \sqrt{4 - x^2}$. What is the domain of this function? Then graph $y = \sqrt{x^2 - 4}$. What is the domain of this function? Do the domains of these two functions overlap? If so, for what values?

Library of Parent Functions: Square Root Function

Radical functions arise from the use of rational exponents. The most common radical function is the *parent square root* function given by $f(x) = \sqrt{x}$. The basic characteristics of the parent square root function are summarized below and on the inside cover of this text.

Graph of $f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Increasing on $(0, \infty)$



Study Tip

Because the square root function is not defined for $x < 0$, you must be careful when analyzing the domains of complicated functions involving the square root symbol.

Example 5 Finding the Domain of a Function

Find the domain of each function.

a. f: $\{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = -3x^2 + 4x + 5$

c. $h(x) = \dfrac{1}{x + 5}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. The domain of g is the set of all *real* numbers.

c. Excluding x-values that yield zero in the denominator, the domain of h is the set of all real numbers x except $x = -5$.

CHECKPOINT Now try Exercise 55.

Example 6 Finding the Domain of a Function

Find the domain of each function.

a. Volume of a sphere: $V = \frac{4}{3}\pi r^3$ **b.** $k(x) = \sqrt{4 - 3x}$

Solution

a. Because this function represents the volume of a sphere, the values of the radius r must be positive (see Figure 1.15). So, the domain is the set of all real numbers r such that $r > 0$.

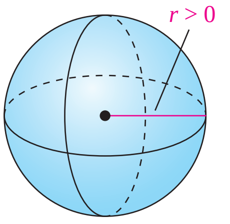


Figure 1.15

b. This function is defined only for x-values for which $4 - 3x \geq 0$. By solving this inequality, you will find that the domain of k is all real numbers that are less than or equal to $\frac{4}{3}$.

CHECKPOINT Now try Exercise 61.

In Example 6(a), note that the *domain of a function may be implied by the physical context*. For instance, from the equation $V = \frac{4}{3}\pi r^3$ you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

For some functions, it may be easier to find the domain and range of the function by examining its graph.

Example 7 Finding the Domain and Range of a Function

Use a graphing utility to find the domain and range of the function $f(x) = \sqrt{9 - x^2}$.

Solution

Graph the function as $y = \sqrt{9 - x^2}$, as shown in Figure 1.16. Using the *trace* feature of a graphing utility, you can determine that the x-values extend from -3 to 3 and the y-values extend from 0 to 3. So, the domain of the function f is all real numbers such that

$$-3 \leq x \leq 3 \qquad \text{Domain of } f$$

and the range of f is all real numbers such that

$$0 \leq y \leq 3. \qquad \text{Range of } f$$

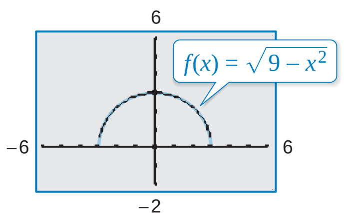


Figure 1.16

CHECKPOINT Now try Exercise 65.

Applications

Example 8 Construction Employees

The number N (in millions) of employees in the construction industry in the United States increased in a linear pattern from 2003 through 2006 (see Figure 1.17). In 2007, the number dropped, then decreased through 2008 in a different linear pattern. These two patterns can be approximated by the function

$$N(t) = \begin{cases} 0.32t + 5.7, & 3 \le t \le 6 \\ -0.42t + 10.5, & 7 \le t \le 8 \end{cases}$$

where t represents the year, with $t = 3$ corresponding to 2003. Use this function to approximate the number of employees for each year from 2003 to 2008. (Source: U.S. Bureau of Labor Statistics)

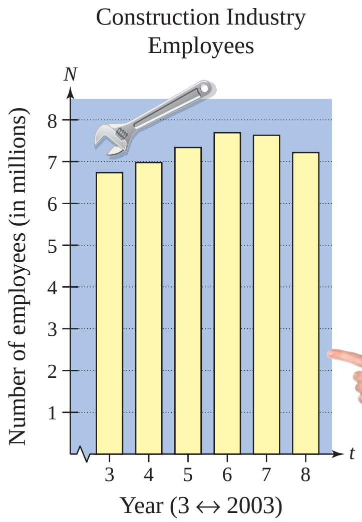


Figure 1.17

Solution

From 2003 to 2006, use $N(t) = 0.32t + 5.7$.

$$\underbrace{6.66,}_{2003} \quad \underbrace{6.98,}_{2004} \quad \underbrace{7.3,}_{2005} \quad \underbrace{7.62}_{2006}$$

From 2007 to 2008, use $N(t) = -0.42t + 10.5$.

$$\underbrace{7.56,}_{2007} \quad \underbrace{7.14}_{2008}$$

CHECKPOINT Now try Exercise 77.

Example 9 The Path of a Baseball

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). Will the baseball clear a 10-foot fence located 300 feet from home plate?

Algebraic Solution

The height of the baseball is a function of the horizontal distance from home plate. When $x = 300$, you can find the height of the baseball as follows.

$$f(x) = -0.0032x^2 + x + 3 \quad \text{Write original function.}$$

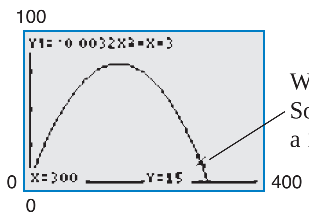
$$f(300) = -0.0032(300)^2 + 300 + 3 \quad \text{Substitute 300 for } x.$$

$$= 15 \quad \text{Simplify.}$$

When $x = 300$, the height of the baseball is 15 feet. So, the baseball will clear a 10-foot fence.

CHECKPOINT Now try Exercise 79.

Graphical Solution



When $x = 300$, $y = 15$. So, the ball will clear a 10-foot fence.

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Difference Quotients

One of the basic definitions in calculus employs the ratio

$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0.$$

This ratio is called a **difference quotient,** as illustrated in Example 10.

Example 10 Evaluating a Difference Quotient

algebra of calculus

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h)-f(x)}{h}$.

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h} \\
&= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h} \\
&= \frac{2xh + h^2 - 4h}{h} \\
&= \frac{h(2x + h - 4)}{h} \\
&= 2x + h - 4, \ \ h \neq 0
\end{aligned}$$

CHECKPOINT Now try Exercise 83.

Study Tip

Notice in Example 10 that h cannot be zero in the original expression. Therefore, you must restrict the domain of the simplified expression by listing $h \neq 0$ so that the simplified expression is equivalent to the original expression.

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function Notation: $y = f(x)$

f is the *name* of the function.
y is the **dependent variable,** or output value.
x is the **independent variable,** or input value.
$f(x)$ is the *value of the function at x.*

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, then f is said to be *defined* at x. If x is not in the domain of f, then f is said to be *undefined* at x.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, then the **implied domain** consists of all real numbers for which the expression is defined.

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

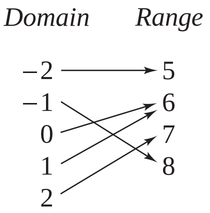
1. A relation that assigns to each element x from a set of inputs, or ________ , exactly one element y in a set of outputs, or ________ , is called a ________ .
2. For an equation that represents y as a function of x, the ________ variable is the set of all x in the domain, and the ________ variable is the set of all y in the range.

3. Can the ordered pairs (3, 0) and (3, 5) represent a function?
4. To find $g(x + 1)$, what do you substitute for x in the function $g(x) = 3x - 2$?
5. Does the domain of the function $f(x) = \sqrt{1 + x}$ include $x = -2$?
6. Is the domain of a piecewise-defined function *implied* or *explicitly described*?

Procedures and Problem Solving

Testing for Functions **In Exercises 7–10, does the relation describe a function? Explain your reasoning.**

7.



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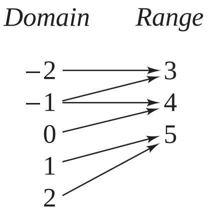

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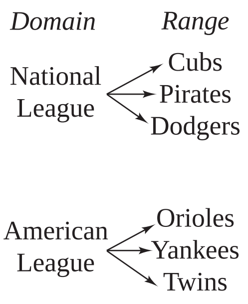
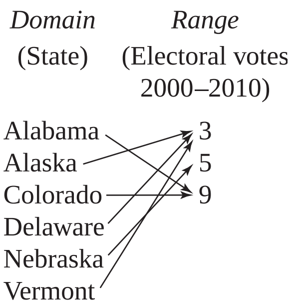


10.



Testing for Functions **In Exercises 11 and 12, decide whether the relation represents y as a function of x. Explain your reasoning.**

✓ **11.**

Input, x	-3	-1	0	1	3
Output, y	-9	-1	0	1	9

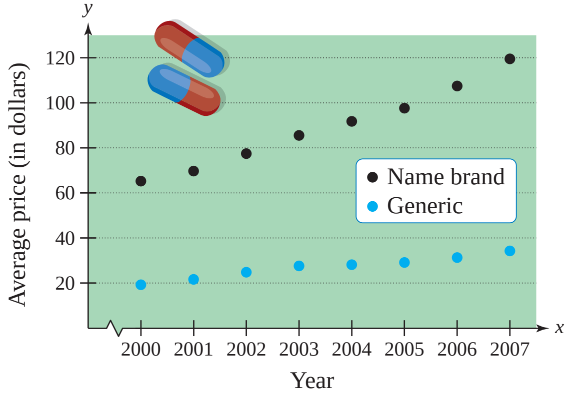
12.

Input, x	0	1	2	1	0
Output, y	-4	-2	0	2	4

Testing for Functions **In Exercises 13 and 14, which sets of ordered pairs represent functions from A to B? Explain.**

13. $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$
    (a) $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$
    (b) $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$
    (c) $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$
14. $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$
    (a) $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$
    (b) $\{(a, 1), (b, 2), (c, 3)\}$
    (c) $\{(1, a), (0, a), (2, c), (3, b)\}$

Pharmacology **In Exercises 15 and 16, use the graph, which shows the average prices of name brand and generic drug prescriptions in the United States.** (Source: National Association of Chain Drug Stores)



15. Is the average price of a name brand prescription a function of the year? Is the average price of a generic prescription a function of the year? Explain.
16. Let $b(t)$ and $g(t)$ represent the average prices of name brand and generic prescriptions, respectively, in year t. Find $b(2007)$ and $g(2000)$.

Testing for Functions Represented Algebraically **In Exercises 17–28, determine whether the equation represents y as a function of x.**

17. $x^2 + y^2 = 4$
18. $x = y^2 + 1$
19. $y = \sqrt{x^2 - 1}$
20. $y = \sqrt{x + 5}$
21. $2x + 3y = 4$
22. $x = -y + 5$
23. $y^2 = x^2 - 1$
24. $x + y^2 = 3$
25. $y = |4 - x|$
26. $|y| = 4 - x$
27. $x = -7$
28. $y = 8$

Evaluating a Function **In Exercises 29–44, evaluate the function at each specified value of the independent variable and simplify.**

29. $f(t) = 3t + 1$
 (a) $f(2)$ (b) $f(-4)$ (c) $f(t + 2)$
30. $g(y) = 7 - 3y$
 (a) $g(0)$ (b) $g\left(\frac{7}{3}\right)$ (c) $g(s + 2)$
31. $h(t) = t^2 - 2t$
 (a) $h(2)$ (b) $h(1.5)$ (c) $h(x + 2)$
32. $V(r) = \frac{4}{3}\pi r^3$
 (a) $V(3)$ (b) $V\left(\frac{3}{2}\right)$ (c) $V(2r)$
33. $f(y) = 3 - \sqrt{y}$
 (a) $f(4)$ (b) $f(0.25)$ (c) $f(4x^2)$
34. $f(x) = \sqrt{x + 8} + 2$
 (a) $f(-4)$ (b) $f(8)$ (c) $f(x - 8)$
35. $q(x) = \dfrac{1}{x^2 - 9}$
 (a) $q(-3)$ (b) $q(2)$ (c) $q(y + 3)$
36. $q(t) = \dfrac{2t^2 + 3}{t^2}$
 (a) $q(2)$ (b) $q(0)$ (c) $q(-x)$
37. $f(x) = \dfrac{|x|}{x}$
 (a) $f(9)$ (b) $f(-9)$ (c) $f(t)$
38. $f(x) = |x| + 4$
 (a) $f(5)$ (b) $f(-5)$ (c) $f(t)$
39. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \ge 0 \end{cases}$
 (a) $f(-1)$ (b) $f(0)$ (c) $f(2)$
40. $f(x) = \begin{cases} 2x + 5, & x \le 0 \\ 2 - x^2, & x > 0 \end{cases}$
 (a) $f(-2)$ (b) $f(0)$ (c) $f(1)$
41. $f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ 2x^2 + 2, & x > 1 \end{cases}$
 (a) $f(-2)$ (b) $f(1)$ (c) $f(2)$
42. $f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 1 - 2x^2, & x > 0 \end{cases}$
 (a) $f(-2)$ (b) $f(0)$ (c) $f(1)$
43. $f(x) = \begin{cases} x + 2, & x < 0 \\ 4, & 0 \le x < 2 \\ x^2 + 1, & x \ge 2 \end{cases}$
 (a) $f(-2)$ (b) $f(1)$ (c) $f(4)$
44. $f(x) = \begin{cases} 5 - 2x, & x < 0 \\ 5, & 0 \le x < 1 \\ 4x + 1, & x \ge 1 \end{cases}$
 (a) $f(-2)$ (b) $f\left(\frac{1}{2}\right)$ (c) $f(1)$

Evaluating a Function **In Exercises 45–48, assume that the domain of f is the set $A = \{-2, -1, 0, 1, 2\}$. Determine the set of ordered pairs representing the function f.**

45. $f(x) = x^2$
46. $f(x) = x^2 - 3$
47. $f(x) = |x| + 2$
48. $f(x) = |x + 1|$

Evaluating a Function **In Exercises 49 and 50, complete the table.**

49. $h(t) = \frac{1}{2}|t + 3|$

t	-5	-4	-3	-2	-1
$h(t)$					

50. $f(s) = \dfrac{|s - 2|}{s - 2}$

s	0	1	$\frac{3}{2}$	$\frac{5}{2}$	4
$f(s)$					

Finding the Inputs That Have Outputs of Zero **In Exercises 51–54, find all values of x such that $f(x) = 0$.**

51. $f(x) = 15 - 3x$
52. $f(x) = 5x + 1$
53. $f(x) = \dfrac{3x - 4}{5}$
54. $f(x) = \dfrac{2x - 3}{7}$

Finding the Domain of a Function **In Exercises 55–64, find the domain of the function.**

55. $f(x) = 5x^2 + 2x - 1$
56. $g(x) = 1 - 2x^2$
57. $h(t) = \dfrac{4}{t}$
58. $s(y) = \dfrac{3y}{y + 5}$
59. $f(x) = \sqrt[3]{x - 4}$
60. $f(x) = \sqrt[4]{x^2 + 3x}$
61. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$
62. $h(x) = \dfrac{10}{x^2 - 2x}$
63. $g(y) = \dfrac{y + 2}{\sqrt{y - 10}}$
64. $f(x) = \dfrac{\sqrt{x + 6}}{6 + x}$

Finding the Domain and Range of a Function **In Exercises 65–68, use a graphing utility to graph the function. Find the domain and range of the function.**

65. $f(x) = \sqrt{4 - x^2}$ **66.** $f(x) = \sqrt{x^2 + 1}$

67. $g(x) = |2x + 3|$ **68.** $g(x) = |x - 5|$

69. Geometry Write the area A of a circle as a function of its circumference C.

70. Geometry Write the area A of an equilateral triangle as a function of the length s of its sides.

71. Exploration An open box of maximum volume is to be made from a square piece of material, 24 centimeters on a side, by cutting equal squares from the corners and turning up the sides (see figure).

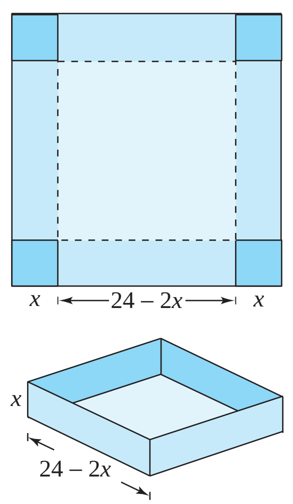
(a) The table shows the volume V (in cubic centimeters) of the box for various heights x (in centimeters). Use the table to estimate the maximum volume.

Height, x	Volume, V
1	484
2	800
3	972
4	1024
5	980
6	864

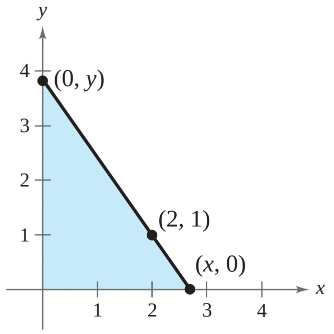
(b) Plot the points (x, V) from the table in part (a). Does the relation defined by the ordered pairs represent V as a function of x?

(c) If V is a function of x, write the function and determine its domain.

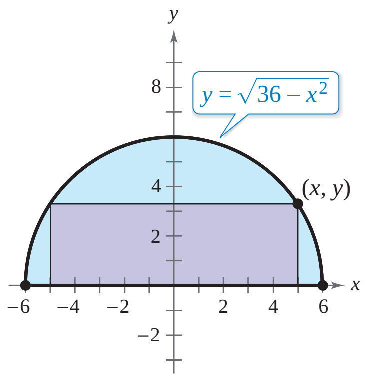
(d) Use a graphing utility to plot the points from the table in part (a) with the function from part (c). How closely does the function represent the data? Explain.



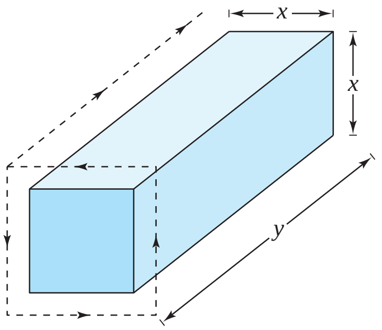
72. Geometry A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(2, 1)$ (see figure). Write the area A of the triangle as a function of x, and determine the domain of the function.



73. Geometry A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$ (see figure). Write the area A of the rectangle as a function of x, and determine the domain of the function.



74. Geometry A rectangular package to be sent by the U.S. Postal Service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure).



(a) Write the volume V of the package as a function of x. What is the domain of the function?

(b) Use a graphing utility to graph the function. Be sure to use an appropriate viewing window.

(c) What dimensions will maximize the volume of the package? Explain.

75. Business A company produces a handheld game console for which the variable cost is \$68.20 per unit and the fixed costs are \$248,000. The game console sells for \$98.98. Let x be the number of units produced and sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Write the profit P as a function of the number of units sold. (*Note:* $P = R - C$.)

76. MODELING DATA

The table shows the revenue y (in thousands of dollars) of a landscaping business for each month of 2010, with $x = 1$ representing January.

Month, x	Revenue, y
1	5.2
2	5.6
3	6.6
4	8.3
5	11.5
6	15.8
7	12.8
8	10.1
9	8.6
10	6.9
11	4.5
12	2.7

The mathematical model below represents the data.

$$f(x) = \begin{cases} -1.97x + 26.3 \\ 0.505x^2 - 1.47x + 6.3 \end{cases}$$

(a) Identify the independent and dependent variables and explain what they represent in the context of the problem.

(b) What is the domain of each part of the piecewise-defined function? Explain your reasoning.

(c) Use the mathematical model to find $f(5)$. Interpret your result in the context of the problem.

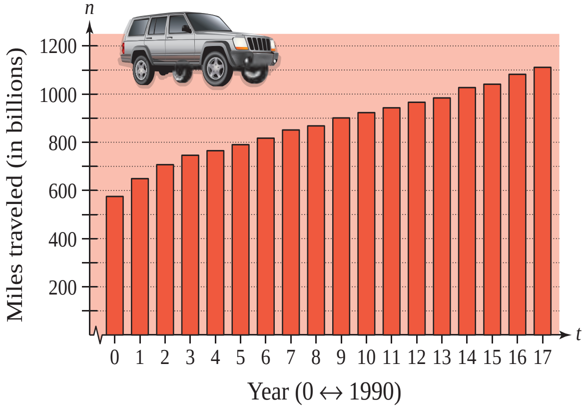
(d) Use the mathematical model to find $f(11)$. Interpret your result in the context of the problem.

(e) How do the values obtained from the models in parts (c) and (d) compare with the actual data values?

✓ **77. Civil Engineering** The numbers n (in billions) of miles traveled by vans, pickup trucks, and sport utility vehicles in the United States from 1990 through 2007 can be approximated by the model

$$n(t) = \begin{cases} -5.24t^2 + 69.5t + 581, & 0 \le t \le 6 \\ 25.7t + 664, & 6 < t \le 17 \end{cases}$$

where t represents the year, with $t = 0$ corresponding to 1990. The actual numbers are shown in the bar graph. (Source: U.S. Federal Highway Administration)



(a) Identify the independent and dependent variables and explain what they represent in the context of the problem.

(b) Use the *table* feature of a graphing utility to approximate the number of miles traveled by vans, pickup trucks, and sport utility vehicles each year from 1990 through 2007.

(c) Compare the values in part (b) with the actual values shown in the bar graph. How well does the model fit the data?

78. ***Why you should learn it*** *(p. 16)* The force F (in tons) of water against the face of a dam is estimated by the function

$$F(y) = 149.76\sqrt{10}y^{5/2}$$

where y is the depth of the water (in feet).

(a) Complete the table. What can you conclude from it?

y	5	10	20	30	40
$F(y)$					

(b) Use a graphing utility to graph the function. Describe your viewing window.

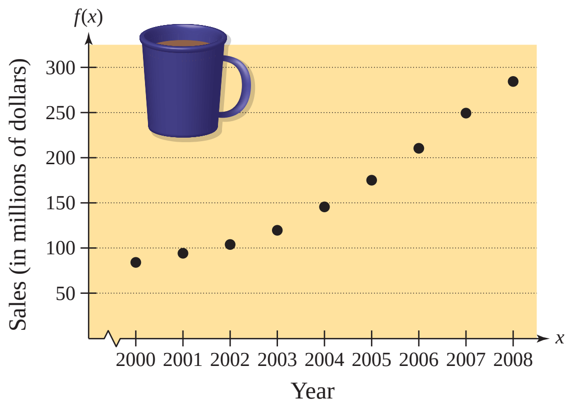
(c) Use the table to approximate the depth at which the force against the dam is 1,000,000 tons. Verify your answer graphically. How could you find a better estimate?

79. **Projectile Motion** A second baseman throws a baseball toward the first baseman 60 feet away. The path of the ball is given by

$$y = -0.004x^2 + 0.3x + 6$$

where y is the height (in feet) and x is the horizontal distance (in feet) from the second baseman. The first baseman can reach 8 feet high. Can the first baseman catch the ball without jumping? Explain.

80. **Business** The graph shows the sales (in millions of dollars) of Peet's Coffee & Tea from 2000 through 2008. Let $f(x)$ represent the sales in year x. (Source: Peet's Coffee & Tea, Inc.)



(a) Find $\dfrac{f(2008) - f(2000)}{2008 - 2000}$ and interpret the result in the context of the problem.

(b) An approximate model for the function is

$$S(t) = 2.484t^2 + 5.71t + 84.0, \quad 0 \le t \le 8$$

where S is the sales (in millions of dollars) and $t = 0$ represents 2000. Complete the table and compare the results with the data in the graph.

t	0	1	2	3	4	5	6	7	8
$S(t)$									

Evaluating a Difference Quotient **In Exercises 81–86, find the difference quotient and simplify your answer.**

81. $f(x) = 2x, \quad \dfrac{f(x + c) - f(x)}{c}, \quad c \neq 0$

82. $g(x) = 3x - 1, \quad \dfrac{g(x + h) - g(x)}{h}, \quad h \neq 0$

83. $f(x) = x^2 - x + 1, \quad \dfrac{f(2 + h) - f(2)}{h}, \quad h \neq 0$

84. $f(x) = x^3 + x, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \neq 0$

85. $f(t) = \dfrac{1}{t}, \quad \dfrac{f(t) - f(1)}{t - 1}, \quad t \neq 1$

86. $f(x) = \dfrac{4}{x + 1}, \quad \dfrac{f(x) - f(7)}{x - 7}, \quad x \neq 7$

Conclusions

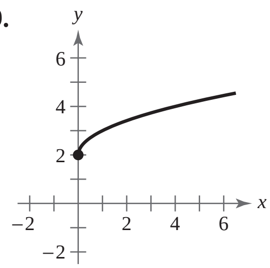
True or False? **In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.**

87. The domain of the function $f(x) = x^4 - 1$ is $(-\infty, \infty)$, and the range of $f(x)$ is $(0, \infty)$.

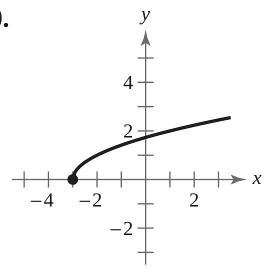
88. The set of ordered pairs $\{(-8, -2), (-6, 0), (-4, 0), (-2, 2), (0, 4), (2, -2)\}$ represents a function.

Think About It **In Exercises 89 and 90, write a square root function for the graph shown. Then, identify the domain and range of the function.**

89.



90.



91. **Think About It** Given $f(x) = x^2$, is f the independent variable? Why or why not?

92. **CAPSTONE**

(a) Describe any differences between a *relation* and a *function*.

(b) In your own words, explain the meanings of *domain* and *range*.

Cumulative Mixed Review

Operations with Rational Expressions **In Exercises 93–96, perform the operation and simplify.**

93. $12 - \dfrac{4}{x + 2}$

94. $\dfrac{3}{x^2 + x - 20} + \dfrac{x}{x^2 + 4x - 5}$

95. $\dfrac{2x^3 + 11x^2 - 6x}{5x} \cdot \dfrac{x + 10}{2x^2 + 5x - 3}$

96. $\dfrac{x + 7}{2(x - 9)} \div \dfrac{x - 7}{2(x - 9)}$

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

1.3 Graphs of Functions

The Graph of a Function

In Section 1.2, some functions were represented graphically by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis. The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember the geometric interpretations of x and $f(x)$.

$x =$ the directed distance from the y-axis

$f(x) =$ the directed distance from the x-axis

Example 1 shows how to use the graph of a function to find the domain and range of the function.

What you should learn

- Find the domains and ranges of functions and use the Vertical Line Test for functions.
- Determine intervals on which functions are increasing, decreasing, or constant.
- Determine relative maximum and relative minimum values of functions.
- Identify and graph step functions and other piecewise-defined functions.
- Identify even and odd functions.

Why you should learn it

Graphs of functions provide visual relationships between two variables. For example, in Exercise 92 on page 39, you will use the graph of a step function to model the cost of sending a package.

Example 1 Finding the Domain and Range of a Function

Use the graph of the function f shown in Figure 1.18 to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

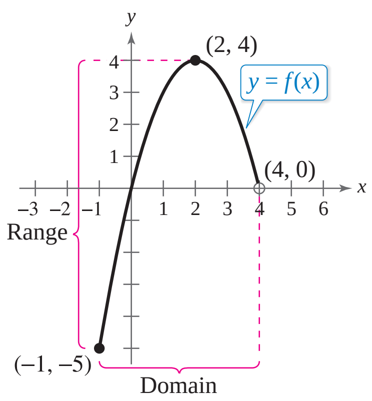


Figure 1.18

Solution

a. The closed dot at $(-1, -5)$ indicates that $x = -1$ is in the domain of f, whereas the open dot at $(4, 0)$ indicates that $x = 4$ is not in the domain. So, the domain of f is all x in the interval $[-1, 4)$.

b. Because $(-1, -5)$ is a point on the graph of f, it follows that

$$f(-1) = -5.$$

Similarly, because $(2, 4)$ is a point on the graph of f, it follows that

$$f(2) = 4.$$

c. Because the graph does not extend below $f(-1) = -5$ or above $f(2) = 4$, the range of f is the interval $[-5, 4]$.

CHECKPOINT Now try Exercise 9.

The use of dots (open or closed) at the extreme left and right points of a graph indicates that the graph does not extend beyond these points. When no such dots are shown, assume that the graph extends beyond these points.

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Example 2 Finding the Domain and Range of a Function

Find the domain and range of

$$f(x) = \sqrt{x - 4}.$$

Algebraic Solution

Because the expression under a radical cannot be negative, the domain of $f(x) = \sqrt{x - 4}$ is the set of all real numbers such that

$$x - 4 \geq 0.$$

Solve this linear inequality for x as follows. (For help with solving linear inequalities, see Appendix E at this textbook's *Companion Website*.)

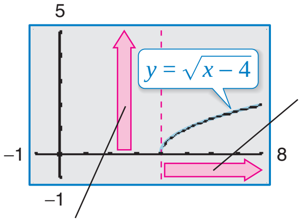
$$x - 4 \geq 0 \qquad \text{Write original inequality.}$$

$$x \geq 4 \qquad \text{Add 4 to each side.}$$

So, the domain is the set of all real numbers greater than or equal to 4. Because the value of a radical expression is never negative, the range of $f(x) = \sqrt{x - 4}$ is the set of all nonnegative real numbers.

CHECKPOINT Now try Exercise 13.

Graphical Solution



The x-coordinates of points on the graph extend from 4 to the right. So, the domain is the set of all real numbers greater than or equal to 4.

The y-coordinates of points on the graph extend from 0 upwards. So, the range is the set of all nonnegative real numbers.

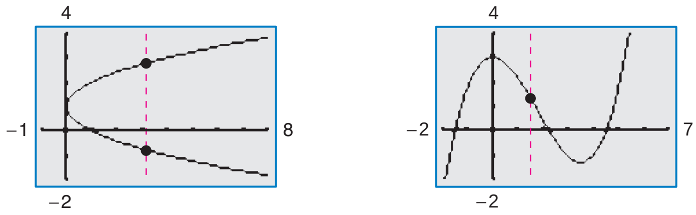
By the definition of a function, at most one y-value corresponds to a given x-value. It follows, then, that a vertical line can intersect the graph of a function at most once. This leads to the **Vertical Line Test** for functions.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.

Example 3 Vertical Line Test for Functions

Use the Vertical Line Test to decide whether the graphs in Figure 1.19 represent y as a function of x.



(a) (b)

Figure 1.19

Solution

a. This is *not* a graph of y as a function of x because you can find a vertical line that intersects the graph twice.

b. This *is* a graph of y as a function of x because every vertical line intersects the graph at most once.

CHECKPOINT Now try Exercise 21.

Technology Tip

Most graphing utilities are designed to graph functions of x more easily than other types of equations. For instance, the graph shown in Figure 1.19(a) represents the equation $x - (y - 1)^2 = 0$. To use a graphing utility to duplicate this graph you must first solve the equation for y to obtain $y = 1 \pm \sqrt{x}$, and then graph the two equations $y_1 = 1 + \sqrt{x}$ and $y_2 = 1 - \sqrt{x}$ in the same viewing window.

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph shown in Figure 1.20. Moving from *left to right*, this graph falls from $x = -2$ to $x = 0$, is constant from $x = 0$ to $x = 2$, and rises from $x = 2$ to $x = 4$.

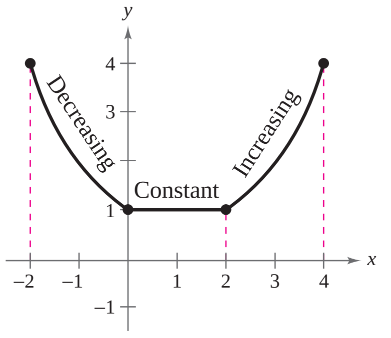


Figure 1.20

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval when, for any x_1 and x_2 in the interval,

$x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval when, for any x_1 and x_2 in the interval,

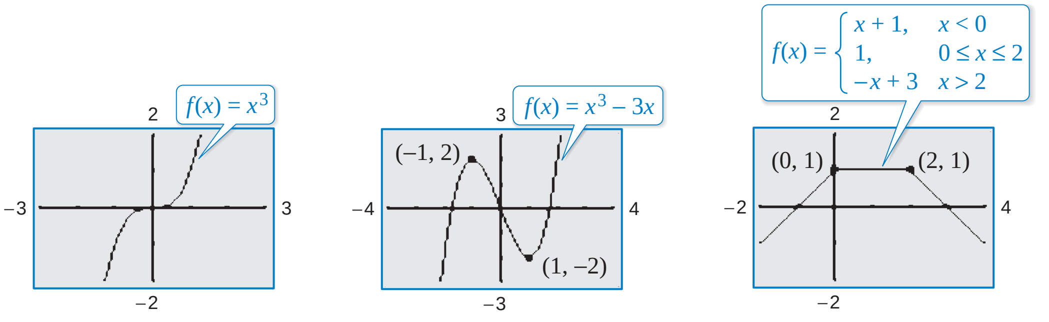
$x_1 < x_2$ implies $f(x_1) > f(x_2)$.

A function f is **constant** on an interval when, for any x_1 and x_2 in the interval,

$f(x_1) = f(x_2)$.

Example 4 Increasing and Decreasing Functions

In Figure 1.21, determine the open intervals on which each function is increasing, decreasing, or constant.



(a) **(b)** **(c)**

Figure 1.21

Solution

a. Although it might appear that there is an interval in which this function is constant, you can see that if $x_1 < x_2$, then $(x_1)^3 < (x_2)^3$, which implies that $f(x_1) < f(x_2)$. So, the function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

CHECKPOINT Now try Exercise 25.

Relative Minimum and Maximum Values

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the relative maximum or relative minimum values of the function.

Definition of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

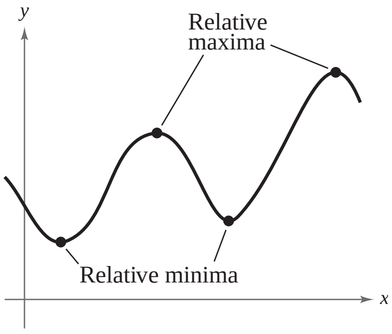


Figure 1.22

Figure 1.22 shows several different examples of relative minima and relative maxima. In Section 2.1, you will study a technique for finding the *exact points* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

Example 5 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function given by $f(x) = 3x^2 - 4x - 2$.

Solution

The graph of f is shown in Figure 1.23. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$(0.67, -3.33)$. See Figure 1.24.

Later, in Section 2.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

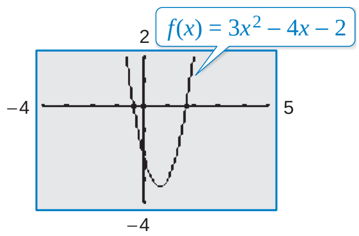


Figure 1.23

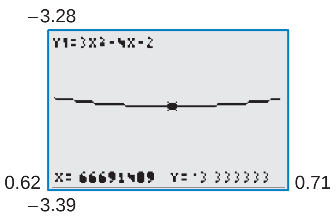


Figure 1.24

✓CHECKPOINT Now try Exercise 35.

Technology Tip

When you use a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *zoom* feature will often produce graphs that are nearly flat, as shown in Figure 1.24. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will stretch vertically when the values of Ymin and Ymax are closer together.

Technology Tip

Some graphing utilities have built-in programs that will find minimum or maximum values. These features are demonstrated in Example 6.

Example 6 Approximating Relative Minima and Maxima

Use a graphing utility to approximate the relative minimum and relative maximum of the function given by $f(x) = -x^3 + x$.

Solution

By using the *minimum* and *maximum* features of the graphing utility, you can estimate that the function has a relative minimum at the point

$(-0.58, -0.38)$ See Figure 1.25.

and a relative maximum at the point

$(0.58, 0.38)$. See Figure 1.26.

If you take a course in calculus, you will learn a technique for finding the exact points at which this function has a relative minimum and a relative maximum.

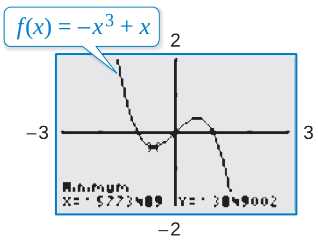


Figure 1.25

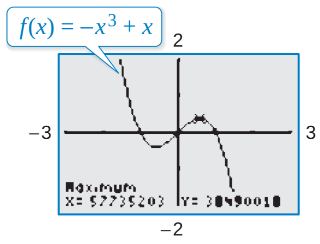


Figure 1.26

CHECKPOINT Now try Exercise 37.

Example 7 Temperature

During a 24-hour period, the temperature y (in degrees Fahrenheit) of a certain city can be approximated by the model

$$y = 0.026x^3 - 1.03x^2 + 10.2x + 34, \quad 0 \le x \le 24$$

where x represents the time of day, with $x = 0$ corresponding to 6 A.M. Approximate the maximum and minimum temperatures during this 24-hour period.

Solution

Using the *maximum* feature of a graphing utility, you can determine that the maximum temperature during the 24-hour period was approximately 64°F. This temperature occurred at about 12:36 P.M. ($x \approx 6.6$), as shown in Figure 1.27. Using the *minimum* feature, you can determine that the minimum temperature during the 24-hour period was approximately 34°F, which occurred at about 1:48 A.M. ($x \approx 19.8$), as shown in Figure 1.28.

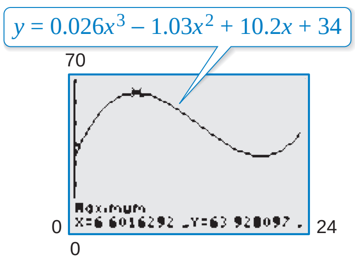


Figure 1.27

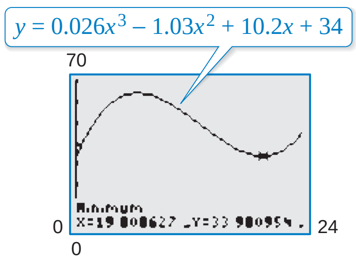


Figure 1.28

CHECKPOINT Now try Exercise 95.

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Step Functions and Piecewise-Defined Functions

Library of Parent Functions: Greatest Integer Function

The *greatest integer function*, denoted by $[\![x]\!]$ and defined as the greatest integer less than or equal to x, has an infinite number of breaks or steps—one at each integer value in its domain. The basic characteristics of the greatest integer function are summarized below.

Graph of $f(x) = [\![x]\!]$

Domain: $(-\infty, \infty)$

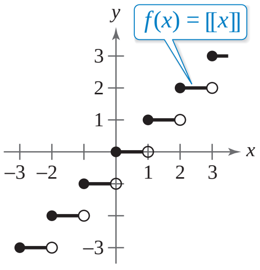
Range: the set of integers

x-intercepts: in the interval $[0, 1)$

y-intercept: $(0, 0)$

Constant between each pair of consecutive integers

Jumps vertically one unit at each integer value



Technology Tip

Most graphing utilities display graphs in *connected mode*, which works well for graphs that do not have breaks. For graphs that do have breaks, such as the greatest integer function, it is better to use *dot mode*. Graph the greatest integer function [often called Int (x)] in *connected* and *dot modes*, and compare the two results.

Because of the vertical jumps described above, the greatest integer function is an example of a **step function** whose graph resembles a set of stairsteps. Some values of the greatest integer function are as follows.

$$[\![-1]\!] = (\text{greatest integer } \le -1) = -1$$

$$\left[\!\!\left[-\tfrac{1}{2}\right]\!\!\right] = \left(\text{greatest integer } \le -\tfrac{1}{2}\right) = -1$$

$$\left[\!\!\left[\tfrac{1}{10}\right]\!\!\right] = \left(\text{greatest integer } \le \tfrac{1}{10}\right) = 0$$

$$[\![1.5]\!] = (\text{greatest integer } \le 1.5) = 1$$

In Section 1.2, you learned that a piecewise-defined function is a function that is defined by two or more equations over a specified domain. To sketch the graph of a piecewise-defined function, you need to sketch the graph of each equation on the appropriate portion of the domain.

Example 8 Sketching a Piecewise-Defined Function

Sketch the graph of

$$f(x) = \begin{cases} 2x + 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}$$

by hand.

Solution

This piecewise-defined function is composed of two linear functions. At and to the left of $x = 1$, the graph is the line given by

$$y = 2x + 3.$$

To the right of $x = 1$, the graph is the line given by

$$y = -x + 4$$

as shown in Figure 1.29. Notice that the point $(1, 5)$ is a solid dot and the point $(1, 3)$ is an open dot. This is because $f(1) = 5$.

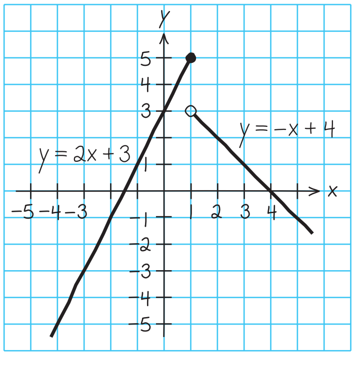


Figure 1.29

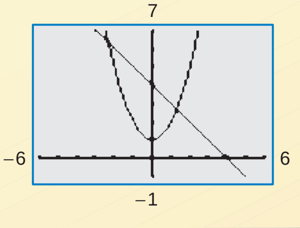
CHECKPOINT Now try Exercise 55.

What's Wrong?

You use a graphing utility to graph

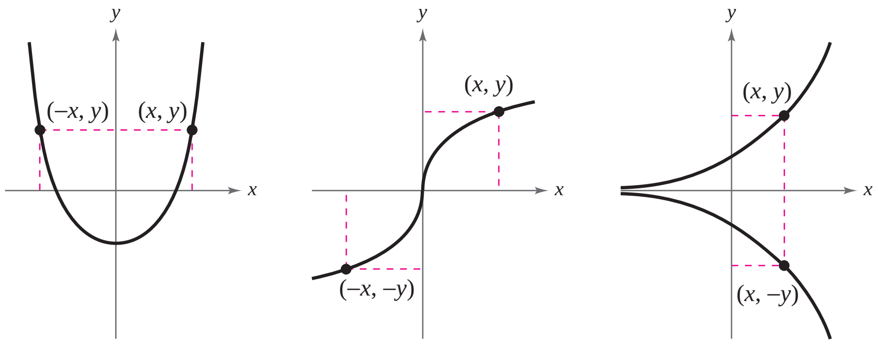
$$f(x) = \begin{cases} x^2 + 1, & x \le 0 \\ 4 - x, & x > 0 \end{cases}$$

by letting $y_1 = x^2 + 1$ and $y_2 = 4 - x$, as shown in the figure. You conclude that this is the graph of f. What's wrong?



Even and Odd Functions

A graph has *symmetry with respect to the y-axis* if whenever (x, y) is on the graph, then so is the point $(-x, y)$. A graph has *symmetry with respect to the origin* if whenever (x, y) is on the graph, then so is the point $(-x, -y)$. A graph has *symmetry with respect to the x-axis* if whenever (x, y) is on the graph, then so is the point $(x, -y)$. A function whose graph is symmetric with respect to the y-axis is an **even function.** A function whose graph is symmetric with respect to the origin is an **odd function.** A graph that is symmetric with respect to the x-axis is not the graph of a function (except for the graph of $y = 0$). These three types of symmetry are illustrated in Figure 1.30.



Symmetric to y-axis
Even function
Figure 1.30

Symmetric to origin
Odd function

Symmetric to x-axis
Not a function

Explore the Concept

Graph each function with a graphing utility. Determine whether the function is *even*, *odd*, or *neither*.

$f(x) = x^2 - x^4$

$g(x) = 2x^3 + 1$

$h(x) = x^5 - 2x^3 + x$

$j(x) = 2 - x^6 - x^8$

$k(x) = x^5 - 2x^4 + x - 2$

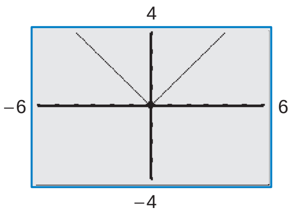
$p(x) = x^9 + 3x^5 - x^3 + x$

What do you notice about the equations of functions that are (a) odd and (b) even? Describe a way to identify a function as (c) odd, (d) even, or (e) neither odd nor even by inspecting the equation.

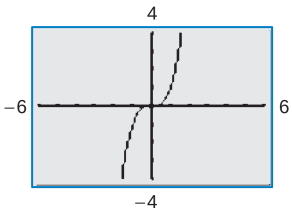
Example 9 Even and Odd Functions

Use the figure to determine whether the function is even, odd, or neither.

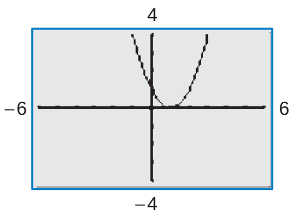
a.



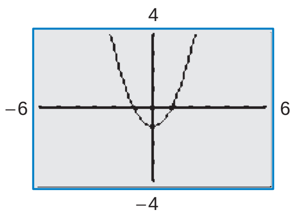
b.



c.



d.



Solution

a. The graph is symmetric with respect to the y-axis. So, the function is even.

b. The graph is symmetric with respect to the origin. So, the function is odd.

c. The graph is neither symmetric with respect to the origin nor with respect to the y-axis. So, the function is neither even nor odd.

d. The graph is symmetric with respect to the y-axis. So, the function is even.

CHECKPOINT Now try Exercise 67.

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Test for Even and Odd Functions

A function f is **even** when, for each x in the domain of f, $f(-x) = f(x)$.
A function f is **odd** when, for each x in the domain of f, $f(-x) = -f(x)$.

Example 10 Even and Odd Functions

Determine whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$

b. $h(x) = x^2 + 1$

c. $f(x) = x^3 - 1$

Algebraic Solution

a. This function is odd because

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -g(x). \end{aligned}$$

b. This function is even because

$$\begin{aligned} h(-x) &= (-x)^2 + 1 \\ &= x^2 + 1 \\ &= h(x). \end{aligned}$$

c. Substituting $-x$ for x produces

$$\begin{aligned} f(-x) &= (-x)^3 - 1 \\ &= -x^3 - 1. \end{aligned}$$

Because

$$f(x) = x^3 - 1$$

and

$$-f(x) = -x^3 + 1$$

you can conclude that

$$f(-x) \neq f(x)$$

and

$$f(-x) \neq -f(x).$$

So, the function is neither even nor odd.

CHECKPOINT Now try Exercise 81.

Graphical Solution

a. In Figure 1.31, the graph is symmetric with respect to the origin. So, this function is odd.

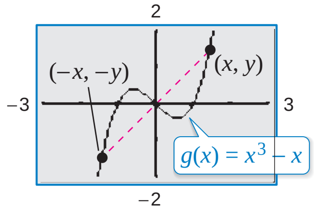


Figure 1.31

b. In Figure 1.32, the graph is symmetric with respect to the y-axis. So, this function is even.

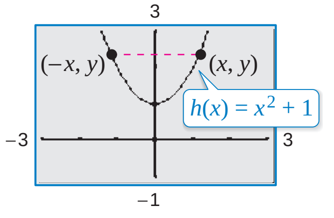


Figure 1.32

c. In Figure 1.33, the graph is neither symmetric with respect to the origin nor with respect to the y-axis. So, this function is neither even nor odd.

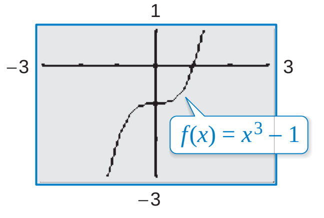


Figure 1.33

To help visualize symmetry with respect to the origin, place a pin at the origin of a graph and rotate the graph 180°. If the result after rotation coincides with the original graph, then the graph is symmetric with respect to the origin.

1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

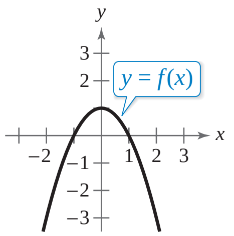
In Exercises 1 and 2, fill in the blank.

1. A function f is _______ on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
2. A function f is _______ when, for each x in the domain of f, $f(-x) = f(x)$.

3. The graph of a function f is the segment from $(1, 2)$ to $(4, 5)$, including the endpoints. What is the domain of f?
4. A vertical line intersects a graph twice. Does the graph represent a function?
5. Let f be a function such that $f(2) \ge f(x)$ for all values of x in the interval $(0, 3)$. Does $f(2)$ represent a relative minimum or a relative maximum?
6. Given $f(x) = [\![x]\!]$, in what interval does $f(x) = 5$?

Procedures and Problem Solving

Finding the Domain and Range of a Function **In Exercises 7–10, use the graph of the function to find the domain and range of f. Then find $f(0)$.**

7.


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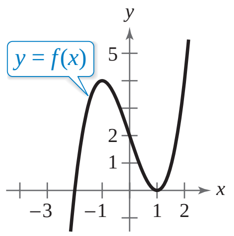

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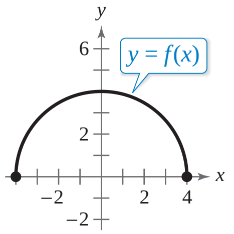

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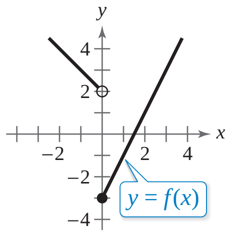


Finding the Domain and Range of a Function **In Exercises 11–16, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.**

11. $f(x) = 2x^2 + 3$
12. $f(x) = -x^2 - 1$
13. $f(x) = \sqrt{x - 1}$
14. $h(t) = \sqrt{4 - t^2}$
15. $f(x) = |x + 3|$
16. $f(x) = -\frac{1}{4}|x - 5|$

Analyzing a Graph **In Exercises 17 and 18, use the graph of the function to answer the questions.**

(a) Determine the domain of the function.

(b) Determine the range of the function.

(c) Find the value(s) of x for which $f(x) = 0$.

(d) What are the values of x from part (c) referred to graphically?

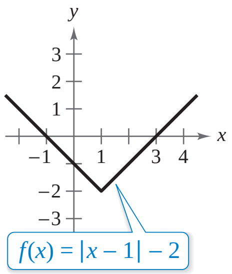
(e) Find $f(0)$, if possible.

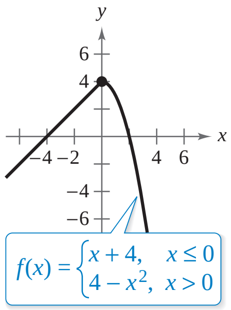
(f) What is the value from part (e) referred to graphically?

(g) What is the value of f at $x = 1$? What are the coordinates of the point?

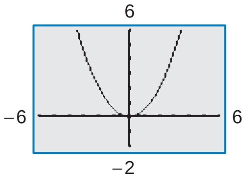
(h) What is the value of f at $x = -1$? What are the coordinates of the point?

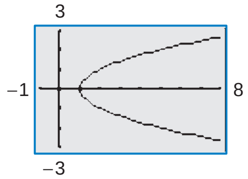
(i) The coordinates of the point on the graph of f at which $x = -3$ can be labeled $(-3, f(-3))$, or $(-3, \quad)$.

17.


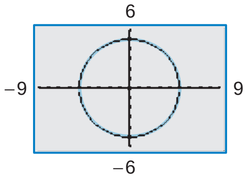
18.


Vertical Line Test for Functions **In Exercises 19–22, use the Vertical Line Test to determine whether y is a function of x. Describe how you can use a graphing utility to produce the given graph.**

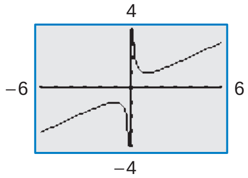
19. $y = \frac{1}{2}x^2$


20. $x - y^2 = 1$


21. $x^2 + y^2 = 25$

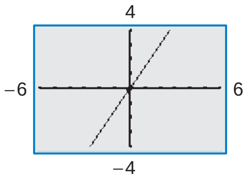


22. $x^2 = 2xy - 1$

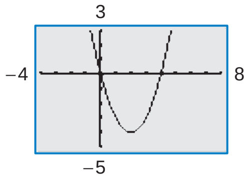


Increasing and Decreasing Functions In Exercises 23–26, determine the open intervals on which the function is increasing, decreasing, or constant.

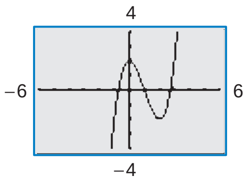
23. $f(x) = \frac{3}{2}x$



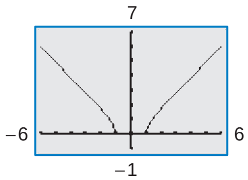
24. $f(x) = x^2 - 4x$



25. $f(x) = x^3 - 3x^2 + 2$



26. $f(x) = \sqrt{x^2 - 1}$



Increasing and Decreasing Functions In Exercises 27–34, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.

27. $f(x) = 3$ **28.** $f(x) = x$

29. $f(x) = x^{2/3}$ **30.** $f(x) = -x^{3/4}$

31. $f(x) = x\sqrt{x+3}$ **32.** $f(x) = \sqrt{1-x}$

33. $f(x) = |x+1| + |x-1|$

34. $f(x) = -|x+4| - |x+1|$

Approximating Relative Minima and Maxima In Exercises 35–46, use a graphing utility to graph the function and to approximate any relative minimum or relative maximum values of the function.

35. $f(x) = x^2 - 6x$ **36.** $f(x) = 3x^2 - 2x - 5$

37. $y = 2x^3 + 3x^2 - 12x$ **38.** $y = x^3 - 6x^2 + 15$

39. $h(x) = (x-1)\sqrt{x}$ **40.** $g(x) = x\sqrt{4-x}$

41. $f(x) = x^2 - 4x - 5$ **42.** $f(x) = 3x^2 - 12x$

43. $f(x) = x^3 - 3x$ **44.** $f(x) = -x^3 + 3x^2$

45. $f(x) = 3x^2 - 6x + 1$ **46.** $f(x) = 8x - 4x^2$

Library of Parent Functions In Exercises 47–52, sketch the graph of the function by hand. Then use a graphing utility to verify the graph.

47. $f(x) = [\![x]\!] + 2$ **48.** $f(x) = [\![x]\!] - 3$

49. $f(x) = [\![x-1]\!] + 2$ **50.** $f(x) = [\![x-2]\!] + 1$

51. $f(x) = [\![2x]\!]$ **52.** $f(x) = [\![4x]\!]$

Describing a Step Function In Exercises 53 and 54, use a graphing utility to graph the function. State the domain and range of the function. Describe the pattern of the graph.

53. $s(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)$

54. $g(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)^2$

Sketching a Piecewise-Defined Function In Exercises 55–62, sketch the graph of the piecewise-defined function by hand.

55. $f(x) = \begin{cases} 2x+3, & x < 0 \\ 3-x, & x \geq 0 \end{cases}$

56. $f(x) = \begin{cases} x+6, & x \leq -4 \\ 2x-4, & x > -4 \end{cases}$

57. $f(x) = \begin{cases} \sqrt{4+x}, & x < 0 \\ \sqrt{4-x}, & x \geq 0 \end{cases}$

58. $f(x) = \begin{cases} 1-(x-1)^2, & x \leq 2 \\ \sqrt{x-2}, & x > 2 \end{cases}$

59. $f(x) = \begin{cases} x+3, & x \leq 0 \\ 3, & 0 < x \leq 2 \\ 2x-1, & x > 2 \end{cases}$

60. $g(x) = \begin{cases} x+5, & x \leq -3 \\ -2, & -3 < x < 1 \\ 5x-4, & x \geq 1 \end{cases}$

61. $f(x) = \begin{cases} 2x+1, & x \leq -1 \\ x^2-2, & x > -1 \end{cases}$

62. $h(x) = \begin{cases} 3+x, & x < 0 \\ x^2+1, & x \geq 0 \end{cases}$

Even and Odd Functions In Exercises 63–72, use a graphing utility to graph the function and determine whether it is even, odd, or neither.

63. $f(x) = 5$ **64.** $f(x) = -9$

65. $f(x) = 3x - 2$ **66.** $f(x) = 5 - 3x$

67. $h(x) = x^2 - 4$ **68.** $f(x) = -x^2 - 8$

69. $f(x) = \sqrt{1-x}$ **70.** $g(t) = \sqrt[3]{t-1}$

71. $f(x) = |x+2|$ **72.** $f(x) = -|x-5|$

Think About It In Exercises 73–78, find the coordinates of a second point on the graph of a function f if the given point is on the graph and the function is (a) even and (b) odd.

73. $\left(-\frac{3}{2}, 4\right)$ **74.** $\left(-\frac{5}{3}, -7\right)$

75. $(4, 9)$ **76.** $(5, -1)$

77. $(x, -y)$ **78.** $(2a, 2c)$

Algebraic-Graphical-Numerical **In Exercises 79–86, determine whether the function is even, odd, or neither (a) algebraically, (b) graphically by using a graphing utility to graph the function, and (c) numerically by using the *table* feature of the graphing utility to compare $f(x)$ and $f(-x)$ for several values of x.**

79. $f(t) = t^2 + 2t - 3$ **80.** $f(x) = x^6 - 2x^2 + 3$

✓ **81.** $g(x) = x^3 - 5x$ **82.** $h(x) = x^3 - 5$

83. $f(x) = x\sqrt{1 - x^2}$ **84.** $f(x) = x\sqrt{x + 5}$

85. $g(s) = 4s^{2/3}$ **86.** $f(s) = 4s^{3/2}$

Finding the Intervals Where a Function is Positive **In Exercises 87–90, graph the function and determine the interval(s) (if any) on the real axis for which $f(x) \geq 0$. Use a graphing utility to verify your results.**

87. $f(x) = 4 - x$ **88.** $f(x) = 4x + 2$

89. $f(x) = x^2 - 9$ **90.** $f(x) = x^2 - 4x$

91. Business The cost of using a telephone calling card is \$1.05 for the first minute and \$0.08 for each additional minute or portion of a minute.

(a) A customer needs a model for the cost C of using the calling card for a call lasting t minutes. Which of the following is the appropriate model?

$$C_1(t) = 1.05 + 0.08[\![t - 1]\!]$$

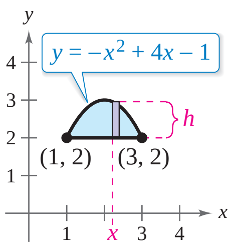
$$C_2(t) = 1.05 - 0.08[\![-(t - 1)]\!]$$

(b) Use a graphing utility to graph the appropriate model. Estimate the cost of a call lasting 18 minutes and 45 seconds.

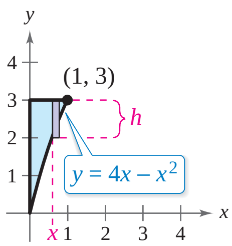
92. ***Why you should learn it*** *(p. 29)* The cost of sending an overnight package from New York to Atlanta is \$18.80 for a package weighing up to but not including 1 pound and \$3.50 for each additional pound or portion of a pound. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds, where $x > 0$. Sketch the graph of the function.

Using the Graph of a Function **In Exercises 93 and 94, write the height h of the rectangle as a function of x.**

93.



94.



✓ **95. MODELING DATA**

The number N (in thousands) of existing condominiums and cooperative homes sold each year from 2000 through 2008 in the United States is approximated by the model

$N = 0.4825t^4 - 11.293t^3 + 65.26t^2 - 48.8t + 578,$
$0 \leq t \leq 8$

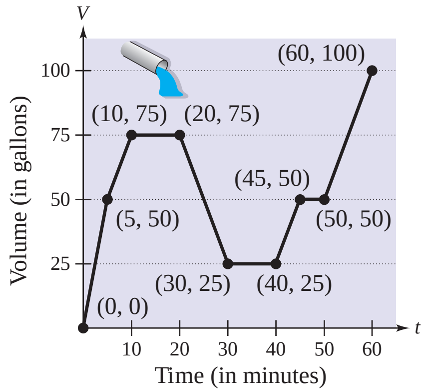
where t represents the year, with $t = 0$ corresponding to 2000. (Source: National Association of Realtors)

(a) Use a graphing utility to graph the model over the appropriate domain.

(b) Use the graph from part (a) to determine during which years the number of cooperative homes and condos was increasing. During which years was the number decreasing?

(c) Approximate the maximum number of cooperative homes and condos sold from 2000 through 2008.

96. Mechanical Engineering The intake pipe of a 100-gallon tank has a flow rate of 10 gallons per minute, and two drain pipes have a flow rate of 5 gallons per minute each. The graph shows the volume V of fluid in the tank as a function of time t. Determine in which pipes the fluid is flowing in specific subintervals of the one-hour interval of time shown on the graph. (There are many correct answers.)



Conclusions

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.**

97. A function with a square root cannot have a domain that is the set of all real numbers.

98. It is possible for an odd function to have the interval $[0, \infty)$ as its domain.

Think About It **In Exercises 99–104, match the graph of the function with the description that best fits the situation.**

(a) The air temperature at a beach on a sunny day

(b) The height of a football kicked in a field goal attempt

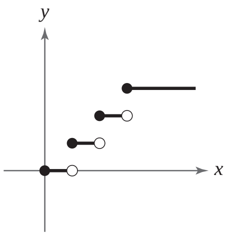
(c) The number of children in a family over time

(d) The population of California as a function of time

(e) The depth of the tide at a beach over a 24-hour period

(f) The number of cupcakes on a tray at a party

99.



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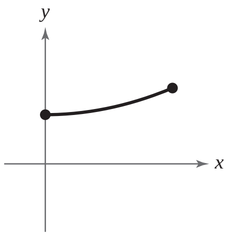

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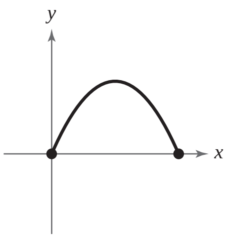

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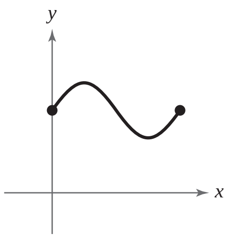
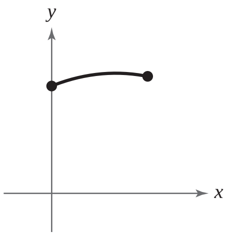


103.



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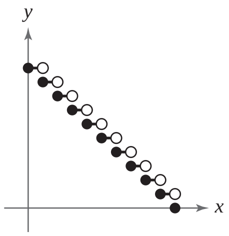


105. **Think About It** Does the graph in Exercise 20 represent x as a function of y? Explain.

106. **Think About It** Does the graph in Exercise 21 represent x as a function of y? Explain.

107. **Think About It** Can you represent the greatest integer function using a piecewise-defined function?

108. **Think About It** How does the graph of the greatest integer function differ from the graph of a line with a slope of zero?

109. Let f be an even function. Determine whether g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = -f(x - 2)$

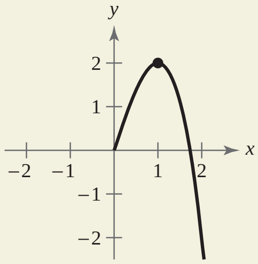
110. **CAPSTONE** Half of the graph of an odd function is shown.

(a) Sketch a complete graph of the function.

(b) Find the domain and range of the function.

(c) Identify the open intervals on which the function is increasing, decreasing, or constant.

(d) Find any relative minimum and relative maximum values of the function.



111. **Proof** Prove that a function of the following form is odd.

$$y = a_{2n+1}x^{2n+1} + a_{2n-1}x^{2n-1} + \cdots + a_3x^3 + a_1x$$

112. **Proof** Prove that a function of the following form is even.

$$y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$$

Cumulative Mixed Review

Identifying Terms and Coefficients **In Exercises 113–116, identify the terms. Then identify the coefficients of the variable terms of the expression.**

113. $-2x^2 + 8x$ **114.** $10 + 3x$

115. $\dfrac{x}{3} - 5x^2 + x^3$ **116.** $7x^4 + \sqrt{2}x^2$

Evaluating a Function **In Exercises 117 and 118, evaluate the function at each specified value of the independent variable and simplify.**

117. $f(x) = -x^2 - x + 3$

(a) $f(4)$ (b) $f(-2)$ (c) $f(x - 2)$

118. $f(x) = x\sqrt{x - 3}$

(a) $f(3)$ (b) $f(12)$ (c) $f(6)$

Evaluating a Difference Quotient **In Exercises 119 and 120, find the difference quotient and simplify your answer.**

119. $f(x) = x^2 - 2x + 9, \dfrac{f(3 + h) - f(3)}{h}, h \neq 0$

120. $f(x) = 5 + 6x - x^2, \dfrac{f(6 + h) - f(6)}{h}, h \neq 0$

1.4 Shifting, Reflecting, and Stretching Graphs

Summary of Graphs of Parent Functions

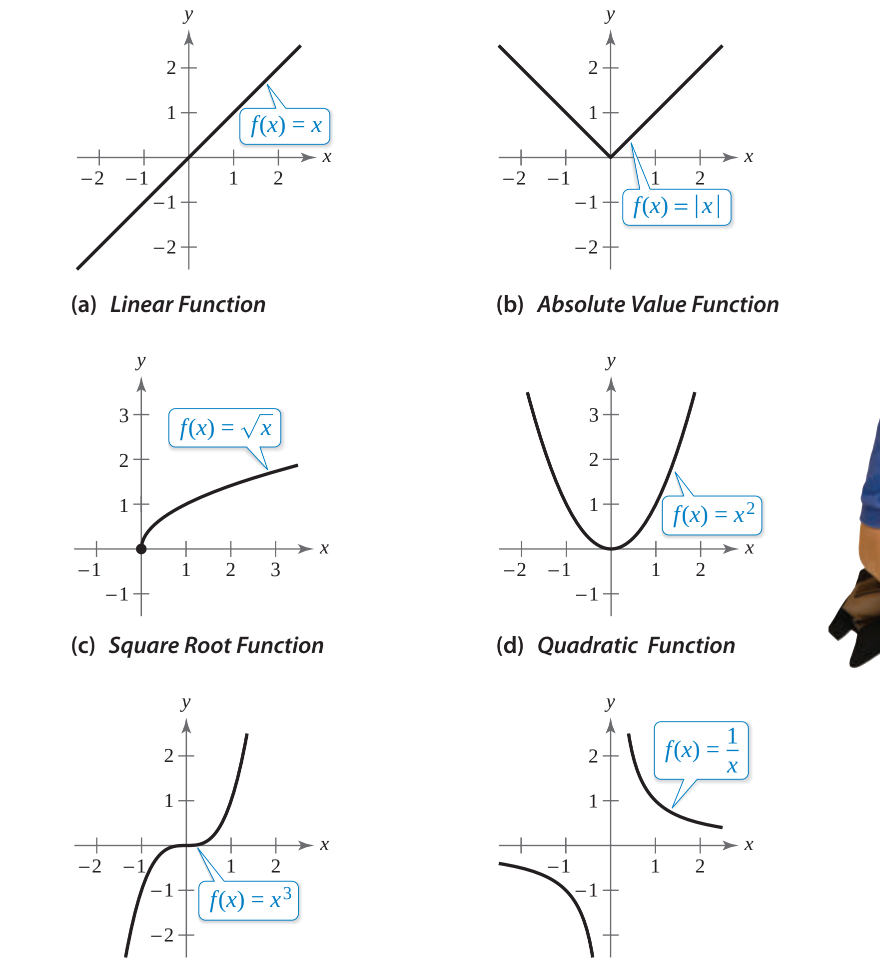
One of the goals of this text is to enable you to build your intuition for the basic shapes of the graphs of different types of functions. For instance, from your study of lines in Section 1.1, you can determine the basic shape of the graph of the parent linear function

$$f(x) = x.$$

Specifically, you know that the graph of this function is a line whose slope is 1 and whose y-intercept is $(0, 0)$.

The six graphs shown in Figure 1.34 represent the most commonly used types of functions in algebra. Familiarity with the basic characteristics of these simple parent graphs will help you analyze the shapes of more complicated graphs.

Library of Parent Functions: Commonly Used Functions



(a) ***Linear Function***

(b) ***Absolute Value Function***

(c) ***Square Root Function***

(d) ***Quadratic Function***

(e) ***Cubic Function***

(f) ***Rational Function***

Figure 1.34

Throughout this section, you will discover how many complicated graphs are derived by shifting, stretching, shrinking, or reflecting the parent graphs shown above. Shifts, stretches, shrinks, and reflections are called *transformations*. Many graphs of functions can be created from combinations of these transformations.

What you should learn

- Recognize graphs of parent functions.
- Use vertical and horizontal shifts and reflections to graph functions.
- Use nonrigid transformations to graph functions.

Why you should learn it

Recognizing the graphs of parent functions and knowing how to shift, reflect, and stretch graphs of functions can help you sketch or describe the graphs of a wide variety of simple functions. For example, in Exercise 66 on page 49, you are asked to describe a transformation that produces the graph of a model for the sales of the WD-40 Company.

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Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the graphs of parent functions summarized in Figure 1.34. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ two units *upward*, as shown in Figure 1.35. In function notation, h and f are related as follows.

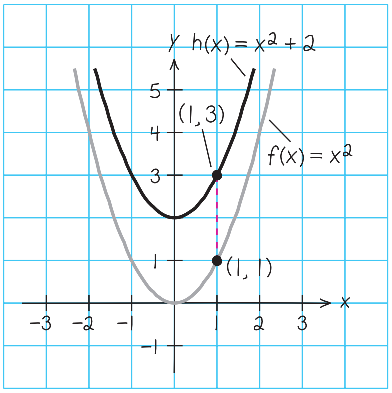
$$\begin{aligned} h(x) &= x^2 + 2 \\ &= f(x) + 2 \qquad \text{Upward shift of two units} \end{aligned}$$

Similarly, you can obtain the graph of

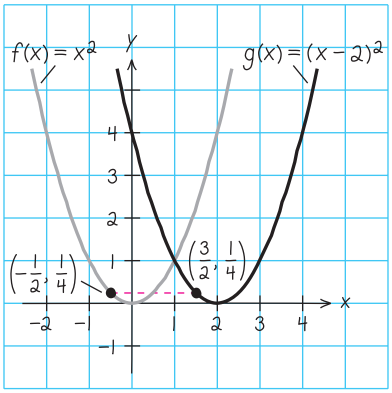
$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ two units to the *right*, as shown in Figure 1.36. In this case, the functions g and f have the following relationship.

$$\begin{aligned} g(x) &= (x - 2)^2 \\ &= f(x - 2) \qquad \text{Right shift of two units} \end{aligned}$$



Vertical shift upward: two units
Figure 1.35



Horizontal shift to the right: two units
Figure 1.36

Explore the Concept

Use a graphing utility to display (in the same viewing window) the graphs of $y = x^2 + c$, where $c = -2, 0, 2,$ and 4. Use the results to describe the effect that c has on the graph.

Use a graphing utility to display (in the same viewing window) the graphs of $y = (x + c)^2$, where $c = -2, 0, 2,$ and 4. Use the results to describe the effect that c has on the graph.

The following list summarizes vertical and horizontal shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *upward*: $h(x) = f(x) + c$
2. Vertical shift c units *downward*: $h(x) = f(x) - c$
3. Horizontal shift c units to the *right*: $h(x) = f(x - c)$
4. Horizontal shift c units to the *left*: $h(x) = f(x + c)$

In items 3 and 4, be sure you see that

$$h(x) = f(x - c)$$

corresponds to a *right* shift and

$$h(x) = f(x + c)$$

corresponds to a *left* shift for $c > 0$.

Example 1 Shifts in the Graph of a Function

Compare the graph of each function with the graph of $f(x) = x^3$.

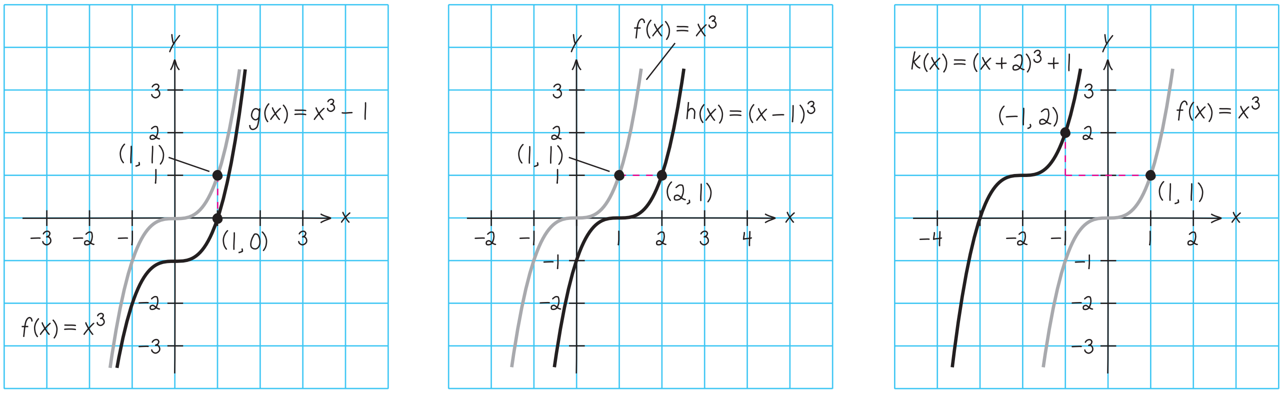
a. $g(x) = x^3 - 1$ **b.** $h(x) = (x - 1)^3$ **c.** $k(x) = (x + 2)^3 + 1$

Solution

a. You obtain the graph of g by shifting the graph of f one unit downward.

b. You obtain the graph of h by shifting the graph of f one unit to the right.

c. You obtain the graph of k by shifting the graph of f two units to the left and then one unit upward.



(a) ***Vertical shift: one unit downward*** **(b)** ***Horizontal shift: one unit right*** **(c)** ***Two units left and one unit upward***

Figure 1.37

CHECKPOINT Now try Exercise 23.

Example 2 Finding Equations from Graphs

The graph of $f(x) = x^2$ is shown in Figure 1.38. Each of the graphs in Figure 1.39 is a transformation of the graph of f. Find an equation for each function.

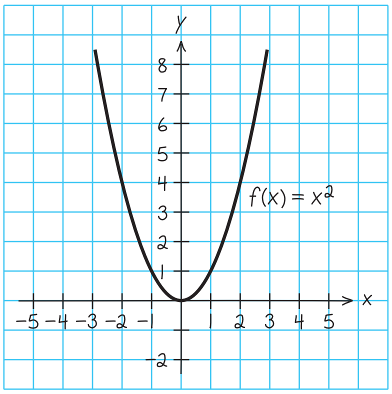
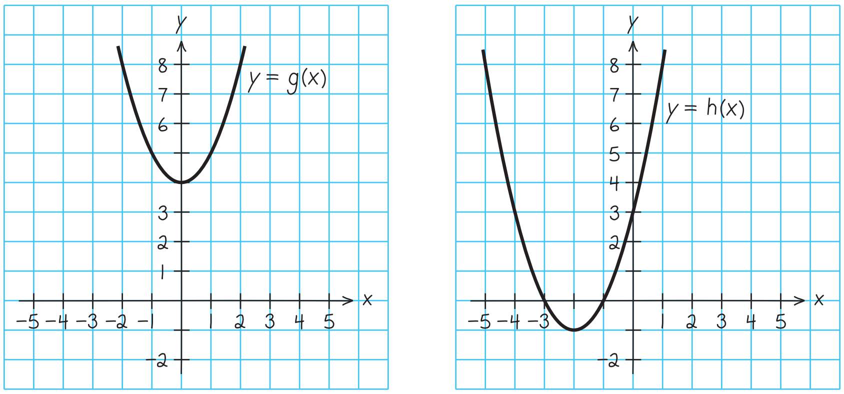


Figure 1.38



(a) **(b)**

Figure 1.39

Solution

a. The graph of g is a vertical shift of four units upward of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = x^2 + 4$.

b. The graph of h is a horizontal shift of two units to the left, and a vertical shift of one unit downward, of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = (x + 2)^2 - 1$.

CHECKPOINT Now try Exercise 31.

Reflecting Graphs

Another common type of transformation is called a **reflection.** For instance, when you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$ (see Figure 1.40).

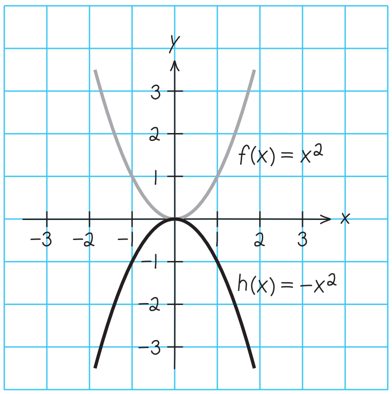


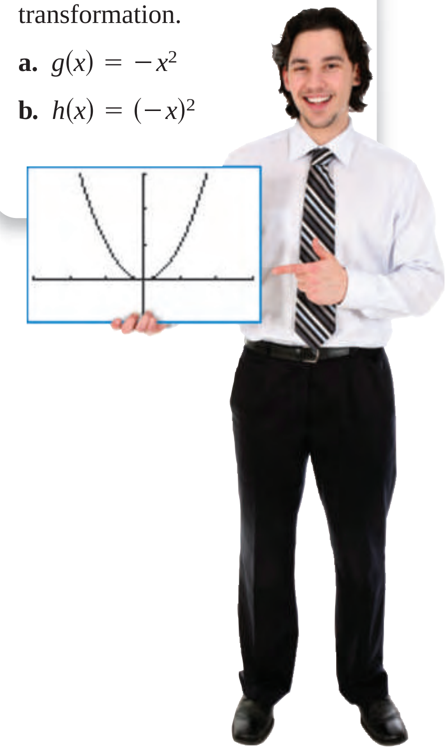
Figure 1.40

Explore the Concept

Compare the graph of each function with the graph of $f(x) = x^2$ by using a graphing utility to graph the function and f in the same viewing window. Describe the transformation.

a. $g(x) = -x^2$

b. $h(x) = (-x)^2$



Reflections in the Coordinate Axes

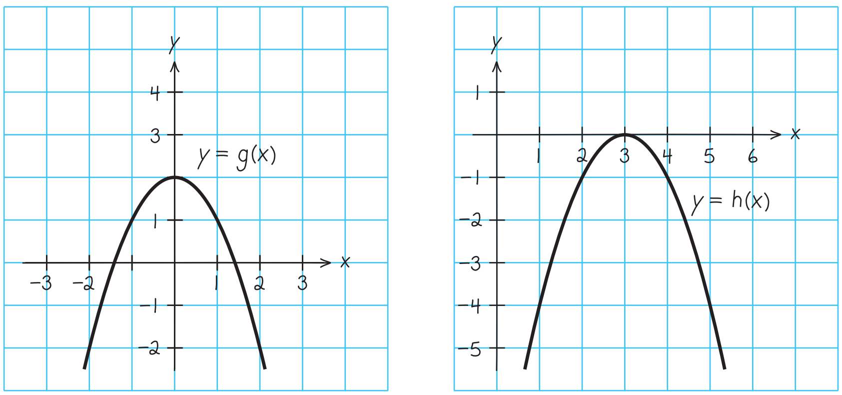
Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. Reflection in the x-axis: $h(x) = -f(x)$

2. Reflection in the y-axis: $h(x) = f(-x)$

Example 3 Finding Equations from Graphs

The graph of $f(x) = x^2$ is shown in Figure 1.40. Each of the graphs in Figure 1.41 is a transformation of the graph of f. Find an equation for each function.



(a) (b)

Figure 1.41

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = -x^2 + 2$.

b. The graph of h is a horizontal shift of three units to the right *followed by* a reflection in the x-axis of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = -(x - 3)^2$.

CHECKPOINT Now try Exercise 33.

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Example 4 Reflections and Shifts

Compare the graph of each function with the graph of

$$f(x) = \sqrt{x}.$$

a. $g(x) = -\sqrt{x}$

b. $h(x) = \sqrt{-x}$

c. $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. Relative to the graph of $f(x) = \sqrt{x}$, the graph of g is a reflection in the x-axis because

$$\begin{aligned} g(x) &= -\sqrt{x} \\ &= -f(x). \end{aligned}$$

b. The graph of h is a reflection of the graph of $f(x) = \sqrt{x}$ in the y-axis because

$$\begin{aligned} h(x) &= \sqrt{-x} \\ &= f(-x). \end{aligned}$$

c. From the equation

$$\begin{aligned} k(x) &= -\sqrt{x+2} \\ &= -f(x+2) \end{aligned}$$

you can conclude that the graph of k is a left shift of two units, followed by a reflection in the x-axis, of the graph of $f(x) = \sqrt{x}$.

CHECKPOINT Now try Exercise 35.

Graphical Solution

a. From the graph in Figure 1.42, you can see that the graph of g is a reflection of the graph of f in the x-axis. Note that the domain of g is $x \geq 0$.

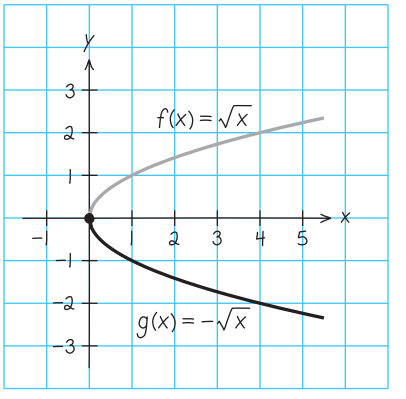


Figure 1.42

b. From the graph in Figure 1.43, you can see that the graph of h is a reflection of the graph of f in the y-axis. Note that the domain of h is $x \leq 0$.

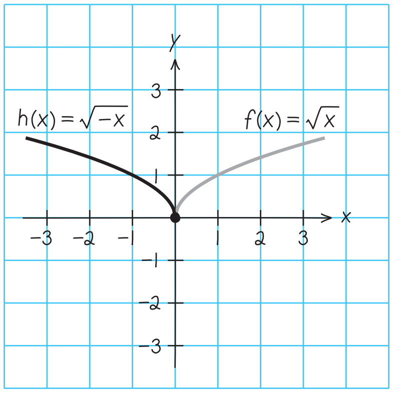


Figure 1.43

c. From the graph in Figure 1.44, you can see that the graph of k is a left shift of two units of the graph of f, followed by a reflection in the x-axis. Note that the domain of k is $x \geq -2$.

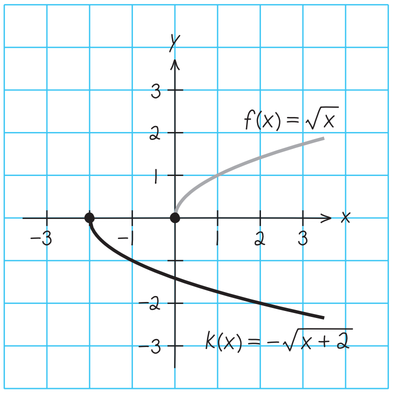


Figure 1.44

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are called **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** when $c > 1$ and a **vertical shrink** when $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** when $c > 1$ and a **horizontal stretch** when $0 < c < 1$.

Example 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$

b. $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of

$$h(x) = 3|x| = 3f(x)$$

is a vertical stretch (each y-value is multiplied by 3) of the graph of f. (See Figure 1.45.)

b. Similarly, the graph of

$$g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$$

is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. (See Figure 1.46.)

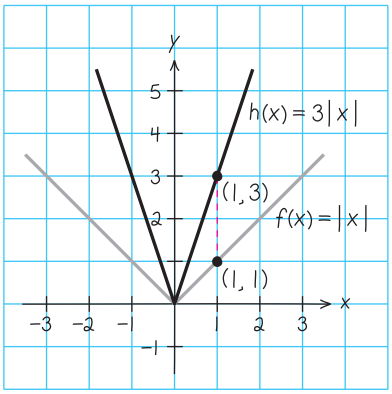


Figure 1.45

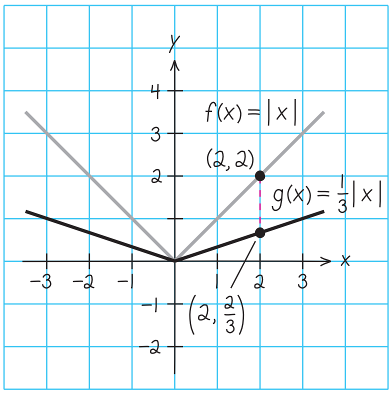


Figure 1.46

CHECKPOINT Now try Exercise 41.

Example 6 Nonrigid Transformations

Compare the graph of $h(x) = f\left(\frac{1}{2}x\right)$ with the graph of $f(x) = 2 - x^3$.

Solution

Relative to the graph of $f(x) = 2 - x^3$, the graph of

$$h(x) = f\left(\tfrac{1}{2}x\right) = 2 - \left(\tfrac{1}{2}x\right)^3 = 2 - \tfrac{1}{8}x^3$$

is a horizontal stretch (each x-value is multiplied by 2) of the graph of f. (See Figure 1.47.)

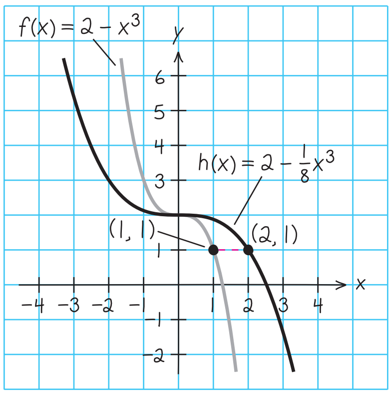


Figure 1.47

CHECKPOINT Now try Exercise 49.

1.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Name three types of rigid transformations.

2. Match the rigid transformation of $y = f(x)$ with the correct representation, where $c > 0$.

(a) $h(x) = f(x) + c$ (i) horizontal shift c units to the left

(b) $h(x) = f(x) - c$ (ii) vertical shift c units upward

(c) $h(x) = f(x - c)$ (iii) horizontal shift c units to the right

(d) $h(x) = f(x + c)$ (iv) vertical shift c units downward

In Exercises 3 and 4, fill in the blanks.

3. A reflection in the x-axis of $y = f(x)$ is represented by $h(x) =$ _______ , while a reflection in the y-axis of $y = f(x)$ is represented by $h(x) =$ _______ .

4. A nonrigid transformation of $y = f(x)$ represented by $cf(x)$ is a vertical stretch when _______ and a vertical shrink when _______ .

Procedures and Problem Solving

Sketching Transformations **In Exercises 5–18, sketch the graphs of the three functions by hand on the same rectangular coordinate system. Verify your results with a graphing utility.**

5. $f(x) = x$
$g(x) = x - 4$
$h(x) = 3x$

6. $f(x) = \frac{1}{2}x$
$g(x) = \frac{1}{2}x + 2$
$h(x) = \frac{1}{2}(x - 2)$

7. $f(x) = x^2$
$g(x) = x^2 + 2$
$h(x) = (x - 2)^2$

8. $f(x) = x^2$
$g(x) = x^2 - 4$
$h(x) = (x + 2)^2 + 1$

9. $f(x) = -x^2$
$g(x) = -x^2 + 1$
$h(x) = -(x - 2)^2$

10. $f(x) = (x - 2)^2$
$g(x) = (x + 2)^2 + 2$
$h(x) = -(x - 2)^2 - 1$

11. $f(x) = x^2$
$g(x) = \frac{1}{2}x^2$
$h(x) = (2x)^2$

12. $f(x) = x^2$
$g(x) = \frac{1}{4}x^2 + 2$
$h(x) = -\frac{1}{4}x^2$

13. $f(x) = |x|$
$g(x) = |x| - 1$
$h(x) = |x - 3|$

14. $f(x) = |x|$
$g(x) = |x + 3|$
$h(x) = -2|x + 2| - 1$

15. $f(x) = \sqrt{x}$
$g(x) = \sqrt{x + 1}$
$h(x) = \sqrt{x - 2} + 1$

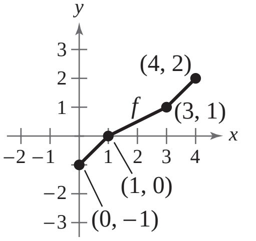
16. $f(x) = \sqrt{x}$
$g(x) = \frac{1}{2}\sqrt{x}$
$h(x) = -\sqrt{x + 4}$

17. $f(x) = \dfrac{1}{x}$
$g(x) = \dfrac{1}{x} + 2$
$h(x) = \dfrac{1}{x - 1} + 2$

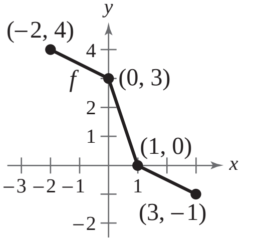
18. $f(x) = \dfrac{1}{x}$
$g(x) = \dfrac{1}{x} - 4$
$h(x) = \dfrac{1}{x + 3} - 1$

Sketching Transformations **In Exercises 19 and 20, use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

19. (a) $y = f(x) + 2$
(b) $y = -f(x)$
(c) $y = f(x - 2)$
(d) $y = f(x + 3)$
(e) $y = 2f(x)$
(f) $y = f(-x)$
(g) $y = f\left(\frac{1}{2}x\right)$

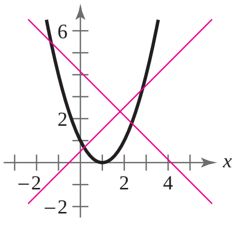


20. (a) $y = f(x) - 1$
(b) $y = f(x + 1)$
(c) $y = f(x - 1)$
(d) $y = -f(x - 2)$
(e) $y = f(-x)$
(f) $y = \frac{1}{2}f(x)$
(g) $y = f(2x)$

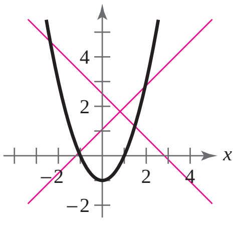


Error Analysis **In Exercises 21 and 22, describe the error in graphing the function.**

21. $f(x) = (x + 1)^2$



22. $f(x) = (x - 1)^2$



Library of Parent Functions **In Exercises 23–28, compare the graph of the function with the graph of its parent function.**

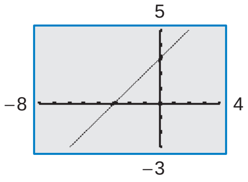
✓ **23.** $y = \sqrt{x} + 2$ **24.** $y = \dfrac{1}{x} - 5$

25. $y = (x - 4)^3$ **26.** $y = |x + 5|$

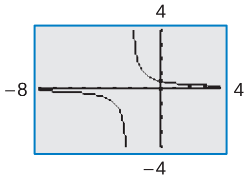
27. $y = x^2 - 2$ **28.** $y = \sqrt{x - 2}$

Library of Parent Functions **In Exercises 29–34, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.**

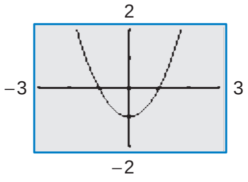
29.



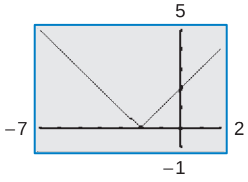
30.



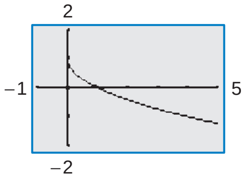
✓ **31.**



32.



✓ **33.**



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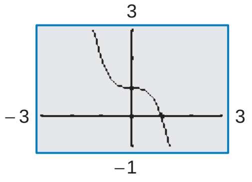

Rigid and Nonrigid Transformations **In Exercises 35–46, compare the graph of the function with the graph of its parent function.**

✓ **35.** $y = -|x|$ **36.** $y = |-x|$

37. $y = (-x)^2$ **38.** $y = -x^3$

39. $y = \dfrac{1}{-x}$ **40.** $y = -\dfrac{1}{x}$

✓ **41.** $h(x) = 4|x|$ **42.** $p(x) = \frac{1}{2}x^2$

43. $g(x) = \frac{1}{4}x^3$ **44.** $y = 2\sqrt{x}$

45. $f(x) = \sqrt{4x}$ **46.** $y = |\frac{1}{2}x|$

Rigid and Nonrigid Transformations **In Exercises 47–50, use a graphing utility to graph the three functions in the same viewing window. Describe the graphs of g and h relative to the graph of f.**

47. $f(x) = x^3 - 3x^2$
$g(x) = f(x + 2)$
$h(x) = \frac{1}{2}f(x)$

48. $f(x) = x^3 - 3x^2 + 2$
$g(x) = f(x - 1)$
$h(x) = f(3x)$

✓ **49.** $f(x) = x^3 - 3x^2$
$g(x) = -\frac{1}{3}f(x)$
$h(x) = f(-x)$

50. $f(x) = x^3 - 3x^2 + 2$
$g(x) = -f(x)$
$h(x) = f(2x)$

Describing Transformations **In Exercises 51–64, g is related to one of the six parent functions on page 41. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to g. (c) Sketch the graph of g by hand. (d) Use function notation to write g in terms of the parent function f.**

51. $g(x) = 2 - (x + 5)^2$ **52.** $g(x) = -(x + 10)^2 + 5$

53. $g(x) = 3 + 2(x - 4)^2$ **54.** $g(x) = -\frac{1}{4}(x + 2)^2 - 2$

55. $g(x) = 3(x - 2)^3$ **56.** $g(x) = -\frac{1}{2}(x + 1)^3$

57. $g(x) = (x - 1)^3 + 2$

58. $g(x) = -(x + 3)^3 - 10$

59. $g(x) = \dfrac{1}{x + 8} - 9$ **60.** $g(x) = \dfrac{1}{x - 7} + 4$

61. $g(x) = -2|x - 1| - 4$ **62.** $g(x) = \frac{1}{2}|x - 2| - 3$

63. $g(x) = -\frac{1}{2}\sqrt{x + 3} - 1$ **64.** $g(x) = -\sqrt{x + 1} - 6$

65. MODELING DATA

The amounts of fuel F (in billions of gallons) used by motor vehicles from 1991 through 2007 are given by the ordered pairs of the form $(t, F(t))$, where $t = 1$ represents 1991. A model for the data is

$$F(t) = -0.099(t - 24.7)^2 + 183.4.$$

(Source: U.S. Federal Highway Administration)

(1, 128.6)
(2, 132.9)
(3, 137.3)
(4, 140.8)
(5, 143.8)
(6, 147.4)
(7, 150.4)
(8, 155.4)
(9, 161.4)
(10, 162.5) (14, 173.5)
(11, 163.5) (15, 174.8)
(12, 168.7) (16, 175.0)
(13, 170.0) (17, 176.1)

(a) Describe the transformation of the parent function $f(t) = t^2$.

(b) Use a graphing utility to graph the model and the data in the same viewing window.

(c) Rewrite the function so that $t = 0$ represents 2000. Explain how you got your answer.

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66. *Why you should learn it* *(p. 41)* The sales S (in millions of dollars) of the WD-40 Company from 2000 through 2008 can be approximated by the function

$$S(t) = 99\sqrt{t + 2.37}$$

where $t = 0$ represents 2000. (Source: WD-40 Company)

(a) Describe the transformation of the parent function $f(t) = \sqrt{t}$.

(b) Use a graphing utility to graph the model over the interval $0 \le t \le 8$.

(c) According to the model, in what year will the sales of WD-40 be approximately 400 million dollars?

(d) Rewrite the function so that $t = 0$ represents 2005. Explain how you got your answer.

Conclusions

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

68. The graphs of $f(x) = |x| + 6$ and $f(x) = |-x| + 6$ are identical.

Exploration **In Exercises 69–72, use the fact that the graph of $y = f(x)$ has x-intercepts at $x = 2$ and $x = -3$ to find the x-intercepts of the given graph. If not possible, state the reason.**

69. $y = f(-x)$

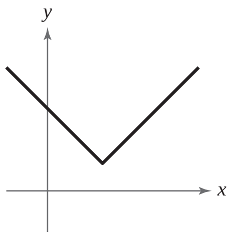
70. $y = 2f(x)$

71. $y = f(x) + 2$

72. $y = f(x - 3)$

Library of Parent Functions **In Exercises 73–76, determine which equation(s) may be represented by the graph shown. There may be more than one correct answer.**

73.



(a) $f(x) = |x + 2| + 1$

(b) $f(x) = |x - 1| + 2$

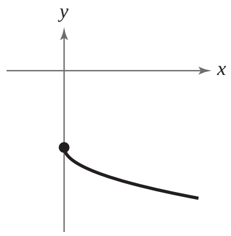
(c) $f(x) = |x - 2| + 1$

(d) $f(x) = 2 + |x - 2|$

(e) $f(x) = |(x - 2) + 1|$

(f) $f(x) = 1 - |x - 2|$

74.



(a) $f(x) = -\sqrt{x} - 4$

(b) $f(x) = -4 - \sqrt{x}$

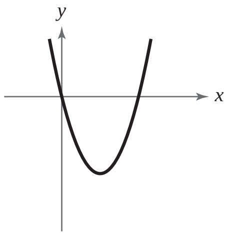
(c) $f(x) = -4 - \sqrt{-x}$

(d) $f(x) = \sqrt{-x} - 4$

(e) $f(x) = \sqrt{-x} + 4$

(f) $f(x) = \sqrt{x} - 4$

75.



(a) $f(x) = (x - 2)^2 - 2$

(b) $f(x) = (x + 4)^2 - 4$

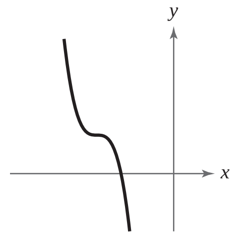
(c) $f(x) = (x - 2)^2 - 4$

(d) $f(x) = (x + 2)^2 - 4$

(e) $f(x) = 4 - (x - 2)^2$

(f) $f(x) = 4 - (x + 2)^2$

76.



(a) $f(x) = -(x - 4)^3 + 2$

(b) $f(x) = -(x + 4)^3 + 2$

(c) $f(x) = -(x - 2)^3 + 4$

(d) $f(x) = (-x - 4)^3 + 2$

(e) $f(x) = (x + 4)^3 + 2$

(f) $f(x) = (-x + 4)^3 + 2$

77. **Think About It** You can use either of two methods to graph a function: plotting points, or translating a parent function as shown in this section. Which method do you prefer to use for each function? Explain.

(a) $f(x) = 3x^2 - 4x + 1$ (b) $f(x) = 2(x - 1)^2 - 6$

78. **Think About It** The graph of $y = f(x)$ passes through the points $(0, 1)$, $(1, 2)$, and $(2, 3)$. Find the corresponding points on the graph of $y = f(x + 2) - 1$.

79. **Think About It** Compare the graph of $g(x) = ax^2$ with the graph of $f(x) = x^2$ when (a) $0 < a < 1$ and (b) $a > 1$.

80. **CAPSTONE** Use the fact that the graph of $y = f(x)$ is increasing on the interval $(-\infty, 2)$ and decreasing on the interval $(2, \infty)$ to find the intervals on which the graph is increasing and decreasing. If not possible, state the reason.

(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = 2f(x)$

(d) $y = f(x) - 3$ (e) $y = f(x + 1)$

Cumulative Mixed Review

Parallel and Perpendicular Lines **In Exercises 81 and 82, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.**

81. L_1: $(-2, -2)$, $(2, 10)$

L_2: $(-1, 3)$, $(3, 9)$

82. L_1: $(-1, -7)$, $(4, 3)$

L_2: $(1, 5)$, $(-2, -7)$

Finding the Domain of a Function **In Exercises 83–86, find the domain of the function.**

83. $f(x) = \dfrac{4}{9 - x}$

84. $f(x) = \dfrac{\sqrt{x - 5}}{x - 7}$

85. $f(x) = \sqrt{100 - x^2}$

86. $f(x) = \sqrt[3]{16 - x^2}$

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1.5 Combinations of Functions

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. When

$$f(x) = 2x - 3 \quad \text{and} \quad g(x) = x^2 - 1$$

you can form the sum, difference, product, and quotient of f and g as follows.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1)$$
$$= x^2 + 2x - 4 \qquad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 - 1)$$
$$= -x^2 + 2x - 2 \qquad \text{Difference}$$
$$f(x) \cdot g(x) = (2x - 3)(x^2 - 1)$$
$$= 2x^3 - 3x^2 - 2x + 3 \qquad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \qquad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient

$$\frac{f(x)}{g(x)}$$

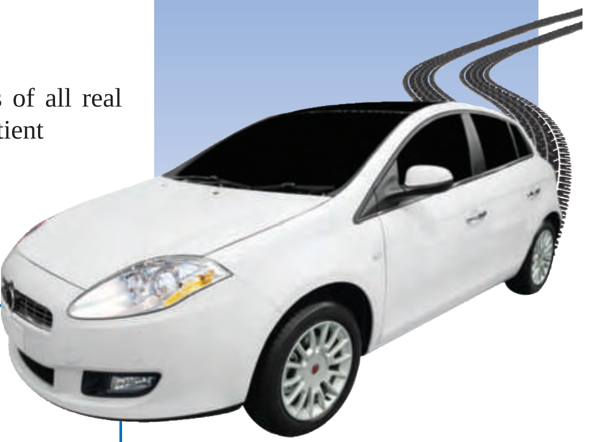
there is the further restriction that $g(x) \neq 0$.

What you should learn

- Add, subtract, multiply, and divide functions.
- Find compositions of one function with another function.
- Use combinations of functions to model and solve real-life problems.

Why you should learn it

You can model some situations by combining functions. For instance, in Exercise 79 on page 57, you will model the stopping distance of a car by combining the driver's reaction time with the car's braking distance.



Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

1. **Sum:** $(f + g)(x) = f(x) + g(x)$
2. **Difference:** $(f - g)(x) = f(x) - g(x)$
3. **Product:** $(fg)(x) = f(x) \cdot g(x)$
4. **Quotient:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

Solution

$$(f + g)(x) = f(x) + g(x)$$
$$= (2x + 1) + (x^2 + 2x - 1)$$
$$= x^2 + 4x$$

When $x = 2$, the value of this sum is $(f + g)(2) = 2^2 + 4(2) = 12$.

CHECKPOINT Now try Exercise 13(a).

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Example 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Algebraic Solution

The difference of the functions f and g is

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (2x + 1) - (x^2 + 2x - 1)\\ &= -x^2 + 2.\end{aligned}$$

When $x = 2$, the value of this difference is

$$\begin{aligned}(f - g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

CHECKPOINT Now try Exercise 13(b).

Graphical Solution

Enter the functions in a graphing utility (see Figure 1.48). Then graph the difference of the two functions, y_3, as shown in Figure 1.49.

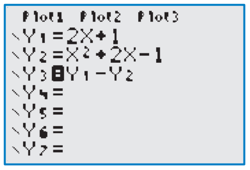


Figure 1.48

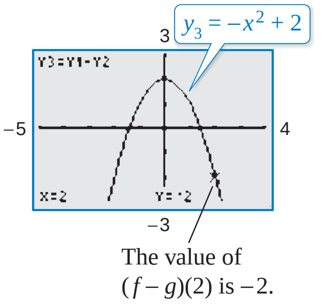


Figure 1.49

Example 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= (x^2)(x - 3)\\ &= x^3 - 3x^2\end{aligned}$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

CHECKPOINT Now try Exercise 13(c).

In Examples 1–3, both f and g have domains that consist of all real numbers. So, the domain of both $(f + g)$ and $(f - g)$ is also the set of all real numbers. Remember that any restrictions on the domains of f or g must be considered when forming the sum, difference, product, or quotient of f and g. For instance, the domain of $f(x) = 1/x$ is all $x \neq 0$, and the domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. This implies that the domain of $(f + g)$ is $(0, \infty)$.

Example 4 Finding the Quotient of Two Functions

Given $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find $(f/g)(x)$. Then find the domain of f/g.

Solution

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domain of f/g is $[0, 2)$.

CHECKPOINT Now try Exercise 13(d).

Compositions of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, when $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$f(g(x)) = f(x + 1)$$
$$= (x + 1)^2.$$

This composition is denoted as $f \circ g$ and is read as "f composed with g."

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.50.)

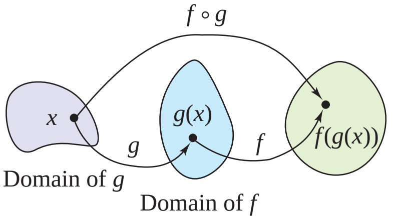


Figure 1.50

Example 5 Forming the Composition of f with g

Find $(f \circ g)(x)$ for

$$f(x) = \sqrt{x},\ x \ge 0, \qquad \text{and} \qquad g(x) = x - 1,\ x \ge 1.$$

If possible, find $(f \circ g)(2)$ and $(f \circ g)(0)$.

Solution

The composition of f with g is

$$(f \circ g)(x) = f(g(x)) \qquad \text{Definition of } f \circ g$$
$$= f(x - 1) \qquad \text{Definition of } g(x)$$
$$= \sqrt{x - 1}, \quad x \ge 1. \qquad \text{Definition of } f(x)$$

The domain of $f \circ g$ is $[1, \infty)$. (See Figure 1.51). So,

$$(f \circ g)(2) = \sqrt{2 - 1} = 1$$

is defined, but $(f \circ g)(0)$ is not defined because 0 is not in the domain of $f \circ g$.

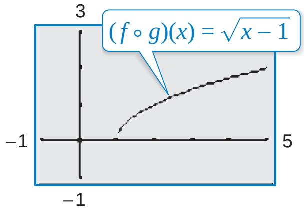


Figure 1.51

CHECKPOINT Now try Exercise 41.

Explore the Concept

Let $f(x) = x + 2$ and $g(x) = 4 - x^2$. Are the compositions $f \circ g$ and $g \circ f$ equal? You can use your graphing utility to answer this question by entering and graphing the following functions.

$$y_1 = (4 - x^2) + 2$$
$$y_2 = 4 - (x + 2)^2$$

What do you observe? Which function represents $f \circ g$ and which represents $g \circ f$?

The composition of f with g is generally not the same as the composition of g with f. This is illustrated in Example 6.

Example 6 Compositions of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, evaluate

(a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$

when $x = 0$ and 1.

Algebraic Solution

a. $(f \circ g)(x) = f(g(x))$ Definition of $f \circ g$

$= f(4 - x^2)$ Definition of $g(x)$

$= (4 - x^2) + 2$ Definition of $f(x)$

$= -x^2 + 6$

$(f \circ g)(0) = -0^2 + 6 = 6$

$(f \circ g)(1) = -1^2 + 6 = 5$

b. $(g \circ f)(x) = g(f(x))$ Definition of $g \circ f$

$= g(x + 2)$ Definition of $f(x)$

$= 4 - (x + 2)^2$ Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$

$= -x^2 - 4x$

$(g \circ f)(0) = -0^2 - 4(0) = 0$

$(g \circ f)(1) = -1^2 - 4(1) = -5$

Note that $f \circ g \neq g \circ f$.

CHECKPOINT Now try Exercise 43.

Graphical Solution

a. and b. Enter $y_1 = f(x)$, $y_2 = g(x)$, $y_3 = (f \circ g)(x)$, and $y_4 = (g \circ f)(x)$, as shown in Figure 1.52. Then use the *table* feature to find the desired function values (see Figure 1.53).

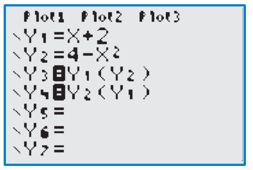


Figure 1.52

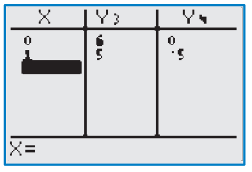


Figure 1.53

From the table you can see that $f \circ g \neq g \circ f$.

Example 7 Finding the Domain of a Composite Function

Find the domain of $f \circ g$ for the functions given by

$f(x) = x^2 - 9$ and $g(x) = \sqrt{9 - x^2}$.

Algebraic Solution

The composition of the functions is as follows.

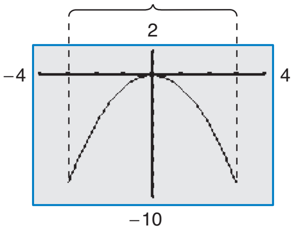
$$(f \circ g)(x) = f(g(x))$$
$$= f\left(\sqrt{9 - x^2}\right)$$
$$= \left(\sqrt{9 - x^2}\right)^2 - 9$$
$$= 9 - x^2 - 9$$
$$= -x^2$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $f \circ g$ is $[-3, 3]$.

CHECKPOINT Now try Exercise 45.

Graphical Solution

The x-coordinates of points on the graph extend from -3 to 3. So, the domain of $f \circ g$ is $[-3, 3]$.



Example 8 A Case in Which $f \circ g = g \circ f$

Given

$$f(x) = 2x + 3 \text{ and } g(x) = \tfrac{1}{2}(x - 3)$$

find each composition.

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{2}(x - 3)\right) \\ &= 2\left[\frac{1}{2}(x - 3)\right] + 3 \\ &= x - 3 + 3 \\ &= x\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= \frac{1}{2}\left[(2x + 3) - 3\right] \\ &= \frac{1}{2}(2x) \\ &= x\end{aligned}$$

CHECKPOINT Now try Exercise 57.

Study Tip

In Example 8, note that the two composite functions $f \circ g$ and $g \circ f$ are equal, and both represent the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You will study this special case in the next section.

Explore the Concept

Write each function as a composition of two functions.

a. $h(x) = |x^3 - 2|$

b. $r(x) = |x^3| - 2$

What do you notice about the inner and outer functions?

In Examples 5–8, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. Basically, to "decompose" a composite function, look for an "inner" and an "outer" function.

Example 9 Identifying a Composite Function

algebra of calculus

Write the function

$$h(x) = \frac{1}{(x - 2)^2}$$

as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x - 2)^2} = (x - 2)^{-2} = f(x - 2) = f(g(x)).$$

CHECKPOINT Now try Exercise 75.

Application

Example 10 Bacteria Count

The number N of bacteria in a refrigerated petri dish is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the petri dish (in degrees Celsius). When the petri dish is removed from refrigeration, the temperature of the petri dish is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time (in hours).

a. Find the composition $N(T(t))$ and interpret its meaning in context.
b. Find the number of bacteria in the petri dish when $t = 2$ hours.
c. Find the time when the bacteria count reaches 2000.

Microbiologist

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

The composite function $N(T(t))$ represents the number of bacteria as a function of the amount of time the petri dish has been out of refrigeration.

b. When $t = 2$, the number of bacteria is

$$N = 320(2)^2 + 420 = 1280 + 420 = 1700.$$

c. The bacteria count will reach $N = 2000$ when $320t^2 + 420 = 2000$. You can solve this equation for t algebraically as follows.

$$\begin{aligned} 320t^2 + 420 &= 2000 \\ 320t^2 &= 1580 \\ t^2 &= \frac{79}{16} \\ t &= \frac{\sqrt{79}}{4} \quad \Longrightarrow \quad t \approx 2.22 \text{ hours} \end{aligned}$$

So, the count will reach 2000 when $t \approx 2.22$ hours. Note that the negative value is rejected because it is not in the domain of the composite function. To confirm your solution, graph the equation $N = 320t^2 + 420$, as shown in Figure 1.54. Then use the *zoom* and *trace* features to approximate $N = 2000$ when $t \approx 2.22$, as shown in Figure 1.55.

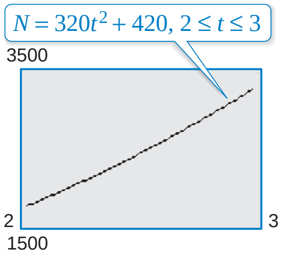


Figure 1.54

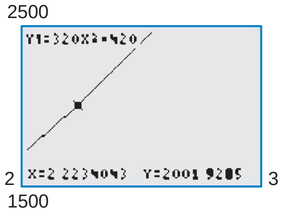


Figure 1.55

CHECKPOINT Now try Exercise 85.

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1.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. Two functions f and g can be combined by the arithmetic operations of _______ , _______ , _______ , and _______ to create new functions.

2. The _______ of the function f with the function g is $(f \circ g)(x) = f(g(x))$.

3. The domain of $f \circ g$ is the set of all x in the domain of g such that _______ is in the domain of f.

4. To decompose a composite function, look for an _______ and an _______ function.

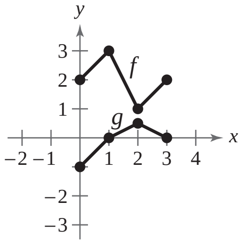
5. Given $f(x) = x^2 + 1$ and $(fg)(x) = 2x(x^2 + 1)$, what is $g(x)$?

6. Given $(f \circ g)(x) = f(x^2 + 1)$, what is $g(x)$?

Procedures and Problem Solving

Finding the Sum of Two Functions **In Exercises 7–10, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

7.



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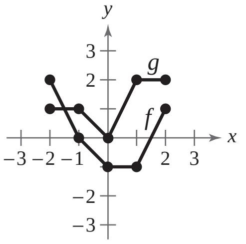

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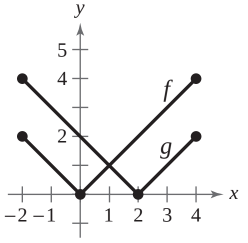

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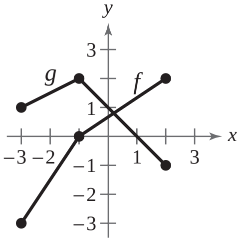


Finding Arithmetic Combinations of Functions **In Exercises 11–18, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?**

11. $f(x) = x + 3, \quad g(x) = x - 3$

12. $f(x) = 2x - 5, \quad g(x) = 1 - x$

13. $f(x) = x^2, \quad g(x) = 1 - x$

14. $f(x) = 2x - 5, \quad g(x) = 5$

15. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1 - x}$

16. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$

17. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$

18. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

Evaluating an Arithmetic Combination of Functions **In Exercises 19–32, evaluate the indicated function for $f(x) = x^2 - 1$ and $g(x) = x - 2$ algebraically. If possible, use a graphing utility to verify your answer.**

19. $(f + g)(3)$

20. $(f - g)(-2)$

21. $(f - g)(0)$

22. $(f + g)(1)$

23. $(fg)(6)$

24. $(fg)(-4)$

25. $(f/g)(-5)$

26. $(f/g)(0)$

27. $(f - g)(2t)$

28. $(f + g)(t - 4)$

29. $(fg)(-5t)$

30. $(fg)(3t^2)$

31. $(f/g)(-t)$

32. $(f/g)(t + 2)$

Graphing an Arithmetic Combination of Functions **In Exercises 33–36, use a graphing utility to graph the functions f, g, and h in the same viewing window.**

33. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1, \quad h(x) = f(x) + g(x)$

34. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4, \quad h(x) = f(x) - g(x)$

35. $f(x) = x^2, \quad g(x) = -2x, \quad h(x) = f(x) \cdot g(x)$

36. $f(x) = 4 - x^2, \quad g(x) = x, \quad h(x) = f(x)/g(x)$

Graphing a Sum of Functions **In Exercises 37–40, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?**

37. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$

38. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$

39. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$

40. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

Compositions of Functions **In Exercises 41–44, find (a) $f \circ g$, (b) $g \circ f$, and, if possible, (c) $(f \circ g)(0)$.**

41. $f(x) = x^2, \quad g(x) = x - 1$
42. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$
43. $f(x) = 3x + 5, \quad g(x) = 5 - x$
44. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

Finding the Domain of a Composite Function **In Exercises 45–54, determine the domains of (a) f, (b) g, and (c) $f \circ g$. Use a graphing utility to verify your results.**

45. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$
46. $f(x) = \sqrt{x + 3}, \quad g(x) = \dfrac{x}{2}$
47. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$
48. $f(x) = x^{1/4}, \quad g(x) = x^4$
49. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$
50. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{2x}$
51. $f(x) = |x - 4|, \quad g(x) = 3 - x$
52. $f(x) = \dfrac{2}{|x|}, \quad g(x) = x - 1$
53. $f(x) = x + 2, \quad g(x) = \dfrac{1}{x^2 - 4}$
54. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

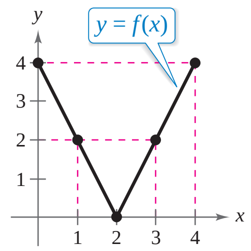
Determining Whether $f \circ g = g \circ f$ **In Exercises 55–60, (a) find $f \circ g$, $g \circ f$, and the domain of $f \circ g$. (b) Use a graphing utility to graph $f \circ g$ and $g \circ f$. Determine whether $f \circ g = g \circ f$.**

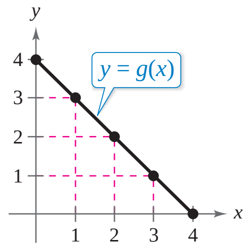
55. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$
56. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3 - 1$
57. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 9$
58. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$
59. $f(x) = x^{2/3}, \quad g(x) = x^6$
60. $f(x) = |x|, \quad g(x) = -x^2 + 1$

Determining Whether $f \circ g = g \circ f$ **In Exercises 61–66, (a) find $(f \circ g)(x)$ and $(g \circ f)(x)$, (b) determine algebraically whether $(f \circ g)(x) = (g \circ f)(x)$, and (c) use a graphing utility to complete a table of values for the two compositions to confirm your answer to part (b).**

61. $f(x) = 5x + 4, \quad g(x) = 4 - x$
62. $f(x) = \frac{1}{4}(x - 1), \quad g(x) = 4x + 1$
63. $f(x) = \sqrt{x + 6}, \quad g(x) = x^2 - 5$
64. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 10}$
65. $f(x) = |x|, \quad g(x) = 2x^3$
66. $f(x) = \dfrac{6}{3x - 5}, \quad g(x) = -x$

Evaluating Combinations of Functions **In Exercises 67–70, use the graphs of f and g to evaluate the functions.**





67. (a) $(f + g)(3)$ (b) $(f/g)(2)$
68. (a) $(f - g)(1)$ (b) $(fg)(4)$
69. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$
70. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

Identifying a Composite Function **In Exercises 71–78, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)**

71. $h(x) = (2x + 1)^2$
72. $h(x) = (1 - x)^3$
73. $h(x) = \sqrt[3]{x^2 - 4}$
74. $h(x) = \sqrt{9 - x}$
75. $h(x) = \dfrac{1}{x + 2}$
76. $h(x) = \dfrac{4}{(5x + 2)^2}$
77. $h(x) = (x + 4)^2 + 2(x + 4)$
78. $h(x) = (x + 3)^{3/2} + 4(x + 3)^{1/2}$

79. ***Why you should learn it*** *(p. 50)* The research and development department of an automobile manufacturer has determined that when required to stop quickly to avoid an accident, the distance (in feet) a car travels during the driver's reaction time is given by

$R(x) = \frac{3}{4}x$

where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is given by

$B(x) = \frac{1}{15}x^2.$

(a) Find the function that represents the total stopping distance T.

(b) Use a graphing utility to graph the functions R, B, and T in the same viewing window for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

80. MODELING DATA

The table shows the total amounts (in billions of dollars) of private expenditures on health services and supplies in the United States (including Puerto Rico) for the years 1997 through 2007. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively. (Source: U.S. Centers for Medicare and Medicaid Services)

Year	y_1	y_2	y_3
1997	162	359	52
1998	175	385	56
1999	184	417	59
2000	193	455	58
2001	200	498	58
2002	211	551	59
2003	225	604	65
2004	235	646	66
2005	247	690	70
2006	255	731	75
2007	269	775	80

The data are approximated by the following models, where t represents the year, with $t = 7$ corresponding to 1997.

$$y_1 = 10.5t + 88$$

$$y_2 = 0.66t^2 + 27.6t + 123$$

$$y_3 = 0.23t^2 - 3.0t + 64$$

(a) Use the models and the *table* feature of a graphing utility to create a table showing the values of y_1, y_2, and y_3 for each year from 1997 through 2007. Compare these models with the original data. Are the models a good fit? Explain.

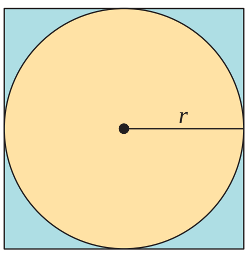
(b) Use the graphing utility to graph y_1, y_2, y_3, and $y_T = y_1 + y_2 + y_3$ in the same viewing window. What does the function y_T represent?

81. Geometry A square concrete foundation was prepared as a base for a large cylindrical gasoline tank (see figure).

(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.



82. Geometry A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outermost ripple is given by $r(t) = 0.6t$, where t is the time (in seconds) after the pebble strikes the water. The area of the circle is given by $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

83. Business A company owns two retail stores. The annual sales (in thousands of dollars) of the stores each year from 2004 through 2010 can be approximated by the models

$$S_1 = 830 + 1.2t^2 \quad \text{and} \quad S_2 = 390 + 75.4t$$

where t is the year, with $t = 4$ corresponding to 2004.

(a) Write a function T that represents the total annual sales of the two stores.

(b) Use a graphing utility to graph S_1, S_2, and T in the same viewing window.

84. Business The annual cost C (in thousands of dollars) and revenue R (in thousands of dollars) for a company each year from 2004 through 2010 can be approximated by the models

$$C = 260 - 8t + 1.6t^2 \quad \text{and} \quad R = 320 + 2.8t$$

where t is the year, with $t = 4$ corresponding to 2004.

(a) Write a function P that represents the annual profits of the company.

(b) Use a graphing utility to graph C, R, and P in the same viewing window.

85. Biology The number of bacteria in a refrigerated food product is given by

$$N(T) = 10T^2 - 20T + 600, \quad 1 \le T \le 20$$

where T is the temperature of the food in degrees Celsius. When the food is removed from the refrigerator, the temperature of the food is given by

$$T(t) = 2t + 1$$

where t is the time in hours.

(a) Find the composite function $N(T(t))$ or $(N \circ T)(t)$ and interpret its meaning in the context of the situation.

(b) Find $(N \circ T)(6)$ and interpret its meaning.

(c) Find the time when the bacteria count reaches 800.

86. Environmental Science The spread of a contaminant is increasing in a circular pattern on the surface of a lake. The radius of the contaminant can be modeled by $r(t) = 5.25\sqrt{t}$, where r is the radius in meters and t is time in hours since contamination.

(a) Find a function that gives the area A of the circular leak in terms of the time t since the spread began.

(b) Find the size of the contaminated area after 36 hours.

(c) Find when the size of the contaminated area is 6250 square meters.

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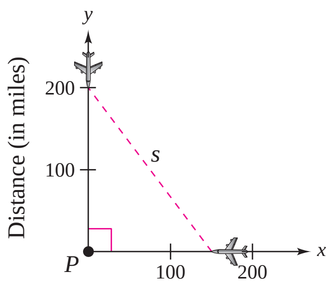
87. **Air Traffic Control** An air traffic controller spots two planes flying at the same altitude. Their flight paths form a right angle at point P. One plane is 150 miles from point P and is moving at 450 miles per hour. The other plane is 200 miles from point P and is moving at 450 miles per hour. Write the distance s between the planes as a function of time t.



88. **Marketing** The suggested retail price of a new car is p dollars. The dealership advertised a factory rebate of \$1200 and an 8% discount.

(a) Write a function R in terms of p giving the cost of the car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(18{,}400)$ and $(S \circ R)(18{,}400)$. Which yields the lower cost for the car? Explain.

Conclusions

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. A function that represents the graph of $f(x) = x^2$ shifted three units to the right is $f(g(x))$, where $g(x) = x + 3$.

90. Given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

Exploration **In Exercises 91 and 92, three siblings are of three different ages. The oldest is twice the age of the middle sibling, and the middle sibling is six years older than one-half the age of the youngest.**

91. (a) Write a composite function that gives the oldest sibling's age in terms of the youngest. Explain how you arrived at your answer.

(b) The oldest sibling is 16 years old. Find the ages of the other two siblings.

92. (a) Write a composite function that gives the youngest sibling's age in terms of the oldest. Explain how you arrived at your answer.

(b) The youngest sibling is two years old. Find the ages of the other two siblings.

93. **Proof** Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

94. **Proof** Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

95. **Proof** Given a function f, prove that $g(x)$ is even and $h(x)$ is odd, where $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and $h(x) = \frac{1}{2}[f(x) - f(-x)]$.

96. (a) Use the result of Exercise 95 to prove that any function can be written as a sum of even and odd functions. (*Hint:* Add the two equations in Exercise 95.)

(b) Use the result of part (a) to write each function as a sum of even and odd functions.

$$f(x) = x^2 - 2x + 1, \quad g(x) = \frac{1}{x + 1}$$

97. **Exploration** The function in Example 9 can be decomposed in other ways. For which of the following pairs of functions is $h(x) = \dfrac{1}{(x - 2)^2}$ equal to $f(g(x))$?

(a) $g(x) = \dfrac{1}{x - 2}$ and $f(x) = x^2$

(b) $g(x) = x^2$ and $f(x) = \dfrac{1}{x - 2}$

(c) $g(x) = (x - 2)^2$ and $f(x) = \dfrac{1}{x}$

98. **CAPSTONE** Consider the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$. Describe the restrictions that need to be made on the domains of f and g so that $f(g(x)) = g(f(x))$.

Cumulative Mixed Review

Evaluating an Equation **In Exercises 99–102, find three points that lie on the graph of the equation. (There are many correct answers.)**

99. $y = -x^2 + x - 5$

100. $y = \frac{1}{5}x^3 - 4x^2 + 1$

101. $x^2 + y^2 = 24$

102. $y = \dfrac{x}{x^2 - 5}$

Finding the Slope-Intercept Form **In Exercises 103–106, find the slope-intercept form of the equation of the line that passes through the two points.**

103. $(-4, -2), (-3, 8)$

104. $(1, 5), (-8, 2)$

105. $\left(\frac{3}{2}, -1\right), \left(-\frac{1}{3}, 4\right)$

106. $(0, 1.1), (-4, 3.1)$

1.6 Inverse Functions

Inverse Functions

Recall from Section 1.2 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4\colon \ \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4\colon \ \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.56. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

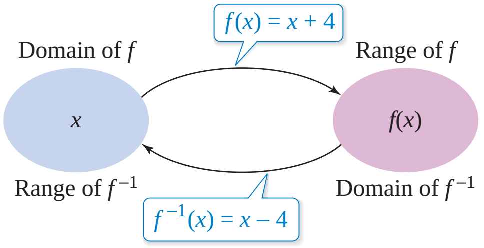


Figure 1.56

Example 1 Finding Inverse Functions Informally

Find the inverse function of

$$f(x) = 4x.$$

Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is given by

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x$$

$$f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

CHECKPOINT Now try Exercise 7.

What you should learn

- Find inverse functions informally and verify that two functions are inverse functions of each other.
- Use graphs of functions to decide whether functions have inverse functions.
- Determine whether functions are one-to-one.
- Find inverse functions algebraically.

Why you should learn it

Inverse functions can be helpful in further exploring how two variables relate to each other. For example, in Exercise 115 on page 69, you will use inverse functions to find the European shoe sizes from the corresponding U.S. shoe sizes.

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Don't be confused by the use of the exponent -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it always refers to the inverse function of the function f and not to the reciprocal of $f(x)$, which is given by

$$\frac{1}{f(x)}.$$

Example 2 Finding Inverse Functions Informally

Find the inverse function of $f(x) = x - 6$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *subtracts* 6 from each input. To "undo" this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is given by

$$f^{-1}(x) = x + 6.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f(x + 6) = (x + 6) - 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(x - 6) = (x - 6) + 6 = x$$

CHECKPOINT Now try Exercise 9.

A table of values can help you understand inverse functions. For instance, the first table below shows several values of the function in Example 2. Interchange the rows of this table to obtain values of the inverse function.

x	-2	-1	0	1	2
$f(x)$	-8	-7	-6	-5	-4

x	-8	-7	-6	-5	-4
$f^{-1}(x)$	-2	-1	0	1	2

In the table at the left, each output is 6 less than the input, and in the table at the right, each output is 6 more than the input.

The formal definition of an inverse function is as follows.

Definition of Inverse Function

Let f and g be two functions such that

$f(g(x)) = x$ for every x in the domain of g

and

$g(f(x)) = x$ for every x in the domain of f.

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

If the function g is the inverse function of the function f, then it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other*.

Example 3 Verifying Inverse Functions Algebraically

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\sqrt[3]{\frac{x+1}{2}}\right) \\ &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(2x^3 - 1) \\ &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

CHECKPOINT Now try Exercise 19.

Technology Tip

Most graphing utilities can graph $y = x^{1/3}$ in two ways:

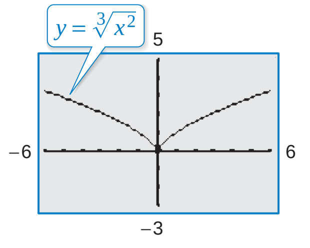
$y_1 = x \wedge (1/3)$ or

$y_1 = \sqrt[3]{x}$.

On some graphing utilities, you may not be able to obtain the complete graph of $y = x^{2/3}$ by entering $y_1 = x \wedge (2/3)$. If not, you should use

$y_1 = (x \wedge (1/3))^2$ or

$y_1 = \sqrt[3]{x^2}$.



Example 4 Verifying Inverse Functions Algebraically

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \qquad \text{or} \qquad h(x) = \frac{5}{x} + 2$$

Solution

By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\left(\dfrac{x-2}{5}\right) - 2} = \frac{25}{x - 12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\dfrac{5}{x} + 2\right) - 2} = \frac{5}{\left(\dfrac{5}{x}\right)} = x.$$

So, it appears that h is the inverse function of f. You can confirm this by showing that the composition of h with f is also equal to the identity function.

CHECKPOINT Now try Exercise 23.

The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point

(a, b)

lies on the graph of f, then the point

(b, a)

must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$, as shown in Figure 1.57.

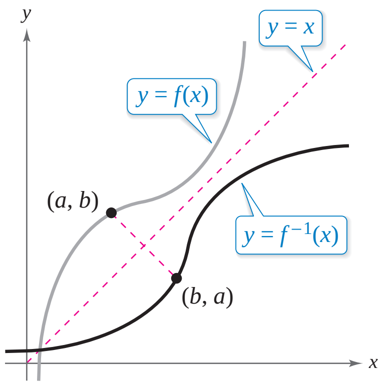


Figure 1.57

Technology Tip

Many graphing utilities have a built-in feature for drawing an inverse function. For instructions on how to use the *draw inverse* feature, see Appendix A; for specific keystrokes, go to this texbook's *Companion Website*.

Example 5 Verifying Inverse Functions Graphically

Verify that the functions f and g from Example 3 are inverse functions of each other graphically.

Solution

From Figure 1.58, you can conclude that f and g are inverse functions of each other.

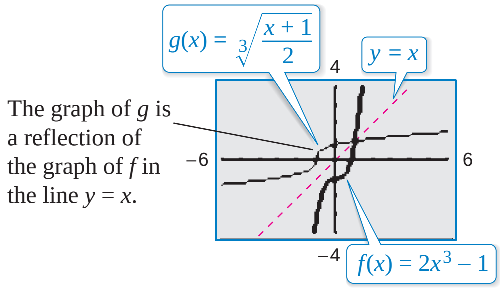


Figure 1.58

CHECKPOINT Now try Exercise 33(b).

Example 6 Verifying Inverse Functions Numerically

Verify that the functions $f(x) = \dfrac{x - 5}{2}$ and $g(x) = 2x + 5$ are inverse functions of each other numerically.

Solution

You can verify that f and g are inverse functions of each other *numerically* by using a graphing utility. Enter $y_1 = f(x)$, $y_2 = g(x)$, $y_3 = f(g(x))$, and $y_4 = g(f(x))$, as shown in Figure 1.59. Then use the *table* feature to create a table (see Figure 1.60).

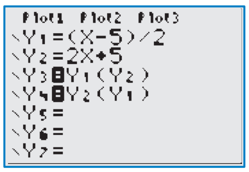


Figure 1.59

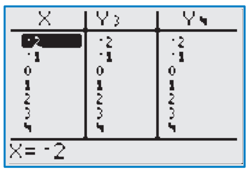


Figure 1.60

Note that the entries for x, y_3, and y_4 are the same. So, $f(g(x)) = x$ and $g(f(x)) = x$. You can conclude that f and g are inverse functions of each other.

CHECKPOINT Now try Exercise 33(c).

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The Existence of an Inverse Function

To have an inverse function, a function must be **one-to-one,** which means that no two elements in the domain of f correspond to the same element in the range of f.

Definition of a One-to-One Function

A function f is **one-to-one** when, for a and b in its domain, $f(a) = f(b)$ implies that $a = b$.

Existence of an Inverse Function

A function f has an inverse function f^{-1} if and only if f is one-to-one.

From its graph, it is easy to tell whether a function of x is one-to-one. Simply check to see that every horizontal line intersects the graph of the function at most once. This is called the **Horizontal Line Test.** For instance, Figure 1.61 shows the graph of $y = x^2$. On the graph, you can find a horizontal line that intersects the graph twice.

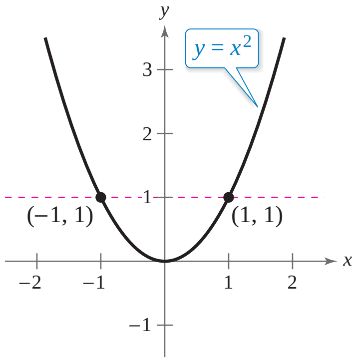


Figure 1.61 *$f(x) = x^2$ is not one-to-one.*

Two special types of functions that pass the Horizontal Line Test are those that are increasing or decreasing on their entire domains.

1. If f is *increasing* on its entire domain, then f is one-to-one.

2. If f is *decreasing* on its entire domain, then f is one-to-one.

Example 7 Testing for One-to-One Functions

Is the function $f(x) = \sqrt{x} + 1$ one-to-one?

Algebraic Solution

Let a and b be nonnegative real numbers with $f(a) = f(b)$.

$$\sqrt{a} + 1 = \sqrt{b} + 1 \qquad \text{Set } f(a) = f(b).$$

$$\sqrt{a} = \sqrt{b}$$

$$a = b$$

So, $f(a) = f(b)$ implies that $a = b$. You can conclude that f *is* one-to-one and *does* have an inverse function.

CHECKPOINT Now try Exercise 67.

Graphical Solution

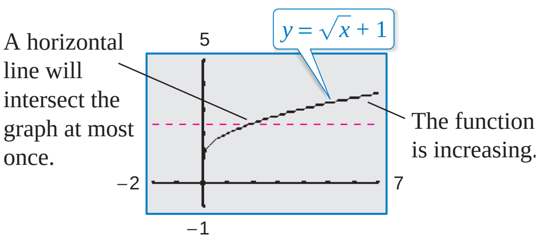


Figure 1.62

From Figure 1.62, you can conclude that f is one-to-one and *does* have an inverse function.

Finding Inverse Functions Algebraically

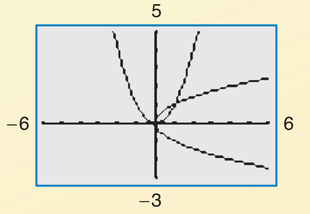
For simple functions, you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

What's Wrong?

You use a graphing utility to graph $y_1 = x^2$ and then use the *draw inverse* feature to conclude that $f(x) = x^2$ has an inverse function (see figure). What's wrong?



Example 8 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \frac{5 - 3x}{2}.$$

Solution

The graph of f in Figure 1.63 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

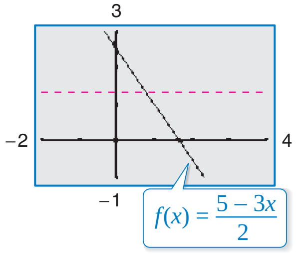


Figure 1.63

$$f(x) = \frac{5 - 3x}{2} \qquad \text{Write original function.}$$

$$y = \frac{5 - 3x}{2} \qquad \text{Replace } f(x) \text{ by } y.$$

$$x = \frac{5 - 3y}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$2x = 5 - 3y \qquad \text{Multiply each side by 2.}$$

$$3y = 5 - 2x \qquad \text{Isolate the } y\text{-term.}$$

$$y = \frac{5 - 2x}{3} \qquad \text{Solve for } y.$$

$$f^{-1}(x) = \frac{5 - 2x}{3} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domains and ranges of f and f^{-1} consist of all real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

CHECKPOINT Now try Exercise 71.

A function f with an implied domain of all real numbers may not pass the Horizontal Line Test. In this case, the domain of f may be restricted so that f does have an inverse function. For instance, when the domain of $f(x) = x^2$ is restricted to the nonnegative real numbers, then f does have an inverse function.

Example 9 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = x^3 - 4.$$

Solution

The graph of f in Figure 1.64 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

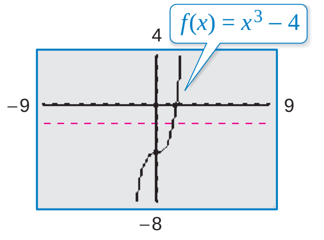


Figure 1.64

$$\begin{aligned} f(x) &= x^3 - 4 && \text{Write original function.} \\ y &= x^3 - 4 && \text{Replace } f(x) \text{ by } y. \\ x &= y^3 - 4 && \text{Interchange } x \text{ and } y. \\ y^3 &= x + 4 && \text{Isolate } y. \\ y &= \sqrt[3]{x + 4} && \text{Solve for } y. \\ f^{-1}(x) &= \sqrt[3]{x + 4} && \text{Replace } y \text{ by } f^{-1}(x). \end{aligned}$$

The domains and ranges of f and f^{-1} consist of all real numbers. You can verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ as follows.

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\sqrt[3]{x + 4}\right) \\ &= \left(\sqrt[3]{x + 4}\right)^3 - 4 \\ &= x + 4 - 4 \\ &= x \end{aligned} \qquad \begin{aligned} f^{-1}(f(x)) &= f^{-1}(x^3 - 4) \\ &= \sqrt[3]{(x^3 - 4) + 4} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

CHECKPOINT Now try Exercise 73.

Example 10 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$

Solution

The graph of f in Figure 1.65 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

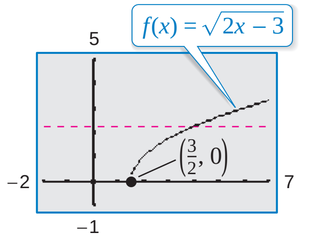


Figure 1.65

$$\begin{aligned} f(x) &= \sqrt{2x - 3} && \text{Write original function.} \\ y &= \sqrt{2x - 3} && \text{Replace } f(x) \text{ by } y. \\ x &= \sqrt{2y - 3} && \text{Interchange } x \text{ and } y. \\ x^2 &= 2y - 3 && \text{Square each side.} \\ 2y &= x^2 + 3 && \text{Isolate } y. \\ y &= \frac{x^2 + 3}{2} && \text{Solve for } y. \\ f^{-1}(x) &= \frac{x^2 + 3}{2}, \quad x \ge 0 && \text{Replace } y \text{ by } f^{-1}(x). \end{aligned}$$

Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

CHECKPOINT Now try Exercise 77.

1.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. If f and g are functions such that $f(g(x)) = x$ and $g(f(x)) = x$, then the function g is the _______ function of f, and is denoted by _______ .

2. The domain of f is the _______ of f^{-1}, and the _______ of f^{-1} is the range of f.

3. The graphs of f and f^{-1} are reflections of each other in the line _______ .

4. To have an inverse function, a function f must be _______ ; that is, $f(a) = f(b)$ implies $a = b$.

5. How many times can a horizontal line intersect the graph of a function that is one-to-one?

6. Can $(1, 4)$ and $(2, 4)$ be two ordered pairs of a one-to-one function?

Procedures and Problem Solving

Finding Inverse Functions Informally **In Exercises 7–14, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

7. $f(x) = 6x$ **8.** $f(x) = \frac{1}{3}x$

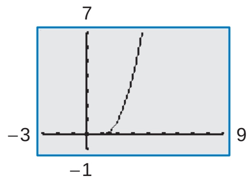
9. $f(x) = x + 7$ **10.** $f(x) = x - 3$

11. $f(x) = 2x + 1$ **12.** $f(x) = (x - 1)/4$

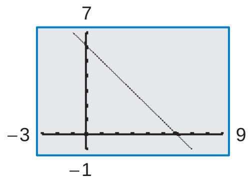
13. $f(x) = \sqrt[3]{x}$ **14.** $f(x) = x^5$

Identifying Graphs of Inverse Functions **In Exercises 15–18, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]**

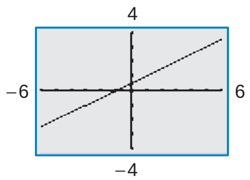
(a)



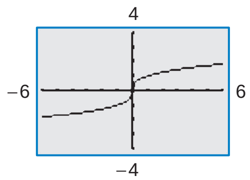
(b)



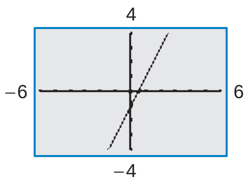
(c)



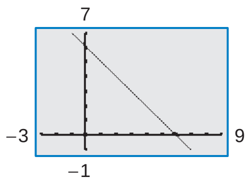
(d)



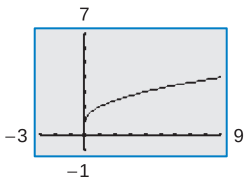
15.



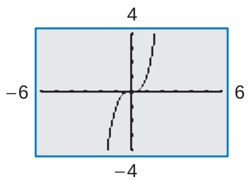
16.



17.



18.



Verifying Inverse Functions Algebraically **In Exercises 19–24, show that f and g are inverse functions algebraically. Use a graphing utility to graph f and g in the same viewing window. Describe the relationship between the graphs.**

19. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$ **20.** $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

21. $f(x) = \sqrt{x - 4}; \quad g(x) = x^2 + 4, \ x \geq 0$

22. $f(x) = 9 - x^2, \ x \geq 0; \quad g(x) = \sqrt{9 - x}$

23. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1 - x}$

24. $f(x) = \dfrac{1}{1 + x}, \ x \geq 0; \quad g(x) = \dfrac{1 - x}{x}, \ 0 < x \leq 1$

Algebraic-Graphical-Numerical **In Exercises 25–34, show that f and g are inverse functions (a) algebraically, (b) graphically, and (c) numerically.**

25. $f(x) = -\dfrac{7}{2}x - 3, \quad g(x) = -\dfrac{2x + 6}{7}$

26. $f(x) = \dfrac{x - 9}{4}, \quad g(x) = 4x + 9$

27. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$

28. $f(x) = \dfrac{x^3}{2}, \quad g(x) = \sqrt[3]{2x}$

29. $f(x) = -\sqrt{x - 8}, \quad g(x) = 8 + x^2, \quad x \leq 0$

30. $f(x) = \sqrt[3]{3x - 10}, \quad g(x) = \dfrac{x^3 + 10}{3}$

31. $f(x) = 2x, \quad g(x) = \dfrac{x}{2}$

32. $f(x) = x - 5, \quad g(x) = x + 5$

33. $f(x) = \dfrac{x - 1}{x + 5}, \quad g(x) = -\dfrac{5x + 1}{x - 1}$

34. $f(x) = \dfrac{x + 3}{x - 2}, \quad g(x) = \dfrac{2x + 3}{x - 1}$

Identifying Whether Functions Have Inverses **In Exercises 35–38, does the function have an inverse? Explain.**

35. *Domain* → *Range*

1 can → \$1
6 cans → \$5
12 cans → \$9
24 cans → \$16

36. *Domain* → *Range*

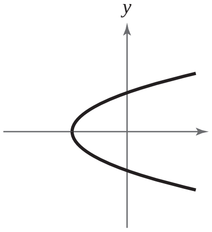
1/2 hour → \$40
1 hour → \$40
2 hours → \$70
4 hours → \$120

37. $\{(-3, 6), (-1, 5), (0, 6)\}$

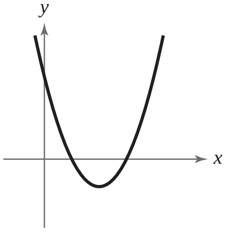
38. $\{(2, 4), (3, 7), (7, 2)\}$

Recognizing One-to-One Functions **In Exercises 39–44, determine whether the graph is that of a function. If so, determine whether the function is one-to-one.**

39.



40.



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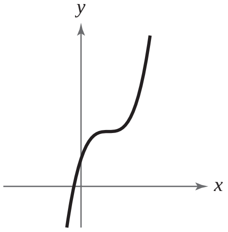
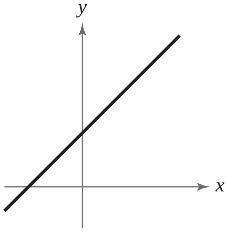


42.



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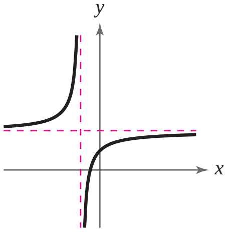

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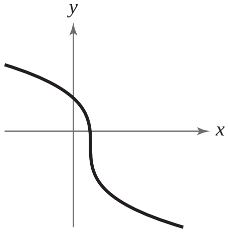


Using the Horizontal Line Test **In Exercises 45–56, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and thus has an inverse function.**

45. $f(x) = 3 - \frac{1}{2}x$

46. $f(x) = \frac{1}{4}(x + 2)^2 - 1$

47. $h(x) = \dfrac{x^2}{x^2 + 1}$

48. $g(x) = \dfrac{4 - x}{6x^2}$

49. $h(x) = \sqrt{16 - x^2}$

50. $f(x) = -2x\sqrt{16 - x^2}$

51. $f(x) = 10$

52. $f(x) = -0.65$

53. $g(x) = (x + 5)^3$

54. $f(x) = x^5 - 7$

55. $h(x) = |x + 4| - |x - 4|$

56. $f(x) = -\dfrac{|x - 6|}{|x + 6|}$

Analyzing a Piecewise-Defined Function **In Exercises 57 and 58, sketch the graph of the piecewise-defined function by hand and use the graph to determine whether an inverse function exists.**

57. $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x, & x > 1 \end{cases}$

58. $f(x) = \begin{cases} (x - 2)^3, & x < 3 \\ (x - 4)^2, & x \ge 3 \end{cases}$

Testing for One-to-One Functions **In Exercises 59–70, determine algebraically whether the function is one-to-one. Verify your answer graphically. If the function is one-to-one, find its inverse.**

59. $f(x) = x^4$

60. $g(x) = x^2 - x^4$

61. $f(x) = \dfrac{3x + 4}{5}$

62. $f(x) = 3x + 5$

63. $f(x) = \dfrac{1}{x^2}$

64. $h(x) = \dfrac{4}{x^2}$

65. $f(x) = (x + 3)^2, \quad x \ge -3$

66. $q(x) = (x - 5)^2, \quad x \le 5$

67. $f(x) = \sqrt{2x + 3}$

68. $f(x) = \sqrt{x - 2}$

69. $f(x) = |x - 2|, \quad x \le 2$

70. $f(x) = \dfrac{x^2}{x^2 + 1}$

Finding an Inverse Function Algebraically **In Exercises 71–80, find the inverse function of *f* algebraically. Use a graphing utility to graph both *f* and f^{-1} in the same viewing window. Describe the relationship between the graphs.**

71. $f(x) = 2x - 3$

72. $f(x) = 3x$

73. $f(x) = x^5$

74. $f(x) = x^3 + 1$

75. $f(x) = x^{3/5}$

76. $f(x) = x^2, \quad x \ge 0$

77. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

78. $f(x) = \sqrt{16 - x^2}, \quad -4 \le x \le 0$

79. $f(x) = \dfrac{4}{x}$

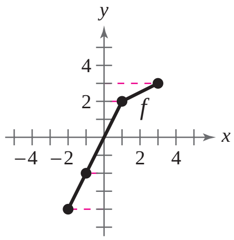
80. $f(x) = \dfrac{6}{\sqrt{x}}$

Think About It **In Exercises 81–90, restrict the domain of the function f so that the function is one-to-one and has an inverse function. Then find the inverse function f^{-1}. State the domains and ranges of f and f^{-1}. Explain your results. (There are many correct answers.)**

81. $f(x) = (x - 2)^2$
82. $f(x) = 1 - x^4$
83. $f(x) = |x + 2|$
84. $f(x) = |x - 2|$
85. $f(x) = (x + 3)^2$
86. $f(x) = (x - 4)^2$
87. $f(x) = -2x^2 + 5$
88. $f(x) = \frac{1}{2}x^2 - 1$
89. $f(x) = |x - 4| + 1$
90. $f(x) = -|x - 1| - 2$

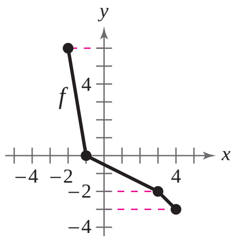
Using the Properties of Inverse Functions **In Exercises 91 and 92, use the graph of the function f to complete the table and sketch the graph of f^{-1}.**

91.



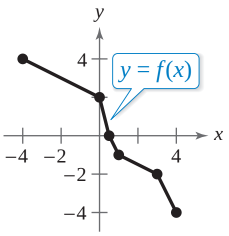
x	$f^{-1}(x)$
−4	
−2	
2	
3	

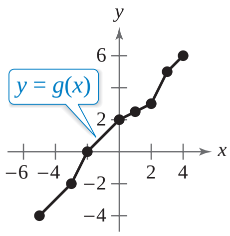
92.



x	$f^{-1}(x)$
−3	
−2	
0	
6	

Using Graphs to Evaluate a Function **In Exercises 93–100, use the graphs of $y = f(x)$ and $y = g(x)$ to evaluate the function.**





93. $f^{-1}(0)$
94. $g^{-1}(0)$
95. $(f \circ g)(2)$
96. $g(f(-4))$
97. $f^{-1}(g(0))$
98. $(g^{-1} \circ f)(3)$
99. $(g \circ f^{-1})(2)$
100. $(f^{-1} \circ g^{-1})(6)$

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Using the *Draw Inverse* Feature **In Exercises 101–104, (a) use a graphing utility to graph the function f, (b) use the *draw inverse* feature of the graphing utility to draw the inverse relation of the function, and (c) determine whether the inverse relation is an inverse function. Explain your reasoning.**

101. $f(x) = x^3 + x + 1$
102. $f(x) = x\sqrt{4 - x^2}$
103. $f(x) = \dfrac{3x^2}{x^2 + 1}$
104. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

Evaluating a Composition of Functions **In Exercises 105–110, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.**

105. $(f^{-1} \circ g^{-1})(1)$
106. $(g^{-1} \circ f^{-1})(-3)$
107. $(f^{-1} \circ f^{-1})(6)$
108. $(g^{-1} \circ g^{-1})(-4)$
109. $(f \circ g)^{-1}$
110. $g^{-1} \circ f^{-1}$

Finding a Composition of Functions **In Exercises 111–114, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified function.**

111. $g^{-1} \circ f^{-1}$
112. $f^{-1} \circ g^{-1}$
113. $(f \circ g)^{-1}$
114. $(g \circ f)^{-1}$

115. *Why you should learn it* *(p. 60)* The table shows men's shoe sizes in the United States and the corresponding European shoe sizes. Let $y = f(x)$ represent the function that gives the men's European shoe size in terms of x, the men's U.S. size.

Men's U.S. shoe size	Men's European shoe size
8	41
9	42
10	43
11	45
12	46
13	47

(a) Is f one-to-one? Explain.
(b) Find $f(11)$.
(c) Find $f^{-1}(43)$, if possible.
(d) Find $f(f^{-1}(41))$.
(e) Find $f^{-1}(f(13))$.

116. **Fashion Design** Let $y = g(x)$ represent the function that gives the women's European shoe size in terms of x, the women's U.S. size. A women's U.S. size 6 shoe corresponds to a European size 38. Find $g^{-1}(g(6))$.

117. **Military Science** You can encode and decode messages using functions and their inverses. To code a message, first translate the letters to numbers using 1 for "A," 2 for "B," and so on. Use 0 for a space. So, "A ball" becomes

1 0 2 1 12 12.

Then, use a one-to-one function to convert to coded numbers. Using $f(x) = 2x - 1$, "A ball" becomes

1 -1 3 1 23 23.

(a) Encode "Call me later" using the function $f(x) = 5x + 4$.

(b) Find the inverse function of $f(x) = 5x + 4$ and use it to decode 119 44 9 104 4 104 49 69 29.

118. **Production Management** Your wage is \$10.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced x is $y = 10 + 0.75x$.

(a) Find the inverse function. What does each variable in the inverse function represent?

(b) Use a graphing utility to graph the function and its inverse function.

(c) Use the *trace* feature of the graphing utility to find the hourly wage when 10 units are produced per hour.

(d) Use the *trace* feature of the graphing utility to find the number of units produced per hour when your hourly wage is \$21.25.

Conclusions

True or False? **In Exercises 119 and 120, determine whether the statement is true or false. Justify your answer.**

119. If f is an even function, then f^{-1} exists.

120. If the inverse function of f exists, and the graph of f has a y-intercept, then the y-intercept of f is an x-intercept of f^{-1}.

Think About It **In Exercises 121–124, determine whether the situation could be represented by a one-to-one function. If so, write a statement that describes the inverse function.**

121. The number of miles n a marathon runner has completed in terms of the time t in hours

122. The population p of a town in terms of the year t from 1990 through 2010 given that the population was greatest in 2000

123. The depth of the tide d at a beach in terms of the time t over a 24-hour period

124. The height h in inches of a human child from age 2 to age 14 in terms of his or her age n in years

125. **Writing** Describe the relationship between the graph of a function f and the graph of its inverse function f^{-1}.

126. **Think About It** The domain of a one-to-one function f is $[0, 9]$ and the range is $[-3, 3]$. Find the domain and range of f^{-1}.

127. **Think About It** The function $f(x) = \frac{9}{5}x + 32$ can be used to convert a temperature of x degrees Celsius to its corresponding temperature in degrees Fahrenheit.

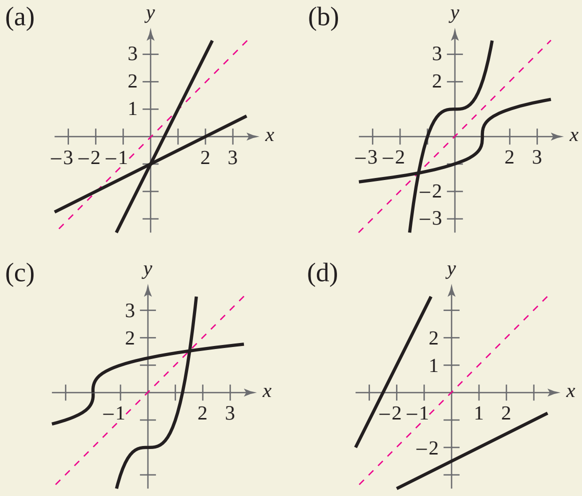
(a) Using the expression for f, make a conceptual argument to show that f has an inverse function.

(b) What does $f^{-1}(50)$ represent?

128. **Think About It** A function f is increasing over its entire domain. Does f have an inverse function? Explain.

129. **Think About It** Describe a type of function that is *not* one-to-one on any interval of its domain.

130. **CAPSTONE** Decide whether the two functions shown in each graph appear to be inverse functions of each other. Explain your reasoning.



131. **Proof** Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

132. **Proof** Prove that if f is a one-to-one odd function, then f^{-1} is an odd function.

Cumulative Mixed Review

Simplifying a Rational Expression **In Exercises 133–136, write the rational expression in simplest form.**

133. $\dfrac{27x^3}{3x^2}$

134. $\dfrac{5x^2y}{xy + 5x}$

135. $\dfrac{x^2 - 36}{6 - x}$

136. $\dfrac{x^2 + 3x - 40}{x^2 - 3x - 10}$

Testing for Functions **In Exercises 137–140, determine whether the equation represents y as a function of x.**

137. $x = 5$

138. $y = \sqrt{x + 2}$

139. $x^2 + y^2 = 9$

140. $x - y^2 = 0$

1.7 Linear Models and Scatter Plots

Scatter Plots and Correlation

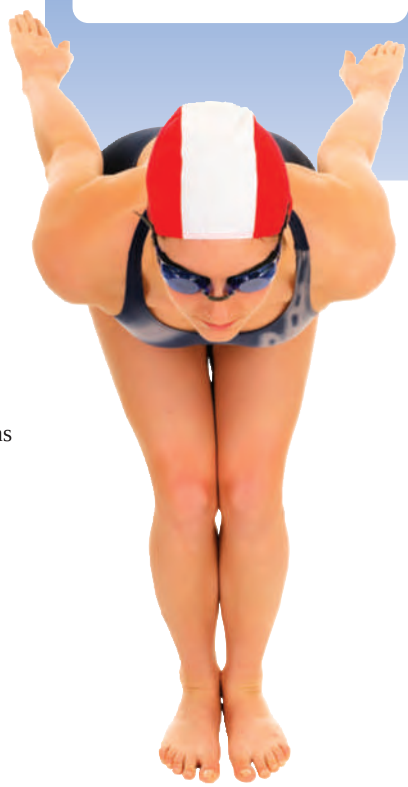
Many real-life situations involve finding relationships between two variables, such as the year and the number of employees in the cellular telecommunications industry. In a typical situation, data are collected and written as a set of ordered pairs. The graph of such a set is called a *scatter plot*. (For a brief discussion of scatter plots, see Appendix B.1.)

What you should learn

- Construct scatter plots and interpret correlation.
- Use scatter plots and a graphing utility to find linear models for data.

Why you should learn it

Real-life data often follow a linear pattern. For instance, in Exercise 25 on page 79, you will find a linear model for the winning times in the women's 400-meter freestyle Olympic swimming event.



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Example 1 Constructing a Scatter Plot

The data in the table show the numbers E (in thousands) of employees in the cellular telecommunications industry in the United States from 2002 through 2007. Construct a scatter plot of the data. (Source: CTIA–The Wireless Association)

Year	Employees in the cellular telecommunications industry, E (in thousands)
2002	192
2003	206
2004	226
2005	233
2006	254
2007	267

Solution

Begin by representing the data with a set of ordered pairs. Let t represent the year, with $t = 2$ corresponding to 2002.

$(2, 192), (3, 206), (4, 226),$

$(5, 233), (6, 254), (7, 267)$

Then plot each point in a coordinate plane, as shown in Figure 1.66.

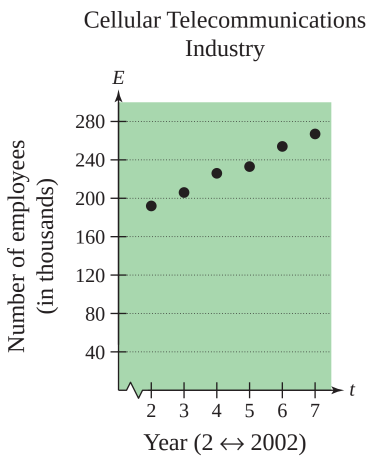


Figure 1.66

CHECKPOINT Now try Exercise 5.

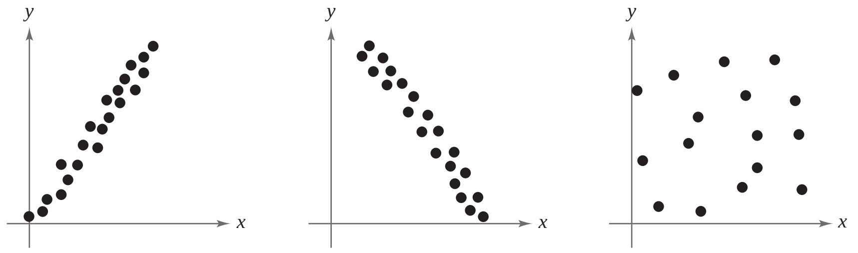
From the scatter plot in Figure 1.66, it appears that the points describe a relationship that is nearly linear. The relationship is not *exactly* linear because the number of employees did not increase by precisely the same amount each year.

A mathematical equation that approximates the relationship between t and E is a *mathematical model*. When developing a mathematical model to describe a set of data, you strive for two (often conflicting) goals—accuracy and simplicity. For the data above, a linear model of the form

$$E = at + b$$

(where a and b are constants) appears to be best. It is simple and relatively accurate.

Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a **positive correlation.** If y tends to decrease as x increases, then the collection is said to have a **negative correlation.** Figure 1.67 shows three examples: one with a positive correlation, one with a negative correlation, and one with no (discernible) correlation.



Positive Correlation ***Negative Correlation*** ***No Correlation***

Figure 1.67

Example 2 Interpreting Correlation

On a Friday, 22 students in a class were asked to record the numbers of hours they spent studying for a test on Monday and the numbers of hours they spent watching television. The results are shown below. (The first coordinate is the number of hours and the second coordinate is the score obtained on the test.)

Study Hours: (0, 40), (1, 41), (2, 51), (3, 58), (3, 49), (4, 48), (4, 64), (5, 55), (5, 69), (5, 58), (5, 75), (6, 68), (6, 63), (6, 93), (7, 84), (7, 67), (8, 90), (8, 76), (9, 95), (9, 72), (9, 85), (10, 98)

TV Hours: (0, 98), (1, 85), (2, 72), (2, 90), (3, 67), (3, 93), (3, 95), (4, 68), (4, 84), (5, 76), (7, 75), (7, 58), (9, 63), (9, 69), (11, 55), (12, 58), (14, 64), (16, 48), (17, 51), (18, 41), (19, 49), (20, 40)

a. Construct a scatter plot for each set of data.

b. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation. What can you conclude?

Solution

a. Scatter plots for the two sets of data are shown in Figure 1.68.

b. The scatter plot relating study hours and test scores has a positive correlation. This means that the more a student studied, the higher his or her score tended to be. The scatter plot relating television hours and test scores has a negative correlation. This means that the more time a student spent watching television, the lower his or her score tended to be.

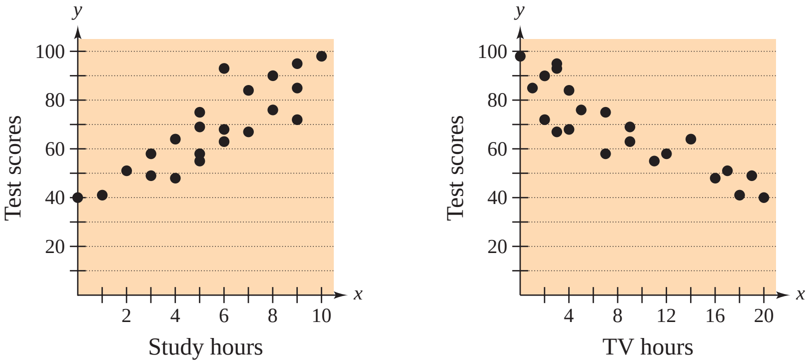


Figure 1.68

CHECKPOINT Now try Exercise 7.

Fitting a Line to Data

Finding a linear model to represent the relationship described by a scatter plot is called **fitting a line to data.** You can do this graphically by simply sketching the line that appears to fit the points, finding two points on the line, and then finding the equation of the line that passes through the two points.

Example 3 Fitting a Line to Data

Find a linear model that relates the year to the number of employees in the cellular telecommunications industry in the United States. (See Example 1.)

Year	Employees in the cellular telecommunications industry, E (in thousands)
2002	192
2003	206
2004	226
2005	233
2006	254
2007	267

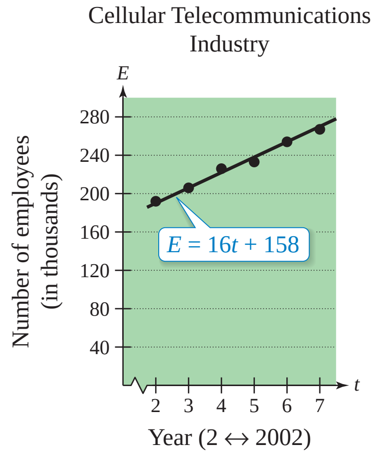


Figure 1.69

Solution

Let t represent the year, with $t = 2$ corresponding to 2002. After plotting the data in the table, draw the line that you think best represents the data, as shown in Figure 1.69. Two points that lie on this line are

$(3, 206)$ and $(6, 254)$.

Using the point-slope form, you can find the equation of the line to be

$$E = 16(t - 3) + 206$$

$$= 16t + 158. \qquad \text{Linear model}$$

CHECKPOINT Now try Exercise 11.

Once you have found a model, you can measure how well the model fits the data by comparing the actual values with the values given by the model, as shown in the following table.

	t	2	3	4	5	6	7
Actual	E	192	206	226	233	254	267
Model	E	190	206	222	238	254	270

The sum of the squares of the differences between the actual values and the model values is called the **sum of the squared differences.** The model that has the least sum is called the **least squares regression line** for the data. For the model in Example 3, the sum of the squared differences is 54. The least squares regression line for the data is

$$E = 15.0t + 162. \qquad \text{Best-fitting linear model}$$

Its sum of squared differences is 37. For more on the least squares regression line, see Appendix C.2 at this textbook's *Companion Website*.

Study Tip

The model in Example 3 is based on the two data points chosen. When different points are chosen, the model may change somewhat. For instance, when you choose $(5, 233)$ and $(7, 267)$, the new model is

$$E = 17(t - 5) + 233$$

$$= 17t + 148.$$

Another way to find a linear model to represent the relationship described by a scatter plot is to enter the data points into a graphing utility and use the *linear regression* feature. This method is demonstrated in Example 4.

Technology Tip

For instructions on how to use the *linear regression* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

Example 4 A Mathematical Model

The data in the table show the estimated numbers V (in thousands) of alternative-fueled vehicles in use in the United States from 2001 through 2007. (Source: Energy Information Administration)

Year	Alternative-fueled vehicles in use, V (in thousands)
2001	425
2002	471
2003	534
2004	565
2005	592
2006	635
2007	696

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 1$ corresponding to 2001.

b. How closely does the model represent the data?

Graphical Solution

a. Use the *linear regression* feature of a graphing utility to obtain the model shown in Figure 1.70.

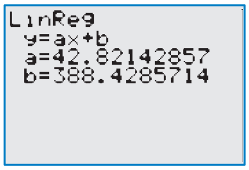


Figure 1.70

You can approximate the model to be $V = 42.8t + 388$.

b. Graph the actual data and the model. From Figure 1.71, it appears that the model is a good fit for the actual data.

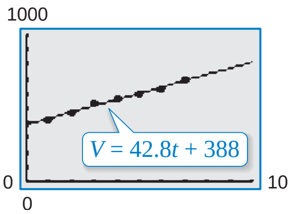


Figure 1.71

CHECKPOINT Now try Exercise 15.

Numerical Solution

a. Using the *linear regression* feature of a graphing utility, you can find that a linear model for the data is $V = 42.8t + 388$.

b. You can see how well the model fits the data by comparing the actual values of V with the values of V given by the model, which are labeled V^* in the table below. From the table, you can see that the model appears to be a good fit for the actual data.

Year	V	V^*
2001	425	431
2002	471	474
2003	534	516
2004	565	559
2005	592	602
2006	635	645
2007	696	688

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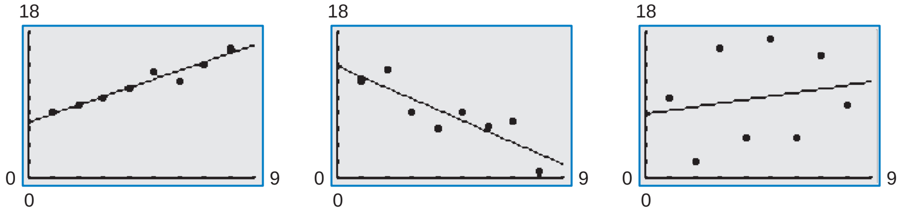
When you use the *regression* feature of a graphing calculator or computer program to find a linear model for data, you will notice that the program may also output an "r-value." For instance, the r-value from Example 4 was $r \approx 0.994$. This r-value is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. The correlation coefficient r varies between -1 and 1. Basically, the closer $|r|$ is to 1, the better the points can be described by a line. Three examples are shown in Figure 1.72.



$r = 0.972$ $r = -0.856$ $r = 0.190$

Figure 1.72

Technology Tip

For some calculators, the *diagnostics on* feature must be selected before the *regression* feature is used in order to see the value of the correlation coefficient r. To learn how to use this feature, consult your user's manual.

Example 5 Finding a Least Squares Regression Line

The following ordered pairs (w, h) represent the shoe sizes w and the heights h (in inches) of 25 men. Use the *regression* feature of a graphing utility to find the least squares regression line for the data.

(10.0, 70.5)	(10.5, 71.0)	(9.5, 69.0)	(11.0, 72.0)	(12.0, 74.0)
(8.5, 67.0)	(9.0, 68.5)	(13.0, 76.0)	(10.5, 71.5)	(10.5, 70.5)
(10.0, 71.0)	(9.5, 70.0)	(10.0, 71.0)	(10.5, 71.0)	(11.0, 71.5)
(12.0, 73.5)	(12.5, 75.0)	(11.0, 72.0)	(9.0, 68.0)	(10.0, 70.0)
(13.0, 75.5)	(10.5, 72.0)	(10.5, 71.0)	(11.0, 73.0)	(8.5, 67.5)

Solution

After entering the data into a graphing utility (see Figure 1.73), you obtain the model shown in Figure 1.74. So, the least squares regression line for the data is

$$h = 1.84w + 51.9.$$

In Figure 1.75, this line is plotted with the data. Note that the plot does not have 25 points because some of the ordered pairs graph as the same point. The correlation coefficient for this model is $r \approx 0.981$, which implies that the model is a good fit for the data.

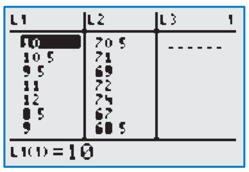


Figure 1.73

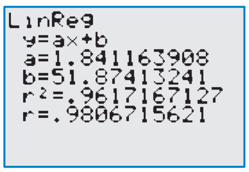


Figure 1.74

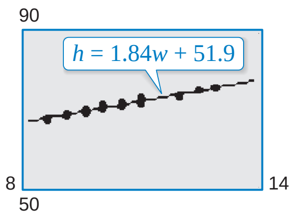


Figure 1.75

CHECKPOINT Now try Exercise 25.

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1.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a _______ correlation.

2. To find the least squares regression line for data, you can use the _______ feature of a graphing utility.

3. In a collection of ordered pairs (x, y), y tends to decrease as x increases. Does the collection have a positive correlation or a negative correlation?

4. You find the least squares regression line for a set of data. The correlation coefficient is 0.114. Is the model a good fit?

Procedures and Problem Solving

5. Constructing a Scatter Plot The following ordered pairs give the years of experience x for 15 sales representatives and the monthly sales y (in thousands of dollars).

(1.5, 41.7), (1.0, 32.4), (0.3, 19.2), (3.0, 48.4), (4.0, 51.2), (0.5, 28.5), (2.5, 50.4), (1.8, 35.5), (2.0, 36.0), (1.5, 40.0), (3.5, 50.3), (4.0, 55.2), (0.5, 29.1), (2.2, 43.2), (2.0, 41.6)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

6. Constructing a Scatter Plot The following ordered pairs give the scores on two consecutive 15-point quizzes for a class of 18 students.

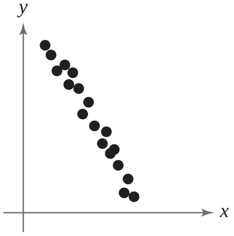
(7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7), (14, 11), (14, 15), (8, 10), (9, 10), (15, 9), (10, 11), (11, 14), (7, 14), (11, 10), (14, 11), (10, 15), (9, 6)

(a) Create a scatter plot of the data.

(b) Does the relationship between consecutive quiz scores appear to be approximately linear? If not, give some possible explanations.

Interpreting Correlation **In Exercises 7–10, the scatter plot of a set of data is shown. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation.**

7.



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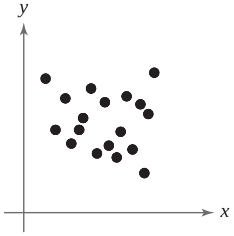

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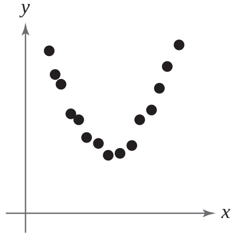

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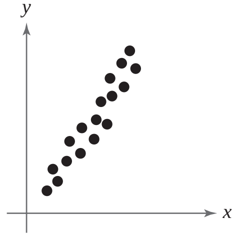


Fitting a Line to Data **In Exercises 11–14, (a) create a scatter plot of the data, (b) draw a line of best fit that passes through two of the points, and (c) use the two points to find an equation of the line.**

11. $(-3, -3)$, $(3, 4)$, $(1, 1)$, $(3, 2)$, $(4, 4)$, $(-1, -1)$

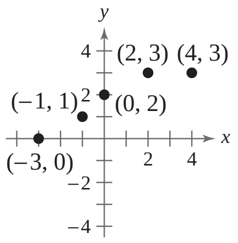
12. $(-2, 3)$, $(-2, 4)$, $(-1, 2)$, $(1, -2)$, $(0, 0)$, $(0, 1)$

13. $(0, 2)$, $(-2, 1)$, $(3, 3)$, $(1, 3)$, $(4, 4)$

14. $(3, 2)$, $(2, 3)$, $(1, 5)$, $(4, 0)$, $(5, 0)$

A Mathematical Model **In Exercises 15 and 16, use the *regression* feature of a graphing utility to find a linear model for the data. Then use the graphing utility to decide how closely the model fits the data (a) graphically and (b) numerically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

15.



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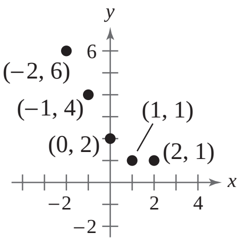
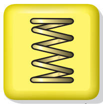


17. MODELING DATA

Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is the measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F kilograms is applied.



Force, F	Elongation, d
20	1.4
40	2.5
60	4.0
80	5.3
100	6.6

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to best fit the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data.

(d) Use the model from part (c) to estimate the elongation of the spring when a force of 55 kilograms is applied.

18. MODELING DATA

The numbers of subscribers S (in millions) to wireless networks from 2002 through 2008 are shown in the table. (Source: CTIA–The Wireless Association)

Year	Subscribers, S (in millions)
2002	140.8
2003	158.7
2004	182.1
2005	207.9
2006	233.0
2007	255.4
2008	270.3

(a) Use a graphing utility to create a scatter plot of the data, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the number of subscribers in 2015. Is your answer reasonable? Explain.

19. MODELING DATA

The total player payrolls T (in millions of dollars) for the Pittsburgh Steelers from 2004 through 2008 are shown in the table. (Source: USA Today)

Year	Total player payroll, T (in millions of dollars)
2004	78.0
2005	84.2
2006	94.0
2007	106.3
2008	128.8

(a) Use a graphing utility to create a scatter plot of the data, with $t = 4$ corresponding to 2004.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the payrolls in 2010 and 2015. Do the results seem reasonable? Explain.

(e) What is the slope of your model? What does it tell you about the player payroll?

20. MODELING DATA

The mean salaries S (in thousands of dollars) of public school teachers in the United States from 2002 through 2008 are shown in the table. (Source: National Education Association)

Year	Mean salary, S (in thousands of dollars)
2002	44.7
2003	45.7
2004	46.6
2005	47.7
2006	49.0
2007	50.8
2008	52.3

(a) Use a graphing utility to create a scatter plot of the data, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the mean salaries in 2016 and 2018. Do the results seem reasonable? Explain.

21. MODELING DATA

The projected populations P (in thousands) of New Jersey for selected years, based on the 2000 census, are shown in the table. (Source: U.S. Census Bureau)

Year	Population, P (in thousands)
2010	9018
2015	9256
2020	9462
2025	9637
2030	9802

(a) Use a graphing utility to create a scatter plot of the data, with $t = 10$ corresponding to 2010.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

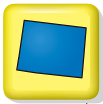
(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of P and the values of P given by the model. How closely does the model fit the data?

(e) Use the model to predict the population of New Jersey in 2050. Does the result seem reasonable? Explain.

22. MODELING DATA

The projected populations P (in thousands) of Wyoming for selected years, based on the 2000 census, are shown in the table. (Source: U.S. Census Bureau)



Year	Population, P (in thousands)
2010	520
2015	528
2020	531
2025	529
2030	523

(a) Use a graphing utility to create a scatter plot of the data, with $t = 10$ corresponding to 2010.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of P and the values of P given by the model. How closely does the model fit the data?

(e) Use the model to predict the population of Wyoming in 2050. Does the result seem reasonable? Explain.

23. MODELING DATA

The table shows the advertising expenditures x and sales volumes y for a company for seven randomly selected months. Both are measured in thousands of dollars.

Month	Advertising expenditures, x	Sales volume, y
1	2.4	202
2	1.6	184
3	2.0	220
4	2.6	240
5	1.4	180
6	1.6	164
7	2.0	186

(a) Use the *regression* feature of a graphing utility to find a linear model for the data.

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to estimate sales for advertising expenditures of \$1500.

24. MODELING DATA

The table shows the numbers T of stores owned by the Target Corporation from 2000 through 2008. (Source: Target Corp.)

Year	Number of stores, T
2000	1307
2001	1381
2002	1475
2003	1553
2004	1308
2005	1397
2006	1488
2007	1591
2008	1682

(a) Use a graphing utility to make a scatter plot of the data, with $t = 0$ corresponding to 2000. Identify two sets of points in the scatter plot that are approximately linear.

(b) Use the *regression* feature of the graphing utility to find a linear model for each set of points.

(c) Write a piecewise-defined model for the data. Use the graphing utility to graph the piecewise-defined model.

(d) Describe a scenario that could be the cause of the break in the data.

25. *Why you should learn it* *(p. 71)* The following ordered pairs (t, T) represent the Olympic year t and the winning time T (in minutes) in the women's 400-meter freestyle swimming event. (Source: International Olympic Committee)

(1952, 5.20)	(1972, 4.32)	(1992, 4.12)
(1956, 4.91)	(1976, 4.16)	(1996, 4.12)
(1960, 4.84)	(1980, 4.15)	(2000, 4.10)
(1964, 4.72)	(1984, 4.12)	(2004, 4.09)
(1968, 4.53)	(1988, 4.06)	(2008, 4.05)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient. Let t represent the year, with $t = 0$ corresponding to 1950.

(b) What information is given by the sign of the slope of the model?

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of y and the values of y given by the model. How closely does the model fit the data?

(e) How can you use the value of the correlation coefficient to help answer the question in part (d)?

(f) Would you use the model to predict the winning times in the future? Explain.

26. MODELING DATA

In a study, 60 colts were measured every 14 days from birth. The ordered pairs (d, l) represent the average length l (in centimeters) of the 60 colts d days after birth: (14, 81.2), (28, 87.1), (42, 93.7), (56, 98.3), (70, 102.4), (84, 106.2), and (98, 110.0). (Source: American Society of Animal Science)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient.

(b) According to the correlation coefficient, does the model represent the data well? Explain.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. How closely does the model fit the data?

(d) Use the model to predict the average length of a colt 112 days after birth.

Conclusions

True or False? **In Exercises 27 and 28, determine whether the statement is true or false. Justify your answer.**

27. A linear regression model with a positive correlation will have a slope that is greater than 0.

28. When the correlation coefficient for a linear regression model is close to -1, the regression line is a poor fit for the data.

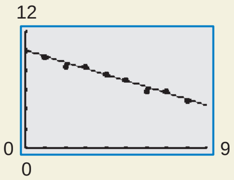
29. Writing Use your school's library, the Internet, or some other reference source to locate data that you think describes a linear relationship. Create a scatter plot of the data and find the least squares regression line that represents the points. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

30. CAPSTONE Each graphing utility screen below shows the least squares regression line for a set of data. The equations and r-values for the models are given.

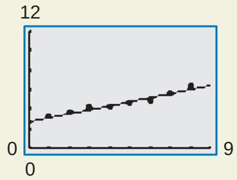
$y = 0.68x + 2.7$
$y = 0.41x + 2.7$
$y = -0.62x + 10.0$

$r = 0.973$
$r = -0.986$
$r = 0.624$

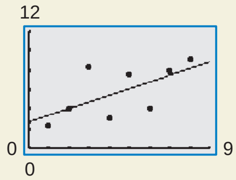
(i)



(ii)



(iii)



(a) Determine the equation and correlation coefficient (r-value) that represents each graph. Explain how you found your answers.

(b) According to the correlation coefficients, which model is the best fit for its data? Explain.

Cumulative Mixed Review

Evaluating a Function **In Exercises 31 and 32, evaluate the function at each value of the independent variable and simplify.**

31. $f(x) = 2x^2 - 3x + 5$

(a) $f(-1)$

(b) $f(w + 2)$

32. $g(x) = 5x^2 - 6x + 1$

(a) $g(-2)$

(b) $g(z - 2)$

Solving Equations **In Exercises 33–36, solve the equation algebraically. Check your solution graphically.**

33. $6x + 1 = -9x - 8$

34. $3(x - 3) = 7x + 2$

35. $8x^2 - 10x - 3 = 0$

36. $10x^2 - 23x - 5 = 0$

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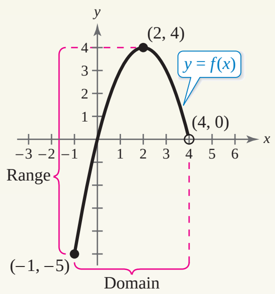
1 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
	Find the slopes of lines *(p. 3)*.	The slope m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$.	1–8
1.1	**Write linear equations given points on lines and their slopes *(p. 5)*.**	The point-slope form of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is $y - y_1 = m(x - x_1)$.	9–16
	Use slope-intercept forms of linear equations to sketch lines *(p. 7)*.	The graph of the equation $y = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$.	17–30
	Use slope to identify parallel and perpendicular lines *(p. 9)*.	**Parallel lines:** Slopes are equal. **Perpendicular lines:** Slopes are negative reciprocals of each other.	31, 32
	Decide whether a relation between two variables represents a function *(p. 16)*.	A function f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain (or set of inputs) of the function f, and the set B contains the range (or set of outputs).	33–42
1.2	**Use function notation and evaluate functions *(p. 18)*, and find the domains of functions *(p. 20)*.**	**Equation:** $f(x) = 5 - x^2$ $\boldsymbol{f(2)}$**:** $f(2) = 5 - 2^2 = 1$ **Domain of $\boldsymbol{f(x) = 5 - x^2}$:** All real numbers x	43–50
	Use functions to model and solve real-life problems *(p. 22)*.	A function can be used to model the number of construction employees in the United States. (See Example 8.)	51, 52
	Evaluate difference quotients *(p. 23)*.	**Difference quotient:** $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$	53, 54
	Find the domains and ranges of functions *(p. 29)*.		55–62
1.3	**Use the Vertical Line Test for functions *(p. 30)*.**	A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.	63–66
	Determine intervals on which functions are increasing, decreasing, or constant *(p. 31)*.	A function f is increasing on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. A function f is decreasing on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$. A function f is constant on an interval when, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.	67–70

	What did you learn?	Explanation and Examples	Review Exercises
1.3	**Determine relative maximum and relative minimum values of functions *(p. 32)*.**	A function value $f(a)$ is called a relative minimum of f when there exists an interval (x_1, x_2) that contains a such that $x_1 < x < x_2$ implies $f(a) \leq f(x)$. A function value $f(a)$ is called a relative maximum of f when there exists an interval (x_1, x_2) that contains a such that $x_1 < x < x_2$ implies $f(a) \geq f(x)$.	71–74
	Identify and graph step functions and other piecewise-defined functions *(p. 34)*.	**Greatest integer:** $f(x) = [\![x]\!]$	75–78
	Identify even and odd functions *(p. 35)*.	**Even:** For each x in the domain of f, $f(-x) = f(x)$. **Odd:** For each x in the domain of f, $f(-x) = -f(x)$.	79–86
1.4	**Recognize graphs of parent functions *(p. 41)*.**	**Linear:** $f(x) = x$; **Quadratic:** $f(x) = x^2$; **Cubic:** $f(x) = x^3$; **Absolute value:** $f(x) = \lvert x \rvert$; **Square root:** $f(x) = \sqrt{x}$; **Rational:** $f(x) = 1/x$ (See Figure 1.34, page 41.)	87–92
	Use vertical and horizontal shifts *(p. 42)*, reflections *(p. 44)*, and nonrigid transformations *(p. 46)* to graph functions.	**Vertical shifts:** $h(x) = f(x) + c$ or $h(x) = f(x) - c$ **Horizontal shifts:** $h(x) = f(x - c)$ or $h(x) = f(x + c)$ **Reflection in the *x*-axis:** $h(x) = -f(x)$ **Reflection in the *y*-axis:** $h(x) = f(-x)$ **Nonrigid transformations:** $h(x) = cf(x)$ or $h(x) = f(cx)$	93–106
1.5	**Add, subtract, multiply, and divide functions *(p. 50)*, find the compositions of functions *(p. 52)*, and write a function as a composition of two functions *(p. 54)*.**	$(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$ $(fg)(x) = f(x) \cdot g(x)$ $(f/g)(x) = f(x)/g(x),\ g(x) \neq 0$ **Composition of functions:** $(f \circ g)(x) = f(g(x))$	107–122
	Use combinations of functions to model and solve real-life problems *(p. 55)*.	A composite function can be used to represent the number of bacteria in a petri dish as a function of the amount of time the petri dish has been out of refrigeration. (See Example 10.)	123, 124
1.6	**Find inverse functions informally and verify that two functions are inverse functions of each other *(p. 60)*.**	Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f. Under these conditions, the function g is the inverse function of the function f.	125–128
	Use graphs of functions to decide whether functions have inverse functions *(p. 63)*.	If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. In short, f^{-1} is a reflection of f in the line $y = x$.	129, 130
	Determine whether functions are one-to-one *(p. 64)*.	A function f is one-to-one when, for a and b in its domain, $f(a) = f(b)$ implies $a = b$.	131–134
	Find inverse functions algebraically *(p. 65)*.	To find inverse functions, replace $f(x)$ by y, interchange the roles of x and y, and solve for y. Replace y by $f^{-1}(x)$.	135–142
1.7	**Construct scatter plots *(p. 71)* and interpret correlation *(p. 72)*.**	A scatter plot is a graphical representation of data written as a set of ordered pairs.	143–146
	Use scatter plots *(p. 73)* and a graphing utility *(p. 74)* to find linear models for data.	The best-fitting linear model can be found using the *linear regression* feature of a graphing utility or a computer program.	147, 148

1 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

1.1

Finding the Slope of a Line **In Exercises 1–8, plot the two points and find the slope of the line passing through the points.**

1. $(-3, 2), (8, 2)$ **2.** $(3, -1), (-3, -1)$

3. $(7, -1), (7, 12)$ **4.** $(8, -1), (8, 2)$

5. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$ **6.** $\left(-\frac{3}{4}, \frac{5}{6}\right), \left(\frac{1}{2}, -\frac{5}{2}\right)$

7. $(-4.5, 6), (2.1, 3)$ **8.** $(-2.7, -6.3), (0, 1.8)$

The Point-Slope Form of the Equation of a Line **In Exercises 9–16, (a) use the point on the line and the slope of the line to find an equation of the line, and (b) find three additional points through which the line passes. (There are many correct answers.)**

	Point	*Slope*
9.	$(2, -1)$	$m = \frac{1}{4}$
10.	$(-3, 5)$	$m = -\frac{3}{2}$
11.	$(0, -5)$	$m = \frac{3}{2}$
12.	$\left(0, \frac{7}{8}\right)$	$m = -\frac{4}{5}$
13.	$(-2, 6)$	$m = 0$
14.	$(-8, 8)$	$m = 0$
15.	$(10, -6)$	m is undefined.
16.	$(5, 4)$	m is undefined.

Finding the Slope-Intercept Form **In Exercises 17–24, write an equation of the line that passes through the points. Use the slope-intercept form, if possible. If not possible, explain why. Use a graphing utility to graph the line (if possible).**

17. $(2, -1), (4, -1)$ **18.** $(0, 0), (0, 10)$

19. $\left(7, \frac{11}{3}\right), \left(9, \frac{11}{3}\right)$ **20.** $\left(\frac{5}{8}, 4\right), \left(\frac{5}{8}, -6\right)$

21. $(-1, 0), (6, 2)$ **22.** $(1, 6), (4, 2)$

23. $(3, -1), (-3, 2)$ **24.** $\left(-\frac{5}{2}, 1\right), \left(-4, \frac{2}{9}\right)$

Using a Rate of Change to Write an Equation **In Exercises 25–28, you are given the dollar value of a product in 2010 *and* the rate at which the value of the item is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2010.)**

	2010 Value	*Rate*
25.	\$12,500	\$850 increase per year
26.	\$3795	\$115 decrease per year
27.	\$625.50	\$42.70 increase per year
28.	\$72.95	\$5.15 decrease per year

29. Business During the second and third quarters of the year, an e-commerce business had sales of \$160,000 and \$185,000, respectively. The growth of sales follows a linear pattern. Estimate sales during the fourth quarter.

30. Accounting The dollar value of a DVD player in 2010 is \$134. The product will decrease in value at an expected rate of \$26.80 per year.

(a) Write a linear equation that gives the dollar value V of the DVD player in terms of the year t. (Let $t = 0$ represent 2010.)

(b) Use a graphing utility to graph the equation found in part (a). Be sure to choose an appropriate viewing window. State the dimensions of your viewing window, and explain why you chose the values that you did.

(c) Use the *value* or *trace* feature of the graphing utility to estimate the dollar value of the DVD player in 2014. Confirm your answer algebraically.

(d) According to the model, when will the DVD player have no value?

Equations of Parallel and Perpendicular Lines **In Exercises 31 and 32, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line. Verify your result with a graphing utility (use a *square setting*).**

	Point	*Line*
31.	$(3, -2)$	$5x - 4y = 8$
32.	$(-8, 3)$	$2x + 3y = 5$

1.2

Testing for Functions **In Exercises 33 and 34, which set of ordered pairs represents a function from A to B? Explain.**

33. $A = \{10, 20, 30, 40\}$ and $B = \{0, 2, 4, 6\}$

(a) $\{(20, 4), (40, 0), (20, 6), (30, 2)\}$

(b) $\{(10, 4), (20, 4), (30, 4), (40, 4)\}$

34. $A = \{u, v, w\}$ and $B = \{-2, -1, 0, 1, 2\}$

(a) $\{(u, -2), (v, 2), (w, 1)\}$

(b) $\{(w, -2), (v, 0), (w, 2)\}$

Testing for Functions Represented Algebraically **In Exercises 35–42, determine whether the equation represents y as a function of x.**

35. $16x^2 - y^2 = 0$ **36.** $x^3 + y^2 = 64$

37. $2x - y - 3 = 0$ **38.** $2x + y = 10$

39. $y = \sqrt{1 - x}$ **40.** $y = \sqrt{x^2 + 4}$

41. $|y| = x + 2$ **42.** $16 - |y| - 4x = 0$

Evaluating a Function **In Exercises 43–46, evaluate the function at each specified value of the independent variable, and simplify.**

43. $f(x) = x^2 + 1$

(a) $f(1)$ (b) $f(-3)$

(c) $f(b^3)$ (d) $f(x - 1)$

44. $g(x) = \sqrt{x^2 + 1}$

(a) $g(-1)$ (b) $g(3)$

(c) $g(3x)$ (d) $g(x + 2)$

45. $h(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 + 2, & x > -1 \end{cases}$

(a) $h(-2)$ (b) $h(-1)$

(c) $h(0)$ (d) $h(2)$

46. $f(x) = \dfrac{3}{2x - 5}$

(a) $f(1)$ (b) $f(-2)$

(c) $f(t)$ (d) $f(10)$

Finding the Domain of a Function **In Exercises 47–50, find the domain of the function.**

47. $f(x) = \dfrac{x - 1}{x + 2}$ **48.** $f(x) = \dfrac{x^2}{x^2 + 1}$

49. $f(x) = \sqrt{25 - x^2}$ **50.** $f(x) = \sqrt{x^2 - 16}$

51. Industrial Engineering A hand tool manufacturer produces a product for which the variable cost is \$5.35 per unit and the fixed costs are \$16,000. The company sells the product for \$8.20 and can sell all that it produces.

(a) Write the total cost C as a function of x, the number of units produced.

(b) Write the profit P as a function of x.

52. Education The numbers n (in millions) of students enrolled in public schools in the United States from 2000 through 2008 can be approximated by

$$n(t) = \begin{cases} 0.76t + 61.4, & 0 \le t \le 4 \\ -0.3333t^3 + 6.6t^2 - 42.37t + 152.7, & 4 < t \le 8 \end{cases}$$

where t is the year, with $t = 0$ corresponding to 2000. (Source: U.S. Census Bureau)

(a) Use the *table* feature of a graphing utility to approximate the enrollment from 2000 through 2008.

(b) Use the graphing utility to graph the model and estimate the enrollment for the years 2009 through 2012. Do the values seem reasonable? Explain.

Evaluating a Difference Quotient **In Exercises 53 and 54, find the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for the given function and simplify your answer.**

53. $f(x) = 2x^2 + 3x - 1$ **54.** $f(x) = x^2 - 3x + 5$

1.3

Finding the Domain and Range of a Function **In Exercises 55–62, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.**

55. $f(x) = 3 - 2x^2$ **56.** $f(x) = 2x^2 + 5$

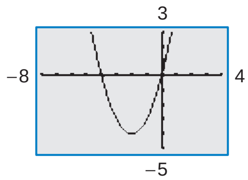
57. $f(x) = \sqrt{x + 3} + 4$ **58.** $f(x) = 2 - \sqrt{x - 5}$

59. $h(x) = \sqrt{36 - x^2}$ **60.** $f(x) = \sqrt{x^2 - 9}$

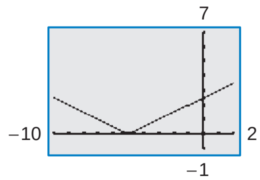
61. $f(x) = |x + 5| + 2$ **62.** $f(x) = |x + 1| - 3$

Vertical Line Test for Functions **In Exercises 63–66, use the Vertical Line Test to determine whether *y* is a function of *x*. Describe how to enter the equation into a graphing utility to produce the given graph.**

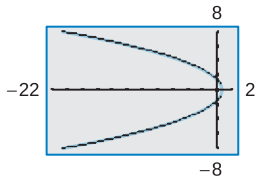
63. $y - 4x = x^2$



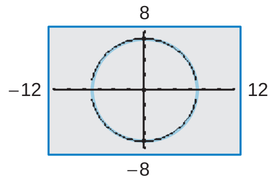
64. $|x + 5| - 2y = 0$



65. $3x + y^2 - 2 = 0$



66. $x^2 + y^2 - 49 = 0$



Increasing and Decreasing Functions **In Exercises 67–70, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.**

67. $f(x) = x^3 - 3x$ **68.** $f(x) = \sqrt{x^2 - 9}$

69. $f(x) = x\sqrt{x - 6}$ **70.** $f(x) = \dfrac{|x + 8|}{2}$

Approximating Relative Minima and Maxima **In Exercises 71–74, use a graphing utility to approximate (to two decimal places) any relative minimum or relative maximum values of the function.**

71. $f(x) = (x^2 - 4)^2$ **72.** $f(x) = x^2 - x - 1$

73. $h(x) = 4x^3 - x^4$ **74.** $f(x) = x^3 - 4x^2 - 1$

Sketching Graphs **In Exercises 75–78, sketch the graph of the function by hand.**

75. $f(x) = \begin{cases} 3x + 5, & x < 0 \\ x - 4, & x \ge 0 \end{cases}$ **76.** $f(x) = \begin{cases} \frac{1}{2}x + 3, & x < 0 \\ 4 - x^2, & x \ge 0 \end{cases}$

77. $f(x) = [\![x]\!] + 3$ **78.** $f(x) = [\![x + 2]\!]$

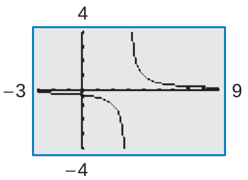
Even and Odd Functions **In Exercises 79–86, determine algebraically whether the function is even, odd, or neither. Verify your answer using a graphing utility.**

79. $f(x) = x^2 + 6$
80. $f(x) = x^2 - x - 1$
81. $f(x) = (x^2 - 8)^2$
82. $f(x) = 2x^3 - x^2$
83. $f(x) = 3x^{5/2}$
84. $f(x) = 3x^{2/5}$
85. $f(x) = \sqrt{4 - x^2}$
86. $f(x) = x\sqrt{x^2 - 1}$

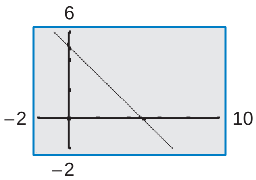
1.4

Library of Parent Functions **In Exercises 87–92, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.**

87.



88.



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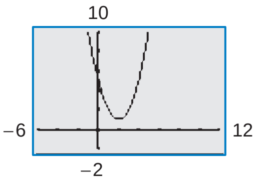

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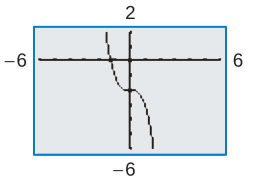

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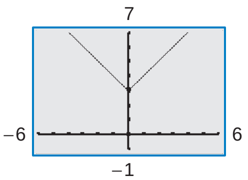
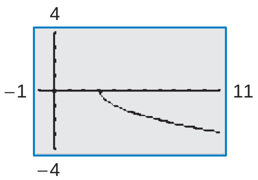
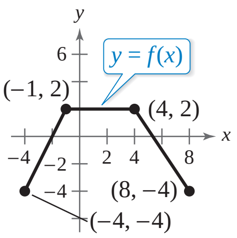


92.



Sketching Transformations **In Exercises 93–96, use the graph of $y = f(x)$ to graph the function.**



93. $y = f(-x)$
94. $y = -f(x)$
95. $y = f(x) - 2$
96. $y = f(x - 1)$

Describing Transformations **In Exercises 97–106, h is related to one of the six parent functions on page 41. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to h. (c) Sketch the graph of h by hand. (d) Use function notation to write h in terms of the parent function f.**

97. $h(x) = \dfrac{1}{x} - 6$
98. $h(x) = -\dfrac{1}{x} - 3$
99. $h(x) = (x - 2)^3 + 5$
100. $h(x) = -(x - 2)^2 - 8$
101. $h(x) = -\sqrt{x} + 6$
102. $h(x) = \sqrt{x - 1} + 4$
103. $h(x) = |x| + 9$
104. $h(x) = |x + 8| - 1$
105. $h(x) = \dfrac{-2}{x + 1} - 3$
106. $h(x) = \dfrac{1}{x + 2} - 4$

1.5

Evaluating a Combination of Functions **In Exercises 107–116, let $f(x) = 3 - 2x$, $g(x) = \sqrt{x}$, and $h(x) = 3x^2 + 2$, and find the indicated values.**

107. $(f - g)(4)$
108. $(f + h)(5)$
109. $(f + g)(25)$
110. $(g - h)(1)$
111. $(fh)(1)$
112. $\left(\dfrac{g}{h}\right)(1)$
113. $(h \circ g)(5)$
114. $(g \circ f)(-3)$
115. $(f \circ h)(-4)$
116. $(g \circ h)(6)$

Identifying a Composite Function **In Exercises 117–122, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)**

117. $h(x) = (x + 3)^2$
118. $h(x) = (1 - 2x)^3$
119. $h(x) = \sqrt{4x + 2}$
120. $h(x) = \sqrt[3]{(x + 2)^2}$
121. $h(x) = \dfrac{4}{x + 2}$
122. $h(x) = \dfrac{6}{(3x + 1)^3}$

Education **In Exercises 123 and 124, the numbers (in thousands) of students taking the SAT (y_1) and the ACT (y_2) for the years 2000 through 2009 can be modeled by**

$$y_1 = -2.61t^2 + 55.0t + 1244 \quad \text{and}$$

$$y_2 = 0.949t^3 - 8.02t^2 + 44.4t + 1056$$

where t represents the year, with $t = 0$ corresponding to 2000. (Source: College Entrance Examination Board and ACT, Inc.)

123. Use a graphing utility to graph y_1, y_2, and $y_1 + y_2$ in the same viewing window.

124. Use the model $y_1 + y_2$ to estimate the total number of students taking the SAT and the ACT in 2010.

1.6

Finding Inverse Functions Informally **In Exercises 125–128, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

125. $f(x) = 6x$
126. $f(x) = x + 5$
127. $f(x) = \dfrac{1}{2}x + 3$
128. $f(x) = \dfrac{x - 4}{5}$

Algebraic-Graphical-Numerical **In Exercises 129 and 130, show that f and g are inverse functions (a) algebraically, (b) graphically, and (c) numerically.**

129. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3 - x}{4}$

130. $f(x) = \sqrt{x + 1}, \quad g(x) = x^2 - 1, x \geq 0$

Using the Horizontal Line Test **In Exercises 131–134, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and an inverse function exists.**

131. $f(x) = \frac{1}{2}x - 3$

132. $f(x) = (x - 1)^2$

133. $h(t) = \dfrac{2}{t - 3}$

134. $g(x) = \sqrt{x + 6}$

Finding an Inverse Function Algebraically **In Exercises 135–142, find the inverse function of *f* algebraically.**

135. $f(x) = \dfrac{1}{2}x - 5$

136. $f(x) = \dfrac{7x + 3}{8}$

137. $f(x) = 4x^3 - 3$

138. $f(x) = 5x^3 + 2$

139. $f(x) = \sqrt{x + 10}$

140. $f(x) = 4\sqrt{6 - x}$

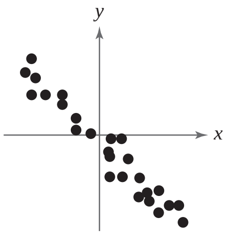
141. $f(x) = \frac{1}{4}x^2 + 1, \quad x \geq 0$

142. $f(x) = 5 - \frac{1}{9}x^2, \quad x \geq 0$

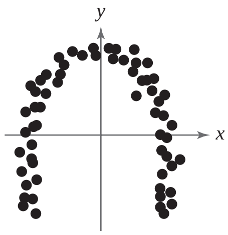
1.7

Interpreting Correlation **In Exercises 143 and 144, the scatter plot of a set of data is shown. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation.**

143.



144.



145. Education The following ordered pairs give the entrance exam scores x and the grade-point averages y after 1 year of college for 10 students.

(75, 2.3), (82, 3.0), (90, 3.6), (65, 2.0), (70, 2.1), (88, 3.5), (93, 3.9), (69, 2.0), (80, 2.8), (85, 3.3)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

146. Industrial Engineering A machine part was tested by bending it x centimeters 10 times per minute until it failed (y equals the time to failure in hours). The results are given as the following ordered pairs.

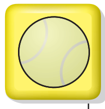
(3, 61), (6, 56), (9, 53), (12, 55), (15, 48), (18, 35), (21, 36), (24, 33), (27, 44), (30, 23)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? If not, give some possible explanations.

147. MODELING DATA

In an experiment, students measured the speed s (in meters per second) of a ball t seconds after it was released. The results are shown in the table.



Time, t	Speed, s
0	0
1	11.0
2	19.4
3	29.2
4	39.4

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to fit the data best.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data and identify the correlation coefficient.

(d) Use the model from part (c) to estimate the speed of the ball after 2.5 seconds.

148. MODELING DATA

The following ordered pairs (x, y) represent the Olympic year x and the winning time y (in minutes) in the men's 400-meter freestyle swimming event. (Source: International Olympic Committee)

(1964, 4.203) (1980, 3.855) (1996, 3.800)
(1968, 4.150) (1984, 3.854) (2000, 3.677)
(1972, 4.005) (1988, 3.783) (2004, 3.718)
(1976, 3.866) (1992, 3.750) (2008, 3.698)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let x represent the year, with $x = 4$ corresponding to 1964.

(b) Use the graphing utility to create a scatter plot of the data. Graph the model in the same viewing window.

(c) Is the model a good fit for the data? Explain.

(d) Is this model appropriate for predicting the winning times in future Olympics? Explain.

Conclusions

True or False? **In Exercises 149–151, determine whether the statement is true or false. Justify your answer.**

149. If the graph of the parent function $f(x) = x^2$ is moved six units to the right, moved three units upward, and reflected in the x-axis, then the point $(-1, 28)$ will lie on the graph of the transformation.

150. If $f(x) = x^n$ where n is odd, then f^{-1} exists.

151. There exists no function f such that $f = f^{-1}$.

1 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

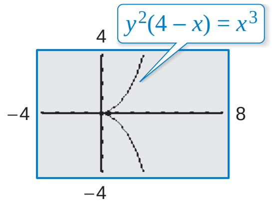


Figure for 3

1. Find the equations of the lines that pass through the point $(0, 4)$ and are (a) parallel to and (b) perpendicular to the line $5x + 2y = 3$.
2. Find the slope-intercept form of the equation of the line that passes through the points $(2, -1)$ and $(-3, 4)$.
3. Does the graph at the right represent y as a function of x? Explain.
4. Evaluate $f(x) = |x + 2| - 15$ at each value of the independent variable and simplify.

   (a) $f(-8)$ (b) $f(14)$ (c) $f(t - 6)$
5. Find the domain of $f(x) = 10 - \sqrt{3 - x}$.
6. An electronics company produces a car stereo for which the variable cost is \$25.60 per unit and the fixed costs are \$24,000. The product sells for \$99.50. Write the total cost C as a function of the number of units produced and sold, x. Write the profit P as a function of the number of units produced and sold, x.

In Exercises 7 and 8, determine algebraically whether the function is even, odd, or neither.

7. $f(x) = 2x^3 - 3x$
8. $f(x) = 3x^4 + 5x^2$

In Exercises 9 and 10, determine the open intervals on which the function is increasing, decreasing, or constant.

9. $h(x) = \frac{1}{4}x^4 - 2x^2$
10. $g(t) = |t + 2| - |t - 2|$

In Exercises 11 and 12, use a graphing utility to graph the functions and to approximate (to two decimal places) any relative minimum or relative maximum values of the function.

11. $f(x) = -x^3 - 5x^2 + 12$
12. $f(x) = x^5 - x^3 + 2$

In Exercises 13–15, (a) identify the parent function *f*, (b) describe the sequence of transformations from *f* to *g*, and (c) sketch the graph of *g*.

13. $g(x) = -2(x - 5)^3 + 3$
14. $g(x) = \sqrt{-x - 7}$
15. $g(x) = 4|-x| - 7$
16. Use the functions $f(x) = x^2$ and $g(x) = \sqrt{2 - x}$ to find the specified function and its domain.

    (a) $(f - g)(x)$ (b) $\left(\dfrac{f}{g}\right)(x)$ (c) $(f \circ g)(x)$ (d) $(g \circ f)(x)$

In Exercises 17–19, determine whether the function has an inverse function, and if so, find the inverse function.

17. $f(x) = x^3 + 8$
18. $f(x) = x^2 + 6$
19. $f(x) = \dfrac{3x\sqrt{x}}{8}$
20. The table shows the average monthly cost C of basic cable television from 2000 through 2008, where t represents the year, with $t = 0$ corresponding to 2000. Use the *regression* feature of a graphing utility to find a linear model for the data. Use the model to estimate the year in which the average monthly cost reached \$50. (Source: SNL Kagan)

Year, t	Average monthly cost, C (in dollars)
0	30.37
1	32.87
2	34.71
3	36.59
4	38.14
5	39.63
6	41.17
7	42.72
8	44.28

Proofs in Mathematics

Conditional Statements

Many theorems are written in the **if-then form** "if p, then q," which is denoted by

$p \rightarrow q$ Conditional statement

where p is the **hypothesis** and q is the **conclusion.** Here are some other ways to express the conditional statement $p \rightarrow q$.

p implies q. p, only if q. p is sufficient for q.

Conditional statements can be either true or false. The conditional statement $p \rightarrow q$ is false only when p is true and q is false. To show that a conditional statement is true, you must prove that the conclusion follows for all cases that fulfill the hypothesis. To show that a conditional statement is false, you need to describe only a single **counterexample** that shows that the statement is not always true.

For instance, $x = -4$ is a counterexample that shows that the following statement is false.

If $x^2 = 16$, then $x = 4$.

The hypothesis "$x^2 = 16$" is true because $(-4)^2 = 16$. However, the conclusion "$x = 4$" is false. This implies that the given conditional statement is false.

For the conditional statement $p \rightarrow q$, there are three important associated conditional statements.

1. The **converse** of $p \rightarrow q$: $q \rightarrow p$
2. The **inverse** of $p \rightarrow q$: $\sim p \rightarrow \sim q$
3. The **contrapositive** of $p \rightarrow q$: $\sim q \rightarrow \sim p$

The symbol ~ means the **negation** of a statement. For instance, the negation of "The engine is running" is "The engine is not running."

Example 1 Writing the Converse, Inverse, and Contrapositive

Write the converse, inverse, and contrapositive of the conditional statement "If I get a B on my test, then I will pass the course."

Solution

a. *Converse:* If I pass the course, then I got a B on my test.

b. *Inverse:* If I do not get a B on my test, then I will not pass the course.

c. *Contrapositive:* If I do not pass the course, then I did not get a B on my test.

In the example above, notice that neither the converse nor the inverse is logically equivalent to the original conditional statement. On the other hand, the contrapositive is logically equivalent to the original conditional statement.

Biconditional Statements

Recall that a conditional statement is a statement of the form "if p, then q." A statement of the form "p if and only if q" is called a **biconditional statement.** A biconditional statement, denoted by

$p \leftrightarrow q$ Biconditional statement

is the conjunction of the conditional statement $p \rightarrow q$ and its converse $q \rightarrow p$.

A biconditional statement can be either true or false. To be true, *both* the conditional statement and its converse must be true.

Example 2 Analyzing a Biconditional Statement

Consider the statement $x = 3$ if and only if $x^2 = 9$.

a. Is the statement a biconditional statement? **b.** Is the statement true?

Solution

a. The statement is a biconditional statement because it is of the form "p if and only if q."

b. The statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If $x = 3$, then $x^2 = 9$.

Converse: If $x^2 = 9$, then $x = 3$.

The first of these statements is true, but the second is false because x could also equal -3. So, the biconditional statement is false.

Knowing how to use biconditional statements is an important tool for reasoning in mathematics.

Example 3 Analyzing a Biconditional Statement

Determine whether the biconditional statement is true or false. If it is false, provide a counterexample.

A number is divisible by 5 if and only if it ends in 0.

Solution

The biconditional statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If a number is divisible by 5, then it ends in 0.

Converse: If a number ends in 0, then it is divisible by 5.

The conditional statement is false. A counterexample is the number 15, which is divisible by 5 but does not end in 0. So, the biconditional statement is false.