UNIT 1 DETERMINANTS

Structure

1.0 INTRODUCTION

In this unit, we shall learn about determinants. Determinant is a square array of numbers symbolizing the sum of certain products of these numbers. Many complicated expressions can be easily handled, if they are expressed as 'determinants'. A determinant of order *n* has *n* rows and *n* columns. In this unit, we shall study determinants of order 2 and 3 only. We shall also study many properties of determinants which help in evaluation of determinants.
Determinants usually arise in connection with linear equations. For example, if the equations $a_1x + b_1 = 0$, and $a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2 b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is a called determinant of second order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

There are many application of determinants. For example, we may use determinants to solve a system of linear equations by a method known as Cramer's rule that we shall discuss in coordinate geometry. For example, in finding are of triangle whose three vertices are given.

1.1 OBJECTIVES

After studying this unit, you should be able to :

- define the term determinant;
- evaluate determinants of order 2 and 3;
- use the properties of determinants for evaluation of determinants;
- use determinants to find area of a triangle;
- use determinants to solve a system of linear equations (Cramer's Rule)

1.2 DETERMINANTS OF ORDER 2 AND 3

We begin by defining the value of determinant of order 2.

Definition : A determinant of order 2 is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where *a*,*b*, *c*, *d* are complex numbers. It denotes the complex number *ad* – *bc*. In other words,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1 : Compute the following determinants :

(a) $\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

(c) $\begin{vmatrix} \alpha + i\beta & \gamma + is \\ -\gamma + is & \alpha - i\beta \end{vmatrix}$ (d) $\begin{vmatrix} \omega & \omega \\ -1 & \omega \end{vmatrix}$

(e) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$ (f) $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix}$

Solutions :

(a) $\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix}$ = 18 – (–10) = 28

(b) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix} = a^2b^2 - (ab)^2 = 0$

(c) $\begin{vmatrix} \alpha + i\beta & \gamma + is \\ -\gamma + is & \alpha - i\beta \end{vmatrix} = \alpha^2 + \beta^2 + \gamma^2 + s^2$

($\because$ $(a + ib)(a - ib) = a^2 + b^2$)

(d) $\begin{vmatrix} \omega & \omega \\ -1 & \omega \end{vmatrix} = \omega^2 + \omega = -1$ because $\omega^2 + \omega + 1 = 0$

(e) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix} = (x-1)(x^2+x+1) - x^3 = x^3 - 1 - x^3 = -1$

(f) $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix} = \left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{4t^2}{(1+t^2)^2}$

$$= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = \frac{(1-t^2)^2}{(1+t^2)^2} = 1 \quad [\because (a-b)^2 + 4ab = (a+b)^2 \ ]$$

1.3 DETERMINANTS OF ORDER 3

Consider the system of Linear Equations :

$a_{11}x + a_{12}y + a_{13}z = b_1$ ………………… (1)

$a_{21}x + a_{22}y + a_{23}z = b_2$ …………………. (2)

$a_{31}x + a_{32}y + a_{33}z = b_3$ ………………… (3)

Where $a_{ij} \in C$ ($1 \le i,\ j \le 3$) and $b_1, b_2, b_3, \in C$ Eliminating x and y from these equation we obtain

$$(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33})z$$
$$= (a_{11}a_{22}b_3 + a_{12}a_{31}b_2 + a_{32}a_{21}b_1 - a_{11}a_{32}b_2 - a_{22}a_{31}b_2 - a_{12}a_{21}b_3).$$

We can get the value of z if the expression $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \neq 0$

The expression on the L.H.S. is denoted by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and is called a determinant of order 3, it has 3 rows, 3 columns and is a complex number.

Definition : A determinant of order 3 is written as $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

where $a_{ij} \in C\ (1 \le i,\ j \le 3)$.

It denotes the complex number

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}$$

Note that we can write

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{23}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Where Δ is written in the last form, we say that it has been expanded along the first row. Similarly, the expansion of Δ along the second row is,

$$\Delta = -a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and the expansion of Δ along the third row is,

$$\Delta = -a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} - a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

We now define a determinant of order 1.

Definition : Let $a \in C$. A determinant of order 1 is denoted by $|a|$ and its value is a.

Example 2 : Evaluate the following determinants by expanding along the first row.

(a) $\begin{vmatrix} 2 & 5 & -3 \\ 4 & -1 & 5 \\ 3 & 6 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & c \end{vmatrix}$

(c) $\begin{vmatrix} x & y & z \\ 1 & 3 & 3 \\ 2 & 4 & 6 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Solutions:

(a) $\begin{vmatrix} 2 & 5 & -3 \\ 4 & -1 & 5 \\ 3 & 6 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & 5 \\ 6 & 2 \end{vmatrix} - 5\begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 4 & -1 \\ 3 & 6 \end{vmatrix}$

$= 2(-2-30) - 5(8-15) - 3(24+3)$

$= 2(-32) - 5(-7) - 3(27)$

$= -64 + 35 - 81 = -110$

(b) $\begin{vmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & c \end{vmatrix} = 2\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 0 & b \\ 1 & 0 \end{vmatrix}$

$= abc - b$

(c) $\begin{vmatrix} x & y & z \\ 1 & 3 & 3 \\ 2 & 4 & 6 \end{vmatrix} = x\begin{vmatrix} 3 & 3 \\ 4 & 6 \end{vmatrix} - y\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} + z\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$

$= x(8-12) - y(6-6) + z(4-6)$

$= 6x-2z$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = 1\begin{vmatrix} b & ca \\ c & ab \end{vmatrix} - a\begin{vmatrix} 1 & ca \\ 1 & ab \end{vmatrix} + bc\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix}$

$= ab^2 - ac^2 - a^2b + a^2c + bc^2 - b^2c$

$= ab^2 - a^2b + bc^2 - b^2c + a^2c + a^2c - ac^2$

$= ab(b-a) + bc(c-b) + ca(a-c)$

Check Your Progress – 1

1. Compute the following determinants :

(a) $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} a & c+id \\ c-id & b \end{vmatrix}$

(c) $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix}$ (d) $\begin{vmatrix} \dfrac{1+t^2}{1-t^2} & \dfrac{2t}{1-t^2} \\ \dfrac{2t}{1-t^2} & \dfrac{1+t^2}{1-t^2} \end{vmatrix}$

2. Show that $\begin{vmatrix} a\alpha + b\gamma & c\alpha + d\gamma \\ a\beta + b\delta & c\beta + d\delta \end{vmatrix} = (ad - bc)(\alpha\delta - \beta\gamma)$

3. Show that $\begin{vmatrix} \dfrac{(1-t)^2}{1+t^2} & \dfrac{2t}{1+t^2} \\ \dfrac{2t}{1+t^2} & -\dfrac{(1+t)^2}{1+t^2} \end{vmatrix} + 1 = 0$

4. Evaluate the following determinants :

(a) $\begin{vmatrix} 2 & -1 & 5 \\ 4 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix}$

5. Show that $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

1.4 PROPERTIES OF DETERMINANTS

Before studying some properties of determinants, we first introduce the concept of minors and cofactors in evaluating determinants.

Minors and Cofactors

Definition : If Δ is a determinant, then the minor M_{ij} of the element a_{ij} is the determinant obtained by deleting ith row and jth column of Δ.

For instance, if

$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$ and

$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Recall that

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Similarly, the expansion of Δ along second and third rows can be written as

$$\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$$

and $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$

respectively.

Definition : The cofactor C_{ij} of the element $a$ij in the determinant Δ is defined to be $(-1)^{i+j}$ M_{ij}, where M_{ij} is the minor of the element a_{ij}.

That is, $C_{ij} = (-1)^{i+j} M_{ij}$

Note that, if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$$\begin{aligned} \Delta &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\ &= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \\ &= a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33} \end{aligned}$$

We can similarly write expansion of Δ along the three columns :

$$\Delta = a_{11}c_{11} + a_{21}c_{21} + a_{31}a_{31}$$

$$= a_{12}c_{12} + a_{22}c_{22} + a_{32}a_{32}$$

$$= a_{13}c_{13} + a_{23}c_{23} + a_{33}a_{33}$$

Thus, the sum of the elements of any row or column of Δ multiplied by their corresponding cofactors is equal to Δ.

Example 3 : Write down the minor and cofactors of each element of the determinant $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$.

Solution: Hence, $\Delta = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$

$M_{11} = |5| = 5$ $\qquad$ $M_{12} = |2| = 2$

$M_{21} = |-1| = -1$ $\qquad$ $M_{22} = |3| = 3$

$C_{11} + (-1)^{1+1}$ $M_{11} = (-1)^2 5 = 5$

$C_{12} + (-1)^{1+2}$ $M_{12} = -2$

$C_{21} + (-1)^{2+1}$ $M_{21} = (-1)^3(-1) = 1$

$C_{22} + (-1)^{2+2}$ $M_{22} = (-1)^4(3) = 3$

Properties of Determinants

The properties of determinants that we will introduce in this section will help us to simplify their evaluation.

1. **Reflection Property**

   The determinant remains unaltered if its rows are changed into columns and the columns into rows.

2. **All Zero Property**

   If all the elements of a row(column) are zero. Then the determinant is zero.

3. **Proportionality (Repetition) Property**

   If the elements of a row(column) are proportional (identical) to the element of the some other row (column), then the determinant is zero.

4. **Switching Property**

   The interchange of any two rows (columns) of the determinant changes its sign.

5. **Scalar Multiple Property**

   If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

6. **Sum Property**

$$\begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

7. **Property of Invariance**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix}$$

   This is, a determinant remains unaltered by adding to a row(column) *k* times some different row (column).

8. **Triangle Property**

   If all the elements of a determinant above or below the main diagonal consists of zerox, then the determinant is equal to the product of diagonal elements.

That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & d_3 \\ 0 & 0 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

Note that from now onwards we shall denote the ith row of a determinant by R_i and its ith column by C_i.

Example 4 : Evaluate the determinant

$$\Delta = \begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{vmatrix}$$

Solution : Applying $R_3 \rightarrow R_3 - R_2$, and $R_2 \rightarrow R_2 - R_1$, we obtain

$$\Delta = \begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_2$, we obtain

$$\Delta = \begin{vmatrix} 0 & 13 & 2 \\ 0 & 3 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = (-1)^{3+1}(1)\begin{vmatrix} 13 & 2 \\ 3 & -3 \end{vmatrix} = -39 - 6 = -45.$$

Example 5 : Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Solution : By applying $R_2 \rightarrow R_2 - R_1$, and $R_3 \rightarrow R_3 - R_1$ we get,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Taking $(b-a)$ common from R_2 and $(c-a)$ common from R_3, we get

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+c \\ 0 & 1 & c+1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)[(c+a)-(b+a)]$$

$$= (b-a)(c-a)(c-b)$$

Example 6 : Evaluate the determinant

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$ where ω is a cube root of unity.

Solution : $$\begin{vmatrix} 1 & \omega & 2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$1 + \omega + \omega^2$

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \quad (\text{By } C_1 \rightarrow C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad (\because 1 + \omega + \omega^2 = 0)$$

= 0 [$\because C_1$ consists of all zero entries].

Example 7 : Show that

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution : Denote the determinant on the L.H.S. by Δ. Then applying $C_1 \rightarrow C_1 + C_2 + C_3$ we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$$

Taking 2 common from C_1 and applying $C_2 \rightarrow C_2 - C_1$, and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} (a+b+c) & -b & -c \\ (a+b+c) & -c & -a \\ (a+b+c) & -a & -b \end{vmatrix}$$

Applying $C_1 \rightarrow C_2 + C_2 + C_3$ and taking (–1) common from both C_2 and C_2, we get

$$\Delta = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Example 8 Show that

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & a \end{vmatrix} = (a^3 - 1)^2$$

Solution :

$$\Delta = \begin{vmatrix} 1+a+a^2 & a & a^2 \\ 1+a+a^2 & 1 & a \\ 1+a+a^2 & a^2 & 1 \end{vmatrix}$$ (By applying $C_1 \rightarrow C_1 + C_2 + C_3$)

$$= (1+a+a^2) \begin{vmatrix} 1 & a & a^2 \\ 1 & 1 & a \\ 1 & a^2 & 1 \end{vmatrix}$$

$$= (1+a+a^2) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a & a-a^2 \\ 0 & a^2-a & 1-a^2 \end{vmatrix}$$ (By applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$)

$$= (1+a+a^2) \begin{vmatrix} 1-a & a-a^2 \\ a^2-a & 1-a^2 \end{vmatrix}$$ (Expanding along C_1)

$$= (1+a+a^2)(1-a^2) \begin{vmatrix} 1 & a \\ -a & 1+a \end{vmatrix}$$ (taking (1– a) common from C_1 and C_2)

$$= (1+a+a^2)(1-a^2)\,(1+a+a^2)$$

$$= (a^3 - 1)^2 \quad (\because a^3 - 1 = (a-1)(a^2+a+1))$$

Example 9 : Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : Taking a, b, and c common four C_1, C_2 and C_3 respectively, we get

$$\Delta = abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$ Taking a, b and c common from R_1, R_2, R_3 respectively, we get.

$$\Delta = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2$ and $C_1 \rightarrow C_2 + C_3$, we get

$$\Delta = a^2b^2c^2 \begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

Expanding along C_1, we get $\Delta = a^2b^2c^2(4) = 4a^2b^2c^2$

Example 10 : Show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : We shall first change the form of this determinant by multiplying R_1, R_2 and R_3 by a, b and c respectively.

Then

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(b^2+c^2) & a^2b & a^2c \\ b^2a & b(c^2+a^2) & b^2c \\ c^2a & c^2b & c(a^2+b^2) \end{vmatrix}$$

Taking a, b and c common from C_1, C_2 and C_3 respectively, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & 2(c^2+a^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

Taking 2 common from R_1 and applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = 2\begin{vmatrix} b^2+c^2 & c^2+a^2 & a^2+b^2 \\ -b^2 & c^2+a^2 & b^2 \\ -c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ we get

$$\Delta = 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

Expanding the determinant along R_1 we get

$$\Delta = -2\,c^2\begin{vmatrix} -c^2 & -b^2 \\ -b^2 & 0 \end{vmatrix} + 2b^2\begin{vmatrix} -c^2 & 0 \\ -b^2 & -a^2 \end{vmatrix}$$

$$= -2\,c^2\,(-\,a^2b^2) + 2b^2a^2c^2$$

$$= 4a^2b^2c^2$$

Check Your Progress – 2

1. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-a)(c-a)(c-b)(a+b+c)$

2. Show that

$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\,(y-x)(z-x)(z-y)$$

3. Show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$$

4. Show that

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\,\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

1.5 APPLICATION OF DETERMINANTS

We first study application of determinants in finding area of a triangle.

Area of Triangle

We begin by recalling that the area of the triangle with vertices A $(x_1\, y_1)$,B $(x_2\, y_2)$, and C$(x_3\, y_3)$, is given by the expression

$$\frac{1}{2}\,|\; x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)\;|$$

The expression within the modulus sign is nothing but the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus, the area of triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Corollary : The three points A(x_1, y_1) , $B\ (x_2, y_2) and\ C(x_3, y_3)$ lie on a straight line if and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Example 11 : Using determinants, find the area of the triangle whose vertices are

(a) $A(1, 4)$, $B(2,3)$ and $C(-5,-3)$

(b) $A(-2,4)$, $B(2,-6)$ and $C(5,4)$

Solution :

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\;\begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ -5 & -3 & 1 \end{vmatrix}\;\right|$$

$$= \frac{1}{2}\left| \begin{vmatrix} 1 & 4 & 1 \\ 1 & -1 & 0 \\ -6 & -7 & 0 \end{vmatrix} \right| \text{(using R1} \rightarrow \text{ R2} - \text{ R1, and R3} \rightarrow \text{ R3} - \text{ R1)}$$

$$= \frac{1}{2}|-7-6|$$

$$= \frac{13}{2} \text{square units}$$

Area of ΔABC $= \frac{1}{2} \left| \begin{vmatrix} -2 & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix} \right|$

$= \frac{1}{2}\ |70|$

= 35 square units

Example 12 : Show that the points (a, b+c), (b, c+a) and (c, a + b) are collinear.

Solution : Let Δ denote the area of the triangle formed by the given points.

$$= \frac{1}{2}\left| \begin{vmatrix} -k+1 & 2k & 1 \\ 2k-1 & 2-4k & 0 \\ -5 & 6-4k & 0 \end{vmatrix} \right|$$

$$= \frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+c+a & 1 \\ c & a+a+b & 1 \end{vmatrix} \quad \text{(using } C_2 \rightarrow C_2 + C_1)$$

$= 0.$ $\qquad$ ($\because C_1$ and C_2 are proportional)

$\therefore$ the given points are collinear.

Cramer's Rule for Solving System of Linear Equation's

Consider a system of 3 linear equations in 2 unknowns :

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \qquad \ldots\ldots(1) \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

A **Solution** of this system is a set of values of x, y, z which make each of three equations true. A system of equations that has one or more solutions is called **consistent.** A system of equation that has no solution is called **inconsistent.**

If $d_1 = d_2 = d_3 = 0$ in (1), the system is said to be **homogeneous system of equations.** If atleast one of d_1, d_2, d_3 *is non* $-$ *zero*, the system is said to be **non homogeneous system of equations.**

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Consider $x\ \Delta$. Using the scalar multiple property we can absorb x in the first column of Δ, that is,

$$x\ \Delta = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + yC_2 + z\, C_3$, we get

$$x\Delta = \begin{vmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta x (\text{say})$$

Note that the determinant Δx can be obtained from Δ by replacing the first column by the elements on the R.H.S. of the system of linear equations that is, by $\begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix}$.

If $\Delta \neq 0$, then $x = \dfrac{\Delta x}{\Delta}$. Similarly, we can show that if $\Delta \neq 0$, then $y = \dfrac{\Delta y}{\Delta}$ and $Z = \dfrac{\Delta z}{\Delta}$, when

Where

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

This method of solving a system of linear equation is known as **Cramer's Rule.**

It must be noted that if $\Delta = 0$ and one of $\Delta_x = \Delta_y = \Delta_z = 0$, then the system has infinite number of solutions and if $\Delta = 0$ and one of $\Delta_x, \Delta_y, \Delta_z$ is non-zero, the system has no solution i.e., it is inconsistent.

Example 13 : Solve the following system of linear equation using Cramer's rule

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 14$$

Solution : We first evaluate Δ, where

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & -4 & 0 \end{vmatrix} = -20 \qquad \text{(expanding along } C_1\text{)}$$

As $\Delta \neq 0$, the given system of linear equations has a unique solution. Next we evaluate Δx, Δy and Δz We have

$$\Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, and $R_3 \to R_3 - R_1$, we get

$$\Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ -5 & 0 & -5 \\ 8 & 0 & 6 \end{vmatrix} = -20 \qquad \text{(expanding along } C_2\text{)}$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6 & 3 \\ 0 & -5 & -5 \\ 0 & -4 & 0 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - 2R_1, \text{ and } R_3 \to R_3 - 3R_1]$$

$$= -20 \qquad [\text{expanding along } C_1]$$

and $\Delta_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & 0 & -5 \\ 0 & -4 & -4 \end{vmatrix}$ [Applying $R_2 \to R_2 - 2R_1$, and $R_3 \to R_3 - 3R_1$]

$= -20$ [expanding along C_1]

Applying Cramer's rule, we get

$$x = \frac{\Delta x}{\Delta} = \frac{-20}{-20} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-20}{-20} = 1 \text{ and}$$

$$z = \frac{\Delta z}{\Delta} = \frac{-20}{-20} = 1$$

Remark : If $d_1 = d_2 = d_3 = 0$ in (1), then $\Delta x = \Delta y = \Delta z = 0$. If $\Delta \neq 0$, then the only solution of the system of linear homogeneous equations.

$a_1 x + b_1 y + c_1 z = 0$

$a_2 x + b_2 y + c_2 z = 0$

$a_2 x + b_2 y + c_2 z = 0$ ……(2)

is $x = 0, y = 0, z = 0$. This is called the trivial solution of the system of equation (2). If $\Delta = 0$, the system (2) has infinite number of solutions.

Example 14 : Solve the system of linear homogeneous equation :

$$2x - y + 3z = 0,$$
$$x + 5y - 7z = 0,$$
$$x - 6y + 10z = 0$$

Solution : We first evaluate Δ. We have

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 5 & -7 \\ 1 & -6 & 10 \end{vmatrix}$$ Applying $R_1 \rightarrow R_1 - 2R_2$ and $R_2 \rightarrow R_2 - R_3$, we get

$= -20$ (expanding along C_1)

$$\Delta = \begin{vmatrix} 0 & -11 & 17 \\ 0 & 11 & -17 \\ 1 & -6 & 10 \end{vmatrix} = 0$$

(because R_1 and R_2 are proportional)

Therefore, the given system of linear homogeneous equations has an infinite number of solutions. Let us find these solutions. We can rewrite the first two equations as :

$$\begin{aligned} 2x - y &= -3z \\ x + 5y &= 7z \end{aligned} \qquad \ldots\ldots (1)$$

Now, we have $\Delta' = \begin{vmatrix} 2 & -1 \\ 2 & 5 \end{vmatrix} = 10 - (-1) = 11.$

As $\Delta' \neq 0$, the system of equation in (1) has a unique solution. We have

$\Delta x = \begin{vmatrix} -3z & -1 \\ 7z & 5 \end{vmatrix} = -15z - (-7z) = -8z$ and

$\Delta x = \begin{vmatrix} 2 & -3z \\ 1 & 7z \end{vmatrix} = 14z - (-3z) = 17z$

By Cramer's Rule, $x = \frac{\Delta x}{\Delta'} = \frac{-8z}{11} = \frac{-8}{11}z$ and $y = \frac{\Delta y}{\Delta'} = \frac{17z}{11} = \frac{17}{11}z.$

We now check that this solution satisfies the last equation. We have

$$x - 6y + 10z = \frac{-8}{11}z = -6\left(\frac{17}{11}z\right) + 10z$$
$$= \frac{1}{11}(-8z - 102z + 110z) = 0.$$

Therefore, the infinite number of the given system of equations are given by

$x = \frac{-8}{11}k,\ \ y = \frac{17}{11}k$ and $z = k,$ where k is any real number.

Check Your Progress – 3

1. Using determinants find the area of the triangle whose vertices are :
   (a) (1,2), (−2,3) and (−3, −4)
   (b) (−3, 5), (3, −6) and (7,2)
2. Using determinants show that (−1,1), (−3, −2) and (−5, −5) are collinear.
3. Find the area of the triangle with vertices at $(-k+1, 2k)$, $(k, 2-2k)$ and $(-4-k, 6-2k)$. For what values of k these points are collinear ?
4. Solve the following system of linear equations using Cramer's rule.
   (a) $x + 2y - z = -1$, $3x + 8y + 2z = 28$, $4x + 9y + z = 14$
   (b) $x + y = 0$, $y + z = 1$, $z + x = 3$
5. Solve the following system of homogeneous linear equations :
   $2x - y + z = 0$, $3x + 2y - z = 0$, $x + 4y + 3z = 0$.

1.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. (a) $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 10 - (-3) = 13$

(b) $\begin{vmatrix} a & c+id \\ c-id & b \end{vmatrix} = ab - (c+id)(c-id) = ab - c - d^2$

(c) $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix} = (n+1)(n-1) - n^2 = n^2 - 1 - n^2 = -1$

(d) $\begin{vmatrix} \dfrac{1-t^2}{1+t^2} & \dfrac{2t}{1-t^2} \\ \dfrac{-2t}{1-t^2} & \dfrac{1-t^2}{1-t^2} \end{vmatrix} = \left(\dfrac{1+t^2}{1-t^2}\right)^2 - \dfrac{4t^2}{(1-t^2)^2}$

$$= \frac{(1+t^2)^2 - 4t^2}{(1-t^2)^2} = \frac{(1-t^2)^2}{(1-t^2)^2} = 1$$

2. $\begin{vmatrix} a\alpha + i\beta & c\alpha + d\gamma \\ a\beta + b\delta & c\beta - d\delta \end{vmatrix} = (a\alpha + b\gamma)\,(c\beta + d\delta) - (a\beta + d\delta)(c\alpha + d\delta)$

$= ad\alpha\delta + bc\gamma\beta - ad\,\beta\gamma - ad\,\beta\gamma - bc\alpha\delta$

$= ad\,(\alpha\delta - \beta\gamma) - bc\,(\alpha\delta - \beta\gamma)$

$= (ad - bc)\,(\alpha\delta - \beta\gamma)$

3. $\begin{vmatrix} \dfrac{(1-t)^2}{1+t^2} & \dfrac{2t}{1-t^2} \\ \dfrac{2t}{1+t^2} & -\dfrac{(1+t)^2}{1+t^2} \end{vmatrix} = \dfrac{(1-t^2)^2}{(1+t^2)^2} - \dfrac{4t^2}{(1+t^2)^2}$

$$= \frac{-[(1-t^2)^2 + 4t^2]}{(1+t^2)^2} = -\frac{(1+t^2)^2}{(1+t^2)^2} = -1$$

$$\therefore \begin{vmatrix} \dfrac{(1-t)^2}{1+t^2} & \dfrac{2t}{1-t^2} \\ \dfrac{2t}{1+t^2} & -\dfrac{(1+t)^2}{1+t^2} \end{vmatrix} + 1 = 0$$

4. (a) $\begin{vmatrix} 2 & -1 & 5 \\ 4 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + 5\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$

$= 2\,(0 - 1) - (8 - 1) + 5(4 - 0)$

$= -2 + 7 + 20 = 25$

(b) $\begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix} = 5\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$

$= 5\,(0 - 2) - 3(6 - 1) + 8(4 - 0)$

$= -10 - 15 + 32 = 7$

5. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$= a(bc - f^2) - h\,(hc - gf) + g(hf - gb)$

$= abc - af^2 - c\,h^2 + fgh + fgh - b\,g^2$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

Check Your Progress 2

1. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$ (Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$= (b-a)\,(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+a^2+ba & c^2+a^2+ca \end{vmatrix}$ (taking $(b-a$ common form C_2 & $(c-a)$ common from C_3)

$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+a^2+ba & c^2+a^2+ca \end{vmatrix}$

$= (b-a)\,(c-a)(\,c^2 + a^2 + ca - b^2 - a^2 - ba)$

$= (b-a)\,(c-a)[(\,c^2 - c^2 + ca - ba)]$

$= (b-a)(c-a)\,[(c-b)(c+b) + (c-b)a]$

2. Taking x, y and z common from C_1, C_2 and C_3 respectively, we get

$$\Delta = xyz\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^2 & y^2 & z^2\end{vmatrix}$$ (Applying $C_2 \to C_2 - C_1$, and $C_3 \to C_3 - C_1$) we get

$$\Delta = xyz\begin{vmatrix}1 & 0 & 0\\ x & y-x & z-x\\ x^2 & y^2-x^2 & z^2-x^2\end{vmatrix}$$

Taking $(y-x)$ common from C_2 and $(z-x)$ from C_2, we get and $(z-x)$ from C_3, we get

$$\Delta = xyz\,(y-x)\,(z-x)\begin{vmatrix}1 & 0 & 0\\ x & 1 & 1\\ x^2 & y+x & z+x\end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = xyz\,(y-x)(z-x)\begin{vmatrix}1 & 1\\ y+x & z+x\end{vmatrix}$$

$$= xyz\,(y-x)\,(z-x)\,(z+x-y-)$$

$$= xyz\,(y-x)\,(z-x)\,(z-y)$$

3. Let $\Delta = \begin{vmatrix}b+c & a & a\\ b & c+a & b\\ c & c & a+b\end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$, we get

$$\Delta = \begin{vmatrix}o & -2c & -2b\\ b & c+a & b\\ c & c & a+b\end{vmatrix} = -2\begin{vmatrix}o & c & b\\ b & c+a & b\\ c & c & a+b\end{vmatrix}$$

(by the scalar multiple property)

Applying $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - R_1$, we get

$$\Delta = -2\begin{vmatrix}o & c & b\\ b & a & o\\ c & o & a\end{vmatrix}$$

Expanding along the first column, we get

$$\Delta = -2\left(-b\begin{vmatrix}c & b\\ o & a\end{vmatrix} + c\begin{vmatrix}c & b\\ a & o\end{vmatrix}\right)$$

$$= -2(-abc - abc) = 4abc.$$

4. We take a, b and c common from C_1, C_2 and C_3 respectively, to obtain

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{b} \end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{a} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Check Your Progress - 3

1. (a) $\Delta = \frac{1}{2} 1 \begin{vmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \\ -3 & -4 & 1 \end{vmatrix} 1$

$= \frac{1}{2} 1 \begin{vmatrix} 1 & 2 & 1 \\ -3 & 1 & 0 \\ -4 & -6 & 0 \end{vmatrix} 1$ (By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$= \frac{1}{2}|(18+4)|$ (Expanding along C_3)

$= \frac{1}{2}|22| = 11$ square units

(b) $\Delta = \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 10 & 2 & 1 \end{vmatrix}$

$= \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 6 & -11 & 0 \\ 10 & -3 & 0 \end{vmatrix}$ (By applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$)

$= \frac{1}{2} 1\ |-18 + 110)|$

$= \frac{1}{2} \times 92 = 46$ square units

2. $\Delta = \begin{vmatrix} -1 & 1 & 1 \\ -3 & -2 & 1 \\ -5 & -5 & 1 \end{vmatrix}$

$= \begin{vmatrix} -1 & 1 & 1 \\ -2 & -3 & 0 \\ -4 & -6 & 0 \end{vmatrix}$ (By applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$)

$= 12 - 12 = 0$

$\therefore$ the given points are collinear.

3. Area of triangle $\Delta = \frac{1}{2} | \begin{vmatrix} -k+1 & 2k & 1 \\ k & 2-2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} |$

$= \frac{1}{2} | \begin{vmatrix} -k+1 & 2k & 1 \\ 2k-1 & 2-4k & 0 \\ -5 & 6-4k & 0 \end{vmatrix} |$

$= \frac{1}{2} 1 \begin{vmatrix} 2k-1 & 2-4k \\ -5 & 6-4k \end{vmatrix} 1$

$= \frac{1}{2}(2k-1)(6-4k) + 5(2-4k)$

$= \frac{1}{2} |- 8k2- 4k + 4|$

$= |4k^2 + 2k - 2|$

These points are collinear if $\Delta = 0$

i.e., if $|4k^2 + 2k - 2| = 0$

i.e., if $2(2k - 1)(k + 1) = 0$

i.e., if $k = -1, \frac{1}{2}$

4. (a) We first evaluate Δ. We have

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 5 \\ 4 & 1 & 5 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - 2C_1 \text{ and } C_3 \to C_3 + C_1]$$

$= 10 - 5 = 5$ (expanding along R_1)

As $\Delta \neq 0$, the given system of equation has a unique solution. We shall now evaluate Δx, Δy and Δz. We have

$$\Delta x = \begin{vmatrix} -1 & 2 & -1 \\ 28 & 8 & 2 \\ 14 & 9 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & -1 \\ 26 & 12 & 0 \\ 13 & 11 & 0 \end{vmatrix}$$

(By applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$ we get)

$$= -\begin{vmatrix} 26 & 12 \\ 13 & 11 \end{vmatrix}$$ (expanding along C_3)

$= -130$

$$\Delta y = \begin{vmatrix} 1 & -1 & -1 \\ 3 & 28 & 2 \\ 4 & 14 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ 5 & 26 & 0 \\ 5 & 13 & 0 \end{vmatrix}$$

(By applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 + C_1$)

$$= -\begin{vmatrix} 5 & 26 \\ 5 & 13 \end{vmatrix}$$ (expanding along R_1)

$= 65$

and $$\Delta z = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & 28 \\ 4 & 9 & 14 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 31 \\ 4 & 1 & 18 \end{vmatrix} \quad (\text{Applying } C_2 \to C_2 - 2C_1 \text{ and } C_3 \to C_3 + C_1)$$

$= 5$ (expanding along R_1)

Hence by Cramer's Rule

$$x = \frac{\Delta x}{\Delta} = \frac{-130}{5} = -26$$

$$y = \frac{\Delta y}{\Delta} = \frac{65}{5} = 13$$

$$z = \frac{\Delta z}{\Delta} = \frac{5}{5} = 1$$

(b) Here,

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1]$$

$= 2$ (Expanding along R_1)

Since $\Delta \neq 0$, $\therefore$ the given system has unique solution, Now, $\Delta x = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2$

$\Delta y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2$

and $\Delta z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4$

Hence by Cramer's Rule

$$x = \frac{\Delta x}{\Delta} = \frac{2}{2} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-2}{2} = -1 \text{ and}$$

$$z = \frac{\Delta z}{\Delta} = \frac{4}{2} = 2$$

5. Here, $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 4 & 3 \end{vmatrix}$

$= \begin{vmatrix} 2 & -1 & 1 \\ 5 & 1 & 0 \\ -5 & 7 & 0 \end{vmatrix}$ (Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 3R_1$)

$= 35 + 5$ (expanding along C_3)

$= 40$

Since $\Delta \neq 0$, $\therefore$ the given system has a unique solution, and the trivial solution $x = y = z = 0$ is the only solution. In fact, $\Delta x = \Delta y = \Delta z = 0$

$\therefore x = y = z = 0.$

$= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = -(1-3) = 2$

$= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = (1-3) = -2$

$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = 3 + 1 = 4$

$$\therefore \quad x = \frac{2}{2} = 1, \qquad y = \frac{-2}{2} = -1, \qquad z = \frac{4}{2} = 2.$$

1.7 SUMMARY

In this unit, first of all, the definitions and the notations for determinants of order 2 and 3 are given. In **sections 1.2 and 1.3** respectively, a number of examples for finding the value of a determinant, are included. Next, properties of determinants are stated. In **section 1.4**, a number of examples illustrate how evaluation of a determinant can be simplified using these properties. Finally, in **section 1.5**, applications of determinants in finding areas of triangles and in solving system of linear equations are explained.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.6**.

UNIT 2 MATRICES - I

Structure

2.0 INTRODUCTION

In this Unit, we shall learn about Matrices. Matrices play central role in mathematics in general, and algebra in particular. A matrix is a rectangular array of numbers. There are many situations in mathematics and science which deal with rectangular arrays of numbers. For example, the following table gives vitamin contents of three food items in conveniently chosen units.

	Vitamin A	Vitamin C	Vitamin D
Food I	0.4	0.5	0.1
Food II	0.3	0.2	0.5
Food III	0.2	0.5	0

The above information can be expressed as a rectangular array having three rows and three columns.

$$\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \\ 0.2 & 0.5 & 0 \end{bmatrix}$$

The above arrangement of numbers is a matrix of order 3×3. Matrices have become an important an powerful tool in mathematics and have found applications to a very large number of disciplines such as Economics, Physics, Chemistry and Engineering.

In this Unit, we shall see how Matrices can be combined thought the arithematic operations of addition, subtraction, and multiplication. The use of Matrices in solving a system of linear equations will also be studied. In Unit 1 we have already studied determinant. It must be noted that a matrix is an arrangement of numbers whereas determinant is number itself. However, we can associate a determinant to every square matrix i.e., to a matrix in which number of rows is equal to the number of columns.

2.1 OBJECTIVES

After studying this Unit, you should be able to :

- define the term matrix;
- add two or more Matrices;
- multiply a matrix by a scalar;
- multiply two Matrices;
- find the inverse of a square matrix (if it exists); and
- use the inverse of a square matrix in solving a system of linear equations.

2.2 MATRICES

We define a matrix as follows :

Def : A $m \times n$ matrix A is a rectangular array of $m\ n$ real (or complex numbers) arranged in m horizontal rows an n vertical columns :

$$A = \begin{array}{c} \left(\begin{array}{cccccc} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1j} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2j} & \ldots\ldots\ldots & a_{2n} \\ : & : & : & : & & \\ : & : & : & : & & \\ a_{i1} & a_{i2} & \ldots\ldots\ldots & a_{ij} & \ldots\ldots\ldots & a_{in} \\ : & : & & & & \\ : & & & & & \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mj} & \ldots\ldots\ldots & a_{mn} \end{array}\right) \begin{array}{l} \\ \\ \\ \\ ith\ row \\ \\ \\ \\ \end{array} \\ \qquad\qquad jth \\ \qquad\qquad column \end{array} \quad \ldots(1)$$

As it is clear from the above definition, the *ith* row of A is $(a_{ij}\ a_{i2}\ \ldots\ a_{in})$ $(1 \le i \le m)$ and the *jth* column is

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ : \\ : \\ a_{mj} \end{pmatrix} \quad (1 \le j \le n)$$

We also note that each element a_{ij} of the matrix has two indices : the row index i and the column index j. a_{ij} is called the $(i,j\ )th$ element of the matrix. For convenience, the Matrices will henceforward be denoted by capital letters and the elements (also called entries) will be denoted by the corresponding lower case letters.

The matrix in (1) is often written in one of the following forms :

A = *[aij]; * A = *(aij)*, A = $(a_{ij})_{m\times n}$ *or* A = $(a_{ij})_{m\times n}$

With i = 1, 2, ………….., m and j = 1, 2, ……….., n

The **dimension** or **order** of a matrix A is determined by the number of rows and columns of the matrix. If *a* matrix A has *m* rows and *n* columns we denote its dimension or order by $m \times n$ read *"m by n".*

For example, A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a 2 × 2 matrix and B = $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ is a 2 × 3 order matrix.

Note a that an *m* × *n* matrix has *mn* elements.

Type of Matrices

1. **Square Matrix :** A **square matrix** is one in which the number of rows is equal to the number of columns. For instance,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 3 & -1 & 6 \\ 8 & 2 & 9 \\ 0 & 7 & 3 \end{bmatrix},\ C = \begin{bmatrix} 5 & -1 & 8 & 3 \\ 2 & 5 & 7 & 8 \\ 3 & 6 & 11 & 0 \\ 0 & -1 & 8 & 7 \end{bmatrix}$$

are square Matrices.

If a square matrix has *n* rows (and thus *n* columns), then *A* is said to be a square matrix of order *n*.

2. **Diagonal Matrix :** A square matrix $A[a_{ij}]_{n\times n}$ for which $a_{ij} = 0$ for $i \neq j,$ is called a **diagonal matrix**.

For instance,

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } E = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

are diagonal Matrices.

If A = $[a_{ij}]_{n\times n}$ is a square matrix of order *n*, then the numbers a_{11}, a_{22}, … a_{nn} are called diagonal elements, and are said to form the **main diagonal** of A. Thus, a square matrix for which every term off the main diagonal is zero is called a diagonal matrix.

3. **Scalar Matrix :** A diagonal matrix A = $[a_{ij}]_{n\times n}$ for which all the terms on the main diagonal are equal, that is $a_{ij} = k$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$ is called a **scalar matrix.**

For instance

$$H = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are scalar Matrices.

4. **Unit or Identity Matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is said to be the **unit matrix** or **identity matrix** if

$$a_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

Note that a unit matrix is a scalar matrix with is on the main diagonal. We denote the unit matrix having n rows (and n columns) by In. For example,

$$I_3 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. **Row Matrix or Column Matrix :** A matrix with just one row of elements is called a **row matrix or row vector**. While a matrix with just one column of elements is called a **column matrix** or **column vector.**

For instance, A = $[2 \ \ 5 \ -15]$ is a row matrix whereas $B = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

is a column matrix.

6. **Zero matrix or Null matrix :** An $m \times n$ matrix is called a **zero matrix** or **null matrix** if each of its elements is zero.

We usually denote the zero matrix by $O_{m \times n}$

$O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $O_{3\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ are example of zero matrices.

Equality of Matrices

Let A = $[a_{ij}]_{m\times n}$ and B =$[b_{ij}]_{r\times s}$ be two Matrices. We say that A and B are equals if

1. $m = r$, i.e., the number of rows in A equals the number of rows in B.

2. $n = s$, *i.e.*, the number of columns in A equals the number of columns in B.
3. $a_{ij} = b_{ij}$ for $I = 1, 2, \ldots\ldots m$ and $j = 1, 2, \ldots\ldots, n$.

We then write A = B, read as "matrix A is equal to B" In other words, two Matrices are equal if their order are equal and their corresponding elements are equal.

Example 1 : Let A and B be two Matrices given by $A = \begin{bmatrix} x+2 \\ 3y-7 \end{bmatrix}$,

$B = \begin{bmatrix} 4-y \\ x-3 \end{bmatrix}$. Find x and y so that $A = B$.

Solution: Both the Matrices are of order 2 ×1. Therefore, by the definition of equality of two Matrices, we have $x + 2 = 4 - y$ and $3y - 7 = x - 3$. That is, $x + y = 2$ and $x - 3y = -4$. Solving these two equations. We get $x = 1/2$ and $y = 3/2$. We can check this solution by substitution in A and B.

$$A = \begin{bmatrix} \left(\frac{1}{2}\right) + 2 \\ 3\left(\frac{3}{2}\right) - 7 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 4 - \left(\frac{3}{2}\right) \\ \left(\frac{1}{2}\right) - 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix}$$

Transpose of a Matrix

Definition: Let $A = [a_{ij}]_{m \times n}$, be a matrix. The **transpose** of A, denoted by A´, is the matrix $A' = [a_{ij}]_{m \times n}$, where $b_{ij} = a_{ji}$ for each i and j.

The transpose of a matrix A is by definition, that matrix which is obtained from A by interchanging its rows and columns.

So, if $A = \begin{bmatrix} -3 & 2 & 5 \\ 0 & 7 & 8 \end{bmatrix}$, then

its transpose is the matrix

$$A' = \begin{bmatrix} -3 & 0 \\ 2 & 7 \\ 5 & 8 \end{bmatrix}.$$

Symmetric and Skew Symmetric Matrices

Definition : A square matrix $A = [a_{ij}]_{n \times n}$, is said to be **symmetric** if A'= A; it is **skew symmetric** if A' = −A.

For example A = $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ is symmetric and B = $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ is a skew – symmetric matrix.

Check Your Progress 1

1. Construct a 2×2 matrix A = $[a_{ij}]_{2\times2}$ where elements are given by
   (a) $a_{ij} = \frac{1}{2}(i + 2j)^2$ (b) $a_{ij} = \frac{1}{2}(i - j)^2$

2. Find x, y when $\begin{bmatrix} x & y \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2x + y & x - y \\ 3 & 5 \end{bmatrix}$.

3. a, b, c and d such that

$$\begin{bmatrix} a - b & 2c + d \\ 2a - b & 2a + d \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}.$$

4. Find the transpose of following Matrices and find whether the matrix is symmetric or skew symmetric.

   (a) A= $\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$

2.3 OPERATION ON MATRICES

Addition

Let A = $[a_{ij}]_{m\times n}$ and B= $[b_{ij}]_{r\times s}$ be two Matrices. We say that A and B are comparable for addition if $m = r$ and $n = s$. That is, A and B are comparable for addition if they have same order.

We define addition of Matrices as follows :

Definition : Let A =$[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ be two Matrices. The sum of A and B is the $m \times n$ matrix C =$[c_{ij}]$ such that

$$C_{ij} = a_{ij} + b_{ij} \qquad (1 \le i \le m, 1 \le j \le n).$$

That is, C is obtained by adding the corresponding elements of A and B. We usually denote C by A + B.

Note that

A + B = $[c_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$

For example, if A $=\begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 5 \end{bmatrix}$ and B $=\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 0+2 & 1+3 \\ 3-1 & 2+0 & 5+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 7 \end{bmatrix}$$

It must be noted that Matrices of different orders cannot be added. For instance,
A = $\begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ Cannot be added.

The following properties of matrix addition can easily be verified.

1. Matrix addition is commutative. That is, if A and B are two $m \times n$ matrices, then

   $$A + B = B + A.$$

2. Matrix addition is associative. That is, if A, B and C are three $m \times n$ matrices, then

   $$(A + B) + C = A + (B + C)$$

3. If A $[a_{ij}]$ = is an $m\times n$ matrix, then

   $$A + O_{m\times n} = O_{m\times n} + A = A,$$

   . where $O_{m\times n}$ is the $m\times n$ null matrix.

4. If A is an $m \times n$ matrix, then we can find an $m \times n$ matrix B such that

   $$A + B = B + A = O_{m\times n}$$

   The matrix B in above property is called 'additive inverse' or 'negative' of A and is denoted by − A.

   Infact, if A = $[a_{ij}]_{m\times n}$ then –A = $[-a_{ij}]_{m\times n}$

   Thus, property 4 can be written as

   $$A + (-A) = (-A) + A = O_{m\times n}$$

We can now define difference of two Matrices.

Definition : Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ two matrices. We define the **difference** A − B to be the $m \times n$ matrix A + (−B).
Note that A − B is of dimension $m\times n$ and $A - B = [a_{ij} - b_{ij}]_{m\times n.}$

For example, if A = $\begin{bmatrix} 4 & -1 & 6 \\ 5 & 8 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -5 & -3 \\ 7 & 0 & 8 \end{bmatrix}$

then A – B = $\begin{bmatrix} 4-2 & -1+5 & 6+3 \\ 5-7 & 8-0 & 3-8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 9 \\ -2 & 8 & -5 \end{bmatrix}$.

Scalar Multiplication

Definition : Let A =$[a_{ij}]_{m\times n}$ be a matrix and let K be a complex number. The scalar multiplication KA of the matrix A and the number K (called the scalar) is the $m\times n$ matrix KA $[ka_{ij}]_{m\times n}$

For example, let $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 5 & 1 \end{bmatrix}$

If K = 4, then kA = 4A = $\begin{bmatrix} 12 & 4 \\ -8 & 0 \\ 20 & 4 \end{bmatrix}$

and if $k = \frac{1}{3}$, then kA $= \frac{1}{3} A = \begin{bmatrix} 1 & 1/3 \\ -2/3 & 0 \\ 5/3 & 1/3 \end{bmatrix}$

Note that if $k = -1$, then (-1)A $= -$ A.

This is one of the properties of scalar multiplication. We list some of these properties without proof.

Properties of Scalar Multiplication

1. Let A = $[a_{ij}]_{m\times n}$ be a matrix and let k_1 and k_2 be two scalars. Then
   (i) $(k_1 + k_2)$ A = k_1A + k_2 A, and
   (ii) k_1 (k_2 A) = ($k_1 k_2$)A.

2. Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ be two matrices and let k be a scalar. Then

   k (A + B) = kA + k B.

Multiplication of two Matrices

Let A = $[aij]_{m\times n}$ and B = $[bij]_{r\times s}$ be two matrices. We say that A and B are **comparable for the product** AB if $n = r$, that is, if the number of columns of A is same as the number of rows of B.

Definition : Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{n\times p}$ be two matrices. Their product AB is the matrix C = $[c_{ij}]_{m\times p}$ such that $c_{ij} = a_{i1}b_{ij} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots\ldots\ldots\ldots + a_{in}b_{nj}$ for $i \le i \le m$, $1 \le j \le p$. Note that the order of AB is $m\times p$.

Example 2 : Let A = $\begin{bmatrix} 2 & 3 & 7 \\ -1 & 5 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 0 \\ 2 & 5 \\ -2 & 7 \end{bmatrix}$

Obtain the product AB.

Solution : Since A is of order 2× 3 and B is or order 3 ×2, therefore, the product AB is defined. Order of AB is 2× 2.

$$AB = \begin{bmatrix} 2 & 3 & 7 \\ -1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 5 \\ -2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 3 + 3\times 2 + 7\times(-2) & 2\times 0 + 3\times 5 + 7\times 7 \\ (-1)\times 3 + 5\times 2 + 2\times(-2) & -1\times 0 + 5\times 5 + 2\times 7 \end{bmatrix}$$

$$= \begin{bmatrix} 6+6-14 & 0+15+49 \\ -3+10-4 & 0+25+14 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 64 \\ 3 & 39 \end{bmatrix}$$

Properties of Matrix Multiplication

Some of the properties satisfied by matrix multiplication are stated below without proof.

1. (Associative Law) : If A = $[a_{ij}]_{m\times n}$, B = $[b_{ij}]_{n\times p}$ and C = $[c_{ij}]_{p\times q}$ are three matrices, then

   (AB) C = A (BC).

2. (Distributive Law): If A = $[a_{ij}]_{m\times n}$, B = $[b_{ij}]_{n\times p}$ and C = $[c_{ij}]_{n\times p}$ are three matrices, then

   A (B + C) = AB + AC.

3. If A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{n\times p}$ are two matrices, and k is a complex number, then

   (kA) B = A(kB) = k(AB).

4. If A = $[a_{ij}]_{m\times n}$ is an $m\times n$ matrix, then

   $I_m A = AI_n = A$,

   Where I_m and I_n are unit matrices of order m and n respectively.

Example 3: Let A = $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 5 & -2 \end{bmatrix}$

Find AB and BA.

Solution : Since A is 3×1 matrix and B is a 1×3 matrix, therefore, AB is defined and its order is 3× 3.

If the number of columns of A is equal to the number of rows of B.

$$AB = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 3 & 1 \times 5 & 1 \times (-2) \\ 2 \times 3 & 2 \times 5 & 2 \times (-2) \\ (-1) \times 3 & (-1) \times 5 & -1 \times (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & -2 \\ 6 & 10 & -4 \\ -3 & -5 & 2 \end{bmatrix}$$

Also, BA is defined and a is 1× 1 matrix

$$BA = \begin{bmatrix} 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= [3 + 10 + 2] = [15]$$

This example illustrates that the matrix multiplication is not commutative. Infact, it may happen that the product AB is defined but BA is not, as in the following case :

$$A = \begin{bmatrix} 1 & 2 \\ 5 & -3 \\ 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$$

We now point out two more matrix properties which run counter to our experience to number systems.

1. It is possible that for two non-zero matrices and A and B, the product AB is a zero matrix.
2. It is possible that for a non-zero matrix A, and two unequal matrices B and C, we have, AB = AC. That is AB = AC, A ≠ 0 may not imply B = C. In other words, cancellation during multiplication does not hold.

These properties can be seen in the following example.

Example 4 : Let $A = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 7 \\ 0 & 0 \end{bmatrix}$.

Show that $AB = O_{2\times 2}$ and $AB = AC$.

Solution : We have

$$AB = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and}$$
$$= O_{2\times 2}$$

$$AC = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= O_{2\times 2}$$

Therefore, AB = AC. We see however, that $A \neq O_{2\times 2}$ and $B \neq C$. Thus, cancellation during multiplication does not hold.

Exponent of a Square Matrix

We now introduce the notion of the exponent of a square matrix. To begin with, we define A^m for any square matrix and for any positive integer *m*.

Let A be a square matrix and *m* a positive integer. We define.

$$A^m = \underbrace{AAA \ldots\ldots A}_{m \text{ times}}$$

More formally, the two equations A' = A and $A^{m+1} = A^m$ A define A^m recursively by defining it first for $m = 1$ and then $m+1$ after it has been defined for *m*, for all $m \geq 1$.

We also define $A^\circ = I_n$, where A is a non−zero square matrix of order *n*.

The usual rules of exponent's namely

$A^m A^n = A^{m+n}$ and $(A^m)^n = A^{mn}$ do hold for matrices if *m* and *n* are non-negative integers.

Example 5 : Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$. Show that $f(A) = O_{2\times 2}$. Use this result to find A^5.

Solution : First, we note that by $f(A)$ we mean $A^2 - 4A + 7I_2$. That is, we replace *x* by A and multiply the constant term by I, the unit matrix. Therefore,

$$f(A) = A^2 - 4A + 7I_2$$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 6+6 \\ -2-2 & -3+4 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times 2}.$$

Hence, $A^2 - 4A - 7I_2$, from which we get

$$A^3 = A^2A = (4A - 7I_2)A$$
$$= 4A^2 - 7I_2A = 4(4A - 7I_2) - 7A \quad [\because I_2 A = A]$$
$$= 9A - 28I_2$$
$$\Rightarrow A^5 = A^2 A^3 = (4A - 7I_2)(9A - 28I_2)$$
$$= 36A^2 - 63I_2A - 112AI_2 + 196 I_2 I_2 \text{ (Distributive Law)}$$
$$= 36(4A - 7I_2) - 63A - 112A + 196 I_2$$
$$= 144A - 252 I_2 - 175A + 196 I_2$$
$$= -31A - 56 I_2$$
$$= -31 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -62 & -93 \\ 31 & -62 \end{bmatrix} - \begin{bmatrix} 56 & 0 \\ 0 & 56 \end{bmatrix}$$
$$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$$

Check Your Progress – 2

1. If P = $\begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}$, Q = $\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, find matrix R such that 5P + 3Q + 2R is a null matrix.

2. If A = $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$, B = $\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$ find a and b.

3. If A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ where $i^2 = -1$ verify $(A + B)^2 = A^2 + B^2$.

4. Let $f(x) = x^2 - 5x + 6$. Find $f(A)$ if

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}.$$

5. If A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that $A^5 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$.

6. If A and B are square matrices of the same order, explain why the following may not hold good in general.
   (a) $(A + B)(A - B) = A^2 - B^2$
   (b) $(A + B)^2 = A^2 + 2AB + B^2$
   (c) $(A - B)^2 = A^2 - 2AB + B^2$.

2.4 INVERTIBLE MATRICES

In this section, we restrict our attention to square matrices and formulate the notion of multiplicative inverse of a matrix.

Definition : An $n \times n$ matrix A is said to be **invertible** or **non-singular** if these exists an $n \times n$ matrix or **non singular** if there exists an $n \times n$ matrix such that AB = BA = I_n.

The matrix B is called an **inverse** of A. If there exists no such matrix B, then A is called **non-invertible** or **singular.**

Example 6 : Find whether A is invertible or not where

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution : (a) We are asked whether we can find a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that AB = I_2 . What we require is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

This would imply that c = 0, d = 1, a = 1 and b = –1, so that matrix

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

does satisfy AB = I_2. Moreover, it also satisfies the equation BA = I_2. This can be verified as follows :

$$BA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 1-1 \\ 0+0 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

This implies that A is invertible and $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is an inverse of A.

(b) Again we ask whether we can find a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that AB = I_2 . What is required in this case is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

This would imply that a=1, b = 0 and the absurdity that 1=0. So no such B exists for this particular A. Hence, A is non invertible.

We will not show that if A is invertible, then B in the above definition is unique.

Theorem : If a matrix has an inverse, then inverse is unique.

Proof : Let B and C be inverses of a matrix A. Then by definition.

$$AB = BA = I_n \qquad ..(1)$$

and $$AC = CA = I_n \qquad .. (2)$$

Now, $B = BI_n$ [property of identify matrix]

$= B(AC)$ [(using 2)]

$= (BA)C$ [associative law]

$= I_nC$ [using (1)]

$= C.$ [property of identity matrix]

This means that we will always get the same inverse irrespective of the method employed. We will write the inverse of A, if it exists, as A^{-1}. Thus

$$AA^{-1} + A^{-1} A = I_n.$$

Definition

Let $A = (a_{ij})_{n\times n}$ be a square matrix of dimension $n\times n$. The cofactors matrix of A is defined to be the matrix $C = (a_{ij})_{n\times n}$ where A_{ij} denotes the cofactor of the element a_{ij} in the matrix A.

For example, if $A= \begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{pmatrix}$,

then $A_{11=} (-1)^{1+1} \begin{vmatrix} 0 & 5 \\ -1 & 2 \end{vmatrix} = 5$

$A_{12=} (-1)^{1+2} \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} = 14$ and

$A_{13=} (-1)^{1+3} \begin{vmatrix} 3 & 0 \\ 4 & -1 \end{vmatrix} = -3.$

Similarly, $A_{21} = 3$, $A_{22} = -2$ $A_{23} = -7$, $A_{31} = -10$, $A_{32} = -2$ and $A_{33} = 6$.

Thus, the cofactor matrix of A is given by $C = \begin{pmatrix} 5 & 14 & -3 \\ 3 & -2 & -7 \\ -10 & -2 & 6 \end{pmatrix}$.

Definition

The adjoint of square matrix $A = (a_{ij})_{n\times n}$ is defined to be the transpose of the cofactor matrix of A. It is denoted by *adj* A.

For example A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then adj A = $\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$

The following theorem will enable us to calculate the inverse of a square matrix. We state the theorem (without proof) for 3×3 matrices only, but it is true for all square matrices of order $n \times n$, where $n \geq 2$.

Theorem : If A is a square matrix of order 3×3, then

A(adj)A = (adjA) A = $|A| I_3$.

In view of this theorem, we note that if $|A| \neq 0$, then

$A\left(\frac{1}{|A|}\text{adj } A\right) = \left(\frac{1}{|A|}\text{adj } A\right)A = I_3$.

Since, the inverse of a square matrix is unique, we see that if $|A| \neq 0$, then

A $\left(\frac{1}{|A|}\text{adj } A\right)$ acts as the inverse of A. That is,

$$\boxed{A^{-1} = \frac{1}{|A|}(\text{adj} A)}$$

Also, a square matrix is invertible (non-singular) if and only if $|A| \neq 0$.

Example 7 : Find the inverse of A = $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

Solution :

We have $A_{11} = (-1)^{1+1} \; |4| = 4$ and $A_{12} = (-1)^{1+2} \; |2| = 2$.

We know that $|A| = a_{11}A_{11} + a_{12}A_{12} = (-3)(4) + 5(-2) = -22$.

Since $|A| \neq 0$ the matrix A is invertible, Also,
$A_{21} = (-1)^{2+1} \; |5| = -5$ and $A_{22} = (-1)^{2+2} \; |-3| = -3$. Therefore,

adj A = $\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix}$

Hence $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-22}\begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -2/11 & 5/22 \\ 1/11 & 3/22 \end{pmatrix}$

Example 8 : If A = $\begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 5 \\ 2 & 5 \end{bmatrix}$,

verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution : Since $|A| = -8 \neq 0$, $\therefore$ A is invertible.

Similarly, $|B| = 20 - 10 = 10 \neq 0$, $\therefore$ B is also invertible.

Let A_{ij} denote the cofactor of a_{ij} – the $(i,j)^{th}$ element of A. Then

$A_{11} = 0$, $A_{12} = -4$, $A_{21} = -2$ and $A_{22} = 3$.

Similarly, if B_{ij} is cofactor of $(i,j)^{th}$ element of B, then

$B_{11} = 5$, $B_{12} = -2$, $B_{21} = -5$ and $B_{22} = 4$

$\therefore$ adj A = $\begin{bmatrix} 0 & -2 \\ -4 & 3 \end{bmatrix}$ and adj B = $\begin{bmatrix} 5 & -5 \\ -2 & 4 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{8}\begin{bmatrix} 0 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1/2 & -3/8 \end{bmatrix}$

and $B^{-1} = \frac{1}{|B|}\text{adj } B = \frac{1}{10}\begin{bmatrix} 5 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/5 & 2/5 \end{bmatrix}$

Let $C = AB = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 12+4 & 15+10 \\ 16+0 & 20+10 \end{bmatrix} = \begin{bmatrix} 16 & 25 \\ 16 & 20 \end{bmatrix}$

We have

$C_{11} = 20$, $C_{12} = -16$, $C_{21} = -25$ and $C_{22} = 16$

Also, $|C| = -80 \neq 0$, $\therefore$ C is invertible.

Also, adj C = $\begin{bmatrix} 20 & -25 \\ -16 & 16 \end{bmatrix}$

$$\Rightarrow C^{-1} = \frac{1}{|C|}\text{adj } C = -\frac{1}{80}\begin{bmatrix} 20 & -25 \\ -16 & 16 \end{bmatrix} = \begin{bmatrix} -1/4 & 5/16 \\ 1/5 & -1/5 \end{bmatrix}.$$

$$\text{Hence, } B^{-1}A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 0 & 1/4 \\ 1/2 & -3/8 \end{bmatrix}$$

$$= \begin{bmatrix} -1/4 & 5/16 \\ 1/5 & -1/5 \end{bmatrix} = C^{-1} = (AB)^{-1}$$

Example 9 : Find the inverse of $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

and verify that $A^{-1}A = I_3$.

Solution : Evaluating the cofactors of the elements in the first row of A, we get

$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$, $A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -3$,

and $A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5$,

$\therefore$ $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= (1)(2) + (2)(-3) + (5)(5) = 21$

Since $|A| \neq 0$, A is invertible. Also,

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3$, $A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6$,

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3$, $A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13$,

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = 9$, $A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$,

$$\therefore \text{ adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{ adj}A = \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$$

To verify that this is the inverse of A, we have

$$A^{-1}A = \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{21}+\frac{6}{21}+\frac{13}{21} & \frac{4}{21}+\frac{9}{21}+\frac{-13}{21} & \frac{10}{21}+\frac{3}{21}+\frac{-13}{21} \\ \frac{-3}{21}+\frac{12}{21}+\frac{9}{21} & \frac{-6}{21}+\frac{18}{21}+\frac{9}{21} & \frac{-15}{21}+\frac{6}{21}+\frac{19}{21} \\ \frac{5}{21}-\frac{6}{21}+\frac{1}{21} & \frac{10}{21}-\frac{9}{21}-\frac{1}{21} & \frac{25}{21}-\frac{3}{21}-\frac{1}{21} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Check Your Progress – 3

1. Find the adjoint of each of the following Matrices :

   (i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

2. For A = $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{bmatrix}$, verify that

   $A\ (\text{adj } A) = (\text{adj}.A)\ A = |A|\ I_3.$

3. Find the inverse of A = $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

4. Let A = $\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, Verify that

   $(AB)^{-1} = B^{-1}A^{-1}$.

5. If A = $\begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^2 = A^{-1}$.

   What is Adj A ?

6. Let A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ Prove that $A^2 - 4A - 5I_3 = 0$.

   Hence, obtain A^{-1}

7. Find the condition under which

   A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. Also obtain the inverse of A.

2.5 SYSTEMS OF LINEAR EQUATIONS

We can use matrices to solve a system of linear equations. Let us consider the following m linear equations in n unknowns :

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots . a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots . a_{2n}x_n = b_2$$

.

.

.

. (1)

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{mn}x_n = b_m$$

where $b_1, b_2 \ldots\ldots b_m$ are not all zero.

The $m\times n$ matrix $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ is called the **coefficient matrix** of the system of linear equations. Using it, we can now write these equations as follows :

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_m \end{pmatrix}$$

We can abbreviate the above matrix equation to $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

and X and B are the $n\times 1$ column vectors.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_m \end{pmatrix}$$

Recall that by a solution of (1) we mean a set of values $x_1.x_2\dots\dots\dots.x_n$ which satisfy all the equations in (1) simultaneously.

For example, $x_1 = 2$, $x_2 = -1$ is a solution of the system of linear equations.

$$3x_1 - 5x_2 = 11$$
$$2x_1 + 3x_2 = 1$$

because $3(2) - 5(-1) = 11$ and $2(2) + 3(-1) = 1$.

Also, recall that the system of linear equations (1) is said to be consistent if it has at least one solution; it is *inconsistent* if it has no solution.

For example, the system of linear equations

$$\begin{aligned} 3x + 2y &= 5 \\ 6x + 4y &= 10 \end{aligned} \qquad (2)$$

is consistent. In fact, $x = k$, $y = \frac{1}{2}(5-2k)$ ($k \in C$) satisfies (2) for all values of $k \in C$. However, the system of linear equations.

$$\begin{aligned} 3x + 2y &= 5 \\ 6x + 4y &= 11 \end{aligned} \qquad (3)$$

is inconsistent. If this system has a solution $x = x_0$, $y = y_0$, then $3x_0 + 2y_0 = 5$ and $6x_0 + 4y_0 = 11$. Multiplying the first equation by 2 and subtracting from the second equation, we get $0 = 1$, which is not possible. Thus, the system in (3) has no solution and hence is an consistent.

Solution of AX= B (A non-singular)

Let us consider the system of linear equations $AX = B$, where A is an $n\times n$ matrix. Suppose that A is non-singular. Then A^{-1} exists and we can pre multiply $AX = B$ by A^{-1} on both sides to obtain

$$A^{-1}(AX) = A^{-1}(B)$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B \quad \text{[associative law]}$$
$$\Rightarrow I_nX = A^{-1}B \quad \text{[property of law]}$$
$$\Rightarrow X = A^{-1}B \quad \text{[property of identity matrix]}$$

Moreover, we have

$$A(A^{-1}B) = (AA^{-1})B \quad \text{[associative law]}$$
$$= I_nB \quad \text{[property of inverse]}$$
$$= B.$$

That is $A^{-1}B$ is a solution of $AX = B$. Thus, if A is non-singular, the system of equations $AX = B$ has a unique solution. This unique solution is given by $X = A^{-1}B$.

Example 10 : Solve the following system of equations by the matrix inverse method :

$$x + 2y = 4, \quad 2x + 5y = 9.$$

Solution : We can put the given system of equations into matrix notation as follows :

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

Here the coefficient matrix is give by $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$.

To check if A^{-1} exists, we note that $A_{11} = (-1)^{1+1}|5| = 5$ and $A_{12} = (-1)^{1+2}|2| = -2$.

$\therefore \ |A| = a_{11}A_{11} + a_{12}A_{12} = (1)(5) + (2)(-2) = 1 \neq 0.$

Since $|A| \neq 0$ A is non- singular (invertible). We also have $A_{21} = (-1)^{2+1}|2| = -2$. $A_{22} = (-1)^{2+2}|1| = 1$. Therefore, the adjoint of A is

$$\text{adj } A = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\therefore X = A^{-1}B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 20 & -18 \\ -8 & +9 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ or } x = 2, \ y = 1.$$

Example 11 : Solve the following system of equations by using matrix inverse :

$$3x + 4y + 7z = 14,\ 2x - y + 3z = 4,\ 2x + 2y - 3z = 0$$

Solution : We can put the given system of equations into the single matrix equation $AX = B$, where

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

The cofactors of |A| are

$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} = -3 \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} = 9$

and $A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$.

$\therefore\ |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (3)(-3) + 4(9) + 7(5) = 62.$

Since $|A| \neq 0$, A is non−singular (invertible). Its remaining cofactors are

$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 7 \\ 2 & -3 \end{vmatrix} = 26, \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 7 \\ 1 & -3 \end{vmatrix} = -16,$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2, \quad A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 7 \\ -1 & -3 \end{vmatrix} = 19,$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 7 \\ 2 & 3 \end{vmatrix} = 5, \quad A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11.$

The adjoint of matrix A is given by

$$\text{adj } A = \begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}$$

$$\text{Also, } X = A^{-1}B = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -42 + 104 \\ 126 - 64 \\ 70 - 8 \end{pmatrix} = \frac{1}{62}\begin{pmatrix} 62 \\ 62 \\ 62 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence $x = 1$, $y = 1$, $z = 1$ is the required solution.

Example 12 : If $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$

are two square matrices, verify that AB = BA = $6I_3$. Hence, solve the system of linear equations : $x - y = 3$, $2x+3y+4z = 17$, $y + 2z = 7$.

Solution :

$$AB = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2+2 & 0-4+10 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6I_3$$

$$\text{and } BA = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4+0 & -2+6-4 & 0+8-8 \\ -4+4+0 & 4+6-4 & 0+8-8 \\ 2-2+0 & -2-3+5 & 0-4+10 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6I_3$$

Thus, AB = BA = $6I_3$

$$\Rightarrow \quad A\left(\frac{1}{6}B\right) = \left(\frac{1}{6}B\right)A = I_3$$

This shows that $A^{-1} = \frac{1}{6}B$. Now the given system of equations can be written as

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

or AX = C, where

$$\therefore \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } C = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$X = A^{-1} C = \frac{1}{6}BC \qquad \left[\because A^{-1} = \frac{1}{6}B\right]$$

$$= \frac{1}{6}\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 34 & -28 \\ -12 & +34 & -28 \\ 2 & -17 & +35 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

Thus, $x = 2$, $y = -1$, $z = 4$ is the required solution.

Solution of a system of Homogeneous Linear Equations :

These are equations of the type $AX = O$. Let us consider the system of n homogeneous equations in n unknowns

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{1n}x_n = 0$$

$$a_{21}\, x_1 + a_{22}\, x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{n1}\, x_1 + a_{n2}\, x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{nn}\, x_n = 0$$

We can write this system as follows

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_n \\ a_{12} & a_{22} & \ldots & a_{2n} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We now abbreviate the above matrix equation to AX = 0, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_n \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & & & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

and X and O are the $n \times 1$ column vectors $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

If A is non-singular, then pre multiplying AX = O by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}O$$

$$\Rightarrow \quad (A^{-1}A)X = O \quad \text{[associative law]}$$
$$\Rightarrow \quad I_nX = O \quad \text{[property of inverse]}$$

$$\Rightarrow \quad X = O \quad \text{[property of identity matrix]}$$
$$\Rightarrow \quad x_1 = 0, x_2 = 0, \ldots\ldots\ldots x_n = 0.$$

Also, note that $x_1 = 0$, $x_2 = 0$, ………, $x_n = 0$ clearly satisfy the given system of homogeneous equations.

Thus, when A is non-singular AX = O has the unique solution. $x_1 = 0$, $x_2 = 0$, ……… $x_n = 0$. This is called the ***trivial solution.***

Important Result

We now state the following results without proof :

1. If A is singular, then AX = O has an infinite number of solutions.
2. Conversely, if AX = 0 has an infinite number of solution, then A is a singular matrix.

Example 13 : Solve the following system of homogeneous linear equations by the matrix method :

$$2x - y + z = 0, \quad 3x + 2y - z = 0, \; x + 4y + 3z = 0$$

Solution :

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 4 & 3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The cofactors of |A| are

$$A_{11=} (-1)^{1+1} \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = 10$$

$$A_{12=} (-1)^{1+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -10$$

and $A_{13=} (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10.$

$$\therefore |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (2)(10) + (-1)(-10) + 1(10) = 40.$$

Since $|A| \neq 0, A$ is non $-$ singular (invertible). This, by known result
$X = O$, that $x = 0, y = 0, z = 0$.

Example 14 : Solve the following system of homogeneous linear equation by the matrix method :

$$2x - y + 2z = 0, \quad 5x + 3y - z = 0, \quad x \ +5y - 5\,z = 0$$

Solution :

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 5 & 3 & -1 \\ 1 & 5 & -5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The cofactors of |A| are

$$A_{11=}\ (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 5 & -5 \end{vmatrix} = -10$$

$$A_{12=}\ (-1)^{1+2} \begin{vmatrix} 5 & -1 \\ 1 & -5 \end{vmatrix} = 24$$

$$\text{and} \quad A_{13=}\ (-1)^{1+3} \begin{vmatrix} 5 & 3 \\ 1 & 5 \end{vmatrix} = 22.$$

$$\therefore\ |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (2)(-10) + (-1)(24) + (2)(22) = 0.$$

Therefore, A is singular matrix. We can rewrite the first two equation as follows: $2x - y = -2z$, $5x + 3y = z$ or in the matrix form as

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -2z \\ z \end{bmatrix}.$$

Now, we have $A_{11} = (-1)^{1+1}|3| = 3$ and $A_{12} = (-1)^{1+2}|5| = -5$.

$$\therefore |A| = a_{11}A_{11} + a_{12}A_{12} = (2)(3) + (-1)(-5) = 11 \neq 0.$$

Thus, A is non singular (invertible). Also, $A_{21} = (-1)^{2+1}|-1| = 1$ and $A_{22} = (-1)^{2+2}|2| = 2$. Therefore, the adjoint of A is given by

$$\text{adj } A = \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}.$$

Therefore, from X = $A^{-1}B$, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} -2z \\ z \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -6z + z \\ 10z + 2z \end{pmatrix} = \begin{bmatrix} -\frac{5}{11}z \\ \frac{12}{11}z \end{bmatrix}$$

$$\Rightarrow x = -\frac{5}{11}z, \quad y = \frac{12}{11}z.$$

Let us check if these values satisfy the third equation. We have

$$x + 5y - 5z = -\frac{5}{11}z + 5\left(\frac{12}{11}z\right) - 5z = \frac{1}{11}z\,(-5z + 60z - 55z) = 0.$$

Thus, all the equation are satisfied by the values

$$\Rightarrow \quad x = -\frac{5}{11}z, \quad y = \frac{12}{11}z, \qquad z = z.$$

Where z is any complex number. Hence, the given system of equation has an infinite number of solutions.

Solutions of AX = B (A Singular)

We state the following result without proof :

If A is a singular, that is |A| = 0, and

1. (adjA) B = 0, then AX = B has an infinite number of solutions (consistent).
2. (adj A) B ≠ 0, then AX = B has no solution (inconsistent).

Example 15 : Solve the following system of linear equation by the matrix method :

$$2x - y + 3z = 5, \quad 3x + 2y - z = 7\ , 4x + 5y - 5z = 9$$

Solution :

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Here, |A| = 0

∴ A is singular matrix. By calculating all the cofactors of A, we can write the adjoint of A. We have

$$adj\ A = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}$$

$$\Rightarrow (adj\ A)B = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, AX = B has an infinite number of solutions. To find these solutions, we write $2x - y = 5 - 3z$, $3x + 2y = 7 + z$ or as a single matrix equation

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -3z \\ 7 & +z \end{bmatrix}$$

Here, |A| = 7 ≠ 0

Since |A| ≠ 0, A is an invertible matrix

Now, adj A $= \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{ adj A} = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

Therefore, from X = $A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5 & -3z \\ 7 & +z \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{17-5z}{7} \\ \dfrac{-1+11z}{7} \end{bmatrix}$$

$$\Rightarrow \; x = \frac{17-5z}{7}, \qquad y = \frac{1}{7}(-1+11z).$$

Let us check that these values satisfy the third equation. We have

$$4x + 5y - 5z = \frac{4}{7}(17 - 5z) + \frac{5}{7}(-1+11z) - 5z$$

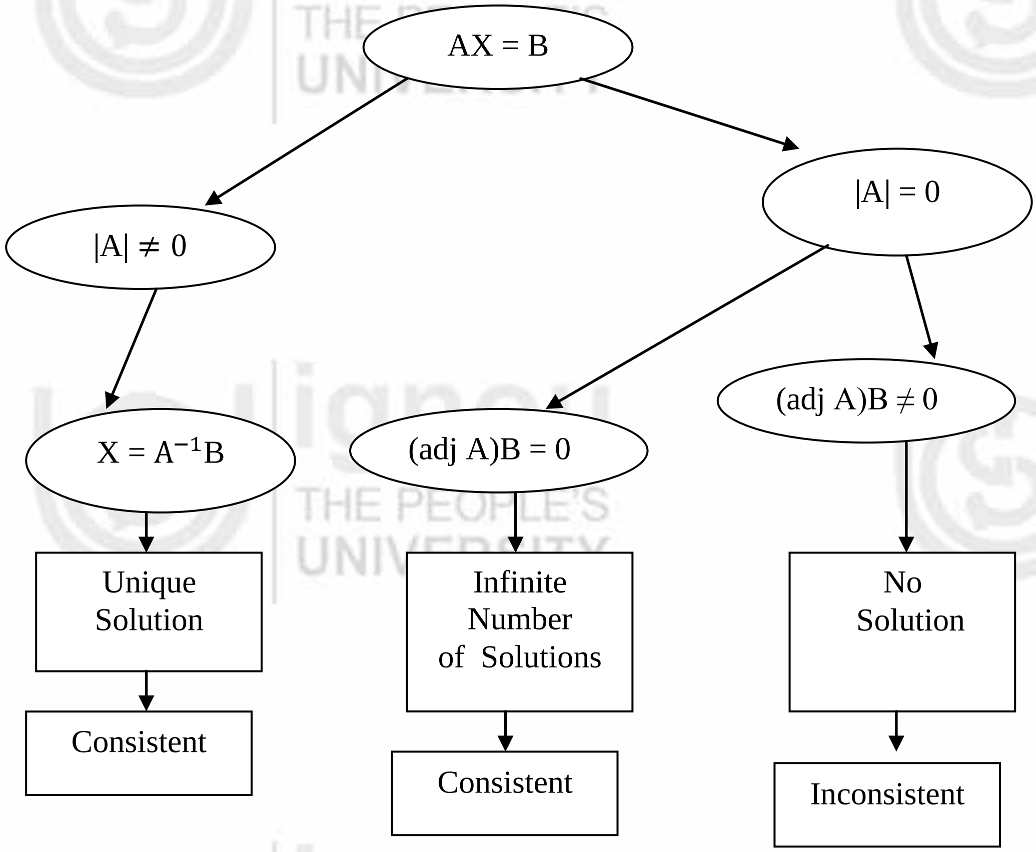
$$= \frac{1}{7}(68 - 20z - 5 + 55z - 35z) = 9.$$

Thus, the values

$$x = \frac{1}{7}(17-5z), y = \frac{1}{7}(-1+11z), \; z = z \quad (z \in C)$$

Satisfy, the given system which therefore has an infinite number of solutions.

In the end, we summarize the results of this section for a square matrix A in the form of a tree diagram.



Check Your Progress 4

1. Solve the following equations by matrix inverse method :
   $4x - 3y = 5,\ 3x - 5y = 1$
2. Use the matrix inverse to solve the following system of equations :
   (a) $x + y - z = 3,\ 2x + 3y + z = 10,\ 3x - y - 7z = 1$
   (b) $8x + 4y + 3z = 18,\ 2x + y + z = 5,\ x + 2y + z = 5$
3. Solve the following system of homogeneous linear equations by the matrix method :
   $3x - y + 2z = 0,\ 4x + 3y + 3z = 0,\ 5x + 7y + 4z = 0$
4. Solve the following system of linear equations by the matrix method :
   $3x + y - 2z = 7$
   $5x + 2y + 3z = 8$
   $8x + 3y + 8z = 11$

2.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. Note that a 2×2 matrix is given by

   $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

   From the formulas given the elements, we have

   (a) $A = \begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

2. From equality of matrices, we have,
   $x = 2x + y,\ y = x - y$

   Solving we get $x = 0$, $y = 0$

3. We have
   $a - b = 5 \quad 2c + d = 3$
   $2a - b = 12 \quad 2a + d = 15$
   Solving we get $a = 7$, $b = 2$, $c = 1$ and $d = 1$.

4. (a) $A' = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} = A$

   $\therefore$ A is symmetric matrix

   (b) $A' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = -A$

   $\therefore$ A is skew – symmetric matrix

(c) $A' = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -5 \\ 3 & 5 & 0 \end{bmatrix} = -A$

$\therefore$ A is skew − symmetric matrix

Check Your Progress - 2

1. Since P and Q are matrices of order 2× 2, 5P + 3Q is a matrix of order 2×2 and therefore R must be a matrix or order 2 × 2.

Let $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$5P + 3Q + 2R = 5\begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix} + 3\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} + 2\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 45 & 5 \\ 35 & 40 \end{bmatrix} + \begin{bmatrix} 3 & 15 \\ 21 & 36 \end{bmatrix} + \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$$

$$= \begin{bmatrix} 48+2a & 20+2b \\ 56+2c & 76+2d \end{bmatrix}$$

Since $5P + 3Q + 2R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we get

$$48 + 2a = 0, \; 20 + 2b = 0, \; 56 + 2c = 0, \; 76 + 2d = 0$$

$$\Rightarrow \quad a = -24, b = -10, c = -28 \text{ and } d = -36.$$

Thus, $R = \begin{bmatrix} -24 & -10 \\ -28 & -36 \end{bmatrix}$

2. We have $(A+B)^2 = (A+B)(A+B)$

$= (A+B)A + (A+B)B$ (Distributive Law)

$= AA + BA + AB + BB$

$= A^2 + BA + AB + B^2$

Therefore, $(A+B)^2 = A^2 + B^2$

$\Rightarrow A^2 + BA + AB + B^2 = A^2 + B^2$

$\Rightarrow BA + AB = 0.$

Thus, we must find a and b such that BA + AB = 0.

We have $BA = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$

and $AB = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

Therefore,

$$BA + AB = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} + \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix}$$

But BA + AB = 0

$\Rightarrow$ $2a - b + 2 = 0$, $-a + 1 = 0$, $2a - 2 = 0$, $-b + 4 = 0$

$\Rightarrow a = 1, b = 4$

3. In view of discussion in solution (2), it is sufficient to show that BA + AB = 0

We have BA = $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

and AB = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Thus, BA + AB = $\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. First, we note that by f(A) we mean

$A^2 - 5A + 6I_3$. We have

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & 2 \end{bmatrix}$$

Therefore,

$$A^2 - 5A + 6I_3 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

5. We have A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Therefore, $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^4 = A^2 \; A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

and $A^5 = A^4 \, A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

6. In general matrix multiplication is not commulative. Therefore, AB may not be equal to BA, even though both of them exist.

Check Your Progress – 3

1. (i) Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The cofactors are

$A_{11=}\ (-1)^{1+1}\ |d| =\ d$

$A_{12=}\ (-1)^{1+2}\ |c| =\ -c$

$A_{21=}\ (-1)^{1+2}\ |b| =\ -b$ and

$A_{22=}\ (-1)^{2+2}\ |a| =\ a$

$\therefore$ adj A = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

(ii) Let A = $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

The cofactors of the elements of A are

$A_{11=}\ (-1)^{1+1}\ \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} =\ -1,$ $A_{12=}\ (-1)^{1+2}\ \begin{bmatrix} 0 & 2 \\ -1 & 5 \end{bmatrix} =\ -2,$

$A_{13=}\ (-1)^{1+3}\ \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} =\ 1,$ $A_{21=}\ (-1)^{2+1}\ \begin{bmatrix} -1 & 3 \\ 3 & 5 \end{bmatrix} =\ 14,$

$A_{22=}\ (-1)^{2+2}\ \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} =\ 13,$ $A_{23=}\ (-1)^{2+3}\ \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} =\ -5,$

$A_{31=}\ (-1)^{3+1}\ \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} =\ -5,$ $A_{32=}\ (-1)^{3+2}\ \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} =\ -4,$

$A_{33=}\ (-1)^{3+3}\ \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} =\ 2.$

$\therefore$ adj A = $\begin{bmatrix} -1 & 14 & 5 \\ -2 & 13 & -4 \\ 1 & -5 & 2 \end{bmatrix}$

2. For the given matrix A, we have

adj A = $\begin{bmatrix} 5 & 3 & -10 \\ 14 & 2 & -2 \\ -3 & -7 & 6 \end{bmatrix}$

Now, A (adj A) = $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & -10 \\ 14 & 2 & -2 \\ -3 & -7 & 6 \end{bmatrix}$

$$= \begin{bmatrix} -26 & 0 & 0 \\ 0 & -26 & 0 \\ 0 & 0 & -26 \end{bmatrix} = -26 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, (adj A) A = $\begin{bmatrix} 5 & 3 & -10 \\ 14 & 13 & -2 \\ -3 & -7 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{bmatrix}$

$$= -26 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also, |A| = – 26

So, A (adj A) = (adj A) A = |A| I_3.

3. Here, |A| = (2) (–1) + (–1) (–2) + (3) (1) = 3

and adj A = $\begin{bmatrix} -1 & 14 & 5 \\ -2 & 13 & -4 \\ 1 & -5 & 2 \end{bmatrix}$ (see solution 1(ii))

$\therefore\ A^{-1} = \dfrac{1}{|A|}$ adj A

$$= \frac{1}{3}\begin{bmatrix} -1 & 14 & 5 \\ -2 & 13 & -4 \\ 1 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -1/3 & 14/3 & 5/3 \\ -2/3 & 13/3 & -4/3 \\ 1/3 & -5/3 & 2/3 \end{bmatrix}.$$

4. We have |A| = –4 and |B| = 20. So, A and B are both invertible.
Also, adj A = $\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix}$ and adj B = $\begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$.

$$\Rightarrow\ A^{-1} = \frac{1}{|A|}\text{ adj } A = -\frac{1}{4}\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1 & -3/4 \end{bmatrix}$$

and $B^{-1} = \dfrac{1}{|B|}\text{ adj } B = \dfrac{1}{20}\begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ -1/10 & 1/5 \end{bmatrix}$

Let C = AB = $\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 16 & 0 \end{bmatrix}$

So, |C| = –80 and adj C = $\begin{bmatrix} 0 & -5 \\ -16 & 14 \end{bmatrix}$

So, $C^{-1} = \dfrac{-1}{80}\begin{bmatrix} 0 & -5 \\ -16 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 1/16 \\ 1/5 & -7/40 \end{bmatrix}$

Hence, $B^{-1}A^{-1} = \begin{bmatrix} 1/4 & 0 \\ -1/10 & 1/5 \end{bmatrix}\begin{bmatrix} 0 & 1/4 \\ 1 & -3/4 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1/16 \\ 1/5 & -7/40 \end{bmatrix} = C^{-1} = (AB)^{-1}$$

5. We have

$$A^2 = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

To show that $A^2 = A^{-1}$, it is enough to show that $A(A^2) = I_3$. We have

$$A(A^2) = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

From $A^{-1} = \frac{1}{|A|}$ adj A, we get adj A $= |A|A^{-1}$. To obtain $|A|$, observe that $A^3 = I_3$ that is, $|A|^3 = |I_3| = 1$ or $|A| = 1$. Therefore adj A $= A^{-1}$

6. We have

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Therefore,

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Also, $|A| = 5 \neq 0$. Therefore, A is invertible pre− multiplying $A^2 - 4A - 5I_3 = 0$ by A^{-1}, we get

$$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I_3 = 0$$

$$\Rightarrow A - 4I_3 - 5A^{-1} = 0$$

$$\therefore \quad 5A^{-1} = A - 4I_3 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$$

7. We have $|A| = ad - bc$. Recall that A is invertible if and only if $|A| \neq 0$. That is A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.

Also, adj A $= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Hence $A^{-1} = \frac{1}{|A|}$ adj $A = \frac{1}{ad - bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Check Your Progress – 4

1. We can put the given system of equations into the single matrix equation.

$$\begin{bmatrix}4 & -3\\3 & -5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}.$$

Here the coefficient matrix is given by A = $\begin{bmatrix}4 & -3\\3 & -5\end{bmatrix}$

Cofactors of |A| are $A_{11} = (-1)^{1+1}(-5)$ and $A_{12} = (-1)^{1+2}|3| = -3.$

$\therefore$ |A| = $a_{11}\,A_{11} + a_{12}\,A_{12} = (4)(-5) + (-3)(-3) = -11$

Since |A| ≠ 0. A is non-singular (invertible). Also $A_{21} = (-1)^{2+1}|-3|$and

$A_{12} = (-1)^{2+2}|4|$ = 4.

$$\therefore \quad \text{adj A} = \begin{pmatrix}-5 & 3\\-3 & 4\end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{-1}{11}\begin{pmatrix}-5 & 3\\-3 & 4\end{pmatrix}$$

$$\therefore \quad X = A^{-1}B = \frac{-1}{11}\begin{pmatrix}-5 & 3\\-3 & 4\end{pmatrix}\begin{bmatrix}5\\1\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \frac{-1}{11}\begin{pmatrix}-25+3\\-15+4\end{pmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

Hence x = 2, y = 1 is the required solution.

2. (a) We can put the above system of equation into the single matrix equation AX = B, where

$$A = \begin{pmatrix}1 & 1 & -1\\2 & 3 & 1\\3 & -1 & -7\end{pmatrix},\ X = \begin{pmatrix}x\\y\\z\end{pmatrix} \text{ and } B = \begin{pmatrix}3\\10\\1\end{pmatrix}$$

The cofactors of |A| are

$$A_{11=}\ (-1)^{1+1}\begin{vmatrix}3 & 1\\-1 & -7\end{vmatrix} = -20 \qquad A_{12=}\ (-1)^{1+2}\begin{vmatrix}2 & 1\\3 & -7\end{vmatrix} = 17$$

and $A_{13\,=}\ (-1)^{1+3}\begin{vmatrix}2 & 3\\3 & -1\end{vmatrix} = -11.$

$\therefore\ a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1(-20) + (1)(17) + (-1)(-11) = 8.$

Since |A| ≠ 0, A is non- singular (invertible). The remaining cofactors are

$$A_{21=}\ (-1)^{2+1}\begin{vmatrix}1 & -1\\-1 & 7\end{vmatrix} = 8 \quad A_{22=}\ (-1)^{2+2}\begin{vmatrix}1 & -1\\3 & -7\end{vmatrix} = -4,$$
$$A_{23=}\ (-1)^{2+3}\begin{vmatrix}1 & 1\\3 & -1\end{vmatrix} = 4 \quad A_{31=}\ (-1)^{3+1}\begin{vmatrix}1 & -1\\3 & 1\end{vmatrix} = 4,$$

$$A_{32=}\ (-1)^{3+2}\begin{vmatrix}1 & -1\\2 & 1\end{vmatrix} = -3 \quad A_{33=}\ (-1)^{3+3}\begin{vmatrix}1 & 1\\2 & 3\end{vmatrix} = 1.$$

$$\therefore \ \text{adj A} = \begin{pmatrix} -20 & 18 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{8} \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}$$

$$\text{Also, X} = A^{-1}B = \frac{1}{8} \begin{pmatrix} -20 & 18 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}$$

$$= \frac{1}{8} \begin{pmatrix} -60 + 80 + 4 \\ 51 - 40 - 3 \\ -33 + 40 + 1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 24 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Thus, $x = 3$, $y = 1$, $z = 1$ is the required solution.

2. (b) We can put the above system of equation into the single matrix equation AX = B, where

$$\text{A} = \begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad \text{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and B} = \begin{pmatrix} 18 \\ 5 \\ 5 \end{pmatrix}.$$

The cofactors of |A| are

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \qquad A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$\text{and} \quad A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

$$\therefore \ |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 8(-1) + 4(-1) + 3(3) = -3.$$

Since |A| ≠ 0, A is non−singular (invertible). The remaining cofactors are

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 2 \qquad A_{22} = (-1)^{2+2} \begin{vmatrix} 8 & 3 \\ 1 & 1 \end{vmatrix} = 5,$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix} = -12 \qquad A_{31} = (-1)^{3+1} \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 1,$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix} = -2 \qquad A_{33} = (-1)^{3+3} \begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix} = 0.$$

$$\therefore \ \text{adj A} = \begin{pmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{-3} \begin{pmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{pmatrix}.$$

$$\text{Also, } X = A^{-1}B = \frac{1}{-3}\begin{pmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{pmatrix}\begin{pmatrix} 18 \\ 5 \\ 5 \end{pmatrix}$$

$$= \frac{1}{-3}\begin{pmatrix} -18+10+5 \\ -18+25-10 \\ 54-60+0 \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} -3 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Thus, $x = 1$, $y = 1$, $z = 2$ is the required solution.

3. We can put the above system of equation into the single matrix equation AX = 0, where

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 3 & 3 \\ 5 & 7 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The cofactors of |A| are

$$A_{11=} (-1)^{1+1}\begin{vmatrix} 3 & 3 \\ 7 & 4 \end{vmatrix} = -9 \qquad A_{12=} (-1)^{1+2}\begin{vmatrix} 4 & 3 \\ 5 & 4 \end{vmatrix} = -1$$

$$\text{and} \quad A_{13=} (-1)^{1+3}\begin{vmatrix} 4 & 3 \\ 5 & 7 \end{vmatrix} = 13.$$

$$\therefore \quad |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (3)(-9) + (-1)(-1) + (2)(13) = 0.$$

Therefore, A is a singular matrix. We can rewrite the first two equations as follows :

$3x - y = -2z$, $4x + 3y = -3z$

or in the matrix form as $\begin{pmatrix} 3 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2z \\ -3z \end{pmatrix}$.

Now, we have $A_{11} = (-1)^{1+1}\,|3| = 3$ and $A_{12} = (-1)^{1+2}\,|4| = -4$.

$$\therefore \quad |A| = a_{11}A_{11} + a_{12}A_{12} = (3)(3) + (-1)(-4) = 13 \neq 0.$$

Thus, A is non- singular (invertible).

Also, $A_{21=} (-1)^{2+1}|-1| = 1$ and $A_{22} = (-1)^{2+2}\,|3| = 3$.

$$\therefore \text{adj}A = \begin{pmatrix} 3 & 1 \\ -4 & 3 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{13}\begin{pmatrix} 3 & 1 \\ -4 & 3 \end{pmatrix}$$

Therefore, from $X = A^{-1}B$, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 3 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} -2z \\ -3z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -6z-3z \\ 8z-9z \end{pmatrix} = \begin{pmatrix} -\frac{9}{13}z \\ -\frac{1}{13}z \end{pmatrix}.$$

$$\Rightarrow x = -\frac{9}{13}z, \qquad y = -\frac{1}{13}z.$$

Let us check if these values satisfy the third equation. We have

$$5x + 7y + 4z = 5\left(-\frac{9}{13}z\right) + 7\left(-\frac{1}{13}z\right) + 4z$$

$$= \frac{z}{13}(-45 - 7 + 52) = \frac{0}{13}z = 0.$$

Thus, all the equations are satisfied by the values

$$x = -\frac{9}{13}z, \qquad y = -\frac{1}{13}z, \qquad z = z.$$

4. We can write the given system of linear equation as the single matrix equation.

AX = B,

Where

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 5 & 2 & 3 \\ 7 & 3 & 8 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 \\ 8 \\ 11 \end{pmatrix}$$

Here, |A| = 0

Therefore, A is a singular matrix.

$$\text{Now adj } A = \begin{pmatrix} 7 & -14 & 7 \\ -19 & 38 & -9 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\Rightarrow (\text{adj } A)B = \begin{pmatrix} 7 & -14 & 7 \\ -19 & 38 & -19 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 14 \\ -38 \\ 2 \end{pmatrix}$$

Since (adj A) B ≠ 0, the given system of equations has no solution (inconsistent).

2.7 SUMMARY

In this unit, first of all, definition and notation of an m x n matrix, are given in **section 2.2**. Next, in this section, special types of matrices, viz., square matrix, diagonal matrix, scalar matrix, unit or identity matrix, row or column matrix and zero or null matrix are also defined. Then, equality of two matrices, transpose of a matrix, symmetric and skew matrices are defined. Each of the above concepts is explained with a suitable example. In **section 2.3**, operations like addition, subtraction, multiplication of two matrices and multiplication of a matrix with a scalar are defined. Further, properties of these matrix operations are stated without proof. Each of these operations is explained with a suitable example. In **section 2.4**, the concepts of an invertible matrix, cofactors of a matrix, adjoint of a square matrix are defined and explained with suitable examples. Finally, in **section 2.5**, method of solving linear equations in n variables using matrices, is given and illustrated with a number of suitable examples. Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.6**.

UNIT 3 MATRICES - II

Structure

3.0 INTRODUCTION

In Unit 2, we have introduced Matrices. In this Unit, we shall study elementary operation on Matrices. There are basically three elementary operations. Scaling, Interchange and Replacement. These operations are called elementary row operations or elementary column operations according as they are performed on rows and columns of the matrix respectively. Elementary operations play important role in reducing Matrices to simpler forms, namely, triangular form or normal form. These forms are very helpful in finding rank of a matrix, inverse of a matrix or in solution of system of linear equations. Rank of a matrix is a very important concept and will be introduced in this unit. We shall see that rank of a matrix remains unaltered under elementary row operations. This provides us with a useful tool for determining the rank of a givne matrix. We have already defined inverse of a square matrix in Unit 2 and discussed a method of finding inverse using adjoint of a matrix. In this unit, we shall discuss a method of finding inverse of a square matrix using elementary row operations only.

3.1 OBJECTIVES

After studying this Unit, you should be able to :

- define elementary row operations;
- reduce a matrix to triangular form using elementary row operations;
- reduce a matrix to normal form using elementary operations;
- define a rank of a matrix;
- find rank of a matrix using elementary operations;
- find inverse of a square matrix usng elementary row operations.

3.2 ELEMENTARY ROW OPERATIONS

Consider the matrices of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Matrices B, C and D are related to the matrix A as follows :

- Matrix B can be obtained from A by multiplying the first row of A by 2;
- Matrix C can be obtained from A by interchanging the first and second rows;
- Matrix D can be obtained from A by adding twice the second row the first row.

Such operations on the rows of a matrix are called elementary operations.

Definitions : An elementary row operations is an operation of any one of the following three types :

1. **Scaling :** Multiplication of a row by a non zero constant.
2. **Interchange :** Interchange of two rows.
3. **Replacement :** Adding one row to a multiple of another row.

We denote scaling by $R_i \to kR_i$, interchange by $R_i \leftrightarrow R_j$ and replacement by $R_i \to R_i + kR_j$.

Thus, the matrices B, C and D are obtained from matrix A by applying elementary row operations $R_1 \to 2R_1$ $R_1 \leftrightarrow R_2$ and $R_1 \to R_1 + 2R_2$ respectively.

Definiton : Two matrices A and B are said to be row equivalent, denoted by A ~ B, if one can be obtained from the other by a finite sequence of elementary row operations.

Clearly, matrices B, C and D discussed above are row equivalent to the matrices A and also to each other by the following remark.

Remark : If A, B and C are three matrices, then the following is obvious.

1. A ~ A
2. If A ~ B, then B ~ A
3. If A ~ B, B ~ C, then A ~ C.

Example 1 : Show that matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is row equivalent to the matrix.

B = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

Solution : We have A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Applying $R_2 \to R_2 - 4\,R_1$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

Applying $R_3 \to R_3 - 7\,R_1$ to the matrix on R. H. S. we get.

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Now Applying $R_3 \to R_3 - 2\,R_2$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = B$$

The matrix B in above example is a triangular matrix.

Definition : A matrix A = $[a_{ij}]$ is called a triangular matrix if $a_{ij}= 0$ whenver $i > j$.

In the above example, we reduced matrix A to the triangular matrix B by elementary row operations. This can be done for any given matrix by the following theorem that we state without proof.

Theorem : Every matrix can be reduced to a triangular matrix by elementary row operations.

Example 2 : Reduce the matrix

$$A = \begin{bmatrix} 5 & 3 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

to triangular form.

Solution : A = $\begin{bmatrix} 5 & 3 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 1 & 2 \end{bmatrix} \quad \text{(by applying } R_1 \leftrightarrow R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & -9 & 0 \end{bmatrix} \quad \text{(by applying } R_3 \rightarrow R_3 - 5R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -25 & -8 \end{bmatrix}$$

which is triangular matrix.

Example 3 : Show that $A = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix}$ is row equivalent to I_3.

Solution :

$$A = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & -5 \\ 1 & 1 & 5 \end{bmatrix} \quad (R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 5 \end{bmatrix} \text{(by } R_2 \rightarrow R_2 \rightarrow 3R_1 \text{ and } R_3 \rightarrow R_3 \rightarrow R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 5 \end{bmatrix} \text{(by } R_1 \rightarrow R_1 - R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 5 \end{bmatrix} \text{(by } R_3 \rightarrow \tfrac{1}{5} R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{(by } R_1 \rightarrow R_1 - 5R_3 \text{ and } R_2 \rightarrow R_2 + 5R_3\text{)}$$

$$= I_3$$

In above example, we have reduced the square matrix A to identity matrix by elementary row operations. Can every square matrix be reduced to identity matrix by elementary row opearations.

The answer, in general, is no, however, if A is a square matrix with $|A| \neq 0$, then A can be reduced to identity matrix by elementary row operations. This we state below without proof.

Theorem : Every non-singular matrix is row equivalent to a unit matrix.

Below we given an algorithm to reduce a non-singular matrix to identity matrix.

1. Make the first element of first column unity by scaling. If the first element is zero the first make use of interchange.
2. Make all elements of first column below the first element zero by using replacement.
3. Now make the second element of second column unity and all other elements zero.
4. Continue the process column by column to get an identity matrix.

The following example illustrate the process.

Example 4 : Reduce the matrix $\begin{bmatrix} 0 & 3 & -3 \\ 2 & -4 & 8 \\ -1 & 3 & -3 \end{bmatrix}$ to I_3.

Solution : $\begin{bmatrix} 0 & 3 & -3 \\ 2 & -4 & 8 \\ -1 & 3 & -3 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & -4 & 8 \\ 0 & 3 & -3 \\ -1 & 3 & -3 \end{bmatrix} \quad \text{(by } R_1 \leftrightarrow R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -3 \\ -1 & 3 & -3 \end{bmatrix} \quad \text{(by (}R_1 \leftrightarrow \tfrac{1}{2} R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{(by } R_3 \rightarrow R_3 + R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{(by } R_2 \rightarrow \tfrac{1}{3} R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{(by } R_1 \rightarrow R_1 + 2R_2 \text{ and } R_3 \rightarrow R_3 - R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(by and } R_3 \rightarrow \tfrac{1}{2} R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(by } R_1 \rightarrow R_1 - 2R_3 \text{ and } R_2 \rightarrow R_2 + R_3\text{)}$$

Check Your Progress – 1

1. Write the Matrices obtained by applying the following elementary row operations on

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(i) $R_1 \leftrightarrow R_3$

(ii) $R_2 \rightarrow R_2 + 3R_1$

(iii) $R_2 \rightarrow R_3$, then $R_2 \rightarrow 2R_2$ and then $R_3 \rightarrow R_3 + 2R_1$

2. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 8 \\ 6 & 7 & 2 \end{bmatrix}$ to triangular form.

3. Show that $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ is row equivalent to I_3.

4. Is the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$ row equivalent to I_3.

5. Which of the following is row equivalent to I_3.

(a) $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \\ -1 & 6 & 2 \end{bmatrix}$

3.3 RANK OF A MATRIX

Suppose A is an $m \times n$ matrix. We can obtain square sub matrices of order r ($0 < r \leq$ least of m and n) from A by selecting the elements in any r rows and r columns of A. We define rank of matrix as follows :

Defintion : Let A be an $m \times n$ matrix. The order of the largest square submatrix of A whose determinant has a non-zero value is called the **'rank'** of the matrix A. The rank of the zero matrix is defiend to be zero.

It is clear from the definition that the rank of a square matrix is r if and only if A has a square submatrix of order r with nonzero determinant, and all square sub matrices of large size have determinant zero.

Example 5 : Find the rank of the matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 3 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution : Since A is a square matrix, A is itself a square submatrix of A.

Also, $|A| = \begin{vmatrix} 0 & 1 & -1 \\ 3 & 1 & 2 \\ 2 & 4 & 6 \end{vmatrix}$

$= -1(18-4) + (-1)(12-2)$

$= -24 \neq 0$

Hence, rank of A is 3.

Example 6 : Determine the rank of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution : Here, $|A| = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$

$= 0$

So, rank of A cannot be 3.

Now $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a square submatrix of A such that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

$\therefore$ rank of A = 2.

Rank and Elementary Operations

The following theorem gives a relationship between rank of a matrix and elementary row operations on the matrix.

Theorem : The rank of a matrix remains unaltered under elementary row operations.

The theorem can be proved by noting that the order of the largest non-singular square submatrix of the matrix is not affected by the elementary row operations. Using properties of determinants, we can see that interchange will only change the sign of determinants of square submatrices, while under scaling values of determinants are multiplied by non zero constant and replacement will not affect the value of the determinant.

Using the above theorem, we can obtain the rank of a matrix A by reducing it to some simpler form, say triangular form or normal form.

Example 7 : Determine the rank of matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

Solution : We first reduce matrix A to triangular form by elementary row operations.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix} \qquad \text{(by } R_1 \leftrightarrow R_2\text{)}$$

$$\sim \begin{bmatrix} 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & 4 & 4 \end{bmatrix} \qquad \text{(by } R_3 \to R_3 - 5R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & 4 \end{bmatrix} \qquad \text{(by } R_3 \to R_3 - 8R_2\text{)}$$

We have thus reduced A to triangular form. The reduced matrix has a square

submatrix $\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$ with non zero

determinant $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{vmatrix} = 1 \times 1 \times (-12) = -12.$

So rank of reduced matrix is 3. Hence rank of A = 3.

Example 8 : Reduce the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 5 \end{bmatrix}$$

to triangular form and hence determine its rank.

Solution : Let us first reduce A to triangular form by using elementary row operations.

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & -6 & 4 & 14 \\ 0 & -9 & 6 & 15 \end{bmatrix} \quad (\text{by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1)$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{by } R_3 \to R_3 - 2R_2,\ R_4 \to R_4 - 3R_2)$$

$$= B$$

Clearly, rank of B cannot B 4; as $|B| = 0$.

Also, $\begin{bmatrix} 2 & 5 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ is a square submatrix

of order 3 of B and $\begin{vmatrix} 2 & 5 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 2 \times (-3) \times 4 = -24 \neq 0$

So, rank of matrix B is 3.

Hence rank of matrix A = 3.

Normal form of a Matrix

We can find rank of a matrix by reducing it to normal form.

Definition : An $m \times n$ matrix of rank r is said to be in normal form if it is of type.

$$\begin{bmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix}$$

For example, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the normal form $\begin{bmatrix} I_2 & O_{2,2} \\ O_{1,2} & O_{1,2} \end{bmatrix}$. We can also

write it as $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Similarly $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is the normal form $[I_3 \quad 0]$.

In section 1, we discussed elementary row operations. We can similarly define elementary column operations also. An elementary operations is either an elementary row operation or an elementary column operation. A matrix A is equivalent to matrix B if B can be obtained from A by a sequence of elementary operations.

Theorem : Every matrix can be reduced to normal form by elementary operations.

We illustrate the above theorem by following example.

Example 9 : Reduce the matrix

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

to normal form by elementary operations.

Solution : $A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 8 \end{bmatrix}$$

Applying $R_3 \to R_3 - 5R_1$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

Applying elementary row operations $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 8R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we apply elementary column operation $C_3 \to C_3 - C_2$, to get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, applying $C_3 \to C_3 - C_1$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have thus reduced A to normal form.

Also, note that the rank of a matrix remains unaltered under elementary operations.

Thus, rank of A in above example is 2 because rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2.

In this regard, we state following theorem without proof.

Theorum : Every matrix of rank r is equivalent to the matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Example 10 Reduce the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$ to its normal form and hence determine its rank.

Solution : We have

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix} \text{ [by } R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 2R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix} \text{ [by } C_3 \to C_3 - 2C_1, \quad C_4 \to C_4 - C_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \text{ [by } R_3 \to R_3 - 3R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ [by } R_3 \to \tfrac{1}{2} R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ [by } C_3 \to C_3 + C_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ [by } C_4 \to C_4 - C_3]$$

Thus, A is reduced to normal form $[I_3 \ 0]$ and hence rank of A is 3.

Check Your Progress – 2

1 By finding a non-zero minor of largest order determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

2 Reduce the matrix $A = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 3 & -1 & 4 & -2 \\ 6 & -1 & 10 & -1 \end{bmatrix}$ to triangular form and hence determine its rank.

3. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$$

by reducing to triangular form.

4. Reduce the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ to its normal form and hence determine its rank.

5. Reduce the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

to its normal form and hence determine its rank.

3.4 INVERSE OF A MATRIX USING ELEMENTARY ROW OPERATIONS

In this section, we shall disucss a method of finding inverse of a square matrix using elementary row operations. We begin by stating the following theorem (without proof) which we require for our discussion.

Theorem : An elementary row operation on the product of two matrices is equivalent to the same elementary row operation on the pre-factor of the product. Recall that an invertible matrix is non singular and that every non singular matrix can be reduced to an identity matrix using elementary row operations only. We now disucss a method of computing inverse of a square matrix using elementary row operations. Let A be an $n \times n$ matrix whose inverse is to be found. Consider the identity $A = I_n A$ where I_n is the identity matrix of order n. Reduce the matrix

A on the L.H.S. to the identity matrix I_n by elementary row operation. Note that this is possible if A is invertible (i.e., non-singular). Now apply all these operations (in the same order) to the pre-factor I_n M the R. H. S. of the identity. In this way, the matrix I_n reduced to some matrix B such that $BA = I_n$. The matrix B so obtained is the inverse of A.

We illustrate the method in the following examples.

Example 11 : Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

using elementary row operations.

Solution : Consider the identity.

$$A = I_2 A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \qquad \text{[by applying } R_2 \rightarrow R_2 - 2R_1\text{]}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} A \qquad \text{[by applying } R_1 \rightarrow R_1 - R_2\text{]}$$

that is, $I_2 = BA$

Where $B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Thus $A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Example 12 : Using elementary row operations find the inverse of the matrix.

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution : Consider

$$A = I_3 A$$

$$\begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \;\; [\text{by } R_1 \rightarrow \;\; R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \;\; [\text{by } R_2 \rightarrow \; R_2 - 2R_1, \;\; R_3 \rightarrow \; R_3 - R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 0 \\ 1 & -2 & 0 \\ 2 & -5 & 1 \end{bmatrix} A \;\; [\text{by } R_1 \rightarrow \; R_1 - 2R_2, \;\; R_3 \rightarrow \; R_3 - 2R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 0 \\ 1 & -2 & 0 \\ -2 & 5 & -1 \end{bmatrix} A \;\; [\text{by } R_3 \rightarrow \; (-1)R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 16 & -40 & 9 \\ -5 & 13 & -3 \\ -2 & -5 & -1 \end{bmatrix} A \; [\text{by } R_1 \rightarrow \; R_1 - 9R_3, R_2 \rightarrow \; R_2 + 3R_3]$$

that is, $I_3 \;\; = BA$

Where B $= \begin{bmatrix} 16 & -40 & 9 \\ -5 & 13 & -3 \\ -2 & 5 & -1 \end{bmatrix} A$

Hence $A^{-1} = \begin{bmatrix} 16 & -40 & 9 \\ -5 & 13 & -3 \\ -2 & 5 & -1 \end{bmatrix}$

Example 13 : Find the inverse, if exists, of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution : Consider

$$A = I_3A$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \;\; [\text{by } R_1 \leftrightarrow \;\; R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \;\; [\text{by } R_3 \rightarrow \; R_3 - 3R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \text{ [by } R_1 \rightarrow R_1 - 2R_2, \quad R_3 \rightarrow R_3 + 5R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \text{ [by } R_3 \rightarrow \tfrac{1}{2}R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \text{ [by } R_1 \rightarrow R_1 + R_3, R_2 \rightarrow R_2 - 2R_3]$$

that is, $I_3 = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$

Hence $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

Example 14 : Find the inverse of A, if it exists, for the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution : Consider the identity

$$A = I_3 A$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Again, applying $R_2 \rightarrow R_2 - 2R_1$ and , $R_3 \rightarrow R_3 + R_1$, we get

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

Again, applying $R_3 \rightarrow R_3 + R_2$, we have

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

Since, we have obtained a row of zeros on the L.H.S., we see that A cannot be reduced to an identity matrix. Thus, A is not invertible, Infact, note that A is a singular matrix as $|A| = 0$.

Check Your Progress – 3

1. Find the inverse of the following Matrices using elementary row operations only.

   (a) $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

2. Using elementary row operations find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$. Find A^{-1} if exists.

4. Find the inverse of matrix A, if it exists, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$.

3.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. (i) $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}$

   (ii) $\begin{bmatrix} 1 & 4 & 7 \\ 5 & 17 & 29 \\ 3 & 6 & 9 \end{bmatrix}$

   (iii) $A \sim \begin{bmatrix} 1 & 4 & 7 \\ 3 & 6 & 9 \\ 2 & 5 & 8 \end{bmatrix}$ (by $R_2 \leftrightarrow R_3$)

   $\sim \begin{bmatrix} 1 & 4 & 7 \\ 6 & 12 & 18 \\ 2 & 5 & 8 \end{bmatrix}$ (by $R_2 \rightarrow 2R_2$)

   $\sim \begin{bmatrix} 1 & 4 & 7 \\ 6 & 12 & 18 \\ 4 & 13 & 22 \end{bmatrix}$ (by $R_3 \rightarrow R_3 \rightarrow 2R_1$)

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 8 \\ 6 & 7 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -7 \\ 0 & -5 & -16 \end{bmatrix} (\text{ by } R_2 \rightarrow R_2 \rightarrow 5R_1 \text{ and } R_3 \rightarrow R_3 - 6R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -16 \end{bmatrix} (\text{ by } R_2 \rightarrow \frac{-1}{7} R_2)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -11 \end{bmatrix} (\text{ by } R_3 \rightarrow R_3 + 5R_2)$$

which is triangular matrix.

Note : There are many other ways and solutions also using different sequence of elementary row operations.

3. Let $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

Then $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ $(\text{ by } R_1 \rightarrow \frac{1}{2} R_1)$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -1 \end{bmatrix} (\text{ by } R_2 \rightarrow R_2 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & -5 & -7 \end{bmatrix} (\text{ by } R_2 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & -12 \end{bmatrix} (\text{ by } R_1 \rightarrow R_1 \rightarrow 2 R_2 \text{ and } R_3 \rightarrow R_3 \rightarrow 5 R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} (\text{ by } R_3 \rightarrow -\frac{1}{12} R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{ by } R_1 \rightarrow R_1 - 5 R_3 \text{ and } R_2 \rightarrow R_2 + R_3)$$

$= I_3$.

4. $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & 3 \end{bmatrix}$

So, $|A| = 1(6 + 2) - 2(-3 - 5) - 3(-2 + 10)$

$= 8 + 16 - 24 = 0$

So, A is a singular matrix. Hence A is not row equivalent to I_3.

5. (a) Let A = $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Then, $|A| = -2(2-0) + 3(0-12)$

$= -40 \neq 0$

So, A is non-singular and hence row-equivalent to I_3.

(b) Let A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \\ -1 & 6 & 2 \end{bmatrix}$

Then, $|A| = -2(6-6) + 0 + 0 = 0$

$\therefore$ A is not row-equivalent to I_3.

Check Your Progress – 2

1. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

So $|A| = 1(1-6) - 2(3-4) + 3(9-2)$

$= -6 + 2 + 21$

$= -17 \neq 0$

So, |A| is largest non zero minor & hence rank A = 3

2. A ~ $\begin{bmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 6 & -1 & 10 & -1 \end{bmatrix}$ (by $R_1 \leftrightarrow R_2$)

~ $\begin{bmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ (by $R_3 \rightarrow R_3 - 2R_1$)

~ $\begin{bmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (by $R_3 \rightarrow R_3 - \frac{1}{2} R_2$)

Where B is a traingular matrix. Clearly every 3– rowed minor of B has value zero.

Also since $\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} = 6 \neq 0$, so rank of B = 2. Hence rank of A = 2.

3. By applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - 3R_1$, we have

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$$

Again, applying $R_4 \to R_4 \frac{-6}{8} R_3$ we have

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Where B is a triangular matrix. Since last row of B consist of zeroes only, therefore rank of B cannot be 4.

$$\text{Also} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -8 \end{bmatrix} = 1 \times 2 \times (-8) = -16 \neq 0$$

$\therefore$ rank of B = 3 and hence, rank of A = 3.

4. $A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ (by $R_1 \leftrightarrow R_2$)

$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$ (by $R_3 \to R_3 - 3R_1$)

$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ (by $R_1 \to R_1 - 2R_2$, by $R_3 \to R_3 + 5R_2$)

$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (by $R_3 - \frac{1}{2} R_3$)

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (by $R_1 \to R_1 + R_3$, $R_2 \to R_2 - 2R_3$)

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (by $C_4 \to C_4 - C_1$)

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{(by } C_4 \to C_4 + C_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{(by } C_4 \to C_4 - C_3\text{)}$$

So normal form of A is $[I_3\ 0]$. Hence rank of A = 3.

5. $$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \quad \text{(by } R_2 \to R_2 - 4R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \end{bmatrix} \quad \text{(by } R_2 \leftrightarrow R_4\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -8 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \text{(by } R_1 \to R_1 + R_2,\ R_3 \to R_3 - 5R_2,\ R_4 \to R_4 - 3R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix} \quad \text{(by } R_3 \leftrightarrow R_4\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -12 \end{bmatrix} \quad \text{(by } R_1 \to R_1 - 2R_3,\ R_4 \to R_4 - 8R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -12 \end{bmatrix} \quad \text{(by } R_4 \to \frac{-1}{12} R_4\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(by } R_1 \to R_1 + 3R_4,\ R_2 \to R_2 - 2R_4,\ R_3 \to R_3 + 2R_4\text{)}$$

$$= I_4 \quad \text{(by } R_4 \to \frac{1}{12} R_4\text{)}$$

So normal form of A is I_4 and hence rank of A = 4.

Check Your Progress – 3

1 (a) A = I_2 A

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix}1 & 2\\0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}A \qquad (\text{by } R_2 \rightarrow R_2 - 3R_1)$$

$$\Rightarrow \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\3/2 & -1/2\end{bmatrix} \qquad (\text{by } R_2 \rightarrow \frac{-1}{2} R_2)$$

$$\Rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1\\3/2 & -1/2\end{bmatrix} \qquad (\text{by } R_1 \rightarrow R_1 - 2R_2)$$

Hence $A^{-1}\begin{bmatrix}-2 & 1\\3/2 & -1/2\end{bmatrix}$

(b) A = I_3 A $\qquad (\text{by } R_2 \rightarrow \frac{-1}{2} R_2)$

$$\begin{bmatrix}1 & 1 & 1\\0 & 2 & 3\\5 & 5 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix}1 & 1 & 1\\0 & 2 & 3\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-5 & 0 & 1\end{bmatrix} A \qquad (\text{by } R_3 \rightarrow R_3 - 5R_1)$$

$$\Rightarrow \begin{bmatrix}1 & 1 & 1\\0 & 1 & 3/2\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1/2 & 0\\-5 & 0 & 1\end{bmatrix} A \qquad (\text{by } R_2 \rightarrow \frac{-1}{2} R_2)$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 3/2\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}1 & -1/2 & 0\\0 & 1/2 & 0\\-5 & 0 & 1\end{bmatrix} A \quad (\text{by } R_1 \rightarrow R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 3/2\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1/2 & 0\\0 & 1/2 & 0\\5/4 & 0 & -1/4\end{bmatrix} A \qquad \left(\text{by } R_3 \rightarrow \frac{1}{4} R_3\right)$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}13/8 & -1/2 & -1/8\\-15/8 & 1/2 & 3/8\\5/4 & 0 & -1/4\end{bmatrix} A \quad \left(\text{by } R_1 \rightarrow R_1 + \frac{1}{2} R_3, R_2 \rightarrow \frac{-3}{2} R_3\right)$$

Hence, $A^{-1}\begin{bmatrix}13/8 & -1/2 & -1/8\\-15/8 & 1/2 & 3/8\\5/4 & 0 & -1/4\end{bmatrix}$

2. $A = I_3 \; A$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad (\text{ by } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{bmatrix} A \qquad (\text{ by } R_3 \to R_3 - 4R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{bmatrix} A \qquad (\text{ by } R_3 \to R_3 + 3R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3/2 & -2 & 1/2 \end{bmatrix} A \qquad \left(\text{by } R_3 \to \frac{1}{2} R_3\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} A \quad (\text{by } R_1 \to R_1 - 3R_3,\ R_2 \to R_2 - 2R_3)$$

3. $A = I_3 \; A$

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1 \\ -3/2 & 1 & 0 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -2 \\ 0 & 0 & 1 \\ -3/2 & 1 & 5 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -2 \\ 0 & 0 & 1 \\ -3/24 & -1/12 & -5/12 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/24 & 5/12 & 1/12 \\ 3/24 & -1/12 & 7/12 \\ 3/24 & -1/12 & -5/12 \end{bmatrix} A$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -3/24 & 5/12 & 1/12 \\ 3/24 & -1/12 & 7/12 \\ 3/24 & -1/12 & -5/12 \end{bmatrix}$$

4. $A = I_3 A$

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & -12 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} A \text{ (by } R_2 \to R_2 - R_1, R_3 \to R_3 - 5R_1)$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} A \quad \text{(by } R_3 \to R_3 - 3R_2)$$

The matrix on L.H.S. has a row of all zeroes. So the matrix A cannot be reduced to an identity matrix. Hence, A is not invertible. Infact note that A is singular as $|A| = 0$.

3.6 SUMMARY

This unit deals with advanced topics on matrices. First of all, in **section 3.2**, the concept of an elementary row operation of a matrix, is given. Then, through examples, it is illustrated how a matrix may be reduced to some standard forms like triangular matrix and identity matrix. In **section 3.3**, a very important concept of rank of matrix, is defined. Through a number of examples, it is explained how rank of a matrix can be found using elementary operations. In **section 3.4**, inverse of an invertible matrix is defined. Finally, through a number of suitable examples, it is explained how inverse of an invertible matrix can be found using elementary operations.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.5**.

UNIT 4 MATHEMATICAL INDUCTION

Structure

4.0 INTRODUCTION

We begin with the following question. What is the sum of first *n* odd natural numbers ?

If *n* equals 1, the sum equals 1, as 1 is the only summand. The answer we seek is a formula that will enable us to determine this sum for each value *n* without having to add the summands.

Table 4.1 lists the sum S_n of the first *n* odd natural numbers, as *n* takes values from 1 to 10.

Table 4.1

n	*Series*	*Sum* (S_n)
1	1	$1=1^2$
2	1+3	$4=2^2$
3	1+3+5	$9=3^2$
4	1+3+5+7	$16=4^2$
5	1+3+5+7+9	$25=5^2$
6	1+3+5+7+9+11	$36=6^2$
7	1+3+5+7+9+11+13	$49=7^2$
8	1 +3 +……..+15	$64=8^2$
9	1 +3 +……..+17	$81=9^2$
10	1 +3 +……..+19	$100=10^2$

Jumping to a Conclusion

Judging from the pattern formed by first 10 sums, we might conjecture that
$S_n = 1 + 3 + 5 + ... + (2n – 1) = n^2$.
Recognizing a pattern and then simply jumping to the conclusion that the pattern must be true for all values of *n* **is not a logically valid method** of proof in

mathematics. There are many instances when the pattern appears to be developing for small values of n and then at some point the pattern fails. Let us look at one example. It was widely believed that $P_n = n^2 + n + 41$ is prime for all natural-numbers. Indeed p_n, is prime for all values of n lying between 1 and 39 as shown in Table 4.2.

But the moment we take $n = 40$, we get

$$P_{40} = 40^2 + 40 + 41$$

$$= 1600+40+41 = 1681 = 41^2$$

which is clearly not a prime.

Table 4.2

n	P_n	n	P_n	n	P_n
1	43	11	173	26	743
		12	197	27	797
2	47	13	223	28	853
3	53	14	251	29	911
		15	281	30	971
4	61	16	313	31	1033
5	71	17	347	32	1097
		18	383	33	1163
6	83	19	421	34	1231
7	97	20	461	35	1301
				36	1373
8	113	21	503	37	1447
9	131	22	547	38	1523
		23	593		
10	151	24	641	39	1601
		25	691		

Just because a rule, pattern or formula seems to work for several values of n, we cannot simply conclude that it is valid for all values of n without going through a *legitimate proof.*

How to Legalize a Pattern?

One way to legalize the pattern is to use the principle of **Mathematical induction.** To see what it is, let us return to our question in the beginning of the chapter. What is the sum of first n odd natural numbers?

We have already seen that the formula

$S_n = 1 + 3 + 5 + ... + (2n - 1) = n^2$ is valid for $n = 1, 2, 3, ..., 10$ (1)

Do we need to compute S_n by adding the first n odd natural numbers ?
A moment's reflection will show that it is not necessary.

Having obtained the value of S_n for some integer n, we can obtain the value of

$$S_{n+1} = S_n + 2n + 1$$

if $S_n = n^2$ for some n, then $S_{n+1} = S_n + 2n + 1 = n^2 + 2n + 1 = (n+1)^2$.

That is, if $S_n = n^2$ for some natural number n, then the formula holds for the next natural number $n + 1$.

Since the formula $S_n = n^2$ holds for $n = 10$, therefore it must hold $n = 11$. Since, it holds for $n = 11$, therefore, it must hold for $n = 12$. Since, it holds for $n = 12$, it holds for $n = 13$, and so on. The principle underlying the foregoing argument is nothing but the principle of mathematical induction. We state this formally in section 4.3.

4.1 OBJECTIVES

After studying this unit, you should be able to:

- use the principle of mathematical induction to establish truth of several formulae and inequalities for each natural number n.

4.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

Let P_n be a statement involving the natural number n. If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1}, for every interger k, then P_n must be true all natural numbers n.

In other words, to prove that a statement P_n holds for all natural numbers, we must go through *two* steps; First, we must prove that P_1 is ture. Second, we must prove that P_{k+1} is true whenever P_k is true.

CAUTION

Just proving P_{k+1} whenever P_k is true will not work.

There is an interesting analogue. Suppose we have "sequence" of dominoes standing in a row, as in Fig. 4.1 Suppose (1) the first domino falls, and (2) whenever any domino falls, then the one next to it (to the right in Fig. 4.2) falls as well. Our conclusion is that each domina will fall (see Fig 4.3). This reasoning closely parallels the ideal of induction.

To apply the principles of mathematical induction, we always need to be able to find P_{k+1} for a given P_k . It is important to acquire some skill in writing P_{k+1} whenever P_k is given.

We now take up some illustrations in which we write some particular terms when we know P_n. We also take up some illustrations in which we write P_{k+1} when we know P_k.

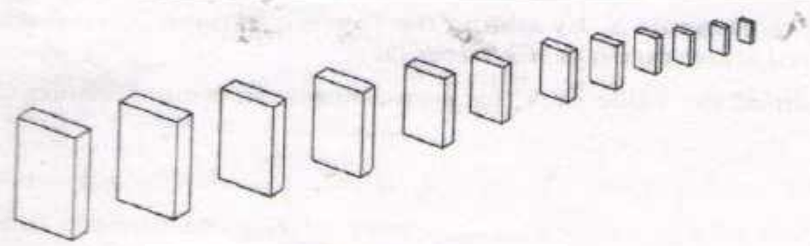


Figure 4.1

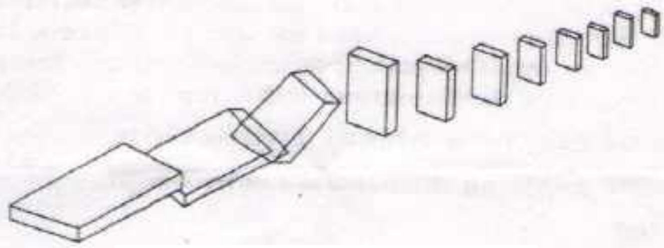


Figure 4.2

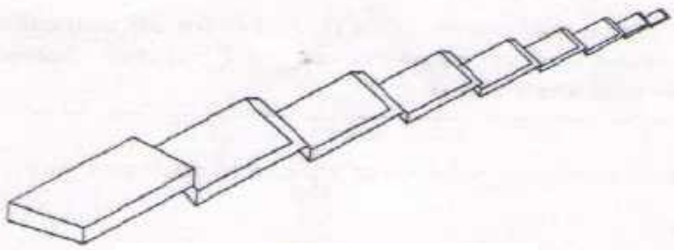


Figure 4.3

Illustration 1 : If P_n is the statement "$n(n + 1)$ is even"., then what is P_4 ? What is P_{10} ?

Solution :

P_4 is the statement "4(4+1) is even", i.e., "20 is even". P_{10} is the statement "10(10+1) is even" i.e., "110 is even".

Ilustration 2 : If P_n is the statement "$n(n + 1)(n + 2)$ is divisible by 12", write $P_{3,}$ P_4 and $P_{5\,.}$ Which one of $P_{3,}$ P_4 and P_5 are true statement ?

Solution

P_3 is "3 (3+1) (3+2) is divisible by 12" i.e., "60 is divisible by 12"

P_4 is "4 (4+1) (4+2) is divisible by 12" i.e., "120 is divisible by 12"

P_5 is "5 (5+1) (5+2) is divisible by 12" i.e., "210 is divisible by 12"

Each of P_3 and P_4 is true. But P_5 is false.

Example 1

(i) If P_n is the statement "$n^3 + n$ is divisible by 3", is the statement P_3 true ? Is the statement P_4 true ?

(ii) If P_n is the statement "$2^{3n} - 1$ is an integral multiple of 7", prove that P_1, P_2 and P_3 are true ?

(iii) If P_1 is the statement "$3^n > n$" are true P_1, P_2 and P_4 true statements ?

(iv) If P_n is the statement "$2^n > n$" what is P_{n+1} ?

(v) If P_n is the statement "$3^n > n$" prove that P_{n+1} is true whenever P_n is true.

(vi) Let P_n is the statement "$n^2 > 100$" prove that P_{k+1} is true whenever P_k is true.

(vii) If P_n is the statement "$2^n > 3n$" and if P_k is true, prove that P_{k+1} is true.

(viii) If P_n is the statement "$2^{3n} - 1$ is a multiple of 7", prove that truth of P_k implies the true of P_{k+1}.

(ix) If P_n is the statement "$10^{n+1} > (n + 1)^5$", prove that P_{k+1} is true whenever P_k is true.

(x) Give an example of a statement $P_{n,}$ such that P_3 is true but P_4 is not true.

(xi) Give an example of statement P_n such that it is not true for any n.

(xii) Give an example of a statement P_n in which $P_{1,}$ $P_{2,}$ P_3 are not true but P_4 is true.

(xiii) Give an example of a statement P_n which is true for each n.

Solution :

(i) P_3 is the statement "$3^3 + 3$ is divisible by 3" i.e., "30 is divisible by 3". which is clearly true.

P_4 is the statement "$4^3 + 4$ is divisible by 4" i.e., "68 is divisible by 3"

This is clearly not true.

(ii) P_1 is the statement "$2^3 - 1$ is an integral multiple of 7", i.e., "7 is an integral multiple of 7". This is a true statement.

P_2 is the statement "$2^6 - 1$ is an integral multiple of 7", i.e., 63 is an integral multiple of 7". This also is a true statement.

P_3 is the statement "$2^9 - 1$ is an integral multiple of 7", i.e., "511 is an integral multiple of 7". This again is a true statement.

(iii) P_1 is $3^1 > 1$", which is clearly true.

P_2 is "$3^2 > 2$". This also is a true statement.

P_4 is "$3^4 > 4$". This again is a true statement.

(iv) P_{n+1} is the statement "$2^{n+1} > n+1$".

(v) We are given that $3^n > n$.

we are interested to show that $3^{n+1} > n + 1$

we have $\dfrac{n+1}{n} = 1 + \dfrac{1}{n} \leq 1 + 1 < 3$

$\Rightarrow n + 1 < 3n < 3.3^n = 3^{n+1}$.

This show that if P_n is true, then P_{n+1} is true.

(vi) We are given that $k^2 > 100$.

we wish to show that $(k+1)^2 > 100$

we have

$(k + 1)^2 = k^2 + 2k + 1 > k^2 > 100$ $\quad [\because 2k+1 > 0]$

$\Rightarrow (k+1)^2 > 100$.

This shows that P_{k+1} is true whenever P_k is true.

(vii) Since P_k is true, we get $2^k > 3k$.

we wish to show that $2^{k+1} > 3(k+1)$

we have

$2^{k+1} = 2.2^k = 2^k + 2^k > 3k + 3k$ $\quad$ [by assumption]

$> 3k + 3$

$\Rightarrow 2^{k+1} > 3(k+1)$

this proves that P_{k+1} is true.

(viii) Since P_k is true we have $2^{3k} - 1$ is a multiple of 7, i.e., there exists an integer m such that $2^{3k} - 1 = 7m$

We wish to show that $2^{3(k+1)} - 1$ is a multiple of 7.

We have

$2^{3(k+1)} - 1 = 2^{3k} . \ 2^3 - 1 = (7m+1). (8) - 1$

$= 56m + 8 - 1 = 56m + 7 = 7(8m + 1)$

This shows that $2^{3(k+1)} - 1$ is a multiple of 7, i.e. P_{k+1} is true.

(ix) Since P_k is true, we have $10^{k+1} > (k+1)^5$

We wish to show that $10^{k+2} > (k+2)^5$

We have

$$\frac{(k+2)^5}{(k+1)^5} = \left(1 + \frac{1}{k+1}\right)^5$$

$$\text{As } k \geq 1, k+1 \geq 2 \Rightarrow \frac{1}{k+1} \leq \frac{1}{2}$$

therefore, $$\left(1 + \frac{1}{k+1}\right)^5 \leq \left(1 + \frac{1}{2}\right)^5 = \left(\frac{3}{5}\right)^5 = \frac{243}{32} < 10$$

Thus, $(k+2)^5 < 10\,(k+1)^5 < 10.10^{k+1} = 10^{k+2}$.

Therefore, P_{k+1} is true.

(x) Let P_n be that statement "$n \leq 3$", then P_3 is true but P_4 is not true.

(xi) Let P_n be the statement "$n\,(n+1)$ is odd". Then P_n is false for every n.

(xii) Let P_n $n \geq 4$.

(xiii) Let P_n be the statement "$n \geq 1$". The P_n is true for each n.

Example 2: Use the principle of mathematical induction to prove that

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots + 2n = n(n+1)$$

for each natural number n.

Solution :

Mathematical induction consists of two distinct parts. First, we must show that the formula holds for $n = 1$.

Let P_n denote the statement

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots + 2n = n(n+1)$$

Step 1. When $n = 1$, P_n becomes $2 = 1(1+1)$ which is clearly true.

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for *some* integer k. The second step is to use this assumption to prove that formula is valid for the next natural number $k+1$.

Step 2. Assume that P_k is true for some $k \in \mathbf{N}$, that is, assume that

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots + 2k = k\,(k+1)$$

is true. We must show that P_{k+1} is true, where P_{k+1}

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots + 2k + 2\,(k+1) = (k+1)(k+2) \qquad (1)$$

Not to Forget

While writing LHS of $P_{k+1,}$ you must remember that not only should you write the last term of the series, but also a term prior to the last term. If you, now supress the last term of the LHS of P_{k+1} what remain of the LHS of P_k.

LHS of (1) = 2 + 4 + 6 + ……………… + $2k$ + 2 (k + 1)

= k (k + 1) + 2(k+1) [induction assumption]

= (k + 1) (k + 2) [taking k +1 common]

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

CAUTION You will lose at least one mark if you do not write this last paragraph.

Example 3 Use the principle of mathematical induction to show that

$$1+4+7+\ldots\ldots+(3n-2)=\frac{1}{2}n(3n-1)$$

for every natural number n.

Solution : Let P_n denote the statement.

$$1+4+7+\ldots\ldots(3n-2)=\frac{1}{2}n(3n-1)$$

When $n = 1$, P_n becomes $1 = \frac{1}{2}(1)[3(1)-1]$ or $1 = 1$

which is clearly true.

This shows that the result holds $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$. That is, assume that

$$1+4+7+\ldots\ldots(3k-2)=\frac{1}{2}k(3k-1)$$

We shall now show that the truth of P_k implies the true of P_{k+1} where P_{k+1} is

$$1+4+7+\ldots\ldots+(3k-2)+\{3(k+1)-2\}=\frac{1}{2}(k+1)(3(k+1)-1)$$

or $$1+4+7+\ldots\ldots+(3k-2)+(3k+1)=\frac{1}{2}(k+1)(3k+2) \qquad (1)$$

LHS of (1)

$= 1+4+7+\ldots\ldots+(3k-2)+(3k+1)$

$= \frac{1}{2}k(3k-1)+(3k+1)$ [induction assumpation]

$= \frac{1}{2}[3k^2 - k + 6k + 2]$

$= \frac{1}{2}(3k^2 + 5k + 2)$

$= \frac{1}{2}[3k^2 + 3k + 2k + 2]$

$= \frac{1}{2}[3k(k+1) + 2(k+1)]$

$= \frac{1}{2}(k+1)(3k+2) =$ RHS of (1)

This shows that the result holds for $n = k + 1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Example 4 : Use the principle of mathematical induction to prove that

$1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$ for every natural number n.

Solution: Let P_n denote the statement

$$1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

When $n = 1$, P_n becomes

$$1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{1}{4}(1^2)(1+1)^2 \text{ or } 1 = 1$$

This shows that the result holds for $n = 1$. Assume that P_k is true for some $k \in \mathbf{N}$. That, is assume that

$1^3 + 2^3 + \ldots\ldots\ldots\ldots + k^3 = \frac{1}{4}k^2(k+1)^2$

We shall now show that the truth of P_k implies the truth of P_{k+1} where P_{k+1} is

$$1^3 + 2^3 + \ldots\ldots\ldots\ldots + k^3 = \frac{1}{4}(k+1)^2(k+1)^2 \qquad (1)$$

LHS of (1)

$= 1^3 + 2^3 + \ldots\ldots\ldots\ldots k^3 = + (k+1)^3$

$= \frac{1}{4}k^2(k+1)^2(k+1)^3$

$= \frac{1}{4}k^2(k+1)^2(k+1)^3$ [induction assumption]

$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)] = \frac{1}{4}(k+1)^2(k+2)^2$

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Example 5 : Use the principle of mathematical induction to prove that

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \dots\dots\dots\dots\dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for every natural number *n*.

Solution: Let P_n denote the statement

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

When $n = 1$, Pn becomes $\frac{1}{1.2} = \frac{1}{1+1}$ or $\frac{1}{2} = \frac{1}{2}$

This shows that the result holds for $n = 1$. Assume that P_k is true for some $k \in \mathbf{N}$.

That, is assume that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \qquad (1)$$

We shall now show that the true of P_k implies the truth of P_{k+1} where P_{k+1} is

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \dots\dots\dots\dots\dots\dots\dots + \frac{1}{k(k+1)} = \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\text{LHS of (1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} \qquad \text{[induction assumption]}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number *n*.

Example 6 : Use the principle of mathematical induction to show that

$$2 + 2^2 + \dots\dots\dots\dots + 2^n = 2^{n+1} - 2$$

for every natural number *n*.

Solution : Let P_n denote the statement

$2 + 2^2 + \ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 2$

When $n = 1$, P_n becomes

$2 = 2^{1+1} - 2$ or $2 = 4 - 2$

This shows that the result holds for $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$.

That is, assume that

$2 + 2^2 + \ldots\ldots\ldots + 2^k = 2^{k+1} - 2$

We shall now show that truth of P_k implies the truth of P_{k+1} is

$2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1} = 2^{k+1} - 2$ (1)

LHS of (1) $= 2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1}$

$= (2^{k+1} - 2) + 2^{k+1}$ [induction assumption]

$= 2^{k+1}(1 + 1) - 2$

$= 2^{k+1}2 - 2 = 2^{k+2} - 2$

$=$ RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Example 7: Show that $2^{3n} - 1$ is divisible by 7 for every natural number n.

Solution : Let P_n denote the statement $7|(2^{3n} - 1)$

For $n = 1$, P_n becomes $7|(2^3 - 1)$

Since $2^3 - 1 = 8 - 1 = 7$, we have $7|7$. This shows that the result is ture for $n = 1$.
Assume that P_k is true for some $k \in \mathbf{N}$.

That is, assume that $7 \mid (2^{3k} - 1)$

That is, assume that $2^{3k} - 1 = 7m$ for some $m \in \mathbf{N}$.

We shall now that that the truth of P_k implies the truth of P_{k+1}, where P_{k+1} is

$7(2^{3(k+1)} - 1)$

Now

$2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 2^{3k}.\, 2^3 - 1$

$= (7m + 1)(8) - 1$ $[\because 2^{3k} - 1 = 7m]$

$= 56m + 8 - 1 = 56m + 7 = 7(8m+1)$

$\Rightarrow 7|[2^{3(k+1)} - 1\,]$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Example 8 Show that $n(n+1)(2n+1)$ is a multiple of 6 for every natural number n.

Solution : Let P_n denote the statement $n(n + 1)(2n + 1)$ is a multiple of 6.
When $n = 1$, P_n becomes $1(1 + 1)((2)(1) + 1) = (1)(2)(3) = 6$ is a multiple of 6.
This shows that the result is true for $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$. That is assume that $k(k + 1)(2k + 1)$ is a mutliple of 6.

Let $\quad k(k + 1)(2k + 1) = 6m$ for some $m \in \mathbf{N}$.

We now show that the truth of P_k implies the truth of P_{k+1}, where P_{k+1} is
$(k + 1)(k + 2)[2(k + 1) + 1] = (k + 1)(k + 2)(2k + 3)$ is a multiple of 6.

We have

$$\begin{aligned}
&(k + 1)(k + 2)(2k + 3) \\
&\quad = (k + 1)(k + 2)[(2k + 1) + 2] \\
&\quad = (k + 1)[k(2k + 1) + 2(2k+1) + 4)] \\
&\quad = (k + 1)[k(2k + 1) + 6(k + 1)] \\
&\quad = k(k + 1)(2k + 1) + 6(k + 1)^2 \\
&\quad = 6m + 6(k + 1)^2 = 6[m + (k + 1)^2]
\end{aligned}$$

Thus $(k + 1)(k + 2)(2k + 3)$ is multiple of 6.

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Example 9 : Show that 11 divides $10^{2n-1} + 1$ for every natural number n.

Solution : Let P_n denote the statement

$$11|(10^{2n-1} + 1).$$

When $n = 1$, P_n becomes $11|(10^{2-1} + 1)$.
As $10^{2-1} + 1 = 10 + 1 = 11$ we have $11|11$.
This shows tht the result is true for $n = 1$

Assume that P_k is true for some $k \in \mathbf{N}$. That is, assume that

$$11(10^{2k-1} + 1).$$

That is, assume that $10^{2k-1} + 1 = 11\,m$ for some $m \in \mathbf{N}$.

We shall now that the truth of P_k implies P_{k+1}, where P_{k+1} is

$$11(10^{2(k+1)-1} + 1)$$

$$\begin{aligned}
\text{Now, } 10^{2(k+1)-1} + 1 &= 10^{2k+2-1} + 1 \\
&= 10^{2k-1+2} + 1 = 10^{2k-1} \,.\, 10^2 \\
&= (11m - 1)\, 10^2 + 1 \qquad [\because 10^{2k-1} + 1 = 11m] \\
&= 1100\,m - 100 + 1 = 1100\,m - 99 = 11(100m - 9) \\
\Rightarrow \quad & 11 \mid (10^{2(k+1)-1} + 1).
\end{aligned}$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Example 10 Show that 133 divides $11^{n+2} + 12^{2n+1}$ for every natural number n.

Solution Let P_n denote the statement

$$133|(11^{n+2} + 12^{2n+1})$$

When $n = 1$, P_n becomes $133|(11^{1+1} + 12^{2+1})$.

$$\begin{aligned}
\text{As} \quad 11^{1+2} + 12^{2+1} &= 11^3 + 12^3 = 1331 + 1728 = 3059 \\
&= (133)\,(29), \text{ we have } 133|\,(11^{1+2} + 12^{2+1})
\end{aligned}$$

This shows that the result is true for $n = 1$. Asumme that P_k is true for some $k \in \mathbf{N}$. That is assume that

$$133|(11^{k+2} + 12^{2k+1})$$

That is, assume that $11^{k+2} + 12^{2k+1} = 133m$ for some $m \in \mathbf{N}$. We shall now show that the truth of P_k implies the truth of P_{k+1}, where P_{k+1} is

$$133 \mid (11^{k+1+2} + 12^{(2k+2+1)})$$

$$\begin{aligned}
\text{Now, } \quad & 11^{k+1+2} + 12^{(2k+2+1)} \\
&= 11^{k+2}\, 11^2 + 12^{2k+1}\, 12^2 \\
&= 11^{k+2}\, 11 + (133m - 11^{2k+1})\, 12^2 \qquad \text{[by induction assumption]} \\
&= 11^{k+2}\, 11 + (133m)(144) - (11^{k+2})\,(144) \\
&= 133(144m) - 133(11^{k+2}) \\
&= 133(144m - 11^{k+2})
\end{aligned}$$

Thus, $133 \mid (11^{k+1+2} + 12^{2(k+1)+1})$

This shows that the result holds for $n = k + 1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so P_n is true for each $n \in \mathbf{N}$.

Example 11 : Show that $14 \mid (3^{4n-2} + 5^{2n-1})$ for all natural number n.

Solution : Let P_n denote the statement $14 \mid (3^{4n-2} + 5^{2n-1})$.

For $n = 1$, we have

$3^{4n-2} + 5^{2n-1} = 3^2 + 5 = 14$ which is divisible by 14.

Assume that P_n is true for some natural number n, say k. That is, assume that $14 \mid (3^{4k-2} + 5^{2k-1})$ is true for some natural number k. Suppose $3^{4k-2} + 5^{2k-1} = 14m$ for some natural number m. We now show that the truth of P_k implies the truth of P_{k+1}, that is, we show that $14 \mid [(3^{4(k+1)-2} + 5^{2(k+1)-1})]$.

We have

$$= 3^{4(k+1)-2} + 5^{2(k+1)-1} = 3^{4k-2}.\,3^4 + 5^{2k-1}5^2$$

$$= (14\,m\,5^{2k-1}).\,3^4 + 5^{2k-1}\cdot 5^2 \qquad [3^{4k-2} = 14m - 5^{2k-1}]$$

$$= (14\,m - (81) + 5^{2k-1}\ (-81+25)$$

$$= (14m)\,(81) - (5^{2k-1})(56) = 14[81m - 4.5^{2k-1}]$$

$$\Rightarrow \quad 14 \mid [3^{4(k+1)-2} + 5^{2(k+1)-1}]$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Check Your Progress – 1

Use the principle of mathematical induction to prove the following formulae.

1. $1 + 3 + 5 + \ldots\ldots\ldots\ldots + (2n-1) = n^2 \qquad \forall\, n \in \mathbf{N}$

2. $1^2 + 2^2 + \cdots \ldots\ldots\ldots\ldots + n^2 = \frac{1}{6}\,n(n+1)(2n+1) \qquad \forall\, n \in \mathbf{N}$

3. $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \qquad \forall\, n \in \mathbf{N}$

4. $1(2^2) + 2\,(3^2) + \ldots\ldots\ldots\ldots + n\,(n+1)^2 \qquad \forall\, n \in \mathbf{N}$

5. $8 \mid (3^n - 1) \qquad \forall\, n \in \mathbf{N}$

6. $24 \mid (5^{2n} - 1) \qquad \forall\, n \in \mathbf{N}$

7. $1 + \frac{1}{2} + \ldots\ldots\ldots\ldots + \frac{1}{2^n} < 2. \qquad \forall\, n \in \mathbf{N}$

8. $1 + 2 + \ldots\ldots\ldots\ldots + n < (2n+1)2 \qquad \forall\, n \in \mathbf{N}$

4.3 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. Let P_n denote the statement

   $1 + 3 + 5 + \ldots\ldots\ldots\ldots\ldots\ldots + (2n - 1) = n^2$

   For $n = 1$, P_n becomes $1 = 1^2$ or $1=1$ which is clearly true.

   Assume that P_k is ture for $k \in \mathbf{N}$

   That is, assume that

   $1 + 3 + 5 + \ldots\ldots\ldots\ldots\ldots\ldots + (2k - 1) = k^2$

   For $n = k + 1$, we have

   $P_{k+1} : 1 + 3 + \ldots\ldots\ldots\ldots\ldots + (2k - 1) + (2k + 1) = (k + 1)^2$

   Now,

   $1 + 3 + \ldots\ldots\ldots + (2k - 1) + (2k+1)$

   $= k^2 + (2k + 1)$ [by induction assumption]

   $= (k + 1)^2$

2. Clearly result is true for $n =1$. Assume that result holds for $n = k$, that is,

   $1^2 + 2^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + k^2 = \frac{1}{6} k (k + 1) (2k + 1)$

   For $n = k+1$,

   LHS $= 1^2 + 2^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + k^2 + (k+1)^2$

   $$= \frac{1}{6} k (k + 1) (2k + 1) + (k + 1)^2$$

   $$= \frac{1}{6}(k + 1) [k (2k + 1) + 6(k + 1)]$$

   $$= \frac{1}{6}(k + 1) [2k^2 + 7k + 6]$$

   $$= \frac{1}{6}(k + 1)[2k^2 + 3k + 4k + 6]$$

   $$= \frac{1}{6}(k + 1) [k(2k + 3) + 2(2k + 3)]$$

   $$= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$$

   $$= \frac{1}{6}(k + 1)(k + 1 + 1)(2k + 1) + 1)$$

   The result holds for $n = k + 1$.

3. Result holds for $n = 1$.

Assume that

$$\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \dots\dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

For $n = k + 1$,

$$\text{LHS} = \frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \dots \frac{1}{(2k-1)\,(2k+1)} + \frac{1}{(2k+1)\,(2k+3)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)\,(2k+3)}$$

$$= \frac{1}{(2k+1)(2k+3)}\,[k(2k+3)+1]$$

$$= \frac{1}{(2k+1)(2k+3)}\,[2k^2+2k+k+1]$$

$$= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} = \frac{(k+1)}{(2k+3)} = \frac{(k+1)}{2(k+1)+1}$$

Result holds for $n = k + 1$.

4. The result holds for $n = 1$.

Assume that

$1(2^2) + 2(3^2) + \dots\dots\dots\dots + k + (k+1)^2 =$

$$\frac{1}{12}\,k(k+1)(k+2)(3k+5)$$

For $n = k + 1$.

LHS $= (1)\,(2)^2 + \dots\dots\dots + k\,(k+1)^2 + (k+1)\,(k+2)^2$

$$= \frac{1}{12}k\,(k+1)\,(k+2)\,(3k+5) + (k+1)\,(k+1)^2$$

$$= \frac{1}{12}k\,(k+1)\,(k+2)\,[k(3k+5) + 12(k+2)$$

$$= \frac{1}{12}k\,(k+1)\,(k+2)\,[3k^2+17k+24]$$

$$= \frac{1}{12}k\,(k+1)(k+2)[3k^2+8k+9k+24]$$

$$= \frac{1}{12}k\,(k+1)\,(k+2)\,[k(3k+8)k+3(k+8)]$$

$$= \frac{1}{12}k\,(k+1)\,(k+2)\,(k+3)(3k+8)$$

$$= \frac{1}{12}k\,(k+1)\,(k+1+1)\,(k+1+2)[(3k+1)+5]$$

The result holds for $n = k+1$.

5. The result holds for $n = 1$.

   Assume that $8 | (3^{2k} - 1)$ for same $k \in \mathbf{N}$.

   Let $3^{2k} - 1 = 8m$ for some $m \in \mathbf{N}$.

   Next, $3^{2(k+1)} - 1 = 3^{2k}.3^2 - 1 = (8m + 1)(9) - 1$

   $= 72m + 9 - 1 = 8(9m + 1)$

   This shows that $8 | (3^{2(k+1)} - 1)$

   The result holds for $n = k+1$.

6. The result holds for $n = 1$

   Assume $24 | (5^{2k} - 1)$ for same $k \in \mathbf{N}$.

   $\Rightarrow 5^{2k} - 1 = 24m$ for same $m \in \mathbf{N}$.

   For $n = k+1$,

   $5^{2k+2} - 1 \quad = 5^{2k}.5^2 - 1$

   $= (24m + 1)(25) - 1$

   $= (24)(25)m + 25 - 1$

   $= 24(25m + 1)$

   Thus, $24 (5^{2k+2} - 1)$

   The result holds for $n = k+1$.

7. The result holds for $n = 1$, as $1 + \frac{1}{2} < 2$

   Assume that

   $1 + \frac{1}{2} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^k} < 2 \ for\ same\ k \in \mathbf{N}.$

   $\Rightarrow \frac{1}{2}(1 + \frac{1}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^k}) < 1$

   $\Rightarrow 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^{k+1}} < 1 + 1 = 2.$

   The result holds for $n = 1$.

8. The result holds for $n = 1$.

   Assume that the result holds for $n = k$, that is ,

   $1 + 2 + \ldots\ldots\ldots\ldots + k < (2k + 1)^2$

   We have,

   $1 + 2 + \ldots\ldots\ldots\ldots + k + (k+1)$

   $< (2k + 1)^2 + (k + 1)$

   $= 4k^2 + 4k + 1 + k + 1$

   $< 4k^2 + 12k + 9 \qquad [\because 7k + 7 > 0]$

   $= (2k + 3)^2$

   Thus, the result holds for $n = k + 1$.

4.4 SUMMARY

The unit is for the purpose of explaining the Principle of Mathematical Induction, one of the very useful mathematical tools. A large number of examples are given to explain the applications of the principle.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 4.3**.

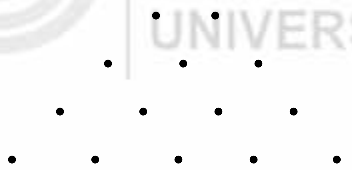
UNIT 1 SEQUENCE AND SERIES

Structure

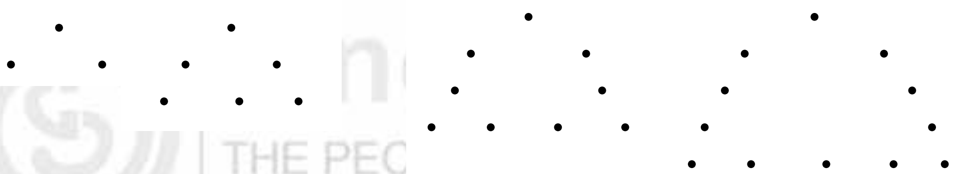
1.0 INTRODUCTION

We begin by looking at some examples which exhibit some pattern.

1. Arrangement of seats in a conference hall. Each row (except the first) contains, one seat more than the number of seats in the row ahead of it. See the following figure.



2. The number of dots used to draw the following triangles :



3. The money in your account in different years when you deposit Rs. 10,000 and at the rate of 10% per annum compounded annually.

10000	11000	12100	13310
$n = 0$	$n = 1$	$n = 2$	$n = 3$

In this unit, we shall study sequences exhibiting some patterns as they grow.

1.1 OBJECTIVES

After studying this unit, you will be able to :

- define an arithmetic progression, geometric progression and harmonic progression;
- find the nth terms of an A.P., G.P., and H.P.;
- find the sum to n terms of an A.P. and G.P.;
- find sum of an infinite G.P.; and
- obtain sum of first n natural numbers, their squares and cubes.

1.2 ARITHMETIC PROGRESSION (A.P.)

An **arithmetic progression** is a sequence of terms such that the difference between any term and the one immediately preceding it is a constant. This difference is called the common difference.

For example, the sequences

(i) 3, 7, 11, 15, 19…………..

(ii) 7, 5, 3, 1, -1…………………..

(iii) $1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}$ ………………

(iv) 2, 2, 2, 2………………..

are arithmetic progressions.

In (i) common difference is 4,

In (ii) common difference is –2,

In (iii) common difference is $-\frac{1}{4}$, and

In (iv) common difference is 0,

In general, an arithmetic progression (A.P.) is given by

$a, \quad a+d, \quad a+2d, \quad a+3d$………………..

We call a as **first term** and d as the **common difference.**

The n^{th} term of the above A.P. is denoted by a_n and is given by

$$a_n = a + (n-1)d$$

Example 1 : Find the first term and the common difference of each of the following arithmetic progressions.

(i) 7, 11, 15, 19, 23………………

(ii) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(iii) $a + 2b, a + b, a, a - b, a - 2b, \ldots\ldots\ldots\ldots\ldots\ldots$

Solution :

	First term	Common difference
(i)	7	4
(ii)	$\frac{1}{6}$	$\frac{1}{3}$
(iii)	$a + 2b$	$-b$

Example 2 : Find the 18th, 23rd and nth terms of the arithmetic progression.

$-11, -9, -7, -5$………………..

Solution: Here $a = -11$, and $d = -9 + 11 = 2$

Thus,

$a_{18} = a + (18 - 1)\, d$

$= -11 + (17)\,(2)$

$= -11 + 34 = 23;$

$a_{23} = a + (23 - 1)d$

$= -11 + (22)\,(2)$

$= -11 + 44 = 33;$ and

$a_n = a + (n - 1)d$

$= -11(n - 1)\,(2)$

$= -11 + 2n - 2 = 2n - 13$

Difference of two Terms of an A.P.

Let the A.P. be

$a, a + d, a + 2d, a + 3d$………….

we have

$a_r - a_s = [a + (r - 1)\, d] - [a + (s - 1)\, d]$

$= a + rd - d - [a + sd - d]$

$= a + rd - d - a - sd + d$

$= (r - s)d$

$\therefore\ a_r - a_s = (r - s)d$

Example 3 : Which term of the A.P. 3, 15, 27, 39, ……….will be 132 more than its 54th term ?

Solution: Common difference of the given A.P. is 12.

If *nth* term is the required term, then

$$a_n - a_{54} = 132$$

$$\Rightarrow \quad (n - 54)(12) = 132$$

$$\Rightarrow \quad n - 54 = \frac{132}{12} = 11$$

$$\Rightarrow \quad n = 54 + 11 = 65$$

Thus, the 65th term of the A.P. is 132 more than the 54th term.

Example 4 : If *pth* term of an A.P. is *q* and its *qth* term is *p*, show that its *rth* term is $p+q-r$. What is its $(p + q)th$ term ?

Solution : If *d* is the common difference of the A.P., then

$$a_p - a_q = (p - q)\, d$$

$$\Rightarrow \; q - p = (p - q)d$$

$$\Rightarrow d = \frac{q-p}{p-q} = -1$$

Now,

$$a_r - a_p = (r - p)\, d = (r - p)(-1)$$

$$\Rightarrow a_r = a_p - r + p$$

$$= q - r + p = p + q - r$$

$$\therefore \; a_{p+q} = p + q - (p + q) = 0 \qquad [\text{ put } r = p + q]$$

Example 5 : If *m* times the *mth* term of an A.P. is *n* times its *nth* term, show that the $(m+n)th$ term of the A.P. is 0.

Solution : We are given that $m\, a_m = n\, a_n$

$$\Rightarrow \quad m\,[a + (m-1)\, d] = n\,[a + (n-1)d]$$

$$\Rightarrow \quad m\,[a + md - d] = n\,[a + nd - d]$$

$$\Rightarrow \quad m\,[\alpha + md] = n[\alpha + nd] \text{ where } \alpha = a - d$$

$$\Rightarrow \quad m\,\alpha + m^2 d = n\,\alpha + n^2 d \Rightarrow (m - n)\,\alpha + (m^2 - n^2)d = 0$$

$$\Rightarrow \quad (m - n)[\,\alpha + (m + n)d] = 0 \quad \Rightarrow \alpha + (m + n)d = 0$$

$$\Rightarrow \quad a + (m + n - 1)d = 0 \qquad [\because \alpha = a - d]$$

Left hand side is nothing but the $(m + n)th$ term of the A.P.

Check Your Progress 1

Find the common difference and write next four terms of the A.P. (1 – 5)

1. 16, 11, 6, 1………………
2. 2, 5, 8, 11, 14…………………..
3. $\frac{1}{n}, \frac{2n+1}{n}, \frac{4n+1}{n}, \frac{6n+1}{n}$ ....
4. $m-1, m-2, m-3, m-4$………………………
5. $\sqrt{3}, \sqrt{27}, \sqrt{48},$ ……………………

6. If a_r denotes the rth term of an AP., show that

$$a_{p+q} + a_{p-q} = 2a_p$$

7. Find k so that $\frac{2}{3} - k, k\ \frac{5}{8} - k$ consecutive are three terms of an A.P.
8. Which term of the sequence

   $17, 16\frac{1}{5}, 15\frac{2}{5}, 14\frac{3}{5} \ldots\ldots\ldots\ldots\ldots$ is the first negative term.

9. The fourth term of an arithmetic progression is equal to 3 times the first term and the seventh term exceeds twice the third term by 1. Find its first term and the common difference.
10. If $(p+1)th$ term of an arithmetic progression is twice the $(q+1)th$ term, show that $(3p+1)th$ term is twice the $(p+q+1)th$ term.
11. If 7 times the 7^{th} term of an arithmetic progression is equal to 11 times its 11^{th} term, show that the 18^{th} term of an arithmetic progression is zero.

1.3 FORMULA FOR SUM TO *N* TERMS OF AN AP

Let

$$S = a+(a+d) +(a+2d) +\ldots\ldots.+ (a+ \overline{n-2}\ d) +(a+(\overline{n-1}\ d) \qquad (1)$$

Writing the expression in the reverse order, we get

$$S = (a + \overline{n-1}\ d) +(a + \overline{n-2}\, d)+ \quad \ldots. +(a + d) + a \qquad (2)$$

Adding (1) and (2) vertically, we get

$$2S = \underbrace{[2a + (n-1)\ d] +[2a + (n-1)d] + \ldots\ldots.+[2a + (n-1)d]}_{n \text{ expressions}}$$

$2S = n[2a + (n-1)d]$

$$\boxed{\Rightarrow \quad S = \frac{n}{2}\ 2a + (n-1)d}$$

Alternative form for the Sum Formula

$$S = \frac{n}{2}\{a + a + (n-1)d\}$$

$$\boxed{S = \frac{n}{2}\{a + l\}}$$

where $l = a + (n-1)d$ is the last term of the AP

Solved Examples

Example 6 : Find the sum of first 100 natural numbers.

Solution : Here $a = 1$, $d = 1$ and $n = 100$.

Using

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

we get

$$S_{100} = \frac{100}{2}[2(1) + (100-1)(1)] = 50(101) = 5050$$

Example 7 : Find the sum of 23 terms and n terms of the A.P. 16, 11, 6, 1……………

Solution : Here $a = 16$, $a + d = 11$ Therefore $d = -5$

Since $S_n = \frac{n}{2}[2a + a + (n-1)d]$ we get

$$S_{23} = \frac{23}{2}[2(16) + (23-1)(-5)] = \frac{23}{2}(32 - 110)$$

$$= \frac{23}{2}(-78) = -23 \times 39 = -897.$$

Also $S_n = \frac{n}{2}[2(16) + (n-1)(-5)] = \frac{n}{2}[32 - 5n + 5] = \frac{n}{2}[37 - 5n]$

Example 9 : How many terms of the A.P. 1,4,7,…….. must be taken so that the sum may be 715 ?

Solution : Here $a = 1$, $d = 3$ and $S_n = 715$.

Using $S_n = \frac{n}{2}[2a + (n-1)d]$, we get $715 = \frac{n}{2}[2(1) + (n-1)(3)]$

$$\Rightarrow 715 = \frac{n}{2}(3n + 1) \quad \Rightarrow \quad 3n^2 + n - 1430 = 0.$$

This is a quadratic equation in n. Its discriminant is positive. We can use the quadratic formula to obtain

$$n = \frac{1 \pm \sqrt{1 + (4)(3)(1430)}}{6} = \frac{1 \pm \sqrt{17161}}{6} = \frac{1 \pm 131}{6} = 22, -\frac{65}{3}.$$

As – 65/3 is a negative fraction, the number of terms cannot be equal to – 65/3
Thus $n = 22$.

Example 10: If in an A.P. $a = 2$ and the sum of first five terms is one–fourth of the sum of the next five terms, show that $a_{20} = -112$.

Solution : Here $a = 2$

We are given that sum of the first five terms is one–fourth of the next five terms.

Think of the A.P. as

$a, a + d, a + 2d, a + 3d,$……….

The sum of the first five terms is

$$a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) = S_5$$

and the sum of the next five terms is

$$(a + 5d) + (a + 6d)\ (a + 7d) + (a + 8d) + (a + 9d).$$

Note that this expression equals $S_{10} - S_5$.

According to the give condition

$$S_{10} = \frac{1}{4}(S_{10} - S_5) \Rightarrow 4S_{10} - S_5 \ or \ 5S_5 = S_{10}$$

$(S_{10} - S_5) =$ or $5\,S_5 = S_{10}$

$$= 5\left[\frac{10}{2}(2a + 9d)\right]$$

$$= 5\left[\frac{10}{2}(2a+9d)\right]$$

$\Rightarrow 5a + 10d = 2a + 9d \Rightarrow d = -3a$

Now, $a_{20} = a + (20-1)d = a + 19d = a + 19(-3a) = a - 57a = -56a$
$= (-56)(2) = -112.$

Example 11 : If sum of the p^{th}, q^{th} and r^{th} terms of an AP are a, b, c respectively, show that

$$(q-r)\frac{a}{p} + (r-p)\frac{b}{q} + (p-q)\frac{c}{r} = 0 \qquad (1)$$

Solution : Let the first term of the AP be A and the common difference be D. We are given :

$$a = S_p = \frac{p}{2}[2A + (p-1)D] \qquad (2)$$

$$b = S_q = \frac{q}{2}[2A + (q-1)D] \qquad (3)$$

$$c = S_r = \frac{r}{2}[2A + (r-1)D] \qquad (4)$$

Form (2), (3) and (4), we get

$$\frac{2a}{p} = (2A - D) + pD \qquad (4)$$

$$\frac{2b}{q} = (2A - D) + qD \qquad (5)$$

$$\frac{2c}{r} = (2A - D) + rD \qquad (6)$$

Multiplying (4) by $q-r$, (5) $r-p$ and (6) by $p-q$, we get

$$2(q-r)\frac{a}{p} + 2(r-p)\frac{b}{q} + 2(p-q)\frac{c}{r}$$

$$= (2A - D)(q-r) + p(q-r)D$$

$$+ (2A - D)(r-p) + q(r-p)D$$

$$+ (2A - D)(p-q) + r(p-q)D$$

$$= (2A - D)\{q - r + r - p + p - q\}$$

$$+ (pq - pr + qr - qp + rp - rq)D$$

$$= (2A - D)(0) + (0)D = 0$$

Dividing both the sides by 2 we get (1).

Example 12: If the sum of the first n terms of an A.P. is given by $S_n = 2n^2 + 5n$, Find the nth term of the A.P.

Solution : We shall use the formula

$a_n = S_n - S_{n-1} \qquad \forall\, n \geq 1,$

Where $S_o = 0$

$$\therefore\ a_n = 2n^2 + 5n - [2(n-1)^2 + 5(n-1)]$$
$$= 2n^2 + 5n - [2\,(n^2 - 2n + 1) + 5n - 5]$$
$$= 2n^2 + 5n - 2\,n^2 + 4n - 2 - 5n + 5$$
$$= 4n + 3$$

Thus, *nth* term of the A.P. is $4n + 3$

Example 13 : Find the sum of all integers between 100 and 1000 which are divisible by 9.

Solution : The first integer greater than 100 and divisible by 9 is 108 and the integer just smaller than 1000 and divisible by 9 is 999. Thus, we have to find the sum of the series.

108 + 117 + 126 + ………….+ 999.

Here $t_1 = a = 108$, $d = 9$ and $l = 999$

Let n be the total number of terms in the series be n. Then

$999 = 108 + 9\,(n - 1) \Rightarrow 111 = 12 + (n - 1) \Rightarrow n = 100$

Hence, the required sum $= \frac{n}{2}(a + l) = \frac{100}{2}(108 + 999)$

$= 50\,(1107) = 55350.$

Example 14 : The interior angels of a convex polygon are in A.P. If the smallest angle is 120^0 and the common difference is 5^0, show that there are 9 sides in the polygon.

Solution : We know that the sum of the interior angles of a convex polygon is $(n-2)(180^0)$ We are given that $a = 120^0$ and $d = 5^0$.

$$\therefore\ \frac{n}{2}[2(120) + (n-1)(5)] = (n-2)(180)$$

$\Rightarrow n[48 + (n - 1] = (n - 2)(72) \Rightarrow n^2 - 25n + 144 = 0$

$\Rightarrow (n - 9)(n - 16) = 0 \Rightarrow n = 9, 16.$

But when $n = 16$, the greatest angle is equal to
$a + (16 - 1)d = 120^0 + 15 \times 5^0 = 195^0$
Which is not possible as in this case one of the interior angle becomes 180^0.
[interior angles are 120^0, 125^0,…………. 175^0, 180^0, 185^0, 190^0, 195^0]
Thus $n = 9$.

Check Your Progress – 2

In questions 1 to 3, find the sum of indicated number of terms.

1. 1,3,5,7, ………….; 100 terms, 200 terms
2. 0.9, 091, 0.92, 0.93…………., 20 terms n terms.
3. $\frac{m-1}{m}, \frac{m-2}{m}, \frac{m-3}{m}$ …………; 10 terms, n terms
4. If the first term of an A.P. is 22, the common difference is – 4, and the sum to n terms is 64, Find n. Explain the double answer.
5. If sum of p terms of an A.P. is $3p^2+4p$, find its nth term.
6. Find the sum of all three digit numbers which leaves remainder (1) when divided by 4.

1.4 GEOMETRIC PROGRESSION (G.P.)

Suppose a ball always rebounds exactly 70 per cent of the distance it falls. For instance, if this ball falls from a height of 100 units, then it will rebound exactly 70 units. As a result the second fall will be from a height of 70 units. See Fig. 1

Now the ball will rebound exactly (0.7) (70) = 49 units

And so, on.

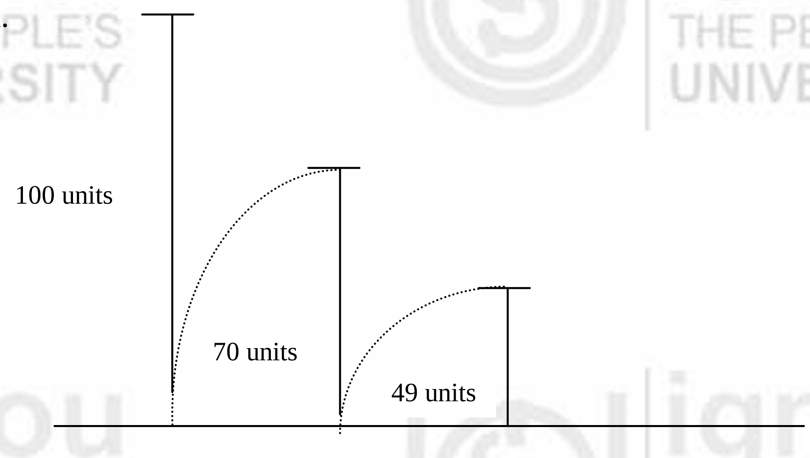


Figure 1

The following table gives the distance through which the ball bounces in the first 5 falls.

Number of fall	Ist fall	2^{nd} fall	3^{rd} fall	4^{th} fall	5^{th} fall
Distance of fall	100	100 (0.7)	$100\ (0.7)^2$	$100\ (0.7)^3$	$100\ (0.7)^4$

If you look carefully you will find that each term in sequence

$100,\ 100\ (0.7),\ 100\ (0.7)^2,\ 100\ (0.7)^3,$ ………..(except the first) is obtained by multiplying the previous term by a fixed constant 0.7.

Such a sequence is called **geometric sequence or geometric progression** or briefly, G.P. In other words, a geometric sequence or geometric progression is a sequence in which each term, except the first, is obtained by multiplying the term immediately preceding it by a fixed, non-zero number. The fixed number is called the **common ratio.**

Definition : A sequence $a_1, a_2, \ldots\ldots a_n, \ldots\ldots$ is called a geometric progression (G.P.). If there exists a constant r, such that

$$a_{k+1} = ra_k \qquad \forall\ k \in \mathbf{N}.$$

Thus, a geometric progression (G.P.) looks as follows :

$a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots$

*where a is called the **first term** of G.P. and r is called as the **common ratio** of the G.P.*

The nth term of the G.P., is given by

$$a_n = ar^{n-1}$$

Some other examples of GPs are

1. 0.1, 0.01, 0.001, 0.0001….$(0.1)^n$,…………….

2. 2, 4, 6, 8, 16,………….., 2^n,……….

3. $-\frac{1}{3}, -\frac{1}{9} - \frac{1}{27} - \frac{1}{81} \ldots\ldots\ldots.. \left(-\frac{1}{3}\right)^n \ldots$

4. $\frac{1}{\sqrt{3},}, 1, \sqrt{3}, 3\ , 3\sqrt{3}, \dots \dots 3^{(n-2)/2}, \dots \dots \dots \dots$

Ratio of two terms of G.P.

Let the G.P. be

$a, ar, ar^2, \dots\dots\dots\dots\dots\dots$

Now,

$$\frac{a_n}{a_m} = \frac{ar^{n-1}}{ar^{m-1}} = r^{n-1-m+1} = r^{n-m}$$

Thus, $\frac{a_n}{a_m} = r^{n-m}$

Solved Examples

Example 15: Find r and the next four terms of the G.P.

$$-3, 1, -\frac{1}{3}, \frac{1}{9}, \dots \dots \dots \dots ..$$

Solution: Here $a = -3$, $ar = 1$

So that

$$r = \frac{ar}{a} = \frac{1}{-3} = -\frac{1}{3}$$

The next four terms of G.P. are

$$a_5 = a_4 r = \left(\frac{1}{9}\right)\left(-\frac{1}{3}\right) = -\frac{1}{27}$$

$$a_6 = a_5 r = -\frac{1}{81}$$

$$a_7 = a_6 r = \left(\frac{1}{81}\right)\left(-\frac{1}{3}\right) = -\frac{1}{243}$$

and $a_8 = a_7 r = \left(-\frac{1}{243}\right)\left(-\frac{1}{3}\right) = -\frac{1}{729}$

Example 16 : Determine the 12^{th} term of a G.P. whose 8^{th} term is 192 and the common ratio is 2.

Solution : We have

$$\frac{a_{12}}{a_8} = r^{12-8} = r^4 = 2^4$$

$$\Rightarrow \quad a_{12} = a_8(2^4) = 192 \times 16 = 3072$$

Example 17 : Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, they form a G.P. find the numbers.

Solution : Let the three numbers in AP be $a - d, a, a + d$. we are given

$$(a-d)+a+(a+d)=15 \Rightarrow 3a=15 \text{ or } a=5$$

According to the given condition $a - d + 1$, $a + 3$ *and* $a + d + 9$ are in GP

$$\Rightarrow \quad \frac{a+3}{a-d+1} = \frac{a+d+9}{a+3}$$

$\Rightarrow (a+3)^2 = (a-d+1)(a+d+9) \Rightarrow (5+3)^2 = (5-d+1)(5+d+9)$

$\Rightarrow 64 = (6-d)(14+d) \Rightarrow 64 = 84 - 8d - d^2$

$\Rightarrow d^2 + 8d - 20 = 0 \Rightarrow (d+10)(d-2) = 0 \Rightarrow d = -10 \text{ or } d = 2$

If $d = -10$, the numbers are 15, 5, –5.

If $d = 2$, the numbers are 3, 5, 7.

Thus, the numbers are 15, 5, –5, or 3, 5, 7.

Check Your Progress - 3

For question 1 to 3 find the common ratio of each of the following G.P.

1. $2, 1, \frac{1}{2}, \frac{1}{4} \ldots\ldots\ldots\ldots.$
2. $-3, 1, -\frac{1}{3}, \frac{1}{9} \ldots\ldots\ldots.$
3. 3, 6, 12, 24…………..

For questions 4 to 6, find the *n*th term of the GP.

4. 128, –96, 72………
5. 100, –110, 121…………….
6. 3, 3.3, 3.63…………….
7. Determine the 18^{th} term of the GP whose 5^{th} term is 1 and common ratio is 2/3.
8. The 5^{th}, 8^{th} and 11^{th} terms of a GP are a, b, c respectively. Show that a, b, c are in GP.

1.5 SUM TO *n* TERMS OF a G.P.

We wish to find the sum of first n terms of the GP whose first term is a and the common ratio r.

Let us denote the sum of first n terms by S_n, that is,

$$S_n = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1}$$

Let us multiply both sides of this equation by r. We write the results of this operation below the original equation and line up vertically the terms of the same exponent (to prepare for a subtraction).

$$S_n = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-2} + ar^{n-1} \quad (1)$$

$$rS_n = ar + ar^2 + ar^3 + \ldots\ldots\ldots\ ar^{n-2} + ar^{n-1} + ar^n \quad (2)$$

Now subtract both the sides of lower equation [equation (2)] from the upper equation [equation (1)].

Most of the summands on the right side cancel. All that is left is the equation.

$$S_n = a - ar^n \Rightarrow (1-r)\, S_n = a(1-r^n)$$

Now, if $r \neq 1$then $S_n = \frac{a(1-r^n)}{1-r}$

Thus, the sum to n terms of a GP is $S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$

Clearly $S_n = na$ where $r = 1$.

Example 18 : Find the sum to 20 terms of a GP 128, – 96, 72, –54,……….

Solution

Here $a = 128$, $ar = -96$, therefore, $r = -96/128 = -3/4$.

Now that $r \neq 1$. Therefore,

$$S_n \frac{128[1-(-3/4)^{20}]}{1-(-3/4)} = 128\left[1-\left(-\frac{3}{4}\right)^{20}\right]\left(\frac{4}{7}\right)$$

$$= \frac{128(4^{20}-3^{20})}{4^{20}}\left(\frac{4}{7}\right) = \frac{2^6(4^{20}-3^{20})(2^2)}{2^{40}(7)} = \frac{4^{20}-3^{20}}{(2^{32})(7)}$$

Example 19 : How many terms of the GP $\sqrt{3}$, 3, 3 $\sqrt{3}$, … Add upto $39 + 13\sqrt{3}$.

Solution : Here $a = \sqrt{3}$, $ar = 3$, so that $r = \sqrt{3}$.

Let $39 + 13\sqrt{3}$ be the sum to n terms of the given GP, that is,

$$S_n = 39 + 13\sqrt{3}$$

$$\Rightarrow \quad \frac{a(r^n-1)}{r-1} = 39 + 13\sqrt{3} \quad \Rightarrow \quad \frac{(\sqrt{3})\left[(\sqrt{3})^n - 1\right]}{\sqrt{3}-1}$$

$$= 13\sqrt{3}\left(\sqrt{3}+1\right)$$

$\Rightarrow \quad 3^{n/2} = 1 + 13(3–1) = 1 + 26 = 27 = 3^3 \Rightarrow n/2 = 3$ or $n = 6$.

Thus, 6 terms of $\sqrt{3}$, 3, 3 $\sqrt{3}$ … … ….
are required to obtain a sum of 39+ 13 $\sqrt{13}$.

Example 20 : Find the sum to n terms of the series 9 + 99 + 999 + …………..

Solution : Note that we can write

$9 = 10 – 1$, $99= 100 – 1 = 10^2 – 1$, $999 = 10^3 – 1$, etc.

The sum to n terms of the series can be written as

$S = (10 – 1) + (10^2 – 1) + (10^3 – 1) + \ldots+(10^n – 1)$

$$= (10 + 10^2 + \cdots. +10^n) – n = \frac{10(10^n - 1)}{10 - 1} - n = \frac{10}{9}(10^n - 1) - n$$

Three Terms in GP

Three terms in GP whose product is given are taken as

$$\frac{a}{r}, a, ar$$

Example 21 : If sum of three numbers in GP is 38 and their product is 1728, find the numbers.

Solution : Let the three number be a/r, a, ar

Then, $\frac{a}{r} + a + ar = 38$ (1)

and $\left(\frac{a}{r}\right)(a)(ar) = 1728$ (2)

We can write (2) as $a^3 =1728 = 12^3 \Rightarrow a = 12$.

Putting $a = 12$ in (1), we get $\frac{12}{r} + 12 + 12r =38$

$$\Rightarrow \frac{12}{r} + 12r = 26 \text{ or } \frac{1}{r} + r = \frac{26}{12} = \frac{13}{6}$$

$\Rightarrow \quad 6 (r^2 + 1) = 13r \Rightarrow 6 r^2 –13r + 6 = 0 \Rightarrow 6 r^2 – 9r – 4r+ 6 = 0$

$\Rightarrow \quad 3r (2r – 3) – 2(2r – 3) = 0 \Rightarrow (3r – 2)(2r – 3) = 0 \Rightarrow r = 2/3$ or $3/2$.

When $a = 12$ and $r = \frac{2}{3}$, $\quad \frac{a}{r} = \frac{12}{2/3} = 18, a = 12, ar = 12\left(\frac{2}{3}\right) = 8.$

When $a = 12$ and $r = \frac{3}{2}$, $\frac{a}{r} = \frac{12}{3/2} = 8, a = 12, ar = 12\left(\frac{3}{2}\right) = 8.$

Hence, the numbers are either 18, 12, 8 or 8, 12, 18.

Sum of the Infinite Number of a G.P.

If $-1 < r < 1$, then sum of the infinite G.P.

$a + ar + ar^2 + \cdots$

is $\frac{a}{1-r}$

Example 22: Find the sum of the infinite G.P.

1, –1/3, 1/9, –1/27…………………………….

Solution: Here $a = 1$, $r = -1/3$

Thus, sum of the infinite G.P. is

$$\frac{a}{1-r} = \frac{a}{1-(-\frac{1}{3})} = \frac{3}{4}$$

Example 23 : The common ratio of a GP is –4/5 and the sum to infinity is 80/9. Find the first term of the GP.

Solution : Here $r = -4/5$

We are given

$$\frac{a}{1-r} = \frac{80}{9}$$

$$\Rightarrow a = \frac{80}{9}(1-r) = \frac{80}{9}\left[1-\left(\frac{4}{5}\right)\right]$$

$$= \frac{80}{9} \times \frac{9}{5} = 16$$

Thus, first term of the G.P. is 16.

Check Your Progress – 4

1. Find the sum of 10 terms and n terms of the G.P.

$1, \frac{2}{3}, \frac{4}{9} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..,$

2. Find the sum of 12 terms and n terms of the G.P.

$2, -\frac{1}{2}, \frac{1}{8}$

3. Find the sum to n terms of the series

   $5 + 55 + 555 + \ldots\ldots\ldots\ldots\ldots$

4. Find the sum to n terms of the series

   $0.6 + 0.66 + 0.666 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

5. Show that

   $$\underbrace{111\ldots\ldots\ldots\ldots1}_{91 \text{ times}}$$

   is not a prime number.

6. The sum of three numbers in G.P. is 31 and sum of their squares is 651. Find the numbers.

7. Find the of the infinite G.P.

   $7,\ -1,\ \frac{1}{7},\ -\frac{1}{49}\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

8. Find the sum of an infinite G.P. whose first term is 28 and the fourth term is 4/99.

1.6 ARITHEMETIC – GEOMETRIC PROGRESSION (A.G.P.)

A sequence is said to be arithmetic geometric sequence if the *nth* term of the sequence is obtained by multiplying the nth term of A.P. and a G.P.

Thus, A.G.P. is given by

$ab, (a + d)\, br, (a + 2d)\, br^2 \ldots\ldots\ldots\ldots\ldots$

nth term of the A. G. P. is given by $a_n = (a + \overline{n-1})br^{n-1}$

Sum to n terms of the A.G.P. is given by

$$S_n = \frac{ab}{1-r} + \frac{bdr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r}, \qquad r \neq 1$$

$$\text{If } r = 1, \text{then } Sn = b\,\frac{n}{2}[2a + (n-1)\,d]$$

Sum of Infinite terms of A.G.P

If $|r| < 1$, $r^{n-1} \to 0, r^n \to 0, nr^n \to 0$

As $n \to \infty$ the sum of infinite A.G.P. is

$$S = \frac{ab}{1-r} + \frac{bdr}{(1-r)^2}$$

Example 23: Find the sum to n terms of the series.

$$1 + \frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution : Here $a = 1$, $d = 3$, $b = 1$, $r = 1/5$

$$S_n = \frac{1}{1-1/5} + \frac{3\left(\frac{1}{5}\right)[1-\left(\frac{1}{5}\right)^{n-1}]}{(1-\frac{1}{5})^2} - \frac{[1+(n-1)(3)]\left(\frac{1}{5}\right)^n}{1-1/5}$$

$$= \frac{5}{4} + \frac{15}{16}\left[1-\left(\frac{1}{5}\right)^{n-1}\right] - \frac{5}{4}[3n-2]\left(\frac{1}{5}\right)^n$$

$$= \frac{5}{4} + \frac{15}{16} - \left[\frac{15}{16} + \frac{3n-2}{4}\right]\left(\frac{1}{5}\right)^{n-1}$$

$$= \frac{35}{16} - \frac{12n+7}{16}\left(\frac{1}{5}\right)^{n-1}$$

Example 24: Find the sum to infinite number of terms of A.G. P.

$$3 + 5\,(1/4) + 7\left(\frac{1}{4}\right)^2 + 9\left(\frac{1}{4}\right)^3 + \ldots\ldots\ldots\ldots$$

Solution : Here $a = 3$, $d = 2$, $b = 1$, $r = 1/4$

Thus, $S = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

$$= \frac{3}{1-1/4} + \frac{2\left[\frac{1}{4}\right]}{\left(1-\frac{1}{4}\right)^2}$$

$$= \frac{3(4)}{3} + \frac{2(4)}{9} = \frac{44}{9}$$

Check Your Progress 5

For Question 1– 2 : Find the sum to n terms of the A.G.P.

1. $3 + \frac{5}{4} + \frac{7}{4^2} + \frac{9}{4^3} + \ldots\ldots\ldots\ldots\ldots$

2. $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots\ldots\ldots\ldots, x \neq 1$

3. Find the sum of the inifinite number of terms of the A.G. P.

   $1 + 5x + 9x^2 + 13x^3 + \ldots\ldots\ldots\ldots\ldots, \ (|x| < 1)$

1.7 HARMONIC PROGRESSION (H.P.)

A sequence a_1, a_2, a_3……………of non-zero numbers is said to a harmonic progression (H.P.) if

$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \ldots\ldots\ldots$

forms an A.P.

Example 25 : Find the 10th term of the H.P.

$$\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution : We find the 10th term of the A.P. 3, 7, 11, 15…………

$$a_{10} = 3 + (10 - 1)(4) = 39$$

$$\text{Thus, tenth term of the H.P. is } \frac{1}{39}$$

1.8 SUM OF FIRST *n* NATRUAL NUMBERS THEIR SQUARES AND CUBES

The sequence of natural numbers

1,2,3,………………..*n*,……………

is an AP with first term as well as the common difference equal to 1. Thus

$1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n}{2}(1 + n)$

Therefore, $S_n = 1 + 2 + \ldots\ldots\ldots n = \frac{n}{2}(n+1)$

We now consider an alternative way of obtaining the sum of first *n* natural numbers.

Sum of Squares of first *n* natural Numbers*

$(r + 1)^2 - (r - 1)^2 = 4r$

By letting $r = 1,2,3, \ldots\ldots\ldots,(n-2), (n-1), n$ successively, we obtain

$2^2 - 0^2 = 4(1);$

$3^2 - 1^2 = 4(2);$

$4^2 - 2^2 = 4(3);$

$5^2 - 3^2 = 4(4);$

……………….

* We could also have started with identity $(r + 1)^2 - r^2 = 2r + 1$

$(n-1)^2 - (n-3)^2 = 4(n-2)$;

$n^2 - (n-2)^2 = 4(n-1)$;

$(n+1)^2 - (n-1)^2 = 4(n)$;

When we add the above identities, we find that all terms except $(n+1)^2, n^2, -1^2$ and -0^2 cancel out from the LHS and we obtain

$(n+1)^2 + n^2 - 1 - 0 = 4(1 + 2 + 3 + \ldots\ldots\ldots + n)$

But $(n+1)^2 + n^2 - 1 = n^2 + 2n + 1 + n^2 - 1 = 2n^2 + 2n = 2n\,(2n+1)$

Therefore, $2n\,(n+1) = 4\,(1 + 2 + 3 + \ldots\ldots + n)$
$\Rightarrow\; 1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = \frac{1}{2}n\,(n+1)$

Sum of Square of First *n* Natural Numbers

In this case we begin with the identity

$(r+1)^3 - (r-1)^3 = 6r^2 + 2$

Letting $r =$ 1, 2, 3,4,……..$(n-2)$, $(n-1)$, n we obtain

$2^3 - 0^2 = 6\,(1^2) + 2$

$3^2 - 1^3 = 6\,(2^2) + 2$

$4^3 - 2^3 = 6\,(3^2) + 2$

$5^3 - 3^3 = 6\,(4^2) + 2$
……………….
$(n-1)^3 - (n-3)^3 = 6(n-2)^2 + 2$
$n^3 - (n-2)^3 = 6(n-1)^2 + 2$
$(n+1)^3 - (n-1)^3 = 6(n)^2 + 2$

Adding, we obtain

$$(n+1)^3 + n^3 - 1^3 - 0^3 = 6(1^2 + 2^2 + \ldots\ldots + n^2) + 2n$$

$$\Rightarrow\; 6(1^2 + 2^2 + \ldots\ldots + n^2) = (n+1)^3 + n^3 - 1 - 2n$$

But $\quad (n+1)^3 + n^3 - 1 - 2n = n^3 + 3n^2 + 3n + 1 + n^3 - 1 - 2n$

$$= 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1) = n(n+1)(2n+1).$$

Thus, $\quad 6(1^2 + 2^2 + \ldots\ldots\ldots\ldots + n^2) = n(n+1)(2n+1)$

$$\Rightarrow\; 1^2 + 2^2 + \ldots\ldots\ldots\ldots\ldots + n^2 = \frac{1}{6}n\,(n+1)(2n+1)$$

Sum of Cubes of First n Natural Numbers

In this case, we begin with the identity

$(r+1)^4 - (r-1)^4 = 8r^3 + 8$

Letting r = 1, 2 ,3, 4,……..(n–2), (n–1), n , successively, we obtain

$$
\begin{aligned}
2^4 - 0^4 &= 8(1^3) + 8(1)\\
3^4 - 1^4 &= 8(2^3) + 8(2)\\
4^4 - 2^4 &= 8(3^3) + 8(3)\\
&\cdots\cdots\cdots\cdots\\
(n-1)^4 - (n-3)^4 &= 8(n-2)^3 + 8(n-2)\\
n^4 - (n-2)^4 &= 8(n-1)^3 + 8(n-1)\\
(n+1)^4 - (n-1)^4 &= 8(n)^3 + 8(n)
\end{aligned}
$$

Adding, we obtain

$(n+1)^4 + n^4 - 1^4 - 0^4 = 8(1^3 + 2^3 + 3^3 \ldots\ldots + n^3) + 8(1+2+3+\ldots+n)$

But $(1 + 2 + 3 + \ldots\ldots + n = \frac{1}{2}n(n+1)$.

Thus $(n+1)^4 + n^4 - 1 = 8(1^3 + 2^3 + 3^3 \ldots\ldots + n^3) + 4n(n+1)$

$$= 8(1^3 + 2^3 + 3^3 + \ldots n^3) = (n+1)^4 + n^4 - 1 - 4n(n+1)$$

But $(n+1)^4 + n^4 - 1 - 4n(n+1)$

$$= n^4 + 4n^3 + 6n^2 + 4n + 1 + n^4 - 1 + 4n^2 - 4n$$

$$= 2n^4 + 4n^3 + 2n^2 = 2n^2(n^2 + 2n + 1) = 2n^2(n+1)^2$$

Thus, $8(1^3 + 2^3 + 2^3 + \ldots + n^3) = 2n^2(n+1)^2$

$\Rightarrow \quad 1^3 + 2^3 + \ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$

In sigma notation the above three identities read as

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1),$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

and $$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$$

Example 26 : Find the sum of the following series

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$$

Solution : Let t_r denote the rth term of $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$, then

$$t_r = (2r-1)^2 = 4r^2 - 4r + 1$$

Thus

$$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} (4r^2 - 4r + 1) = 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

But $\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$, $\sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$ and $\sum_{r=1}^{n} 1 = n$.

Therefore, $\sum_{r=1}^{n} t_r = 4\{\frac{1}{6} n(n+1)(2n+1)\} - 4\{\frac{1}{2} n(n+1)\} + n$

$$= \frac{2}{3} n(n+1)(2n+1) - 2n(n+1) + n.$$

We now take $\frac{1}{3} n$ common from each on the right side, so that

$$\sum_{r=1}^{n} t_r = \frac{1}{3} n[2(n+1)(2n+1) - 6(n+1) + 3)]$$

$$= \frac{1}{3} n[2(2n^2 + 2n + n + 1) - (6n + 6) + 3]$$

$$= \frac{1}{3} n[(4n^2 + 6n + 2 - 6n - 6 + 3) = \frac{1}{3} n(4n^2 - 1)$$

Example 27 : Find the sum of the series

$$(1)(2^2) + (2)(3^2) + (3)(4^2) + \ldots\ldots\ldots \text{upto } n \text{ terms}$$

Solution : Let t_r denote the r^{th} term of the given series, then

$$t_r = (r)(r+1)^2 = r(r^2 + 2r + 1) = r^3 + 2r^2 + r$$

Thus, $\sum t_r = \sum_{r=1}^{n} (r^3 + 2r^2 + r) = \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = 1$

$$= \frac{1}{4} n^2(n+1)^2 + 2\{\frac{1}{6} n(n+1)(2n+1)\} + \frac{1}{2} n(n+1)$$

$$= \frac{1}{4} n^2(n+1)^2 + \frac{1}{3} n(n+1)(2n+1) + \frac{1}{2} n(n+1)$$

$$= \frac{1}{12} n(n+1) \quad [3n(n+1) + 4(2n+1) + 6]$$

$$= \frac{1}{12} n(n+1) \quad [3n^2 + 3n + 8n + 4 + 6] = \frac{1}{12} n(n+1)(3n^2 + 11n + 10)$$

$$= \frac{1}{12} n(n+1)(3n^2 + 6n + 5n + 10) = \frac{1}{12} n(n+1)[3n(n+2) + 5(n+2)]$$

$$= \frac{1}{12} n(n+1)(n+2)(3n+5).$$

Check Your Progress 6

1. Find the nth term of the H. P. $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \ldots\ldots\ldots\ldots$
2. $2^2 + 4^2 + 6^2 + \ldots\ldots + (2n)^2$
3. 1.2.3 + 2.3.4 + 3.4.5 +………upto n terms
4. 1.3.5 + 3.5.7 + 5.7.9 + ……. upto n terms

1.9 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. $d = 11 - 16 = -5$

   Next four terms are

   $-4, -9, -14, -19$
2. $d = 5 - 2 = 3$

   Next four terms are

   17, 26, 23, 26
3. $d = \frac{2n+1}{n} - \frac{1}{n} = 2 + \frac{1}{n} - \frac{1}{n} = 2$

   Next four terms are

   $$\frac{6n+1}{n} + 2,\ \frac{6n+1}{n} + 2 + 2,\ \frac{6n+1}{n} + 3\,(2),\ \frac{6n+1}{n} + 4\,(2)$$

   $$\text{or } \frac{8n+1}{n},\ \frac{10n+1}{n},\ \frac{12n+1}{n},\ \frac{14n+1}{n}$$

4. $d = (m-2) - (m-1) = -1$

   next four terms are

   $m-5,\ m-6,\ m-7,\ m-8$

5. The given A.P, is

   $\sqrt{3},\ \sqrt[2]{3},\ \sqrt[3]{3},\ \sqrt[4]{3}$

   $\therefore d = \sqrt{3}$

   Next four terms are

   $\sqrt[5]{3},\ \sqrt[6]{3},\ \sqrt[7]{3},\ \sqrt[8]{3},$

6. We have $a_r = a(r-1)d$, so that

$$a_{p+q} + a_{p-q} = a + (p+q-1) + a + (p-q-1)d$$
$$= 2a + (p+q-1+p-q-1)d$$
$$= 2a + 2(p-1)\,d = 2ap$$

7. $k - \left(\frac{2}{3} - k\right) = \left(\frac{5}{8} - k\right) - k$

$\Rightarrow\ 2k - 2/3 = 5/8 - 2k$
$\Rightarrow 4k = 2/3 + 5/8 = 31/24$
$\Rightarrow k = 31/96$

8. $d = 16\frac{1}{5} - 17 = -\frac{4}{5}$

Let *nth* term be the first negative term we have

$a_n = a + (n-1)\,d = 17\,(n-1)\left(-\frac{4}{5}\right) < 0$

$\Rightarrow\ 17 < (n-1)\left(\frac{4}{5}\right) \Rightarrow n-1 > \frac{85}{4}$

$\Rightarrow n > \left(\frac{89}{4}\right) = 22\frac{1}{4}$

Thus, first negative is the 23^{rd} term.

9. $a_4 = 3a_1$
$\Rightarrow\quad a + 3d = 3a$
$\Rightarrow\quad 3\,d = 2a$ (i)

Also, $a_7 - 2a_3 = 1$
$\Rightarrow\quad a + 6d - 2[a + 2d] = 1$
$\Rightarrow\quad a + 6d - 2a - 4d = 1$
$\Rightarrow\quad -a + 2\,d = 1$ (ii)

Putting $d = 2a/3$ (from (i)) in (ii), we get

$-a + \frac{4a}{3} = 1 \Rightarrow \frac{a}{3} = 1 \Rightarrow a = 3$

From (i), $3d = 2(3) \Rightarrow\ d = 2$

10. $a_{p+1} = 2a_{q+1}$
$\Rightarrow\quad a + (p + 1 - 1)d = 2[a + ((q+1)-1)d]$
$\Rightarrow\quad a + pd = 2[a + qd]$

$\Rightarrow \quad a = (p - 2q)d$

Now, $a_{3p+1} = a + (3p + 1 - 1)d$

$= a + 3pd = (p - 2q)d + 3pd$

$= (4p - 2q)d = 2(2p - q)d$

and $a_{p+q+1} = a + (p + q + 1 - 1)d$

$= (p - 2q)\, d + (p + q)d$

$= (p - 2q + p + q)d$

$= (2p - q)d$

Thus, $a_{3p+1} = 2a_{p+q+1}$

11. $7a_7 = 11a_{11}$

$\Rightarrow \quad 7[a + 6d] = 11[a + 10d]$

$\Rightarrow \quad 7a + 42d = 11a + 110\, d$

$\Rightarrow \quad 4a + 68d = 0 \quad \Rightarrow \quad a + 17d = 0$

$\Rightarrow \quad a_{18} = 0$

Check Your Progress 2

1. Here $a = 1$, $d = 2$

$S_{100} = \frac{100}{2}[2a + (100 - 1)d]$

$= 50\,[2(1) + (99)2] = (50)(200)$

$= 10{,}000$

$S_{200} = \frac{200}{2}[2a + (200 - 1)d]$

$= 100\,[2(1) + (200 - 1)2] = (100)(400)$

$= 40{,}000$

2. $a = 0.9$, $d = 0.01$

$S_{20} = \frac{20}{2}[2a + (20 - 1)\, d]$

$= 10\,[2\,(0.9) + (19)\, 0.01] = 19.9$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$= \frac{n}{2}\,[2(0.9) + (n - 1)(0.01)]$

$= n\,(n + 179)/200$

3. $a = \dfrac{m-1}{m}, d = \dfrac{m-2}{m} - \dfrac{m-1}{m} = -\dfrac{1}{m}$

$S_{10} = \frac{10}{2}[2a + (10-1)d]$

$= 5\left[2\frac{(m-1)}{m} + 9\left(\frac{-1}{m}\right)\right]$

$= 5(2m-11)/m$

$$S_n = \frac{n}{2}[2a + (n-1)]d = \frac{n}{2}\left[2\frac{(m-1)}{m} + \frac{(n-1)(-1)}{m}\right]$$

$$= \frac{n}{2m}[2m - 2 - n + 1] = \frac{n}{2m}(2m - n - 1)$$

4. $a = 22, d = -4$

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[2(22) + (n-1)(-4)].$

$= \frac{n(4)}{2}[11 - n + 1] = 2n(12 - n)$

Now, $2n(12-n) = 64$

$\Rightarrow \quad n^2 - 12n + 32 = 0$

$\Rightarrow \quad (n-4)(n-8) = 0$

$\Rightarrow \quad n = 4, n = 8$

Double answer occurs as

$a_5 + a_6 + a_7 + a_8$

$= \quad 6 + 2 - 2 - 6 = 0$

5. $S_p = 3p^2 + 4p$

$a_n = S_n - S_{n-1}$

$= \quad 3n^2 + 4n - [3(n-1)^2 + 4(n-1)]$

$= \quad 3n^2 + 4n - [3n^2 - 2n + 1 + 4n - 4]$

$= \quad 3n^2 + 4n - [3n^2 - 6n + 3 + 4n - 4]$

$= \quad 6n + 1$

6. The smallest 3 digit number which leaves remainder 1 when divided by 4 is 101 and the last 3 digit number with this property is 997. Let 997 be the nth term of the A.P. then

$997 = 101 + (n-1)(4) \Rightarrow n = 225$

Required sum $= \frac{n}{2}[a + l] = \frac{225}{2}(101 + 997) = 123525$

Check Your Progress – 3

1. $r = \frac{1}{2}$

2. $r = \frac{1}{(-3)} = -\frac{1}{3}$

3. $r = \frac{6}{3} = 2$

4. $r = \frac{-96}{128} = -\frac{3}{4}$

   $a_n = a \quad r^{n-1} = 128\left(-\frac{3}{4}\right)^{n-1}$

5. $r = \frac{-110}{100} = -1.1$

   $a_n = a\,r^{n-1} = 100(-1.1)^{n-1}$

6. $r = \frac{3.3}{3} = 1.1.$

   $a_n = r^{n-1} = 3(1.1)^{n-1}$

7. $a_{18} = a\,r^{18-1} = a\,r^{17}$

   $a_5 = a\,r^{5-1} = a\,r^4 = 1$

   Now, $\frac{a_{18}}{a_5} = \frac{a\,r^{17}}{a\,r^4} = r^{13}$

   $\Rightarrow \quad a_{18} = a_5 r^{13} = 1\left(\frac{2}{3}\right)^{13}$

8. $a = a_5 = AR^4$

   $b = a_8 = AR^7$

   and $c = a_{11} = AR^{10}$

   We have $\frac{b}{a} = R^3 = \frac{c}{b}$

   $\Rightarrow$ *a,b, c are in G.P.*

Check Your Progress – 4

1. Here $a = 1$, $r = 2/3$

$$S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{1-\left(\frac{2}{3}\right)^{10}}{1-\left(\frac{2}{3}\right)} = 3\left[1-\left(\frac{2}{3}\right)^{10}\right]$$

$$\text{and } s_n = \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{2}{3}\right)^n}{1-2/3} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

2. $a = 2, \quad r = \dfrac{-1/2}{2} = -\dfrac{1}{4}$

$$S_{12} = \frac{a(1-r^{12})}{1-r} = \frac{2\left(1-\left(-\frac{1}{4}\right)^{12}\right)}{1-\left(\frac{-1}{4}\right)} = \frac{8}{5}\left[1-\left(\frac{1}{4}\right)^{12}\right]$$

$$\text{and } S_n = \frac{a(1-r^n)}{1-r} = \frac{2\left(1-\left(-\frac{1}{4}\right)^{n}\right)}{1-\left(\frac{-1}{4}\right)} = \frac{8}{5}\left[1-\left(-\frac{1}{4}\right)^{n}\right]$$

3. Let s_n = 5 + 55 + 555 + ……….+ upto n terms

= 5 [1 + 11 + 111 + ………… + upto n terms]

$$= \frac{5}{9}[9 + 99 + 999 + \dots\dots\dots\dots + \text{upto } n \text{ terms}]$$

$$= \frac{5}{9}[(10 - 1) + (10_2 - 1) + (10_3 - 1) + \dots\dots\dots\dots + (10n - 1)]$$

$$= \frac{5}{9}[(10 + 10_2 + \cdots \dots\dots\dots\dots\dots + 10_n]$$

$$= \frac{5}{9}\left[\frac{10(10^n-1)}{10-1} - n\right] = \frac{5}{9}\left\{\frac{10(10^n-1)}{10-1}\right\} - n]$$

4. Let S$_n$ = 0.6 + 0.66 + 0.666 + ……………… + upto n terms

= 6[0.1 + 0.11 + 0.111 + ………… + upto n terms]

$$= \frac{6}{9}[0.9 + 0.99 + 0.999 + \dots\dots\dots\dots + \text{upto } n \text{ terms}]$$

$$= \frac{2}{3}[(1 - 0.1) + (1- 0.01) + (1- 0.001) + \dots\dots\dots\dots (1 - (0.1)^n)]$$

$$= \frac{2}{3}[n - \{0.1 + (0.1)^2 + \dots\dots\dots\dots + (0.1)^n\}]$$

$$= \frac{2}{3}[n - \{0.1 + (0.1)^2 + \dots\dots\dots\dots + (0.1)^n\}]$$

$$= \frac{2}{3}\left[n - \frac{(0.1)(1-(0.1)^n)}{1-0.1}\right]$$

$$= \frac{2}{3}[n - \frac{1}{9}(1-(0.1)^n)$$

5. We have

$$A = \underbrace{111\ldots\ldots\ldots\ldots1}_{91 \text{ times}} = \frac{1}{9}(\underbrace{99\ldots\ldots\ldots9}_{91 \text{ times}})$$

$$= \frac{1}{9}\,(10^{91} - 1)$$

$$= \frac{10^{91}-1}{10^{13}-1}.\frac{10^{13}-1}{10-1}$$

$$= [1+10^{13} + 10^{26}+ \ldots\ldots\ldots\ldots\ldots 10^{78}]$$

$$[1+10\ldots\ldots\ldots 10^{12}]$$

$\Rightarrow$ a is not a prime number.

6. Let three numbers in G. P. be $\frac{a}{r}, a, ar$

We have

$$\frac{a}{r} + a + ar = 31$$

and $\frac{a^2}{r^2} + a^2 + a^2 r^2 = 651$

$$\Rightarrow a\,(r + \tfrac{1}{r} + 1) = 31$$

and $a^2(r^2 + \frac{1}{r^2} + 2 - 1) = 651$

$$\Rightarrow \quad a\left(r + \frac{1}{r} - 1\right) = 31$$

and $a^2\left[\left(r + \frac{1}{r}\right)^2 - 1\right] = 651$

$$\Rightarrow \quad a\left(r + \frac{1}{r} + 1\right) = 31$$

and $a\left(r + \frac{1}{r} + 1\right) = \frac{651}{31} = 21$

Subtracting we get $2a = 10 \Rightarrow a = 5.$

Also, $5\left(r + \frac{1}{r} + 1\right) = 31$

$$\Rightarrow r + \frac{1}{r} = \frac{26}{5} \Rightarrow r = 5, \frac{1}{5}$$

Thus, numbers are

1, 5, 25 or 25, 5, 1

7. $S = \dfrac{a}{1-r} = \dfrac{7}{1-\left(-\frac{1}{7}\right)} = \dfrac{49}{8}$

8. $a = 28, \quad ar^3 = \dfrac{4}{49}$

$$\Rightarrow r^3 = \frac{4}{49} \times \frac{1}{28} = \frac{1}{7^3}$$

$\Rightarrow r = 1/7$

$$\text{Thus}, S = \frac{a}{1-r} = \frac{28}{1-1/7} = \frac{28 \times 7}{6} = \frac{98}{3}$$

Check Your Progress – 5

1. Here, $a = 3$, $d = 2$, $r = 1/4$

Thus

$$S_n = \frac{3}{1-1/4} + \frac{(2)\left(\frac{1}{4}\right)\left[1-\left(\frac{1}{4}\right)^{n-1}\right]}{\left(1-\frac{1}{4}\right)^2} - \frac{3+(n-1)(2)\ (1/4)^n}{1-\frac{1}{4}}$$

$$= 4 + \frac{8}{9}\left[1-\left(\frac{1}{4}\right)^{n-1}\right] - \frac{1}{3}(2n-1)\left(\frac{1}{4}\right)^{n-1}$$

$$= 4 + \frac{8}{9}\left[\frac{8}{9} + \frac{2n+1}{3}\right]\left(\frac{1}{4}\right)^{n-1}$$

$$= \frac{44}{9} - \frac{6n+11}{9}\left(\frac{1}{4}\right)^{n-1}$$

2. Here, $a = 1$, $d = 2$, $r = x$ Thus,

$$S_n = \frac{1}{1-x} - \frac{(2)(x)(1-x^{n-1})}{(1-x)^2} - \frac{(1+2n-1)x^n}{1-x}$$

$$= \frac{1}{1-x} + \frac{2x}{(1-x)^2} + \frac{2x^n}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$$

$$= \frac{1-3x}{(1-x)^2} + \frac{2x^n}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$$

3. Here $a = 1$, $d = 4$, $r = x$

Thus,

$$S = \frac{1}{1-x} + \frac{4x}{(1-x)^2} = \frac{1-x+4x}{(1-x)^2}$$

$$= \frac{1-3x}{(1-x)^2}$$

Check Your Progress 6

1. 2,7,12,17 ……… are in A.P.

$$a_n = a + (n-1)d = 2 + (n-1)(5) = 5n - 3$$

Thus, nth term of the H. P. is $\frac{1}{5n-3}$

2. We have

$$2^4 + 2^4 \ldots\ldots\ldots\ldots\ldots\ldots + (n^2)^2$$

$$= 2^2\,[1^2 + 2^2 + \ldots\ldots\ldots..2^n]$$

$$= 4\frac{n(n+1)(2n+1)}{6} = \frac{2}{3}n(n+1)(2n+1)$$

3. The rth term of 1.2.3. + 2 3.4 + 3.4.5 + …. is given by

$$t_r = r(r+1)(r+2) = r(r^2 + r + 2r + 2) = r^3 + 3r^2 + 2r$$

Thus $\sum_{r=1}^{n} \mathrm{t_r} = \sum_{r=1}^{n}(r^3 + 3r^2 + 2r) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$$= \frac{1}{4}n^2(n+1)^2 + 3\left\{\left\{\frac{1}{6}n(n+1)(2n+1)\right\} + 2\left\{\frac{1}{2}n(n+1)\right\}\right\}$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + n(n+1)$$

$$= \frac{1}{4}n(n+1)[n(n+1) + 2(2n+1) + 4] = \frac{1}{2}n(n+1)[n^2 + n + 4n + 2 + 4]$$

$$= \frac{1}{4}n(n+1)(n^2 + 5n + 6) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

4. The rth term of the series is given by

$$t_r = (2r-1)(2r+1)(2r+3) = 8r^3 + 12r^3 - 2r - 3$$

Therefore, S_n the sum to n terms of the series is given by

$$\mathrm{S}_n = 8\sum_{r=1}^{n} r^3 + 12\sum_{r=1}^{n} r^2 + 8\sum_{r=1}^{n} r - 3n$$

$$= 8\left[\frac{n(n+1)}{2}\right]^2 + 12\,\frac{n\,(n+1)(2n+1)}{6} - 2\,\frac{n(n+1)}{2} - 3n$$

$= 2n^2(n+1)^2 + 2n(n+1)(2n+1) - n(n+1) - 3n$

$= n(2n^3 + 4n^2 + 2n + 4n^2 + 6n + 2 - n - 1 - 3)$

$= n(2n^3 + 8n^2 + 7n - 2)$

1.10 SUMMARY

In this unit, the well-known concepts of arithmetic progression (A.P.), geometric progression (G.P.), arithmetico-geometric progression (A.G.P.) and harmonic progression as special cases of sequences of numbers, are discussed. First of all, each of these concepts is defined. Then for each concept, through suitable examples, methods for finding nth term and sum upto n terms are explained. A.P is discussed in **sections 1.2 and 1.3,** G.P is discussed in **sections 1.4 and 1.5,** A.G.P is discussed in **section 1.6** and H.P. is discussed in **section 1.7**. The derivation of formulae for the sum of natural numbers, their squares and cubes are discussed in **section 1.8.**

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.9**.

UNIT 2 COMPLEX NUMBERS

Structure

2.0 INTRODUCTION

All the numbers with which we have dealt so far were real numbers. However, some solutions in mathematics, such as solving quadratic equations require a new set of numbers. This new set of numbers is called the set of ***complex numbers***.

If we solve the equation $x^2 = 4$ for x, we find the equation has two solutions.
$x^2 = 4 \Rightarrow x = \sqrt{4} = 2$ or $x = -\sqrt{4} = -2$.

If we solve the equation $x^2 = -1$ in a similar way, we would expect it to have two solutions also.

$$x^2 = -1 \text{ should imply } x = \sqrt{-1} \text{ or } x = -\sqrt{-1}.$$

Each proposed solution of the equation $x^2 = -1$ involves the symbol $\sqrt{-1}$. For years it was believed that square roots of negative numbers denoted by $\sqrt{-5}, \sqrt{-2}$ and $\sqrt{-6}$ were nonsense. In the 17th century, these symbols were termed *imaginary numbers* by Rene Descartes (1596-1650). Now, the imaginary numbers are no longer thought to be impossible. In fact imaginary numbers have important uses in several branches mathematics and physics.

The number $\sqrt{-1}$ occurs so often in mathematics, that we give it a special symbol. We use better i to denote $\sqrt{-1}$. Since i stand for $\sqrt{-1}$, it immediately follows that $i^2 = -1$. The power of i with natural exponent produces an interesting pattern, as follows :

$i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$

also $i^{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$

2.1 OBJECTIVES

After studying this unit, you will be able to :

- define complex number and perform algebraic operations such as addition, substraction, multiplication and division on the complex numbers;
- find modulus, argument and conjugate of a complex number;
- represent complex numbers in the argand plane;
- write polar form of a complex number;
- use Demoivre's theorem; and
- find cube roots of unity and verify some of the identities involving them.

2.2 COMPLEX NUMBERS

Definition : A *complex number* is any number that can be put in the form $a+bi$, where a and b are real number and $i = \sqrt{-1}$. The form $a + bi$ is called **standard form** for complex number. The number a is called the real part of the complex number. The number b is called imaginary part of the complex number.

We usually denote a complex number by z. We write $z = a + bi$. The real part of z is denoted by Re (z) and the imaginary part of z is denoted by Im(z).

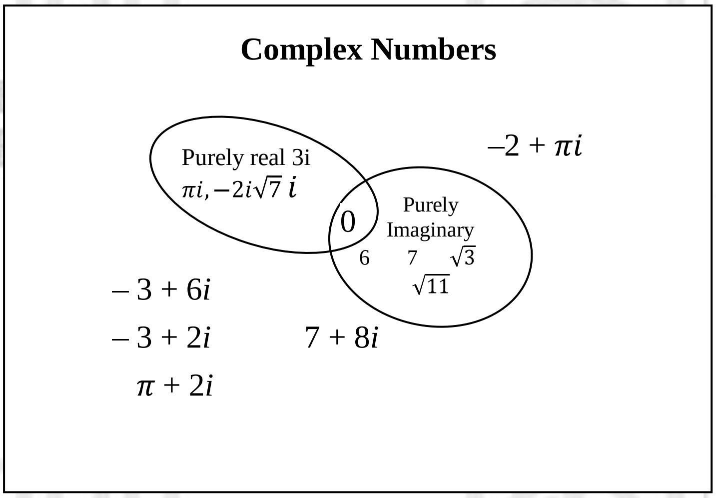


Figure 1

If $b = 0$, the complex number $a + bi$ is the real number a. Thus, any real number is a complex number with zero imaginary part. In other words, the set of real numbers is a subset of the set of complex numbers.

Equality of two Complex Numbers

Two complex numbers are equal if and only if their real parts are equal and also their imaginary parts are equal.

Thus if, $z_1 = a + bi$ *and* $z_2 = c + di$ are two complex numbers, then $z_1 = z_2$, that is, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Example 1 (a) Find x and y if $3x + 4i = 12 - 8y\, i$

(b) Find a and b if $(4a - 3) + 7i = 5 + (2b - 1)i$

Solution :

(a) Since the two complex numbers are equal, their real parts are equal and their imaginary parts are equal :

$$3x = 12 \text{ and } 4 = 8y \Rightarrow x = 4 \text{ and } y = -1/2$$

(b) The real parts are $4a - 3$ and 5. The imaginary parts are 5 and $2b - 1$.

$4a - 3 = 5$ and $7 = 2b - 1 \Rightarrow 4a = 8$ and $2b = 8 \Rightarrow a = 2$ and $b = 4$.

2.3 ALGEBRA OF COMPLEX NUMBERS

Addition of two Complex Numbers

Two complex numbers such as $z_1 = a + bi$ and $z_2 = c + di$ are added as if they are algebraic binomials:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)\, i$$

Observe that $a + bi = (a + 0i) + (0 + bi)$. In other words, $a + bi$ is the sum of the real number a and the imaginary number bi.

Also observe that $z_1 + z_2$ is a complex number.

Illustration

(i) $(3 + 4i) + (7 - 6i) = (3 + 7) + (4 - 6)\, i = 10 - 2i$

(ii) $(8 - 3i) + (6 - 2i) = (8 + 6) + (-3 - 2)\, i = 14 - 5i$

Subtraction of Complex Numbers

If $z_1 = a + bi$ and $z_2 = c + di$, we define $z_1 - z_2$ as $z_1 + (-z_2)$.

That is, $z_1 - z_2 = (a + bi) + ((-c) + (-d)i) = (a - c) + (b - d)\, i$

Example 2

Fill in the blanks

(i) $(-4 + 10\,i) + (-1 + 2\,i) = \ldots.$ (ii) $(-6 + 17i) + (4 - 11\,i) = \ldots$

(ii) $(-4 + 2\,i) + (7 - 2\,i) = \ldots.$ (iv) $(3 - 5\,i) + (-3 + 5\,i) = \ldots$

Solution

(i) $-5 + 12\mathrm{i}$ (ii) $-2 + 6i$

(iii) 3 (iv) 0

Example 3

Fill in the blanks

(i) $-(3 + 4i) = \ldots\ldots\ldots\ldots$ (ii) $(3 - 2i) - (4 - 3i) = \ldots\ldots\ldots\ldots.$

(iii) $(2 + 3i) - (i)$ ………….. (iv) $(5 + 2i) - 2 = \ldots\ldots\ldots\ldots$

Solution

(i) $-3 - 4i$ (ii) $-1 + i$

(iii) $2 + 2i$ (iv) $3 + 2i$

Multiplication of Complex Numbers

Two complex numbers such as $z_1 = a + bi$ and $z_2 = c + di$ are multiplied as if they were algebraic binomials, with $i^2 = -1$;

$$z_1\ z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

By definition, product of two complex numbers is again a complex number. Also observe that $yi = (y + 0i)\ (0 + 1i)$ and is, thus the product of the real number y and the imaginary number i.

Illustration 1

$(3 + 2i)\ (4 + 5i) = 12 + 5i + 8i + 10i^2 = 12 + 15i + 8i - 10 = 2 + 23i$ $[\because i^2 = -1]$

and $(2 + 5i)(7 + 3i) = 14 + 6i + 35i + 15i^2 = 14 + 6i + 35i - 15 = -1 + 41i$

$[\because i^2 = -1]$

Example 4 Perform the indicated operations and write the results in the form of $a + bi$

(i) $(2 + 3i)^2$ (ii) $(1 + i)^3$

(iii) $(\sqrt{5} + 7i)(\sqrt{5} - 7i)$

Solution

(i) $(2 + 3i)^2 = (2 + 3i)(2 + 3i) = (2)(2) + (2)(3i) + (2)(3i) + (3i)(3i)$

$$= 4 + 6i + 6i + 9i^2 = 4 + 12i - 9 = -5 + 12i$$

(ii) $(1+i)^3 = (1+i)(1+i)(1+i) = (1+i+i+i^2)(1+i) = (1+i+i-1)(1+i)$

$$= 2i(1+i) = 2i - 2i^2 = -2 + 2i$$

(iii) $(\sqrt{5} + 7i)(\sqrt{5} - 7i) = (\sqrt{5})(\sqrt{5}) - (\sqrt{5})(7i) + (\sqrt{5})(7i) - (7i)(7i)$

$$= 5 + 7(\sqrt{5}i) - 7(\sqrt{5}i) - 49i^2 = 5 + 49 = 54$$

Multiplicative Inverse of a Non-Zero Complex Number

If $a + ib \neq 0$ is any complex number, then there exists a complex number $x + iy$ such that

$(a + ib)(x + iy) = 1 + 0i =$ the multiplicative identity in C.

The number $x + iy$ is called the multiplicative inverse of $(a + ib)$ in C.

Now, $(a + ib)(x + iy) = 1 + 0i \Rightarrow (ax - by) + i(ay - bx) = 1 + 0i$
[multiplication of complex numbers]

$\Rightarrow$ $ax - by = 1$ and $ay + bx = 0$ [equality of two complex numbers]

$\Rightarrow$ $ax - by - 1 = 0$ and $ay + bx = 0$

Solving these equations for x and y, we have

$$x = \frac{a}{a^2 + b^2} \qquad (1)$$

$$y = \frac{-b}{a^2 + b^2} \qquad (2)$$

both of which exist in **R,** because $(a + ib) \neq 0$ i.e., at least one of a, b is different from zero.

Thus, the multiplicative inverse is of $a + ib$ is

$$x + iy = \frac{a}{a^2 + b^2} - i\frac{a}{a^2 + b^2} = \frac{a - ib}{a^2 + b^2}$$

Thus, every non-zero complex number has a multiplicative inverse in **C.**

Division in Complex Numbers

If $Z_1 = x + i\,y$ and $Z_2 = a + ib \neq 0$,

then

$$\frac{Z_1}{Z_2} = \frac{x + iy}{a + ib} = (x + iy)\frac{1}{(a + ib)}$$

$$= (x + i\,y)\frac{(a - ib)}{(a^2 + b^2)}$$

$$= \frac{ax + by}{a^2 + b^2} + i\frac{bx - ay}{a^2 + b^2}$$

Example 5 If $\left(\frac{1 - i}{1 + i}\right)^{100} = a + ib$, then show that $a = 1$ and $b = 0$

Solution: We have

$$\frac{1 - i}{1 + i} = \frac{(1 - i)}{(1 - i)}\frac{(1 - i)}{(1 + i)} = \frac{(1 - i)^2}{1^2 - i^2}$$

$$= \frac{1 - 2i + i^2}{2} = \frac{1 - 2i - 1}{2} = -i$$

Thus, $\left(\frac{1 - i}{1 + i}\right)^{100} = (-i)^{100} = 1$

$\therefore\ a + ib = 1 \Rightarrow a = 1$ and $b = 0$

Example 6 : If $x = -2 - \sqrt{3}\,i$, find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$.

Solution :

$x = -2 - \sqrt{3}\,i, \Rightarrow x + 2 = -\sqrt{3}\,i \Rightarrow (x + 2)^2 = (-\sqrt{3}\,i)^2$

$\Rightarrow \quad x^2 + 4x + 4 = -3$ or $x^2 + 4x + 7 = 0$

We now divide $2x^4 + 5x^3 + 7x^2 - x + 41$ by $x^2 + 4x + 7$

$$
\begin{array}{r|l|l}
x^2 + 4x + 7 & 2x^4 + 5x^3 + 7x^2 - x + 41 & 2x^2 - 3x + 5 \\
 & 2x^4 + 8x^3 + 14x^2 & \\
 & \ominus \quad \ominus \quad \ominus & \\ \hline
 & \quad -3x^3 - 7x^2 - x + 41 & \\
 & \quad -3x^3 - 12x^2 - 21x & \\
 & \quad \oplus \qquad \oplus \qquad \oplus & \\ \hline
 & \qquad 5x^2 + 20x + 41 & \\
 & \qquad 5x^2 + 20x + 35 & \\
 & \qquad \ominus \quad \ominus \quad \ominus & \\ \hline
 & \qquad\qquad\qquad 6 &
\end{array}
$$

Thus, $2x^4 + 5x^3 + 7x^2 - x + 41 = (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$

$= (0)(2x^2 - 3x + 5) + 6 = 6$

$\therefore$ value of $2x^4 + 5x^3 + 7x^2 - x + 41$ for $x = -2 - \sqrt{3}\, i$ is 6.

Check Your Progress - 1

1. Is the following computation correct ?

$$\sqrt{-5}\,\sqrt{-7} = \sqrt{(-5)(-7)} = \sqrt{35}$$

2. Express each one of the following in the standard form $a + ib$.

(i) $\dfrac{1}{5-4i}$ (ii) $\dfrac{7+2i}{2-7i}$ (iii) $\dfrac{1}{\cos\theta + \text{i Sin } \theta}$ (iv) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}$

3. Find the multiplicative inverse of

(i) $\dfrac{1+i}{1-i}$ (ii) $(1 + \sqrt{3}\, i)^2$ (iii) $(1 + i)(1 + 2i)$

4. Find the value of $x^4 - 4x^3 + 4x^2 + 8x + 40$

when $x = 3 + 2i$.

5. If $(x + iy)^{1/3} = a + ib$, prove that

$$\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$$

6. Find the smallest positive integer for which

$$\left(\frac{1+i}{1-i}\right)^n = 1$$

2.4 CONJUGATE AND MODULUS OF A COMPLEX NUMBER

Conjuate of a Complex Number

Definition : If $z = x + i\,y$, $x, y \in$ **R** is a complex number, then the complex number $x - iy$ is called conjugate of z and is denoted by $\bar{z}$.

For instance,

$$\overline{2+3i} = 2 - 3\,i\,,\ \overline{3-4i} = 3 + 4\,i,\ \bar{i} = -\,i \text{ and}$$

$$\bar{3} = \overline{3+0i} = 3 - 0i = 3$$

Some properties of Complex Conjugates

1. $\bar{\bar{z}} = z$
2. $\overline{z_1 z_2} = \bar{z}_1\ \bar{z}_2$
3. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\ if\ z_2 \neq 0$

5 If $z = a + ib$, then

$z + \bar{z} = 2\,a = 2\,(z)$

and $z - \bar{z} = 2\,ib = 2i\ \text{Im}\,(z)$

6. $z = \bar{z} \quad \Leftrightarrow \quad z$ is real

7. $z = -\,\bar{z} \quad \Leftrightarrow \quad z$ is imaginary

Modulus of a Complex Number

Definition : If $z = x + iy$, $x, y \in$ **R** is a complex number, then the real number $\sqrt{x^2 + y^2}$ is called the modulus of the complex number z, and is denoted by $|z|$.

For instance, if $z = 2 + 3i$, then $|z| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

and if $z = 5 - 12i$, then $|z| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

Note that

$|z| = |-z| = |-\bar{z}| = |\bar{z}|$.

and if c is a real number, then $|cz| = |c|\,|z|$

Some properties of Modulus of complex numbers

1. $|z|^2 = z\,\bar{z}$
2. $|z| = 0 \Leftrightarrow z = 0$

3. $\frac{1}{Z} = \frac{\bar{Z}}{|Z|^2}$ if $Z \neq 0$ 4. $|Z_1Z_2| = |Z_1|\,|Z_2|$

5. $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$ if $Z_2 \neq 0$ 6. $-|Z| \leq Z \leq |Z|$

7. $|Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2 + Z_1\bar{Z}_2 + \bar{Z}_1Z_2$

$= |Z_1|^2 + |Z_2|^2 + 2Re(Z_1\bar{Z}_2)$

Example 7: If $a + ib \neq 0$, show that

$$\left|\frac{a-ib}{a+ib}\right| = 1$$

Solution : Let $Z = a + ib$, then $\bar{Z} = a - ib$

Since $|Z| = |\bar{Z}|$, we get

$1 = \frac{|\bar{Z}|}{|Z|} = \left|\frac{\bar{Z}}{Z}\right| = \left|\frac{a-ib}{a+ib}\right|$ $\left[\because \left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}\right]$

Example 8: If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then $x^2 + y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

Solution : $(x+iy)^2 = \frac{a+ib}{c+id}$

$\Rightarrow |(x+iy)^2| = \left|\frac{a+ib}{c+id}\right|$

$\Rightarrow |(x+iy)|^2 = \left|\frac{a+ib}{c+id}\right|$ $\left[\because \frac{|Z_1|}{|Z_2|} = \left|\frac{Z_1}{Z_2}\right|\right]$

$\Rightarrow \left(\sqrt{x^2+y^2}\right)^2 = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

$\Rightarrow x^2 + y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

Example 9: If $(a - ib)(x + iy) = (a^2 + b^2)i$ and $a + ib \neq 0$, show that $x = b$ and $y = a$.

Solution: Let $Z = a + ib$, then $Z\bar{Z} = a^2 + b^2$

Now, $(a+ib)(x - iy) = (a^2 + b^2)i$

$\Rightarrow Z(x + iy) = Z\bar{Z}i$

$\Rightarrow x + iy = \bar{Z}i = (a - ib)i = ai + b$

$\Rightarrow x = b, \quad y = a$ [by definition of equality of Compex Numbers]

Check Your Progress – 2

1. Let $Ƶ = x + iy$ and $\omega = \frac{1 - iƵ}{Ƶ - i}$. If $|\omega| = 1$, show that Ƶ is purely real.

2. If $|Ƶ| = 1$, $Ƶ \neq -1$ show that
$$\frac{Ƶ - 1}{Ƶ + 1} \text{ is purely imaginary}$$

3. If $|Ƶ - i| = |Ƶ + i|$, show that Im (Ƶ) = 0.

4. If $(a + bi)(3 + i) = (1 + i)(2 + i)$, find a and b.

5. If $(\text{Cos}\,\theta + i\,\text{Sin}\,\theta)^2 = x + iy$, that show $x^2 + y^2 = 1$.

2.5 REPRESENTATION OF A COMPLEX NUMBERS AS POINTS IN A PLANE AND POLAR FORM OF A COMPLEX NUMBER

Let OX and OY be two rectangular axes in a plane with their point of intersection as the origin.

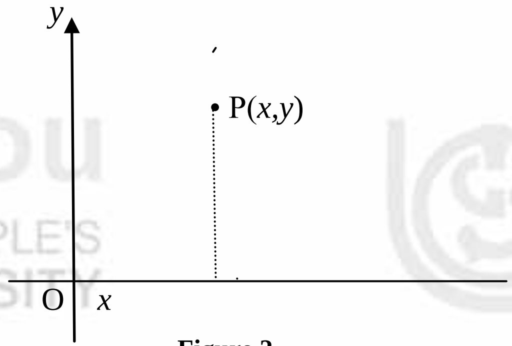


Figure 2

To each ordered pair (x,y) there corresponds a point P in the plane such that the x-coordiante of P is x and the y – coordinate of P is y. Thus, to a complex number $z = x + iy$ where corresponds a point $P\,(x,y)$ in the plane. Conversely, to every point $P'(x', y')$ there corresponds a complex number $x' + iy'$.

Thus, there is one-to-one correspondence between the set C of all complex numbers and the set of all the points in a plane.

For Example, the complex number $4 + 3i$ is represented by the point (4, 3) and the point (– 3, – 4) represents the complex number $– 3 –4i$.

We note that the points corresponding to the complex numbers of the type a lie on the x-axis and the complex numbers of the type bi are represented by points on the y-axis.

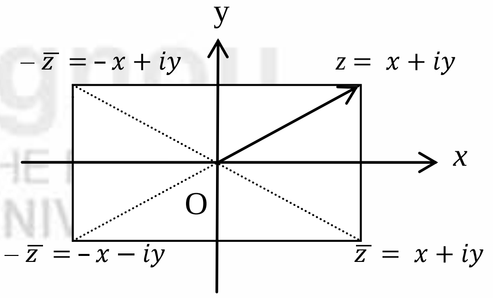


Figure 3

Note that the points *z* and – *z* are symmetric with respect to point *O*, while points z and – $\bar{z}$ are symmetric with respect to the real axis, since if *z* = *x* + *iy*, then –*z* – (– *x*) + *i*(–*y*) and $\bar{z}$ = *x* + *i*(–*y*). See Figure 3.

Remark : Since the points on the *x*-axis represent complex number *z* with *I(z)* = 0, the *x*-axis is also known as the real axis. Points on the y-axis represent complex numbers z with R(*z*) = 0, the *y*-axis is also known as the imaginary axis. The plane is called as the *Argand plane*, *Argand diagram*, *complex plane or Gaussian plane.*

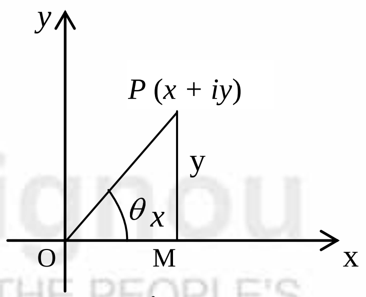


Figure 4

Note that *OP* = $\sqrt{x^2 + y^2} = |z|$

Polar Representation of Complex Numbers

Let *P(z)* represents the complex number *z* = *x* + *iy* as shown in the complex plane. Recall that the modulus or the absolute value of the complex number *z* is defined as the length *OP*. It is denoted by |*z*|. Thus if *r* = *OP*; we have

r = |*z*| = *OP*

= $\sqrt{OM^2 + PM^2} = \sqrt{x^2 + y^2}$

= $\sqrt{[Re(z)]^2 + [Im(z)]^2}$

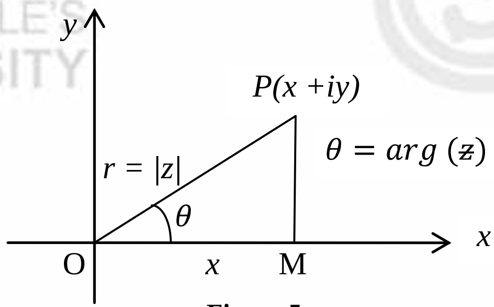


Figure 5

If θ be the angle which OP makes with OX in anticlockwise sense, then θ is called the *argument* or *amplitude* of the complex number *z* = *x* + *iy*.

Now in the right triangle ΔOMP,

x = OM = OP $\cos\theta = r\cos\theta$ (1)

y = MP = OP $\sin\theta = r\sin\theta$ (2)

Thus, the complex number z can be written as

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

This, is known as the *polar* form of the complex number.
Squaring and adding (1) and (2) we have

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2.1 = r^2$$

[Pythagorean identity]

Thus $r^2 = x^2 + y^2$ or $r = \sqrt{x^2 + y^2}$

which is the *modulus* of the complex number $z = x + iy$.

Dividing (2) and (1) , we have

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \tan\theta = \frac{y}{x}.$$

θ is the argument of the complex number $z = x + iy$.

The value of θ $(-\pi < \theta \leq \pi)$ is called the *principal* value of the argument or amplitude of z. We denote it by Arg z instead of arg z.

2.6 POWERS OF COMPLEX NUMBERS

Product of *n* Complex Numbers

We first take up product of complex numbers.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, ………..
$z_n = r_n(\cos\theta_n + i\sin\theta_n)$, then

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)]$$

However, we shall not prove this statement.
When $r_1 = r_2 = \ldots\ldots\ldots\ldots r_n = 1$, we get

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots\ldots\ldots\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1 + \theta_2 + \ldots\ldots\ldots\theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots\ldots\theta_n) \quad (1)$$

Corollary 1. $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$ and

2. $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

Proof From (1), above we have

$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$

Equating real and imaginary parts, we get

$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

and $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

De Moivre's Theorem (for Integral Index)

Taking $\theta_1 = \theta_2 \ldots\ldots\theta_n = \theta$ in (1) we obtain

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This proves the result for positive integral index.

However, it is valid for every integer *n*.

Example 10 : Use De Moivre's theorem to find $(\sqrt{3} + i)^3$.

Solution : We first put $\sqrt{3} + i$ in the polar form.

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow \quad (\sqrt{3})^2 + 1^2 = r^2(cos^2\theta + sin^2\theta)$

$\Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = 2$

Thus, $\sqrt{3} + i = 2(cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = 2\cos\theta$ and $1 = 2\sin\theta$

$\Rightarrow \quad 2\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = 30°$.

Now, $(\sqrt{3} + i)^3 = [2\cos(30°) + i\sin(30°)]^3$

$= 2^3[\cos(30°) + I\sin(30°)]^3$

$= \quad 8\,[\cos(3\times 30°) + i\sin(3\times 30°))]$ [De Moivre's theorem]

$= \quad 8\,(\cos 90° + i\sin 90°) = 8(0+i)$

$= \quad 8i$

Cube Roots of Unity

Let $x = (1)^{1/3}$

$\Rightarrow x^3 = 1 \Rightarrow x^3 - 1 = 0 \quad \Rightarrow (x-1)(x^3 + x + 1) = 0$

Therefore, either $x - 1 = 0 \Rightarrow (x-1)(x^2 + x + 1) = 0$

$$\Rightarrow either\ x = 1\ or\ x = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm \sqrt{(-3)}}{2} = \frac{-1 \pm i\sqrt{(3)}}{2}$$

Thus, the three cube roots of unity are, $1, \frac{-1}{2} + i\,\frac{\sqrt{3}}{2}, \frac{-1}{2} - \frac{i\sqrt{3}}{2}$

Hence, there are three cube roots of unity.

Out of these one root (i.e., 1) is real and remaining two viz.,

$\frac{-1 + i\sqrt{(3)}}{2}\ and\ \frac{-1 - i\sqrt{(3)}}{2}$ are complex.

We ususally denote the cube root $\frac{-1}{2} + \frac{\sqrt{3}}{2}\,i$ by ω note that

$$\omega^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)^2 = \frac{1}{4} - \frac{3}{4} - \frac{2\sqrt{3}}{4}\,\mathrm{i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$$

Hence, the cube roots of unity are 1, ω, ω^2.

Also, note that $\omega^3 = 1$.

Some properties of Cube Roots of Unity

1. $1 + \omega + \omega^2 = 0$
2. $\omega^3 = 1$
3. $\frac{1}{\omega} = \omega^2$ and $\frac{1}{\omega^2} = \omega$

Example 11: If 1, ω, ω^2 are cube roots of unity, show that

(i) $(1+\omega)^2 - (1+\omega)^3 + \omega^2 = 0$

(ii) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$

Solution : (i) As $1+\omega+\omega^2 = 0$, we get

$1+\omega = -\omega^2$ and $1+\omega^2 = -\omega$

Thus,

$(1+\omega)^2 - (1+\omega^2)^3 + \omega^2$

$= (-\omega^2)^2 - (-\omega)^3 + \omega^2$

$= \omega^4 + \omega^3 + \omega^2 = \omega^3\omega + 1 + \omega^2$

$= \omega + 1 + \omega^2 = 0$

(ii) Since $\omega^{10} = (\omega^3)^3\ \omega = \omega$

and $\omega^{11} = (\omega^3)^3\ \omega^2 = \omega^2$,

Thus $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4-2\omega-2\omega^2+\omega^3]^2$

$= [4-2(\omega+\omega^2)+1]^2$

$= [4-2(-1)+1]^2$ $\qquad [\because \omega+\omega^2 = -1]$

$= 7^2 = 49$

Example 12: If $x = a+b$, $y = a\omega + b\omega^2$

and $z = a\omega^2 + b\omega$, show that

$xyz = a^3 + b^3$

Solution: xyz

$= (a+b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= (a+b)(a^3\omega^3 + ab\omega^4 + ab\,\omega^2 + b^2\omega^3)$

$= (a+b)(a^2 + ab(\omega^3\omega + \omega^2) + b^2)$ $\quad [\because \omega^3 = 1]$

$= (a+b)[a^2 + ab(-1) + b^2)]$

$= (a+b)(a^2 - ab + b^2)$

$= a^3 + b^3$ $\qquad [\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Check Your Progress – 3

1. Calculate

   (i) $(\cos 30° + i \sin 30°)(\cos 60° + i \sin 60°)$

   (ii) $(\cos \theta + i \sin\theta)(\cos \theta - i \sin\theta)$

   (iii) $(\cos 45° + i \sin 45°)^2$

2. Use identities

   $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

   $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$ to obtain values of

   (i) $\cos(75°)$ (ii) $\sin 75°$

   (iii) $\cos(90° + \theta)$ (iv) $\sin(90° + \theta)$

   (v) $\cos(105°)$ (vi) $\sin(105°)$

3. Using the identities in Question 2, show that

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

4. If 1, ω, ω^2 are three cube roots of unity, show that

   (i) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^6)(1 + \omega^8) = 2$

   (ii) $(1 - \omega^2 + \omega^2)^5 + (1 - \omega^2 - \omega^2)^5 = 32$

   (iii) $(2 + 3\omega + 2\omega^2)^9 = (2 + 3\omega + 3\omega^2)^9 = 1$

5. If $x = a + b$, $y = a\omega + b\omega^2$ and

   $z = a\omega^2 + b\omega$, show that

   (i) $x + y + z = 0$ (ii) $x^2 + y^2 + z^2 = 6ab$

   (ii) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$

2.7 ANSWERS TO CHECK YOUR PROGRESS

1. No.

   The formula $\sqrt{a}\sqrt{b} = \sqrt{ab}$

   holds when at least one of $a, b \geq 0$.

2. (i) $\dfrac{1}{5 - 4i} = \dfrac{5 + 4i}{(5 - 4i)(5 + 4i)} = \dfrac{(5 + 4i)}{25 + 16}$

$$= \frac{5}{41} + \frac{4}{41} i$$

   (ii) $\dfrac{7 + 2i}{2 - 7i} = \dfrac{7 + 2i}{-2i^2 - 7i} = \dfrac{7 + 2i}{(-i)(7 + 2i)} = \dfrac{-1}{i} = \dfrac{i^2}{i} = i$

(iii) $$\frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{(\cos^2\theta - i^2\sin^2\theta)} = \frac{\cos\theta - i\sin\theta}{(\cos^2\theta + i^2\sin^2\theta)} = \cos\theta - i\sin\theta$$

(iv) $$\frac{2-\sqrt{-25}}{1-\sqrt{-16}} = \frac{2-5i}{1-4i} = \frac{(2-5i)}{(1-4i)}\frac{(1+4i)}{(1+4i)}$$

$$= \frac{2-5i+8i-20i^2}{1-16i^2}$$

$$= \frac{22+3i}{17} = \frac{22}{17} + \frac{3}{17}i$$

3. (i) Multiplicative inverse of $\frac{1+i}{1-i}$ is

$$\frac{1-i}{1+i} = \frac{1-i}{1+i}\frac{1-i}{1-i} = \frac{(1-i)^2}{1^2-i^2} = \frac{1+i^2-2i}{1+1} = \frac{1-1-2i}{2} = -i$$

(i) Multiplicative inverse of $(1+\sqrt{3}\,\mathrm{i})^2$is

$$\frac{1}{(1+\sqrt{3}\,i)^2} = \frac{(1-\sqrt{3}\,\mathrm{i})^2}{\left((1+\sqrt{3}\,i)\;\;(1-\sqrt{3}\,i)\;\;\right)^2} = \frac{1-2\sqrt{3}\,i+3i^2}{(1+3)^2} = \frac{1-2\sqrt{3}\,i-3}{16}$$

$$= \frac{-2-2\sqrt{3}\,i}{16} = -\frac{1}{8}(1+\sqrt{3}i)$$

(ii) We have

$$(1+i)(1+2i) = 1+1i+2i+2i = 1+3i-2 = -1+3i$$

Its multiplicative inverse is

$$\frac{1}{-1+3i} = \frac{-1-3i}{(-1+3i)(-1-3i)}$$

$$= \frac{-1-3i}{1-9i^2} = \frac{-1-3i}{1+9} = -\frac{1}{10} - \frac{3}{10}i = -\frac{1}{10}(1+3i)$$

4. $x = 3 + 2i \Rightarrow x - 3 = 2i$

$$\Rightarrow (x-3)^2 = (2i)^2 \Rightarrow x^2 - 6x + 9 = -4$$
$$\text{or}\quad x^2 - 6x + 13 = 0$$

Let's divide $x^4 - 4x^3 + 4x^2 + 8x + 39$ by $x^2 - 6x + 13$.

$$
\begin{array}{r|l}
x^2 - 6x + 13 & x^4 - 4x^3 + 4x^2 + 8x + 40 \;\big(\; x^2 + 2x + 3 \\
 & x^4 - 6x^3 + 13x^2 \\
 & \ominus \quad \oplus \quad \ominus \\
\hline
 & 2x^3 - 9x^2 + 8x + 40 \\
 & 2x^3 - 12x^2 + 26x \\
 & \ominus \quad \oplus \quad \ominus \\
\hline
 & 3x^2 - 18x + 40 \\
 & 3x^2 - 18x + 39 \\
 & \ominus \quad \oplus \quad \ominus \\
\hline
 & 1
\end{array}
$$

Thus, $x^4 - 4x^3 + 4x^2 + 8x + 40$

$= (x^2 - 6x + 13)(x^2 + 2x + 3) + 1$

$= 0 + 1 = 1$

5. $x + iy = (a + ib)^3 = a^3 + i^3b^3 + 3a(ib)(a + ib)$

$= (a^3 - 3a^2b) + i(3a^2b - b^3)$

$\Rightarrow x = a^3 - 3a^2b$ and $y = 3a^2b - b^3$

$\Rightarrow \frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = 3a^2 - b^2$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = (a^2 - 3b^2) + (3a^2 - b^2) = 4(a^2 - b^2)$

6. We have $\frac{1+i}{1-i} = \frac{-i^2+i}{1-i} = \frac{i(1-i)}{1-i} = i$

$\left(\frac{1+i}{1-i}\right)^n = i^n$

$\therefore$ The smallest value of n is 4.

Check Your Progess – 2

1. Let Ƶ $= x + iy$

Now, $|\omega| = 1 \Rightarrow |1 - i$ Ƶ$| = |$Ƶ $- i|$

$\Rightarrow |1 - i(x + iy)| = |x + iy - i|$

$\Rightarrow |(1 + y) - ix| = |x + (y - 1)i|$

$\Rightarrow |(1 + y) - ix|^2 = |x + (y - 1)i|^2$

$\Rightarrow (1 + y)^2 + x^2 = x^2 + (y - 1)^2$

$\Rightarrow 1 + 2y + y^2 = y^2 - 2y + 1 \Rightarrow 4y = 0$ or $y = 0$

$\therefore$ Ƶ $= x \Rightarrow$ Ƶ is purely real.

2. Let Ƶ $= x + iy$

As |Ƶ| = 1, we get $x^2 + y^2 = 1$

Now, $\dfrac{Z-1}{Z+1} = \dfrac{(x-1)+iy}{(x+1)+iy}$

$$= \frac{[(x-1)+iy][(x+1)-iy]}{(x+1)^2 + y^2}$$

$$= \frac{(x^2-1) + y^2 + iy\,(x+1-x+1)}{x^2 + 2x + 1 + y^2}$$

$$= \frac{(1-1) + 2i\,xy}{2(x\ +\ 1)} = \frac{xy}{x\ +\ 1}i$$

$\Rightarrow \dfrac{z-1}{z+1}$ is purely imaginary.

3. Let $Z = x + iy$

$$|Z - i| = |Z + i|$$

$\Rightarrow |x + iy - i| = |x + iy + i|$

$\Rightarrow |x + i\,(y-1)|^2 = |x + i\,(y+1)|^2$

$\Rightarrow \quad x^2 + (y-1)^2 = x^2 + (y+1)^2$

$\Rightarrow \quad (y-1)^2 - (y+1)^2 = 0$

$\Rightarrow -4y = 0 \Rightarrow y = 0$

Thus, Im $(Z) = 0$

4. $a\ +\ bi\ = \dfrac{(1+i)(2+i)}{3+i} = \dfrac{2-1+3i}{3+i}$

$$= \frac{1+3i}{3+i} = \frac{(1+3i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{3+3+(9-1)i}{9+1} = \frac{6+8i}{10}$$

$$= \frac{3}{5} + \frac{4}{5}i \Rightarrow a = \frac{3}{5}, \quad b = \frac{4}{5}$$

5. $|(cos\,\theta + i\sin\theta)^2| = |x + iy|$

$|cos\,\theta + i\sin\theta|^2 = |x + iy|$

$\Rightarrow |cos\,\theta + i\sin\theta|^2 = \sqrt{x^2 + y^2}$

$\Rightarrow \left(\sqrt{cos^2\theta + \sin^2\theta}\right)^2 = \sqrt{x^2 + y^2}$

$\Rightarrow \ x^2 + y^2 = 1$

Check Your Progress – 3

1. (i) cos (30° + 60°) + i sin (30° + 60°)

   = cos 90° + i sin90°= i

   (ii) $\left(\cos\theta\right)^2 - i^2\sin^2\theta = \sin^2\theta + \sin^2\theta = 1$

   (iii) cos (2(45°) + i sin (2 (45°))

   = cos 90° + i sin 90° = i

2. (i) cos 75° = cos (45°+ 30°)

   $= \cos 45° \cos 30° - \sin 45° \sin 30°$

   $$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

   $$= \frac{(\sqrt{3}-1)\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

   (ii) sin 75° = sin (45° + 30°)

   $= \sin 45° \cos 30° - \sin 45° \sin 30°$

   $$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}$$

   $$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{4}}{4}$$

   (iii) cos (90° + θ) = cos 90° cos θ – sin 90° sin θ
   = (0) (cos θ) – (1) sin θ = – sin θ

   (iv) sin (90° + θ) = sin 90° cos θ + cos 90° sin θ
   = (1) (cos θ) + (0) sin θ = cos θ

   (v) cos (105°) = cos (60° + 45°)

   = cos 60° cos 45° – sin 60°sin 45°

   $$= \frac{1}{2}.\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} = -\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

   $$= -\frac{\sqrt{6}-\sqrt{2}}{4}$$

   (vi) sin (105°) = sin 60° + 45°)

$= \sin 60° \cos 45° (\cos\theta) + \cos 60° \sin 45°$

$$= \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} - \frac{1}{2}.\frac{1}{\sqrt{2}} = -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$= -\frac{\sqrt{6}+\sqrt{2}}{4}$$

3. $\tan\left(\theta_1 + \theta_2\right) = \dfrac{sin(\theta_1 + \theta_2)}{cos(\theta_1 + \theta_2)}$

$$= \frac{\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2}{\cos\theta_1 \cos\theta_2 - \sin\theta_{1_1} \sin\theta_2}$$

Divide the numerator and denominator by $\cos\theta_1 \cos\theta_2$ to obtain

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

4. (i) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^6)(1+\omega^8)$

$= (1+\omega)(1+\omega^2)(1+\omega)(1+1)(1+\omega^2)$

$= 2((1+\omega)(1+\omega^2))^2 = 2((-\omega^2)(-\omega))^2 = 2\omega^6 = 2$

(ii) $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5$

$= (-\omega-\omega)^5 + (-\omega^2-\omega^2)^5$

$= (-2)^5\omega^5 + (-2)^5(\omega^2)^5$

$= -32\omega^2 - 32\omega = -32(\omega^2+\omega)$

$= (-32)(-1) = 32$

(iii) $(2+3\omega+2\omega^2)^9$

$= (2+2\omega+2\omega^2+\omega)^9 = (0+\omega)^9 = \omega^9 = 1$

and $(2+2\omega+2\omega^2)^9 = (2+2\omega+2\omega^2+\omega^2)^9$

$= (0+\omega^2)^9 = \omega^{18} = 1$

5. (i) $x + y + z = a(1+\omega^2+\omega)\, b(1+\omega^2+\omega)$

$= (0) + b(0) = 0$

(ii) $x^2 + y^2 + z^2$

$= (a^2 + b^2 + 2ab) + (a^2\omega^2 + b^2\omega^4 + 2ab\,\omega^3) + (a^2\omega^4 + b^2\omega^2 + 2ab\,\omega^3)$

$= a^2(1 + \omega^2 + \omega^2) + b^2(1 + \omega^4 + \omega^2) + 2ab(1 + \omega^3 + \omega^3)$

$= a^2(0) + b^2(0) + 2ab\,(1 + 1 + 1) = 6ab$

We know that

$x^3 + y^3 + z^3 - xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)$

$= 0$

Thus, $x^3 + y^3 + z^3 = 3xyz.$

Also, $xyz = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= (a+b)\,[a^3\omega^3 + b^2\omega^3 + ab\,(\omega^2 + \omega^4)]$

$= (a+b)(a^2 + b^2 - ab) = a^3 + b^3$

Thus,

$$x^3 + y^3 + z^3 - 3xyz = 3\,(a^3 + b^3)$$

2.8 SUMMARY

In this unit, first of all, in **section 2.2**, the concept of complex number is defined. In **section 2.3**, various algebraic operations, viz., addition, subtraction, multiplication and division of two complex numbers are defined and illustrated with suitable examples. In **section 2.4**, concepts of conjugate of a complex number and modulus of a complex number are defined and explained with suitable examples. The properties of conjugate and modulus operations are stated without proof. In **section 2.5**, representation of a complex number as a point in a plane, in cartesian and polar forms, are explained. Finally, in **section 2.6**, DeMoivre's Theorem for integral index, for finding nth power of a complex number, is illustrated with a number of examples. Also, some properties of cube roots of unity are discussed.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.7**.

UNIT 3 EQUATIONS

Structure

3.0 INTRODUCTION

Solution of equations lies at the very heart of algebra. It has enormous applications. The importance of equations stems from the fact that they provide a means by which many complicated relationships in real–life problems can be written down in a clear and concise form.

In earlier classes you studied how to solve a first degree (linear) equation.

$bx + c = 0$ $(b \neq 0)$.

Recall that you had obtained its roots as $x = -c/b$.

In this unit, we shall take up solving second, third and fourth degree equations in one variable.

3.1 OBJECTIVES

After studying this unit, you will be able to:

- solve a quadratic equation, cubic and biquadractic equations;
- find values of symmetric expressions involving roots of a cubic and quadratic equation;
- form equations whose roots are known; and
- use equations to solve several real life problems.

3.2 SOLUTION OF QUADRATIC EQUATIONS

Definition : Any equation that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b and c are real number and a not equal to 0 ($a \neq 0$), is called a quadratic equation. The form $ax^2 + bx + c = 0$ is called the standard form for a quadratic equation.

3.3 QUADRATIC FORMULA

We now use the method of completing the square to obtain the formula for roots of a quadratic equation. Towards this end we first list the steps for completing the square.

Steps for completing the Square

Step 1 : Write an equivalent equation with only the x^2 term and the x term on the left side of the equation. The coefficient of the x^2 term must be 1.

Step 2 : Add the square of one–half the coefficient of the x term to both sides of the equation.

Step 3 : Express the left side of the equation as a perfect square.

Step 4 : Solve for x.

Theorem (The Quadratic Formula)

For any quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$, the two solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Proof : We shall prove this quadratic theorem by completing the square on $ax^2 + bx + c = 0$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{[divide by } a\text{]}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{1}{2}.\frac{b}{a}\right) = -\frac{c}{a} + \left(\frac{1}{2}.\frac{b}{a}\right) \quad [add \ \left(\frac{1}{2}.\frac{b}{a}\right) \text{ to each side}]$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2\right]$$

$$\Rightarrow x^2 + \frac{b}{a}x = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad [\text{if } x^2 = k, x = \pm\sqrt{k}\,]$$

$$\Rightarrow \quad x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a} \qquad [\text{solve for } x]$$

$$\Rightarrow x = \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x = - \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

This complete the proof. Let us see what have proved. If our equation is in the form $ax^2 + bx + c = 0$(standard form), where $a \neq 0$, the two solutions are always given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is known as the *quadratic formula*. If we substitute the coefficient *a*, *b* and *c* of any quadratic equation in standard form in the formula, we need only perform some basic arithmetic to arrive at the solution set.

The Nature of Solutions

Since the solution set for every quadratic equation is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\},$$

the solutions can be expressed as

$$\alpha = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } \beta = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Discriminant

The expression D = $b^2 - 4ac$ is called discriminant because it determines the nature of the solutions of a quadratic equation.

1. If $b^2 - 4ac = 0$ then α = β and the equation will have one real root.

2. If $b^2 - 4ac > 0$ then α and β will be two distinct real numbers, and the equation will have two, unequal real roots.

3. If $b^2 - 4ac < 0$ then α and β will be two distinct complex numbers, and the equation will have no real roots.

If D = $b^2 - 4ac < 0$ then $4ac - b^2 > 0$. In this case, the complex number $\omega_1 = i\sqrt{4ac - b^2}$ and $\omega_2 = -i\sqrt{4ac - b^2}$ are such that $\omega^2 = b^2 - 4ac$ and no other complex number z is such that $z^2 = 4ac - b^2$. In this case the two roots may written as

$$\alpha = \frac{-b + \omega_1}{2a}, \text{and} \quad \beta = \frac{-b - \omega_2}{2a}$$

Sum and Product of the roots

We have

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$$\text{and } \alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \left(\frac{-b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 \qquad [(a+b)(a-b) = a^2 - b^2]$$

$$= \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Thus, if α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Forming Quadratic Equations with given Roots

An equation whose roots are α and β can be written as $(x - \alpha)(x - \beta) = 0$

or $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or $x^2 -$ (Sum of roots) x + (Product of the roots) = 0

Example 1: If α and β are the roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$ find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ (ii) $\alpha^2 + \beta^2$ (iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (iv) $\alpha^3 + \beta^3$

(v) $\alpha^2 + \beta^2$ (vi) $\alpha^{-3} + \beta^{-3}$ (vii) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$ (viii) $\alpha^2 - \beta^2$

Solution:

Since α and β are the roots of $ax^2 + bx + c = 0$, $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

(i) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-b/a}{c/a} = \frac{-b}{c}$.

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{-b}{a}\right)^2 - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$.

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \left(\frac{b^2 - 2ac}{a^2}\right)\left(\frac{a}{c}\right) = \frac{b^2 - 2ac}{ac}$. [see(ii)]

(iv) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ $[a^3 + b^3 = (a + b)^3 - 3ab(a + b)]$

$$= \left(\frac{-b}{a}\right)^3 - \frac{3c}{a}\left(\frac{-b}{a}\right) = \frac{-b^3}{a^3} + \frac{3bc}{a^3} = \frac{3abc - b^3}{a^3}.$$

(v) $\alpha^6 + \beta^6 = (\alpha^3)^2 + (\beta^3)^2 = (\alpha^2 + \beta^2)^3 - 3\alpha^2\beta^2(\alpha^2 + \beta^2)$

$[a^3 + b^3 = (a + b)^3 - 3ab(a + b)]$

$$= [(\alpha + \beta)^2 - 2\alpha\beta]^3 - 3(\alpha\beta)^2[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= \left[\left(\frac{-b}{a}\right)^2 - \frac{2c}{a}\right]^2 - 3\left(\frac{c}{a}\right)^2\left[\left(\frac{-b}{a}\right)^2 - \frac{2c}{a}\right]$$

$$= \left[\frac{b^2}{a^2} - \frac{2c}{a}\right]^3 - 3\left(\frac{c}{a}\right)^2\left[\frac{b^2}{a^2} - \frac{2c}{a}\right]$$

$$= \frac{(b^2 - 2ac)^3 - 3a^2c^2(b^2 - 2ac)}{a^6}$$

$$= \frac{1}{a^6}[b^6 + 8a^3c^3 + 3\,b^2\,(2ac)^2 - 3\,b^4\,(2ac) - 3\,a^2\,b^2\,c^2 + 6\,a^3c^3]$$

$$= \frac{1}{a^6}\;[b^6 - 9\,a^2\,b^2\,c^2 - 6a\,b^4c\; - \;2\,a^3\,c^3\,]$$

(vi) $\alpha^{-3} + \beta^{-3} \quad = \dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} = \dfrac{\alpha^3 + \alpha^3}{(\alpha\beta)^3} = \dfrac{(\alpha+\beta)^3 \;\; - \;\alpha\beta\,(\alpha + B)}{(\alpha\beta)^3}$

$$= \frac{(-b/a)^3 - 3\left(\frac{c}{a}\right)(-b/a)}{(c/a)^3}$$

$$= \frac{-\frac{b^3}{a^3} + 3(bc/a^2)}{c^3/a^3} = \frac{3abc - b^3}{c^3}$$

(vii) $\left(\dfrac{\alpha}{\beta} - \dfrac{\beta}{\alpha}\right)^2 = \left(\dfrac{\alpha^2 - \beta^2}{\alpha\beta}\right)^2 = \dfrac{[(\alpha-\beta)(\alpha+\beta)]^2}{\alpha^2\beta^2}$

$$= \frac{(\alpha+\beta)^2[(\alpha+\beta)^2 - 2\alpha\beta]}{(\alpha\beta)^2} = \frac{b^2}{a^2}\left[\frac{(b^2 - 2ac)/a^2}{c^2/a^2}\right]$$

$$= \frac{b^4 - 2\,ab^2c}{a^2c^2}$$

(viii) $\alpha^2 - \;\alpha^2\;(\alpha - \beta)(\;\alpha + \beta)$

But $(\alpha\; - \beta)^2 = (\alpha+\beta)^2 - \;4\;\alpha\beta = \left(\frac{-b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2 - 4ac}{a}$

$$\Rightarrow\; \alpha - \beta = \frac{\pm\sqrt{b^2 - 4ac}}{a}$$

Thus, $\quad \alpha^2 - \beta^2 = \dfrac{\pm\sqrt{b^2 - 4ac}}{a} . \left(\dfrac{-b}{a}\right) = \dfrac{\pm b\sqrt{b^2 - 4ac}}{b^2}$.

Example 2: Find the quadratic equations with real coefficients and with the following pairs of roots(s) :

(i) 3/5, –4/3 (ii) $1 + \sqrt{3}$, $1 - \sqrt{3}$

(iii) $\dfrac{1}{10 - \sqrt{72}}$, $\dfrac{1}{10 + 6\sqrt{2}}$ (iv) $\dfrac{m-n}{m+n}, -\dfrac{m+n}{m-n}$

(v) 2, $1+\sqrt{3}$ (vi) $2 - 3i$, $2 + 3i$

Solution :

(i) We have, sum of the roots $\frac{3}{5} + \left(\frac{-4}{3}\right) = \frac{9-20}{15} = \frac{-11}{15}$

and product of the roots $= \left(\frac{3}{5}\right)\left(\frac{-4}{3}\right) = -\frac{4}{5}$

Thus, the quadratic equation whose roots are 3/5, –4/3 is

$x^2 - (-11/15)x + (-4/5) = 0$ or $15x^2 + 11x - 12 = 0.$

(ii) We have, sum of the roots $(1+\sqrt{3}) + (1-\sqrt{3}) = 2.$

and product of the roots $(1+\sqrt{3})(1-\sqrt{3}) = 1-3 = -2.$

Thus, the quadratic equation whose roots are $1+\sqrt{3}$, $1-\sqrt{3}$ = is

$$x^2 - 2x - 2 = 0$$

(iii) Note that $\sqrt{72} = \sqrt{6^2.2} = 6\sqrt{2}.$

$$\text{Sum of the roots} = \frac{1}{10-\sqrt{72}} + \frac{1}{10+6\sqrt{2}}$$

$$= \frac{10+6\sqrt{2}+10-6\sqrt{2}}{100-72} = \frac{20}{28} = \frac{5}{7}$$

$$\text{and product of the roots} = \left(\frac{1}{10-\sqrt{72}}\right)\left(\frac{1}{10+6\sqrt{2}}\right) = \frac{1}{100-72} = \frac{1}{28}$$

Thus, the quadratic equation whose roots are $\frac{1}{10-\sqrt{72}}$ and $\frac{1}{10+6\sqrt{2}}$ is

$x^2 - (5/7)x + 1/28 = 0$ or $28x^2 - 20x + 1 = 0.$

(iv) We have sum of the roots $= \frac{m-n}{m+n} + \left(-\frac{m+n}{m-n}\right)$

$$= \frac{(m-n)(m-n) - (m+n)(m+n)}{m^2-n^2} = \frac{(m-n)^2 - (m+n)^2}{(m+n)^2} = -\frac{4mn}{(m+n)^2}$$

$$\text{and product of the roots} = \left(\frac{m-n}{m+n}\right)\left[-\frac{m+n}{m-n}\right] = -1.$$

Thus, the quadratic equation whose roots are $\frac{m-n}{m+n}, -\frac{m+n}{m-n}$ is

$$x^2 - \left(-\frac{4mn}{m^2-n^2}\right)x - 1 = 0 \text{ or } (m^2-n^2)\,x^2 + 4mnx - (m^2-n^2) = 0.$$

(iv) Sum of the roots = $2 + 1 + \sqrt{3} = 3 + \sqrt{3}$

and product of the roots = $2(1+\sqrt{3})$.

Thus, the required equation is $x^2 - (3+\sqrt{3})x + 2(1+\sqrt{3}) = 0$

(vi) Sum of the roots = $(2–3i) + (2–3i) = 4$

and product of the roots = $(2–3i)(2–3i) = 4 + 9 = 13$

Thus, the required quadratic equation is $x^2 - 4x + 13 = 0$.

Example 3

(i) The two roots r_1 and r_1 of the quadratic equation $x^2 + kx + 12 = 0$ are such that $|r_1 - r_2| = 1$. Find k.

(ii) The roots r_1 and r_2 of the quadratic equation $5x^2 - px + 1 = 0$ are such that $|r_1 - r_2| = 1$. Determine p.

(iii) If α and β be the roots of the equation $x^2 - 3ax + a^2 = 0$. Determine a if $\alpha^2 + \beta^2 = 7/4$.

(iv) Determine k is one of the roots of the equation $k(x-1)^2 = 5x - 7$ is double the other.

(v) If p and q are roots of the quadratic equation $x^2 + px + q = 0$. Find p and q.

(vi) If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $p : q$, show that $ac(p+q)^2 = b^2pq$.

(vii) If one root of the quadratic equation $ax^2 + bx + c = 0$ is square of the other. Prove that $b^3 + a^2c + ac^2 = 3abc$.

Solution :

(i) We have $r_1 + r_2 = -k$ and $r_1 r_2 = 12$.

Now, $|r_1 - r_2| = 1 \Rightarrow |r_1 - r_2|^2 = 1$

But $|r_1 - r_2| = 1 \Rightarrow (r_1 + r_2)^2 - 4r_1r_1 = (-k)^2 = -4(12) = k^2 - 48$

Thus, $k^2 - 48 = 1$ or $k^2 = 49 \Rightarrow k = \pm 7$.

(ii) We have $r_1 + r_2 = p/5$ and $r_1r_1 = 1/5$.

Now $|r_1 - r_2| = 1 \Rightarrow |r_1 - r_2|^2 = 1$

But $|r_1 - r_2|^2 = (r_1 + r_2)^2 - 4r_1r_2 = \left(\frac{p}{5}\right)^2 - 4\left(\frac{1}{5}\right) = \frac{p^2 - 20}{25}$

Thus, $\dfrac{p^2 - 20}{25} = 1$ or $p^2 - 20 = 25 \Rightarrow p^2 = 45$ or $p = \pm 3\sqrt{5}$.

(iii) We have $\alpha + \beta = 3a$ and $\alpha\beta = a^2$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9a^2 - 2a^2 = 7a^2$

$\Rightarrow \quad 7/4 = 7a^2 \text{ or } a^2 = 1/4 \text{ or } a = \pm 1/2.$

(iv) We first write the given quadratic equation in standard form, as

$k(x-1)^2 = 5x - 7 \Rightarrow k(x^2 - 2x + 1) = 5x - 7$

or $\quad kx^2 - (2k+5)x + k + 7 = 0.$

Let the roots of this equation be α and 2α. Then

$$\alpha + 2\alpha = \frac{2k+5}{k} \text{ and } \alpha.\ 2\alpha = \frac{k+7}{k} \Rightarrow 3\alpha = \frac{2k+5}{k} \text{ and } 2\alpha^2 = \frac{k+7}{k}.$$

Thus, $\alpha = \frac{2k+5}{3k}$. Putting this value in $2\alpha^2 = \frac{k+7}{k}$, we get

$$2\left(\frac{2k+5}{3k}\right)^2 = \frac{k+7}{k}$$

$\Rightarrow \quad 2(2k+5)^2 = 9k(k+7) \quad \Rightarrow \quad 2(4k^2 + 20k + 25) = 9k^2 + 63k$

$\Rightarrow \quad k^2 + 23k - 50 = 0 \quad \Rightarrow \quad k^2 + 25k - 2k - 50 = 0$

$\Rightarrow \quad k(k+25) - 2(k+25) = 0 \quad \Rightarrow \quad (k-2)(k+25) = 0$

$\Rightarrow \quad k = 2 \text{ or } k = -25.$

Thus, the required values of k are 2 and −25.

A Piece of Advise

It is always advisable to use the given condition and reduce the number of unknown. Here instead of beginning with α and β and putting β = 2 α, it is advisable to begin with α and 2α.

When one root is three times other, take the roots as α, 3α; when one is square of the other, take roots as α, α^2.

(v) We have $p + q = -p$ and $pq = q$

Now, $pq = q \Rightarrow q(p-1) = 0 \Rightarrow q = 0 \text{ or } p = 1$

If $q = 0$ then $p = -p$ or $2p = 0$ or $p = 0$.

If $p = 1$ then $1 + q = -1$ or $\Rightarrow q = -2$

Hence, the two solutions are $p = 0$, $q = 0$ and $p = 1$ $q = -2$.

(vi) Let the roots of the equation be $p\alpha$ and $q\alpha$. Then

$p\alpha + q\alpha = -b/a$ and $(p\alpha)(q\alpha) = c/a$

$\Rightarrow \quad (p+q)\alpha = -\frac{b}{a} \text{ and } pq\,\alpha^2 = c/a$

$$\Rightarrow \quad \alpha = \frac{-b}{a\,(P+q)} \text{ and } pq\,\alpha^2 = \frac{c}{a}$$

Putting the value of α from the first relation to the second relation, we get

$$pq\left(\frac{-b}{a(p+q)}\right)^2 = \frac{c}{a} \Rightarrow \frac{pqb^2}{a^2(p+q)} = \frac{c}{a} \Rightarrow b^2pq = ac(p+q)^2.$$

(vii) Let the roots of the equation be α and α^2 . Then

$\alpha + \alpha^2 = -b/a$ *and* $\alpha\,\alpha^2 = c/a \Rightarrow \alpha + \alpha^2 = -b/a$ *and* $\alpha^3 = c/a$

Cubing both the sides of the relation $\alpha + \alpha^2 = -b/a$ we get

$$(\alpha + \alpha^2)^3 = (-b/a)^3$$

$\Rightarrow \alpha^3 + (\alpha^2)^3 + 3\,\alpha\,\alpha^2\,(\alpha + \alpha^2) = -b^3/a^3$

$[\because (x+y)^3 = x^3 + y^3 + 3xy(x+y)]$

$\Rightarrow \alpha^3 + \alpha^6 + 3\alpha^3\,(\alpha + \alpha^3) = -b^3/a^3$

Substituting the value of α^3 *and* $\alpha + \alpha^2$, we get

$$\frac{c}{a} + \left(\frac{c}{a}\right)^2 + 3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right) = \frac{-b^3}{a^3} \Rightarrow \frac{c}{a} + \frac{c^2}{a^2} - \frac{3bc}{a^2} = \frac{-b^3}{a^3}$$

Multiplying both the sides by a^3, *we obtain*

$$a^2c + ac^2 - 3abc = -b^3$$

$$\Rightarrow \quad b^3 + a^2c + a\,c^2 = 3abc$$

Example 4

(i) The sum of the squares of two numbers is 233 and one of the numbers is 3 less than twice the other. Find the numbers.

(ii) A positive number exceeds its positive square root by 12. Find the number.

(iii) The sum of the squares of three consecutive natural numbers is 110. Find the natural numbers.

(iv) Two numbers are such that their sum is 54 and product is 629. Find the numbers.

(v) The length of a rectangular field is greater than its width by 10 metres. If the area of the field is 144 sq. m. find its dimensions.

(vi) The number of straight lines y that can connect x points in a plane is given by :
$y = (x/2)\,(x - 1)$.
How many points does a figures have if only 15 lines can be drawn connecting its vertices ?

(vii) Rachit wishes to start a 100 sq. m rectangular (not a square) vegetable garden. Since he has not only 30 metres barbed wire for fencing, he fences three sides of the rectangle, letting his house wall act the fourth side. How wide is vegetable garden ?

Solution :

(i) Since one of the numbers is 3 less than twice the other, we take the numbers to be x and $2x - 3$. According to the given problem.

$$x^2 + (2x - 3)^2 = 233$$

$$\Rightarrow \quad x^2 + 4x^2 + 9 - 12x - 233 = 0 \text{ or } 5x^2 - 12x - 244 = 0$$

$$\therefore \quad x = \frac{12 \pm \sqrt{44 - 4(5)(-224)}}{10} = \frac{12 \pm \sqrt{144 + 4480}}{10}$$

$$x = \frac{12 \pm \sqrt{4264}}{10} = \frac{12 \pm 68}{10} = 8 \text{ or } \frac{-28}{5}$$

When $x = 8$, the two numbers are 8 and $2 \times 8 - 3 = 13$

When $x = \frac{-28}{5}$, the two numbers are $\frac{-28}{5}$ and $2\left(\frac{-28}{5}\right) - 3 = \frac{-71}{5}$.

(ii) Let the positive number be x. According to the given condition

$$x - \sqrt{x} = 12 \Rightarrow x - 12 = \sqrt{x}$$

Squaring both the sides, we $x^2 - 24x + 144 = x$

$$\Rightarrow \quad x^2 - 25x + 144 = 0 \quad \Rightarrow \quad x^2 - 9x - 16x + 144 = 0$$

$$\Rightarrow \quad x(x-9) - 16(x-9) = 0 \quad \Rightarrow \quad (x - 16)(x - 9) = 0$$

$$\Rightarrow \quad x = 16 \text{ or } 9.$$

Putting $x = 16$ in (1), we get $16 - 4 = 12$ which is true.

Putting $x = 9$ (1), we get $9 - 3 = 12$ or $- 6 = 12$ which is not true.

Thus, $x = 16$.

Alternative solution

Putting $\sqrt{x} = y$ *in* (1) we get $y^2 - y = 12$ or $y^2 - y - 12 = 0$

$$\Rightarrow \quad (y - 4)(y + 3) = 0 \quad \Rightarrow \quad y = 4 \text{ or } y = -3 \quad \sqrt{x} = 4 \text{ or } \sqrt{x} = -3$$

Since $\sqrt{x} \geq 0$, we reject $\sqrt{x} = -3$

Thus, $\sqrt{x} = 4 \Rightarrow x = 16$.

(iii) Let three consecutive natural numbers be x, $x+1$, and $x + 2$. According to the given condition.

$x^2 + (x+1)^2 + (x+2)^2 = 110$

$\Rightarrow \quad x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 = 110 \quad \Rightarrow \quad 3x^2 + 6x + 5 = 110$

$\Rightarrow \quad 3x^2 + 6x - 105 = 0 \qquad \Rightarrow \quad x^2 + 2x - 35 = 0$

$\Rightarrow \quad x^2 + 7x - 5x - 35 = 0 \qquad \Rightarrow \quad x(x + 7) - 5(x + 7) = 0$

$\Rightarrow \quad (x - 5)(x + 7) = 0 \qquad \Rightarrow \quad x = 5, -7$

Since -7 is not a natural number, so we take $x = 5$

Thus, the required natural numbers are 5, 6,7.

Remark : If you take three numbers as $(x - 1)$, x and $(x + 1)$, then the calculation is much simpler. Try it as an exercise of your self and see.

(iv) Since the sum of the numbers is 54, we let the numbers be x and $54-x$.

According to the given problem $x(54 - x) = 629$

or $\quad 54x - x^2 = 629 \qquad$ or $\quad x^2 - 54x + 629 = 0$
or $\quad x^2 - 37x - 17x + 629 \qquad$ or $\quad x(x - 37) - 17(x - 37) = 0$
or $\quad (x - 37)(x - 17) = 0 \qquad$ or $\quad x = 37, 17$
when $x = 37$, the numbers are 37 and $54 - x = 54 - 37 = 17$

when $x = 17$, the numbers are 17 and $54 - 17 = 37$

Hence, the numbers are 17 and 37.

(v) Let the width of the rectangular field be x metres, then length is $x + 10$ metres.

Area of the field = length × breadth = $(x + 10)\, x\text{m}^2 = (x^2 + 10x)\ \text{m}^2$

Thus $x^2 + 10x = 144 \Rightarrow x^2 + 10x - 144 = 0$

$$\Rightarrow x = \frac{-10 \pm \sqrt{(10)^2 - 4 \times 1 \times (-144)}}{2 \times 1}$$

$$= \frac{-10 \pm \sqrt{100 + 576}}{2 \times 1} = \frac{-10 \pm \sqrt{676}}{2} = \frac{-10 \pm 26}{2} = 8, -18$$

Since x cannot be negative, we take $x = 8$. Thus, the dimension of the field is $8\ m \times 18m$.

(vi) Since, the number of lines is given to be 15. We must have

$$\frac{x}{2}(x-1) = 15 \text{ or } x(x-1) = 30 \text{ or } x^2 - x - 30 = 0$$

$$\Rightarrow x^2 - 6x + 5x - 30 = 0 \Rightarrow (x-6)(x+5) = 0 \Rightarrow x = 6 \text{ or } -5.$$

As x cannot be negative, we have $x = 6$.

(vii) Let the dimension of the rectangular fixed be x metres be y metres. Suppose the house is along the side having length y metres [see Figure 1]

Then $x + x + y = 30$.

and $xy = 100$

$\Rightarrow$ $y = 30 - 2x$ and $xy = 100$

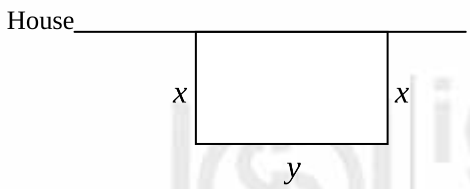


Figure 1

Putting $y = 30 - 2x$ in $xy = 100$, we get $x(30 - 2x) = 100$

$\Rightarrow$ $2x(15 - x) = 100$ or $15x - x^2 + 50 = 0$

$\Rightarrow x^2 - 15x + 50 = 0 \Rightarrow (x-5)(x-10) = 0 \Rightarrow x = 5$ or $x = 10$.

When $x = 5$, $y = 30 - 2(5) = 20$

When $x = 10$, $y = 30 - 2(10) = 10$

As the garden is rectangular and not a square, the dimension of the vegetable garden has to be 5 m $\times$ $20m$.

Example 5 : (i) If α and β are the roots of the equation $2x^2 - 3x - 5 = 0$ from a quadratic equation whose roots are α^2, β^2.

(ii) If α, β are the roots of the equation $2x^2 - 3x + 1 = 0$ form an equation whose roots are α/β and β/α.

(iii) If α, β are the roots of $x^2 - 4x + 5 = 0$ form and equation whose roots are α^2+2, β^2+2.

(iv) If α and β be the roots of the equation $x^2 - px + q = 0$ form and equation whose roots are

$$\frac{1}{a\,\alpha + b} \text{ and } \frac{1}{a\,\beta + b}.$$

Solution (i) Since α and β are roots of $2x^2 - 3x - 5 = 0$

$\alpha + \beta = 3/2$ and $\alpha\beta = -5/2$

We are to form a quadratic equation whose roots are α^2, β^2.

Let S = Sum of roots = $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$.

$$= \left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right) = \frac{9}{4} + 5 = \frac{29}{4}$$

P = Product of roots = $\alpha^2\ \beta^2 = (\alpha\beta)^2 = \left(-\frac{5}{2}\right)^2 = \frac{25}{4}$.

Putting values of S and P in $x^2 - Sx + P = 0$, the required equation is

$x^2 - (29/4)x + 25/4 = 0$ or $4x^2 - 29x + 25 = 0$.

(ii) Since α and β are roots of $2x^2 - 3x + 1 = 0$, $\alpha + \beta = 3/2$ and $\alpha\beta = ½$.

We are to form an equation whose roots are α/β and β/α.

$$\text{Let S} = \text{Sum of roots} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}.$$

$$= \frac{\left(\frac{3}{2}\right)^2 - 2\left(\frac{1}{2}\right)}{\frac{1}{2}}$$

$$= \left(\frac{5}{4}\right)\left(\frac{2}{1}\right) = \frac{5}{2}$$

$$\text{P} = \text{Product of roots} = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$$

Thus, the required quadratic equation is

$x^2 - (5/2)x + 1 = 0$ or $2x^2 - 5x + 2 = 0$

(iii) Since α, β are roots of $x^2 - 4x + 5 = 0$ $\alpha + \beta = 4$ and $\alpha\beta = 5$.

The roots of the required equation are $\alpha^2 + 2$ and $\beta^2 + 2$.

Let S = Sum of the roots = $(\alpha^2 + 2) + (\beta^2 + 2) = \alpha^2 + \beta^2 + 4$

$= (\alpha + \beta)^2 - 2\alpha\beta + 4 = (4)^2 - 2 \times 5 + 4 = 10$

and P = Product of the roots = $(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4$

$= \alpha^2\beta^2 + 2[(\alpha + \beta)^2 - 2\alpha\beta)] + 4 = 25 + 2[16–10] + 4 = 41$.

Thus, the required equation is $x^2 - Sx + P = 0$ or $x^2 - 10x + 41 = 0$

(iv) Since α, β are the roots of $x^2 - px + q = 0$ $\alpha + \beta = p$ and $\alpha, \beta = q$.

$$\text{The roots of the required equation are } \frac{1}{a\alpha + b} \text{ and } \frac{1}{a\beta + b}.$$

$$\text{Let S} = \text{Sum of the roots} = \frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$$

$$= \frac{a(\alpha+\beta)+ 2b}{a^2\alpha\beta + ab(\alpha+\beta)+ b^2}$$

$$= \frac{ap+2b}{a^2q+abp+ b^2}$$

$$\text{P} = \text{Product of the roots } \frac{1}{a\alpha+b} \times \frac{1}{a\beta+b} = \frac{1}{\alpha^2\alpha\beta + ab(\alpha+\beta)+ b^2}$$

$$\frac{1}{\alpha^2q+abp+ b^2}$$

The required equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

or $x^2 - \frac{ap+2b}{a^2q+abp+ b^2} x + \frac{1}{a^2+abp+ b^2} = 0$

$\Rightarrow \ (a^2q+abp+ b^2) \ x^2 - (ap+2b)x + 1 = 0$

Check Your Progress – 1

1. If α and B are the roots of the equation $ax^2 + bx + c = 0$ $a \neq 0$, find the value of

   (i) $\dfrac{\alpha+\beta}{\frac{1}{1/\alpha}+\frac{1}{1/\beta}}$ (ii) $\alpha^4 + \alpha^4 + \alpha^2\beta^2$

   (iii) $\dfrac{\alpha^3}{\beta}+\dfrac{\beta^3}{\alpha}$ (iv) $\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right)^3$

   (v) $\alpha^3\beta+\alpha\beta^3$ (vi) $\alpha^{-4}+ \beta^{-4}$

2. Find the quadratic equation with real coefficients and with the following pairs of root(s) :

   (i) 2/7, –3/7 (ii) $2+\sqrt{2}, 2-\sqrt{2}$

   (iii) $\dfrac{1}{5-\sqrt{6}}, \dfrac{1}{5+\sqrt{6}}$ (iv) $\dfrac{m}{n}, \quad \dfrac{n}{m},$

   (v) $3, 1-\sqrt{2}$ (vi) $1-3i, \ 1+3i$

3. If α, β are the roots of the equation $3x^2 - 4x + 1 = 0$, form an equation whose roots are α^2/β *and* β^2/α.

4. If α, β are roots of $x^2 - 2x + 3 = 0$, form an equation whose roots are α + 2, β + 2.

5. If α, β are the roots of $2x^2 - 3x + 5 = 0$ find the equation whose roots are α + 1/β and β + 1/α.

6. If α, β be the roots of $2x^2 - 3x + 1 = 0$ find an equation whose roots are

$$\frac{\alpha}{2\beta+3}, \frac{\beta}{2\alpha+3}.$$

7. If α, β are the roots of $ax^2 + bx + c = 0$ form an equation whose roots form an equation whose roots are α + β and αβ /(α + β).

8. If p, q be the roots of $3x^2 - 4x + 1 = 0$, show that b and c are the roots are

$$x^2 - (p + q - pq)x - pq\,(p + q) = 0.$$

3.4 CUBIC AND BIQADRATIC EQUATIONS

A polynomial equation in x, in which the highest exponent is 3 is called a **cubic** equation, and if the highest exponent is 4, it is called a biquadratic equation. Thus, a cubic equation look as

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \quad a_0 \neq 0 \qquad (1)$$

and *a* biquadratic equation looks as

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \quad a_0 \neq 0 \qquad (2)$$

Relation between Roots and Coefficients

If α, β, γ are the roots of (1), then

$$\alpha + \beta + \gamma = - a_1/a_0 = - \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\beta\gamma + \gamma\alpha + \alpha\beta = a_1/a_0 = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$and\ \alpha\beta\gamma = \frac{-a_3}{a_0} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

If α, β, γ and δ are the roots of (2), then

$$\alpha + \beta + \gamma + \delta = \frac{-a_1}{a_0} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = -\frac{a_2}{a_0} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{a_3}{a_0} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

$$\alpha\beta\gamma\delta = \frac{a_4}{a_0} = \frac{\text{Constant term}}{\text{Coefficient of } x^4}$$

Remarks : It will be difficult to solve a cubic equation and biquadrate equation just by knowing the relation between roots and coefficients. However, if we know one more relation between the roots, it becomes easier for us to solve the equation.

Solved Examples

Example 6 : Solve the equation

$$2x^3 - 15\,x^4 + 37x - 30 = 0 \qquad (1)$$

If the roots of the equation are in A.P.

Solution : Recall three numbers in A.P. can be taken as $\alpha - \beta,\ \alpha,\ \alpha + \beta$.

If $\alpha - \beta,\ \alpha,\ \alpha + \beta$ are roots of (1), then $(\alpha - \beta) + \alpha + (\alpha + \beta) = 15/2 \Rightarrow 3\alpha = 15/2$

$\Rightarrow \alpha = 5/2$

Next,

$\alpha(\alpha - \beta) + \alpha(\alpha + \beta)\ (\alpha - \beta)(\alpha + \beta) = 37/2$

$\Rightarrow\ \alpha^2 - \alpha\beta + \alpha^2 + \alpha\beta + \alpha^2 - \beta^2 = 37/2$

$\Rightarrow 3\alpha^2 - \beta^2 = 37/2$

$$\Rightarrow \beta^2 = 3\alpha^2 - \frac{37}{2} = 3 \times \frac{25}{4} - \frac{37}{2} = \frac{1}{4}$$

$$\Rightarrow \beta = \pm\frac{1}{2}$$

When $\beta = ½$, the roots are

$$\frac{5}{2} - \frac{1}{2}, \frac{5}{2}, \frac{5}{2} + \frac{1}{2}\ \text{, or } 2, \frac{5}{2}, 3$$

When $\beta = -\frac{1}{2}$, the roots are 3, 5/2 2.

It is easily to check that these are roots of (1).

Example 7 : Solve the equation

$$6x^3 - 11x^2 - 3x + 2 = 0 \qquad (1)$$

Given that the roots are in H.P.

Solution : Let the roots of the equation be

$$\frac{1}{\alpha - \beta}, \frac{1}{\alpha}, \frac{1}{\alpha + \beta}$$

We have

$$\frac{1}{\alpha-\beta} + \frac{1}{\alpha} + \frac{1}{\alpha+\beta} = \frac{11}{6} \qquad (2)$$

$$\frac{1}{\alpha-\beta} \cdot \frac{1}{\alpha} + \frac{1}{\alpha} \cdot \frac{1}{\alpha+\beta} + \frac{1}{\alpha-\beta} \cdot \frac{1}{\alpha+\beta} = \frac{-3}{6} = \frac{-1}{2} \qquad (3)$$

$$\text{and } \frac{1}{\alpha-\beta} \frac{1}{\alpha} \quad \frac{1}{\alpha+\beta} = \frac{-2}{6} = \frac{-1}{3} \qquad (4)$$

From (3), we obtain

$$\frac{(\alpha+\beta)+\alpha+(\alpha-\beta)}{(\alpha-\beta)\alpha(\alpha+\beta)} = \frac{-1}{2}$$

$$\Rightarrow 3\alpha = \frac{3}{2} \Rightarrow \alpha = \frac{1}{2}$$

Putting this in (4), we obtain

$$\left(\frac{1}{2}-\beta\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}+\beta\right) = -3$$

$$\Rightarrow \frac{1}{4} - \beta^2 = -6$$

$$\Rightarrow \beta^2 = \frac{1}{4} + 6 = \frac{25}{4}$$

$$\Rightarrow \beta^2 = \pm \frac{5}{2}.$$

If $\beta^2 = \frac{5}{2}$, roots become $-\frac{1}{2}$, 2 and $\frac{1}{3}$

If β = –5/2, we get these roots in the reverse order.

Example 8: Solve the equation

$$8x^3 - 14x^2 + 7x - 1 = 0 \qquad (1)$$

The roots being in G.P.

Solution : As the roots are in G.P. we may take them as

$$\frac{\alpha}{r}, \alpha, \alpha r$$

Now,

$$\text{Now,} \frac{\alpha}{r} + \alpha + \alpha r = \frac{14}{8} = \frac{7}{4}, \qquad (2)$$

$$\left(\frac{\alpha}{r}\right)(\alpha) + \alpha(\alpha r) + \left(\frac{\alpha}{r}\right)(\alpha r) = \frac{7}{8} \qquad (3)$$

$$\left(\frac{\alpha}{r}\right)(\alpha)(\alpha r) = \frac{1}{8} \qquad (4)$$

From (4), we get $\alpha^3 = \frac{1}{8} \Rightarrow \alpha = \frac{1}{2}$

Putting this in (2), we get

$$\frac{1}{2}\left(\frac{1}{r} + r\right) = \frac{7}{4} - \frac{1}{2} = \frac{5}{4}$$

$\Rightarrow \quad r + \frac{1}{r} = \frac{5}{2} \qquad \Rightarrow \quad \frac{r^2+1}{r} = \frac{5}{2}$

$\Rightarrow \quad 2r^2 - 5r + 2 = 0 \qquad \Rightarrow \quad (2r-1)(r-2) = 0$

$\Rightarrow \quad r = 1/2, r = 2$

When $r = 2$, we get the roots as

$\frac{1}{4}, \frac{1}{2}, 1$

When $r = 1/2$, we get these roots in the reverse order.

It is easy to verify that these roots satisfy the equation (1)

Example 9: Solve the equation

$$x^2 - 13x^2 + 15x + 189 = 0 \qquad (1)$$

being given that one root exceeds the other by 2.

Solution : As one root exceeds the other by 2, we may take the root as α, α+2 and β.

Now.

$\alpha + (\alpha + 2) + \beta = 13 \qquad (2)$

$\alpha(\alpha + 2) + \alpha\beta + (\alpha + 2)\beta = 15 \qquad (3)$

and $\alpha(\alpha + 2)\beta = -189 \qquad (4)$

From (2), $\beta = 11 - 2\alpha$.

Putting this in (3), we get

$\alpha(\alpha + 2) + (2\alpha + 2)(11 - 2\alpha) = 15$

$\Rightarrow \quad \alpha^2 + 2\alpha + 22\alpha + 22 - 4\alpha^2 - 4\alpha = 15$

$\Rightarrow \quad 3\alpha^2 - 20\alpha - 7 = 0$

$\Rightarrow \quad (3\alpha + 1)(\alpha - 7) = 0$

$\Rightarrow \quad \alpha = -1/3, 7$

when $\alpha = -\frac{1}{3}, \beta = 11 + \frac{2}{3} = \frac{35}{3}$.

But $\alpha = \frac{1}{3}$, $\quad \beta = \frac{35}{3}$ does not satisfy equation (4)

When $\alpha = 7$, $\beta = -3$

These values satisfy equation (4)

Thus, roots of (1) are 7,9 and –3

Example 10 : Solve the equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0 \qquad (1)$$

being given that it has two roots equal in magnitude but opposite sign.

Solution : Let the roots of (1) be

$\alpha, -\alpha, \beta$ and γ.

We have

$\alpha + (-\alpha) + \beta + \gamma = 2 \qquad (2)$

$[\alpha + (-\alpha)(\beta + \gamma) + \alpha(-\alpha) + \beta\gamma = 4 \qquad (3)$

$(\alpha + (-\alpha))\beta\gamma + \alpha(-\alpha)(\beta + \gamma) = -6 \qquad (4)$

$\alpha(-\alpha)\beta\gamma = -21 \qquad (5)$

(2) gives $\beta + \gamma = 2$

Putting this in (4), we get $\alpha^2 = 3$ or $\alpha = \pm\sqrt{3}$.

From (3), we get

$$0(\beta + \gamma) + (-3) + \beta\gamma = 4$$

$$\Rightarrow \beta\gamma = 7$$

The quadratic equation whose roots are β and γ is

$$x^2 - (\beta + \gamma)x + \beta\gamma = 0$$

or $\quad x^2 - 2x + 7 = 0$

$(x-1)^2 + 6 = 0$

$\Rightarrow x = 1 \pm \sqrt{6}\,i$

Thus, roots of (1) are

$\sqrt{3}, -\sqrt{3}, 1 + \sqrt{6}i,\ 1 - \sqrt{6}i$

Example 11: Solve the equation

$$3x^4 - 25\,x^3 + 50\,x^2 - 50\,x^2 + 12 = 0 \qquad (1)$$

The product of two of roots being 2.

Solution : Let roots of (1) be

$\alpha, \beta, \gamma, \delta,$ where $\gamma\,\delta = 2.$

We have

$$\alpha + \beta + \gamma + \delta = \frac{25}{3} \qquad (2)$$

$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\,\delta = \frac{50}{3} \qquad (3)$$

$$(\alpha + \beta)\,\gamma\delta + \alpha\beta + (\gamma + \delta) = \frac{50}{3} \qquad (4)$$

and $\alpha\beta\gamma\delta = -12/3 = 4$ (5)

As $\gamma\,\delta = 2$, from (5) we get $\alpha\beta = +2$.

Putting $\alpha\beta = 2$, $\gamma\delta = 2$ in (3), we get

$$(\alpha + \beta)(\gamma + \delta) = \frac{50}{3} - 4 = \frac{38}{3}$$

Equation whose rots are $\alpha + \beta$ and $r + \delta$ is

$$x^2 - (\alpha + \beta + \gamma + \delta)x + (\alpha + \beta)(\gamma + \delta) = 0$$

$$\Rightarrow \quad x^2 - \frac{25}{3}x + \frac{38}{3} = 0$$

$$\Rightarrow \quad 3x^2 - 25x + 38 = 0$$

$$\Rightarrow \quad 3x^2 - 6x - 19x + 38 = 0$$

$$\Rightarrow \quad 3x(x-2) - 19\,(x-2) = 0$$

$\Rightarrow \quad (3x - 19)(x - 2) = 0 \;\Rightarrow x = \dfrac{19}{3}, x = 2$

Let $\alpha + \beta = 19/3$ and $r + \delta = 2$

Equation whose roots are α and β is

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or $x^2 - \dfrac{19}{3}x + 2 = 0$

$\Rightarrow \; 3x^2 - 19x + 6 = 0$

$\Rightarrow (3x - 1)(x - 6) = 0$

$\Rightarrow \; x = 1/3, 6$

Equation whose roots are γ and δ is

$x^2 - (r + \delta)x + r\delta = 0$

$\Rightarrow \quad x^2 - 2x + 2 = 0$

$\Rightarrow \quad (x-1)^2 + 1 = 0 \quad \Rightarrow (x-1)^2 = i^2$

$\Rightarrow \quad x - 1 = \pm i, \qquad x = 1 \pm i$

Thus, roots of (1) are 1/3, 6, $1 + i$, $1 - i$.

Example 12 : The Product of two of the roots of the equation

$x^4 - 5x^3 + 10x^3 - 10x + 4 = 0$ is equal to the product of the other two. (1)

Solution : Let roots of (1) be

$\alpha, \beta, \gamma, \delta$

where $\quad \alpha\beta = \gamma\,\delta$

We have

$\alpha + \beta + \gamma + \delta = 5$ (2)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\,\delta = 10$ (3)

$\alpha\beta(\gamma + \delta) + \gamma\,\delta\,(\alpha + \beta) = 10$ (4)

$\alpha\beta\gamma\delta = 4$ (5)

Since $\alpha\beta = \gamma\delta$ (5) gives $\alpha\beta = \gamma\,\delta = \pm 2$.

But $\alpha\beta = \gamma\,\delta = -2$, gives

$-2(\gamma + \delta + \alpha + \beta) = 10$ [from (4)]

$\alpha + \beta + \gamma + \delta = -5$

This contradicts (2)

Thus, $\alpha\beta = \gamma\,\delta = 2$.

Putting these values in (3), we obtain

$(\alpha + \beta)(\gamma + \delta) = 6$.

A quadratic equation whose roots are $\alpha+\beta$, $\gamma + \delta$ is

$x^2 - (\alpha + \beta + \gamma + \delta)x + (\alpha + \beta)(\gamma + \delta) = 0$

or $x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

Let $\alpha + \beta = 2$ and $\gamma + \delta = 3$.

A quadratic equation whose roots are α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - 2x + 2 = 0 \Rightarrow (x - 1)^2 = -1 = i^2$

$\Rightarrow x = 1 \pm i$

A quadratic equation whose roots are γ and δ is

$x^2 - (\gamma + \delta)x + \gamma\delta = 0$

or $x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

Hence, roots of (1) are

$1, 2, 1+i, 1-i$

Example 13: Solve the equation

$x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$, the sum of two of the roots being equal to the sum of the other two.

Solution : Let roots of the equation be α, β, γ and δ where $\alpha + \beta = \gamma + \delta$.

We have

$\alpha + \beta + \gamma + \delta = 8$ (2)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\,\delta = 21$ (3)

$\alpha\beta(\gamma + \delta) + \gamma\,\delta(\alpha + \beta) = 20$ (4)

$\alpha\,\beta\,\gamma\;\;\delta = 5$ (5)

As $\alpha+\beta = \gamma + \delta$, from (2), we get

$2\,(\alpha + \beta) = 8 \Rightarrow \alpha+\beta = 4$

Thus, $\alpha+\beta = \gamma + \delta = 4$

Putting this in (3), we get

$\alpha\beta + \gamma\,\delta = 5$

A quadratic equation whose roots are $\alpha\beta$ and $\gamma\,\delta$ is

$x^2 - (\alpha\beta + \gamma\,\delta)x + (\alpha\beta)\,(\gamma\,\delta) = 0$

$\Rightarrow x^2 - 5x + 5 = 0$

$$\Rightarrow \quad x = \frac{5 \pm \sqrt{25-20}}{2} = \frac{1}{2}\,(5 \pm \sqrt{5}\,)$$

$$\text{Let } \alpha\beta = \frac{1}{2}\,(5 + \sqrt{5}\,) \text{ and } \gamma\,\delta = \frac{1}{2}\left(5 - \sqrt{5}\,\right)$$

A quadratic equation whose roots are α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$$or \qquad x^2 - 4x + \frac{1}{2}(5 + \sqrt{5}\,) = 0$$

$$\Rightarrow \; x = \frac{4 \pm \sqrt{16 - 10 - 2\sqrt{5}}}{2} = \frac{4 \pm \sqrt{6 - 2\sqrt{5}}}{2}$$

$$= \frac{4 \pm \sqrt{(\sqrt{5}-1)^2}}{2} = \frac{4 \pm (\sqrt{5}-1)}{2}$$

$$= \frac{1}{2}\left(3 + \sqrt{5}\right), \frac{1}{2}(5 - \sqrt{5}),$$

A quadratic equation whose roots are γ, δ is

$x^2 - (\gamma + \delta)x + \gamma\delta = 0$

or $x^2 - 4x + \frac{1}{2}(5 - \sqrt{5}) = 0$

$$\Rightarrow\ x = \frac{4 \pm \sqrt{16 - 10 + 2\sqrt{5}}}{2}$$

$$= \frac{4 \pm \sqrt{(\sqrt{5}+1)^2}}{2} = \frac{4 \pm (\sqrt{5}+1)}{2}$$

$$= \frac{3-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}$$

Thus, roots of (1) are

$\frac{1}{2}(3\pm\sqrt{5}),\ \frac{1}{2}(5\pm\sqrt{5})$

Example 14 : If roots of $ax^3 + bx^2 + cx + d = 0$ are in A.P. show that

$$2b^3 - 9abc + 27a^2d = 0 \qquad (1)$$

Solution : Let roots of

(1) be $\alpha - \beta$, α $\alpha+\beta$

we have

$$(\alpha-\beta) + \alpha + (\alpha+\beta) = -\frac{b}{a}$$

$$\Rightarrow 3\alpha = -\frac{b}{a} \Rightarrow \alpha = -\frac{b}{3a}$$

As α is a roots of (1), we get

$$a\left(-\frac{b}{3a}\right)^3 + b\left(-\frac{b}{3a}\right)^2 + c\left(-\frac{b}{3a}\right) + d = 0$$

$$\Rightarrow \frac{-b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + \text{d} = 0$$

$$\Rightarrow -b^3 + 3b^3 - 9abc + 27a^2d = 0$$

$$\Rightarrow 2b^3 - 9abc + 27a^2d = 0$$

Example 15 : If α, β, γ be the roots of the equation $x^3 - px^2 + qx - \gamma = 0$ (1)

Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$ (ii) $\beta^2 r^2 + \alpha^2 + \alpha^2\beta^2$

(iii) $\alpha^3 + \beta^3 + \gamma^3$ (iv) $\sum \alpha^2\beta$

Solution : We have

$\alpha + \beta + \gamma = p$

$\beta\gamma + \gamma\alpha + \alpha\beta = q,$

and $\alpha\beta\gamma = \gamma$

Now,

(i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\,(\beta\gamma + \gamma\alpha + \alpha\beta)$

$= p^2 - 2q$

(ii) $\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2$

$= (\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2[(\beta\gamma)(\gamma\alpha) + (\gamma\alpha)(\alpha\beta) + (\beta\gamma)(\alpha\beta)]$

$= (\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2\alpha\beta\gamma\,(\alpha + \beta + \gamma)$

$= q^2 - 2\gamma p$

(iii) $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)\ (\alpha^2 + \beta^2 + \gamma^2 - \beta r - \gamma\alpha - \alpha\beta)$

$= (\alpha + \beta + \gamma)\ [(\alpha + \beta + \gamma)^2 - 3(\beta\gamma + \gamma\alpha + \alpha\beta)]$

$\Rightarrow\ \alpha^3 + \beta^3 + \gamma^3 - 3\gamma = p\,[p^2 - 3q]$

$\Rightarrow\ \alpha^3 + \beta^3 + \gamma^3 = p^3 - 3pq + 3\gamma$

(iv) $\sum \alpha^3\beta = \alpha^3(\beta + \gamma) + \alpha^3(\gamma + \alpha) + \alpha^3(\alpha + \beta)$

$= (\alpha + \beta + \gamma)\,(\beta\gamma + \gamma\alpha + \alpha\beta) - 3\alpha\beta\gamma$

$= pq - 3\gamma$

Example 16 : α, β, γ are the roots of the equation $x^3 + px + q = 0,$ (1)

Then, show that

$$\frac{1}{7}(\alpha^7 + \beta^7 + \gamma^7) = \frac{1}{7}(\alpha^2 + \beta^2 + \gamma^2)\,\frac{1}{5}(\alpha^5 + \beta^5 + \gamma^5)$$

Solution : We have

$\alpha + \beta + \gamma = 0$

$\beta r + \alpha\gamma + \alpha\beta = p$

and $\alpha\beta\gamma = q$

Now, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$$= 0 - 2p = -2p \qquad (2)$$

Next

$$\alpha^3 + \beta^3 + \gamma^3 = -p(\alpha + \beta + \gamma) - 3q = -3q \qquad (3)$$

[using (1)]

Multiplying (1) by x^2, we get

$$x^5 + px^3 + qx^3 = 0$$

As α, β, γ satisfy this equation,

$$\alpha^5 + \beta^5 + \gamma^5 = -p(\alpha^3 + \beta^3 + \gamma^3) - q(\alpha^2 + \beta^2 + \gamma^2)$$

$$= -p(-3q) - q(-2p) = 5pq \qquad (4)$$

Multiplying (1) by x^4, we get

$$x^7 + px^5 + qx^4 = 0$$

As α, β, r satisfy this equation, we get

$$\alpha^7 + \beta^7 + \gamma^7 = -p(\alpha^5 + \beta^5 + \gamma^5) - q(\alpha^4 + \beta^4 + \gamma^4) \qquad (5)$$

But $\alpha^5 + \beta^5 + \gamma^5 = 5pq$ (6)

Multiplying (1) by x we get

$$x^4 + px^2 + +qx = 0.$$

As α, β, γ satisfy this equation, we obtain

$$\alpha^4 + \beta^4 + \gamma^4 = -p(\alpha^2 + \beta^2 + \gamma^2) - q(\alpha + \beta + \gamma)$$

$$= -p(-2p) - 9(0) = 2p^2 \qquad (7)$$

From (5), (6) and (7) we get

$$\alpha^7 + \beta^7 + \gamma^7 = -p(5pq) - q(2p^2) = -7p^2q$$

$$\Rightarrow \frac{1}{7}(\alpha^7 + \beta^7 + \gamma^7) = -p^2q \qquad (8)$$

Also, $\frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)\ \ \frac{1}{5}(\alpha^5 + \beta^5 + \gamma^5)$

$$= \frac{1}{2}(-2p).\frac{1}{5}5(pq) = -p^2q \qquad (9)$$

From (8) and (9), we get

$$\frac{1}{7}(\alpha^7 + \beta^7 + \gamma^7) = \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)\,\frac{1}{5}(\alpha^5 + \beta^5 + \gamma^5)$$

Example 17 : If $\alpha, \beta, \gamma, \delta$ are the roots of the equation.

$$x^4 + px^3 + qx^2 + rx + s = 0 \qquad (1)$$

Find the value of

(i) $\sum \alpha^2$ (ii) $\sum x^2$

Solution : As α, β, r, δ are the roots of (1)

$$\sum \alpha = -p$$
$$\sum \alpha\beta = q$$
$$\sum \alpha\beta\gamma = -r$$

and $\alpha\beta\gamma\delta = s$

Now,

$\sum \alpha^2 \beta = \text{q} = (\sum \alpha)(\sum \alpha\beta) - \sum \alpha\beta\gamma$

$= (-p)(q) - (-\gamma) = -pq + \gamma$

$Next\ \sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

$= (-p)^2 - 2q = p^2 - 2q$

Check Your Progress – 2

1. Solve the equation $32x^2 - 48x^2 + 22x - 3 = 0$, given that the roots are in A.P.
2. Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$, given that the roots are in G.P.
3. Solve the equation $3x^3 + 11x^2 + 12x + 4 = 0$, given that the roots are in H.P.
4. Solve the equation $32x^3 - 48x^2 + 22x - 3 = 0$, given that sum of two roots is 1.
5. Solve the equation $x^3 - 9x^2 + 23x - 15 = 0$, two of the roots being in the ration 3:5.
6. Solve the equation $x^3 - 13x^2 + 15x + 189 = 0$, given that difference between two of its roots is 2.
7. Solve the equation $27x^4 - 195x^3 + 494x^2 - 520x + 192 = 9$, given that the roots are in G.P.
8. Solve the equation $8x^2 - 2x^2 - 27x^2 + 6x + 9 = 0$, given that two roots are equal in magnitude but opposite in signs.
9. Solve the equation $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$, given that sum of two of the roots being equal to the sum of the other two.
10. Solve the equation $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, given that product of two of its roots equals the product of the other two.

11. Solve the equation $x^4 - 10x^3 + 42x^2 - 82x + 65 = 0$, given that product of two of its roots is 13.
12. If roots of $ax^3 + bx^3 + cx + d = 0$ are in G. P. Show that $ac^3 = b^3 d$.
13. If the roots of $x^3 - px^2 + qx - r = 0$ $(r \neq 0)$are in H.P. show that $27r^3 - 9pq\,\gamma + 2q^3 = 0$
14. If α, β, γ are the roots of $x^3 - px^2 + 9x - r = 0$ $(r \neq 0)$

    Find the value of

    (i) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$ (ii) $\sum \frac{\alpha}{\beta}$ (iii) $\sum \frac{1}{\alpha^2}$
15. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - px^3 + qx^2 - rx + s = 0$, $s \neq 0$

    Find the value of (i) $\sum \alpha^2$ (ii) $\sum \frac{1}{\alpha}$

3.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1 We have α + β = –*b/a*, αβ = *c/a*

(i) $$\frac{\alpha+\beta}{\frac{1}{\alpha}+\frac{1}{\beta}} = \frac{\alpha+\beta}{(\beta+\alpha)/\alpha\beta} = \alpha\beta = c/a$$

(ii) $$\alpha^4 + \beta^4 + \alpha^2\beta^2 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$$

$$= [(\alpha+\beta)^2 - 2\alpha\beta]^2 - \alpha^2\beta^2$$

$$= \left[\left(\frac{-b}{a}\right)^2 - 2\,\frac{c}{a}\right]^2 - \left(\frac{c}{a}\right)^2$$

$$= \left(\frac{b^2-2ac}{a^2}\right)^2 - \left(\frac{c}{a^2}\right)^2$$

$$= \frac{(b^2-2ac)^2 - a^2c^2}{a^4}$$

(iii) $$\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha} = \frac{\alpha^4 + \alpha^4}{\alpha\beta} = \frac{(\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2}{\alpha\beta}$$

$$= \frac{[(\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2] - 2\alpha^2\beta^2}{\alpha\beta}$$

$$= \frac{[(-b/a)^2 - 2c/a]^2 - 2(c/a)^2}{c/a}$$

$$= \frac{(b^2 - 2ac)^2/a^4 - 2c^2/a^2}{c/a}$$

$$= \frac{(b^2 - 2ac)^2 - 2a^2c^2}{a^3c}$$

(iv) We have $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$$= \frac{(-b/a)^2 - 2c/a}{c/a}$$

$$= \frac{b^2 - 2ac}{ac}$$

Thus, $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)^3 = \frac{(b^2 - 2ac)^3}{a^3c^3}$

(v) $\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2)$

$$= \alpha\beta[(\alpha+\beta)^2 - 2\alpha\beta]$$

$$= \frac{c}{a}\left[\left(\frac{-b}{a}\right)^2 - 2\,\frac{c}{a}\right]$$

$$= \frac{c(b^2 - 2ac)}{a^3}$$

(vi) $\alpha^{-4} + \beta^{-4} = \frac{1}{\alpha^4} + \frac{1}{\beta^4} = \frac{\alpha^4 + \beta^4}{\alpha^4\beta^4}$

$$= \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha^4\beta^4}$$

$$= \frac{[\alpha^2 + \beta^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2}{(\alpha\beta)^4}$$

$$= \frac{[(-\frac{b}{a})^2 - 2c/a]^2 - 2(\frac{c}{a})^2}{(c/a)^4}$$

$$= \frac{(b^2 - 2ac)^2/(a)^4 - 2c^2/a^2}{c^4/a^4}$$

$$= \frac{(b^2 - 2ac)^2 2c^2a^2(a)^4 - 2c^2/a^2}{c^4}$$

2. (i) Sum of the roots $= \frac{2}{7} + \frac{(-3)}{7} = -\frac{1}{7}$

and product of roots $\left(\frac{2}{7}\right)\left(-\frac{3}{7}\right) = -\frac{6}{49}$

Required equation is $x^2 - \left(-\frac{1}{7}\right)x - \frac{6}{49} = 0$

$\Rightarrow 49x^2 + 7x - 6 = 0$

(ii) Sum of the roots = $(2 + \sqrt{2}) + (2 - \sqrt{2}) = 4$

and product of roots = $(2 + \sqrt{2})(2 - \sqrt{2}) = 4 - 2 = 2$

$\therefore$ required equation is

$x^2 - 4x + 2 = 0$

(iii) Sum of the roots $= \frac{1}{5-\sqrt{6}} + \frac{1}{5+\sqrt{6}}$

$$= \frac{(5+\sqrt{6}) + (5-\sqrt{6})}{(5-\sqrt{6})(5+\sqrt{6})} = \frac{10}{25-6} = \frac{10}{19}$$

and product of the roots $= \frac{1}{5-\sqrt{6}} \cdot \frac{1}{5+\sqrt{6}} = \frac{1}{19}$

$\therefore$ required equation is $x^2 - \frac{10}{19}x + \frac{1}{19} = 0$

or $19x^2 - 10x + 1 = 0$

(iv) Sum of the roots $\frac{m}{n} + \left(-\frac{n}{m}\right) = \frac{m^2 - n^2}{mn}$

and proudct of the roots $= \left(\frac{m}{n}\right)\left(-\frac{n}{m}\right) = -1$

$\therefore$ required equation is $x^2 - \frac{m^2 - n^2}{mn}x - 1 = 0$

$\Rightarrow mnx^2 - (m^2 - n^2)x - 1 = 0$

(v) Sum of the roots = $3 + (1 - \sqrt{2}) = 4 - \sqrt{2}$,

and product of the roots = $3(1 - \sqrt{2})$

$\therefore$ required equation is

$x^2 - (4 - \sqrt{2})x + 3(1 - \sqrt{2}) = 0$

(vi) Sum of the roots $= (1-3i) + (1+3i) = 2$

Product of the roots $= (1-3i)(1+3i) = 1 + 9 = 10$

$\therefore$ required equation is

$$x^2 - 2x + 10 = 0$$

3. We have

$\alpha + \beta = 4/3, \quad \alpha\beta = 1/3$

Now, $$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{(4/3)^3 - 3(1/3)(4/3)}{1/3}$$

$$= 3\left[\frac{64}{27} - \frac{4}{3}\right] = \frac{64}{9} - 4 = \frac{28}{9}$$

and $$\left(\frac{\alpha^2}{\beta}\right)\left(\frac{\beta^2}{\alpha}\right) = \alpha\beta = 1/3$$

$\therefore$ required equation is

$$x^2 - \frac{28}{9}x + \frac{1}{3} = 0$$

or $$9x^2 - 28x + 3 = 0$$

4. $\alpha + \beta = 2$, $\alpha\beta = 3$.

Now, $(\alpha + 2) + (\beta + 2) = \alpha + \beta + 4 = 6$

and $(\alpha + 2)(\beta + 2) = \alpha\beta + 2(\alpha + \beta) + 4$

$= 3 + 2(2) + 4 = 11$

Thus, required equation is

$$x^2 - 6x + 11 = 0.$$

5. $\alpha + \beta = 3/2$ and $\alpha\beta = 5/2$

Now, $$\left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right) = (\alpha + \beta) + \left(\frac{1}{\beta} + \frac{1}{\alpha}\right)$$

$$= (\alpha + \beta) + \frac{(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{3}{2} + \frac{\frac{3}{2}}{\frac{5}{2}} = \frac{3}{2} + \frac{3}{5} = \frac{21}{10}$$

and $(\alpha + \frac{1}{\beta})(\beta + \frac{1}{\alpha}) = \alpha\beta + \frac{1}{\alpha\beta} + 2$

$$= \frac{5}{2} + \frac{2}{5} + 2$$

$$= \frac{49}{10}$$

$\therefore$ required equation is

$$x^2 - \frac{21}{10}x + \frac{49}{10} = 0.$$

Or $10x^2 - 21x + 49 = 0$

6. $\alpha + \beta = 3/2$, $\alpha\beta = ½$

Now, $$\frac{\alpha}{2\beta+3} + \frac{\beta}{2\alpha+3} = \frac{2\,\alpha^2 + 3\alpha + 2\beta^2 + 3\beta}{(2\beta+3)(2\alpha+3)}$$

$$= \frac{2(\alpha+\beta)^2 + 3(\alpha+\beta)}{4\,\alpha\beta + 6(\alpha+\beta) + 9}$$

$$= \frac{2[(\alpha^2 + \beta^2) - 2\alpha\beta] + 3(\alpha+\beta)}{4\,\alpha\beta + 6(\alpha+\beta) + 9}$$

$$= \frac{2[9/4 - 2\left(\frac{1}{2}\right)] + 3(3/2)}{4\,(1/2) + 6(3/2) + 9}$$

$$= \frac{7}{2+9+9} = \frac{7}{20}$$

and $$\left(\frac{\alpha}{2\beta+3}\right)\left(\frac{\beta}{2\alpha+3}\right) = \frac{\alpha\beta}{4\alpha\beta + 6(\alpha+\beta) + 9}$$

$$= \frac{1/2}{4(1/2) + 6\,(3/2) + 9}$$

$$= \frac{1/2}{20} = \frac{1}{40}$$

Thus, required equation is

$$x^2 - \frac{7}{20}x + \frac{1}{40} = 0$$

$$40x^2 - 14x + 1 = 0$$

7. $\alpha + \beta = -b/a, \quad \alpha\ \beta = c/a$

$$\text{Now, } \alpha + \beta + \frac{\alpha\beta}{\alpha+\beta} = \frac{(\alpha+\beta)^2 + \alpha\beta}{\alpha+\beta}$$

$$= \frac{(-b/a)^2 + ac}{-b/a}$$

$$= \frac{b^2 + ac}{-ab}$$

$$\text{and } (\alpha+\beta)\left(\frac{\alpha\beta}{\alpha+\beta}\right) = \alpha\beta = \frac{c}{a}$$

Thus, required equation is

$$x^2 + \frac{b^2+ac}{ab}x + \frac{c}{a} = 0$$

$$\Rightarrow ab\,x^2 + (b^2+ac)x + bc = 0$$

8. We have $p + q = -b$ and $pq = c$.

Now, $b + c = -(p+q) + pq$

and $bc = -(p+q)pq$

Thus, required equation is

$$x^2 + (p+q-pq)x - pq\,(p+q) = 0.$$

Check Your Progress – 2

1. Let the roots be $\alpha - \beta, \alpha, \alpha + \beta$. Then

$(\alpha - \beta) + \alpha + (\alpha + \beta) = 48/32 \Rightarrow 3\alpha \Rightarrow = 3/2 \Rightarrow \alpha = ½$

$(\alpha - \beta)\alpha + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = 22/32$

$$\Rightarrow 3\alpha^2 - \beta^2 = \frac{21}{32}$$

$$\Rightarrow ¾ - \beta^2 = 22/32$$

$$\Rightarrow \beta^2 = \frac{3}{4} - \frac{22}{32} = \frac{2}{32} = \frac{1}{16}$$

$$\Rightarrow \beta = \pm\frac{1}{4}$$

Thus, required roots are

$$\frac{1}{2}-\frac{1}{4},\frac{1}{2},\frac{1}{2}+\frac{1}{4} \text{ or } \frac{1}{4},\frac{1}{2},\frac{3}{4}$$

These roots satisfy the relation product of roots = 3/32.

2. Let the roots be α/β, α, $\alpha\beta$

We have product of the roots $= \dfrac{8}{27}$

$$\Rightarrow (\alpha/\beta)\,(\alpha)\,(\alpha\beta) = \frac{8}{27}$$

$$\Rightarrow \alpha^3 = \frac{8}{27} \Rightarrow \alpha = \frac{2}{3}$$

Also,

$$\frac{\alpha}{\beta}+\alpha+\alpha\beta = \frac{-42}{27}$$

$$\frac{2}{3}\left(\beta+\frac{1}{\beta}\right)+\frac{2}{3} = -\frac{42}{27}$$

$$\Rightarrow \frac{2}{3}\left(\beta+\frac{1}{\beta}\right) = \frac{-14}{9}-\frac{2}{3} = \frac{-20}{9}$$

$$\Rightarrow \beta+\frac{1}{\beta} = \frac{-10}{3} \Rightarrow \beta = -3, -\frac{1}{3}$$

These values of α, β satisfy the relation

$$\frac{\alpha}{\beta}.\alpha + \alpha.\alpha\beta + \frac{\alpha}{\beta}.\alpha\beta = -\frac{-28}{27} \text{ (verify)}$$

3. Let the roots be $\dfrac{1}{\alpha-\beta},\dfrac{1}{\alpha},\dfrac{1}{\alpha+\beta}$

We have

$$\frac{1}{\alpha-\beta}.\frac{1}{\alpha}+\frac{1}{\alpha}.\frac{1}{\alpha+\beta}+\frac{1}{\alpha-\beta}.\frac{1}{\alpha+\beta} = \frac{12}{3}$$

and $\dfrac{1}{(\alpha-\beta)\alpha\,(\alpha+\beta)} = \dfrac{-4}{3}$

$$\Rightarrow \frac{(\alpha+\beta)+(\alpha-\beta)+\alpha}{\alpha(\alpha+\beta)(\alpha-\beta)} = 4$$

and $(\alpha-\beta)\alpha(\alpha+\beta) = \frac{-3}{4}$

Thus, 3 α = –3 ⇒ α = –1

Also, (–1–β) (–1) (–1+β) = –3/4

$$\Rightarrow 1-\beta^2 = \frac{3}{4} \quad \Rightarrow \beta^2 = \frac{1}{4} \Rightarrow \beta \pm \frac{1}{2}$$

$$\text{Thus, roots are } \frac{1}{-1-\frac{1}{2}}, \frac{1}{-1}, \frac{1}{-1+\frac{1}{2}}$$

or $-\frac{2}{3}, -1, 2$

It is easy to check these are roots of the equation.

4. Let the roots be α, β and γ. Suppose $\alpha + \beta = 1$

We have $\alpha + \beta + \gamma = 48/32 = 3/2 \Rightarrow r = ½$

Also, $\alpha\beta + \beta\gamma + \gamma\alpha = 22/32$

$$\Rightarrow \alpha\beta + (\alpha+\beta)\gamma = \frac{22}{32}$$

$$\Rightarrow \alpha\beta + (1)\left(\frac{1}{2}\right) = \frac{22}{32} \Rightarrow \alpha\beta = 3/16$$

∴ equation whose roots are α and β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - x + 3/16 = 0 \Rightarrow 16x^2 - 16x + 3 = 0$

$\Rightarrow 16x^2 - 4x - 12x + 3 = 0$

$\Rightarrow 4x(4x - 1) - 3(4x - 1) = 0$

$\Rightarrow (4x - 3)(4x - 1) = 0$

$\Rightarrow x = \frac{1}{4}, \frac{3}{4}$

It is easy to verify that $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ are roots of the equation.

5. Let the roots of the equation be 3α, 5α, and β.

   We have

   $3\alpha + 5\alpha + \beta = 9$ (i)

   $(3\alpha)(5\alpha) + (3\alpha)\beta + (5\alpha)\beta = 23$ (ii)

   and $(3\alpha)(5\alpha)\beta = 15$ (iii)

   From (i), $\beta = 9–8\alpha$

   Putting this in (ii) we get

   $15\alpha^2 + 8\alpha(9 – 8\alpha) = 23$

   $\Rightarrow \quad 49\alpha^2 - 72\alpha + 23 = 0$
   $\Rightarrow \; 49\alpha^2 - 49\alpha - 23\alpha + 23 = 0$
   $\Rightarrow \quad 49\alpha(\alpha - 1) - 23\alpha(\alpha - 1) = 0$
   $\Rightarrow \quad (49\alpha – 1)(\alpha – 1) = 0 \Rightarrow = 1/49, 1$

   When $\alpha = 1, \beta = 1$

   Note that these values satisfy (iii)

   When $\alpha = 1/49$, $\beta = 9–8/49 = 433/49$

   These values do not satisfy the relation (iii)

   Thus, roots are 3,5, and 1

6. Let the roots be α, $\alpha+2$, and β

   We have

   $\alpha + (\alpha + 2) + \beta = 9$ (i)

   $\alpha(\alpha + 2) + \alpha\beta + (\alpha + 2)\beta = 23$ (ii)

   $\alpha(\alpha + 2)\beta = 15$ (iii)

   From (i) we get $\beta = 7 – 2\alpha$

   Putting this in (ii), we obtain

   $\alpha(\alpha + 2) + (\alpha + \alpha + 2)(7 – 2\alpha) = 23$

   $\Rightarrow \; \alpha^2 + 2\alpha + 14 + 10\alpha - 4\alpha^2 = 23$
   $\Rightarrow \; 3\alpha^2 - 12\alpha + 9 = 0 \Rightarrow \; \alpha^2 - 4\alpha + 3 = 0$
   $\Rightarrow \; (\alpha - 1)(\alpha - 3) = 0 \;\; \Rightarrow \alpha = 1, 3$

   When $\alpha = 1$, $\beta = 5$

These values satisfy (iii)

Thus roots are 1,3,5

When α = 3, β = 1

These values satisfy (iii)

In this case roots are 3,5, 1

7. Let the roots be $\alpha, \beta\, \gamma, \delta$

As these are in G.P.

$$\frac{\alpha}{\beta} = \frac{\gamma}{\beta} = \frac{\delta}{\gamma} \Rightarrow \gamma\delta = \beta\gamma$$

We have $\alpha + \beta + \gamma + \delta = \dfrac{195}{27}$ (i)

$$(\alpha + \delta)(\beta + \gamma) + \gamma\,\delta + \beta\,\gamma = \frac{494}{27} \qquad \text{(ii)}$$

$$\gamma\,\delta\ (\beta + \gamma) + \beta\gamma\,(\alpha + \delta) = \frac{520}{27} \qquad \text{(iii)}$$

$$(\alpha\,\delta)(\beta\gamma) = \frac{192}{27} \qquad \text{(iv)}$$

As $\gamma\,\delta = \beta\,\gamma$, from (iv), we get

$(\gamma\delta)^2 = 64/9 \Rightarrow \gamma\,\delta = 8/3$ [Use (i) and (iii)]

Putting these values in (iii), we get

$(\alpha + \delta)\,(\beta + \gamma) = 350/27$ (v)

From (i) and (v), we get equation whose roots are $\alpha + \delta$ and $\beta + \gamma$, is

$$x^2 - \frac{195}{27}x + \frac{350}{27} = 0$$

or $27x^2 - 195x + 350 = 0$

$\Rightarrow 27x^2 - 90x - 105x + 350 = 0$

$\Rightarrow 9x(3x - 10) - 35x\,(3x - 10) = 0$

$\Rightarrow (9x - 35)\,(3x - 10) = 0$

$\Rightarrow x = \dfrac{35}{9}, \dfrac{10}{3}$

Let $\alpha + \delta = 35/9$, $\beta + \gamma = 10/3$

Equation whose roots are α, δ is

$$x^2 - \frac{35}{9}x + \frac{8}{3} = 0$$

$\Rightarrow \quad 9x^2 - 35x + 24 = 0$

$\Rightarrow \quad 9x^2 - 27x - 8x + 24 = 0$

$\Rightarrow \quad 9x(x-3) - 8(x-3) = 0$

$\Rightarrow \quad (9x-3)(x-3) = 0$

$$x = \frac{8}{9}, 3$$

Next equation whose roots β and γ is

$x^2 - (\beta + \gamma)x + \beta\gamma = 0$

$$\Rightarrow \quad x^2 - \frac{10}{3}x + \frac{8}{3} = 0$$

$\Rightarrow \quad 3x^2 - 10x + 8 = 0$

$\Rightarrow \quad (3x-4)(x-2) = 0$

$\Rightarrow \quad x = 4/3, 2$

Thus, roots are

8/9, 4/3, 2, 3 (verify)

8. Let the roots be

$\alpha, -\alpha, \beta$ and γ

We have

$\alpha + (-\alpha) + \beta + \gamma = 2/8 = ¼$ (i)

$[\alpha + (-\alpha)](\beta + \gamma) + \alpha(-\alpha) + \beta\gamma = -27/8$ (ii)

$[\alpha + (-\alpha)]\beta\gamma + \alpha(-\alpha)(\beta + \gamma) = -6/8 = -3/4$ (iii)

$\alpha(-\alpha)\beta\gamma = 9/8$ (iv)

From (i) $\beta + \gamma = ¼$

From (iii) $-\alpha^2(1/4) = -3/4 \Rightarrow \alpha^2 = 3 \Rightarrow \alpha = \pm\sqrt{3}$

From (ii) $-3 + \beta\gamma = -27/8 \Rightarrow \beta\gamma = -\frac{3}{8}$

Equation whose roots are β and γ is

$x^2 - (\beta + \gamma)x + \beta\gamma = 0$

or $\quad x^2 - (1/4)x - 3/8 = 0$

$\Rightarrow\ 8x^2 - 2x - 3 = 0$

$\Rightarrow\ x = 3/4, -1/2$

Thus, roots are $\sqrt{3}, -\sqrt{3}, \frac{3}{4}, \frac{-1}{2}$ (verify)

9. Let the roots be $\alpha, \beta\, \gamma$ and δ, and suppose that

$\alpha + \beta = \gamma + \delta$ (i)

We have

$\alpha + \beta + \gamma + \delta = -2$ (ii)

$(\alpha+\beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = -21$ (iii)

$(\alpha + \beta)\gamma\,\delta + \alpha\beta\,(\gamma + \delta) = 22$ (iv)

$\alpha\ \beta\ \gamma\ \delta = 40$ (v)

From (i) and (ii),

$\alpha + \beta = \gamma + \delta = -1$

From (iii), we get $(-1)(-1) + \alpha\beta + \gamma\,\delta = -21$

$\Rightarrow \alpha\beta + \gamma\delta = -22$

Equation whose roots are $\alpha\beta$ and $\gamma\delta$ *is*

$x^2 - (\alpha\beta + r\,\delta)x + (\alpha\beta)(r\,\delta) = 0$

$\Rightarrow\ x^2 + 22x + 40 = 0$

$\Rightarrow\ (x + 2)(x + 20) = 0 \Rightarrow\ x = -2, 20$

Let $\alpha\beta = -2\ \gamma\,\delta = -20$

Equation whose roots are α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or $x^2 + x - 2 = 0 \Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = -2, 1$

Equation whose roots are γ, δ

$x^2 - (\gamma + \delta)x + (\gamma\delta) = 0$

or $x^2 + x - 20 = 0 \Rightarrow (x + 5)(x - 4) = -4, 5$

$\Rightarrow\ x = -5, 4$

Thus, roots are –2, 1, –5, 4

10. Let roots be $\alpha, \beta\ \gamma$ and δ, and assume that

$\alpha\beta = \gamma\delta$ (i)

We have

$\alpha + \beta + \gamma + \delta = \frac{15}{2}$ (ii)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\,\delta = \frac{35}{2}$ (iii)

$(\alpha + \beta)\,\gamma\delta + \alpha\beta\,(\gamma + \delta) = \frac{30}{2} = 15$ (iv)

$\alpha\beta\gamma\delta = \frac{8}{2} = 4$ (v)

From (i) and (iv)

$\alpha\beta = \gamma\,\delta = \pm 2$

From (ii) and (iv), it follows that

$\alpha\beta = \gamma\,\delta = 2$

From (iii), $(\alpha + \beta)(\gamma + \delta) = \frac{35}{2} - 4 = 27/2$

Thus, equation whose roots are
$\alpha + \beta$ and $\gamma + \delta$ *is*

$$x^2 - (\alpha + \beta + \gamma + \delta)x + (\alpha + \beta)(\gamma + \delta) = 0$$

or $x^2 - (15/2)x + 27/2 = 0$ or $2x^2 - 15x + 27 = 0$

$\Rightarrow\ 2x^2 - 9x - 6x + 27 = 0 \quad \Rightarrow x(2x - 9) - 3(2x - 9) = 0$
$\Rightarrow\ (x - 3)(2x - 9) = 0 \Rightarrow x = 3, 9/2$

Let $\alpha + \beta = 3,\ \gamma + \delta = 9/2$

Equation whose roots are α and β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - 3x + 2 = 0 \Rightarrow\ (x - 1)(x - 2) = 0$
$\Rightarrow x = 1, 2$

Equation whose roots are γ and δ is

$\Rightarrow\ x^2 - (\gamma + \delta)x + \gamma\,\delta = 0$

$\Rightarrow \quad x^2 - \left(\frac{9}{2}\right)x + 2 = 0$

$\Rightarrow \quad 2x^2 - 9x + 4 = 0$

$\Rightarrow \quad (2x - 1)(x - 4) - 0$

$\Rightarrow \quad x = ½, 4$

Thus, roots are

½, 1, 2, 4

11. Let roots be $\alpha, \beta\ \gamma$ and δ where $\alpha\beta = 13$ we have

$\alpha + \beta + \gamma + \delta = 10$ (i)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = 42$ (ii)

$(\alpha + \beta)\gamma\delta + \alpha\beta(\gamma + \delta) = 82$ (iii)

$\alpha\beta\gamma\ \delta = 65$ (iv)

As $\alpha\beta = 13$, from (iv) we get $\gamma\delta = 5$ from (iii)

$5(\alpha + \beta) + 13(\gamma + \delta) = 82$ (v)

From (i) and (v) $8(\gamma + \delta) = 32$

$\Rightarrow \quad \gamma + \delta = 4$

From (i) $\alpha + \beta = 6$

Equation whose roots are α and β is

$x^2 - 6x + 13 = 0 \quad \Rightarrow \ x = 3 \pm 2i$

Equation whose roots are γ and δ is

$x^2 - 4x + 5 = 0 \quad \Rightarrow \ x = 2 \pm i$

Thus, roots are $2 \pm i$, $3 \pm 2\,i$

12. Let the roots be $\frac{\alpha}{\beta}, \alpha, \alpha\beta$

We have

$\frac{\alpha}{\beta} + \alpha + \alpha\beta = -\frac{b}{a}$ (i)

$\frac{\alpha}{\beta}.\alpha + \alpha.\alpha\beta + \frac{\alpha}{\beta}.\alpha\beta = \frac{c}{a}$ (ii)

$$\frac{\alpha}{\beta} . \alpha . \alpha\beta = -\frac{d}{a} \qquad \text{(iii)}$$

(iii) ⇒ $\alpha^3 = -d/a$

From (i) $\alpha\,(\beta + \frac{1}{\beta} + 1) = -\frac{b}{a}$

and from (ii)

$$\alpha^2\,(\beta + \frac{1}{\beta} + 1) = \frac{c}{a}$$

$$\Rightarrow \alpha = -\frac{c}{b} \quad \Rightarrow \alpha^3 = \frac{-c^3}{b^3}$$

$$\therefore -\frac{d}{a} = -\frac{c^3}{b^3} \Rightarrow b^3 d = ac^3$$

13. Let roots be $\frac{1}{\alpha - \beta}, \frac{1}{\alpha}, \frac{1}{\alpha + \beta}$

We have

$$\frac{1}{\alpha - \beta} + \frac{1}{\alpha} + \frac{1}{\alpha + \beta} = p \qquad \text{(i)}$$

$$\frac{1}{(\alpha - \beta)\alpha} + \frac{1}{\alpha(\alpha + \beta)} \frac{1}{(\alpha - \beta)(\alpha + \beta)} = q \qquad \text{(ii)}$$

and $\frac{1}{(\alpha - \beta)(\alpha)(\alpha + \beta)} = r$ (iii)

From (i) and (iii), we get

and $\frac{(\alpha + \beta) + \alpha + (\alpha - \beta)}{(\alpha - \beta)(\alpha)(\alpha + \beta)} = q$

and $\frac{1}{\alpha - \beta(\alpha)(\alpha + \beta)} = r$

$\therefore\ 3\alpha r = q \quad \Rightarrow r = q/3r$

As $1/\alpha$ is a root of $x^3 - px^2 + qx - r = 0$,

We get

$$\frac{1}{\alpha^3} - \frac{p}{\alpha^2} + \frac{q}{\alpha} - r = 0$$

$\Rightarrow\ 1 - p\alpha + q\ \alpha^2 - r\ \alpha^3 = 0$

$\Rightarrow\ 1 - \dfrac{pq}{3r} + q\ \dfrac{q^2}{qr^2} - r\dfrac{q^3}{27r^3} = 0$

$\Rightarrow 27\ r^3 - 9pqr + 3q^3 - q^3 = 0$

$\Rightarrow\ 27\ r^3 - 9pqr + 2q^3 = 0$

14. We have

$\alpha + \beta + \gamma = p$

$\beta\gamma + \alpha\gamma + \alpha\beta = q$

$\alpha\beta\gamma = \gamma$

Now, $(\beta + \gamma)\ (\gamma + \alpha)\ (\alpha + \beta)$

$= (p - \alpha)\ (p - \beta)\ (p - \gamma)$

$= p^3 - pp^3 + qp - \gamma = pq - \gamma$

$(ii)\quad \sum\frac{\alpha}{\beta} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta}$

$$= \frac{\beta + \alpha}{\alpha} + \frac{\gamma + \alpha}{\beta} + \frac{\alpha + \beta}{\gamma}$$

$$= \frac{p - \alpha}{\alpha} + \frac{p - \beta}{\beta} + \frac{p - \gamma}{\gamma}$$

$$= p\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 3$$

$$= \frac{pq}{r} - 3$$

(iii) $\sum\frac{1}{\alpha^2} = \frac{\beta^2\gamma^2 + \gamma^2\alpha^2 + \beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$

$$= \frac{(\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)^2}{(\alpha\beta\gamma)^2}$$

$$= \frac{q^2 - 2rp}{r^2}$$

15. $\sum \alpha = -p, \sum \beta = q, \sum \alpha\,\beta\gamma = -r$ and

$\alpha\beta\gamma\delta = s$

Now,

(i) $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$= p^2 - 2q$

(ii) $\sum\frac{1}{\alpha} = \frac{\sum\alpha\beta r}{\alpha\beta\gamma\delta} = \frac{-r}{s}$

3.6 SUMMARY

This unit deals with solutions of equations of degrees 2, 3 and 4 in a single variable. In **sections 3.2 and 3.3**, first of all, the method of solving quadratic equations is given and then nature of these solutions is discussed. Next, method of forming quadratic equations for which the two roots are known, is discussed. In **section 3.4**, relations between roots and coefficients of cubic and biquadratic equations are mentioned. Using these relations, methods of solving special type of cubic and biquadratic equations are discussed. All the above-mentioned methods are illustrated with suitable examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.5**.

UNIT 4 INEQUALITIES

Structure

4.0 INTRODUCTION

We are familiar with inequality symbols $>$, $\geq$, $<$ and $\leq$. We have used these symbols to compare two real numbers and subsets of real numbers. For instance, recall that the inequality $x \geq 3$ denote the set of all real numbers x that are greater than or equal to 3. In this unit, we shall extend our work to include more complex inequalities such as :

$$5x - 3 > 2x + 7$$

4.1 OBJECTIVES

After going through this unit, you will be able to:

- find the solution set of a linear inequality and draw its graph on number line;
- find the solution set of a compound inequality;
- find the solution set of an inequality involving absolute value;
- graph solution set of linear inequality in two variables;
- graph solution set of system of simultaneous inequality in two variables; and
- find solution set of quadratic and non-linear inequalities.

4.2 SOLVING A LINEAR INEQUALITY

A linear inequality contains a linear expression on one side of the inequality and a linear expression or a constant on the other side.

For instance

$2x + 5 > 3 - 7x;$

$3x + 7 \geq 8;$ and

$2x - 5 < 6 + 4x$

are linear inequalities.

By solving a linear inequality in one variable, we mean finding all the values of the variable for which the inequality becomes true.

We first list some tools for solving inequalities.

Tools for Solving Inequalities

The solution set of an inequality will not change if we perform any of the following:

Addition

Add (or subtract) the same number or the both sides of the inequality.

Multiplication by Positive Number

Multiply (or divide) both sides by the same *positive* number.

Multiplication by Negative Number

Multiply (Or divide) both sides by the *negative* number and *reverse* the direction of the inequality symbol.

Solved Examples

Example 1: Solve the inequality

$$x + \frac{2}{3} \geq \frac{7}{2}$$

and graph its solution set.

Solution : The given in equalities is

$$x + \frac{2}{3} \geq \frac{7}{2}$$

$$\Leftrightarrow x + \frac{2}{3} + \left(-\frac{2}{3}\right) \geq \frac{7}{2} + \left(-\frac{2}{3}\right) \qquad \left[add \left(-\frac{2}{3}\right) \text{ to both the sides}\right]$$

$$\Leftrightarrow x \geq \frac{7}{2} - \frac{2}{3}$$

$$\Leftrightarrow x \geq \frac{17}{6}$$

The Solution set is $\{ x \mid x \geq 17/6\} = [\, 17/6, \infty]$

The graph of this set is

Example 2: Solve the inequality

$4x < 28$

and graph its solution set.

Solution : The given inequality is

$$4x < 28$$

$$\Leftrightarrow \quad \frac{1}{4}(4x) < \frac{1}{4}(28) \qquad \text{[Multiply both the sides by 1/4]}$$

$$\Leftrightarrow \quad x < 7$$

CAUTION As 4 is positive, the direction of the inequality is not reversed.

The solution set is $\{ x \mid x < 7 \} = (-\infty, 7)$

The graph of the set is

Example 3 : Solve the inequality

$$-3x \geq \frac{5}{6}$$

Solution : The given inequality is

$$-3x \geq \frac{5}{6}$$

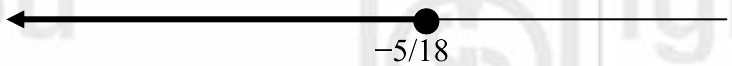
$$\Leftrightarrow \quad \left(-\frac{1}{3}\right)(-3x) \leq \left(-\frac{1}{3}\right)\left(\frac{5}{6}\right)$$

[Multiple both sides by (− 1/3) and reverse the direction of the inequality]

$$\Leftrightarrow \quad x \leq -\frac{5}{18}$$

The solution set is $\{ x \mid x \leq -5/18 \} = (-\infty, -5/18]$

The graph of this set is



CAUTION : Since − 1/3 is a negative number the direction of the inequality must be reversed.

Example 4: Find the solution set of the inequality

$$7x + 4 \leq 4x + 16$$

Solution : The given inequality

$$7x + 4 \leq 4x + 16$$

We first bring the like terms on one side. This can be achieved by adding $-4x$ to both the sides of the inequality.

$$(-4x) + 7x + 4 \leq (-4x) + 4x + 16$$

$\Leftrightarrow \quad 3x + 4 \leq 16$

$\Leftrightarrow \quad 3x + 4 + (-4) \leq 16 + (-4)$ [add -4 to both the sides]

$\Leftrightarrow \quad 3x \leq 12$

$\Leftrightarrow \quad \frac{1}{3}(3x) \leq \left(\frac{1}{3}\right)(12)$ [Multiply both the sides by 1/3]

$\Leftrightarrow \quad x \leq 4$

Thus, the solution set is $\{x \mid x \leq 4\} = (-\infty, 4]$

Example 5 : Find the solution set of the inequality

$$2(x - 1) \geq 7x$$

Solution : The given inequality is

$$2(x - 1) \geq 7x$$

We first remove the parentheses. This can be achieved by using distributive property.

$$2x - 2 \geq 7x$$

As in the previous problem our next objective is to collect x − terms on one side of the inequality. Towards this end we add $-2x$ to both the sides of the inequality.

$$(-2x) + (2x - 2) \geq (-2x) + 7x \Leftrightarrow -2 \geq 5x$$

$\Leftrightarrow \quad \frac{1}{5}(-2) \geq \frac{1}{5}(5x)$ [Multiply both sides by 1/5]

$\Leftrightarrow \quad -\frac{2}{5} \geq x.$

Thus, the solution set is $\{x \mid x \leq -2/5\} = (-\infty, 2/5]$

Example 6: Solve the inequality

$$-(x - 3) + 4 < -2x + 5$$

and graph the solution set.

Solution : Remove the parentheses to obtain

$$-x + 3 + 4 < -2x + 5$$

$\Leftrightarrow 2x - x < 5 - 7$

$\Leftrightarrow x < -2$

$\therefore$ Solution set is $\{x | \, x < -2\} = (-\infty, -2)$

The Graph of the solution set is

Example 7 : Solve the inequality

$$\frac{3}{5}(x - 2) \le \frac{5}{3}(2 - x)$$

and graph the solution set.

Solution : We first clear fractions by multiply both sides by 15

$$\frac{3(x-2)}{5} \le \frac{5(2-x)}{3} \Leftrightarrow 3[3(x - 2] \le 5[5(2-x)]$$

$\Leftrightarrow \quad 9x - 18 \le 50 - 25x$

$\Leftrightarrow \quad 9x + 25x \le 50 + 18$

$\Leftrightarrow \quad 34x \le 68$ or $x \le 2$

$\therefore$ Solution set is $\{x | \, x \le 2\} = (-\infty, 2]$

The graph of the inequality is

Example 8: Solve the inequality

$$\mathbf{\frac{1}{3}(4 - 2\boldsymbol{x}) \ge \frac{\boldsymbol{x}}{2} - 3}$$

and graph the solution set.

Solution : We first clear fractions by multiplying both sides by 6.

$$6\left(\frac{4-2x}{3}\right) \ge 6\left(\frac{x}{2} - 3\right)$$

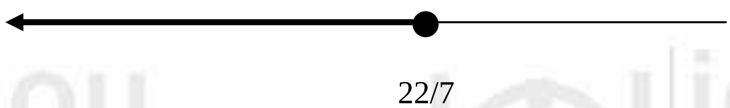
$\Leftrightarrow \quad 2(4 - 2x) \ge 3x - 18$

$\Leftrightarrow \quad 8 - 4x \ge 3x - 18$ [remove paranthesis]

$\Leftrightarrow \quad 8 + 18 \ge 3x + 4x$ or $22 \ge 7x$ or $7x \le 22$ or $x \le 22/7$

$\therefore$ Solution set is $\{x | \, x \le 22/7\} = (-\infty, \frac{22}{7}]$

The graph of the solution set is



Example 9 : Solve the inequality

$$-3 < 4 - 7x < 18$$

and graph the solution set.

Solution : The given inequality

$$-3 < 4 - 7x < 18$$

is equivalent to the inequality

$$-3 - 4 < 7x < 18 - 4 \text{ or } -7 < -7x < 14$$

We divide this compound inequality by − 7 and reverse the inequality signs to obtain

$$\frac{-7}{-7} > \frac{-7x}{-7} > \frac{14}{-7} \text{ or } -2 < x < 1$$

$\therefore$ Solution set is $\{ x \mid -2 < x < 1 \} = (-2, 1)$

The graph of this set is

Example 10 : Solve the inequality

$$-2 < \frac{1}{5}(4 - 3x) \leq 8$$

and graph the solution set.

Solution : We first multiply the given inequality by 5 to obtain

$$(-2)(5) < 4 - 3x \leq (8)(5)$$

$$\Leftrightarrow \quad -10 - 4 < -3x \leq 40 - 4$$

$$\Leftrightarrow \quad \frac{-14}{-3} > x \geq \frac{-44}{4} \quad \Leftrightarrow \quad -11 \leq x < 14/3$$

$\therefore$ Solution set is $\{ x \mid -11 \leq x \ \frac{14}{3} \} = [-11, \frac{14}{3}]$

Graph of the solution set is

Example 11: Solve the inequality

$$\frac{2x - 5}{x + 2} < 2$$

and graph its solution

Solution : The inequality $\frac{2x - 5}{x + 2} < 2$ is

equivalent to
$$\frac{2x + 5}{x + 2} - 2 < 0 \Leftrightarrow \frac{2x - 5 - 2x - 4}{x + 2} < 0$$
$$\Leftrightarrow \frac{-9}{x + 2} < 0$$

But $\frac{a}{b} < 0, a < 0 \;\Leftrightarrow\; b > 0.$

Thus, $\frac{-9}{x+2} < 0,\; -9 < 0 \;\Leftrightarrow\; x + 2 > 0 \Leftrightarrow x > -2$

$\therefore$ Solution set of the inequality is $(-2, \infty)$

The graph of the solution set is

Example 12 : Solve the inequality

$$\frac{\mathbf{6}}{\boldsymbol{x-3}} < 5$$

Solution : To solve this inequality, we have to consider two cases.

Case 1: $x < 3$

As $x < 3$, $x - 3 < 0$, therefore $\frac{6}{x-3} < 0$ and $\frac{6}{x-3} < 5$

$\therefore$ In this case, solution set of the inequality is $(-\infty, \mathbf{3})$

Case 2: $x > 3$

As $x > 3$, $x - 3 > 0$

$$\therefore \frac{\mathbf{6}}{\boldsymbol{x-3}} < 5,\; x - 3 > 0$$
$$\Leftrightarrow \mathbf{6} < 5x - 15$$
$$\Leftrightarrow \mathbf{21} < 5x \Leftrightarrow \boldsymbol{x} > 21/5$$

Thus, in this case solution set of the inequality is

$\{ x \mid x > 21/5 \} = (21/5, \infty)$

Hence, solution of the inequality is $(-\infty, 3) \cup (21/5, \infty)$.

Check Your Progress 1

Solve the following inequalities (1 – 4) and graph the solution set.

1. $2x + 1 > -3$
2. $3x - 2 \geq 4$
3. $2 - 5x \leq 4$
4. $3 + 2x < 7$

Find the solution set of the inequalities (5–8)

5. $3x + 2 \leq 2x + 5$
6. $3(x - 2) \geq 5 - 2x$
7. $-(x - 4) < 2x + 4$
8. $\frac{1}{5}(x-3) \geq \frac{7}{3}(3-x)$

Find the solution set of the inequalities (9–10)

9. $-2 \leq 5 - 4x \leq 7$
10. $-3 \leq \frac{1}{4}(x - 2) \leq 4$
11. $\frac{x+3}{x-1} < 0$
12. $\frac{x+8}{x+1} > 1$

4.3 INEQUALITIES AND ABSOLUTE VALUE

Inequalities often occur in combination with absolute value, such as $|x - 7| < 2$ or $|x + 1| > 2$. In order to see how to solve these, we first list some useful properties of absolute value.

We first recall that if $x \in \mathbf{R}$, then

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Note that $|x| = \max\{-x, x\}$.

Properties of Absolute Values

1. $|-x| = |x| \quad \forall x \in \mathbf{R}$
2. $|xy| = |x||y| \quad \forall x, y \in \mathbf{R}$
3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \quad \forall x, y \in \mathbf{R}\ y \neq 0$
4. $|x| = \sqrt{x^2} \quad \forall x \in |\mathbf{R}$
5. If $a > 0$, then

$|x| \leq a \Leftrightarrow -a \leq x \leq a$

$|x| \geq a \Leftrightarrow -x \leq a$ or $x \geq a$

6. If $a > 0$, then $|x - y| < a$

$$\Leftrightarrow -a < x - y < a \Leftrightarrow y - a < x < y + a$$

7. Triangle Inequality

$$|x + y| \leq |x| + |y| \quad \forall x, y \in \boldsymbol{R}$$

The equality holds if and only if $xy > 0$

Solved Examples

Example 13 : Solve the inequality

$$\left|\frac{3x-4}{2}\right| \leq \frac{5}{12} \text{ and graph the solution set.}$$

Solution : We use the property $|y| \leq b \Leftrightarrow -b \leq y \leq b$. Thus

$$\left|\frac{3x-4}{2}\right| \leq \frac{5}{12} \quad \Leftrightarrow \quad -\frac{5}{12} \leq \frac{3x-4}{2} \leq \frac{5}{12}$$

$$\Leftrightarrow 2\left(-\frac{5}{12}\right) \leq 3x - 4 \leq 2\left(\frac{5}{12}\right) \Leftrightarrow \frac{-5}{6} + 4 \leq 3x \leq 4 + \frac{5}{6}$$

$$\Leftrightarrow \frac{-19}{6} \leq 3x \leq \frac{29}{6} \Leftrightarrow \frac{-19}{18} \leq x \leq \frac{29}{18}$$

Thus, solution set of the inequality is $\left[\frac{-19}{18}, \frac{29}{18}\right]$

The graph of this set is

Example 14 : Find solution set of the inequality

$$\left|x + \frac{1}{4}\right| > \frac{7}{4}$$

Solution : We use the property $|y| > b \Leftrightarrow y > b \text{ or } y < -b$. *Thus,*

$$\left|x + \frac{1}{4}\right| > \frac{7}{4} \quad \Leftrightarrow \quad x + \frac{1}{4} > \frac{7}{4} \text{ or } x + \frac{1}{4} \leq -\frac{7}{4}$$

$$\Leftrightarrow x > \frac{7}{4} - \frac{1}{4} \text{ or } x < -\frac{7}{4} - \frac{1}{4} \Leftrightarrow x > \frac{3}{2} \text{ or } x < -2$$

$\therefore$ Solution set of the inequality is

$$\left\{x \middle| x > \frac{3}{2} \text{ or } x < -2\right\} = (-\infty, -2) \cup \left(\frac{3}{2}, \infty\right)$$

Example 15 : Solve the inequality

$$\frac{2}{|x-3|} > 5 \text{ and graph its solution.}$$

Solution : Note that the domain of the inequality is{ $x \mid x \neq 3$)

For $x \neq 3$, $|x - 3| > 0$

Thus, the given inequality

$$\frac{2}{|x-3|} > 5 \Leftrightarrow 2 > 5|x-3|$$

$$\Leftrightarrow |x–3| < \frac{2}{5} \Leftrightarrow \frac{-2}{5} < x - 3 < \frac{2}{5}$$

$$\Leftrightarrow 3–\frac{2}{5} < x < 3 + \frac{2}{3} \Leftrightarrow \frac{13}{5} < x < \frac{17}{5}$$

$\therefore$ the solution set of the inequality is

$\{x|13/5 < x < 17/5\} = (13/5, 17/5)$

The graph of this set is

Application of inequalities

Inequalities arise in applications just as equations do. We use the following phrases for inequalities.

Statement	**Symbols**
a is more than b	$a > b$
a is less than b	$a < b$
a is not less than b	$a \geq b$
a is not more than b	$a \leq b$
a is at most b	$a \leq b$
a is at least b	$a \geq b$

Example 16 : Translate the following statements into an inequality.

(i) x is a most 7 (ii) – 4 is less than y

(iii) Seven more than z is at least 7 (iv) Twice x is greater than y

(v) The square root of y is at least 7 (vi) The opposite of z is less than y

Solution :

(i) x is a most 7 :

$$x \le 7$$

(ii) – 4 is less than y

$$-4 < y$$

(iii) Seven more than z (i.e., $7 + z$) is at least 15 :

$$7 + z \ge 15$$

(iv) Twice of x (i.e., $2x$) is greater than y :

$$2x > y$$

(v) The square root of y is (i.e., $\sqrt{y}$) is at least 7 :

$$\sqrt{y} \ge 7$$

(vi) The opposite of z (i.e., $-z$) is less than y

$$-z \le y$$

Example 17 : Vishi obtained 73, 67 and 72 marks in her first three mathematics tests. How much marks should she get in her fourth test so as to have an average of at least 75 ?

Solution: Let x denote her marks in the fourth test. The sum of marks obtained by her in four tests divided by 4 is her average marks. This, average is to be at least 75. So we have the inequality

$$\frac{73 + 67 + 72 + x}{4} \ge 75 \iff \frac{212 + x}{4} \ge 75$$

$$\iff 4\left(\frac{212 + x}{4}\right) \ge 4\,(75) \iff 212 + x \ge 300$$

$$\iff 212 + x - 212 \ge 300 - 212 \iff x \ge 88$$

Thus, she must get at least 88 marks in her fourth test.

Example 18 : The sum of three consecutive integers be no more than 12. What are the integer.

Solution : Let three consecutive integers be $x - 1$, x and $x + 1$. Sum of these integers is no more than 12 means:

$$(x - 1) + x + (x + 1) \le 12 \iff 3x \le 12 \iff x \le 4$$

Thus, the integers must be $x - 1$, x and $x + 1$ with $x \le 4$

Check Your Progress – 2

Solve the following inequalities :

1. $|x-3| \geq 2$

2. $\left|\frac{2x-5}{3}\right| \leq 1$

3. $\left|\frac{x-5}{3}\right| < 6$

4. $\frac{5}{|x-3|} < 7$

5. In drilling world's deepest hole, it was found that the temperature T in degree Celsius x km below the surface of the earth, was given by the formula

   $T = 30 + 25\ (x-3),\ \ 3 \leq x \leq 15$

   At what depth your expect to find temperature between 200℃ and 300℃.

6. IQ (Intelligence Quotient) of a person is given by the formula

   $\text{IQ} = \frac{MA}{CA} \times 100$

   Where MA stands for the mental age of the person CA for the chronological age. If $80 \leq$ IQ ≤ 140 for a group of 12 year old children, find the range of their mental age.

7. The cost of manufacturing x telephones by Josh Mobiles is given by C = 3000 + 200x, and the revenue from selling these are given by R = 300x. How many telephones must be produced and sold in order to realize a profit ?

   [**Hint :** Profit = revenue − cost]

8. A doctor has prescribed 3 cc of medication for a patient. The tolerance (that is, the amount by which the medication may differ from the acceptable amount) is 0.005 cc. Find the lower and upper limits of medication to be given.

4.4 LINEAR INEQUALITIES IN TWO VARIABLES

A Linear inequality in two variables is an expression that can be put in the form

$$ax + by < c$$

where a, b and c are real numbers (where a and b are not both 0's). The inequality symbol can be any one of the following four :

$<\ ,\ \leq,\ >, \geq$

Some examples of linear inequalities are

$x + 3y < 6$, $y - x > 1$, $x + 2y \geq 1$, $2x + 3y \geq 6$

The solution set for a linear inequality is a section of the coordinate plane. The boundary for the section is found by replacing the inequality symbol with an equal sign and representing the resulting equation in graphical form. The boundary is included in the solution set (and represented with a solid line) if the inequality symbol used originally is $\leq$ or $\geq$ The boundary is not included (and is represented with a broken line) if the original symbol is < or >.

Let us look at some illustrations.

Illustration : Represent graphically the solution set for $2x + 3y \geq 6$.

Solution

The boundary for the graph is the plot of the equation $2x + 3y = 6$. for which the x–intercept and y–intercept are 3 and 2, respectively. (Recall that x–intercept is obtained by letting $y = 0$ and y–intercept by letting $x = 0$). The boundary is included in the solution set, because the inequality symbol is $\geq$.

The graph of the boundary is given in Figure 4.1

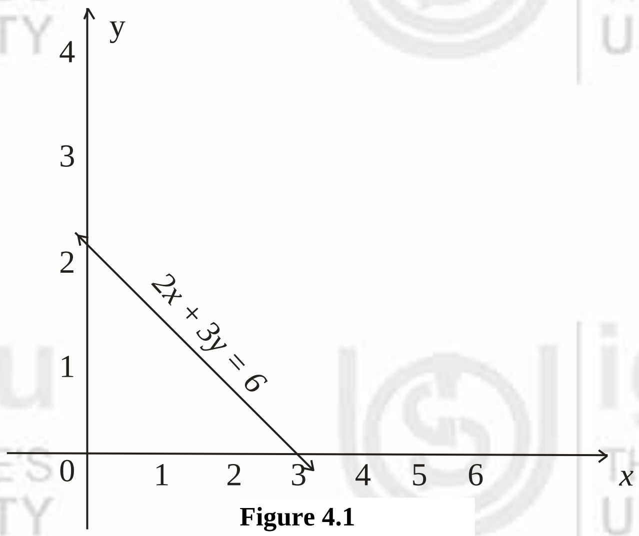


Figure 4.1

The boundary splits the coordinate plane into two sections or regions, the region above the boundary and the region below the boundary. The solution set for $2x + 3y \geq 6$ is one of these two regions, along with boundary. To find the correct region, we simply choose any convenient point this is not on the boundary. We substitute the coordinates of the point in the original inequality $2x + 3y \geq 6$. If the point we choose satisfies the inequality, then it is a member of the solution set and all points on the same side of the boundary as that of the chosen points are also in the solution set. If the coordinates of our point do not satisfy the original inequality, then solution set lies on the other side of the boundary.

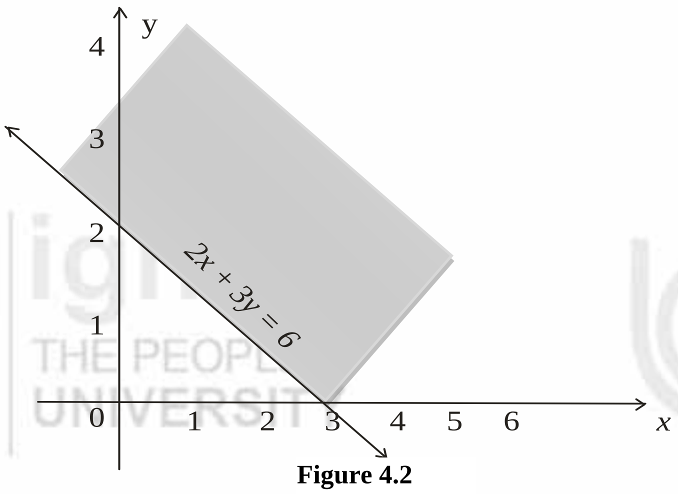


Figure 4.2

In this illustration, a convenient point off the boundary is the origin.

Substituting (0,0)

into $2x + 3y \geq 6$

gives us $2(0) + 3(0) \geq 6$ or $0 \geq 6$ (a false statement)

Since the origin is *not* a solution of the inequality $2x + 3y \geq 6$ and it lies below the boundary, the solution set must lies on the other side of the boundary.

The graph of $2x + 3y \geq 6$ is given by the shaded part of Figure 4.2.

4.5 PROCEDURE TO GRAPH THE SOLUTION SET OF LINEAR INEQUALITIES IN TWO VARIABLES

Step 1 : Replace the inequality with an equal sign. The resulting equation represents the boundary for the solution set.

Step 2: Represent graphically the boundary found is step 1 using the solid line, if the boundary is included in the solution set (that is, if the original inequality symbol was either $\geq$ or $\leq$). Use a *broken line* to draw the graph of the boundary, if it is not included in the solution set (that is, if the original inequality sign was either $<$ or $>$).

Step 3 Choose any convenient point not on the boundary and substitute the coordinates into the original inequality. (if the origin does not lie on the boundary, it is preferable to use the origin). If the resulting statement is *true*, the graph lies on the same side of the boundary as the chosen point. If the resulting statement is false, the solution set lies on the *opposite side* of the boundary.

Example 19 Draw the graph of the inequality $y < 2x - 3$

Solution

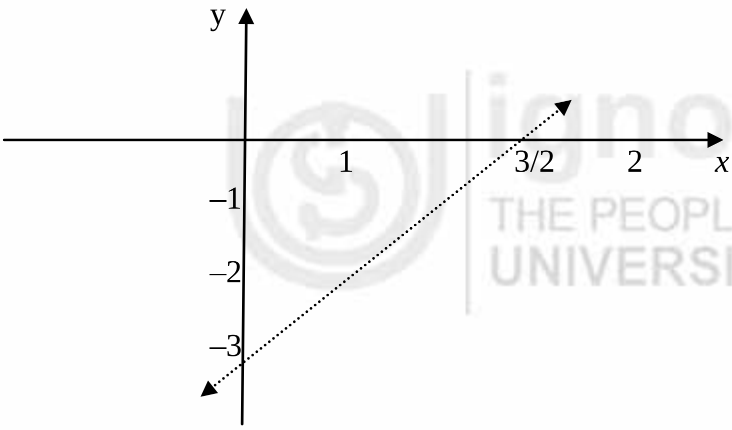


Figure 4.3

Solution

The boundary is the graph of $y = 2x - 3$. It is a line which makes x – intercept of 3/2 and y–intercept of −3. Since the original symbol is <, the boundary is not included in the set. Therefore, we use a broker line to represent the boundary. (See Fig 4.3).

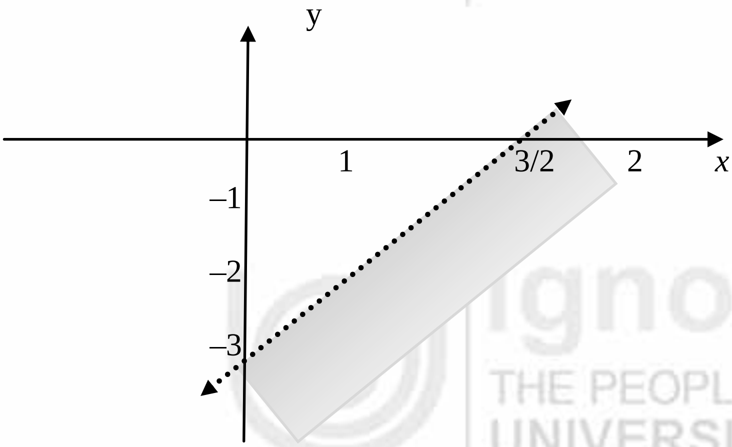


Figure 4.4

We use the origin as a test point. Substituting (0,0) into $y < 2x - 3$, we have $0 < 2(0) - 3$ or $0 < -3$, which is a false statement.

Thus, the origin is not in the solution. Since it lies above the boundary, the solution set lies on the other side of the boundary.

The graph is given in Figure 4.4

Example 20 Draw the graph of the inequality $x + y \le 0$

Solution : The boundary is the graph of $x + y = 0$ It is a line that passes through (0,0) and (−1, 1). Since the boundary passes through the origin, we use (1,0) as a test point. Since (1,0) does not satisfy the inequality $x + y \le 0$ and (1,0) lies above the boundary, the solution set must on the other side of the boundary. (See figure 4.5)

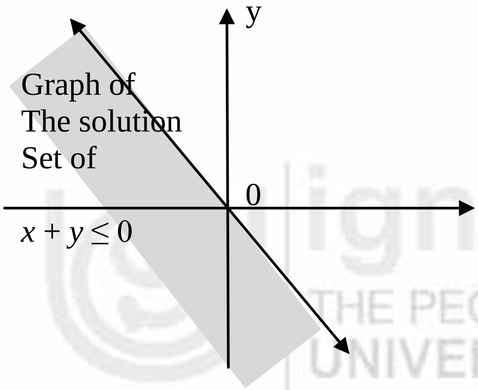


Figure 4.5

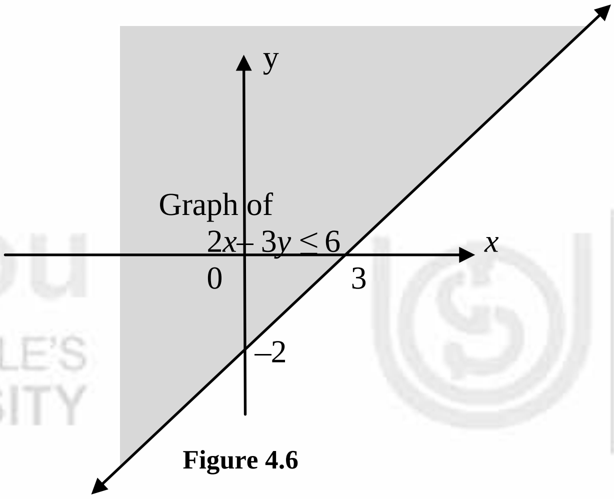


Figure 4.6

Example 21: Draw the graph of the inequality $2x - 3y \le 6$

Solution : The double intercept form of the boundary is $\frac{x}{3} + \frac{y}{-2} = 1$. Test point (0,0) satisfies the inequality. The graph of the inequality is given in Figure 4.6

Example 22: Draw the graph of the following inequalities

(i) $x > 3$ (ii) $y \le 4$

Solution: The boundary is $x = 3$ which is a vertical line. All the points to the right of $x = 3$ have x–coordinate greater than 3 and all the points to the left of $x = 3$ have x coordinate less than 3 (See figure 4.7)

(ii) The boundary is $y = 4$, which is a horizontal line. All the points of the line $y = 4$ and below this line have y coordinate ≤ 4 See Figure 4.8.

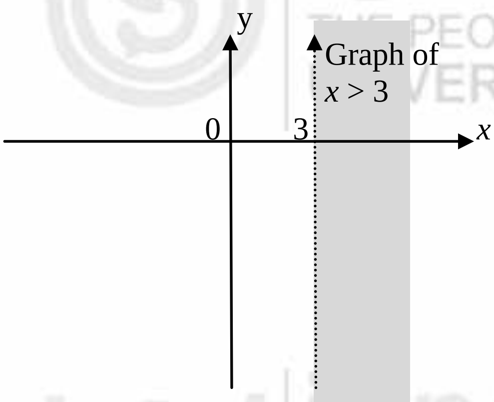


Figure 4.7

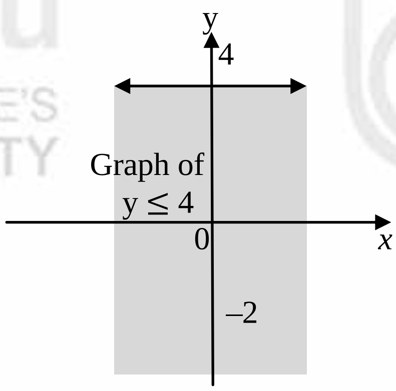


Figure 4.8

4.6 SYSTEM OF LINEAR INEQUALITIES IN TWO VARIABLES

Consider the system of linear inequalities in two variables

$3x + 2y \leq 6$ and $x + 2y \geq 4$

First of all we draw the boundaries $3x + 2y = 6$ (that is, $\left(that\ is, \frac{x}{2} + \frac{y}{3} = 1\right)$

and $x + 2y = 4$ $\left(\text{that is,} \frac{x}{4} + \frac{y}{2} = 1\right)$. We indicate the solution sets of the inequalities by drawing arrows as show in Figure 4.9 and then shade the common (i.e. solution set). See Figure 4.9.

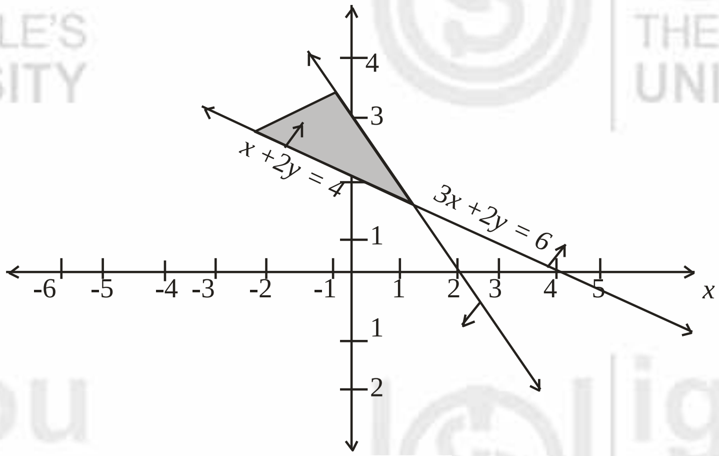


Figure 4.9

Example 23 Draw the graph of the solution set of the following inequalities on the same graph.

$2x + y \geq 8$, $x + 2y \geq 8$ and $x + y \leq 6$.

Solution : We first draw the graphs of the boundaries $2x + y = 8$.

$\left(i.e., \frac{x}{4} + \frac{y}{8} = 1\right), x + 2y = 8$ $\left(i.e., \frac{x}{8} + \frac{y}{4} = 1\right)$ and $x + y$

$= 6$ $\left(i.e., \frac{x}{6} + \frac{y}{6} = 1\right)$

We indicate the solution set of each of the inequalities by two arrows and then shade the common region (i.e., shows solution set) in Fig. 4.10.

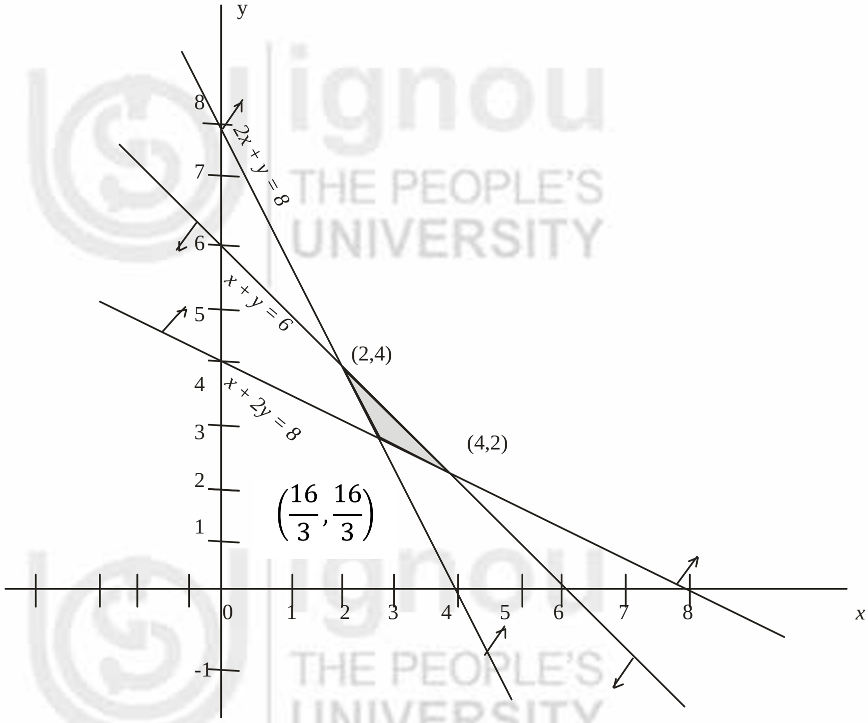


Figure 4.10

Example 24 Draw the solution set of the following inequalities on the same graph.

$$3x - 5y \le 15, \; 3x + 2y \ge 6, \; x \le y \; and \; x \ge 0$$

Solution : As $x \ge 0$ we shall restrict ourselves to the right of y–axis (including the y–axis). We draw the boundaries $3x - 5y = 15$, $3x + 2y = 6$

$\frac{1}{2}$ and $x = y$. Note that $x = 0$ is y–axis. We next indicate the solution

$\left(i.e., \frac{x}{2} + \frac{y}{3} = 1\right)$ set of each of the inequalities by drawing the arrows as shown in Figure 4.11. We finally shade the common region (i.e., solution set)

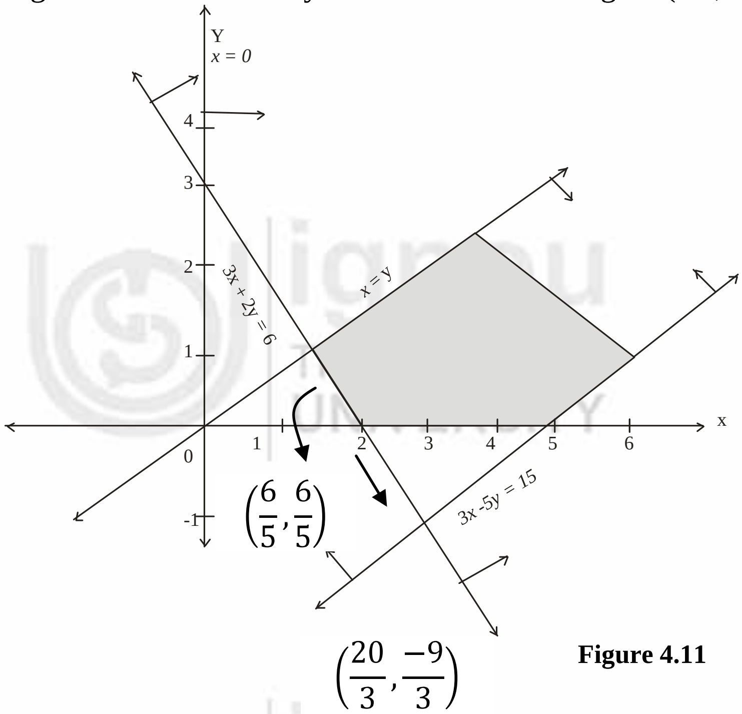


Figure 4.11

Check Your Progress 3

Draw the graph of the following inequalities

1. $x - y \geq -3$
2. $2x + 3y \leq 6$
3. $y < 2x$
4. $x \leq -2$
5. $y > 2$

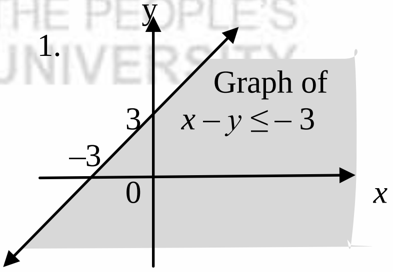
For each of the question 6 to 7 draw the graph of the following inequalities on the same graph.

6. $2x + y \geq 10, \quad x + 4y \geq 12, \quad 6x + y \geq 18, \quad x \geq 0, \quad y \geq 0$

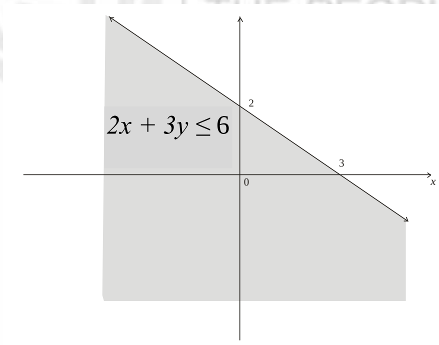
7. $2x + 3y \leq 6, \quad x + 4y \leq 4, \quad x \geq 0, \quad y \geq 0.$

Answers

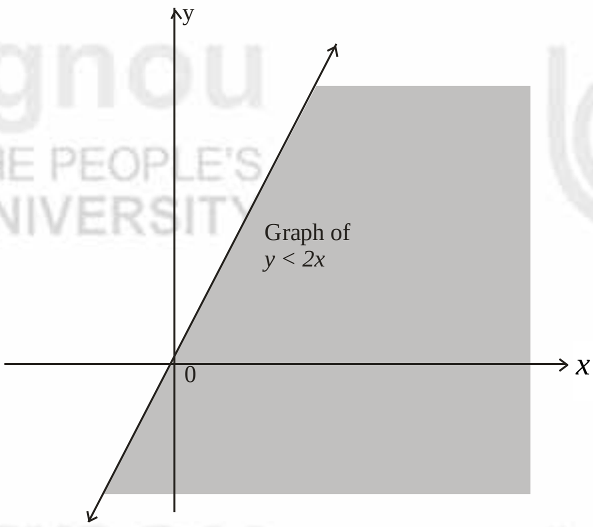
1.



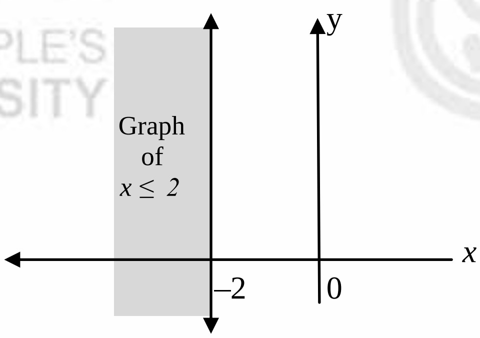
2.



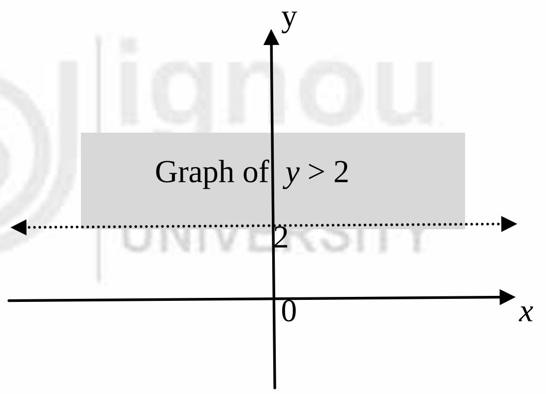
3.

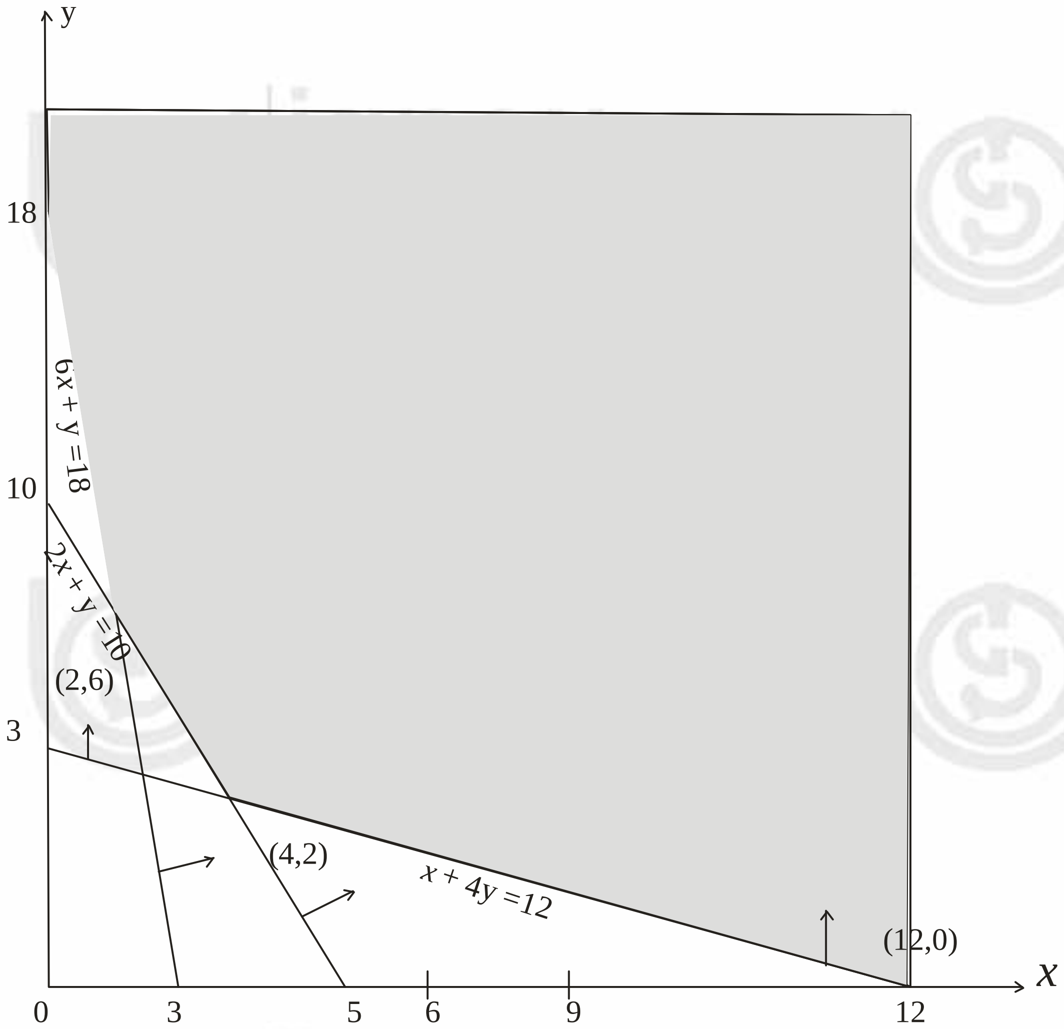


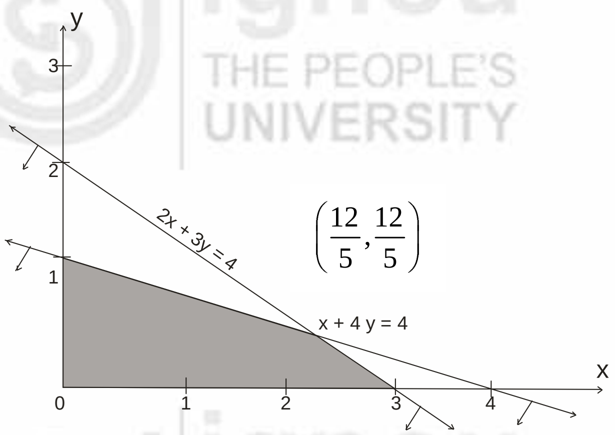
4.



5.


y
Graph of y > 2
2
0
x


y
18
10
3
6x + y =18
2x + y =10
(2,6)
(4,2)
x + 4y =12
(12,0)
X
0
3
5
6
9
12


y
3
2
1
2x + 3y = 4
$\left(\frac{12}{5}, \frac{12}{5}\right)$
x + 4 y = 4
x
0
1
2
3
4

4.7 QUADRATIC AND OTHER NON–LINEAR INEQUALITIES

In this section, we examine quadratic inequalities in one variable. Consider the inequality $x^2 - 5x + 6 \leq 0$.

The solution set of this inequality is the set of all real x which make the inequality a true statement.

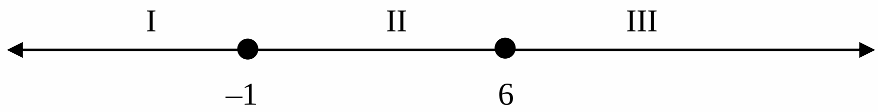
In general, a quadratic inequality is an inequality of the form $ax^2 + bx + c > 0$ where $a \neq 0$. The sign $>$ can be replaced by $\geq$, $<$ or $\leq$.

We illustrate the method of solution of a quadratic inequality by an illustration.

Illustration Solve $x^2 - 5x - 6 < 0$.

First factorise the left–hand side : $(x + 1)(x - 6) < 0$

Next show on a number line the points for which either factor is zero. These points divide into the line into three regions as follows :



Now test each region to see if the inequality is satisfied there. To do this it is sufficient to substitute one value from the each region.

Region I Using $x = -2$ as a test value we see that both $(x + 1)$ and $(x - 6)$ are negative, so that their product is positive. Thus, the inequality $x^2 - 5x - 6 < 0$ is not satisfied in this region. As such we reject region I.

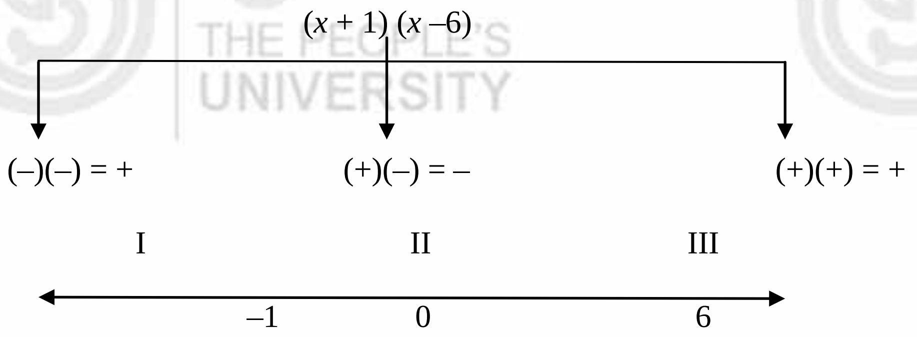
Region II Using $x = 0$ as a test value we see that $(x + 1)$ is positive and $(x - 6)$ is negative. Thus, the inequality $x^2 - 5x - 6 < 0$ is satisfied in this region. That is, this region is a part of the solution.

Region III Using $x = 7$ as a test value, we see that both $(x + 1)$ and $(x - 6)$ are positive, so that their product is positive. Thus, the inequality $x^2 - 5x - 6 < 0$ is not satisfied in the region. As such we reject the region III.

Finally, we check the critical numbers themselves (critical numbers are the number, for which the expression on the left–hand side becomes zero). As $x = -1$ and $x = 6$ make $x^2 - 5x - 6$ equal to zero, these are not parts of the solution. Hence, the solution set is region II :

$$\{x | -1 < x < 6\}$$

In practice, we do much of the work mentally and only need to show a sketch as follows :



(It is a straight line)
This shows that the solution set is

$$\{x \mid -1 < x < 6\}$$

CAUTION : Sometimes students attempt to solve a quadratic inequality as if it were a quadratic equation and write

$$(x + 1)(x - 6) < 0$$
$$\Leftrightarrow \; x + 1 < 0 \text{ and } x - 6 < 0$$
$$\Leftrightarrow \; x < -1 \text{ and } x < 6$$

The method is not correct and should never be used.

The above procedure may be used to solve any quadratic inequality and also any other inequality in which one side is zero and other can be written as product and /or quotient of linear factors.

Procedure to Solve a Quadratic Inequality

Step 1 Write the inequality in such a way that one side of the inequality is zero.

Step 2 Write the quadratic expression as product of linear factors.

Step 3 Replace the inequality sing by the equality sign and solve the resulting equation to obtain critical numbers.

Step 4 Plot the critical numbers on the number line and name the different region as I, II, III and so, on.

Step 5 Take a value from each region and check if the inequality is satisfied for that value. If it does include the region into the solution set otherwise reject the region.

Step 6 Check each of the critical to see if it is a part of the solution.

Step 7 Write the solution set.

Example 25 : Solve the following inequalities.

(i) $15x^2 + 4x - 4 \geq 0$ (ii) $x^2 + 9x \leq -18$ (iii) $x(x+2) < -1$
(iv) $20x - 25 - 4x^2 \leq 0$ (v) $x^2 + x + 1 > 0$ (vi) $3x^2 - 8x < 3$

Solution :

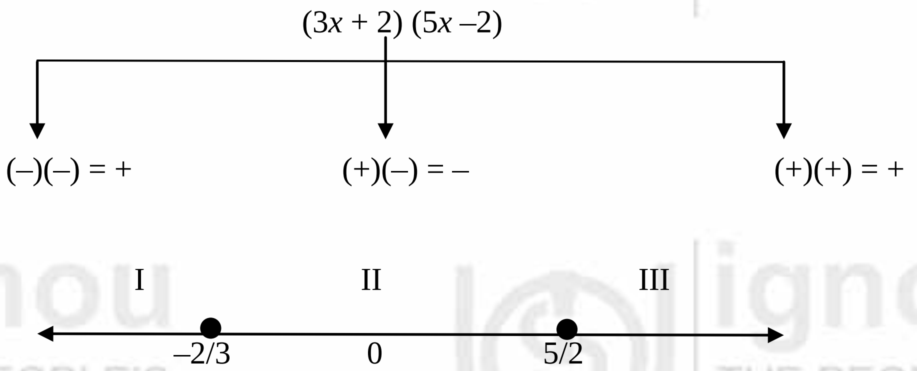
As one side of the inequality is already zero, we go to the second step and resolve the LHS into quadratic factors :

$$15x^2 + 4x - 4 = 15x^2 + 10x - 6x - 4 = 5x(3x+2) - 2(3x+2)$$

$$= (3x+2)(5x-2)$$

In this case, the critical numbers can be obtained by solving $3x + 2 = 0$ and $5x - 2 = 0 \Rightarrow x = -2/3, 2/5$.

We now work as explained in the illustration.



Thus, region I and region III are included in the solution set.

For $x = -2/3$, $3x + 2 = 0$

$\Rightarrow (3x+2)(5x-2) = 0 \leq 0$

$\therefore$ $-2/3$ is included in the solution set.

Similarly, $x = 5/2$ is included in the solution set.

Hence, the solution set of the inequality is

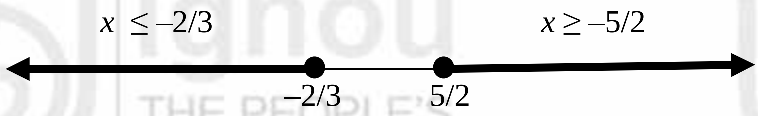
$\{x | x \leq -2/3\} \cup \{x | x \geq 5/2\} \cup \{-2/3, 5/2\}$

If we combine $x = -2/3$ with $x < -2/3$, we obtain $x \leq -2/3$ and combine $x = 5/2$ with $x > 5/2$ to obtain $x \geq 5/2$.

Thus, the required solution set is

$\{x | x \leq -2/3\} \cup \{x | x \geq 5/2\}$

We may represent it on the number line as follows :

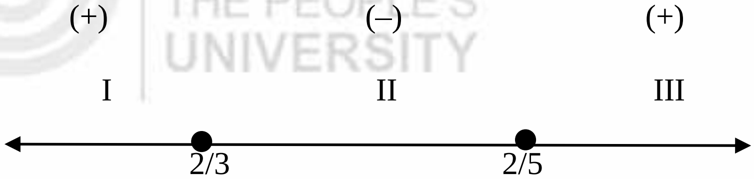


Simplification (In case of Distinct Linear Factors): If one side of the inequality is product of distinct linear factors and the other side is zero, then it is sufficient to know the sign the of the inequality in just one region. The signs in the remaining regions can be obtained by the following rule :

"Two adjacent regions must have opposite signs."

For instance, if we put $x = 0$ in $(3x+2)\ (5x–2)$, we obtain –4 which is negative, Therefore, sign of $(3x+2)\ (5x–2)$ is negative in region II.

By the above rule, the signs in region I and region III must be positive.



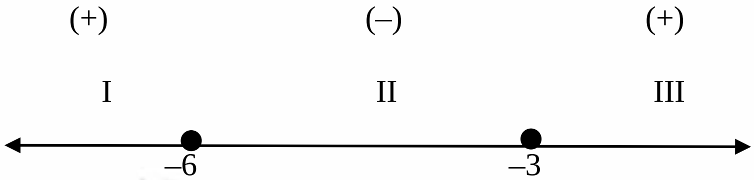
This is the same thing as we obtained earlier.

(ii) We rewrite the inequality as

$x^2 + 9x + 18 \le 0$ or $(x + 6)\ (x + 3) \le 0$

The critical numbers are –6 and –3. As distinct linear factors are involved we use the above simplified procedure.

By taking $x = 0$, we find $(x + 6)(x + 3) = 18 > 0$ and hence has a positive sign in region III. Thus, sign in region II must be negative and hence, in region I it must be positive.



Thus region II is a part of the solution.

Also, $x = –6$ and $x = –3$ satisfy the given inequality.

Therefore, solution of the inequality is

$\{x | -6 < x < -3\} \cup \{-6, -3\} = \{x | -6 \le x < -3\}$

(iii) We rewrite the inequality as $x^2 + 2x + 1$, 0 or $(x + 1)^2 < 0$.

But there is no real x for which $(x + 1)^2 < 0$

Hence, the given inequality has no solution.

(iv) We multiply the inequality by a negative sign to obtain the inequality

$4x^2 - 20x + 25 \geq 0$ or $(2x - 5)^2 \geq 0$.

This inequality is satisfied for all real x. Therefore, the solution set of the inequality is **R** is the set of real numbers.

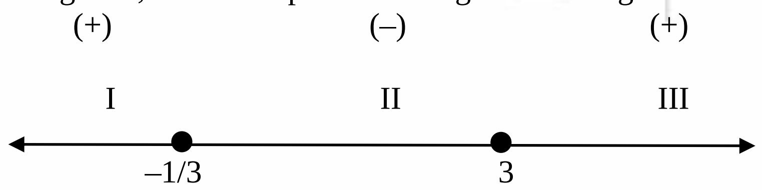
(v) We complete the square using the quadratic and linear factors :

$$x^2 + x + 1 = x^2 + 2\left(\frac{1}{2}\right)x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

As $(x + ½)^2 > 0$ for each real x, we get $x^2 + x + 1 > 0$ for each real x. Hence, solution set of the inequality is **R**

(vi) We rewrite the inequality as $3x^2 - 8x - 3 < 0$ or $(3x + 1)(x - 3) < 0$. The critical numbers are $x = -1/3$ and $x = 3$.

As distinct linear factors are involved, we require the sign of inequality in just one region. We can easily obtain this sign in region II by taking $x = 0$. As the sign in region II is negative, it must be positive in region I and region III.



Also $x = -1/3$ and $x = 3$ do not satisfy the given inequality. Hence, the solution set of this inequality is

$$\{x \mid -1/3 < x < 3\}.$$

Using Absolute Value to Solve Quadratic Inequalities

Sometimes, it is useful to convert a quadratic inequality into an inequality involving absolute value. The following are the two rules :

1. For $a > 0$

$$x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow (x < -a \text{ or } x > a)$$

2. For $a > 0$,

$$x^2 > a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$$

The above rules hold if we replace '>' by '≥' and '<' by '≤'

CAUTION $x^2 > a^2$ *does not imply* $x > \pm a$

and $x^2 < a^2$ does not imply $x < \pm a$.

Example 26 : Solve the following inequalities.

(i) $9 - x^2 \geq 0$ (ii) $4 - x^2 < 0$

(iii) $x^2 - 4x - 21 \leq 0$ (iv) $(x + 6)^2 > 49$

Solution

(i) $9 - x^2 \geq 0 \Leftrightarrow 9 \geq x^2 \Leftrightarrow x^2 \leq 9 \Leftrightarrow |x| \leq 3 \Leftrightarrow -3 \leq x \leq 3.$

(ii) $4 - x^2 < 0 \Leftrightarrow x^2 > 4 \Leftrightarrow |x| > 2 \Leftrightarrow x < -2$ *or* $x > 2$

(iii)We first complete the squares using first two terms :

$x^2 - 4x - 21 = x^2 - 2(2)x + 2^2 - 21 - 2^2 = (x-2)^2 - 25.$

Now, $x^2 - 4x - 21 \leq 0 \Leftrightarrow |x-2| \leq 5 \Leftrightarrow -5 \leq x - 2 \leq 5 \Leftrightarrow -3 \leq x \leq 7.$

(iv) $(x+6)^2 > 49 \Leftrightarrow (x+6)^2 > 7^2$

$\Leftrightarrow |x+6| > 7 \Leftrightarrow x + 6 < -7$ *or* $x + 6 > 7$

$\Leftrightarrow x < -13$ or $x > 1$

Algebraic Method to Solve Quadratic Inequality

We explain the technique in the following example :

Example 27 : Solve the following quadratic inequalities :

(i) $x^2 + x - 12 \leq 0$ (ii) $6x^2 + 7x - 3 > 0$

Solution

(i) We first factorise the LHS

$$x^2 + x - 12 = x^2 + 4x - 3x - 12 = x(x+4) - 3(x+4)$$
$$= (x+4)(x-3)$$

Thus, the give inequality becomes $(x + 4)(x - 3) \leq 0$.

But $ab \leq 0 \Leftrightarrow (a \leq 0, b \geq 0)$ or $(a \geq 0, b \leq 0)$

Therefore, $(x + 4)(x - 3) \leq 0$

$\Leftrightarrow$ $x + 4 \leq 0$ and $x - 3 \geq 0$ *or* $x + 4 \geq 0$ *and* $x - 3 \leq 0$

$\Leftrightarrow$ $x \leq -4$ and $x \geq 3$ *or* $x \geq -4$ and $x \leq 3$

But $x \leq -4$ and $x \geq 3$ is not possible.

Also, $x \leq -4$ and $x \leq 3 \Leftrightarrow -4 \leq x \leq 3$.

Hence, solution set of the given inequality is $\{x \mid -4 \leq x \leq 3\}$.

We first factorise the LHS

$6x^2 + 7x - 3 = 6x^2 + 9x - 2x - 3 = 3x(2x-3) - (2x+3)$

$= (2x+3)(3x-1)$

Thus, the given inequality becomes $(2x + 3)(3x - 1) > 0$

But $ab > 0 \Leftrightarrow (a > 0, b > 0)$ or $(a < 0, b < 0)$

Therefore, $(2x + 3)(3x - 1) > 0$

$\Leftrightarrow$ $2x + 3 < 0$ and $3x - 1 < 0$
or $2x + 3 > 0$ and $3x - 1 > 0$
$\Leftrightarrow$ $x < -3/2$ and $x < 1/3$

or $x > -3/2$ and $x > 1/3$

Now, $x < -3/2$ and $x < 1/3$ can hold simultaneously if $x < -3/2$ whereas $x > -3/2$ and $x > 1/3$ can hold simultaneously if $x > 1/3$.

Thus, the solution set of the inequality is $\{x | x < -3/2 \text{ or } > 1/3\}$.

Check Your Progress 4

Solve the following inequalities :

1. $x^2 + 5x \le -6$
2. $6x - x^2 - 8 \ge 0$
3. $x(x + 4) < -3$
4. $x(x + 2) \ge 0$
5. $x^2 - x + 1 > 0$
6. $2x^2 - 5x + 2 > 0$

Solve the following inequalities by algebraic method.

7 $x^2 - 5x + 6 \ge 0$

8 $3x^2 - 10x + 3 < 0$

4.8 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. $2x + 1 > -3 \quad \Leftrightarrow 2x > -1-3 \Leftrightarrow 2x > -4 \quad \Leftrightarrow x > -2$

$\therefore$ Solution set of the inequality is $\{x | x > -2\} = (-2, \infty)$

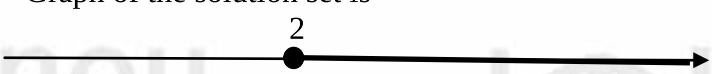
Graph of the solution set is

2. $3x - 2 \ge 4 \Leftrightarrow 3x \ge 2 + 4 = 6 \Leftrightarrow 3x \ge 6 \quad \Leftrightarrow x \ge 2$

$\therefore$ Solution set of the inequality is

$\{x | x \ge 2\} = [2, \infty)$

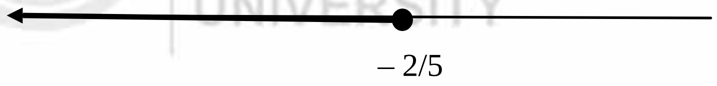
Graph of the solution set is



3. $2 - 5x \le 4 \iff -5x \ge 4 - 2 \iff -5x \ge 2 \iff x \le -2/5\}$

$\therefore$ Solution set is $\{x \mid x \le -2/5\} = (-\infty, -\mathbf{2/5}]$

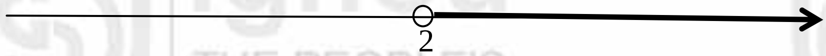
Graph of the solution set is



4. $3 + 2x < 7 \iff 2x < 7 - 3 \iff 2x < 4 \iff x < 2$

$\therefore$ Solution set is $\{x \mid x < 2\} = (2, -\infty)$

Graph of the solution set is



5. $3x + 2 \le 2x + 5 \iff 3x - 2x \le 5 - 2 \iff x \le 3$

Thus, the solution set is $\{x \mid x \le 3\} = [3, \infty)$

6. $3(x - 2) \ge 5 - 2x \iff 3x - 6 \ge 5 - 2x$

$\iff 5x \ge 11 \iff x \ge 11/6$

Thus, the solution set is $\{x \mid x \ge 11/6\} = [11/6, \infty)$

7. $-(x - 4) < 2x + 4 \iff -x + 4 < 2x + 4 \iff -3x < 0$

$\iff \boldsymbol{x > 0}$

Thus, the solution set is $\{x \mid x > 0) = (0, \infty)$

8. Multiply both the side by 15 to obtain

$3(x - 3) \ge 35(3 - x) \iff 3x - 9 \ge 105 - 35x$

$\iff 38x \ge 114 \iff x \ge 114/38 = 3$

Thus, the solution set is $\{x \mid x \ge 3\} = [3, \infty)$

9. $-2 \le 5 - 4x \le 7 \iff -2 - 5 \le -4x \le -5 + 7$

$\iff -7 \le -4x \le 2 \iff -2 \le 4x \le 7$

$\iff -2/4 \le x \le 7/4 \iff -1/2 \le x \le 7/4$

Thus, the solution set is $\{x \mid -1/2 \le x \le 7/4\} = [-½, 7/4]$

10. Multiply both the sides of the inequality by 4 to obtain

$-12 \leq x - 2 \leq 16 \Leftrightarrow -12 + 2 \leq x \leq 16 + 2$

$\Leftrightarrow -10 \leq x \leq 18$

Thus, the solution set is $\{ x \mid -10 \leq x \leq 18 = [-10, 18]$

11. $\frac{x+3}{x-1} \leq 0 \Leftrightarrow x + 3$ and $x - 1$ must be of opposite signs.

Case 1 $x + 3 < 0$ and $x - 1 > 0$

$\Leftrightarrow x < -3$ and $x > 1$

No such x exists.

Case 2 $x + 3 > 0$ and $x - 1 < 0$

$\Leftrightarrow x > -3$ and $x < 1$

Thus, $-3 < x < 1$

$\therefore$ Solution set of the inequality is $(-3, 1)$

12. $\frac{x+8}{x+1} > 1 \Leftrightarrow \frac{x+8}{x+1} - 1 > 0$

$\Leftrightarrow \frac{x+8-x-1}{x+1} > 0 \Leftrightarrow \frac{7}{x+1} > 0.$

Now, $\frac{7}{x+1} > 0, \quad 7 > 0 \Leftrightarrow x + 1 > 0, \quad \Leftrightarrow x > -1$

Thus, solution set of the inequality is $(-1, \infty)$

Check Your Progress 2

1. $|x-3| \geq 2 \Leftrightarrow x - 3 \leq -2$ or $x - 3 \geq 2$

$\Leftrightarrow x \leq 1$ *or* $x \geq 5$

Thus, Solution set is $(-\infty, 1] \cup [5, \infty)$

2. $\left|\frac{2x-3}{3}\right| \leq \Leftrightarrow -1 \leq \frac{1}{3}(2x-5) \leq 1$

$\Leftrightarrow \quad -3 \leq 2x - 5 \leq 3 \qquad \Leftrightarrow \quad 2 \leq 2x \leq 8$

$\Leftrightarrow 1 \leq x \leq 8/3$

Thus, Solution set is $[1, 8/3]$

3. $\left|\frac{x-5}{3}\right| < 6 \quad \Leftrightarrow \ -6 < \frac{x-5}{3} < 6$

$\Leftrightarrow -18 < x - 5 < 18 \quad \Leftrightarrow \quad -13 < x < 23$

$\therefore$ Solution set is $(-13, 23)$

4. $\frac{5}{|x-3|} < 7 \ \Leftrightarrow \ 5 < 7|x-3|, x \neq 3$

$\Leftrightarrow \ |x-3| > \frac{5}{7}, x \neq 3$

$\Leftrightarrow \ x-3 < -\frac{5}{7}$ or $x-3 > \frac{5}{7}, x \neq 3$

$\Leftrightarrow \ x < \frac{16}{7}$ or $x > 26/7, x \neq 3$

Thus, Solution set is $(-\infty, 16/7) \cup (\frac{26}{7}, \infty) - \{3\}$
$= (-\infty, 16/7) \cup (\frac{26}{7}, \infty)$.

5. We wish to find x so that $200 < T < 300$, that is,

$200 < 30 + 25(x - 3) < 300$

$\Leftrightarrow \quad 200 - 30 < 25(x - 3) < 300 - 30 \Leftrightarrow \quad 170 < 25(x - 3) < 270$

$\Leftrightarrow \quad 170/25 < x - 3 < 270/50 \Leftrightarrow \ 6.8 + 3 < x < 10.8 + 3$

$\Leftrightarrow \quad 9.8 < x < 13.8$

Thus, the required depth lies between 9.8 and 13.8 km.

6. We are given CA = 12 and $80 \leq IQ \leq 140$. Thus,

$80 \leq \frac{MA}{12}(100) \leq 140 \ \Leftrightarrow \ \frac{80 \times 12}{100} \leq MA \leq \frac{140 \times 12}{100}$

$\Leftrightarrow \ 9.6 \leq MA \leq 16.8$

7. Profit $=$ Revenue $-$ Cost
$\quad = 300x - (3000 + 200x)$
$\quad = 100x - 3000$

In order to have a profit, we must have

$$\begin{aligned} & \text{Profit} > 0 \\ \Leftrightarrow\ & 100\,x - 3000 > 0 \\ \Leftrightarrow\ & 100\,x > 3000 \\ \Leftrightarrow\ & x > 30 \end{aligned}$$

As x is a natural number, $x \geq 31$

Thus, Josh mobiles must manufacture at least 31 mobile sets.

8. Let m = amount of medication (in CC) given to the patient,

   We must have

   $| m - 3 | < 0.005$

   $\Leftrightarrow -0.005 < m - 3 < 0.005$

   $\Leftrightarrow\ 2.995 < m < 3.005$

Check Your Progress 3

1. The double – intercept form of the boundary is $\frac{x}{-3} + \frac{y}{3} = 1$. Test point (0,0) satisfies the inequality. The graph is given in the answers.

2. The double – intercept form of the boundary is $\frac{x}{3} + \frac{y}{2} = 1$. Test point (0,0) satisfies the inequality. The graph is given in the answers.

3. The boundary $y = 2x$ represents the line through the origin and (1,2). Note that (0,2) does not satisfy the inequality $y < 2x$. So, shade the half plane not containing (0,2).

4. $x = -2$ is a vertical line through $(-2, 0)$.

5. $y = 2$ is a horizontal line through (0,2).

6. Since $x \geq 0$ and $y \geq 0$ we restrict ourselves to the first quadrant only. We draw the boundaries.

   $2x + y = 10 \left(i.e., \frac{x}{5} + \frac{y}{10} = 1\right)$, $x + 4y = 12 \left(i.e., \frac{x}{12} + \frac{y}{3} = 1\right)$,

   and $6x + y = 18 \left(i.e., \frac{x}{3} + \frac{y}{18} = 1\right)$

   Next, we indicate the solution set of each of the inequalities by drawing the arrows, and then finally shade the common region (i.e., the solution set).

7. We once again restrict ourselves to the first quadrant. We draw the boundaries.

   $2x + 3y = 6 \left(i.e., \frac{x}{3} + \frac{y}{2} = 1\right)$ and $x + 4y = 4 \left(i.e., \frac{x}{4} + \frac{y}{1} = 1\right)$.

   We indicate the solution set of each of the inequalities by arrows and shade the common region, i.e., the solution set.

Check Your Progress 4

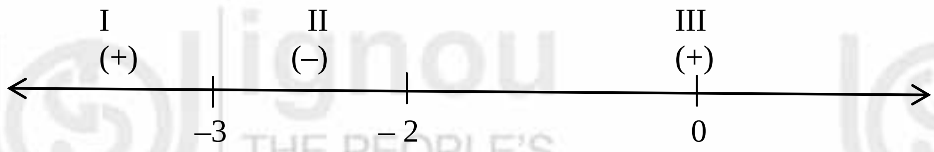
1 $x^2 + 5x \le -6$ $\Leftrightarrow x^2 + 5x + 6 \le 0$

$\Leftrightarrow x^2 + 2x + 3x + 6 \le 0$ $\Leftrightarrow x\ (x+2) + 3\ (x+2) \le 0$

$\Leftrightarrow\ (x+3)(x+2) \le 0$

The critical numbers are –3 and – 2

As distinct linear factors are involved, we required sign of the inequality in one of the three regions. For $x = 0$, $x^2 + 5x + 6 = 6 > 0$. Signs in different regions are shown below.



Thus, solution set is [– 3, –2]

2. $x^2 - 6x + 8 \le 0$

$\Leftrightarrow x^2 - 6x + 8 \le 0$ $\Leftrightarrow x^2 - 2x - 4x + 8 \le 0$

$\Leftrightarrow x\,(x-2) - 4\,(x-2) \le 0$ $\Leftrightarrow (x-2)(x-4) \le 0$

As explained in Question 1, $2 \le x \le 4$.

See the figure below.

Thus, solution set is [2,4]

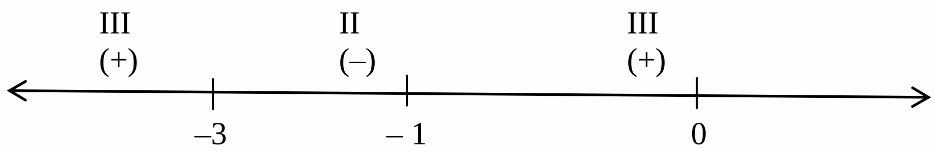
3. $x\,(x+4) < -2$ $\Leftrightarrow x^2$

$\Leftrightarrow x^2 + 3x + x + 3 < 0$ $\Leftrightarrow x(x+3) + (x+3) < 0$

$\Leftrightarrow (x+3)\,(x+1) < 0$

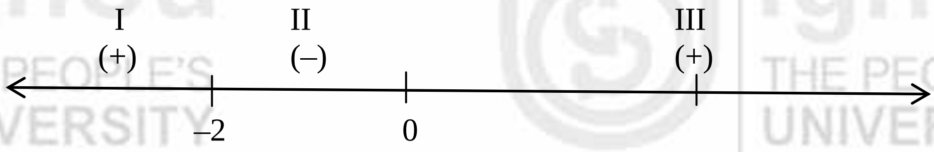
Critical numbers are – 3 and – 1.

Sign of the expression $x^2 - 4x + 3$ in different regions in given below.



Required solution set is (–3, –1)

4. Critical numbers are –2 and 0 sign of $x(x+2)$ in different regions is shown below.



Required solution set is $(-\infty, -2] \cup [0, \infty)$

5. $x^2 - x + 1 = x^2 - 2\left(\frac{1}{2}\right)x + \frac{1}{4} + 1 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} 0$

For each real x.

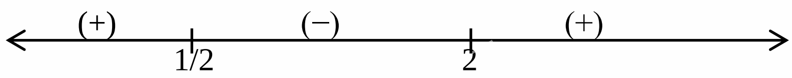
Thus, solution set is $\boldsymbol{R}$

6. $2x^2 - 5x + 2 = 2x^2 - 4x - x + 2$

$= 2x(x-2) - (x-2)$

$= (2x-1)(x-2)$

Critical numbers are ½ and 2.



$2x^2 - 5x + 2 > 0 \quad \Leftrightarrow \quad x < ½$ or $x > 2$

$\therefore$ Solution set is $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$

7. $x^2 - 5x + 6 = x^2 - 2x - 3x + 6 = x(x-2) - 3(x-2) = (x-2)(x-3)$

Now, $x^2 - 5x + 6 \geq 0$

$\Leftrightarrow (x-2)(x-3) \geq 0$

$\Leftrightarrow x - 2 \geq 0, \ x - 3 \geq 0$ or

$x - 2 \leq 0, x - 3 \leq 0$

$\Leftrightarrow (x \geq 2, x \geq 3)$ or$(x \leq 2, x \leq 3)$

$\Leftrightarrow x \geq 3 \text{ or } x \leq 2.$

Thus, solution set is $(-\infty, 2) \cup (3, \infty)$

8. $3x^2 - 10x + 3 = 3x^2 - 9x - x + 3$

$= 3x(x-3) - (x-3)$

$= (3x-1)(x-3)$

Now, $3x^2 - 10x + 3 < 0$

$\Leftrightarrow (3x - 10)(x - 3) < 0$

$\Leftrightarrow (3x - 1 < 0, x - 3 > 0) \text{ or } (3x - 1 > 0, x - 3 < 0)$

$\Leftrightarrow$ ($x < 1/3$, $x > 3$) or ($x > 1/3$, $x > 3$)

$$\Leftrightarrow \frac{1}{3} < x < 3$$

Thus, solution set is (1/3, 3).

4.9 SUMMARY

In this unit, to begin with, in **section 4.2**, a number of tools for solving inequalities are given. Then, a number of examples are given to illustrate how to use these tools in solving inequalities in one variable and also how to draw graph of the solution, on the number line. In **section 4.3**, solutions of inequalities involving absolute values are discussed. In **sections 4.4, 4.5 and 4.6**, first the concept of linear inequalities in two variables, is discussed and then method of solving these inequalities graphically, are explained with suitable examples. In **section 4.7**, some methods of solving quadratic and other non-linear inequalities, are first discussed and then illustrated with suitable examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 4.8**.

UNIT 1 DIFFERENTIAL CALCULUS

Structure

1.0 INTRODUCTION

In this Unit, we shall define the concept of limit, continuity and differentiability.

1.1 OBJECTIVES

After studying this unit, you should be able to :

- define limit of a function;
- define continuity of a function; and
- define derivative of a function.

1.2 LIMITS AND CONTINUITY

We start by defining a function. Let A and B be two non empty sets. A function f from the set A to the set B is a rule that assigns to each element x of A a unique element y of B.

We call y the image of x under f and denote it by $f(x)$. The domain of f is the set A, and the co−domain of f is the set B. The range of f consists of all images of elements in A. We shall only work with functions whose domains and co–domains are subsets of real numbers.

Given functions f and g, their sum $f + g$, difference $f - g$, product $f.\,g$ and quotient f / g are defined by

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$

$$(f\,.\,g)(x) = f(x)\,g(x)$$
and $$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

For the functions $f + g$, $f - g$, $f.\,g$, the domain is defined to be intersections of the domains of f and g, and for f / g the domain is the intersection excluding the points where $g(x) = 0$.

The composition of the function f with function g, denoted by fog, is defined by $(fog)(x) = f(g(x))$.

The domain of fog is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Limit of a Function

We now discuss intuitively what we mean by the limit of a function. Suppose a function f is defined on an open interval (α, β) except possibly at the point $a \in (\alpha, \beta)$ we say that

$$f(x) \to L \text{ as } x \to a$$

(read $f(x)$ approaches L as x approaches a), if $f(x)$ takes values very, very close to L, as x takes values very, very close to a, and if the difference between $f(x)$ and L can be made as small as we wish by taking x sufficiently close to but different from a.

As a mathematical short hand for $f(x) \to L$ as $x \to a$, we write

$$\lim_{x \to a} f(x) = L.$$

Example 1 : Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

Solution : Let $f(x) = \dfrac{x^2 - 9}{x - 3}$. This function is defined for each x except for $x = 3$. This function is defined for each x except for $x = 3$. Let us calculate the value of f at $x = 3 + h$, where $h \neq 0$. We have

$$f\,(3 + h) = \frac{(3 + h)^2 - 9}{3 + h - 3} = \frac{9 + 6h + h^2 - 9}{h} = \frac{h(6 + h)}{h} = 6 + h$$

We now note that as x takes values which are very close to 3, that is, h takes values very close to 0, $f(3 + h)$ takes values which are very close to 6. Also, the difference between $f\,(3 + h)$ and 6 (which is equal to h) can be made as small as we wish by taking h sufficiently close to zero.

Thus,

$$\lim_{x \to 3} f(x) = 6$$

Properties of Limits

We now state some properties of limit (without proof) and use them to evaluate limits.

Theorem 1 : Let a be a real number and let $f(x) = g(x)$ for all $x \neq a$ in an open interval containing a. If the limit $g(x)$ as $x \to a$ exists, then the limit of $f(x)$ also exists, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

Theorem 2 : If c and x are two real numbers and n is a positive integer, then the following properties are true :

(1) $\lim\limits_{x\to a} c = c$

(2) $\lim\limits_{x\to a} x = a$

(3) $\lim\limits_{x\to a} x^n = a^n$

Theorem 3 : Let c and a be two real numbers, n a positive integer, and let f and g be two functions whose limit exist as $x \to a$. Then the following results hold :

1. $\lim\limits_{x\to a} [c\, f(x)] = c\left[\lim\limits_{x\to a} f(x)\right]$

2 $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

3. $\lim\limits_{x\to a}[f(x)g(x)] = [\lim\limits_{x\to a} f(x)]\,[\lim\limits_{x\to a} g(x)]$

4. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, provided $\lim\limits_{x\to a} g(x) \neq 0,$

5. $\lim\limits_{x\to a} [f(x)]^n = [\lim\limits_{x\to a} f(x)]^n$

6. If $\lim\limits_{x\to a} f(x) = f(a)$, then $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}$

Example 2: Evaluate $\lim\limits_{x\to 3} (4x^2 + 7)$

Solution : $\lim\limits_{x\to 3} (4x^2 + 7) = \lim\limits_{x\to 3} 4x^2 + \lim\limits_{x\to 3} 7$

$= 4\lim\limits_{x\to 3} x^2 + \lim\limits_{x\to 2} 7$

$= 4(3)^2 + 7 = 4 \times 9 + 7$

$= 43$

Note : If $p(x)$ is a polynomial, then $\lim\limits_{x\to a} p(x) = p(a)$.

If $q(x)$ is also a polynomial and $q(a) \neq 0$, then

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

Example 3: Evaluate the following limits :

(i) $\lim_{x\to 2} [(x- 1)2 + 6]$ (ii) $\lim_{x\to 0} \frac{ax+b}{cx+d}$ $(d \neq 0)$

(iii) $\lim_{x\to 2} \frac{x^2+5x+7}{x^2+8}$ (iv) $\lim_{x\to -1} \sqrt{x+17}$

Solution: (i) $\lim_{x\to 2}$ $[(x-1)^2 + 6] = (2-1)^2 + 6 = 1+ 6 = 7$

(ii) Since $\lim_{x\to 0} cx + d = d \neq 0$,

$$\lim_{x\to 0} \frac{ax+b}{cx+d} = \frac{a(0)+b}{c(0)+d} = \frac{b}{d}$$

(iii) Since $\lim_{x\to 3}$ $(x^2 +8) = 3^2+ 8 = 17 \neq 0$,

$$\therefore \lim_{x\to 3} \frac{x^2 + 5x+7}{x^2 + 8} = \frac{3^2 + 5(3)+7}{3^2 + 8} = \frac{31}{17}$$

(iv) Since $\lim_{x\to -1}$ $x + 17 = -1 + 17 = 16$, we have

$$\lim_{x\to -1} \sqrt{x+17} = \sqrt{16} = 4$$

Example 4: Evaluate the following limits.

(i) $\lim_{x\to 5} \frac{x^2-7x+10}{x-5}$ (ii) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(iii) $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$ (iv) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution : (i) Here, $\lim_{x\to 5} (x-5) = 0$. So direct substitution will not work.

We can proceed by cancelling the common factor $(x-5)$ in numerator and denominator and using theorem 1, as shown below :

$$\lim_{x\to 5} \frac{x^2-7x+10}{x-5} = \lim_{x\to 5} \frac{(x-2)(x-5)}{(x-5)}$$

$$= \lim_{x\to 5} (x-2), \quad \text{for } x \neq 5$$

$$= 5 - 2 = 3$$

(ii) Since $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ for $x \neq 1$,

therefore by theorem 1, we have

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 1+1 = 2.$$

(iii) Once again we see that direct substitution fails because itsleads to indeterminate form $\frac{0}{0}$. In this case, rationalising the numerator helps as follows. For $x \neq 0$,

$$\frac{\sqrt{x+2}-\sqrt{2}}{x} = \left(\frac{\sqrt{x+2}-\sqrt{2}}{x}\right)\left(\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right)$$

$$= \left(\frac{x+2-2}{\sqrt{x+2}+\sqrt{2}}\right) = \frac{x}{x(\sqrt{x+2}+\sqrt{2})}$$

$$= \frac{1}{\sqrt{x+2}+\sqrt{2}}$$

Therefore, by Theorem 1, we have

$$\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \lim_{x\to 0}\frac{1}{\sqrt{0+2}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(iv) For $x \neq 0$, we have

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \left(\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\right)\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \frac{2x}{x\ \sqrt{1+x}-\sqrt{1-x}} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$\therefore$ by theorem 1, we have

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{2} = 1$$

An important limit

Example 5 : Prove that $\lim\limits_{x\to 0}\dfrac{x^n - a^n}{x-a} = na^{n-1}$ where n is positive integer

Solution : We know that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ xa^{n-2} + a^{n-1})$$

Therefore, for $x \neq a$, we get

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ xa^{n-2} + a^{n-1}$$

Hence by Theorem 1, we get

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ xa^{n-2} + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ aa^{n-2} + a^{n-1}$$

$$= n\,a^{n-1}$$

Note : The above limit is valid for negative integer n, and in general for any rational index n provided $a > 0$. The above formula can be directly used to evaluate limits.

Example 6 : Evaluate $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x^2 - 9}$

Solution : $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x^2 - 9} = \lim\limits_{x \to 3} \dfrac{x^3 - 3^3}{x^2 - 3^2}$

$$= \lim_{x \to 3} \frac{\dfrac{x^3 - 3^3}{x - 3}}{\dfrac{x^2 - 3^2}{x - 3}}$$

$$= \frac{3.3^{3-1}}{2.3^{2-1}} \quad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \right)$$

$$= \frac{27}{6} = \frac{9}{2}$$

One–sided Limits

Definition : Let f be a function defiend on an open interval $(a - h, a + h)$ $(h > 0)$. A number L is said to be the **Left Hand Limit (L.H.L.)** of f at a if $f(x)$ takes values very close to L as x takes values very close to a on the left of a $(x \neq a)$. We then write

$$\lim_{x \to a} f(x) = L$$

We similarly define L to be the **Right Hand Limit** if $f(x)$ takes values close to L as x takes values close to a on the right of a and write $\lim\limits_{x \to a+} f(x) = L$

Note that $\lim\limits_{x \to a} f(x)$ exists and is equal to L if and only if $\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$ both exist and are equal to L.

$$\lim_{x \to a-} f(x) = L = \lim_{x \to a+} f(x) = \lim_{x \to a} f(x)$$

Example 7 : Show that $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution : Let $f(x) = \frac{|x|}{x}$, $x \neq 0$.

$$\text{Since } |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$\therefore f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

So, $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0} (1) = 1$ and

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0} (-1) = -1$$

Thus $\lim\limits_{x \to 0} f(x)$ does not exist.

Continuous Functions

Definition : A function f is said to be **continuous** at $x = a$ if the following three conditions are met :

(1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ *exists*

(3) $\lim_{x \to a} f(x) = f(a)$

Example 8: Show that $f(x) = |x|$ is continuous at $x = 0$

Solution: Recall that

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

To show that f is continuous at $x = 0$, it is sufficient to show that

$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} (x) = f(0)$ and

We have

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0+} f(0 - h) = \lim_{h \to 0+} f(-h)$$

$$= \lim_{h \to 0+} -(-h)$$

$$= \lim_{h \to 0+} h = 0$$

and $\lim_{x \to 0+} f(x) = \lim_{h \to 0+} f(0 + h) = \lim_{h \to 0+} f(h)$

$$= \lim_{h \to 0+} (h) = 0.$$

Thus, $\lim_{h \to 0-} f(x) = \lim_{h \to 0+} f(x) = 0$

Also, $f(0) = 0$

Therefore, $\lim_{x \to 0+} f(x) = 0 = f(0)$

Hence, f is continuous at $x = 0$.

Example 9: Check the continuity of f at the indicated point

$$(i) \quad f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{at} \quad x = 0$$

$$(ii)\quad f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 2 & x = 1 \end{cases} \qquad \text{at} \quad x = 1$$

Solution : (i) We have already seen in Example 7 that $\lim\limits_{x\to 0}\dfrac{|x|}{x}$ does not exist.

Hence, f is not continous at $x = 0$

(*ii*) Here, $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$

$$= \lim_{x\to 1}(x+1)\measuredangle$$

$$= 2$$

Also, $f(1) = 2$

$$\therefore \lim_{x\to 1} f(x) = f(1)$$

Hence, f is continuous at $x = 1$.

Definition : A function is said to be **continuous on an open interval** (a,b) if it is continuous at each point of the interval. A function which is continuous on the entire real line $(-\infty,\infty)$ is said to be **everywhere continuous.**

Algebra of Continuous Functions

Theorum : Let c be a real number and let f and g be continuous at $x = a$. Then the functions cf, $f+g$, $f-g$, fg are also continuous at $x = a$. The functions $\frac{1}{g}$ and $\frac{f}{g}$ are continuous provided $g(a) \neq 0$.

Remark: It must be noted that polynomial functions, rational functions, trigonometric functions, exponential and logarithmic function are continuous in their domains.

Example 10 : Find the points of discontinuity of the following functions :

(i) $f(x) = \begin{cases} x^2 & if\ x > 0 \\ x+3 & x \leq 0 \end{cases}$

(ii) $f(x) = \begin{cases} x & if\ x \neq 0 \\ 1 & x = 0 \end{cases}$

Solution : (i) Since x^2 and $x + 3$ are polynomial functions, and polynomial functions are continuous at each point in R, f is continuous at each x € R except possibly at $x = 0$. For $x = 0$, we have

$$\lim_{x\to 0-} f(x) = \lim_{h\to 0+} f(0-h) = \lim_{h\to 0+}(-h+3) = 0+3 = 3$$

$$\text{and} \lim_{x\to 0+} f(x) = \lim_{h\to 0+} f(0+h) = \lim_{h\to 0+} f(h) = \lim_{h\to 0+} h^2 = 0.$$

Therefore, since $\lim\limits_{x\to 0-} f(x) f \lim\limits_{x\to 0+} f(x), f$ is not continuous $at\ x = 0$

(ii) Since, polynomial functions are continuous at each point of R, f is also continuous at each x € R except possibly at x = 0. At this point, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0 \neq f(0).$$

Thus, f is not continuous at x = 0

Check Your Progress – 1

1. Evaluate the following limits:

(i) $\lim_{x\to 2} (3x^3 + 2x + 1)$ (ii) $\lim_{x\to 2} \frac{x-2}{x+2}$

(iii) $\lim_{x\to 2} \frac{x^2 - 5x + 2}{x - 1}$ (iv) $\lim_{x\to 2} \sqrt[3]{3x^2 - 19}$

2. Evaluate the following limits:

(i) $\lim_{x\to 2} \frac{x^2 - 4}{x + 2}$ (ii) $\lim_{x\to 5} \frac{\sqrt{x-1} - 2}{x - 5}$

3. Evaluate the following limits:

(i) $\lim_{x\to a} \frac{x^{7/6} - a^{7/6}}{x^{3/5} - a^{3/5}}$ $(a > 0)$

(ii) $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n}$ $(m, n \text{ are rational numbers}, a > 0)$

4. Check the continuity of f at the indicated point where

$f(x) = \begin{cases} 2 - x & if\ x < 0 \\ x + x & if\ x \geq 0 \end{cases}$ at x =0

5. For what value of constant k the function f is continuous at x = 5 ?

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & if\ x \neq 5 \\ k & if\ x = 5 \end{cases}$$

1.3 DERIVATIVE OF A FUNCTION

Definition: A function f is said to be **differentiable** at x if and only if

$$\lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists. If this limit exsits, it is called the derivative of f at x and is denoted by

$f^1(x)$ or $\frac{dy}{dx}$.

i.e., $\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f^1(x)$

A function is said to be **differentiable on an open interval I** if it is differentiable at each point of I.

Example 11: Differentiate $f(x) = x^2$ by using the definition.

Solution : We first find the difference quotient as follows :

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{x^2 + 2\,x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{\Delta x(2x+\Delta x)}{\Delta x}$$

$$= 2x + \Delta x$$

It follows that

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}(2x+\Delta x) = 2x$$

Remark: It can be easily proved that if f is differentiable at a point x, then f is continuous at x. Thus, if f is not continuous at x, then f is not differentiable at x.

Some differentiation Rules

We now develop several "rules" that allow us to calculate derivatives without the direct use of limit definition.

Theorem 1 (Constant Rule). The derivative of a constant is zero. That is,

$$\frac{d}{dx}[c] = 0$$

where c is a real number.

Proof : Let $f(x) = c$ then

$$\frac{d}{dx}[c] = f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{c-c}{\Delta x} = 0$$

Theorem 2 : (Scalar Multiple Rule). If f is differentiable function and c is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Proof : By definiton

$$\frac{d}{dx}[cf(x)] = \lim_{\Delta x\to 0}\frac{cf(x+\Delta x)-c\,f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} c\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] = cf'(x)$$

Theorem 3 : (Sum and Difference Rule). If f and g are two differentiable functions, then

Sum Rule $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Proof : We have

$$\frac{d}{dx}[f(x) + g(x)] = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) + g(x+\Delta x) - [f(x) + g(x)]}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

We can similarly prove the difference rule.

Theorum 4 : (Product Rule). If *f* and *g* are two differentiable functions, then

$$\frac{d}{dx}[f(x)\, g(x)] = f(x) + g'(x) + f'(x) + g(x)$$

Proof: We have $\frac{d}{dx}[f(x)\, g(x)] = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)\, g(x+\Delta x) - f(x)\, g(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)\, (g(x+\Delta x) - f(x+\Delta x)\, g(x) + f(x+\Delta x) g(x) - f(x) g(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)\, (g(x+\Delta x) - g(x))}{\Delta x} + g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x)}{\Delta x}\right]\left[\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + g(x)\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

(using the product and scalar multiple rules of limits). Now, since f is differentiable at *x*, it is also continuous at *x*.

$$\therefore \lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$$

Thus $\frac{d}{dx}[f(x)\, g(x)] = f(x) g'(x) + g(x)\, f'(x)$

Theorem 5 : (Power Rule) If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

For n =1, we have

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(x) = \lim_{\Delta x \to 0} f(x)\frac{x + \Delta x - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

$$= 1 = 1x^0 = nx^{n-1}.$$

If $n > 1$, then the binomial expansion produces

$$\frac{d}{dx}(x^n) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{{}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2}(\Delta x)^2 + \ldots\ldots + {}^nC_n\, (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots\ldots\ldots\ldots\ldots\ldots + (\Delta x)^{n-1} \right]$$

$$= nx^{n-1}.$$

Theorem 6: (Reciprocal Rule). If f is differentiable function such that $f(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-f'(x)}{[f(x)]^2}$$

Proof $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{1}{f(x + \Delta x)} - \frac{1}{f(x)}\right]$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x) - f(x + \Delta x)}{f(x + \Delta x) f(x)}\right]$$

$$= \lim_{\Delta x \to 0} \left[-\left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right)\right]\left[\left(\frac{1}{f(x + \Delta x) f(x)}\right)\right]$$

$= -f'(x).\frac{1}{f(x)f(x)}$ ($\because \lim_{\Delta x \to 0} (f(x + \Delta x) = f(x)$
as f being diff. at x is continuous at x)

$$= \frac{-f'(x)}{[f(x)]^2}$$

Theorem 7 : (Quotient Rule) : If f and g are two differentiable function such that $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Proof: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{d}{dx}\left[f(x)\dfrac{1}{g(x)}\right]$

$$= \frac{1}{g(x)}\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right] \text{ [Product Rule]}$$

$$= \frac{1}{g(x)} + f'(x) + f(x)\left[\frac{-g'(x)}{[g(x)]^2}\right]$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Remark : The power rule can be extended for any integer. Indeed, if $n = 0$, we have

$$\frac{d}{dx}(x^{n)=}\frac{d}{dx}(1) = 0 = 0x^{-1} \qquad x \neq 0,$$

and if n is a negative integer, then by using reciprocal rule we can prove

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Thus we have

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{for any integer } n.$$

Example 2 : Find the derivatives of the following function .

(i) $y = 2x^5 - 3x$ (ii) $y = \dfrac{1}{x^2 + 3}$

(iii) $y = \dfrac{x}{x+2}$ (iv) $y = \dfrac{x^2}{x^2 - 5}$

Solution : (i) $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x^5 - 3x)$

$$= 2\frac{d}{dx}(x^5) - 3\frac{d}{dx}(x)$$

$$= 2.(5x^4) - 3.1$$

$$= 10x^4 - 3$$

(ii) $\dfrac{dy}{dx} = \dfrac{-\dfrac{d}{dx}[x^2 + 3]}{[x^2 + 3]^2}$ [using reciprocal rule]

$$= \frac{-2x}{(x^2 + 3)^2}$$

(iii) $$\frac{dy}{dx} = \frac{(x+2)\frac{d}{dx}(x) - x\frac{d}{dx}(x+2)}{(x+2)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{(x+2).1 - x.1}{(x+2)^2}$$

$$= \frac{2}{(x+2)^2}$$

(iv) $$\frac{dy}{dx} = \frac{(x^2-5)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x^2-5)}{(x^2-5)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{(x^2-5)(2x\ ) - x^2(2x)}{(x^2-5)^2}$$

$$= \frac{2x^3 - 10x - 2x^3}{(x^2-5)^2} = \frac{-10x}{(x^2-5)^2}$$

Derivative of Exponential and Logarithmic Functions

To find the derivatives of the natural exponential function e^x and the natural logarithmic function lnx, we need the following limits.

(1) $$\lim_{\Delta x \to 0} \frac{e^x - 1}{x} = 1$$

(2) $$\lim_{\Delta x \to 0} \frac{\ln(1+x)}{x} = 1$$

Theorem 8 : The derivative of the natural exponential function is given by

$$\frac{d}{dx}(e^x) = e^x \qquad (x \epsilon R)$$

Proof : By definition

$$\frac{d}{dx}(e^x) = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x(1)$$

$$= e^x$$

Theorum 9 : The derivative of the natural logarithmic function is given by

$$\frac{d}{dx}(lnx) = \frac{1}{x} \quad (x > 0)$$

Proof : By definition

$$\frac{d}{dx}(lnx) = \lim_{\Delta x \to 0} \frac{ln(x+\Delta x) - lnx}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} ln\frac{(x+\Delta x)}{x} (\because lna - lnb = ln\frac{a}{b})$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} ln\left(1+\frac{\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x/x}$$

$$= \frac{1}{x}\lim_{\Delta x \to 0} \frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x/x} = \frac{1}{x}(1) = \frac{1}{x}$$

Corollary : If $a > 0$ and $a \neq 1$, then the derivative of the general logarithemic function is

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$$

Proof : We know that

$$\log_a x = (lnx)(\log_a e)$$

$$\Rightarrow \frac{d}{dx}(\log_a x) = \frac{d}{dx}[(logx)(\log_a e)]$$

$$= \log_a e \frac{d}{dx}(logx)$$

$$= \frac{1}{x}(\log_a e)$$

Remark : Similar to the proof of theorem, we can prove that if $a > 0$, and $a \neq 1$, then the derivative of the general exponential function is

$$\frac{d}{dx}(a^x) = a^x lna \quad (x \in R)$$

Example 13 : Find the derivative of the following functions.

(i) x^2e^x (ii) $\frac{lnx}{x}$

(iii) $\frac{e^x}{x^2+3}$ (iv) $5^x lnx$

Solution : (i) Using the product rule

$$\frac{d}{dx}\,(x^2e^x) = \frac{d}{dx}\,(x^2)e^x + x^2\frac{d}{dx}\,(e^x)$$

$$= 2x\,e^x + x^2e^x = (2x + \,x^2)\,e^x$$

(i) Using the quotient rule, we have

$$\frac{d}{dx}\,\frac{lnx}{x} = \frac{x\frac{d}{dx}(lnx) - \,lnx\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\,.\frac{1}{x} - \,(lnx)(1)}{x^2}$$

$$= \;\frac{1 - lnx}{x^2}$$

(ii) Using the quotient rule, we have

$$\frac{d}{dx}\left(\frac{e^x}{x^2+3}\right) = \frac{(x^2+3)\frac{d}{dx}\,(e^x) - \,e^x\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$= \frac{(x^2+3)\,(e^x) - \,e^x(2x)}{(x^2+3)^2}$$

$$= \frac{(x^2-2x+3)e^x}{(x^2+3)^2}$$

(iii) Using the product rule, we have

$$\frac{d}{dx}(5^x lnx) = \;\frac{d}{dx}(5^x)lnx + \;5^x\frac{d}{dx}(lnx)$$

$$= \;(5^x lnx5) + \;5^x\left(\frac{1}{x}\right)$$

$$= \;5^x(ln5)lnx + \;\left(\frac{5^x}{x}\right).$$

Check Your Progress – 2

1. Find the derivative of each of the following functions.

(i) $y \;=\; x^5 - \;3x^4 + \;2\,x - 1$ (ii) $y = \dfrac{2x-1}{\pi^2}$

(iii) $\dfrac{3x+5}{2x+7}$ (iv) $y \;=\; \dfrac{x^3-4}{x^3}$

2. Find the derivative of each of the following functions.

(i) $e^x \ln x$ (ii) $\dfrac{e^x}{x^2}$

(iii) $\dfrac{\ln x}{x^2}$ (iv) $2^x + x^2 + 2^2$

(v) $\dfrac{e^x}{x^2+7}$ (vi) $5^x e^x$

3. Using the limit $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = lna,$ prove that $\frac{d}{dx}(a^x) = a^x \, lna,$ where $a > 0$ and $a \neq 1$.

1.4 THE CHAIN RULE

We now discuss one of the most powerful rules in differential calculus, the chain rule, which deals with composite of functions.

Theorem 10: If $y = f(u)$ is differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or, equivalently, $\frac{d}{dx}[f(gx))] = f'(g(x))g'(x)$.

Proof : Let $(F(x) = f(g(x))$. We have to show that for $x = c$,

$$F'(c) = f'(g((c))g'(c).$$

An important consideration in this proof is the behaviour of g as x approaches c. A problem occurs if there are values of x other than c such that $g(x) = g(c)$. However, in this proof we shall assume that $g(x) \neq g(c)$ for values of x other than c. Thus, we can multiply and divide by the same (non– zero) quantity $g(x) - g(c)$. Note that as g is differentiable, it is continuous and it follows that $g(x) \to g(c)$ as $x \to$ c.

$$F'(c) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c}$$

$$= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \frac{g(x) - g(c)}{x - c}\right] \qquad [\because g(x) \neq g(c)]$$

$$= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \lim_{x \to c} \frac{g(x) - g(c)}{x \to c}$$

$$= f'(g(c)g'(c)$$

Remark : We can extend the chain rule for more than two functions. For example, if $F(x) = f[g(h(x))]$, then

$$F'(c) = f'[g(h(c))]g'(h(c))h'(c).$$

In other words

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example 14 : Find the derivatives of the following functions.

(i) $y = (x^2+1)^3$ (ii) $y = e^{x^2}$

(iii) $y = ln\,(2x^2 + e^x)$ (iv) $y = (x + lnx)^2$

Solution : (i) Put $x^2 +1 = u$

Then $y = u^3$ where $u = x^2 + 1$

$$\therefore \quad \frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 2x$$

Then by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2)(2x)$$

$$= 6x\,(x^2+1)^2$$

(ii) In this case we take $x^2 = u$, so that $y = e^u$

Then by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)(2x) = 2xe^{x^2}.$$

(iii) Take $u = 2x^2 + e^x$, so that $y = lnu$.

Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{u}(4x + e^x)$$

$$= \frac{4x + e^x}{2x^2 + e^x}$$

(iv) Take $u = x + \ln x$, so that $y = u^2$

Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2u)\left(1 + \frac{1}{x}\right)$$

$$= 2(x + \ln x)\left(1 + \frac{1}{x}\right)$$

We will now extend the power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

to real exponents. We will do this in two stages – first to rational exponents and then to real exponents. We shall use the chain rule.

Theorem 11 : For rational values of *n*

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof : Let $n = {}^{p}/_{q}$, where *p*, *q* are integers and $q > 0$. Then $nq = p$ is an integer.
Let $u = x^n$ and consider the equation.

$$(x^n)^q = x^{nq} = x^p \ \ or\ u^q = x^p \ \ ……………….(1)$$

Now differentiate (1) using the chain rule on the left and the power rule (for integers) on the right to obtain

$$qu^{q-1}\frac{du}{dx} = p\,x^{p-1}$$

$$\Rightarrow \frac{d}{dx}(x^n) = \frac{px^{p-1}}{qu^{q-1}}$$

But $u^{q-1} = {}^{u^q}/_{u} = {}^{x^p}/_{x^n}$, because $u = x^n$. Thus

$$\frac{d}{dx}(x^n) = \frac{px^{p-1}}{q\,{}^{x^p}/_{x^n}} = nx^{p-1+n-p} = nx^{n-1}$$

Theorem 12 : For a real number *n*

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof : Recall that if *n* is real, then by definition

$$x^n = e^{nlnx}$$

Now put $u = nlnx$, so that $x^n = e^u$. Then by the chain rule

$$\frac{d}{dx}(x^n) = \frac{d}{du}(e^u)\frac{du}{dx} = (e^u)\frac{d}{dx}(nlnx) = (e^{nlnx})\left(\frac{n}{x}\right)$$

$$= \frac{nx^n}{x} = nx^{n-1}$$

Example 15 : Find the derivative of each of the following functions:

(i) $y = (x^2+2)^{2/3}$ (ii) $y = e^{\sqrt{x}}$

(iii) $y = \ln(1+\sqrt{1+x^2})$ (iv) $y = x^2\,e^{x^2}$

Solution : (i) Putting $u = x^2 + 2$, we have

$$\frac{dy}{dx} = \frac{2}{3}(x^2+2)^{\frac{2}{3}-1}\frac{d}{dx}(x^2+2)$$

$$= \frac{2}{3}(x^2+2)^{-1/3}(2x)$$

$$= \frac{4x}{3(x^2+1)^{1/3}}$$

(ii) Putting $u = \sqrt{x}$, we have

$$\frac{dy}{dx} = e^{\sqrt{x}} \frac{d}{dx}(\sqrt{x}) = e^{\sqrt{x}} \frac{1}{\sqrt[2]{x}} = \frac{e^{\sqrt{x}}}{\sqrt[2]{x}}$$

(iii) $$\frac{dy}{dx} = \frac{1}{1+\sqrt{1+x^2}} \frac{d}{dx}\left(1+\sqrt{1+x^2}\right)$$

$$= \frac{1}{1+\sqrt{1+x^2}} \frac{1}{2\sqrt{1+x^2}} \frac{d}{dx}(1+x^2)$$

$$= \left(\frac{1}{1+\sqrt{1+x^2}}\right)\left(\frac{1}{2\sqrt{1+x^2}}\right)(2x)$$

$$= \left(\frac{x}{(1+\sqrt{1+x^2})\sqrt{1+x^2}}\right)$$

(iv) $$\frac{dy}{dx} = \frac{d}{dx}(x^2)\, e^{x^2} + x^2 \frac{d}{dx}\left(e^{x^2}\right)$$

$$= 2x\, e^{x^2} + x^2 e^{x^2} \frac{d}{dx}(x^2)$$

$$= 2x\, e^{x^2} + x^2 e^{x^2}(2x)$$

$$= 2x\, e^{x^2}(1+x^2)$$

Example 16 : Find the derivatives of following functions :

(i) $y = \ln\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$ (ii) $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(iii) $y = \sqrt[3]{x(x+1)(x+2)}$

Solution : (i) Rewriting the argument of the log, we have

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \, \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$= \frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(1+x)-(1-x)}$$

$$= \frac{(1+x)+(1-x)-\sqrt[2]{1+x}\,\sqrt{1-x}}{2x}$$

$$= \frac{2-\sqrt[2]{1-x^2}}{2x} = \frac{1-\sqrt{1-x^2}}{x}$$

Therefore, $y = ln\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$= ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)$$

$$= ln\ (1-\sqrt{1-x^2}) - lnx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\sqrt{1-x^2}}\frac{d}{dx}\left(1-(1-x^2)^{-1/2}\right) - \frac{1}{x}$$

$$= \left[\frac{1}{1-\sqrt{1-x^2}}\left\{\frac{d}{dx}0 - \frac{1}{2}(1-x^2)^{-1/2}(-2x)\right\} - \frac{1}{x}\right]$$

$$= \frac{1}{1-\sqrt{1-x^2}}\frac{x}{\sqrt{1-x^2}} - \frac{1}{x}$$

$$= \frac{x^2 - [\sqrt{1-x^2}(1-\sqrt{1-x^2})]}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$= \frac{x^2 - \sqrt{1-x^2} + (1-x^2)}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$= \frac{1-\sqrt{1-x^2}}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})} = \frac{1}{x\sqrt{1-x^2}}$$

(i) One can apply the quotient rule in this case. However, we will avoid it by rewriting the given expression.

$$Y = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^{2x}-1+2}{e^{2x}-1}$$

$$= 1 + \frac{2}{e^{2x}-1} = 1+2\,(e^{2x}-1)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = 0 + 2(-1)(e^{2x}-1)^{-2}\frac{d}{dx}(e^{2x}-1)$$

$$= \frac{-2}{(e^{2x}-1)^2}(2e^{2x}) = \frac{-4e^{2x}}{(e^{2x}-1)^2}$$

(ii) We have $y = [x(x+1)(x+2)]^{1/3}$

So, $\frac{dy}{dx} = \frac{1}{3}[x(x+1)(x+2)]^{\frac{1}{3}-1}\frac{d}{dx}[x(x+1)(x+2)]$ (chain Rule)

$$= \frac{1}{3}[x(x+1)(x+2)]^{-\frac{2}{3}}\frac{d}{dx}[x(x+1)(x+2)] \text{ (product rule)}$$

$$= \frac{1}{3}[x(x+1)(x+2)]^{-\frac{2}{3}}\frac{d}{dx}[(x+1)(x+2) + x(x+2) + x(x+1)]$$

$$= \frac{1}{3}[x(x+1)(x+2)]^{-\frac{2}{3}}\ x(x+1)(x+2)\left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}\right]$$

$$= \frac{1}{3}[x(x+1)(x+2)]^{1/3}\ \left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}\right]$$

Check Your Progress 3

1. Find the derivatives of each of the following functions :

(i) $y = (x^3 + x)^{3/2}$ (ii) $y = ln\left(\frac{x^2}{2}\right)$

(iii) $y = e^{(x^2+2x)}$ (iv) $y = ln\ (x+\sqrt{x})$

2. Find $\frac{dy}{dx}$ *where*

(i) $y = \frac{1-e^x}{e^{2x}}$ (ii) $y = \frac{x}{\sqrt{x^2-1}}$

(iii) $y = 2^{x/lnx}$

3. Differentiate each of the following functions :

(i) $y = ln\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right]$ (ii) $y = \sqrt{\frac{1-x}{1+x}}$

(iii) $y = \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1-\sqrt{x^2-1}}}$

1.5 DIFFERENTIATION OF PARAMETRIC FORMS

Suppose *x* and *y* are given as functions of another variable *t*. We call *t*, the variable in which *x* and *y* are expressed as parameter. In this case, we find $\frac{dy}{dx}$ as follows :

Let $x = f(t)$ and $y = g(t)$,where *f* and *g* are differentiable functions of *t* and $f'(t) \neq 0\ \forall$ t. Let Δx and Δy be the increments and *x* and *y* respectively, corresponding to the increment Δt in *t*. That is $\Delta x = f(t+\Delta t) - f(t)$ and $\Delta y = g(t+\Delta t) - g(t)$

Since $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

and $\Delta x \to 0$ as $\Delta t \to 0$, we can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{g(t+\Delta t) - g(t)}{f(t+\Delta t) - f(t)}$$

Dividing both the numerator and denominator by Δt, we can use the differentiability of *f* and *g* to conclude that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\left[\frac{g(t+\Delta t) - g(t)}{\Delta t}\right]}{\left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right]}$$

$$= \frac{g'(t)}{f'(t)} = \frac{dy/dt}{dx/dt}$$

Example 17 : Find $\frac{dy}{dx}$ when

(a) $x = at^2$, $y = 2at$ (b) $= ct$, $y = {}^{c}/_{t}$

(c) $x = lnt, y = {}^{1}/_{t}$ (d) $y = \frac{3at}{1+t^2}$

Solution: (a) We have

$$\frac{dy}{dx} = \frac{d}{dt}[at^2] = 2at$$

and $\frac{dy}{dt} = \frac{d}{dt}[2at] = 2a$

$$so, \frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

(b) We have

$$\frac{dy}{dx} = \frac{d}{dt}[ct] = c, and\ \frac{dy}{dx} = \frac{d}{dt}\left[\frac{c}{t}\right] = \frac{d}{dt}[ct^{-1}]$$

$$= [c(-1)t^{-2}] = \frac{c}{t^2}$$

since, $\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}}$, we get

$$\frac{dy}{dx} = \frac{-c/t^2}{c} = -\frac{1}{t^2}$$

(c) We have $\frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = -\frac{1}{t^2}$

$$\therefore \frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = (-1)\ \frac{1/t}{1/t^2} = -\frac{1}{t}$$

(d) We have

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{3at}{1+t^2}\right]$$

$$= 3a\frac{(1+t^2)\frac{dx}{dt} - t\,\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= 3a\frac{(1+t^2)(1) - t\,(2t)}{(1+t^2)^2}$$

$$= 3a\frac{(1-t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{3at^2}{(1+t^2)}\right] \text{ and}$$

$$= 3a\frac{(1+t^2)\frac{d}{dt}(t^2) - (t^2)\,\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= 3a\frac{(1+t^2)(2t) - (t^2)\,(2t)}{(1+t^2)^2}$$

$$= \frac{6at}{(1+t^2)^2}$$

Since, $\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

we get

$$\frac{dy}{dx} = \frac{6at\big/(1+t^2)^2}{3a(1-t^2)\big/(1+t^2)^2}$$

$$= \frac{2t}{1-t^2}$$

Second Order Derivatives

Let *y = f(x)* be a function. If *f* is a differentiable function, then its derivative is a function. If the derivative is itself differentiable we can differentiate it and get another function called the second derivative. The second derivative is denoted by y or $f\ (x)$ or $\dfrac{d^2y}{dx^2}$

Thus

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example 18 : If $y = \dfrac{lnx}{x}$, show that $\dfrac{d^2y}{dx^2} = \dfrac{2lnx - 3}{x^3}$

Solution : we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{lnx}{x}\right] = \frac{d}{dx}[x^{-1}lnx]$$

$= \frac{d}{dx}(x^{-1}) lnx + \ x^{-1}\frac{d}{dx}\ (lnx)$ (product rule)

$= \ (-\ 1)x^{-2}lnx + \ x^{-1}\frac{1}{x}$

$= x^{-2}$ [1 – ln$x$]

Differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\ [x^{-2}][1 - lnx] + \ x^{-2}\frac{d}{dx}[1 - lnx]$$

$= (-2)\ x^{-3}(1 - lnx) + \ x^{-2}\left(0 - \frac{1}{x}\right)$

$= -2x^{-3}(1 - lnx) + \ x^{-3}$

$= -x^{-3}(2-2\ln x + 1)$

$= \frac{2lnx - 3}{x^3}$

Example 19 : If y = $ae^{mx} + \ be^{-mx}$, show that $\frac{d^2y}{dx^2} = \ m^2y$

Solution : We have $y = ae^{mx} + \ be^{-mx}$

Differentiating both sides with respect to x, we get $\frac{dy}{dx} = \ \frac{d}{dx}\ (ae^{mx} + \ be^{-mx})$
$= \ ame^{mx} - \ bme^{-mx}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(ame^{mx} - \ bme^{-mx})$$

$= am^2e^{mx} - \ bm(-m)e^{-mx}$

$= am^2e^{mx} + \ bm^2e^{-mx}$

$= m^2(ae^{mx} + \ be^{-mx})$

$= m^2y$

Example 20 : If $y = ln\ (x + \sqrt{x^2 + 1})$, prove that

$$(x^2 + 1)\ \frac{d^2y}{dx^2} + x\ \frac{dy}{dx} = 0$$

Solution : We have $y = ln\ (x + \sqrt{x^2 + 1})$

Differentiating both sides, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}}\frac{d}{dx}\left[x + \ (x^2 + 1)^{\frac{1}{2}}\right] \quad \text{(chain rule)}$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}\,(x^2+1)^{-\frac{1}{2}}(2x)\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-\frac{1}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = \left(-\frac{1}{2}\right)(x^2+1)^{-\frac{3}{2}}\frac{d}{dx}[(x^2+1)]$$

$$= -\frac{1}{2}\frac{1}{(x^2+1)^{\frac{3}{2}}}\;2x = \frac{-x}{(x^2+1)^{\frac{3}{2}}}$$

Now, $(x^2+1)\,\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$

$$= (x^2+1)\left[\frac{-x}{(x^2+1)^{\frac{3}{2}}}\right] + x\frac{1}{\sqrt{x^2+1}}$$

$$= -\frac{x}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}} = 0$$

Thus, $(x^2+1)\,\dfrac{d^2y}{dx^2} + x\,\dfrac{dy}{dx} = 0$

Check Your Progress – 4

1. Find $\frac{dy}{dx}$ when

1. Find $\dfrac{dy}{dx}$ when

   (a) $x = \dfrac{1}{2}(e^{\theta} - e^{-\theta})$ and $y = \dfrac{1}{2}(e^{\theta} - e^{-\theta})$

   (b) $x = a\left(t - \dfrac{1}{t}\right)$ and $y = a\left(t + \dfrac{1}{t}\right)$

   (c) $x = \dfrac{a(1-t^2)}{(1+t^2)}$ and y = $\dfrac{2bt}{1+t^2}$

2. If $y = \sqrt{1+x^2}$, find $\frac{d^2y}{dx^2}$

3. If $y = ln\,(\sqrt{x-1} + \sqrt{x+1})$, prove that

   $$(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

4. If $y = ax + \frac{b}{x}$, show that $\frac{xd^2y}{dx^2} + \frac{xdy}{dx} - y = 0$

1.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. (i) $\lim_{x\to 3} (3\,x^3 + 2x + 1) = 3\,.(2)^3 + 2(2) + 1 = 29$

(ii) $\lim_{x\to 2} \frac{x-2}{x+2} = \frac{2-2}{2+2} = \frac{0}{4} = 0$

(iii) $\lim_{x\to 2} \frac{x^2 - 5x + 2}{x-1} = \frac{2^2 - 5(2) + 2}{2-1} = -2$

(iv) $\lim_{x\to 3} \sqrt[3]{3x^2 - 19} = \sqrt[3]{3(3)^2 - 19} = \sqrt[3]{27 - 19} = \sqrt[3]{8} = 2$

2. (i) $\lim_{x\to 2} \frac{x^2 - 4}{x+2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x+2} = \lim_{x\to -2} (x-2) = -2 - 2 = -4$

(ii) $\lim_{x\to 5} \frac{\sqrt{x-1} - 2}{x-5} = \lim_{x\to 5} \left[\left(\frac{\sqrt{x-1} - 2}{x-5}\right)\left(\frac{\sqrt{x-1} + 2}{\sqrt{x-1} + 2}\right)\right]$

$$= \lim_{x\to 5} \frac{(x-1) - 4}{(x-5)(\sqrt{x-1} + 2)}$$

$$= \lim_{x\to 5} \frac{(x-5)}{(x-5)(\sqrt{x-1} + 2)}$$

$$= \lim_{x\to 5} \frac{1}{(\sqrt{x-1} + 2)} = \lim_{x\to 5} \frac{1}{(\sqrt{5-1} + 2)} = \frac{1}{4}$$

3. (i) $\lim_{x\to a} \frac{x^{7/6} - a^{7/6}}{x^{3/5} - a^{3/5}} = \lim_{x\to a} \frac{\frac{x^{7/6} - a^{7/6}}{x-a}}{\frac{x^{3/5} - a^{3/5}}{x-a}}$

$$= \frac{\lim_{x\to a} \frac{x^{7/6} - a^{7/6}}{x-a}}{\lim_{x\to a} \frac{x^{3/5} - a^{3/5}}{x-a}} = \frac{(7/6)}{(3/5)} \frac{a^{7/6-1}}{a^{3/5-1}} \left(\because \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right)$$

$$= \frac{35}{18} \frac{a^{1/6}}{a^{-2/5}} = \frac{35}{18} a^{\frac{17}{30}}$$

(ii) $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} \quad = \lim_{x \to a} \frac{(x^m - a^{m)/(x-a)}}{(x^n - a^n)/(x-a)}$

$$= \frac{\lim_{x \to a} \frac{x^m - a^m}{x - a}}{\lim_{x \to a} \frac{x^n - a^n}{x - a}}$$

$$= \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n}\, a^{m-n}$$

4. We have

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0^+} f(0 - h) = \lim_{h \to 0^+} f(-h)$$

$$= \lim_{h \to 0^+} [2 - (-h)] = \lim_{h \to 0^+} (2 + h) = 2$$

and $\lim_{x \to 0^+} f(x) = \lim_{h \to 0^+} f(0 + h) = \lim_{h \to 0^+} (f(h)) = \lim_{h \to 0^+} (2 + h) = 2$

Thus, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 2 \Rightarrow \lim_{x \to 0^+} f(x) = 2$

Also, $f(0) = 2 + 0 = 2$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Hence, f is continuous at $x = 0$.

5. For f to be continuous at $x = 5$, we must have

$f(5) = \lim_{x \to 5} f(x)$

$$\Rightarrow k = \lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+5)}{x - 5}$$

So, $k = \lim_{x \to 5} (x + 5) = 5 + 5 = 10$

Thus, $k = 10$

Check Your Progress 2

1. (i) $\frac{dy}{dx} = \frac{d}{dx}(x^5 - 3x^4 + 2x - 1) = 5x^4 - 12x^3 + 2$

(ii) $\frac{dy}{dx} = \frac{d}{dx} \frac{2x - 1}{\pi^2} = \frac{1}{\pi^2} \frac{d}{dx}(2x - 1) = \frac{2}{\pi^2}$

(iii) $\frac{dy}{dx} = \frac{(2x+7)\frac{d}{dx}(3x+5) - (3x+5)\frac{d}{dx}(2x+7)}{(2x+7)^2}$ (Quotient Rule)

$$= \frac{(2x+7).3 - (3x+5).2}{(2x+7)^2}$$

$$= \frac{11}{(2x+7)^2}$$

(iv) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3-4}{x^3}\right) = \frac{(x^3)\frac{d}{dx}(x^3-4) - (x^3-4)\frac{d}{dx}(x^3)}{(x^3)^2}$

$$= \frac{x^3(3x^2) - (x^3-4)(3x^2)}{x^6}$$

$$= \frac{4x^2}{x^6} = \frac{4}{x^4}$$

2. (i) $\frac{d}{dx}(e^x lnx) = \frac{d}{dx}(e^x) lnx + e^x \frac{d}{dx} lnx$

$$= (e^x lnx) + \frac{e^x}{x} = e^x(lnx + \frac{1}{x})$$

(ii) $\frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{x^2\frac{d}{dx}(e^x) - (e^x)\frac{d}{dx}(x^2)}{x^4} = \frac{e^x(x-2)}{x^3}$

(iii) $\frac{d}{dx}\left(\frac{lnx}{x^3}\right) = \frac{x^3\frac{d}{dx}(lnx) - (lnx)\frac{d}{dx}(x^3)}{(x^3)^2}$

$$= \frac{x^3\frac{1}{x} - (lnx)\frac{d}{dx}(3x^2)}{x^6}$$

$$= \frac{x^2(1-3lnx)}{x^6} = \frac{1-3lnx}{x^4}$$

(iv) $\frac{d}{dx}(2^x + x^2 + 2^2) = \frac{d}{dx}(2^x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(2^2)$

$$= 2^x ln2 + 2x + 0$$

$$= 2^x ln2 + 2x$$

(v) $\frac{d}{dx}\left(\frac{e^x}{x^2+7}\right) = \frac{(x^2+7)\frac{d}{dx}(e^x) - (e^x)\frac{d}{dx}(x^2+7)}{(x^2+7)^2}$

$$= \frac{(x^2+7)e^x - e^x(2x)}{(x^2+7)^2}$$

$$= \frac{e^x[x^2 - 2x + 7]}{(x^2+7)^2}$$

(vi) $\frac{d}{dx}(5^x e^x) = \frac{d}{dx}(5^x)e^x + 5^x \frac{d}{dx}(e^x)$

$= 5^x \ln 5 \; e^x + 5^x \; e^x$

$= 5^x e^x (ln5 + 1)$

3. $$\frac{d}{dx}(a^x) = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x}$$

$$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$= a^x lna$ (using the given limit)

Check Your Progress – 3

1. (i) $$\frac{dy}{dx} = \frac{3}{2}(x^3 + x)^{\frac{3}{2}-1} \frac{d}{dx}(x^3 + x)$$

$$= \frac{3}{2}(x^3 + x)^{1/2}(3x^2 + 1)$$

(ii) $$\frac{dy}{dx} = \frac{1}{(x^2/2)} \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{2}{x^2}\left(\frac{2x}{2}\right) = \frac{2}{x}$$

(iii) $$\frac{dy}{dx} = e^{(x^2+2x)} \frac{d}{dx}(x^2 + 2x) = e^{(x^2+2x)}(2x + 2)$$

$$= 2(x + 1)e^{(x^2+2x)}$$

(iv) $$\frac{dy}{dx} = \frac{1}{x + \sqrt{x}} \frac{d}{dx}(x + \sqrt{x}) = \frac{1}{x + \sqrt{x}}\left(1 + \frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{x} + 1}{2\sqrt{x}(x + \sqrt{x})}$$

2. (i) $$\frac{dy}{dx} = \frac{\frac{d}{dx}(1 - e^x)e^{2x} - (1 - e^x)\frac{d}{dx}(e^{2x})}{(e^{2x})^2}$$

$$= \frac{e^{2x}(-e^x) - (1 - e^x)(2e^{2x})}{e^{4x}} = \frac{e^x - 2}{e^{2x}}$$

(ii) $$\frac{dy}{dx} = \frac{\sqrt{(x^2-1}\frac{d}{dx}(x) - x\frac{d}{dx}\sqrt{(x^2-1}}{(\sqrt{(x^2-1)}^2}$$

$$= \frac{\sqrt{(x^2-1} - x\left(\frac{1}{2\sqrt{(x^2-1)}}\right)2x}{x^2-1}$$

$$= \frac{(x^2-1) - x^2}{(x^2-1)\sqrt{x^2-1}} = \frac{-1}{(x^2-1)^{3/2}}$$

(iii) $$\frac{dy}{dx} = 2x^{x/ln2} ln2 \frac{d}{dx}\left(\frac{x}{lnx}\right)$$

$$= 2x^{x/lnx}\left[\frac{1.\, lnx - x.\frac{1}{x}}{(lnx)^2}\right]$$

$$= \frac{2x^{x/lnx} ln2 (lnx - 1)}{(lnx)^2}$$

3. (i) Rewriting the given expression, we have

$$y = ln\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right]$$

$$= In\, e^x + ln\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}} \qquad [ln(ab) = lna + lnb]$$

$$= xlne\ \frac{3}{4} + ln\left(\frac{x-2}{x+2}\right) \qquad [lna^x = xlna]$$

$$= x + \frac{3}{4}[ln(x-2) - ln(x+2)] \qquad [ln(e) = 1 \text{ and } ln(a/b) = lna - lnb]$$

$$\Rightarrow \frac{dy}{dx} = 1' + \frac{3}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right]$$

$$= 1 + \frac{3}{4}\left[\frac{(x+2) - (x-2)}{(x-2)(x+2)}\right]$$

$$= 1 + \frac{3}{4}\left[\frac{x+2-x+2}{x^2-4}\right]$$

$$= 1 + \frac{3}{x^2-4}$$

$$= \frac{x^2-4+3}{x^2-4} = \frac{x^2-1}{x^2-4}$$

(ii) $$y = \left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{1-x}{1+x}\right) \quad \text{(Chain Rule)}$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\ \frac{-2}{(1+x)^2}$$

$$= \frac{-1}{(1+x)^2}\sqrt{\frac{1+x}{1-x}}$$

(iii) Rewriting the given expression, we have

$$y = \frac{\sqrt{(x^2+1)} + \sqrt{(x^2-1)}}{\sqrt{(x^2+1)} - \sqrt{(x^2-1)}} = \frac{\sqrt{(x^2+1)} + \sqrt{(x^2-1)}}{\sqrt{(x^2+1)} - \sqrt{(x^2-1)}}\ \frac{\sqrt{(x^2+1)} + \sqrt{(x^2-1)}}{\sqrt{(x^2+1)} - \sqrt{(x^2-1)}}$$

$$= \frac{\left(\sqrt{(x^2+1)} + \sqrt{(x^2-1)}\right)^2}{(x^2+1) - (x^2-1)} = \frac{(x^2+1) + (x^2-1) + \sqrt[2]{(x^2+1)(x^2-1)}}{2}$$

$$= \frac{2x^2 + \sqrt[2]{x^4-1}}{2} = x^2 + (x^4-1)^{1/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}[((x^4-1)^{\frac{1}{2}}]$$

$$= 2x + \frac{1}{2}(x^4-1)^{-\frac{1}{2}}\frac{d}{dx}(x^4-1)$$

$$= 2x + \frac{1}{\sqrt[2]{(x^4-1)}}(4x^3)$$

$$= 2x + \frac{2x^3}{\sqrt{(x^4-1)}}$$

Check Your Progress 4

1. (a) $$\frac{dx}{d\theta} = (e^{\theta} + e^{-\theta})/2$$

$$\frac{dx}{d\theta} = (e^{\theta} - e^{-\theta})/2$$

$$\therefore \frac{dx}{dy} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{1}{2}(e^{\theta} - e^{-\theta)}}{\frac{1}{2}(e^{\theta} + e^{-\theta})} = \frac{x}{y}$$

(b) $$\frac{dx}{dt} = a\left(1 + \frac{1}{t^2}\right), \quad \frac{dy}{dt} = b\left(1 - \frac{1}{t^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b(1-\frac{1}{t^2})}{a(1+\frac{1}{t^2}),} = \frac{b(t^2-1)}{a(t^2+1)}$$

(c) $$\frac{dx}{dt} = \frac{a(1+t^2)(-2t) - a(1-t^2)\,(2t)}{(1+t^2)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{a[-2t - 2t^3 - 2t + 2t^3]}{(1+t^2)^2}$$

$$= \frac{-4at}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2b\frac{(1-t^2)(1) - t(2t)}{(1+t^2)^2}$$

$$= \frac{2b(1-t^2)}{-4at}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(1-t^2)}{-4at} = \frac{-b(t^2-1)}{(1+t^2)^2}$$

2. $$\frac{dy}{dx} = \frac{d}{dx}[(1+x^2] = \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1+x^2)$$

$$= \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}(2x)$$

$$= x(1+x^2)^{-1/2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x(1+x^2)^{-\frac{1}{2}}\right] = \frac{d}{dx}(x)(1+x^2)^{-\frac{1}{2}} + x\frac{d}{dx}(1+x^2)^{-\frac{1}{2}}$$

$$= 1.\,(1+x^2)^{-\frac{1}{2}} + x\left[-\frac{1}{2}(1+x^2)^{-\frac{1}{2}-1}(2x)\right]$$

$$= (1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-3/2}$$

$$= (1+x^2)^{-\frac{1}{2}}\left[1 - \frac{x^2}{1+x^2}\right] = \frac{(1+x^2)^{-\frac{1}{2}}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

3. We have y = ln ($\sqrt{x-1} + \sqrt{x+1}$)

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x-1}+\sqrt{x+1}} \frac{d}{dx}(\sqrt{x-1} + \sqrt{x+1}) \quad \text{(Chain Rule)}$$

$$= \frac{1}{\sqrt{x-1}+\sqrt{x+1}} \left(\frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x-1}}\right)$$

$$= \frac{(\sqrt{x-1}+\sqrt{x+1})}{2(\sqrt{x-1}+\sqrt{x+1})\sqrt{x-1}\sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x^2-1}} = \frac{1}{2}(x^2-1)^{-1/2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\frac{d}{dx}\left[(x^2-1)^{-\frac{1}{2}}\right]$$

$$= -\frac{1}{4}[(x^2-1)^{-\frac{1}{2}-1}(2x)] \quad \text{(Chain Rule)}$$

$$= -\frac{1}{2}x(x^2-1)^{-3/2}$$

$$\therefore (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = (x^2-1)\left[-\frac{1}{2}x(x^2-1)^{-3/2}\right] + x\left[\frac{1}{2}(x^2-1)^{-1/2}\right]$$

$$= -\frac{1}{2}x(x^2-1)^{-1/2} + \frac{1}{2}x(x^2-1)^{-1/2} = 0.$$

4. When have y = $ax + \frac{b}{x}$

$$\therefore \frac{dy}{dx} = a - \frac{b}{x^2} \text{ and } \frac{d^2y}{dx^2} = \frac{d}{dx}(a - bx^{-2}) = 2bx^{-3} = \frac{2b}{x^3}$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\left(\frac{2b}{x^3}\right) + x\left(a - \frac{b}{x^2}\right) - \left(ax + \frac{b}{x}\right)$$

$$= \frac{2b}{x} + ax - \frac{b}{x} - ax - \frac{b}{x}$$

= 0

1.7 SUMMARY

In **section 1.2** of the unit, to begin with, the concept of limit of a function is defined. Then, some properties of limits are stated. Next, the concept of one-sided limit is defined. Then, the concept of continuity of a function is defined. Each of these concepts is illustrated with a number of examples.

In **section 1.3**, the concepts of differentiability of a function at a point and in an open interval are defined. Then, a number of rules for finding derivatives of simple functions are derived. In **section 1.4,** chain rule of differentiation is derived and is explained with a number of examples. In **section 1.5**, the concept of differentiation of parametric forms is defined followed by the definition of the concept of second order derivative. Each of these concepts is explained with a number of suitable examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.6**.

UNIT 2 SIMPLE APPLICATION OF DIFFERENTIAL CALCULUS

Structure

2.0 INTRODUCTION

In the previous unit, we defined the derivative of a function. In this Unit, we shall discuss several applications of the derivative of a function. Derivatives have been used in various spheres of Physics, Chemistry, Medical Sciences and Economics.

Derivative of y with respect to x determines the rate of change of y with respect to x. This can be used to find rate of change of a quantity with respect to another quantity. Chain rule can be used to find rates of change of two quantities that are changing with respect to the same quantity.

The concept of derivative can also be used to determine whether a function is increasing or decreasing in an interval. Indeed, if the derivative of a function is positive in an interval, then the function increases in that interval. Similarly, if the derivative is negative then the function decreases. If the derivative is zero in an interval, then the function remains constant in that interval.

Derivatives can also be used to find maximum and minimum values of a function in an interval. The maximum and minimum values are called extreme values of a function. The extreme values can be absolute or can be local. The first derivative test and the second derivative tests are used to determine the points of local extrema.

2.1 OBJECTIVES

After studying this unit, you should able to :

- compute the rate of change of one quantity, knowing the rate of change of some other related quantity;

- determine when a function increases or decreases;
- find the absolute maximum and absolute minimum of a continuous function on the closed interval $[a,b]$; and
- find local maximum and local minimum of a function.

2.2 RATE OF CHANGE OF QUANTITIES

Let us consider a differentiate function $y = f(x)$. We can give a physical interpretation to $\frac{dy}{dx}$ the derivative $\frac{dy}{dx} = f'(x)$ represents the rate of change of y with respect to x, when y varies with x according to the rule $y = f(x)$.

Thus, $f'(x) \left(or \left. \frac{dy}{dx} \right]_{x=x_0} \right)$ represents the rate of change of y with respect to x at $x = x_0$.

For example, Let s be the distance of a particle from origin at a time t. Then s is a function of t and the derivative $\frac{ds}{dt}$ represents speed which is the rate of change of distance with respect to time t.

We can use chain rule to find the rates of two or more related variables that are changing with respect to the same variable. Suppose x and y are both varying with t (i.e., x, y are function of t). Then, by chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = f'(x)\frac{dx}{dt}$$

Thus, the rate of change one variable can be calculated if the rate of change of the other variable is known.

Example 1: A spherical balloon is being inflated at the rate of 900 cubic centimeters per second. How fast is the radius of the balloon increasing when the radius is 15 cm ?

Solution : We are given the rate of change of volume and are asked the rate of change of the radius. Let r cm be the radius and V cubic centimeters be the volume of the balloon at instant t.

$$\text{Thus, } V = \frac{4}{3}\pi r^3$$

Differentiating both the sides of this equation with respect to t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Note that have used the chain rule on the right hand side

It is given that $\frac{dV}{dt} = 900 \text{ and } r = 15$. Thus,

$$900 = 4\pi\,(15)^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi\,(225)} = \frac{1}{\pi}\ cm/\ sec.$$

$$\text{Thus, the radius is increasing at the rate of } = \frac{1}{\pi}\ \ cm/\ sec.$$

Example 2 : A rock is thrown into a lake producing a circular ripple. The radius of the ripple is increasing at the rate of 3 m/s. How fast is the area inside the ripple increasing when the radius is 10 m ?

Solution : We are given the rate of change of the radius and we are asked to find the rate of change of area.

Let r m be the radius and A square meters be the area at time t. Thus, $A = \pi\, r^2$.

Differentiating both the sides with respect to t, we get

$$\frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = 2\,\pi\, r\ \frac{dr}{dt}$$

We are given when $r\ =\ 10, \frac{dr}{dt} = 3$, therefore $\frac{dA}{dt} = 2\,\pi(10)(3) = 60\,\pi$

Thus, the area is increase at the rate of 60 $\pi\ sq.m/s$.

Example 3 : If a mothball evaporates at a rate proportional to its surface area 4 πr^2, show that its radius decreases at a constant rate.

Solution : Let r be the radius V be the volume of the mouthball at time t. Then

$$V = \frac{4}{3}\pi r^3$$

We are given that the mothball evaporates at a rate proportional to its surface area. It means that the rate of decrease of volume V of the mothball is proportional to 4 πr^2.

$$\text{Thus,} \frac{dV}{dt}\ = -k\ (4\pi r^2)$$

where $k > 0$ is a constant. (Negative sign has been introduced to show that the volume is decreasing).

$$\text{But}\ \ \frac{dV}{dt} = 4\pi r^2\ \ \frac{dr}{dt}.\ \ \text{Threfore } 4\pi r^2\frac{dr}{dt} =\ -k\ (4\pi r^2) or \frac{dr}{dt} =\ -k$$

This shows that the radius decreases at a constant rate. [Decrease is due to the negative sign].

Example 4 : A young child is flying kite which is at a height of 50 m. The wind is carrying the kite horizontally away from the child at a speed of 6.5 m/s. How fast must the kite string be let out when the string is 130 m ?

Solution : Let *h* be the horizontal distance of the kite from the point directly over the child's head 5. Let *l* be the length of kite string from the child to the kite at time *t*. [See Fig. 1] Then

$l^2 = h^2 + 50^2$

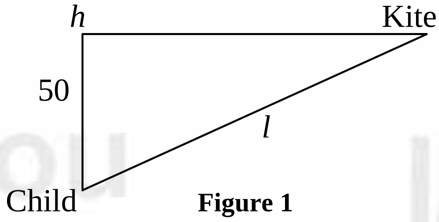


Figure 1

Differentiating both the sides with respect to *t*, we get

$$2l\,\frac{dl}{dt} = 2h\,\frac{dh}{dt} \text{ or } l\,\frac{dl}{dt} = h\,\frac{dh}{dt}.$$

We are given $\frac{dh}{dt} = 6.5\ m/s.$ We are interested to find *dl/dt* when *l* = 130.
But when *l* = 130, $h^2 = l^2 - 50^2 = 130^2 - 50^2 = 14400$ or $h = 120.$

$$\text{Thus, } \frac{dl}{dt} = \frac{120}{130} \times 6.5 = 5\ m/s.$$

This shows that the string should be let out at a rate of 6 m/s.

Example 5 : A ladder 13 m long leans against a house. The foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5 *m/s*. How fast is the angle ∝ between the ladder and the ground is changing when the foot of ladder is 12 m away from the house.

Solution : Let *y* be the distance of the top of the ladder from the ground, and let *x* be the distance of the bottom of the ladder from the base of the wall at time (Figure 2) *t*. We have $\tan\alpha = \frac{y}{x}$

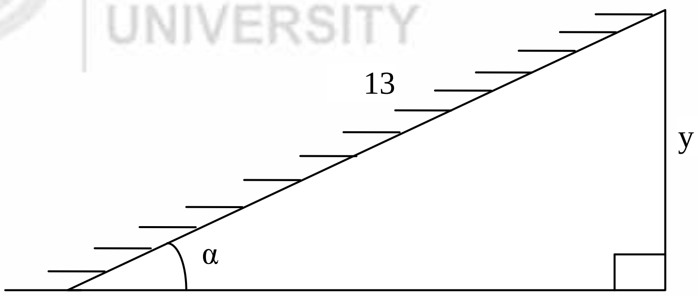


Figure 2

Differentiating both the sides with respect to *t*, we get

$$Sec^2\,\alpha\,\frac{d\alpha}{dt} = \frac{x\frac{dy}{dt} - y\,\frac{dx}{dt}}{x^2}$$

Now by the Pythagorean theorem $x^2 + y^2 = 13^2$

Differentiating both the sides with respect to t, we get $\quad 2x\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

$$\text{Thus,} \frac{d\alpha}{dt} = \frac{\frac{-x^2}{y}\frac{dx}{dt} - y\,\frac{dx}{dt}}{x^2\text{Sec}^2\,\alpha} = -\frac{x^2+y^2}{x^2\,y[1+tan^2\alpha]}\frac{dx}{dt}$$

$$= -\frac{x^2+y^2}{x^2\,y\left[1+\frac{y^2}{x^2}\right]}\;\frac{dx}{dt} = -\frac{1}{y}\,\frac{dy}{dt}$$

we are given that $\frac{dx}{dt} = 1.5$ and $x = 12$. For $x = 12$, we have

$$y = \sqrt{13^2 - x^2} = \sqrt{13^2 - 12^2} = 5$$

$$\text{Therefore,} \frac{d\alpha}{dt} = -\frac{1}{5}\;(1.5) = -\;0.3$$

Thus, α is decreasing at the rate of 0.3 radians per second.

Example 6 : Water, at the rate of 15 cubic centimetres per minute is pouring into a leaking cistern whose shape is a cone 16 centimetres deep and 8 centimetres in diameter at the top. At the time, the water is 12 centimetres deep, the water level is observed to be rising 0.5 centimetres per minutes. How fast is the water leaking out ?

Solution : Let *h* be the depth of the water, and *r* be the radius of the water surface at time *t*. [see fig 3]. Let *V* be the volume of the water at time *t*,

Then $\qquad V = \frac{1}{3}\pi r^2 h$

Now, since ΔOAB is similar to ΔOCD, we have

$$\frac{AB}{OA} = \frac{CD}{OC} \Rightarrow \frac{r}{h} = \frac{4}{16} \text{ or } r = \frac{1}{4}\,h.$$

$$\text{Thus,} V = \frac{1}{3}\,\pi\left(\frac{1}{4}\,h\right)^2 h = \frac{1}{48}\,\pi\,r^3$$

Differentiating both the sides with respect to *t*, we get

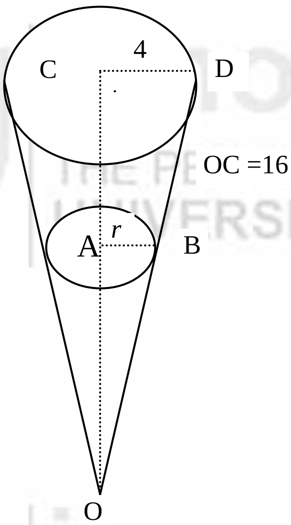


Figure 3

$$\frac{dV}{dt} = \frac{1}{48}\,\pi(3\,h^2)\frac{dh}{dt} = \frac{\pi h^2}{16}\frac{dh}{dt}$$

We are given that when $h = 12, \frac{dh}{dt} = \frac{1}{2}$ Hence, at that moment,

$$\frac{dV}{dt} = \frac{\pi(12)^2}{16} \times \frac{1}{2} = 4.5\,\pi.$$

Since, the rate at which the water is pouring in is 15, the rate of leakage is $(15 - 4.5\pi)$ cm^3 /min.

Example 7 : Sand is being poured into a conical pile at the constant rate of 50 cubic centimetres per minute. Frictional forces in the sand are such that the height of the cone is always one half of the radius of its base. How fast is the height of the pile increasing when the sand is 5 cm deep ?

Solution : Let r the radius, h be height and V be the volume of the cone (Fig. 4) of sand at time t.

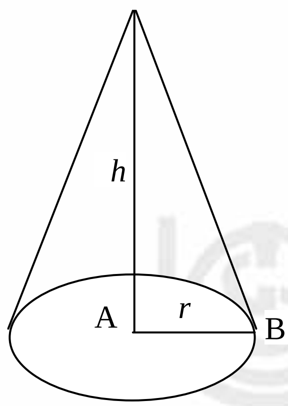


Figure 4

Then $V = \frac{1}{3}\pi r^2 h$

We are given that $h = \frac{1}{2} r$ *or* $r = -2h$. Thus,

$$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3.$$

Differentiating both the sides with respect to t, we get

$$\frac{dV}{dt} = \frac{4}{3}\,\pi(3\,h^2)\frac{dh}{dt} = 4\pi h^2\frac{dh}{dt}.$$

we are given that $\frac{dV}{dt} = 50$, thus $50 = 4\pi h^2 \frac{dh}{dt}$.

$$\Rightarrow \frac{dh}{dt} = \frac{50}{4\pi h^2}$$

When $h = 5. \frac{dh}{dt} = \frac{50}{4\pi(5^2)} = \frac{50}{100\pi} = \frac{1}{2\pi}$.

Hence, the height of the cone is rising at the rate of $(1/2\pi)$ cm/min.

Check Your Progress – 1

1. The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.
2. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference ?

3. A man 180 cm tall walks at a rate of 2 m/s away from a source of light that is 9 m above the ground. How fast is the length of his shadow increasing when he is 3 m away from the base of light ?
4. A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second. The height of the funnel is 20 centimetres and the radius of the top is 4 centimeters. How fast is the fluid level dropping when the level stands 5 centimetres above the vertex of the cone ?

2.3 INCREASING AND DECREASING FUNCTIONS

In this section, we shall study how the derivative can be used to obtain the interval in which the function is increasing or decreasing. We begin with the following definitions.

Definition : A function f is said to be **increasing*** on an interval I if for any two number x_1 and x_2 in I.

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function f is said to be **decreasing**** on an interval I if for any two numbers x_1 and x_1 in I,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

From this definition, we can see that a function is increasing if its graph moves up as x moves to the right and a function is decreasing if its graph moves down as x moves to the right. For example, the function in Fig 5 is decreasing on the interval $(-\infty, a]$ is constant on the interval $[a,b]$ and is increasing on the interval $[b, \infty)$.

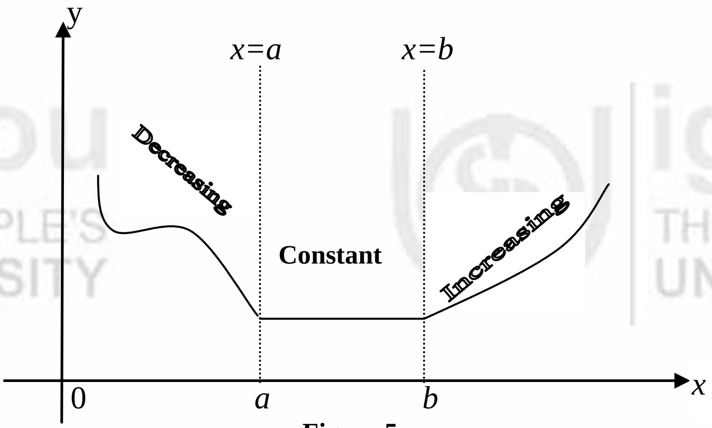


Figure 5

* some authors call such a function strictly increasing.

** some authors call such a function strictly decreasing.

Example 8 : Show that $f(x) = 3x + 2$ is an increasing function on **R**.

Solution :

(i) Let $x_1, x_2 \in \mathbf{R}$ and suppose $x_1 < x_2$
Since $x_1 < x_2$, we have $3x_1 < 3x_2$ ($\because 3>0$)

$\Rightarrow 3x_1 + 2 < 3x_2 + 2 \Rightarrow f(x_1) < f(x_2)$.

Thus, f is increasing on **R.** See also Figure 6.

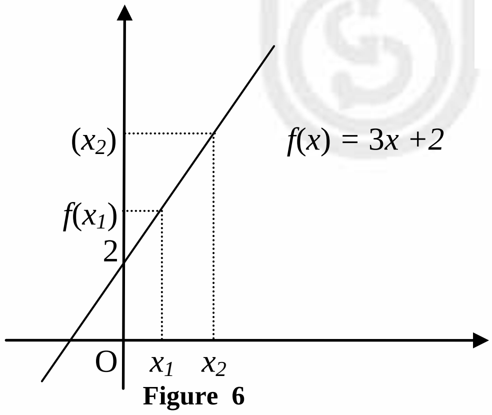


Figure 6

Example 9 : Show that $f(x) = x^2$ is a decreasing function on the interval $(-\infty, 0]$.

Solution : Note that the domain of f consist of all non–positive real numbers.

Suppose $x_1 < x_2 \leq 0$.

Since $x_1 < x_2$ *and* $x_1 < 0$, it follows that $x^2_1 > x_1 x_2$ ……… (1)

Again $x_1 < x_2$ *and* $x_1 \leq 0$, it follows that $x_1 x_2 \geq x_2^2$………..(2)

Combining (1) and (2) we get $x^2_1 > x_1 x_2 \geq x_2^2$

i.e., $x^2_1 > x^2_2$.

which means $f(x_1) > f(x_2)$

Hence, f is a decreasing function on $(-\infty, 0)$. See also Fig 7

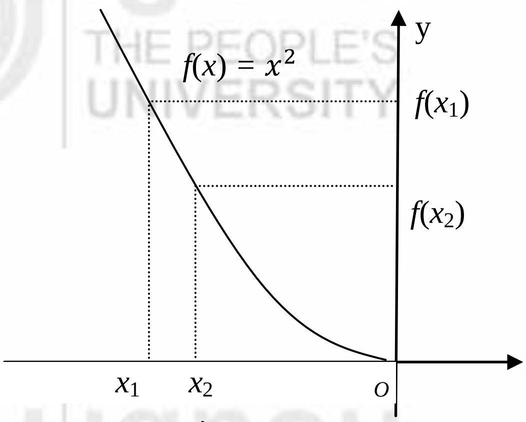


Figure 7

Example 10 : Show that $f(x) = x^2$, $x \in \mathbf{R}$ is neither increasing nor decreasing.

Solution : Let's take two points x_1 and x_2 such that $x_1 < 0 < x_2$. Then $f(x_1) = x^2_1 > 0 = f(0)$ *and* $f(0) = 0 < x^2_2 = f(x_2)$. Since $x_1 < 0$ and $0 < x_2$ implies $f(x_1) > f(0)$ *and* $f(0) < f(x_2)$, it follows that f is neither decreasing nor increasing. See also Fig 8.

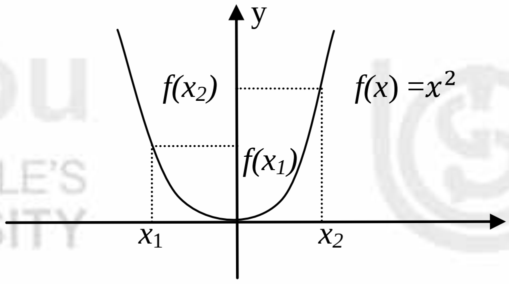


Figure 8

Use of Derivative to check Increasing, Decreasing

We now see how can use derivative f' to determine where a function f is increasing and where it is decreasing.

The graph of $y = f(x)$ in Figure 9 indicates that if the slope of a tangent line is positive in an open interval I (that is, if $f'(x) > 0$ for every $x \in I$), then f is increasing on I. Similarly, it appears that if the slope is negative (that is, $f'(x) < 0$), then f is decreasing

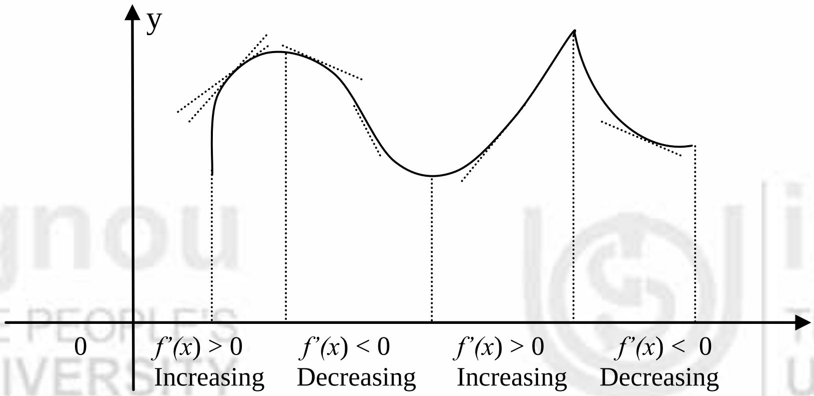


Figure 9

What we have observed intuitively are actually true as a consequence of the following theorem.

Theorem : Let f be continuous on $[a,b]$ and differentiable on (a, b)

(1) If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on $[a,b]$.

(2) If $f'(x) > 0$ for all $x \in (a,b)$, then f is decreasing on $[a,b]$.

We may also note that if $f'(x) > 0$ throughout an infinite interval $(-\infty, a]$ or $[b, \infty)$, then f is increasing on $(-\infty, a]$ or $[b, \infty)$, respectively, provided f is continuous on these intervals. An analogus result holds for decreasing function if $f'(x) < 0$.

Example 11 : Determine for which values of x the following functions are increasing and for which values they are decreasing.

(i) $f(x) = 16x^2 + 3x + 2$ (ii) $f(x) = -5x^2 + 7x + 8$

(iii) $f(x) = x^3 - 3x$ (iv) $f(x) = -2x^3 + 24x + 7$

Solution :

(i) We have $f'(x) = 32x + 3$.

Now, $f'(x) > 0$ if $32x + 3 > 0$, that is, if $x > -3/32$ and $f'(x) < 0$ if $32x + 3 < 0$ that is, if $x < -3/32$. Thus, $f(x)$ decreases on $(-\infty, -3/32]$ and increases on $[-3/32, \infty]$.

(ii) We have $f'(x) = -10x + 7$.

Now, $f'(x) > 0$ if $-10x + 7 > 0$, that is, if $x < 7/10$ and $f'(x) < 0$ if $-10x + 7 < 0$, that is, if $x > 7/10$. Thus, $f(x)$ decreases on $[7/10, \infty)$ and increases on $(-\infty, 7/10]$.

(iii) We have $f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$.

See figure 10 and note that if $x < -1$, then both $x + 1$ and $x - 1$ are negative and therefore, $f'(x) > 0$.

If $-1 < x < 1$, then $x + 1 > 0$ and $x - 1 < 0$, therefore $f'(x) < 0$. Finally, if $x > 1$, then both $x + 1$ and $x - 1$ are positive and therefore $f'(x) > 0$.

Thus, $f(x)$ increases on $(-\infty, -1] \cup [1, \infty)$ and decrease on $[-1, 1]$.

		-1		1		
Sign of $(x+1)$	$- - -$		$+ + +$		$+ + +$	
Sign of $(x-1)$	$- - -$		$- - -$		$+ + +$	
$f'(x)$	$+ + +$	0	$- - -$	0	$+ + +$	x

Figure 10

(iv) We have $f'(x) = -6x^2 + 24 = -6(x + 2)(x - 2)$.

		-2		2		
Sign of -6	$- - -$		$- - -$		$- - -$	
Sign of $(x+1)$	$- - -$		$+ + +$		$+ + +$	
Sign of $(x-1)$	$- - -$		$- - -$		$+ + +$	
$f'(x)$	$- - -$	0	$+ + +$	0	$- - -$	x

$f(x)$

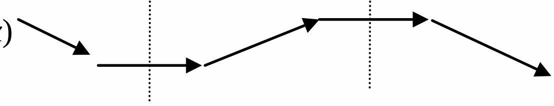


Figure 11

See figure 11 and note that

if $x < -2$ then $x + 2 < 0$, $x - 2$, $0 < 0$ and therefore $f'(x) < 0$.

if $-2 < x < 2$ then $x + 2 > 0$, $x - 2 < 0$, $-6 < 0$ and therefore $f'(x) > 0$.

if $x > 2$, then $x + 2 > 0$, $x - 2 > 0$, $-6 < 0$ and therefore $f'(x) < 0$.

Thus, $f(x)$ decreases on $(-\infty, -2] \cup [2, \infty)$ and increases on $[-2, 2]$.

Example 12 : Determine the values of x for which the following functions are increasing and for which they are decreasing.

(i) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

(ii) $f(x) = (x-1)(x-2)^2$

(iii) $f(x) = (x-4)^3(x-3)^2$

Solution : (i) We have $f'(x) = 4x^3 - 24x^2 + 44x - 24$

$$= 4(x^2 - 6x^2 + 11x - 6)$$

$$= 4(x-1)(x-2)(x-3)$$

Sign of $(x-1)$	$---$	$+++$	$+++$	$+++$
Sign of $(x-2)$	$---$	$---$	$+++$	$+++$
Sign of $(x-3)$	$---$	$---$	$---$	$+++$

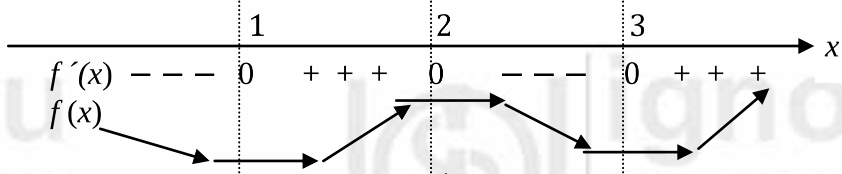


Figure 12

From Figure12 it is clear that

if $x < 1$, then $x - 1 < 0$, $x - 2 < 0$, $x - 3 < 0$ and therefore, $f'(x) < 0$.

if $1 < x < 2$, then $x - 1 > 0$, $x - 2 < 0$, $x - 3 < 0$ and therefore, $f'(x) > 0$,

if $2 < x < 3$, then $x - 1 > 0$, $x - 2 > 0$, $x - 3 < 0$ and therefore, $f'(x) < 0$,

if $x > 3$, then $x - 1 > 0$, $x - 2 > 0$, $x - 3 > 0$ and therefore, $f'(x) > 0$.

Thus, $f(x)$ increases on $[1,2] \cup [3,\infty)$ and decreases on $(-\infty,1] \cup [2,3]$.

(ii) We have $f'(x) = (1)(x-2)^2 + (x-1)(2)(x-2)$

$= (x-2)(x-2+2x-2) = (x-2)(3x-4) = 3(x-4/3)(x-2)$.

Sign of $(x - 4/3)$	$---$	$+++$	$+++$
Sign of $(x-2)$	$---$	$---$	$+++$

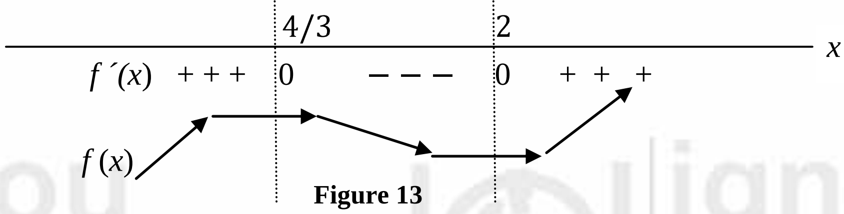


Figure 13

From Figure 13 we note that

if $x < 4/3$, then $x - 4/3 < 0$, $x - 2 < 0$ and therefore, $f'(x) > 0$.

if $4/3 < x < 2$, then $x - 4/3 > 0$, $x - 2 < 0$ and therefore $f'(x) < 0$.

if $x > 2$, then $x - 4/3 > 0$, $x - 2 > 0$ and therefore $f'(x) > 0$.

Thus, $f(x)$ increase on $(-\infty, 4/3] \cup [2, \infty)$ and decreases on $[4/3, 2]$.

(iii) We have $f'(x) = 3(x-4)^2(x-3)^2 + (x-4)^3(2)(x-3)$

$$= (x-4)^2(x-3)\,[3(x-3) + 2(x-4)]$$

$$= (x-4)^2(x-3)(5x-17) = 5(x-4)^2(x-3)(x - 17/5)$$

Since $(x-4)^2 > 0$ for each $x \neq 0$ the sign of $f'(x)$ depends on the sign of $(x-3)(x-17/5)$.

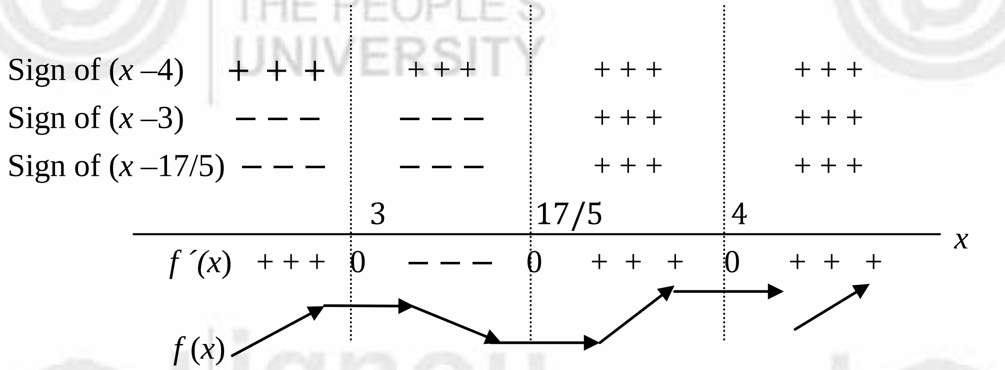


Figure 14

From figure 14 we note that

if $x < 3$, then $x - 3 < 0$, $x - 17/5 < 0$ and therefore, $f'(x) > 0$

if $3 < x < 17/5$, then $x - 3 > 0$, $x - 17/5 < 0$ and therefore $f'(x) < 0$

if $x > 17/5$, $x \neq 4$, then $x - 3 > 0$. $x - 17/5 > 0$ and therefore, $f'(x) > 0$.

Also $f'(4) = 0$.

Thus, $f(x)$ increase on $(-\infty, 3]$, $[17/5, 4]$ and $[4, \infty)$ and decreases $[3, 17/5]$.

Hence, $f(x)$ increase on $(-\infty, 3] \cup [17/5, \infty)$ and decreases on $[3, 17/5]$.

Example 13 : Determine the intervals in which the following functions are increasing or decreasing.

(i) $f(x) = e^{1/x}, x \neq 0$

(ii) $f(x) = \dfrac{1 + x + x^2}{1 - x + x^2}$, $x \in \mathbf{R}$

Solution :

(i) We have, for $x \neq 0$ $f'(x) = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\, e^{\frac{1}{x}}$

As $e^{1/x} > 0$ and $x^2 > 0\ \forall\, x \neq 0$, we get $f'(x) < 0\ \forall\, x \neq 0$.

Thus, $f(x)$ decreases on $(-\infty, 0) \cup (0, \infty)$.

(ii) We have $f'(x) = \dfrac{(1+2x)(1-x+x^2)-(-1+2x)(1+x+x^2)}{(1-x+x^2)^2}$

$$= \frac{2(1-x^2)}{(1-x+x^2)^2}$$

Since $(1-x+x^2)^2 > 0$ $f'(x) > 0$ if $1 - x^2 > 0$ or $-1 < x < 1$ and $f'(x) < 0$ if $1 - x^2 < 0$ that is, if $x < -1$ or $x > 1$.

Thus, $f'(x)$ increases on $[-1,1]$ and decreases on $(-\infty, -1] \cup [1,\infty)$.

Check Your Progress – 2

1. Show that $f(x) = x^2$ is a decreasing function on the interval $[0, \infty)$ using the definition of decreasing function.
2. Let $f(x) = ax + b$, where a and b are real constants. Prove that
   (i) If $a > 0$, then f is an increasing function on **R**.
   (ii) If $a < 0$, then f is an decreasing function on **R**.
3. Determine for which values of x the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing and for which it is decreasing.

4. Determine for which values of x the following functions are increasing and for which they are decreasing.
   (i) $f(x) = x^8 + 6x^2$ (ii) $f(x) = \frac{x}{1+x^2}$
   (iii) $f(x) = x^4 - 4x$ (iv) $f(x) = (x-1)e^x + 2$
5. Determine the values of x for which the following functions are increasing and for which they are decreasing.
   (i) $f(x) = 5x^{3/2} - 3x^{5/2}$ $x > 0$ (ii) $f(x) = \frac{x}{x+3}$ $x \neq -3$
   (ii) $f(x) = x + 1/x$ $x \neq 0$ (iv) $f(x) = x^3 + 1/x^3$ $x \neq 0$

2.4 MAXIMA AND MINIMA OF FUNCTIONS

In this section, we shall study how we can use the derivative to solve problems of finding the maximum and minimum values of a function on an interval.

We begin by looking at the definition of the minimum and the maximum values of a function on an interval.

Definition : Let f be defined on an interval I containing 'c'

1. $f(c)$ is the (absolute) **minimum of f on I** if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the (absolute) **maximum of f on** I if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are called the **extreme values** or **extreme,** of the function on the interval.

Remark : A function need not have a minimum or maximum on an interval. For example $f(x) = x$ has neither a maximum nor a minimum on open interval (0,1). Similarly, $f(x) = x^3$ has neither any maximum nor any minimum value in **R**. See figures 15 and 16.

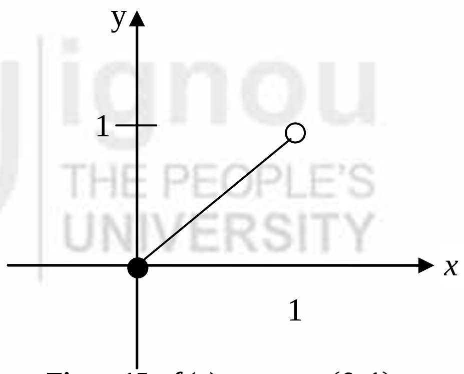


Figure15 : $f(x) = x,\ x \in (0, 1)$

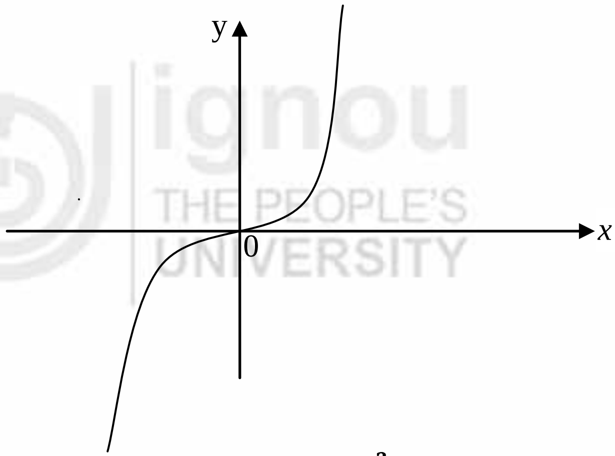


Figure 16 : $f(x)\ : x^3,\ x \in R$

2. If f is a continuous function defined on a closed and bounded interval $[a,b]$, then f has both a minimum and a maximum value on the interval $[a,b]$. This is called the extreme value theorem and its proof is beyond the scope of our course.

   Look at the graph of some function $f(x)$ in figure 17.

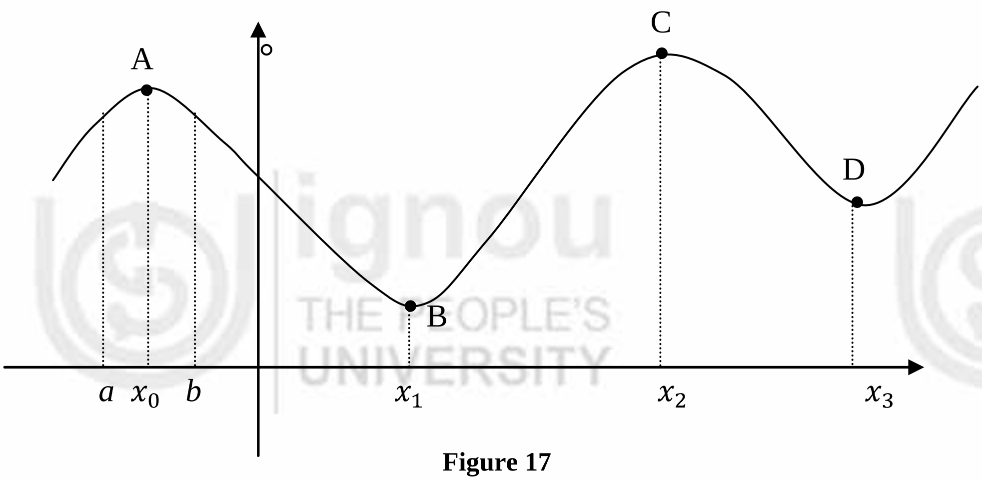


Figure 17

Note that at $x = x_0$, the point A on graph is not an absolute maximum because $f(x_2) > f(x_0)$. But if we consider the interval (a,b), then f has a maximum value at $x = x_0$ in the interval (a,b). Point A is a point of local maximum of f.
Similarly f has a local minimum at point B.

Definition : Suppose f is a function defined on an intervals I. f is said to have a local (relative maximum at $c \in I$ for each $x \in I$ for which $c - h < x < c + h, x \neq c$ we have $f(x) > f(c)$.

Definition : Suppose f is a function defined on an interval I. f is said to have a local (relative minimum at $c \in$ I if there is a positive number h such that for each $x \in I$ for which $c - h < x < c + h$, $x \neq c$ we have $f(x) > f(c)$.

Again Fig. 18 suggest that at a relative extreme the derivative is either zero or undefined. We call the x–values at these special points critical numbers.

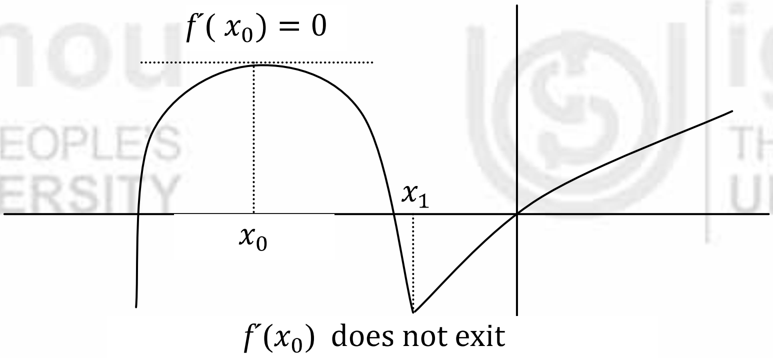


Figure 18

Definition : If f is defined at c, then c is called a critical number if f if $f'(c) = 0$ or f' is not defined at c.

The following theorem which we state without proof tells us that relative extreme can occur only at critical points.

Theorem: If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

If f is a continuous function on interval $[a,b]$, then the absolute extrema of f occur either at a critical number or at the end points a and b. By comparing the values of f at these points we can find the absolute maximum or absolute minimum of f on $[a,b]$.

Example 14 : Find the absolute maximum and minimum of the following functions in the given interval.

(i) $f(x) = x^2$ on $[-3,3]$

(ii) $f(x) = 3x^4 - 4x^3$ on $[-1,2]$

Solution : (i) $f(x) = x^2 \quad x \in [-3,3]$

Differentiating w.r.t. x., we get

$$f'(x) = 2x$$

To obtain critical numbers we set $f'(x) = 0$. This gives $2x = 0$ or $x = 0$ which lies in the interval $(-3,3)$.

Since f' is defined for all x, we conclude that this is the only critical number of f. Let us now evaluate f at the critical number and at the end of points of $[-3,3]$.

$f(-3) = 9$
$f(0) = 0$
$f(3) = 9$

This shows that the absolute maximum of f on $[-3,3]$ is $f(-3) = f(3) = 9$ and the absolute minimum is $f(0) = 0$

(ii) $f(x) = 3x^4 - 4x^3 \quad x \in [-1, 2]$

$f'(x) = 12x^3 - 12x$

To obtain critical numbers, we set $f'(x) = 0$
or $12x^3 - 12x = 0$

which implies $x = 0$ or $x = 1$.

Both these values lie in the interval $(-1, 2)$

Let us now evaluate f at the critical number and at the end points of $[-1,2]$

$f(-1) = 7$
$f(0) = 0$
$f(1) = -1$
$f(2) = 16$

This shows that the absolute maximum 16 of f occurs at $x = 2$ and the absolute minimum -1 occurs at $x = 1$.

First Derivative Test

How do we know whether f has a local maximum or a local minimum at a critical point c ? we shall study two tests to decide whether a critical point c is a point of local maxima or local minima. We begin with the following result which is known **as first derivative test.** This result is stated without any proof.

Theorem : Let c be a critical point for f, and suppose that f is continuous at c and differentiable on some interval I containing c, except possibly at c itself. Then

(i) if f' changes from positive to negative at c, that is, if there exists some $h > 0$ such that $c - h < x < c$, implies $f'(x) > 0$ and $c < x < c + h$ implies $f'(x) < 0$, then f has a local maximum at c.

(ii) if f' changes sign from negative to positive at c, that is, if there exists some $h > 0$ such that $c - h < x < c$ implies $f'(x) < 0$ and $c < x < c + h$ implies $f'(x) > 0$ then f has a local minimum at c.

(iii) if $f'(x) > 0$ or if $f'(x) < 0$ for every x in I except $x = c$ then $f(c)$ is not a local extremum of f.

See the figures 19 and 20 below.

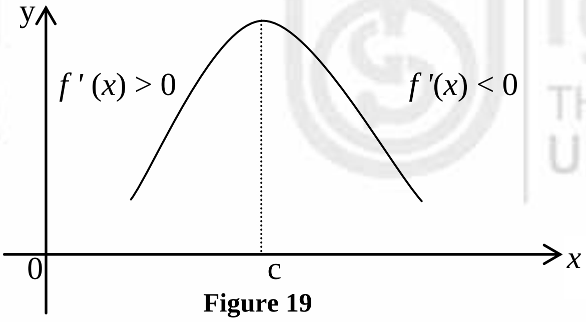


Figure 19

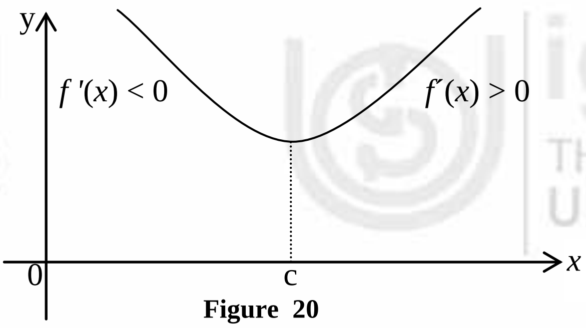


Figure 20

As an illustration of ideas involved, imagine a blind person riding in a car. If that person could feel the car travelling uphill then downhill, he or she would know that the car has passed through a high point of the highway.

Essentially, the sign of derivative $f'(x)$ indicates whether the graph goes uphill or downhill. Therefore, without actually seeing the picture we can deduce the right conclusion in each case.

We summarize the first derivative test for local maxima and minima in the following box.

First Derivative Test for Local Maxima and Minima

Let c be a critical number of f i.e., $f'(c) = 0$

If $f'(x)$ changes sign from positive to negative at c then $f(c)$ is a local maximum. See fig 21.

If $f'(x)$ changes sign from negative to positive at c then $f(c)$ is a local minimum. See fig 22.

Note : $f'(x)$ does not change, sign at c, then $f(c)$ is neither a local maximum nor local minimum.

Local Maximum at c

Case 1

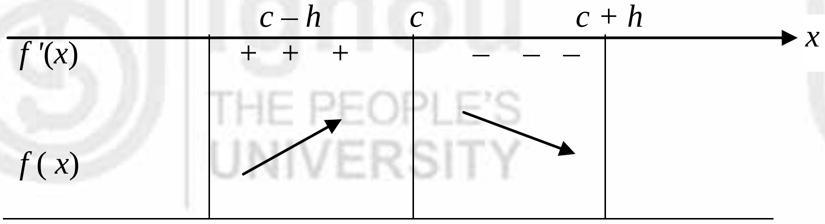


Figure 21

Local Minimum at c

Case 12

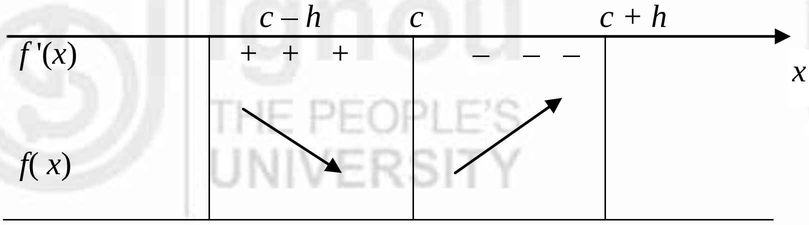


Figure 22

Example 15 : Find the local (relative) extrema of the following functions

(i) $f(x) = 2x^3 + 3x^2 - 12x + 7$ (ii) $f(x) = 1/(x^2 + 2)$

(iii) $f(x) = x\,e^x$ (iv) $f(x) = \frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$

Solution

(i) f is continuous and differentiable on **R**, the set of real numbers. Therefore, the only critical values of f will be the solutions of the equation $f'(x) = 0$.
Now, $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1)$

Setting $f'(x) = 0$ we obtain $x = -2, 1$

Thus, $x = -2$ and $x = 1$ are the only critical numbers of f. Figure 23 shows the sign of derivative f' in three intervals.

From Figure 23 it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 1$, $f'(x) < 0$ and if $x > 1$, $f'(x) > 0$.

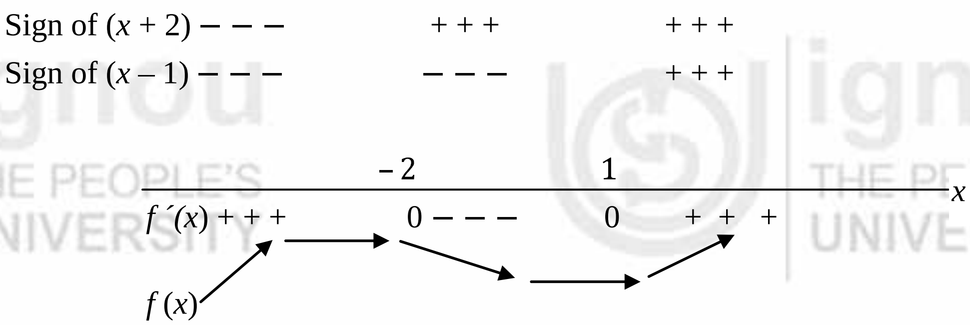


Figure 23

Using the first derivative test we conclude that $f(x)$ has a local maximum at $x = -2$ and $f(x)$ has local minimum at $x = 1$.

Now, $f(-2) - 2 = 2(-2)^3 + 3(-2^2) - 12(-2) + 7 = -16 + 12 + 24 + 7 = 27$ is the value of local maximum at $x = -2$ and $f(1) = 2 + 3 - 12 + 7 = 0$ is the value of local minimum at $x = 1$.

(ii) Since $x^2 + 2$ is a polynomial and $x^2+2 \neq 0 \; \forall \; x \in \boldsymbol{R}$ $f(x) = \frac{1}{x^2+2}$ is continuous and differentiable on **R**, the set of real numbers. Therefore, the only critical values of f will be the solutions of the equation $f'(x) = 0$

$$\text{Now, } f'(x) = \frac{-2x}{(x^2+2)^2}$$

Setting $f'(x) = 0$ we obtain $x = 0$. Thus, $x = 0$ is the only critical number of f. Figure 24 shows the sign of derivative in two intervals.

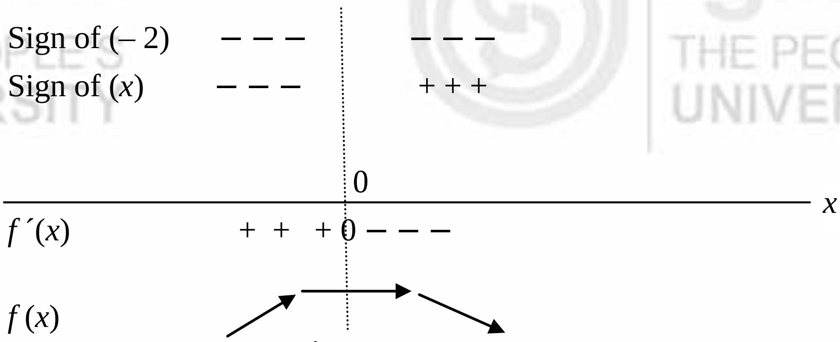


Figure 24

From Figure 24 it is clear that $f'(x) > 0$ if $x < 0$ and $f'(x) < 0$ if $x > 0$

Using the first derivative test, we conclude that $f(x)$ has a local maximum at $x = 0$.

Now since $f(0) = \frac{1}{0^2+2} = \frac{1}{2}$ the value of local maximum at $x = 0$ is $1/2$.

(iii) Since x and e^x are continuous and differentaible on $\boldsymbol{R}$, $f(x) = xe^x$ is continuous and differentiable on **R.**

Therefore, the only critical values of f will be solutions of $f'(x) = 0$.
Now, $f'(x) = x\,e^x + (1)e^x = (x+1)e^x$

Since $e^x > 0 \; \forall \; x \in R$ $f'(x) = 0$ gives $x = -1$. Thus, $x = -1$ is the only critical number of f.

Figure 25 shows the sign of derivative f' in two intervals

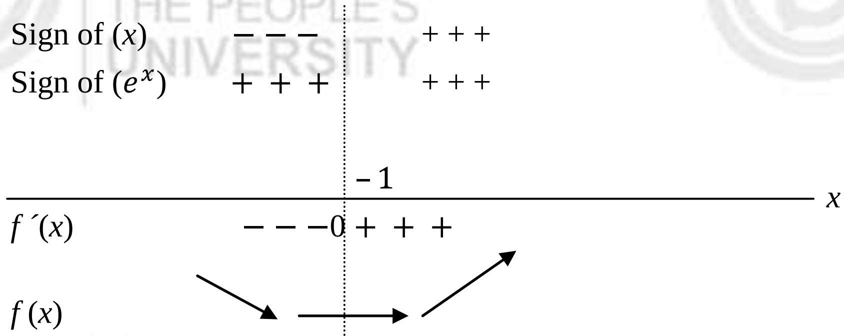


Figure 25

From Figure 25 it is clear that $f'(x) < 0$ if $x < -1$ and $f'(x) > 0$ if $x > -1$

Using the first derivative test we conclude that $f(x)$ has a local minimum at $x = -1$ and the value of local minimum is $f(-1) = (-1)\, e^{-1} = -1/e$.

(iv) Since f is a polynomial function, f is continuous and differentiable on **R.** Therefore, the only critical numbers of f are the solutions of the equation $f'(x) = 0$.

$$\text{We have } f'(x) = -3x^3 - 24x^2 - 54x$$

$$= -3(x^2 + 8x + 15)x$$

$$= -3(x+5)(x+3)x$$

Setting $f'(x) = 0$, we obtain $x = -5, -3, 0$ Thus, $-5, -3$ and 0 are the only critical numbers of f.

Figure 26 shows the sign of derivative f' in four intervals.

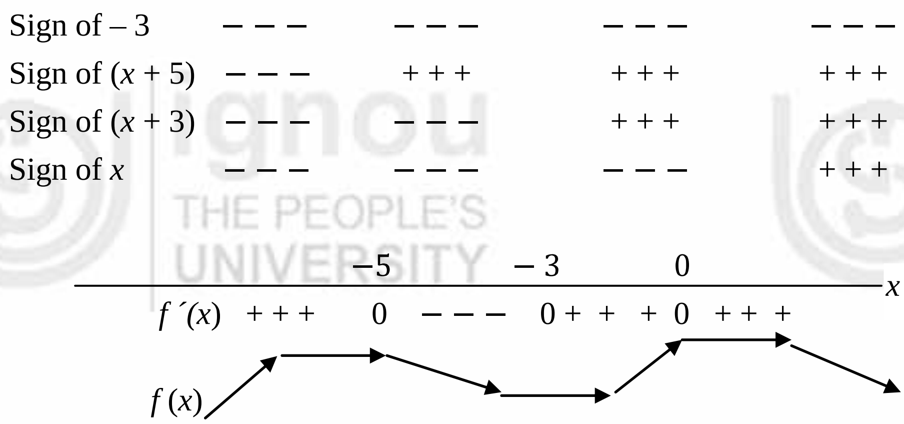


Figure 26

It is clear from figure 26 that $f'(x) > 0$ for $x < -5$, $f'(x) < 0$ for $-5 < x < -3$, $f'(x) > 0$ for $-3 < x < 0$ and $f'(x) < 0$ for $x > 0$

Using the first derivative test we get that $f(x)$ has a local maximum at $x = -5$, a local minimum at $x = -3$ and a local maximum at $x - 0$.

Values of local maximum at $x = -5$ is $f(-5) = 73.75$ and value of local minimum at $x = -3$ is $f(-3) = 57.75$ and the value of local maximum at 0 is $f(0) = 105$.

Second Derivative Test

The first derivative test is very useful for finding the local maxima and local minima of a function. But it is slightly cumbersome to apply as we have to determine the sign of f' around the point under consideration. However, we can avoid determining the sign of derivative f' around the point under consideration, say c, if we know the sign of second derivative f at point c. We shall call it as the second derivative test.

Theorem : (Second Derivative Test)

Let $f(x)$ be a differentiable function on I and let $c \in I$. Let $f'(x)$ be continuous at c. Then

1. c is a point of local maximum if both $f'(c) = 0$ and $f''(c) < 0$.
2. c is a point of local minimum if both $f'(c) = 0$ and $f''(c) > 0$.

Remark : If $f'(c) = 0$ and $f''(c) = 0$, then the second derivative test fails. In this case, we use the first derivative test to determine whether c is a point of local maximum or a point of a local minimum.

We summarize the second – derivative test for local maxima and minima in the following table.

Second Derivative Test for Local Maxima and Minima

$f'(c)$	$f''(c)$	$f(c)$
0	+	Local Minimum
0	–	Local Maximum
0	0	Test Fails

We shall adopt the following guidelines to determine local maxima and minima.

Guidelines to find Local Maxima and Local Minima

The function f is assumed to posses the second derivative on the interval *I*.

Step 1 : Find $f'(x)$ and set it equal to 0.

Step 2 : Solve $f'(x) = 0$ to obtain the critical numbers of *f*.

Let the solution of this equation be α, β, γ ……….

We shall consider only those values of *x* which lie in *I* and which are not end points of *I*.

Step 3 : Evaluate $f''(\alpha)$

If $f''(\alpha) < 0$, $f(x)$ has a local maximum at $x = \alpha$ and its value if $f(\alpha)$

If $f''(\alpha) > 0$, $f(x)$ has a local minimum at $x = \alpha$ and its value if $f(\alpha)$

If $f''(\alpha) = 0$, apply the first derivative test.

Step 4 : If the list of values in Step 2 is not exhausted, repeat step 3, with that value.

Example 16 : Find the points of local maxima and minima, if any, of each of the following functions. Find also the local maximum values and local minimum values.

(i) $f(x) = x^3 - 6x^2 + 9x + 1$, $\quad x \in \boldsymbol{R}$

(ii) $f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$, $\quad x \in \boldsymbol{R}$

(iii) $f(x) = x^3 - 2ax^2 + a^2 x \ (a > 0)$, $\quad x \in \boldsymbol{R}$

Solution : $f'(x) = x^3 - 6x^2 + 9x + 1$

Thus, $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$.

To obtain critical number of *f*, we set $f'(x) = 0$ this yields $x = 1, 3$.

Therefore, the critical number of *f* are $x = 1, 3$.

Now $f''(x) = 6x - 12 = 6(x-2)$

We have $f''(1) = 6(1-2) = -6 < 0$ and $f''(3) = 6(3-2) = 6 > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 3$. The value of local maximum at $x = 1$ is $f(1) = 1 - 6 + 9 + 1 = 5$ and the value of local minimum at $x = 3$ is $f(3) = 3^3 - 6(3^2) + 9(3) + 1 = 27 - 54 + 27 + 1 = 1$.

(ii) We have $f'(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

Thus, $f'(x) = x^5 - 20x^4 + 100x^3 = x^3(x^2 - 20x + 100) = x^3(x-10)^2$

As $f'(x)$ is defined for every value of *x*, the critical number *f* are solutions of $f'(x) = 0$. Setting $f'(x) = 0$, we get $x = 0$ *or* $x = 10$.

Now $\quad f''(x) = 3x^2(x-10)^2 + x^3(2)(x-10)$

$$= x^2(x-10)[3(x-10) + 2x] = 5x^2(x-10)(x-6)$$

We have $f\,'(0) = 0$ and $f\,'(10) = 0$. Therefore we cannot use the second derivative test to decide about the local maxima and minima. We, therefore use the first derivative test.

Figure 27 shows the sign of derivative $f\,'$ in three intervals.

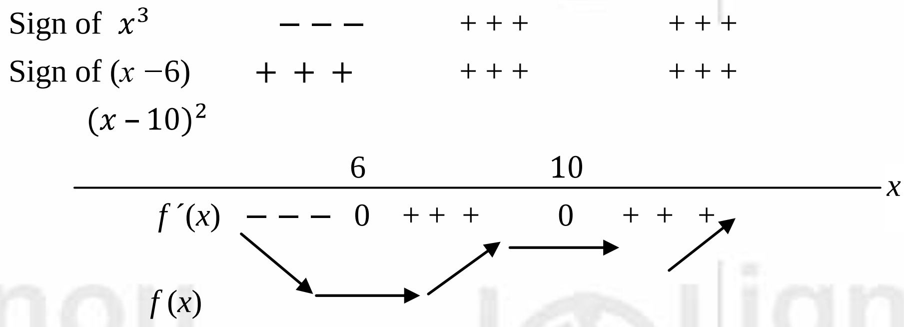


Figure 27

From figure 27 it is clear that $f\,'(x) < 0$ for $x < 0$, $f\,'(x) > 0$ for $0 < x < 10$ and $f\,'(x) > 0$ for $x > 10$

Thus, $f(x)$ has a local minimum at $x = 0$ and its value is $f(0) = 0$. But $f(x)$ has neither a local maximum nor a local minimum at $x = 10$.

(iii) We have $f(x) = x^3 - 2ax^2 + a^2 x \ (a > 0)$,

Thus, $f(x) = 3x^2 - 4ax + a^2 = (3x - a)(x - a)$

As $f\,'(x)$ is defined for each $x \in \boldsymbol{R}$, to obtain critical number of f we set $f\,'(x) = 0$. This yields $x = a/3$ or $x = a$. Therefore, the critical numbers of f and $a/3$ and a. Now, $f\,'(a) = 6x - 4a$.

We have $f\,'\left(\frac{a}{3}\right) = 6\left(\frac{a}{3}\right) - 4a = 2a - 4a = -2a < 0$ and
$f(a) = 6a - 4a = 2a > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = a/3$ and a local minimum at $x = a$. The value of local maximum at

$$x = \frac{a}{3} \ is\ f\left(\frac{a}{3}\right) = \left(\frac{a}{3}\right)^3 - 2a\left(\frac{a}{3}\right)^2 + a^2\left(\frac{a}{3}\right) = \frac{4}{27}a^3$$

and the value of local minimum at $x = a$ is $f(a) = a^3 - 2a.\ a^2 + a.a = 0$

Example 17 : Show that $f(x) = x^2 \ln\left(\frac{1}{x}\right)$ has a local maximum $at\ x = 1/\sqrt{e}$.

Solution : We have $f(x) = x^2 \ln\left(\frac{1}{x}\right) = -x^2 \ln x \quad x > 0$.

Thus, $f\,'(x) = -2x \ln x - x^2\left(\frac{1}{x}\right) = -2x \ln x - x$.

Note that $f\,'(x)$ is defined for each $x > 0$. Since, we have to show that $f(x)$ has a local maximum at $x = 1/\sqrt{e}$, it is sufficient to show that $f\,'(1/\sqrt{e}) < 0$ and

$f'(1/\sqrt{e}) < 0$. We have

$$f'\left(\frac{1}{\sqrt{e}}\right) = 2\left(\frac{1}{\sqrt{e}}\right) \ln\left(e^{-\frac{1}{2}}\right) - \frac{1}{\sqrt{e}} = \frac{-2}{\sqrt{e}}\left(-\frac{1}{2}\right) \ln(e) - \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}} = 0.$$

Now, $f'(x) = -2(1) \ln x - 2x(1/x) - 1 = -2 \ln x - 3$

$$\Rightarrow f'\left(\frac{1}{\sqrt{e}}\right) = -2\ln\left(\frac{1}{\sqrt{e}}\right) - 3 = -2\ln\left(e^{-\frac{1}{2}}\right) - 3 = \ln e - 3 = 1 - 3 = -2 < 0.$$

Thus, $f(x)$ has a local maximum at $x = 1/\sqrt{e}$ and its value is

$$f'\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2e}.$$

Check Your Progress – 3

1. Find the absolute maximum and minimum of the following functions in the given intervals.
   (i) $f(x) = 4x^2 - 7x + 3$ on $[-2, 3]$
   (ii) $f(x) = \frac{x^3}{x+2}$ on $[-1,1]$
2. Using first derivative test find the local maxima and minima of the following functions.
   (i) $f(x) = x^3 - 12x$
   (ii) $f(x) - \frac{x}{2} + \frac{2}{x}, x > 0$

3. Use second derivative test to find the local maxima and minima of the following functions.
   (i) $f(x) = x^3 - 2x^2 + x + 1$ $\quad x \in \mathbf{R}$
   (ii) $f(x) = x + 2\sqrt{1-x}$ $\quad x \leq 1$

2.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. We are given the rate of change of the side of the square and require the rate of change of the perimeter.

   Let s cm be the side of the square and P cm be the perimeter of the square at time t. Thus
   $$P = 4s$$
   Differentiating both the sides with respect to t, we get

   $$\frac{dP}{dt} = 4\frac{ds}{dt}$$

   But $\frac{ds}{dt} = 0.2$ therefore, $\frac{dP}{dt} = 4(0.2) = 0.8$

   Thus, the perimeter of the square is increasing at the rate of 0.8 cm/s.

2. We are given that the rate of change of radius and require the rate of change of the circumference.

   Let r cm be the radius of the circle and C cm be circumference of the circle at time t.

   Thus, $C = 2\pi r$

   Differentiating both sides with respect to t, we get

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}. \text{ But } \frac{dr}{dt} = 0.7, \text{ therefore, } \frac{dC}{dt} = 2\pi(0.7) = 1.4\pi.$$

   Thus, the circumference of the circle is increasing at the rate of $1.4\pi \; cm/s$.

3. Let x be the distance of the man from the base of the light post, and y be the length of the shadow (see figure 28)

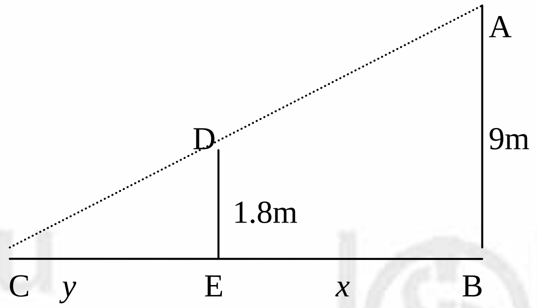


Figure 28

$$\Delta \text{ABC is simliar to } \Delta \text{ DEC. Hence, } \frac{AB}{DE} = \frac{BC}{CE} \text{ which implies } \frac{9}{1.8} = \frac{x+y}{x}$$

$\Rightarrow 5y = x + y$ *or* $4y = x$ *or* $y = x/4$.

Differentiating both the sides with respect to t, we get $\frac{dy}{dt} = \frac{1}{4}\frac{dx}{dt}$.

But we are given that $\frac{dx}{dt} = 2$. Thus, $\frac{dy}{dt} = \frac{1}{4}(2) = 0.5$

   This shows that the shadow is lengthening at the rate of 0.5 m/s.

4. The fluid in the funnel forms a cone with radius r, height h and volume V (figure 29). Recall that the volume of the cone is given by $\frac{1}{3}\pi r^2 h$.

   Therefore, $V = \frac{1}{3}\pi r^2 h$

Note that ΔAOB and ΔADC are similar.

Therefore,

$$\frac{AO}{OB} = \frac{AD}{DC}$$

$$or\ \frac{20}{4} = \frac{h}{r}$$

$$or\ r = \frac{1}{5}\ h$$

$$\text{Thus, } V = \frac{1}{3}\pi\left(\frac{1}{5}h\right)^2 = \frac{1}{75}\pi h^3$$

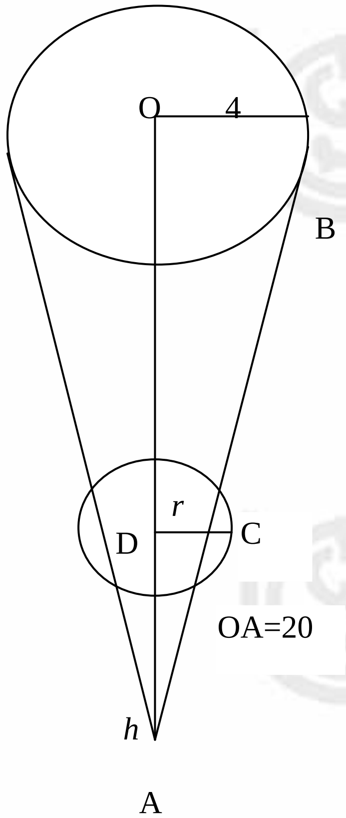


Figure 29

Differentiating both the sides with respect to t, we get

$$\frac{dV}{dt} = \frac{1}{75}\ \pi(3h^2)\ \frac{dh}{dt} = \frac{\pi h^2}{25}\frac{dh}{dt}$$

We are given that $\frac{dV}{dt} = -12$ (Negative sign is due to decrease in volume)

$$\text{Therefore,} -12 = \frac{\pi h^2}{25}\frac{dh}{dt}\ \text{Or}\ \frac{dh}{dt} = \frac{-300}{\pi h^2}$$

$$\text{Thus, when } h = -5, \quad \frac{dh}{dt} = \frac{-300}{\pi 25} = \frac{-12}{\pi}$$

This shows, that the fluid is dropping at the rate of $12/\pi$cm/s.

Check Your Progress 2

1. First note that the domain of f consists of non–negative real number. Now, let $0 \leq x_1 < x_2$.

   Since $x_1 < x_2$ and $x_1 \geq 0$, it follows that
   $$x_1^2 \leq x_1 x_2 \qquad \text{....(1)}$$

   Again, $x_1 < x_2$ and $x_2 > 0$, it follows that
   $$x_1 x_2 < x_2^2 \qquad \text{....(2)}$$

   From (1) and (2), we get
   $$x_1^2 < x_1 x_2 < x_2^2$$
   $$i.e., \quad x_1^2 < x_2^2$$
   which means $f(x_1) < f(x_2)$

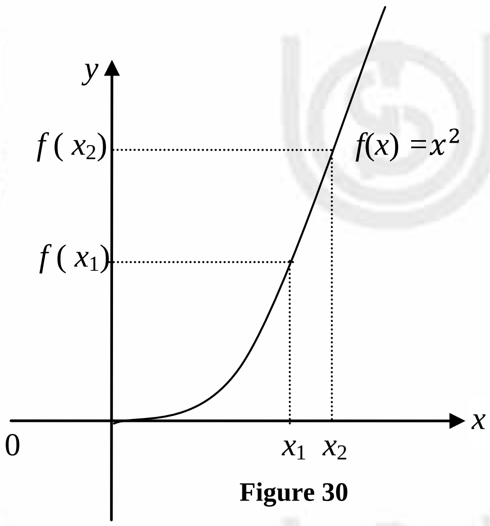


Figure 30

   Hence, f is an increasing function on the interval $(0, \infty)$. See also figure 30

2. (a) Let $a > 0$

Suppose $x_1, x_1 \in \mathbf{R}$ and $x_1 > x_2$ As $a > 0$,

we get $a\,x_1 > a\,x_2$

$\Rightarrow a\,x_1 + b > a\,x_2 + b \;\Rightarrow\; f(x_1) > f(x_2)$

Thus, f increase on $\mathbf{R}$. See figure 31.

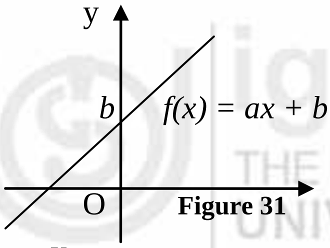


Figure 31

(b) Next, let $a < 0$.

Suppose $x_1, x_2 \in \mathbf{R}$ *and* $x_1 > x_2$. As $a < 0$.

we get $a\,x_1 < ax_2 \quad \Rightarrow$

$a\,x_1 + b < a\,x_2 + b \Rightarrow f(x_1) < f(x_2)$.

Thus, f decreases on $\mathbf{R}$. See figure 32.

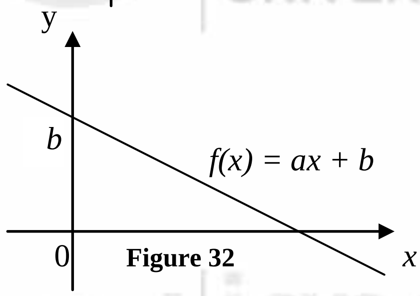


Figure 32

3. We have $f'(x) = 6\,x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$

	$x < 2$		$2 < x < 3$		$x > 3$
Sign of $(x-2)$	$- - -$		$+ + +$		$+ + +$
Sign of $(x-3)$	$- - -$		$- - -$		$+ + +$
x		2		3	
$f'(x)$	$+ + +$	0	$- - -$	0	$+ + +$

$f(x)$

Figure 33

See figure 33 and note that

If $x < 2$, then $x - 2 < 0$, $x - 3 < 0$ and therefore, $f'(x) > 0$,

If $2 < x < 3$, then $x - 2 > 0$, $x - 3 < 0$ and therefore, $f'(x) < 0$,

If $x > 3$, then $x - 2 > 0$, $x - 3 > 0$ and therefore, $f'(x) > 0$.

Thus, $f(x)$ increases on $(-\infty, 2] \cup [3, \infty)$ and decreases on $[2,3]$.

4. (i) We have $f'(x) = 8\,x^7 + 12\,x = 4x\,(2\,x^6 + 3)$.

Since $2x^6 + 3 > 0\ \forall\, x \in R$ $f'(x) < 0$ and $x < 0$ and $f'(x) > 0$ for $x > 0$.

Thus, $f(x)$ decreases for $x \le 0$ and increases for $x \ge 0$.

(ii) We have $f'(x) = \dfrac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2} = \dfrac{-(x+1)(x-1)}{(1+x^2)^2}$

	$x < -1$		$-1 < x < 1$		$x > 1$
Sign of (-1)	$- - -$		$- - -$		$- - -$
Sign of $(x-1)$	$- - -$		$+ + +$		$+ + +$
Sign of $(x-1)$	$- - -$		$- - -$		$+ + +$
x		-1		1	
$f'(x)$	$- - -$	0	$+ + +$	0	$- - -$

$f(x)$

Figure 34

Referring to Figure 33, we get $f'(x) < 0$ *if* $x < -1$ *or* $x > 1$
and $f'(x) > 0$ if $-1 < x < 1$
Thus, $f'(x) > 0$ if $x < -1$ *or* $x > 1$ and $f'(x) < 0$ if $-1 < x < 1$.
Hence, $f(x)$ decreases on $(-\infty, -1] \cup [1, \infty)$ and increases on $[-1,1]$.

(iii) We have $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(x^2 + x + 1)$

[using $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$]

$$= 4(x-1)[x^2 + 2\left(\frac{1}{2}\right)x + 1] = 4(x-1)[(x+\frac{1}{2})^2 + \tfrac{3}{4}]$$

Since $(x + 1/2)^2 + 3/4 > 0 \ \forall\, x \in \mathbf{R}$, *we get* $f'(x) < 0$ if $x < 1$
and $f'(x) > 0$ if $x > 1$.
Thus, $f'(x)$ decrease on $(-\infty, 1]$ and increase on $[1, \infty)$

(iv) We have $f'(x) = (x-1)\,e^x + (1)\,e^x = (x - 1 + 1)\,e^x = x\,e^x$.

As $e^x > 0 \ \forall\ x \in \boldsymbol{R}$ $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ *for* $x < 0$.

Thus, $f(x)$ increases for $x \geq 0$ and decreases for $x \leq 0$

5. (i) We have, $x > 0$, $f'(x) = \frac{15}{2}x^{1/2} - \frac{15}{2}x^{3/2} = \frac{15}{2}\sqrt{x}\,(1-x)$.

As $\sqrt{x} > 0$, for $x > 0$ we get $f'(x) > 0$ if $0 < x < 1$ and $f'(x) < 0$ if $x > 1$.

Thus, $f'(x) > 0$ if $0 < x < 1$ and $f'(x) < 0$ if $x > 1$.

Hence, $f(x)$ increases for $0 < x \leq 1$ and decreases for $x \geq 1$.

(ii) We have $x \neq -3$ $f'(x) = \dfrac{(x+3)(1) - x(1)}{(x+3)^2} = \dfrac{3}{(x+3)^2}$

$\Rightarrow f'(x) > 0$ for $x \neq -3$. Thus, $f(x)$ increases on $(-\infty, -3) \cup (-3, -\infty)$

(iii) We have for $x \neq 0$ $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2} = \dfrac{(x+1)(x-1)}{x^2}$

Since $x^2 > 0$ for $x \neq 0$ $f'(x) > 0$ if $(x+1)(x-1) > 0$ $x \neq 0$
and $f'(x) < 0$ if $(x+1)(x-1) < 0$, $x \neq 0$.

Note that $f'(x) > 0$ if $x < -1$ *or* $x > 1$ and $f'(x) < 0$ if $-1 < x < 0$ *or* $0 < x < 1$.

Thus, $f(x)$ increases on $(-\infty, -1] \cup (1, \infty)$ and decreases on
$[-1, 0] \cup (0,1]$

(iv) We have $f'(x) = 3x^2 - \dfrac{3}{x^4} = \dfrac{3(x^6 - 1)}{x^4} = \dfrac{3(x^2-1)(x^4 + x^2 + 1)}{x^4}$

[$\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)$]

Since $x^4 + x^2 + 1 > 0$ and $x^4 > 0 \; for \; x \neq 0,$ we get $f'(x) > 0$ $if \; x^2 - 1 > 0$ and $f'(x) < 0 \; if \; x^2 - 1 < 0$. As in (iii), we get *f(x)* increases on (–∞, –1] ∪ [1, ∞) and decreases on [–1, 0] ∪ (0,1].

Check Your Progress 3

1. (i) *f(x)* = $4\,x^2 - 7\,x + 3,\;\; x \in$ [–2,3]

Differentiating with respect to *x*, we get

f '(x) = 8*x* – 7

To obtain critical numbers, we set *f ' (x)* = 0. This gives 8*x* –7 = 0 or *x* = 7/8, which lies in the interval (–2,3). Since *f '* is defined for all *x*, we conclude that this is the only critical number of *f*. We now evaluate *f* at the critical number and at the end points of [–2, 3].

We have

f (–2) = $4(-2)^2$ + 7(–2) + 3 = 33

$$f\left(\frac{7}{8}\right) = f\left(\frac{7}{8}\right)^2 - 7\left(\frac{7}{8}\right) + 3 = -\frac{1}{16}$$

f(3) = $4(3)^2 - 7(3)$ + 3 = 18

This shows that the absolute maximum of *f* on [–2,3] is *f* (–2) = 33 and the absolute minimum is *f* (7/8) = $-\frac{1}{16}$.

(ii) $f(x) = \dfrac{x^3}{x+2},\;\; x \in [-1,1]$

Differentiating w.r.t. *x*, we get

$$f(x) = \frac{2x^2(x+3)}{(x+2)^2}$$

Note that *f '(x)* is not defined for *x* = –2. However, it does not lie in the interval (–1,1).

To obtain critical number of *f* we set *f* '(*x*) = 0. This give*s* $2x^2(x+3) = 0$ which implies *x* = *0* or *x* = –3 Since –3 does not lie in (–1, 1), 0 is the only critical number of *f*. We therefore evaluate *f* at –1, 0 and 1. Now

f (–1) = –1
f (0) = 0
and *f* (1) = 1/3

This shows that *f* has the absolute maximum at *x* = 1 and the absolute minimum at *x* = –1.

2. (i) $f(x) = x^3 - 12x$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 2(x-2)(x+2)$

Setting $f'(x) = 0$, we obtain $x = 2, -2$ Thus, $x = -2$, and $x = 2$ are the only critical numbers of f. Fig. 35 shows the sign of derivative f' in three intervals.

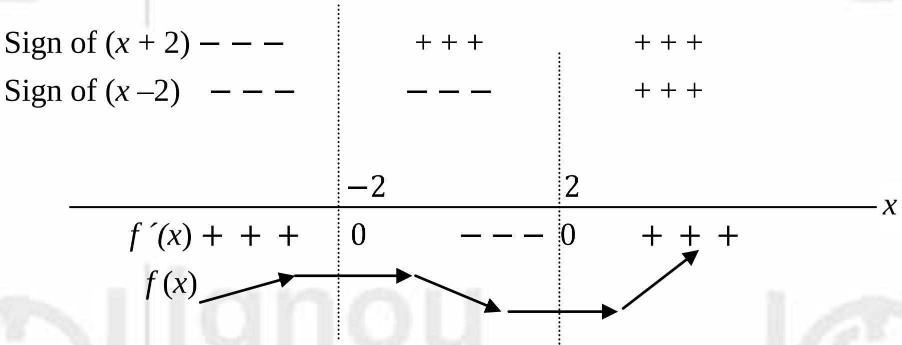


Figure 35

From figure 35 it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 2$, $f'(x) < 0$ and if $x > 2$, $f'(x) > 0$.

Using the first derivative test, we conclude that

$f(x)$ has a local maximum at $x = -2$ and a local minimum *at* $x = 2$.

Now, $f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$ is the value of local maximum at $x = -2$ and $f(2) = 2^3 - 12(2) = 8 - 24 = -16$ is the value of the local minimum at $x = 2$.

(ii) We have $f(x) = \dfrac{x}{2} + \dfrac{2}{x} = \dfrac{1}{2}x + 2x^{-1}, x > 0.$

Thus, $f'(x) = \dfrac{1}{2} + 2(-1)x^{-2} = \dfrac{1}{2} - \dfrac{2}{x^2} = \dfrac{x^2 - 4}{2x^2} = \dfrac{(x+2)(x-2)}{2x^2}$

Note that $f'(x)$ is defined for each $x > 0$, therefore to obtain the critical numbers of f, we set $f'(x) = 0$. This gives us $(x + 2)(x - 2) = 0$ or $x = -2, 2$. As domain of f is $\{x | x > 0\}$, the only critical number of f lying in the domain of f is 2.

Also, since $2x^2 > 0$ and $x + 2 > 0$ $\forall$ $x > 0$, the sign of $f'(x)$ is determined by the sign of the factor $x - 2$.

Figure 36 give the sign of $f'(x)$ in two intervals.

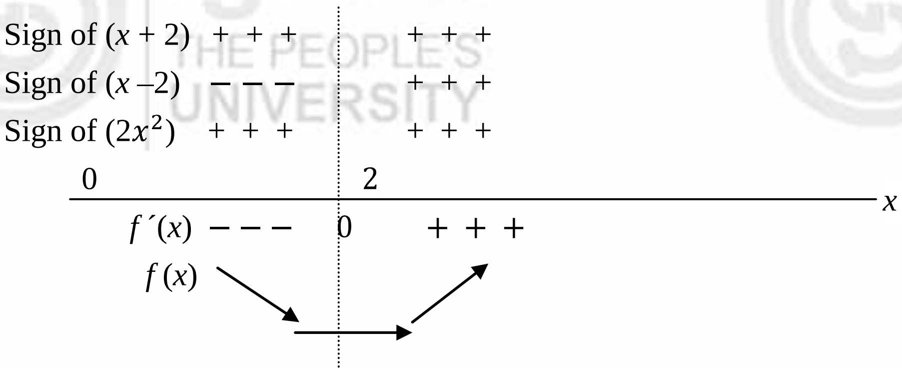


Figure 36

From figure 36 it is clear that $f'(x) < 0$ for $0 < x < 2$ and $f'(x) > 0$ for $x > 2$. Thus, $f(x)$ has a local minimum at $x = 2$ and its value is

$$f(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2.$$

3. (i) $f(x) = x^2 - 2x^2 + x + 1, \quad x \in R$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 4x + 1$
$= (3x - 1)(x - 1)$

For obtaining critical numbers, we set $f'(x) = 0$

or $(3x - 1)(x - 1) = 0$

or $x = 1/3$ and $x = 1$

So, $x = 1$ and $x = 1/3$ are the only critical number of f.

Now $f(x) = 6x - 4$

We have

$f'(1) = 6(1) - 4 = 2 > 0$

and $f'(1/3) = 6(1/3) - 4 = -2 < 0$
Using the second derivative test we see that $f(x)$ has a local maximum at $x = 1/3$ and a local minimum at $x = 1$.

Now $f'(1/3) = 31/27$ and $f(1) = 1$

Hence, f has a local maximum 31/27at $x = 1/3$ and a local minimum 1 at $x = 1$.

(ii) We have $f(x) = x + 2\sqrt{1 - x} = x + (1 - x)^{1/2} \quad x \le 1$

Thus, $f'(x) = 1 - (1 - x)^{-1/2} \ (-1) = 1 - \frac{1}{\sqrt{1-x}}$
Note that $f'(x)$ is defined for each $x < 1$

To obtain the critical numbers of f, we set $f'(x) = 0$ but $f'(x) = 0$ implies
$1 - \frac{1}{\sqrt{1-x}}$ or $\sqrt{1-x} = 1$ or $1 - x = 1$ or $x = 0$.

This lies in the domain of f. Therefore, the only critical of f is $x = 0$.

Now, $\quad f(x) = 0 - \left(-\frac{1}{2}\right)(1 - x)^{-3/2}(-1) = \dfrac{-1}{(1 - x)^{-3/2}}$

We have $f(0) = \dfrac{-1}{2(1 - 0)^{-3/2}} = -\dfrac{1}{2} < 0$
Using the second derivative test we see that $f(x)$ has a local maximum at $x = 0$. The value of local maximum at $x = 0$ $f(0) = 0 + \sqrt{1 - 0} = 1$

2.6 SUMMARY

The unit is, as suggested by the title, on applications of differential calculus. In **section 2.2**, the concept of 'rate of change' of a derivable variable/quantity is introduced and illustrated with a number of examples. In **section 2.3**, concepts of 'increasing function' and 'decreasing function' are discussed and explained with a number of examples. In **section 2.4**, methods for finding out (local) maxima and minima, are discussed and explained with examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.5**.

UNIT 3 INTEGRATION

Structure

3.0 INTRODUCTION

In Unit 1, we were primarily concerned with the problem of **finding the derivative of given function.** In this unit, we take up the inverse problem, that of finding the original function when we are given the derivative of a function. For instance, we are interested in finding the function F if we know that $F'(x) = 4x^3$. From our knowledge of derivative, we can say that

$$F(x) = x^4 \text{ because } \frac{d}{dx}[x^4] = 4x^3$$

We call the function F an antiderivative of F′ or F(x) is an antiderivative of f. Note that antiderivative of a function is not unique. For instance, x^4+1, x^4+23 are also antiderivatives of $4x^3$. In general, if f(x) is an antiderivative of f (x), then F(x) + c, where C is an arbitrary constant is also an antiderivative of *f*.

3.1 OBJECTIVES

After studying this Unit, you should able to:

- define antiderivative of a function;
- use table of integration to obtain antiderivative of some simple functions;
- use substitution to integrate a function; and
- use formula for integration by parts.

3.2 BASIC INTEGRATION RULES

If F(x) is an antiderivative of f(x) we write

$$\int \underbrace{f(x)}_{\text{Integrand}} \underbrace{dx}_{\text{Variable of Integration}} = F(x) + C \longleftarrow \text{Constant of Integration}$$

We read $\int f(x)dx$ is the antiderivative of f with respect to x. The differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

Note that

$$\int F'(x)dx = F(x) + c \quad \text{and}$$

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

In this sense the integration is the inverse of the differentiation and differentiation is the inverse of integration.

We use the above observations to obtain the following basic rules of integration.

Basic Integration Rules

Table

Differentiation Formula	Integration Formula
1. $\frac{d}{dx}\ k\ = 0$	1. $\int 0\ dx = k$
2. $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	2. $\int x^n\ dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$
3. $\frac{d}{dx}\ In\,\lvert x\rvert\ = \frac{1}{x}$	3. $\int \frac{1}{x}\ dx = In\,\lvert x\rvert + c$
4. $\frac{d}{dx}\left[e^x\right] = e^x$	4. $\int e^x dx = e^x + c$
5. $\frac{d}{dx}\left[a^x\right] = a^x \ln a$	5. $\int a^x dx = \frac{a^x}{\ln a}, a > 0, a \neq 1$
6. $\frac{d}{dx}\ kf(x)\ = kf'(x)$	6. $\int kf(x) = k\int f(x) + c$
7. $\frac{d}{dx}\ f(x) \pm g(x)$ $= f'(x) \pm g(x)$	7. $\int f(x) \pm k\int f(x) + c =$ $\int f(x)dx \pm \int g(x)dx$

The general pattern of integration is as follows:

Given integral
↓
Rewrite
↓
Integrate
↓
Simplify

Illustration

$$\int \left(\frac{3}{x^4} + \frac{2}{x^2} - \frac{4}{x}\right)\ dx$$

$$= 3\int x^{-4}dx + 2\int x^{-2}dx - 4\int \frac{1}{x}dx \qquad \text{[Rewrite]}$$

$$= \frac{3x^{-4+1}}{-4+1} + 2\frac{x^{-2+1}}{-2+1} - 4\ln|x| + \text{c} \qquad \text{[Integrate]}$$

$$= -\frac{1}{x^3} - \frac{2}{x} - 4\ln|x| + c \qquad \text{[Simplify]}$$

Solved Examples

Example 1: Evalutate

$$\int (2x^{1/2} + 3x^{1/3} - 4x^{1/4})\ dx$$

Solution :

$$\int (2x^{1/2} + 3x^{1/3} - 4x^{1/4})\ dx$$

$$= 2\int x^{\frac{1}{2}}dx + 3\int x^{\frac{1}{3}}dx - 4\int x^{\frac{1}{4}}dx \qquad \text{[Rewrite]}$$

$$= 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 3\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 4\frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1} + c \qquad \text{[Integrate]}$$

$$= \frac{4}{3}x^{\frac{3}{2}} + \frac{9}{4}x^{\frac{4}{3}} - \frac{16}{5}x^{\frac{5}{4}} + c \qquad \text{[Simplify]}$$

Example 2 : Evalutate

$$\int \frac{\left(\sqrt{x} + x^{1/3}\right)^2}{x} dx$$

Solution :

$$\int \frac{\left(\sqrt{x} + x^{1/3}\right)^2}{x} dx$$

$$= \int \frac{1}{x}\left[\left(\sqrt{x}\right)^2 + 2\left(\sqrt{x}\right)(x^{\frac{1}{3}}) + \left(x^{\frac{1}{3}}\right)^2\right] dx$$

$$= \int \frac{1}{x}\left[x + 2x^{\frac{1}{2}+\frac{1}{3}} + x^{\frac{2}{3}}\right] dx$$

$$= \int \left[1 + 2x^{\frac{5}{6}-1} + x^{\frac{2}{3}-1}\right] dx$$

$$= \int [1 + 2x^{-1/6} + x^{-1/3}] dx$$

$$= x + \frac{2x^{--\frac{1}{6}+1}}{(-\frac{1}{6}+1)} + \frac{x^{-\frac{1}{3}+1}}{(-\frac{1}{3}+1)} + c$$

$$= x + \frac{12}{5}x^{5/6} + \frac{3}{2}x^{2/3} + c$$

Example 3 : Evalaute

$$\int \frac{2^x + 3^x}{5^x} dx$$

Solution:

$$\int \frac{2^x + 3^x}{5^x} dx = \int \left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right) dx$$

$$= \int \left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right] dx$$

$$= \frac{\left({}^2/_5\right)^x}{\ln\left({}^2/_5\right)} + \frac{\left({}^3/_5\right)^x}{\ln\left({}^3/_5\right)} + c$$

Example 4 : Evaluate

$$\int \frac{(a^x + b^x)^2}{a^x b^x}\, dx$$

Solution We have

$$\frac{(a^x + b^x)^2}{a^x b^x} = \frac{(a^x)^2 + (b^x)^2 + 2a^x b^x}{a^x b^x}$$

$$= \frac{(a^x)^2}{a^x b^x} + \frac{(b^x)^2}{a^x b^x} + \frac{2a^x b^x}{a^x b^x}$$

$$= \frac{a^x}{b^x} + \frac{b^x}{a^x} + 2$$

$$= \left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2$$

Thus,

$$\int \frac{(a^x + b^x)^2}{a^x b^x} dx = \int \left[\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2\right] dx$$

$$= \frac{\left(\frac{a}{b}\right)^x}{\ln\left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\ln\left(\frac{b}{a}\right)} + 2x + c$$

Example 5 : Evaluate

$$\int (e^{a \ln x} + e^{x \ln a})\, dx$$

Solution : We know that

$$e^{a \ln x} = x^a$$

and $e^{x \ln a} = x^a$

Thus, $\int (e^{a \ln x} + e^{x \ln a})\, dx$

$$= \int (x^a + a^x) dx$$

$$= \frac{x^{a+1}}{a+1} + \frac{a^x}{\ln a} + c$$

Check Your Progress – 1

Integrate the following functions.

1. $x^3 + 2^x$ 2. $x^e + e^x$

3. $(\sqrt{x} + x^2)/x^2$ 4. $(2^x + 3^x)^2/5^x$

5. $3^x + x^7 - 2/x^4$ 6. $(3^x + 5^x)/7^x$

Answers

1. $\frac{1}{4}x^4 + \frac{2^x}{ln2} + c$ 2. $\frac{x^{e+1}}{e+1} + e^x + c$

3. $\ln|x| + x + 4x^{1/2} + c$

4. $\left(\frac{4}{5}\right)^x \frac{1}{\ln(4/5)} + \left(\frac{9}{5}\right)^x \frac{1}{\ln(9/5)} + 2(6/5)^x \frac{1}{\ln(6/5)} + c$

5. $\frac{3^x}{\ln 3} + \frac{x^8}{8} + \frac{2}{3x^3} + c$ 6. $\left(\frac{3}{7}\right)^x \frac{1}{\ln(3/7)} + \left(\frac{5}{7}\right)^x \frac{1}{\ln(5/7)} + c$

3.3 INTEGRATION BY SUBSTITUTION

If the integrand is of the form $\int f(g(x))g'(x)dx,$ we can integrate it by substituting g(x) = t. We illustrate the technique in the following illustration.

Illustration: Integrate $e^x(e^x + 2)^7$ To integrate this function, we put

$e^x + 2 = t \Rightarrow e^x dx = dt$

Thus,

$$\int e^x(e^x + 2)^7 dx = \int t^7 \, dt$$

$$= \frac{1}{8}t^8 + c$$

$$= \frac{1}{8}(e^x + 2)^8 + c$$

Solved Examples

Example 6 : Evaluate

$$\int \sqrt{7x - 2}\ dx$$

Solution : To evaluate this integral,

We put $7x - 2 = t^2$

$$\Rightarrow 7dx = 2tdt \text{ or } dx = \frac{2}{7}\, t\, dt$$

$$\therefore \int \sqrt{7x-2}\, dx = \int \sqrt{t^2}\, \frac{2}{7} t\, dt = \frac{2}{7} \int t^2\, dt$$

$$= \frac{2}{7}\left(\frac{1}{3}\right) t^3 + c = \frac{2}{21}\, t^3 + c$$

$$= \frac{2}{21} (7x-2)^{3/2} + c$$

Example 7 : Evaluate

$$\int x^2 \sqrt{5x-3}\, dx$$

Solution : In this case, again, we put

$5x - 3 = t^2 \Rightarrow 5\, dx = 2tdt$

$$\therefore dx = \frac{2}{5}\, t\, dt$$

Also, $x = \frac{1}{5}\, (t^2 + 3)$

Thus,

$$\int x^2 \sqrt{5x-3}\, dx = \frac{1}{5} \int (t^2 + 3)\sqrt{t^2}\, \frac{2}{5}\, t\, dt$$

$$= \frac{2}{25} \int (t^2 + 3) t^2 dt$$

$$= \frac{2}{25} (t^4 + 3t^2) dt$$

$$= \frac{2}{25}\left(\frac{1}{5} t^5 + \frac{3t^3}{3}\right) + c$$

$$= \frac{2}{125} (t^5 + 5t^3) + c$$

$$= \frac{2}{125} \left[(5x-3)^{5/2} + 5(5x-3)^{3/2}\right] + c$$

Example 8 : Evaluate

$$\mathrm{I} = \int \frac{dx}{(3x-2)^2}$$

Solution : Put $3x - 2 = t \Rightarrow 3dx = dt$, so that

$$I = \frac{1}{3}\int \frac{dt}{t^2} = \frac{1}{3}\int t^{-2}\ dt$$

$$= \frac{1}{3}\int \frac{t^{-2+1}}{-2+1} + c = -\frac{1}{3t} + c$$

$$= -\frac{1}{3(3x-2)} + c$$

Example 9 : Evaluate

$$\int (x+1)\, e^x (xe^x + 3)^4 dx$$

Solution : Put $x\, e^x + 3 = t$

$\Rightarrow (x\, e^x + e^x)\, dx = dt$

or $(x+1)\, e^x\, dx = dt$

Thus,

$$\int (x+1) e^x\, (xe^x + 3)^4 dx$$

$$= \int t^4\ \mathrm{dt} = \frac{1}{5} t^5 + \mathrm{c} = \frac{1}{5}(x\, e^x + 3)^5 + c$$

Example 10 : Evaluate the integral

$$\int \frac{2e^x + 3e^{-x}}{3\, e^x + 4\, e^{-x}}\, dx$$

Solution : **Remark** To evaluate an integral of

$$\int \frac{ae^x + be^{-x}}{c\, e^x + d\, e^{-x}}\, dx$$

We write

Numerator = α (Denominator) $+ \beta \frac{d}{dx}$ (Denominator)

and obtain values of α and β, by equating coefficients of e^x *and* e^{-x}

In the present case, we write

$2\, e^x + 3\, e^{-x} = \alpha\, (3e^x + 4\, e^{-x}) + \beta \frac{d}{dx} (3\, e^x - 4\, e^{-x})$

$\Rightarrow 2\, e^x + 3\, e^{-x} = \alpha\, (3e^x + 4\, e^{-x}) + \beta\, (3\, e^x - 4\, e^{-x})$

Equating coefficients of e^x and e^{-x}, we obtain

$$2 = 3\alpha + 3\beta$$

and $\quad 3 = 4\alpha - 4\beta$

$\Rightarrow \quad \alpha + \beta = 2/3$ and $\alpha - \beta = 3/4$

Adding, we obtain

$$2\alpha = \frac{2}{3} + \frac{3}{4} \quad or\ \alpha = \frac{17}{24}$$

$$\therefore \quad \beta = \frac{2}{3} - \alpha = \frac{2}{3} - \frac{17}{24} = -\frac{1}{24}$$

Thus,

$$\int \frac{3e^x - 4e^{-x}}{3e^x + 4e^{-x}}\,dx = \int \frac{\left(\frac{17}{24}\right)(3e^x + 4e^{-x}) + \left(-\frac{1}{24}\right)(3e^x - 4e^{-x})}{3e^x + 4e^{-x}}\,dx$$

$$= \left(\frac{17}{24}\right)\int dx - \left(\frac{1}{24}\right)\int \frac{3e^x - 4e^{-x}}{3e^x + 4e^{-x}}\,dx$$

$$= \frac{17}{24}\,x - \frac{1}{24}\,I_1$$

Where $I_1 = \int \frac{2e^x + 3e^{-x}}{3e^x + 4e^{-x}}\,dx$

$Put\ \ 3e^x + 4e^{-x} = t$

$\Rightarrow (3e^x + 4e^{-x})dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \text{In}\,|t|$

$= \ln\,(3e^x + 4e^{-x})$.

$$\text{Hence,} \int \frac{2e^x + 3e^{-x}}{3e^x + 4e^{-x}}\,dx = \frac{17}{24}\,x - \frac{1}{24}\ln(3e^x + 4e^{-x}) + c$$

Check Your Progress – 2

Evaluate the following integrals.

1. $\int \frac{x}{\sqrt{x+1}}\,dx$ 2. $\int \frac{e^{3x}}{e^{3x}+4}\,dx$

3. $\int \frac{4x-7}{(2x^2-7x+8)^2}\,dx$ 4. $\int x\sqrt{x+1}\,dx$

5. $\int \frac{dx}{\sqrt{x} + x}$

6. $\int 2^{4-5x}\, dx$

7. $\int \frac{e^x + 3e^{-x}}{2e^x + e^{-x}}\, dx$

8. $\int \frac{x^3}{\sqrt{x^2 - 1}}$

Answers

1. $\frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$

2. $\frac{1}{3}\ln(e^{3x} + 4) + c$

3. $\frac{-1}{(2x^2 - 7x + 8)^2} + c$

4. $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$

5 $2\ln(\sqrt{x} + 1) + c$

6. $-\frac{1}{5\ln 2} 2^{4-5x} + c$

7. $\frac{5}{4}x + \frac{7}{4}\ln|2e^x - e^{-x}| + c$

8. $\frac{1}{3}(x^2 - 1)^{\frac{3}{2}} + \sqrt{x^2 - 1} + c$

3.4 INTEGRATION OF RATONAL FUNCTIONS

A function $R(x)$ is said to be rational if $R(x)$ is of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials in x. For instance, $\frac{x - 3}{x^2 + 1}$ and $\frac{2x + 1}{x^2 - 3x + 5}$ are rational functions.

A rational function $R(x) = \frac{P(x)}{Q(x)}$ is said to be **proper** if $\deg(p(x)) < deg\,(Q(x))$ and is said to be **improper** if $\deg(P(x)) \geq \deg(Q(x))$.

In case $R(x) = \frac{P(x)}{Q(x)}$ is **improper** rational function, we can write it as

$$R(x) = A(x) + \frac{B(x)}{Q(x)}$$

where $A(x)$ is a polynomial and $\frac{B(x)}{Q(x)}$ is a proper rational functions

Partial Fractions

Recall when we add two rational functions, we get a rational function. For instance, when we add

$\frac{2}{2x-3}$ and $\frac{1}{1-x}$

we get $\frac{2}{2x-3}+\frac{1}{1-x} = \frac{2(1-x)+ 2x-3}{(2x-3)(1-x)} = \frac{-1}{(2x-3)(1-x)}$

We call $\frac{2}{2x-3}$ and $\frac{1}{1-x}$

as partial fractions of $\frac{-1}{(2x-3)(1-x)}$

Methods of Splitting a Rational Function into Partial Fractions

Case 1 : When denominator consists of distinct Linear factors

We illustrate the method in the following illustration.

Illustraton: Resolve

$$\frac{x}{(2x-1)(x+1)(x-2)}$$

into partial fractions.

We write

$$\frac{x}{(2x-1)(x+1)(x-2)} = \frac{A}{2x-1}+\frac{B}{x+1}+\frac{C}{x-2}$$

where A, B and C are constants.

$\Rightarrow$ $x = A(x+1)(x-2) + B(2x-1)(x-2) + C(2x-1)(x+1)$

Put $x = ½$, -1 and 2 to obtain

$\frac{1}{2} = A\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) \Rightarrow A = -\frac{2}{9};$

$-1 = B(-3)(-3) \Rightarrow B = -\frac{1}{9};$

$2 = C(3)(3) \Rightarrow C = \frac{2}{9}$

Thus

$$\frac{x}{(2x-1)(x+1)(x-2)} = -\frac{2}{9}\frac{1}{2x-1}-\frac{1}{9}\frac{1}{x+1}+\frac{2}{9}\frac{1}{x-2}$$

Case 2: When Denominator consists of repeated Linear Factors

Illustration: Resolve

$$\frac{x}{(2x-1)(x+1)^2}$$

into partial fractions.

Write

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{2x-1} + \underbrace{\frac{B}{x+1} + \frac{C}{(x+1)^2}}_{\text{Note carefully}}$$

where A, B and C are constants.

$\Rightarrow x = A(x+1)^2 + B(x-1)(x+1) + C\,(x-1)$

Put x =1 and -1, to obtain

$1 = 4A \Rightarrow A = 1/4$; and

$-1 = -2C \Rightarrow C = 1/2$.

Next, we compare coefficients of x^2 on both the sides to obtain

$0 = A + B \Rightarrow B = -A = -\frac{1}{4}.$

Case 3 : When the Denominator consists of irreducible Quadratic Factor.

Illustration : Resolve

$$\frac{x}{(x+1)(x^2+x+1)}$$

Into partial fractions.

Write

$$\frac{x}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

where A, B and C are constants.

$\Rightarrow x = A(x^2+x+1) + (Bx + C)\,(x + 1)$

Put $x = -1$ to obtain $A = -1$. Comparing coefficients, we obtain

$O = A + B \Rightarrow B = -A = 1$

Next, put $x = 0$ to obtain

$0 = A + C \Rightarrow C = -A = 1$

Thus,

$$\frac{x}{(x+1)(x^2+x+1)} = \frac{-1}{x+1} + \frac{x+1}{x^2+x+1}$$

Solved Examples

Example 11 : Evaluate the integral

$$\int \frac{x}{(x+1)(2x-1)}\, dx$$

Solution : We first resolve the integrand into partial fractions. Write

$$\frac{x}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$$

$\Rightarrow x = A(2x-1) + B(x+1)$

Put $x = ½$ and -1 to obtain

$$\frac{1}{2} = B\left(\frac{1}{2}+1\right) \Rightarrow B = \frac{1}{3}$$

$$-1 = A(-3) \Rightarrow A = \frac{1}{3}$$

Thus,

$$\int \frac{x}{(x+1)(2x-1)}\, dx = \frac{1}{3}\int \frac{dx}{(x+1)} + \frac{1}{3}\int \frac{dx}{(2x-1)}$$

$$= \frac{1}{3}\log|x+1| + \frac{1}{3}.\frac{1}{2}\log|2x-1| + c$$

$$= \frac{1}{3}\log|x+1| + \frac{1}{6}\log|2x-1| + c$$

Example 12: Integrate

(i) $\dfrac{1}{x^2-a^2}$ (ii) $\dfrac{1}{a^2-x^2}$

Solution: (i) We write $\dfrac{1}{x^2-a^2}$ as $\dfrac{1}{(x-a)(x+a)}$ and split into partial fractions.

Write

$$\frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$\Rightarrow 1 = A(x+a) + B(x-a)$

Put $x = a$ and $-a$ to obtain

$$1 = A(2a) \Rightarrow A = \frac{1}{2a};$$

$$1 = B(-2a) \Rightarrow B = \frac{1}{2a};$$

Thus,

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \left|\frac{1}{x+a} - \frac{1}{x-a}\right| dx$$

$$= \frac{1}{2a} [\log|x+a| - \log|x-a|] + c$$

$$= \frac{1}{2a} log \left[\frac{x+a}{x-a}\right] + c$$

(ii) Note that

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} = \int \left[\frac{1}{a+x} + \frac{1}{a-x}\right] dx$$

$$= \frac{1}{2a} [\log|a+x| - \log|a-x|] + c$$

$$= \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c$$

Two Important Formulae

1. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x+a}{x-a}\right| + c$

2. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c$

Remark : Above two formulae may be used as standard formulae.

Example 13 : Evaluate the integral.

$$\int \frac{x\,dx}{(x-1)(x+5)(2x-1)}$$

Solution : We write

$$\frac{x}{(x-1)(x+5)(2x-1)} = \frac{A}{x-1} + \frac{B}{x+5} + \frac{C}{2x-1}$$

$\Rightarrow x = A\,(x+5)(2x-1) + B(x-1)(2x-1) + C(x-1)(x+5)$

Put $x = 1, -5$ and ½ to obtain

$1 = 6A \Rightarrow A = 1/6$

$-5 = 66B \Rightarrow B = -5/66$

$$\frac{1}{2} = -\frac{11}{4}\,\mathrm{C} \Rightarrow \mathrm{C} = -2/11$$

Thus,

$$\int \frac{x}{(x-1)(x+5)(2x-1)} = \frac{1}{6}\int \frac{dx}{x-1} - \frac{5}{66}\int \frac{dx}{x+5} - \frac{2}{11}\int \frac{dx}{2x-1}$$

$$= \frac{1}{6}\log|x-1| - \frac{5}{66}\log|x+5| - \frac{1}{11}\log|2x-1| + c$$

Example 14 : Evaluate the integral

$$\mathrm{I} = \int \frac{dx}{1+3e^x+2e^{2x}}$$

Solution: Put $e^x = t$, so that $e^x dx = dt$, and

$$\mathrm{I} = \int \frac{dt}{t(1+3t+2t^2)}$$

$$= \int \frac{dt}{t(1+t)(1+2t)}$$

We now split

$$\frac{1}{t(1+t)(1+2t)}$$

into partial fractions, to obtain

$$\frac{1}{t(1+t)(1+2t)} = \frac{A}{t} + \frac{B}{1+t} + \frac{C}{1+2t}$$

$\Rightarrow 1 = \mathrm{A}(1+t)(1+2t) + Bt(1+2t) + \mathrm{C}t(1+t)$

Put t = 0, –1 and –1/2 to obtain

1 = A $\Rightarrow$ A = 1;

1 = B $\Rightarrow$ B = 1;

1 = −C/4 $\Rightarrow$ C = −4

Thus,

$$\int \frac{dt}{t(1+t)(1+2t)} = \int \frac{dt}{t} + \int \frac{dt}{(1+t)} - 4\int \frac{dt}{(1+2t)}$$

$= \log|t| + \log|1+t| - 2\log|1+2t| + c$

$= \log(e^x) + \log(e^x+1) - 2\log(2e^x+1) + c$

$$= x + log\frac{e^x+1}{(2e^x+1)^2} + c$$

Example 15 : Evalaute the integral

$$\mathrm{I} = \int \frac{x^2}{(x+1)^3}\, dx$$

Solution : To evaluate an integral of the form

$$\int \frac{P(x)}{(a+bx)^r}\, dx, \text{we put } a + bx = t.$$

$$\text{So, we put } x + 1 = t \Rightarrow dx = dt$$

$$\text{and I} = \int \frac{(t+1)^2}{t^3}\, dt = \int \frac{t^2+2t+1}{t^3}\, dt$$

$$= \int \left(\frac{1}{t} - 2t^{-2} + t^{-3}\right) dt$$

$$= \log|\mathrm{t}| - \frac{2t^{-1}}{-1} + \frac{t^{-2}}{-2} + c$$

$$= \log|\mathrm{t}| + \frac{2}{t} - \frac{1}{2t^2} + c$$

$$= \log|x+1| - \frac{2}{x+1} + \frac{1}{2(x+1)^2} + c$$

Example 16 : Evaluate the integral

$$\mathrm{I} = \int \frac{(x+1)^2}{(x-1)^2}\, dx$$

Solution : Put $x - 1$ = t, so that

$$\mathrm{I} = \int \frac{(t+1+1)^2}{t^2} dt = \int \frac{(t+2)^2}{t^2}\, dt$$

$$= \int \frac{(t+4t+4)^2}{t^2}\, dt$$

$$= \int \left[1 + \frac{4}{t} + 4t^{-2}\right] dt$$

$$= t + 4log|t| - \frac{4}{t} + c$$

$$= x - 1 + 4log|x-1| - \frac{4}{x-1} + c$$

$$= x + 4log|x-1| - \frac{4}{x-1} + c \text{ [absorb } -1 \text{ in the constant of integration]}$$

Example 17 : Evaluate the integral

$$I = \int \frac{3x-1}{(x+1)^2(2x-1)}\,dx$$

Solution :

We write

$$\frac{3x-1}{(x+1)^2(2x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x-1}$$

$\Rightarrow 3x-1 = A(x+1)(2x-1) + B(2x-1) + C\,(x+1)^2$

Put $x = -1$ and ½ to obtain

$-4 = -3B \Rightarrow B = 4/3$

$$\frac{3}{2} - 1 == C\,(-1/2+1)^2 \Rightarrow \frac{1}{2} = \frac{1}{4}C \Rightarrow C = 2$$

Comparing coefficient of x^2, we get

$0 = 2A + C \Rightarrow 2A = -C = -2$

$\Rightarrow A = -1$

Thus,

$$\int \frac{3x-1}{(x+1)^2(2x-1)}\,dx = -\int \frac{dx}{x+1} + \frac{4}{3}\int (x+1)^{-2}\,dx + 2\int \frac{dx}{2x-1}$$

$$= -\log|x+1| + \frac{4}{3}\,\frac{(x+1)^{-2+1}}{(-2+1)} + \frac{2log|2x-1|}{2} + c$$

$$= \log\left|\frac{2x-1}{x+1}\right| - \frac{4}{3}\,\frac{1}{x+1} + c$$

Example 18 : Evaluate the integral

$$I = \int \frac{dx}{(e^x-1)^2}$$

Solution :

Put $e^x - 1 = t$, so that $e^x dx = dt$, and

$$I = \int \frac{dt}{t^2(t+1)}$$

We split $\frac{1}{t^2(t+1)}$ into partial fractions

$$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

$\Rightarrow 1 = At(t+1) + B(t+1) + Ct^2$

Put $t = 0$, $t = -1$ to obtain

$1 = B \Rightarrow B = 1$

$1 = C \Rightarrow C = 1$

Comapring coefficient at t^2, we obtain

$0 = A + C \Rightarrow A = -C = -1$

Thus,

$$I = \int \left[-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right] dt$$

$$= -\log|t| - \frac{1}{t} + \log|t+1| + c$$

$$= \log\left|\frac{t+1}{t}\right| - \frac{1}{t} + c$$

$$= log\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x} + c$$

Check Your Progress 3

Integrate the following functions

1. $\frac{x^2+1}{(2x+1)(x-1)(x+1)}$
2. $\frac{x^2+1}{x(x^2-1)}$
3. $\frac{2x-3}{(x^2-1)(2x+3)}$
4. $\frac{x}{x(1+4x^3+3x^6)}$
5. $\frac{e^x}{e^x-3e^{-x}+2}$
6. $\frac{x^2}{(x+2)^3}$
7. $\frac{x^2}{(x-1)^3(x+1)}$
8. $\frac{e^x}{(e^x-1)^3}$

Answers

1. $-\frac{5}{6}\log|2x+1|+\frac{1}{3}\log|x-1|+\log|x+1|+c$

2. $\log\left|\frac{x^1-1}{x}\right|+c$

3. $\frac{5}{2}\log|x+1|+\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+c$

4. $\log|x|+\frac{1}{6}\log|1+x^3|-\frac{1}{2}\log|1+3x^3|+c$

5. $\frac{1}{4}\log\frac{|e^x-1|}{(e^x+1)^3}+c$

6. $\log|x+2|+\frac{4}{x+2}-\frac{2}{(x+1)^2}+c$

7. $\frac{1}{8}\log\frac{|x-1|}{|x+1|}-\frac{3}{4}\frac{2}{x-1}-\frac{1}{4}\frac{1}{(x+1)^2}+c$

8. $-\frac{1}{2}\frac{1}{(e^x-1)^2}+c$

3.5 INTEGRATION BY PARTS

Recall the product rule for the derivative

$$\frac{d}{dx}[uv] = uv' + vu'$$

$$\Rightarrow uv = \int uv' + vu'\, dx$$

$$\Rightarrow= \int uv'\,dx = uv - \int vu'\,dx$$

We can write the above formula as

$$\int u(x)v(x)dx = u(x)\int v(x)dx - \int\left[\frac{du}{dx}\int v(x)dx\right]dx$$

In words, the above formula state

Integral of the product of two functions

= First function × integral of the second function – Integral of (the derivative of the first function × integral of the second function)

For instance, to evaluate $\int x.e^x$, we take e^x as second function and x as the first function.

By the above formula

$$\int xe^x\ dx\ = xe^x - \int \frac{d}{dx}[x]\ e^x dx$$

$$= xe^x - \int 1.e^x = xe^x - e^x + c$$

Solved Examples

Example 18 Integrate $x \log x$

Soluton : We take x as the second function and log x as the first function.

$$\int x\ logx\ dx = \int (logx)x\ dx$$

$$= (\log x)\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + c$$

Example 19 Evaluate

$$\int \sqrt{x}\ logx\ dx$$

Solution : We take $\sqrt{x}$ as the second function and log x as the first function. We have

$$\int \sqrt{x} \log x dx = \int (\log x)x^{1/2}dx$$

$$= (\log x)\frac{x^{3/2}}{3/2} - \int \frac{1}{x}\frac{x^{3/2}}{3/2}dx$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int \frac{x^{3/2}}{3/2} + c$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{4}{9}\int x^{3/2} + c$$

Example 20 : Evaluate

$$\int \frac{logx}{x^2}\, dx$$

Solution : We take x^{-2} as the second function and log x as the first function.

$$\text{I} = \int x^{-2}\, logx\, dx$$

$$= \frac{x^{-2+1}}{-2+1}\log x - \int \frac{1}{x} . \left(\frac{x^{-1}}{-1}\right) dx$$

$$= -\frac{1}{x}\log x + \int x^{-2}\, dx$$

$$= -\frac{1}{x}\log x + \frac{x^{-1}}{-1} + c$$

$$= -\frac{1}{x}\log x - \frac{1}{x} + c$$

Evaluate 21: Evaluate

$$\int xe^{-x}\, dx$$

Solution: We take e^{-x} as the second function and x as the first function. We have

$$\int xe^{-x} = x\left(\frac{e^{-x}}{-1}\right) - \int (1)\frac{e^{-x}}{-1}\, dx$$

$$= -xe^{-x} + \int e^{-x}\, dx$$

$$= -x\, e^{-x} - e^{-x} + c$$

$$= -(x+1)\, e^{-x} + c$$

Example 22 : Evaluate

$$\int \log(1+x)^{1+x}\, dx$$

Solution : We write $\log(1+x)^{1+x}$ = (1+ x) log (1+ x) and (1+ x) as the second function. We have

$$\text{I} = \int (1+x)\log(1+x) \quad dx$$

$$= \frac{1}{2}(1+x)^2\log(1+x) - \int (1+x)^2\frac{1}{1+x}dx$$

$$= \frac{1}{2}(1+x)^2 \log(1+x) - \int (1+x)dx$$

$$= \frac{1}{2}(1+x)^2 \log(1+x) - \frac{1}{4}(1+x)^2 + c$$

Example 23 : Evaluate

$$\int \log x \, dx$$

Solution : We write log x =1. log x and take 1 as the 2nd function and log$x$ as the first function.

$$\int \log x dx = \int 1.\log x dx$$

$$= x \log x - \int (x)\frac{1}{x} dx$$

$$= x \log x - \int dx$$

$$= x \log x - x + c$$

$$= x (\log x - 1) + c$$

Example 24 ; Evaluate

$$\int x^3 (\log x)^2 \, dx$$

Solution : We take x^3 as the second function. We have

$$\int x^3 (\log x)^2 \, dx = \frac{1}{4} x^4 (\log x)^2 - \frac{1}{4}\int x^4 2(\log x)\frac{1}{x} dx$$

$$= \frac{1}{4} x^4 (\log x)^2 - \frac{1}{2}\int x^3 (\log x) \, dx$$

$$= \frac{1}{4} x^4 (\log x)^2 - \frac{1}{2}\left[\frac{1}{4} x^4 \log x - \frac{1}{4}\int x^4 \frac{1}{x} \, dx\right]$$

$$= \frac{1}{4} x^4 (\log x)^2 - \frac{1}{2}\left[\frac{1}{4} x^4 \log x - \frac{1}{4}\int x^3 \, dx\right]$$

$$= \frac{1}{4} x^4 (\log x)^2 - \frac{1}{2}\left[\frac{1}{4} x^4 \log x - \frac{1}{16} x^4\right] + c$$

$$= \frac{1}{4} x^4 (\log x)^2 - \frac{1}{8} x^4 \log x + \frac{1}{32} x^4 + c$$

$$= \frac{1}{32} x^4 [8(\log x)^2 - 4 \log x + 1] + c$$

Remark : If an integrand is of the form $e^x(f(x) + f'(x))$, we write it as $e^x f(x) + e^x f'(x)$, and just integrate the first function. We have

$$I = \int e^x f(x) + f'(x))dx$$

$$= \int e^x (f(x))dx + \int e^x f'(x)dx$$

$$= e^x f(x) - \int e^x f'(x) + \int e^x f'(x)dx$$

$$= e^x f(x) + c$$

Example 25 : Evaluate the integral

$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

Solution : We write

$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \int e^x \frac{1}{x}\, dx - \int e^x \frac{1}{x^2} dx$$

$$= e^x \frac{1}{x} - \int e^x \frac{(-1)}{x^2}\, dx - \int e^x \frac{1}{x^2}\, dx$$

$$= \frac{e^x}{x} + c$$

Check Your Progress 4

Integrate the followings:

1. $x^2 e^x$ 2. $x\log(1+x)\, dx$

3. $e^{\sqrt{x}}$ 4. $e^x (\log x + \frac{1}{x})$

5. $e^x \frac{x+1}{(x+2)^2}$ 6. $\log\sqrt{x}$

7. $\log(1+x)$ 8. $(1-x)^2 \log x$

Answers

1. $(x^2 - 2x + 2)\, e^x + c$

2. $\frac{1}{2}(x^2 - x)\log(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + c$

3. $2(\sqrt{x} - 1)\, e^{\sqrt{x}} + c$

4. $e^x \log x + c$

5. $\dfrac{e^x}{x+2} + c$

6. $\frac{1}{2}\,(xlogx - x) + c$

7. $(x+1)\log(1+x) - x + c$

8. $\left(x - \frac{1}{3}x^3\right)\log x - x + \frac{1}{9}x^3 + c$

3.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. $\int (x^3 + 2^x)dx = \frac{1}{4}x^4 + \frac{1}{ln2}2^x + c$

2. $\int (x^e + e^x)\,dx = \frac{1}{e+1}e^x + e^x + c$

3. $\int \frac{(\sqrt{x} + x)^2}{x^2}\,dx = \int \frac{x + 2x^{1/2}x + x^2}{x^2}dx$

$$= \int (\frac{1}{x} + 2x^{-\frac{1}{2}} + 1)\,dx$$

$$= \ln|x| + 4x^{1/2} + x + c$$

4. $\int \frac{(2^x + 3^x)^2}{5^x}dx = \int \frac{(2^x)^2 + 2(2^x) + 3^x\,(3^x)^2}{5^x}\,dx$

$$= \int \left[\left(\frac{4}{5}\right)^x + 2\left(\frac{6}{5}\right)^x + \left(\frac{9}{5}\right)^x\right]dx$$

$$= \frac{(4/5)^x}{\ln(4/5)} + 2\,\frac{(6/5)^x}{\ln(6/5)} + \frac{(9/5)^x}{\ln(9/5)} + c$$

5. $\int (3^x + x^7 - 2\,x^{-4})\,dx = \frac{3^x}{\ln 3} + \frac{x^8}{8} - \frac{2x^{-3}}{(-3)} + c$

$$= \frac{3^x}{\ln 3} + \frac{1}{8}x^8 + \frac{2}{3x^3} + c$$

6. $\int \left(\frac{3^x + 5^x}{7^x}\right)\,dx = \int \left[\left(\frac{3}{7}\right)^x + \left(\frac{5}{7}\right)^x\right]dx$

$$= \frac{(3/7)^x}{\ln(3/7)} + \frac{(5/7)^x}{\ln(5/7)} + c$$

Check Your Progress 2

1. Put $x + 1 = t^2$ So that $x = t^2 - 1$ *and* $dx = 2t\,dt$

$$\therefore \int \frac{x}{\sqrt{x+1}} = \int \frac{t^2-1}{t} 2tdt = 2\left[\frac{t^3}{3} - t\right] + c$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + 2\sqrt{x+1} + c$$

2. Put $e^{3x} + 4 = t$, *so that* $3e^{3x}dx = dt$

$$\int \frac{e^{3x}}{e^{3x} + 4} dx = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3}\ln|t| + c$$

$$= \frac{1}{3}\ln(e^{3x} + 4) + c$$

3. Put $2x^2 - 7x + 8 = t$, so that $(4x - 7)dx = dt$

$$\therefore \int \frac{4x-7}{(2x^2-7x+8)^2} dx = \int \frac{dt}{t^2} = \int t^{-2}\,dt$$

$$= -\frac{1}{t} + c = \frac{-1}{(2x^2-7x+8)} + c$$

4. Put $\sqrt{x+1} = t^2$ so that $x = t^2 - 1$, $dx = 2tdt$

$$\therefore \int x\sqrt{x+1}\,dx = \int (t^2-1)t.2t\,dt$$

$$= \frac{2}{5}t^5 - \frac{2}{3}t^3 + c$$

$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$$

5. Put $\sqrt{x} = t$ or $x = t^2$, so that dx = 2tdt

$$\therefore \int \frac{dx}{\sqrt{x} + x} = \int \frac{2tdt}{t + t^2} = 2\int \frac{dt}{t+1}$$

$= 2\ln(t+1) + c = 2\ln(\sqrt{x} + 1) + c$

6. Put $4 - 5x = t$, so that $-5dx = dt$

$$\text{Thus, } \int 2^{4-5x}\,dx = -\frac{1}{5}\int 2^t dt = -\frac{1}{5}\frac{2^t}{ln2} + c$$

$$= -\frac{1}{5}\frac{2^{4-5x}}{ln2} + c$$

7. Write

$e^x + 3e^{-x} = \alpha\,(2\,e^x - e^{-x}) + \beta\,\frac{d}{dx}(2\,e^x - e^{-x})$

$\Rightarrow e^x + 3e^{-x} = \alpha\,(2e^x - e^{-x}) + \beta\,(2\,e^x + e^{-x})$

Equating coefficients of e^x and e^{-x}, we obtain

$1 = 2\alpha + 2\beta$ and $3 = -\alpha + \beta$

$\Rightarrow \alpha + \beta = \frac{1}{2}$ and $-\alpha + \beta = 3$

Solving, we obtain $\alpha = -\frac{5}{4},\ \beta = \frac{7}{4}$

Thus,

$$\int \frac{e^x + 3e^{-x}}{2e^x + e^{-x}}\,dx = \int \frac{\left(-\frac{5}{4}\right)(2e^x - e^{-x}) + \left(\frac{7}{4}\right)(2e^x + e^{-x})}{(2e^x - e^{-x})}\,dx$$

$$= \left(-\frac{5}{4}\right)\int dx + \frac{7}{4}\int \frac{2e^x + e^{-x}}{2e^x - e^{-x}}\,dx$$

$$= -\frac{5}{4}x + \frac{7}{4}I_1$$

where $I_1 = \int \frac{2e^x + e^{-x}}{2e^x - e^{-x}}\,dx$

Put $2e^x + e^{-x} = t$, so that

$(2e^x + e^{-x})dx = dt$

$\therefore I_1 = \int \frac{dt}{t} = \ln|t|$

$= \ln|2e^x - e^{-x}|$

Thus,

$\int \frac{e^x + 3e^{-x}}{2e^x - e^{-x}}\,dx$

$= -\frac{5}{4}x + \frac{7}{4}\ln|2e^x - e^{-x}| + c$

8. Put $x^2 - 1 = t^2$, so that $2x dx = 2t dt$.

Now,

$$\int \frac{x^3}{\sqrt{x^2-1}}\,dx = \int \frac{x^2 x\,dx}{\sqrt{x^2-1}} = \int \frac{(t^2+1)t\,dt}{t}$$

$$= \int (t^2+1)dt = \frac{1}{3}t^3 + t + c$$

$$= \frac{1}{3}(x^2+1)^{3/2} + (x^2+1)^{1/2} + c$$

Check Your Progress 3

1. We split $\frac{x^2+1}{(2x+1)(x-1)(x+1)}$ into partial fractions.

We write

$$\frac{x^2+1}{(2x+1)(x-1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\Rightarrow x^2 + 1 = A(x-1)(x+1) + B(2x+1)(x+1) + C\,(2x+1)(x-1)$$

Put $x = -1/2, 1, -1$ to obtain

$$\frac{1}{4} + 1 = A\left(\frac{-3}{2}\right)\left(\frac{1}{2}\right) \Rightarrow A = \frac{-5}{3};$$

$2 = B(3)(2) \Rightarrow B = 1/3$; and

$2 = C(-1)(-2) \Rightarrow C = 1$

Thus,

$$\int \frac{x^2+1}{(2x+1)(x-1)(x+1)}\,dx = -\frac{5}{3}\int \frac{dx}{2x+1} + \frac{1}{3}\int \frac{dx}{x-1} + \int \frac{dx}{x+1}$$

$$= -\frac{5}{3} \times \frac{1}{2}\log|2x+1| + \frac{1}{3}\log|x-1| + \log|x+1| + c$$

$$= -\frac{5}{6}\log|2x+1| + \frac{1}{3}\log|x-1| + \log|x+1| + c$$

2. Write

$$\frac{x^2+1}{x(x^2-1)} = \frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$\Rightarrow x^2+1 = A(x^2-1) + Bx\,(x+1) + Cx(x-1)$

Put x = 0, 1, –1 to obtain

1= –A $\Rightarrow$ A = –1;

2 = 2B $\Rightarrow$B = 1; and

2 = 2C $\Rightarrow$C = 1

$$\int \frac{x^2+1}{x(x^2-1)}\,dx = -\int \frac{dx}{x} + \int \frac{dx}{x-1} + \int \frac{dx}{x+1}$$

= log|x| + log |x – 1| + | log |x + 1| + c

$$= log\left|\frac{x^2-1}{x}\right| + c$$

3. Write

$$\frac{2x-3}{(x^2-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+3)}$$

$\Rightarrow$2x – 3 = A(x + 1)(2 x+ 3) + B(x – 1)(2x + 3) + C (x^2-1)

Put x = 1, −1 and −3/2 to obtain

−1 = A(2)(5) $\Rightarrow$ A = −1/10

−5 = −2B $\Rightarrow$ B = 5/2

−6 = 5C/4 $\Rightarrow$ C= −24/5

Thus,

$$\int \frac{2x-3}{(x^2-1)(2x+3)}\,dx = -\frac{1}{10}\int \frac{dx}{x-1} + \frac{5}{2}\int \frac{dx}{x+1} - \frac{24}{5}\int \frac{dx}{2x+3}dx$$

$$= -\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + c$$

4. Multiply the numerator and denominator by x^2 to obtain

$$I = \int \frac{x^2}{x(1+4x^3+3x^6)}dx = \int \frac{x^2}{x^3(1+4x^3+3x^6)}dx$$

Put $x^3 = t,$ so that

$$I = \frac{1}{3}\int \frac{dt}{t(1+4t+3t^2)} = \frac{1}{3}\int \frac{dt}{t(1+t)(1+3t)}$$

Now, write

$$\frac{1}{t(1+t)(1+3t)} = \frac{A}{t} + \frac{B}{1+t} + \frac{C}{1+3t}$$

$\Rightarrow 1 = A(1+t)(1+3t) + Bt(1+3t) + Ct(1+t)$

Put $t = 0, -1$ and $-1/3$ to obtain

A = 1, B=1/2, C= –9/2

Thus,

$$\frac{1}{3}\int \frac{dt}{t(1+t)(1+3t)} = \frac{1}{3}\int \left[\frac{1}{t} + \frac{1}{2(1+t)} - \frac{9}{2(1+3t)}\right] dt$$

$$= \frac{1}{3}\left[\log|t| + \frac{1}{2}\log|1+t| - \frac{9}{2}\times\frac{1}{3}\log|1+3t|\right] + c$$

$$= \left[\log|x| + \tfrac{1}{6}\log|1+x^3| - \tfrac{1}{2}\log|1+3x^3|\right] + c$$

5. Write

$$\frac{e^x}{e^x - 3e^{-x} + 2} = \frac{e^x}{e^x - 3/e^x + 2} = \frac{e^{2x}}{e^{2x} + 2e^x - 3}$$

$$\text{Let I} = \int \frac{e^x e^x}{e^{2x} + 2e^x - 3} dx$$

Put $e^x = t$, so that $e^x dx$ and

$$\text{I} = \int \frac{t}{t^2 + 2t - 3}\, dt = \int \frac{t}{(t-1)(t+3)} dt$$

Split $\frac{t}{(t-1)(t+3)}$ into portial fractions, to obtain

$$\frac{t}{(t-1)(t+3)} = \frac{1}{4}\frac{1}{t-1} - \frac{3}{4}\frac{1}{t+3}$$

$$\Rightarrow \int \frac{t}{(t-1)(t+3)} dt = \frac{1}{4}\log|t-1| - \frac{3}{4}|t+3| + c$$

$$= \frac{1}{4} log\left|\frac{t-1}{(t+3)^3}\right| + c$$

$$\text{Thus, I} = \frac{1}{4}\log\left|\frac{e^x - 1}{(e^x+3)^3}\right| + c$$

6. Put $x + 2 = t$, so that

$$\text{I} = \int \frac{x^2}{(x+2)^2} dx = \frac{(t-2)^2}{t^3} dt$$

$$= \int \frac{t^2 - 4t + 4}{t^3} dt$$

$$= \int \left[\frac{1}{t} - \frac{4}{t^2} + \frac{4}{t^3} \right] dt$$

$$= \log|t| + \frac{4}{t} - \frac{2}{t^2} + c$$

$$= \log|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + c$$

7. Write

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$$

$$\Rightarrow x^2 = \text{A}(x-1)^2(x+1) + \text{B}(x-1)(x+1) + \text{C}(x+1) + \text{D}(x-1)^3$$

Put x =1 and –1 to obtain

1 = 2 C $\Rightarrow$C=1/2 and 1 = – 8D $\Rightarrow$D = –1/8

Comparing coefficient of x^3, we obtain

0 = A + D $\Rightarrow$A = – D = 1/8

Next, put $x= 0$ to obtain

0 = A – B+C – D $\Rightarrow$ B = A + C – D = ¾

Thus

$$\text{I} = \int \frac{x^2\, dx}{(x-3)^3(x+1)} = \frac{1}{8}\log\left|\frac{x-1}{x+1}\right| - \frac{3}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{(x-1)^2} + c$$

8. Put $e^x - 1 = t,$ so that

$$\text{I} = \int \frac{dt}{t^3} = \int t^{-3} dt = \frac{t^{-3+1}}{-3+1} + C$$

$$= -\frac{1}{2t^2} + c = -\frac{1}{2}\frac{1}{(e^{x-1})^2} + c$$

Check Your Progress 4

1. $\int x^2 e^x dx = x^2 e^x - \int 2xe^x$

$$= x^2e^x - 2[xe^x - \int (1)\, e^x dx]$$

$$= x^2e^x - 2[xe^x - e^x] + c$$

$$= (x^2 - 2x + 2)e^x + c$$

2. $$\int x \log(1 + x) dx = \frac{1}{2}x^2 \log(1 + x) - \frac{1}{2}\int \frac{x^2}{1 + x}\, dx$$

$$\text{Let } I_1 = \int \frac{x^2}{1 + x}\, dx = \int \frac{x^2 - 1 + 1}{1 + x}\, dx$$

$$= \int \left[x - 1 + \frac{1}{1 + x}\right] dx$$

$$= \frac{1}{2}x^2 - x + \log(1 + x)$$

Thus,

$$\int x \log(1 + x) dx = \frac{1}{2}x^2 \log(1 + x) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}x\log(1 + x) + c$$

3. Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\, dt$

Thus,

$$I = \int e^{\sqrt{x}} dx = 2\int t\, e^t dt$$

$$= 2[te^t - \int (1)\, e^t dt]$$

$$= 2[te^t - e^t] + c$$

$$= 2(\sqrt{x} - 1)\, e^{\sqrt{x}} + c$$

4. $$I = \int e^x \log x + \int e^x \frac{1}{x}\, dx$$

$$= e^x \log x - \int e^x \frac{1}{x}\, dx + \int e^x \frac{1}{x} dx$$

$$= e^x \log x + c$$

5. We write $$\frac{x + 1}{(x + 2)^2} \quad \frac{x + 2 - 1}{(x + 2)^2} = \frac{1}{x + 2} - \frac{1}{(x + 2)^2}$$

We have

$$= \int e^x \frac{x + 1}{(x + 1)^2} dx \quad \int e^x \left[\frac{1}{x + 2} - \frac{1}{(x + 1)^2}\right] dx$$

$$= \int e^x (x+2)^{-1}\ dx - \int e^x \frac{1}{(x+2)^2}\ dx$$

$$= e^x (x+2)^{-1}\, dx - \int e^x\,(-1)(x+2)^{-2}\ dx - \int e^x \frac{1}{(x+2)^2}\ dx$$

$$= \frac{e^x}{x+2} + \int \frac{e^x}{(x+2)^2}\ dx - \int e^x \frac{1}{(x+2)^2}\ dx$$

$$= \frac{e^x}{x+2} + c$$

6. $\int \log\sqrt{x}\ dx = \frac{1}{2}\int \log x\, dx = \frac{1}{2}\int (1)\log x\, dx$

$$= \frac{1}{2}[x\log x - \int (x)\frac{1}{x}dx]$$

$$= \frac{1}{2}\ [x\log x - x] + c$$

7. $\int \log(1+x)\, dx = \int (1) \log(1+x)\, dx$

$$= x\log(1+x) \int x\ \frac{1}{1+x}dx$$

$$= x\log(1+\ x) - \int \frac{x+1-1}{x+1}\ dx$$

$$= x\log(1+\ x) - \int \left[1 - \frac{1}{1+x}\right] dx$$

$= x\log(1+x) - [x - \log(1+x)] + c$

$= (x+1)\log(1+x) - x + c$

8. $\int (1-x^2)\ \log x\, dx$

$$= \left(x - \frac{x^3}{3}\right)\log x - \int \left(x - \frac{x^3}{3}\right)\frac{1}{x}dx$$

$$= \left(x - \frac{1}{3}x^3\right)\log x - \int \left(1 - \frac{x^2}{3}\right)\ dx$$

$$= \left(x - \frac{1}{3}x^3\right)\log x - \left(x - \frac{x^3}{9}\right) + c$$

$$= \left(x - \frac{1}{3}x^3\right)\log x - \ x + \frac{x^3}{9}\ + c$$

3.7 SUMMARY

The unit discusses integration of a function as inverse of the derivative of the function. In **section 3.2**, basic integration rules are derived using corresponding differentiation rules. A number of examples are included to explain application of the rules. In **section 3.3**, for finding integral of complex functions in terms of simpler functions, the method of substitution is discussed through suitable examples. In **section 3.4**, methods for integration of rational functions, are introduced and explained. In **section 3.5**, method of integration by parts for finding integral of product of two functions in terms of the integrals of the functions is discussed.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.6**.

UNIT 4 APPLICATIONS OF INTEGRAL CALCULUS

Structure

4.0 INTRODUCTION

In the previous unit, we introduced the concept of indefinite integral (anti-derivative). We now look at a new problem – that of finding the area of a plane region. At first glance these two ideas seem, to be unrelated, but we shall see in this unit that ideas are closely related by an important theorem called the Fundamental Theorem of Calculus. We shall also learn how to use integration to find the length of curves.

4.1 OBJECTIVES

After studying this unit you should be able to :

- evaluate the definite integral $\int_a^b f(x)dx$;
- use definite integral to find area under a curve and area between two curves; and
- use definite integral to find length of a curve.

4.2 DEFINITE INTEGRAL

Suppose f is a continuous function defined on the interval $[a,b]$, and let F be an antiderivative of f, then we write

$$\int_a^b f(x)dx = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$$

For instance

1. $\int_1^2 xdx = \left[\frac{x^2}{1}\right]_1^2 = \frac{1}{2}(2^2 - 1^1) = \frac{3}{2}$

2. $\int_2^3 \frac{1}{x} dx \quad [\log|x|]_2^3 = \log 3 - \log 2 = \log(3/2)$

3. $\int_0^2 e^x dx = [e^x]_0^2 = e^2 - e^0 = e^2 - 1$

4. $\int_0^1 \frac{dx}{x+1} = [\log|x+1|]_0^1 = \log_2 - \log_1 = \log_2$

Some Properties of Definite Integral

We list these properties without proof.

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(t)dt = \int_a^b f(u)du = \int_a^b f(x)dx$

4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

5. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

6. If f is an odd function, that is $f(-x) = -f(x)$ $\forall\, x \in [-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 0$$

For instance,

$$\int_{-1}^{1} x^3 dx = 0$$

7. If f is an even function, that is, $f(-x) = f(x)$ $\forall\, x \in [-a, a]$, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

8. $\int_0^{2a} f(x)dx = \int_0^a [f(x) + f(2a-x)]dx$

9. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Fundamental Theorem of Calculus

Let f be a continuous function defined on a closed and bounded interval $[a,b]$.
Let

$$\mathrm{A}(x) = \int_0^x f(t)dt,$$

Then $\mathrm{A}'(x) = f(x) \quad \forall\, x \in [a,b]$.

Illustration

Let $f(x) = x$ for $1 \leq x \leq 3$.

Let $\mathrm{A}(x) = \int_1^x f(t)dt = \int_1^x t\,dt = \left[\frac{1}{2}t^2\right]_1^x = \frac{1}{2}(x^2 - 1)$

$\Rightarrow \mathrm{A}'(x) = \frac{1}{2}(2x) = x = f(x)$ for $1 \leq x \leq 3$.

Remark : In fact when $f(x) \geq 0$, $\mathrm{A}(x) = \int_a^x f(t)dt$ represents the area bounded by $y = f(x)$, the x – axis and the ordinates $x = a$ and $x = b$. See Figure 4.1.

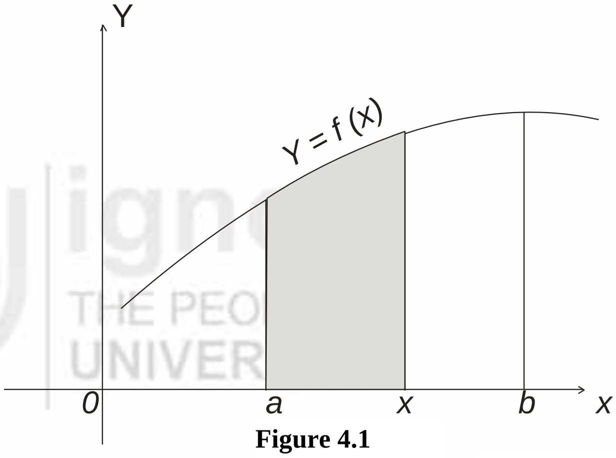


Figure 4.1

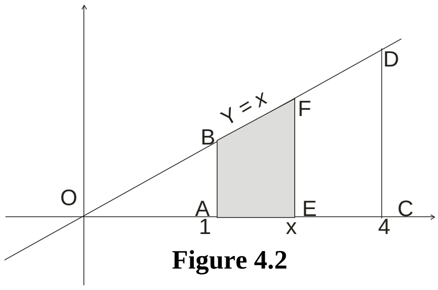


Figure 4.2

We illustrate this fact in the following illustration.

Illustration

Let $f(x) = x$, $1 \leq x \leq 4$.

Area of the trapezium ABFE $= \frac{1}{2}$(sum of parallel sides) × (height)

$= \frac{1}{2}$ (AB + EF) (AE)

$$= \frac{1}{2}(1 + x)(x - 1) = \frac{1}{2}(x^2 - 1)$$

Also, $\mathrm{A}(x) =$

$$\int_1^x t\,dt = \frac{1}{2}\left[t^2\right]_1^x = \frac{1}{2}(x^2 - 1)$$

Thus, $\mathrm{A}(x)$ = area bounded by $y = x$, the *x-axis* and the ordinates $x = 1$, $x = 4$.

Solved Examples

Example 1 : Evaluate the definite integral

$$\int_1^4 \frac{dx}{\sqrt{x}}$$

Solution : We have

$$\int_1^4 \frac{dx}{\sqrt{x}} = \int_1^4 x^{-1/2}\, dx = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^4$$

$$= \left[2\sqrt{x}\right]_1^4 = 2\,(\sqrt{4} - \sqrt{1}\,)$$

$$= 2\,(2 - 1) = 2$$

Example 2 : Evaluate the definite integral

$$\int_3^5 \frac{x}{x^2 - 5}\, dx$$

Solution : We first evaluate the integral

$$\int \frac{x}{x^2 - 5}\, dx$$

We put $x^2 - 5 = t,$ so that $2x dx = dt$.

Thus,

$$\int \frac{x\, dx}{x^2 - 5} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| = \frac{1}{2}\log|x^2 - 5|$$

$$\Rightarrow \int_3^5 \frac{x\, dx}{x^2 - 5} = \left[\frac{1}{2}\log|x^2 - 5|\right]_3^5$$

$$= \frac{1}{2}\left[\log(25 - 5) - \log(9 - 5)\right]$$

$$= \frac{1}{2}\log\frac{20}{4} = \frac{1}{2}\log 5$$

Example 3 Evaluate the definite integral

$$\int_1^3 \frac{dx}{x^2(x+1)}$$

Solution

We first evaluate the integral

$$\int \frac{dx}{x^2(x+1)}$$

Towords this end, we first split $\dfrac{1}{x^2(x+1)}$ into partial fractions

Write $$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow\ 1 = Ax(x+1) + B(x+1) + Cx^2$$

Put $x = 0$ and $x = -1$, to obtain

1 = B and 1 = C.

Next, let us compare coefficient of x^2 to obtain

0 = A + C $\Rightarrow A = -C = -1$

Thus,

$$\int \frac{dx}{x^2(x+1)} = \int \left[-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right] dx$$

$$= -\log|x| - \frac{1}{x} + \log|x+1|$$

$$= \log\left|\frac{x+1}{x}\right| - \frac{1}{x}$$

$$\therefore \int_1^3 \frac{dx}{x^2(x+1)} = \left[\log\left|\frac{x+1}{x}\right| - \frac{1}{x}\right]_1^3$$

$$= \log\frac{4}{3} - \frac{1}{3} - \log 2 + 1$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Example 4 : Evaluate the definite integral

$$\int_{-3}^{-1} \frac{dx}{x}$$

Solution : We have

$$\int_{-3}^{-1} \frac{dx}{x} = [\log\ |x|]_{-3}^{-1}$$

$$= \log 1 - \log 3 = \log\left(\frac{1}{3}\right)$$

Example 5 : Evaluate the definite integral

$$\int_2^4 \frac{dx}{x^2 - 9}$$

Solution : Recall

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left[\frac{x + a}{x - a}\right]$$

Thus,

$$\int_2^4 \frac{dx}{x^2 - 9} = \frac{1}{(2)(3)} \left[log \left|\frac{x + 3}{x - 3}\right|\right]_2^4$$

$$= \frac{1}{6} [\log 7 - log 5]$$

$$= \frac{1}{6} \log \frac{7}{5}$$

Example 6 : Evaluate the definite integral

$$\mathrm{I} = \int_{-1}^{2} \frac{x}{(x^2 + 1)^2}\, dx$$

Solution : We first evaluate the integral

$$\mathrm{I} = \int \frac{x}{(x^2 + 1)^2}\, dx$$

Put $x^2 + 1 = t \Rightarrow 2xdx = dt$

$$\therefore \ \mathrm{I} = \frac{1}{2} \int \frac{\mathrm{dt}}{t^2} = \frac{1}{2} \int \mathrm{t}^{-2}\, \mathrm{dt}$$

$$= \frac{1}{2} \frac{\mathrm{t}^{-2+1}}{-2+1} = -\frac{1}{2\mathrm{t}} = -\frac{1}{2(x^2 + 1)}$$

Thus,

$$\int_{-1}^{2} \frac{x}{(x^2 + 1)^2}\, dx = \left[-\frac{1}{2(x^2 + 1)}\right]_{-1}^{2}$$

$$= -\frac{1}{2(4 + 1)} + \frac{1}{2(1 + 1)}$$

$$= \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

Example 7: Evaluate the definite integral

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$$

Solution : We first evaluate the integral

$$\text{I} = \int \frac{x}{\sqrt{2x-1}}\,dx$$

We put $2x - 1 = t^2 \Rightarrow x = \frac{1}{2}(t^2+1)$

$$\Rightarrow dx = \tfrac{1}{2}(2t)dt = tdt$$

Thus,

$$\text{I} = \int \frac{\frac{1}{2}(t^2+1)}{t}\, t\,\text{dt} = \frac{1}{2}\int (t^2+1)\,dt$$

$$= \frac{1}{2}\left[\frac{1}{3}t^3 + t\right] = \frac{1}{2}\left[\frac{1}{3}\,(2x-1)^{3/2} + \sqrt{2x-1}\right]$$

$$\therefore \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \left[\frac{1}{6}(2x-1)^{\frac{3}{2}} + \frac{1}{2}(2x-1)^{\frac{1}{2}}\right]_1^5$$

$$= \frac{1}{6}\left[9^{\frac{3}{2}} - 1\right] + \frac{1}{2}\left[9^{\frac{1}{2}} - 1\right]$$

$$= \frac{1}{6}(26) + \frac{1}{2}(2)$$

$$= \frac{13}{3} + 1 = \frac{16}{3}$$

Example 8 : Evaluate the definite integral

$$\int_0^1 \frac{24x^3}{(1+x^2)^4}\,dx$$

Solution : We first evaluate the integral

$$\text{I} = \int \frac{24x^3}{(1+x^2)^4}\,dx = 12\int \frac{2x^2}{(1+x^2)^4}\,x\,dx$$

Put $1 + x^2 = t \Rightarrow 2xdx = \text{dt}$,

So, that

$$\text{I} = 12\int \frac{(t-1)}{t^4}\,dt = 12\int (t^{-3} - t^{-4})\,dt$$

$$= 12\left[\frac{t^{-2}}{-2} - \frac{t^{-3}}{-3}\right] = 12\left(\frac{1}{3t^3} - \frac{1}{2t^2}\right)$$

$$= \frac{4}{t^3} - \frac{6}{t^2} = \frac{4}{(1+x^2)^3} - \frac{6}{(1+x^2)^2}$$

$$\text{Thus, } \int_0^1 \frac{24x^3}{(1+x^2)^4}\, dx = \left[\frac{4}{(1+x^2)^3} - \frac{6}{(1+x^2)^2}\right]_0^1$$

$$= \frac{4}{2^3} - \frac{6}{2^2} - 4 + 6$$

$$= \frac{1}{2} - \frac{3}{2} + 2 = 1$$

Check Your Progress – 1

Evaluate the following definite integrals.

1. $\int_0^2 \frac{x}{x+1}\, dx$ 2. $\int_1^2 \frac{dx}{(x+1)(x+2)}$

3. $\int_0^3 (e^x + x)\quad dx$ 4. $\int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$

5. $\int_0^2 x\sqrt{3x-2}\, dx$ 6. $\int_0^2 \frac{dx}{x+4-x^2}$

7. $\int_a^b \frac{logx}{x}\, dx$ 8. $\int_0^a \frac{x^3}{\sqrt{a^2-x^2}}\, dx$

Answers

1. 2 – log2 2. $\log\left(\frac{9}{8}\right)$

3. $(e^3 - 1) + 9/2$ 4. 1 5. 326/135

6. $\frac{1}{\sqrt{17}}\log\left(\frac{21-5\sqrt{17}}{4}\right)$ 7. $\frac{1}{2}\left(\log\frac{b}{a}\right)\ \log ab$

8. $\frac{2}{3}a^3$

4.3 AREA UNDER THE CURVE

Suppose f is continuous and

$f(x) \geq 0\ \forall\ x \in [a,b]$, then area bounded by $y = f(x)$, the x – *axis* and the ordinates $x = a$, and $x = b$ is given by

$$\int_a^b f(x)dx$$

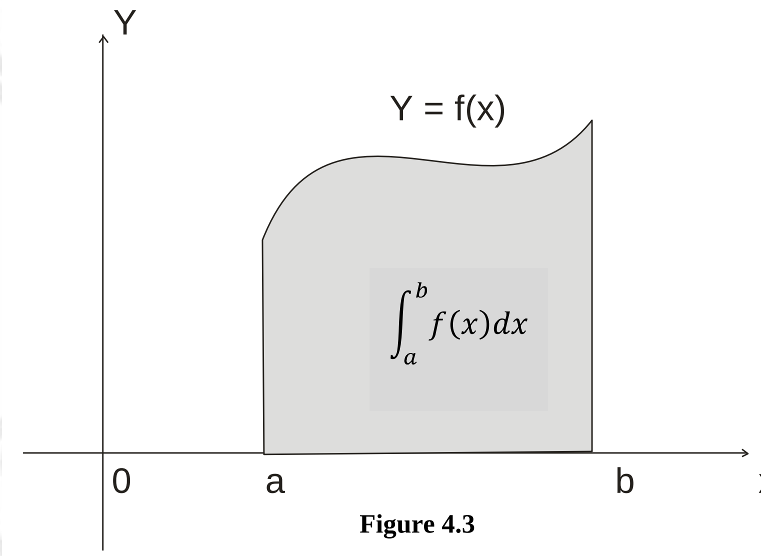


Figure 4.3

Area between two Curves

Suppose f and g be two continuous functions defined on $[a,b]$ such that $f(x) \geq g(x)\ \forall\ x \in [a,b]$ then area bounded by $y = f(x)$, $y = g(x)$ and the ordinates $x = a$ and $x = b$, is

$$\int_a^b [f(x) - g(x)]dx$$

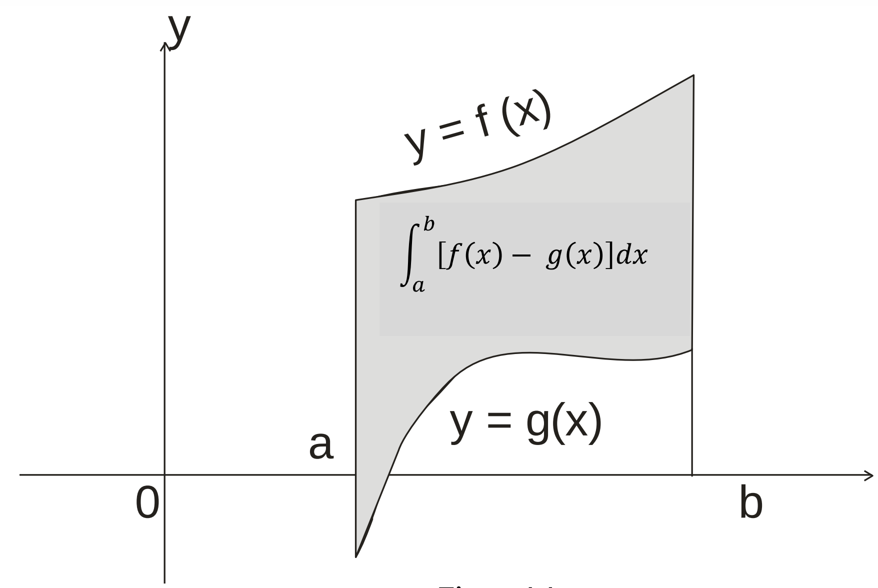


Figure 4.4

Solved Examples

Example 9 : Find the area bounded by the x –axis and $y = 2x - 3$, $2 \leq x \leq 5$.

Solution : $y = 2x - 3$ represents a straight line *passing* through (2, 1) and (5, 7)

Required area

$$= \int_2^5 (2x - 3)dx$$

$$= \left[x^2 - 3x\right]_2^5$$

$= 25 - 3(5) - (4 - 6)$

$= 12$ sq. units

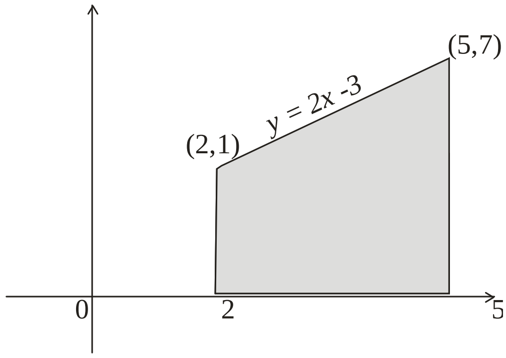


Figure 4.5

Example 10: Find the area bounded by the x–*axis*, $y = x^2$, $0 \leq x \leq 3$.

Solution $y = x^2$ represents a parabola that open upwards.
Required area

$$= \int_0^3 x^2 \, dx$$

$$= \left[\frac{1}{3}x^3\right]_0^3 = 9 \text{ sq. units}$$

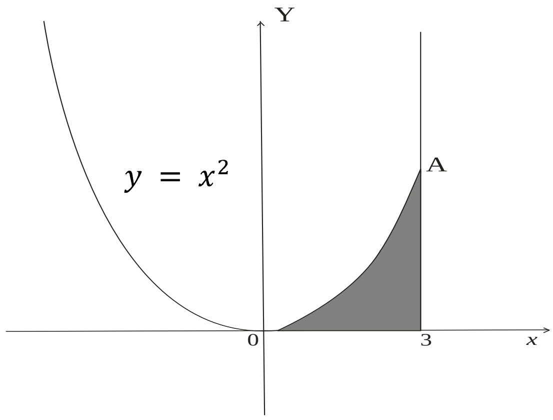


Figure 4.6

Example 11: Find the area bounded by the x– *axis*, $y = \sqrt{x}$ and $1 \leq x \leq 4$.

Solution : $y = \sqrt{x}$ represents part of parabola whose axis is the x-axis.

Required area

$$= \int_1^4 \sqrt{x}dx = \int_1^4 x^{1/2}dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2/3\,(4^{3/2} - 1^{3/2})$$

$$= \frac{14}{3} \text{ sq. units}$$

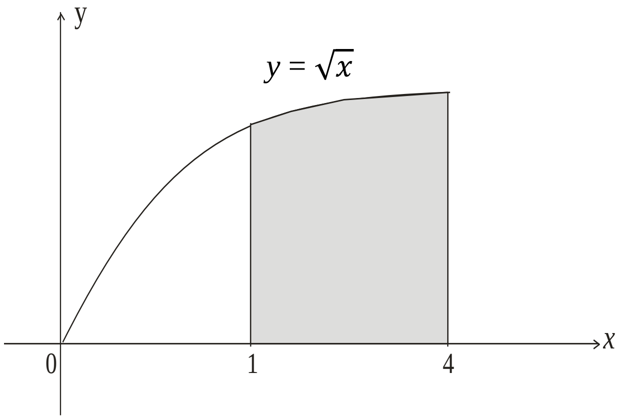


Figure 4.7

Example 12 Find the area bounded by the curves y = x^2 and $y = x$.

Solution : To obtain point of intersection of y = x^2 and $y = x$, we set

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1.$

Thus, the two curves intersect in (0,0) and (1,1). See Fig 4.8

Required area

$= \int_0^1 (x - x^2)dx$

$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$

$= \left(\frac{1}{2} - \frac{1}{3}\right)$ sq. units

$= \frac{1}{6}$ sq. units.

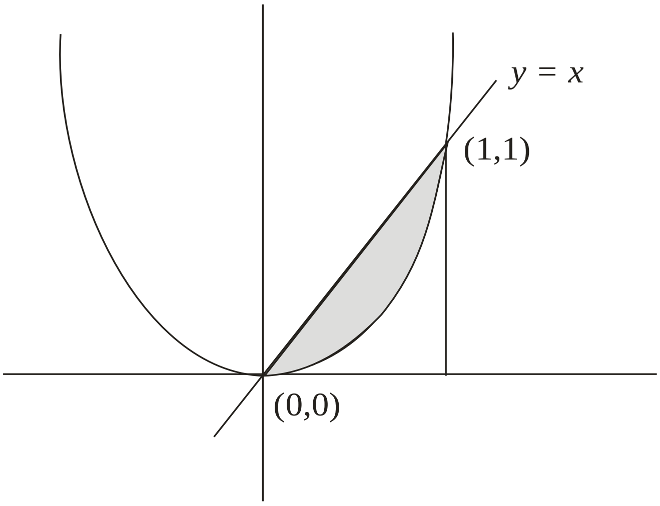


Figure 4.8

Example 13 : Find the area bounded by the curves $y = \sqrt{x}$ *and* $y = x$.

Solution : To obtain the point of intersection,

We set $\sqrt{x} = x \Rightarrow \sqrt{x}(1 - \sqrt{x}) = 0$

$\Rightarrow \sqrt{x} = 0$ or $1 - \sqrt{x} = 0 \Rightarrow x = 0$ or $x = 1$.

Required area

$= \int_0^1 (\sqrt{x} - x)dx$

$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$

$= \left(\frac{2}{3} - \frac{1}{2}\right)$sq.units

$= \left(\frac{1}{6}\right)$sq.units

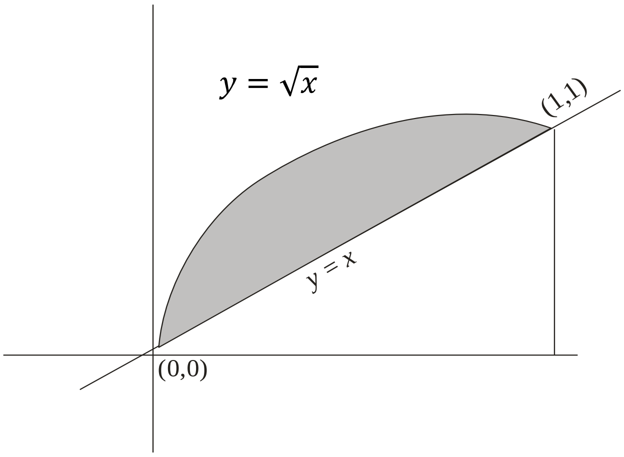


Figure 4.9

Example 14 : Find the area lying between

$y = 2x + 1, y = 3x, \quad 1 \leq x \leq 4.$

Solution : The two straight lines intersect in (1,3)

Required area

$$= \int_1^4 [3x - (2x+1)]\, dx$$

$$= \int_1^4 [(x-1)dx = \left[\frac{1}{2}x^2 - x\right]_1^4 \, dx$$

$$= \left[\frac{1}{2}(4^2) - 4 - \frac{1}{2}(1^2) + 1\right] \text{ sq. units.}$$

= 4.5 sq. units.

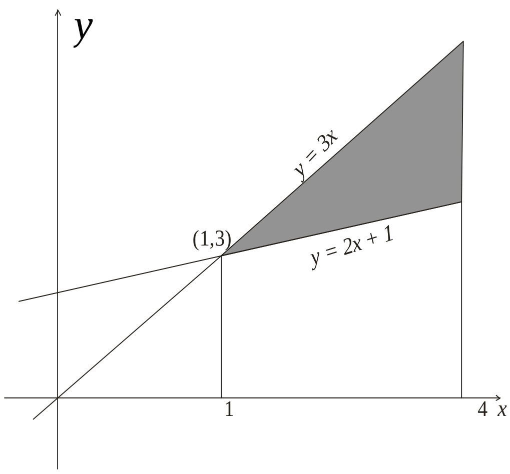


Figure 4.10

Example 15: Find the area bounded by the x – axis, $y = \frac{1}{x}$, $1 \le x \le 4$.

Solution :

$$\text{Required area} = \int_1^4 \frac{1}{x}\, dx$$

$$= \left[\log x\right]_1^4$$

= log 4 – log 1

= 2log 2 sq. units

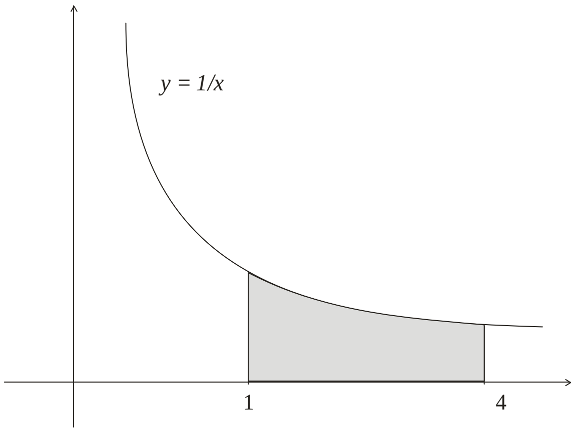


Figure 4.11

Check Your Progress – 2

1. Find the area bounded by the x–axis, $y = 5 - 2x$ and the y – axis.

2. Find the area bounded by the x–axis, $y = 2 + 3x$ and the ordinates $x = 0$ and $x = 3$

3. Find the area bounded lying between the lines $y = 3 + 2x$, $y = 3 - x$, $0 \le x \le 3$.

4. Find the area bounded by the curves y = x^2 and $y^2 = x$.

5. Find the area bounded by the line $y = x$ and the parabola $y^2 = x$.

6. Find the area bounded by the curve y = e^x, the x–axis and the ordinates $x = 1$ and $x = 3$.

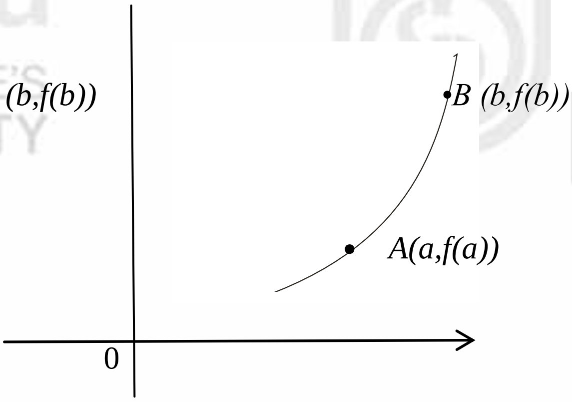
7. Find the area lying in the first quadrant and bounded by the x-axis, the line $y = x$ the curve $y = 1/x$ and the ordinate $x = 2$.

4.4 LENGTH OF CURVES (RECTIFICATION)

To find length of curve

Y = *f(x)*, from the point of A*(a, f(a))* and B *(b,f(b))*

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

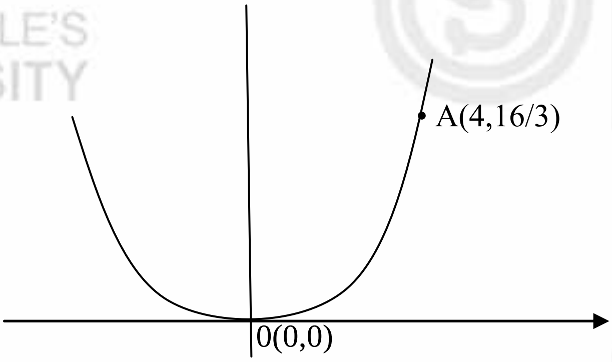


Solved Examples

Example 16 Find the length of the curve

$$y = \frac{2}{3}x^{3/2} \text{ from } (0,0) \text{to } (4, 16/3)$$

Solution : We have



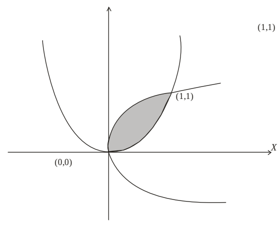
$$\frac{dy}{dx} = \frac{2}{3}.\frac{3}{2}x^{1/2} = x^{1/2}$$

Required length of the curve

$$= \int_0^4 \sqrt{1 + (x^{1/2})^2}\ dx$$

$$\int_0^4 \sqrt{1+x}dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^4$$

$$= \frac{2}{3}\left[5\sqrt{5} - 1\right]\ units$$



Example 17 : Find the length of the curve y = 2*x* + 3

Solution : We have

$$\frac{dy}{dx} = 2$$

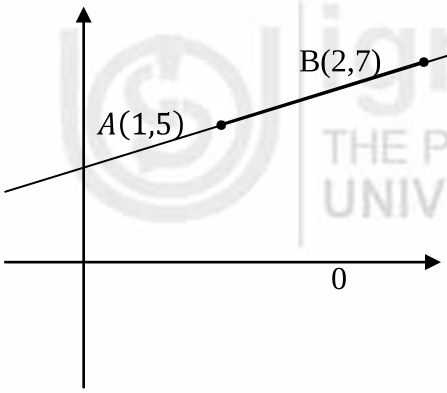
Required length

$$= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

$$= \int_1^2 \sqrt{1+4}\; dx$$

$$= \left[\sqrt{5}\,x\right]_1^2$$

$$= 2\sqrt{5} - \sqrt{5} = \sqrt{5}\; units$$



Check Your Progress – 3

1. Find the length of the curve y = 3 – x from (–1, 4) to (3,0).
2. Find the length of the curve y = 3 + $\frac{1}{2}x$ from (0,3) to (2, 4).
3. Find the length of the curve y = $2x^{3/2}$ from point (1,2) to (4,16).

4.7 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. $\int_0^2 \frac{x}{x+1}\,dx = \int_0^2 \frac{x+1-1}{x+1}\,dx \int_0^2 \left[1 - \frac{1}{x+1}\right] dx$

$$= x - \log(x+1) \Big|_0^2$$

$= [2 – \log2] – [0 – \log1]$

$= 2 – \log2$

2. $\int_1^2 \frac{dx}{(x+1)(x+2)} = \int_0^2 \left[\frac{1}{x+1} - \frac{1}{x+2}\right] dx$

[split into partial fractions]

$$= \left[\log\left|\frac{x+1}{x+2}\right|\right]_1^2 = \log\frac{3}{4} - \log\frac{2}{3}$$

$$= \log\left(\frac{9}{8}\right)$$

3. $\int_0^3 (e^x + x)dx = \left[\left(e^x + \frac{1}{2}x^2\right)\right]_0^3$

$$= (e^3 - 1) + \frac{9}{2}$$

4. To evaluate $\quad I = \int \frac{x}{\sqrt{1-x^2}}\,dx$

put $1 - x^2 = t^2$, so that $-xdx = t\,dt$, and

$$I = \int \frac{-t}{t}\,dt = -\int dt = -t = -\sqrt{1-x^2}$$

$$Thus, \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \left[-\sqrt{1-x^2}\,\right]_0^1 = -0 + 1 = 1$$

5. To evaluate $\quad I = \int x\sqrt{3x-2}\,dx$, put

$3x - 2 = t^2$, so that $3dx = 2t\,dt$

$$\therefore I = \int \frac{1}{3}(t^2 + 2)\frac{2}{3}t.tdt = \frac{2}{9}\int (t^4 + 2t^2)dt$$

$$= \frac{2}{9}\left(\frac{1}{5}t^5 + \frac{2}{3}t^3\right)$$

$$= \frac{2}{45}(3x-2)^{\frac{5}{2}} + \frac{4}{27}(3x-2)^{\frac{3}{2}}$$

Thus, $\int_1^2 x\sqrt{3x-2}\,dx$

$$= \left[\frac{2}{45}(3x-2)^{\frac{5}{2}} + \frac{4}{27}(3x-2)^{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{45}\left(4^{5/2} - 1^{5/2}\right) + \frac{4}{27}\left(4^{3/2} - 1^{3/2}\right)$$

$$= \frac{2}{45}(2^5 - 1) + \frac{4}{27}(8-1)$$

$$= \frac{62}{45} + \frac{28}{27} = \frac{326}{135}$$

6. We have $x + 4 - x^2 = -(x^2 - x - 4)$

$$= -\left[x^2 - 2\left(\frac{1}{2}\right) + \frac{1}{4} - 4 - \frac{1}{4}\right]$$

$$= -\left[\left(x - \frac{1}{2}\right)^2 - \frac{17}{4}\right] = \frac{17}{4} - \left(x - \frac{1}{2}\right)^2$$

We have

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{\frac{17}{4}-(x-\frac{1}{2})^2}$$

$$= \left[\frac{1}{\sqrt{17}} log \left| \frac{\sqrt{\frac{17}{4}}-(x-\frac{1}{2})^2}{\sqrt{\frac{17}{4}}+(x-\frac{1}{2})^2} \right|\right]_0^2$$

$$= \frac{1}{\sqrt{17}} \log \left|\frac{(\sqrt{17}-3)/2}{(\sqrt{17}+3)/2}\right| - \log \left|\frac{(\sqrt{17}+1)/2}{(\sqrt{17}-1)/2}\right|$$

$$= \frac{1}{\sqrt{17}} \log \left|\frac{(\sqrt{17}-3)(\sqrt{17}-1)}{(\sqrt{17}+3)(\sqrt{17}+1)}\right|$$

$$= \frac{1}{\sqrt{17}} \log \left|\frac{17+3-4\sqrt{17}}{17+3+4\sqrt{17}}\right| = \log \left|\frac{5-\sqrt{17}}{5+\sqrt{17}}\right|$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{(5-\sqrt{17})^2}{25-17}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{25+17-10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{21-5\sqrt{17}}{4}\right)$$

7. Put $\log x = t$, so that $\frac{1}{x} dx = dt$.

$$\therefore \int \frac{\log x}{x} dx = \int t\, dt = \frac{1}{2} t^2 = \frac{1}{2} (log x)^2$$

Thus,

$$\int_a^b \frac{\log x}{x} dx = \left[(\log x)^2\right]_b^a$$

$$= \frac{1}{2}[(\log b)^2 - (\log a)^2]$$

$$= \frac{1}{2} [\log b - \log a)\ (\log b + \log a)]]$$

$$= \frac{1}{2} (\log \frac{b}{a})\ (\log(ab)).$$

8. Put $a^2 - x^2 = t^2$, so that $-2x\,dx = 2t\,dt$

$$\int \frac{x^3}{\sqrt{a^2 - x^2}}\,dx = \int \frac{x^2}{\sqrt{a^2 - x^2}}\,x\,dx$$

$$= \int \frac{a^2 - t^2}{t}\,(-t)\,dt$$

$$= \int (t^2 - a^2)\,dt = \frac{1}{3}t^3 - a^2 t$$

$$= \frac{1}{3}(a^2 - x^2)^{3/2} - a^2(a^2 - x^2)^{1/2}$$

Thus,

$$\int_0^a \frac{x^3}{\sqrt{a^2 - x^2}}\,dx$$

$$= \left[\frac{1}{3}(a^2 - x^2)^{3/2} - a^2(a^2 - x^2)^{1/2}\right]_0^a$$

$$= 0 - \frac{1}{3}a^3 + a^3 = \frac{2}{3}a^3$$

Check Your Progress – 2

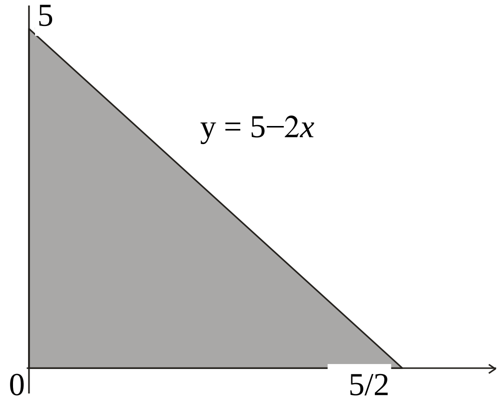
1. The line y = 5 – 2x meets the x–axis at (5/2, 0) and the y–axis at (0,5).

Required area

$$= \int_0^{5/2}(5 - 2x)\,dx$$

$$= \left[5x - x^2\right]_0^{5/2}$$

$$= \frac{25}{4}\ \text{sq. units}$$

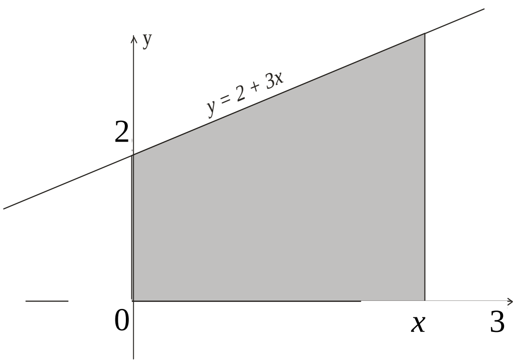


2. Required area

$$= \int_0^3 (2 + 3x)\,dx$$

$$= \left[2x + \frac{3}{2}x^2\right]_0^3$$

$$= 6 + \frac{3}{2}(9) = \frac{39}{2}\ \text{sq. units}$$

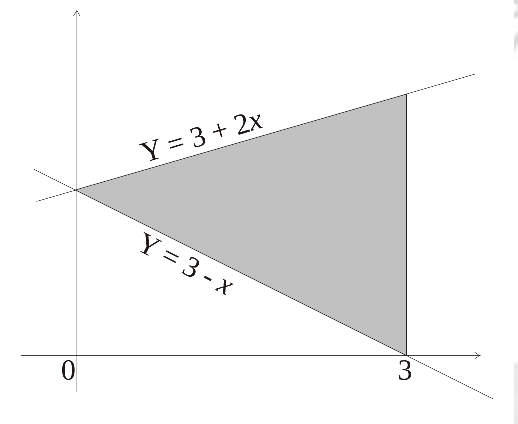


3. $y = 3 + 2x$ represents a straight line passing through (0,3) and sloping upwards, and $y = 3 - 2x$ represents a straight line passing through (0,3) and sloping downwards and meeting the x –axis at (3,0)

Required area

$$= \int_0^3 [3 + 2x - (3 - x)]dx$$

$$= \int_0^3 3x dx \; = \left[\frac{3}{2}x^2\right]_0^3 = \frac{27}{2} \text{ sq. units}$$



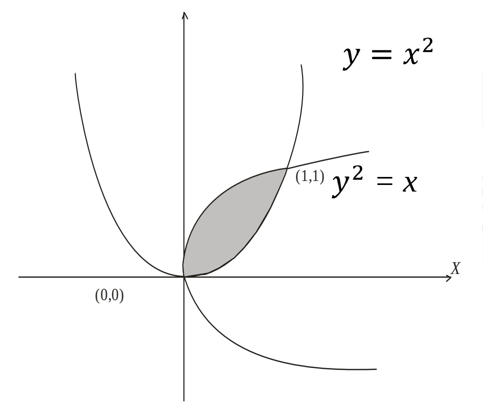
4. We first find the points of intersection of $y = x^2$ and $y^2 = x$. We have $x = y^2 = (x^2)^2$

$\Rightarrow x = x^4$

$\Rightarrow x(1 - x^3) = 0$

$\Rightarrow x = 0$ or $x = 1$

Required area



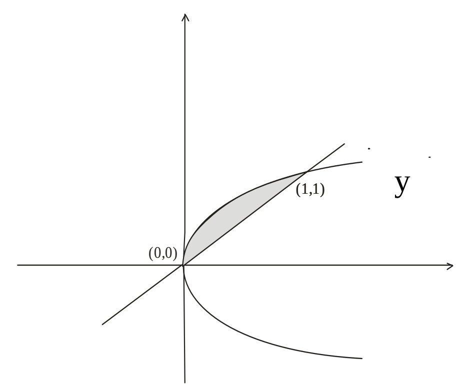
$$= \int_0^1 (\sqrt{x} - x^2)\, dx = \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units}$$

5. We first find the point of intersections of $y = x$ and $y^2 = x$. We have $x^2 = x$

$\Rightarrow x(x - 1) = 0$

$\Rightarrow x = 0$ or $x = 1$

Required area



x

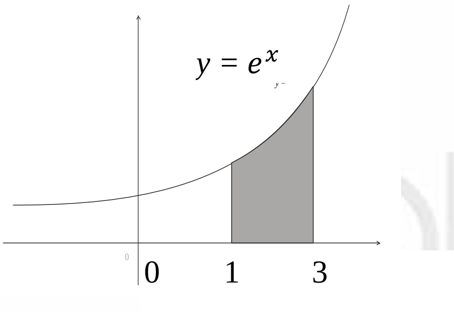
$$= \int_0^1 (\sqrt{x} - x)\, dx = \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{6} \text{ sq. units}$$

6. Required area

$= \int_1^3 e^x \, dx$

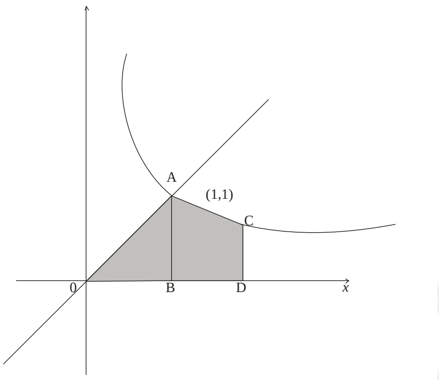
$= \left[e^x \right]_1^3 = (e^3 - e)\ sq.units$



7. To obtain the points of intersection of

y = *x* and *y* = *1/x*,

We set
$x = 1/x$
$\Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$
We take $x = 1$, as we are interested in the a



Required area

= area (OAB) + area (BACD)

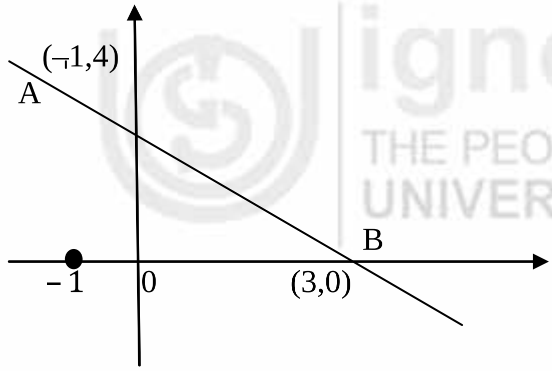
$$\int_0^1 x\ dx + \int_1^2 \frac{1}{x}\, dx$$

$$= \left[\frac{1}{2} x^2 \right]_0^1 + \log x \Big|_1^2$$

$$= \left(\frac{1}{2} + \log 2 \right) \text{sq.units}$$

Check Your Progress – 3

1. We have



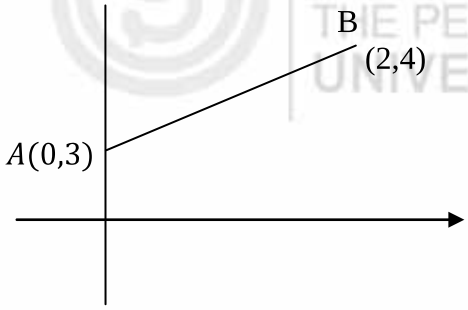
$$\frac{dy}{dx} = -1$$

Required length

$$= \int_{-1}^{3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_{1}^{3} \sqrt{1+\ 1}\ dx = \sqrt{2}\ x\Big|_{-1}^{3} = \sqrt{2}\,(3+1) = 4\sqrt{2}\ \text{units}$$

2. We have



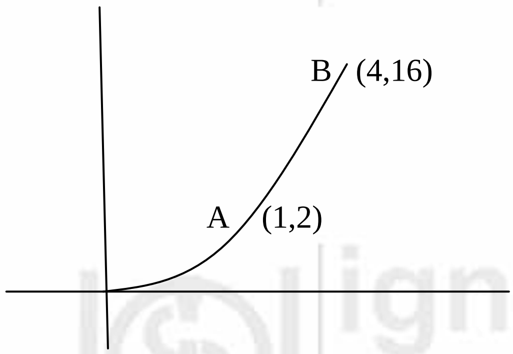
$$\frac{dy}{dx} = \frac{1}{2}$$

Required length

$$= \int_{0}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_{0}^{2} \sqrt{1 + \frac{1}{4}}\ dx\ = \frac{\sqrt{5}}{2}\int_{0}^{2} dx\ = \frac{\sqrt{5}}{2}\ \ x\Big|_{0}^{2} = \ \sqrt{5}\ \text{units}$$

3. We have



$$\frac{dy}{dx} = 3x^{1/2}$$

Required length

$$= \int_{1}^{4} \sqrt{1+\ 9x}\ dx = \left[\frac{(1+9x)^{3/2}}{9\left(\frac{3}{2}\right)}\right]_{1}^{4} = \frac{2}{27}\left[37\sqrt{37} - \ 10\sqrt{10}\ \right]\text{units}$$

4.6 SUMMARY

In this unit, as the title of the unit suggests, applications of Integral Calculus, are discussed. In **section 4.2**, first, the concept of 'definite integral' is introduced and then methods of finding the value of a definite integral, are illustrated through examples. Methods of finding the area under a curve is illustrated in **section 4.3**. Method of finding the length of a curve is discussed in **section 4.4** .

Answers/Solutions to questions/problems/exercises given sections of the unit are available in **section 4.5**.

UNIT 1 VECTORS – 1

Stucture

1.0 INTRODUCTION

In this unit, we shall study vectors. A vectors is a quantity having both magnitude and direction, such as displacement, velocity, force etc. A scalar is a quantity having magnitude only but no direction, such as mass, length, time etc. A vector is represented by a directed line segment. A directed line segment is a portion of a straight line, where the two end-points are distinguished as initial and terminal.

Scalars are represented using single real numbers as complex numbers only. A vector in plane is represented using two real numbers. This is done by considering a rectangular coordinate system by which every point in place is associated with a pair of numbers (x,y). Then the vector whose initial point is origin and terminal pont is (x,y) is $x\,\hat{\imath} + y\,\hat{\jmath}$. This is component form of a vector. Similarly, component form of a vector in space is $x\,\hat{\imath} + y\,\hat{\jmath} + z\hat{k}$. We shall discuss it in detail in this Unit. Further, we shall learn to add and subtract vectors and to multiply a vector by a scalar. There are many applications of vectors in geometry. We shall prove the section formula for vectors and solve many problems in geometry using vectors.

1.1 OBJECTIVES

After studying this unit, you will be able to :

- define the terms scalar and vector;
- define position vector of a point;
- find direction cosines and direction ratios of a vector;
- find sum and difference of two vectors;
- multiply a vector by a scalar;
- write a vector in component form; and
- use selection formula in geometrical problems.

1.2 VECTORS AND SCALARS

Many quantities in geometry and physics, such as area, volume, mass, temperature, speed etc. are characterized by a single real number called magnitude only. Such quantities are called **scalars.** There are, however, other physical quantities such as displacement, force, velocity, acceleration etc. which cannot be characterized by a single real number only. We require both magnitude and direction to specify them completely. Such quantities are called **vectors.**

Definition : A quantity having both magnitude and direction is called a vector. Graphically, a vector is represented by a directed line segement. A directed line segment with initial point A and terminal point B is denoted by $\overrightarrow{AB}$.

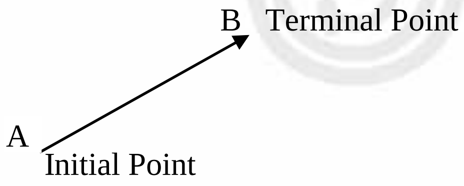


Figure 1

We usually denote vectors by lower case, bold face letters $\bar{a}, \bar{b}$, etc. or by letters with arrows above them such as $\bar{a}, \bar{b}$,etc.

If $\vec{a}$ is a vector represented by directed line segment $\overrightarrow{AB}$, then magnitude of $\vec{a}$ is the length of $\overrightarrow{AB}$ and is dentoed by $|\vec{a}|$or $|\overrightarrow{AB}|$.

Two vectors are said to be equal if they have same magnitude and the same direction.

A vector with zero magnitude (i.e., when initial point and terminal point coincide) is called a **zero vector** or **null vector.** A vector whose magnitude is unity (i.e., 1 unit) is called a **unit vector.**

Position Vector and Direction Cosines

The position vector $\vec{r}$ of any point P with respect to the origin of reference O is a vector $\overrightarrow{OP}$. Recall that a point P in space is uniquely determined by three coordinates. If P has coordinates (x,y, z) and O is the origin of the rectangular coordinates system, then the magnitude of the position vector $\overrightarrow{OP}$ is given by

$$|\vec{r}| = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

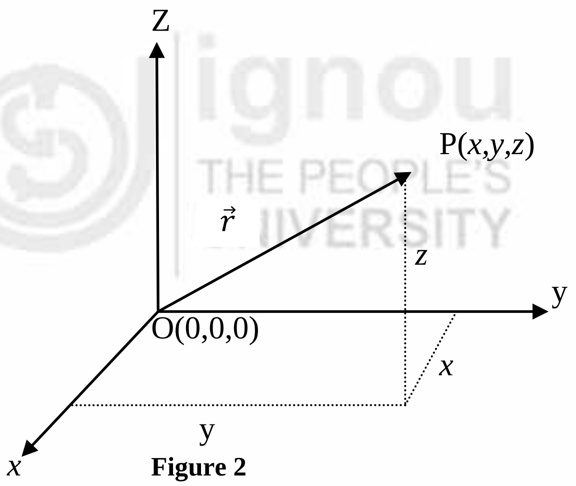


Figure 2

Suppose $\overrightarrow{OP}$ makes angles α, β and γ with the positive directions of *x*, *y* and *z* – axes respectively. These angles are called **direction angles** of $\overrightarrow{OP}$ $(or\ \vec{r}\ )$. The cosines of these angles i.e., $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called **direction cosines** of the vector $\vec{r}$ and usually denoted by *l*, *m* and *n* respectively.

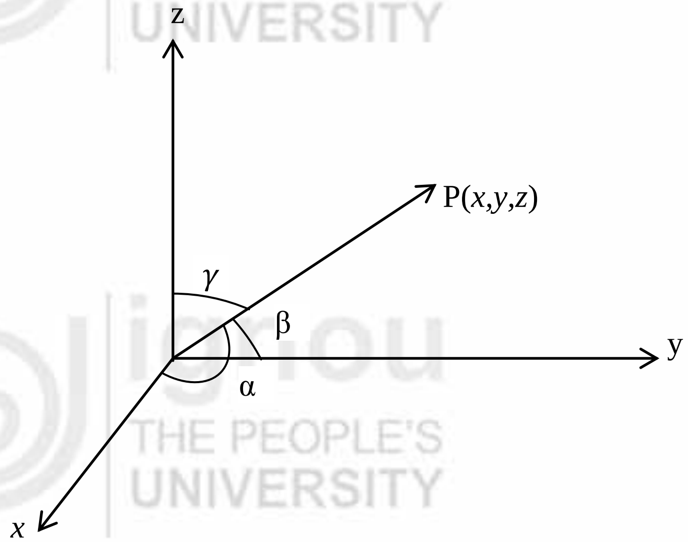


Figure 3

It must be noted that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

i.e., $l^2 + m^2 + n^2 = 1$

In fact, note that $\cos\alpha = \frac{x}{r}$ (here, $r = |\vec{r}|$). Similarly, $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$.

Thus, the coordinates of the point P may also be expressed as (*lr*, *mr*, *nr*). The numbers *lr*, *mr* and *nr*, proportional to the direction cosines are called **direction ratios** of vector $\vec{r}$, and denoted by *a*, *b* and *c* respectively.

Example 1 : Find the magnitude and direction angles of the position vector of the point P(1,2,−2).

Solution : Let $\vec{r} = \overrightarrow{\mathrm{OP}}$. We have

$$r = |\vec{r}| = |\overrightarrow{\mathrm{OP}}| = \sqrt{1^2 + 2^2 + (-2)^2} = 3$$

and $\cos\alpha = \dfrac{x}{r} = \dfrac{1}{3}$

and $\cos\beta = \dfrac{y}{r} = \dfrac{2}{3}$

$$\cos\Upsilon = \frac{z}{r} = \frac{-2}{3}.$$

Hence $\alpha = \cos^{-1}\dfrac{1}{3} \approx 70^0 30'$

$$\beta = \cos^{-1}\frac{2}{3} \approx 48^0 10'$$

$$\gamma = \cos^{-1}\left(\frac{2}{3}\right) \approx 131^0 50'$$

Thus, the vector $\vec{r}$ forms acute antles with the *x*-axis and *y*-axis, and an obtuse angle with the *z*-axis.

Coinitial, Collinear and Coplanar Vectors

Two or more vector having the same initial points are called **coinitial Vectors.** Two vectors are said to be **collinear** if they are parallel to the same line, irrespective of their magnitude and directions. A set of vectors is said to be c**oplanar** if they lie on the same plane, or the planes in which the different vectors lie are all parallel to the same plane.

Addition of Vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors. We position them so that the initial point of $\vec{b}$ is the terminal point of $\vec{a}$. Then the vector extending from the intial point of $\vec{a}$ to the terminal point of $\vec{b}$ is defined as the **sum** of $\vec{a}$ and $\vec{b}$ and is denoted as $\vec{a} + \vec{b}$.

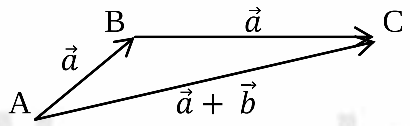


Figure 4

In the Figure 4 $\vec{a} = \overline{AB}$, $\vec{b} = \overline{BC}$ and

$\vec{a} + \vec{b} = \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

This is called **triangle law of addition of** vectors.

If the vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides of a parallelogram, then their sum $\vec{a} + \vec{b}$ is the vector represented by the diagonal of the parallelogram through their common point. **This is called parallelogram law of vector addition.**

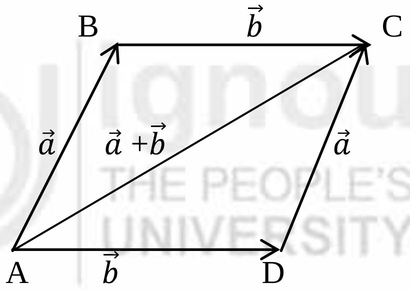


Figure 5

In above figure 5, by triangle law

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

But $\overrightarrow{BC}=\overrightarrow{AD}$

$\therefore \quad \overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$

which is the paralellogram law.

Properties of Vector Addition

1. Addition of vectors is commulative i.e.,
   $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
   for any vectors $\vec{a}$ and $\vec{b}$.

Proof : Let $\overrightarrow{\mathrm{OA}}=\vec{a}$ and $\overrightarrow{\mathrm{AB}}=\vec{b}$. We have $\vec{a}+\vec{b}=\overrightarrow{\mathrm{OB}}$.

Completing the parallelogram OABC having OA and OB as adjacent sides (see Figure 6), we have

$\overrightarrow{OC}=\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{CB}=\overrightarrow{OA}=\vec{a}$ = $\overrightarrow{AB}$=$\vec{a}$ and $\overrightarrow{CB}=\overrightarrow{OA}=\vec{a}$

So, we have

$\overrightarrow{OC}=\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{CB}=\overrightarrow{OA}=\vec{a}$

Here, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

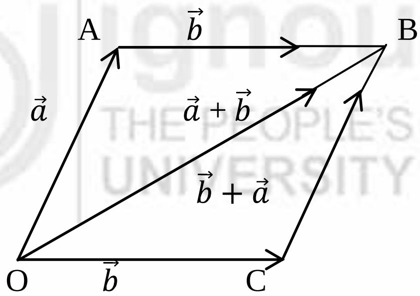


Figure 6

2. Addition of vectors is associative, i.e.,

$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$, where $\vec{a},\vec{b},\vec{c}$ are any three vectors.

Proof : Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$ respectively.

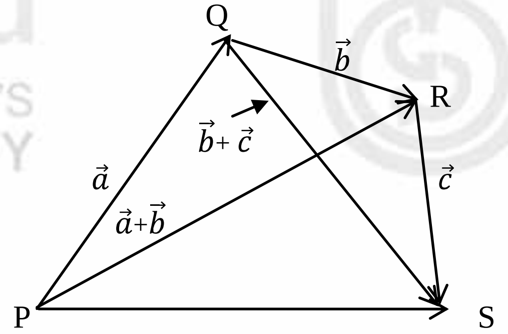


Figure 7

Then $\vec{a}+\vec{b}=\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{PR}$

$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

Again, $\vec{b}+\vec{c}=\overrightarrow{QR}+\overrightarrow{RS}=\overrightarrow{QS}$

$\Rightarrow \quad \vec{a}+(\vec{b}+\vec{c})=\overrightarrow{PQ}+\overrightarrow{QS}=\overrightarrow{PS}$

Hence $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

3. If $\vec{a}$ is any vector, and $\vec{0}$ is the zero vector then $\vec{a}+\vec{0}=\vec{a}$.
   Here, zero vector $\vec{0}$ is the additive identity for vector addition.

Difference of Vectors

Let $\vec{a}$ be a vector represented by $\overrightarrow{AB}$. Then ***negative*** of $\vec{a}$, is denoted by $-\vec{a}$ is defined by the vector $\overrightarrow{BA}$. So, $-\vec{a}$ is a vector having same magnitude as $\vec{a}$ but direction opposite to that of $\vec{a}$. It is also clear that

$\vec{a}+(-\vec{a})=\overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{AA}=\vec{0}$

If $\vec{a}$ and $\vec{b}$ are two vectors, then their difference is defined by

$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

Geometrically, let $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{BC}$ and construct $\overrightarrow{BC'}$, so that its magnitude is same as the vector $\overrightarrow{BC}$, but the direction opposite to $\overrightarrow{BC}$ $i.e.,\overrightarrow{BC'}=-\overrightarrow{BC}$, as shown in figure below (Figure 8).

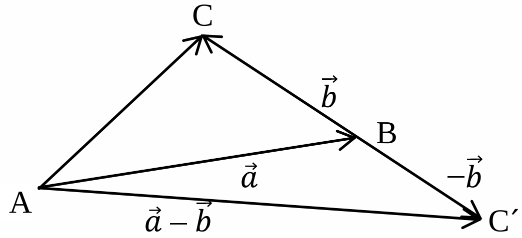


Figure 8

Then, by triangle law

$$\overrightarrow{AC}' = \overrightarrow{AB} + \overrightarrow{BC}' = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}.$$

The vector $\overrightarrow{AC'}$ represents difference of $\vec{a}$ and $\vec{b}$.

Multiplication of a Vector by a Scalar

The product of vector $\vec{a}$ by a scalar, λ denoted by $\lambda\vec{a}$ is defined as a vector whose length is $|\lambda|$ times that of $\vec{a}$ and whose direction is the same or opposite as that of $\vec{a}$ according as λ is positive or negative.

If $\lambda = -1$ then $\lambda\vec{a} = (-1)\,\vec{a} = -\vec{a}$, which is negative of vector $\vec{a}$. Also, note that $\vec{a} + (-1)\vec{b} = \vec{a} - \vec{b}$ and $-\lambda\vec{a} = \lambda\,(-\vec{a})$.

Let $\vec{a}$ be a non zero vector. If we take $\lambda = \frac{1}{|\vec{a}|}$, then $\lambda\,\vec{a}$ is a unit vector take because

$$|\lambda\,\vec{a}| = |\lambda||\,\vec{a}| = \frac{1}{|\,\vec{a}|}|\,\vec{a}| = 1.$$

This vector $\lambda\vec{a}$ is denoted by $\hat{a}$, and is the unit vector in the direction of $\vec{a}$.

Thus, $\hat{a} = \frac{1}{|\vec{a}|}$.

Let $\vec{a}$ be a vector and λ_1 and λ_2 are scalars. Then it is easy to prove that

$$\lambda_1(\lambda_2\,\vec{a}\,) = (\lambda_1\lambda_2)\vec{a}$$

Distributive Property : Let $\vec{a}$ and $\vec{b}$be vectors and α and β be any scalars. Then

(*i*) $(\vec{a} + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$

(*ii*) $\alpha\ (\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}.$

Proof of (i)

Let $\overrightarrow{AB}$ be vector $\vec{a}$ as shown in figure below (Figure 9)

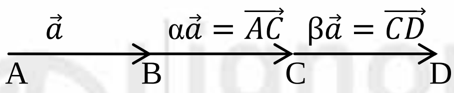


Figure 9

First assume that α and β are positive. Then $\alpha\vec{a}$ is given by $\overrightarrow{AC}$ which is in same direction as $\overrightarrow{AB}$ such that $|\overrightarrow{AC}| = \alpha\,|\overrightarrow{AB}|$. To consider $\beta\vec{a}$, We choose the initial point C. If $\overrightarrow{CD} = \beta\vec{a}$, then by triangle law of vector addition,

$$\alpha\vec{a} + \beta\vec{a} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$$

Now, $|\overrightarrow{AD}| = AC + CD = \alpha\,|\vec{a}| + \beta\,|\vec{a}|$
$= (\alpha + \beta\,)\,|\vec{a}|$

Also, the direction of $\overrightarrow{AD}$ is the same as that of $\vec{a}$. Thus, $\overrightarrow{AD} = (\alpha + \beta)\,\vec{a}$ by definition of scalar multiple of a vector. Hence, we have

$$(\alpha + \beta)\,\vec{a} = \alpha\,\vec{a} + \beta\,\vec{a}.$$

Also, it is clear that the property holds when one or both of α, β are negative since $-\alpha\vec{a} = \alpha\vec{a}$.

Proof of (ii)

As shown in figure (Fig. 10) below, Let $\vec{a}$ & $\vec{b}$ be represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively. Then $\overrightarrow{AC} = \vec{a} + \vec{b}$.

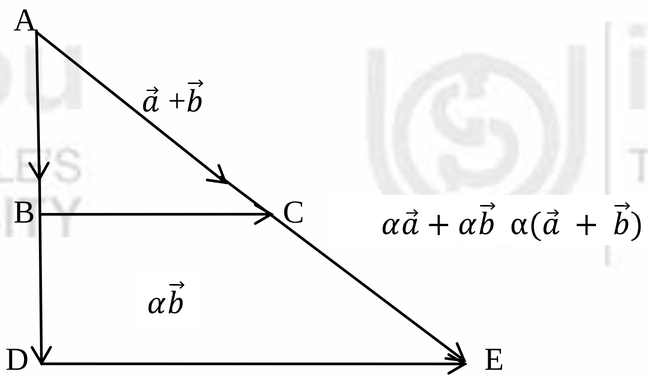


Figure 10

We assume α to be positive. Take a point D on AB such that AD = α AB and point E on AC such that AE = α AC. Join D and E.

But, $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1}$

Now, Δ ABC ~Δ ADE (by construction)

$$\therefore \frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} = \alpha$$

and DE is parallel to BC.

$\Rightarrow DE = \alpha$ BC

i.e., $\overrightarrow{DE} = \alpha\ \overrightarrow{BC} = \alpha\,\vec{b}$

Now from ΔADE,

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$

i.e., $\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}.$

Check Your Progress – 1

1. Classify the following quantities as vectors or scalars.

   (a) distance (b) force

   (c) Velocity (d) workdone

   (e) temperature (f) length

   (g) Speed (h) acceleration

2. Prove that $l^2 + m^2 + n^2 = 1$, where l, m and n are direction cosines of a vector.
3. Prove that in a triangle ABC,

   $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$
4. Find the magnitude and direction of the position vector of point P(1, –1, $\sqrt{2}$)
5. If the position vector of the point P (x, 0,3) has magnitude 5, find the value of x.

1.3 COMPONENT OF A VECTOR

Consider the points A(1,0,0) B(0,1,0) and C(0,0,1) on x, y and z–*axes* respectively. Then, the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are unit vectors because and $\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR} = \vec{b} - \vec{r}$

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are called **units vectors along the *x-axis, y, axis* and *z–axis*** respectively and are denoted by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively.

Suppose P (x, y , z) is a point in space and consider the position vector $\overrightarrow{OP}$ shown in following figure (Figure 11). Let P_1 be the foot of the perpendicular from P on the xy plane.

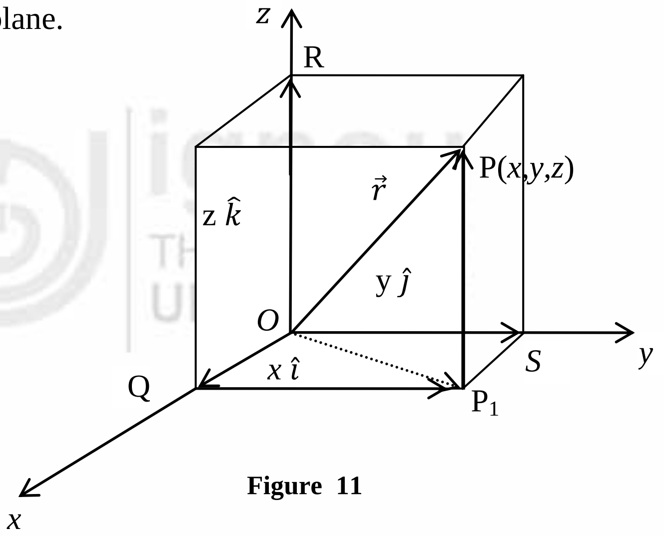


Figure 11

Since $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vectors along x, y and z-*axes* respectively and P has coordinates (x,y, z), therefore

$\overrightarrow{OQ} = x\hat{i}$, $\overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OR} = z\hat{k}$.

So, $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$.

Now, $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P}$

But, $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1}$

$\therefore \quad \overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{OQ} + \overrightarrow{QP_1}$

$or, \ \overrightarrow{OP}(or\ \vec{r}) = y\hat{j} + z\hat{k}$

This form of any vector is called **component form.** Here, x, y and z are called scalar components of $\vec{r}$ and $x\hat{\imath}$, $y\hat{\jmath}$, $z\hat{k}$ and called vector components of $\vec{r}$.

Also note that length of vector $\vec{r} = x\hat{\imath}, + y\hat{\jmath} + z\hat{k}$ is

$|\vec{r}\,| = |\overrightarrow{\mathrm{OP}}| = \sqrt{x^2 + y^2 + z^2}$.

Let $\vec{a} = a_1\hat{\imath}, + a_2\,\hat{\jmath} + a_3\hat{k}$ and

$\vec{b} = b_1\hat{\imath}, + b_2\,\hat{\jmath} + b_3\hat{k}$ be component form of two vectors. Then

1. The vectors $\vec{a}$ and $\vec{b}$ are equal if and only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$,
2. The sum of vectors $\vec{a}$ and $\vec{b}$ is given by
   $\vec{a} + \vec{b} = (a_1 + b_1)\,\hat{\imath}, + (a_2 + b_2)\,\hat{\jmath} + (a_3 + b_3)\,\hat{k}$
3. The difference of vectors $\vec{a}$ and $\vec{b}$ is given by
   $\vec{a} - \vec{b} = (a_1 - b_1)\,\hat{\imath}, + (a_2 - b_2)\,\hat{\jmath} + (a_3 - b_3)\,\hat{k}$
4. The multiplication of vector $\vec{a}$ by any scalar λ is given by
   $\lambda\,\vec{a} = (\lambda a_1)\,\hat{\imath} + (\lambda a_2)\,\hat{\jmath} + (\lambda a_3)\,\hat{k}$

We may observe that vectors $\vec{a}$ and λ $\vec{a}$ are always collinear, whatever be the value of λ. In fact, two vectors $\vec{a}$ and $\vec{b}$are collinear if and only if there exists a non zero scalar λ such that $\vec{b} = \lambda\,\vec{a}$.

If $\vec{a} = a_1\,\hat{\imath} + a_2)\,\hat{\jmath} + a_3\,\hat{k}$ and

$\vec{b} = b_1\,\hat{\imath} + b_2)\,\hat{\jmath} + b_3\,\hat{k}$,

then the two vectors are collinear if and only if

$(b_1\hat{\imath} + b_2)\hat{\jmath} + b_3\,\hat{k} = \lambda\,(a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k})$

$\Leftrightarrow\quad b_1 = \lambda\,a_1,\ b_2 = \lambda\,a_2\,,\ b_3 = \lambda\,a_3$

$$\Leftrightarrow\quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda.$$

Example 2 : Let $\vec{a} = 2\ \hat{\imath} + 3\,\hat{\jmath} - \hat{k}$ and

$\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Evaluate

(i) $2\,\vec{a} + 3\,\vec{b}$ (ii) $\vec{a} - 2\,\vec{b}$

Solution : (i) $2\,\vec{a} = 2(2\,\hat{\imath} + 3\,\hat{\jmath} - \hat{k}\,) = 4\hat{\imath} + 6\,\hat{\jmath} - 2\,\hat{k}$

$3\vec{b} = 3(\hat{\imath} + 2\,\hat{\jmath} + 3\hat{k}\,) = 3\hat{\imath} + 6\,\hat{\jmath} + 9\,\hat{k}$

$\therefore\quad 2\vec{a} + 3\ \vec{b} = 7\hat{\imath} + 12\,\hat{\jmath} + 7\,\hat{k}$

(ii) $\vec{a} - 2\vec{b} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) - 2(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

$= (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) - (2\hat{\imath} + 4\hat{\jmath} + 6\hat{k})$

$= 0\hat{\imath} - \hat{\jmath} - 7\hat{k}$

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{5}(3\hat{i} - 4\hat{k}) = \left(\frac{3}{5}\right)\hat{i} - \left(\frac{4}{5}\right)\hat{k}$$

Example 3 : Find a unit vector in the direction of the vector

$$\vec{a} = 3\hat{\imath} - 4\hat{k}$$

Solution : $\vec{a} = 3\hat{\imath} - 4\hat{k} = 3\hat{\imath} + 0\hat{\jmath} - 4\hat{k}$

$$\therefore |\vec{a}| = \sqrt{3^2 + 0^2 + (-4)^2} = \sqrt{25} = 5.$$

So, unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{5}(3\hat{i} - 4\hat{k}) = \left(\frac{3}{5}\right)\hat{i} - \left(\frac{4}{5}\right)\hat{k}$$

Example 4 : Find a unit vector in the direction of $(\vec{a} - \vec{b})$ where

$\vec{a} = -\hat{i} + \hat{j} + \hat{k}$ and

$\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}.$

Solution : Here, $\vec{a} - \vec{b} = (-\hat{\imath} + \hat{\jmath} + \hat{k}) - (2\hat{\imath} + \hat{\jmath} - 3\hat{k})$

$= -3\hat{\imath} + 4\hat{k}$

and $|\vec{a} - \vec{b}| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$

$\therefore$ unit vector in the direction of $\vec{a} - \vec{b}$

$$= \frac{1}{|\vec{a} - \vec{b}|}(\vec{a} - \vec{b})$$

$$= \frac{1}{5}(-3\hat{\imath} + 4\hat{k}) = \frac{-3}{5}\hat{\imath} + \frac{4}{5}\hat{k}$$

Example 5 : Let $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{b} = -\hat{\imath} + \hat{\jmath}$

Find a vector in the direction of $\vec{a} + \vec{b}$ that has magnitude 7 units.

Solution : Here, $\vec{a} + \vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k}) + (-i + j)$

$= 3\hat{j} + 3\hat{k} = \vec{c}$

The unit vector in the direction of $\vec{c} = \vec{a} + \vec{b}$ is

$$\vec{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{18}}(3\hat{\jmath} + 3\hat{k})$$

$$= \frac{3}{3\sqrt{2}}\hat{\jmath} + \frac{3}{3\sqrt{2}}\hat{k} = \frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\vec{c}$ is

$$= 7\left(\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = \frac{7}{\sqrt{2}}\hat{j} + \frac{7}{\sqrt{2}}\hat{k} = 7\hat{c}.$$

Example 6 : If $\vec{a}$ and $\vec{b}$ are position vectors of the points (1, –1) and (–2, *m*) respectively, then find the value of *m* for which $\vec{a}$ and $\vec{b}$ are collinear.

Solution : Here $\vec{a} = \hat{\imath} - \hat{\jmath}$

$$\vec{b} = -2\hat{i} + m\hat{j}$$

$\vec{a}$ and $\vec{b}$ are collinear if $\vec{a} = \vec{b}$ where λ is a real number

i.e., $\hat{i} - \hat{j} = \lambda(-2\hat{i} - m\hat{j}) = -2\lambda + m\lambda j$

Comparing the component on both sides, we get

$$1 = -2\lambda \Rightarrow \lambda = \frac{-1}{2} \text{ also, } \lambda m = -1 \Rightarrow \frac{-1}{2}m = -1 \Rightarrow m = 2.$$

Check Your Progress – 2

1. Let $\vec{a} = 4\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{b} = -2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$.

   Find the component form and magnitude of the following vectors:

   (a) $\vec{a} + \vec{b}$ (b) $\vec{a} - \vec{b}$

   (b) $\frac{5}{2}\vec{a}$ (d) $2\vec{a} + 3\vec{b}$

2. Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - \hat{k}$.

   Find

   (a) $|\vec{a} - \vec{b}|(\vec{a} + \vec{b})$ (b) $|2\vec{a} - 3\vec{b}|$

3. Find a unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ *and* $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

4. Write the direction ratio's of the vector $\vec{r} = 2\hat{i} = \hat{j} - \hat{k}$ and hence calculate its direction cosines.

5. Show that vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $4\hat{i} + 6\hat{j} - 2\hat{k}$ are collinear.

1.4 SECTION FORMULA

In this section, we shall discuss section formula and its applications. Before that let us find component form of a vector joining two points.

Vector Joining Two Points

Let P(x_1, y_1, z_1) and Q (x_2, y_2, z_2) *be* any two points. The position vectors of P and Q are

$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and

$\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

The vector joining P and Q is $\overrightarrow{PQ}$

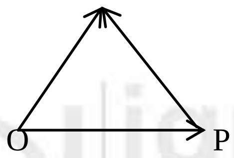


Figure 12

By triangle law, we have

$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})$

$= (x_2 - x_2)\hat{i} + (y_2 - y_2)\hat{k} + (z_2 - z_2)\hat{k}$

The magnitude of Vector $\overrightarrow{PQ}$ is given by

$$\overrightarrow{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 7 : Find the vector $\overrightarrow{PQ}$ where P is the point (5, 7, –1) and Q is the point (2, 9, −2)

Solution : $\overrightarrow{PQ}$= (– 2 – 5)$\hat{\imath}$ + (9–7) $\hat{\jmath}$ + (– 2 –1) $\hat{k}$ = – 7 $\hat{\imath}$ + 2 $\hat{\jmath}$ –3 $\hat{k}$.

Section Formula : To find the positon vector of the point which divides the line joining two given points in a given ratio.

Let P and Q be two points with position vectors : $\vec{a}$ and $\vec{b}$ respectively. Let O be the origin of reference so that

$\overrightarrow{OP} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

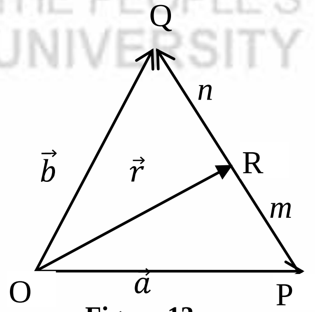


Figure 13

Let R be a point which divides PQ in the ratio *m : n.*

Let : $\overrightarrow{QR} = \vec{r}$.

Now, $\dfrac{PR}{RQ} = \dfrac{m}{n}$ i.e., $nPR = m\,RQ$ which gives the vector equality

$n\overrightarrow{PR} = m\overrightarrow{RQ}$

From above figure, (Figure 13) we have

$= \dfrac{1}{2}(1+t)\vec{a}.$

$\therefore \quad \overrightarrow{PQ} = \overrightarrow{SR}$

Therefore,

$n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$

$$\Rightarrow \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Hence, the position vector of the point R which dividies P and Q in the ratio of $m : n$ internally is given by

$$\overrightarrow{OR} = \frac{m\,\vec{b} + n\,\vec{a}}{m + n}.$$

Corollary : If R is the midpoint of PQ, then $m = n$. Therefore, the position vector of the midpoint of the joint of two points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{r} = \frac{1}{2}\left(\vec{a} + \vec{b}\right)$$

Remark : If R divides the line segment PQ in the ratio $m : n$ externally, then the position vector of R is given by

$$\overrightarrow{OR} = \frac{m\,\vec{b} - n\vec{a}}{m+n}.$$

Example 8 : Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ in the ratio 2:1

(i) internally (ii) externally

Solution :

(i) Position vector of R which divides PQ in the ratio 2:1 internally is

$$\overrightarrow{OR} = \frac{2\left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) + 1\left(2\hat{\imath} + \hat{\jmath} - \hat{k}\right)}{2+1}$$

$$= \frac{4\hat{\imath} + 5\hat{\jmath} + \hat{k}}{3} = \frac{4}{3}\hat{\imath} + \frac{5}{3}\hat{\jmath} + \frac{1}{3}\hat{k}$$

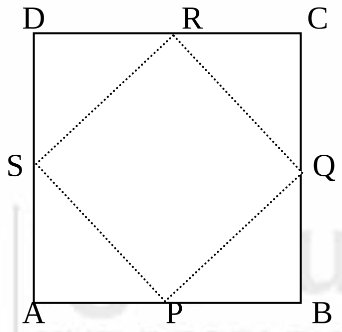
(ii) Position vector of R which divides PQ in the ratio 2:1 externally is

$$\overrightarrow{OR} = \frac{2(\hat{\imath} + 2\hat{\jmath} + \hat{k}) - (2\hat{\imath} + \hat{\jmath} - \hat{k})}{2-1}$$

$$= 3\hat{j} + 3\hat{k}$$

Example 9 : If the mid-points of the consecutive sides of a quadrilateral are joined, then show by using vectors that they form a parallelogram.

Solution : Let $\vec{a}$, $\vec{b}$, $\vec{c}$ $\vec{d}$ be the positon vectors of the vertices A, B, C, D of the quadrilateral ABCD. Let *P*, *Q*, *R*, *S* be the mid-points of sides AB, BC, CD, DA respectively. Then the position vectors of P, Q, R and S are $\frac{1}{2}(\vec{a} + \vec{b}), \frac{1}{2}(\vec{b} + \vec{c}), \frac{1}{2}(\vec{c} + \vec{d})$ and $\frac{1}{2}(\vec{d} + \vec{a})$ respectively.



$$Now, \overrightarrow{PQ} = \overrightarrow{PQ} - \overrightarrow{PQ} = \frac{1}{2}(\vec{b}+\vec{c}) - \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}(\vec{c}-\vec{a})$$

or $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ $\quad (\because\ \overrightarrow{CA} = -\overrightarrow{AC})$

$\therefore$ $\overrightarrow{\text{PQ}} = \overrightarrow{SR}$

$\Rightarrow$ PQ = SR and also PQ || SR.

Since a pair of opposite sides are equal and parallel, therefore, PQRS is a parallelogram.

Example 10 : Prove that the three medians of a triangle meet at a point called the centroid of the triangle which divides each of the medians in the ratio 2:1.

Solution : Let the position vectors of the vertices A, B, C of a triangle ABC with respect to any origin O be $\vec{a}, \vec{b}, \vec{c}$. The position vectors of the mid-points D, E, F of the sides are

$$\frac{1}{2}(\vec{b} + \vec{c}),\ \frac{1}{2}(\vec{c} + \vec{a}),\ \frac{1}{2}(\vec{a} + \vec{b}),$$ respectively.

Let G be the point on the median AD such that AG : GD = 2:1.

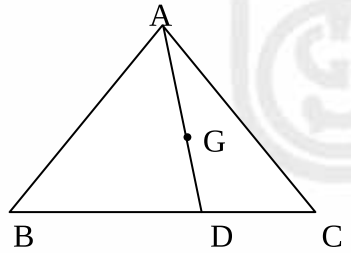


Figure 15

Then by the section formula, the position vector of G is given by

$$\overrightarrow{OG} = \frac{2\overrightarrow{OD} + \overrightarrow{OA}}{2+1} = \frac{2\left[\frac{1}{2}\ \vec{b} + \vec{c}\ \right] + \vec{a}}{3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

If G_1 were the point on the median BE such that $BG_1 : G_1E = 2:1$, the same argument would show that

$$\overrightarrow{OG_1} = \frac{\vec{a}\ + \vec{b} + \vec{c}}{3}$$

In other words, G and G_1 coincide. By symmetry, we conclude that all the three medians pass through the point G such that

$$\overrightarrow{OG} = \frac{\vec{a}\ + \vec{b} + \vec{c}}{3}$$

and which divides each of the medians in the ratio 2:1.

Example 11 : Prove that the straight line joining the mid-points of two non parallel sides of a trapezium is parallel to the parallel sides and half of their sum.

Solution : Let OABC be a trapezium with parallel sides OA and CB. Take O as the origin of reference.

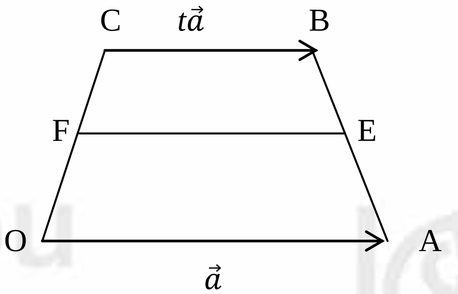


Figure 16

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{c}$, so that $\vec{a}$, $\vec{c}$ are the position vectors of the points A and C respectively.

As CB is parallel to OA, the Vector $\overrightarrow{CB}$ must be a product of the vector $\overrightarrow{OA}$ by some scalar, say, t.

So, $\overrightarrow{CB} = t \quad \overrightarrow{OA} = t\vec{a}$ (1)

$\therefore$ The position vector $\overrightarrow{OB}$ of B is

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = c + t\vec{a}$

Since F is the mid point of OC,

$\therefore$ position vector of F $= \dfrac{\vec{o}+\vec{c}}{2} = \dfrac{1}{2}\vec{c}$

Similarly, positon vector of midpoint E of AB

$$= \frac{\vec{a} + (\vec{c} + t\vec{a})}{2} = \frac{(1+t)\vec{a} + \vec{c}}{2}$$

We have

$$\overrightarrow{FE} = \overrightarrow{OE} - \overrightarrow{OF}$$
$$= \frac{(1+t)\vec{a} + \vec{c}}{2} - \frac{\vec{c}}{2}$$
$$= \frac{1}{2}(1+t)\vec{a}.$$

$$i.e., \overrightarrow{FE} = \frac{1}{2}(1+t)\overrightarrow{OA}$$

So, $\overrightarrow{FE}$ is a scalar multiple of $\overrightarrow{OA}$

$$\Rightarrow \overrightarrow{FE} \parallel \overrightarrow{OA} \text{ and } \overrightarrow{FE} = \frac{1}{2}(1+t)\overrightarrow{OA}$$

Also, from (1) we have

CB = t OA

$\therefore$ OA + CB = (1 + t) OA = 2FE.

Example 12 : Show that the three points with position vectors
$-2\vec{a} + 3\vec{b} + 5\vec{c}, \quad \vec{a} + 2\vec{b} + 3\vec{c}, \quad 7\vec{a} - \vec{c}$
are collinear.

Solution : Let us denote the three points by A, B and C respectively.

We have

$$\overrightarrow{AB} = (\vec{a} + 2\vec{b} + 3\vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c})$$
$$= 3\vec{a} - \vec{b} - 2\vec{c}.$$

$$\overrightarrow{AC} = (7\vec{a} - \vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c})$$
$$= (9\vec{a} - 3\vec{b} - 6\vec{c}) = 3\overrightarrow{AB}$$

Thus, the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are collinear. These vectors are also cointial, therefore, the points A, B and C are collinear.

Check Your Progress – 3

1. Find a unit vector in the direction of vector $\overrightarrow{PQ}$ joining the points P (1, 2, 3) and Q (–1, 1, 2).

2. (i) Let P and Q be two points with position vectors $\overrightarrow{QP}= 3\vec{a} - 2\vec{b}$ and OQ= $\vec{a} + \vec{b}$. Find the position vector of a point R which divides the line joining P and Q in the ratio 2 :1. (i) internally, and (ii) externally.

3. Show that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is half of its length.

4. Show that diagonals of a quadrilateral bisect each other if and only if it is a parallelogram.

5. ABCD is a parallelogram and P is the point of intersection of its diagonals, O is the origin prove that

$$4\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

6. Show that the three points : A(6, –7, –1), B(2, –3, 1) and C(4, –5, 0) are collinear.

1.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1.
(a)	Scalar	(b)	Vector
(c)	Vector	(d)	Scalar
(e)	Scalar	(f)	Scalar
(g)	Scalar	(h)	Vectors

2. If $\overrightarrow{QP} = \vec{r}$ makes angles α, β and γ with positive direction of *x, y* and *z* *axis* respectively then $l = \cos\alpha$, m = cos β, n = cos γ

If P has coordinates (x, y, z), then $x = r\cos\alpha$, $y = r\cos\beta$ and $z = r\cos\gamma$

$$\text{where} \quad r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

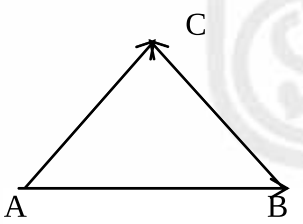
$$\text{So}, \cos\alpha = \frac{x}{r}, \cos\beta = \frac{y}{r}, \cos\gamma = \frac{z}{r},$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1.$$

3. From adjoining figure, we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ using triangle law of vector addition.

$\therefore\ \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \vec{0}$



or $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ $\quad (\because\ \overrightarrow{CA} = -\overrightarrow{AC})$

Figure 17

4. $|\overrightarrow{OP}| = r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (-1)^2 + \sqrt{2}^2} = \sqrt{4} = 2$

$$\cos\alpha = \frac{x}{r} = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$$

$$\cos\beta = \frac{y}{r} = -\frac{1}{2} \Rightarrow \beta = \frac{2\pi}{3}$$

$$\cos\gamma = \frac{z}{r} = \frac{\sqrt{2}}{z} = \frac{1}{\sqrt{2}} \Rightarrow \gamma = \frac{\pi}{4}.$$

5. $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore\ 5 = \sqrt{x^2 + 0^2 + 3^2}$

$25 = x^2 + 9$

$\therefore\ x^2 = 16 \qquad \Rightarrow x = \pm 4.$

Check Your Progress – 2

1. (a) $\vec{a} + \vec{b} = 2\hat{i} + 2\hat{j} + 4\hat{k}$

$$|\vec{a} + \vec{b}| = \sqrt{2^2 + 2^2 + 4^2} = \sqrt{24} = 2\sqrt{6}$$

(b) $\vec{a} - \vec{b} = 6\hat{i} - 8\hat{j} - 2\hat{k}$

$$|\vec{a} - \vec{b}| = \sqrt{6^2 + (-8)^2 + (-2)^2} = \sqrt{104} = 2\sqrt{26}$$

(c) $\frac{5}{2}\vec{a} = \frac{5}{2}(4\hat{\imath} + 3\hat{\jmath} + \hat{k}) = 10\hat{\imath} - \frac{15}{2}\hat{\jmath} + \frac{5}{2}\hat{k}$

$$\left|\frac{5}{2}\vec{a}\right| = \sqrt{100 + \frac{225}{4} + \frac{25}{4}} = \sqrt{\frac{650}{4}} = \frac{5}{2}\sqrt{26}$$

$r = 2\hat{i} + \hat{j} - \hat{k},$

$3\vec{b} = -\ 6\hat{i} + 15\hat{j} + 9\hat{k}$

$\vec{r} = a\hat{i} + b\hat{j} + c\hat{k},$

$\therefore\ |2\vec{a} + 3\vec{b}| = \sqrt{4 + 81 + 121} = \sqrt{206}$

2. (a) $\vec{a}+\vec{b}=3\hat{i}+2\hat{j}+\hat{k}$

i.e., $\overrightarrow{OB}+\overrightarrow{OB}=\overrightarrow{OB}$ and

$\therefore\ |\vec{a}-\vec{b}| = \sqrt{(-1)^2+(-6)^2+\ 3^2}\ \ = \sqrt{46}$

$\therefore\quad |\vec{a}-\vec{b}|(\vec{a}+\vec{b}) = \sqrt{46}(3\hat{i}+2\hat{j}+\hat{k})$

(b) $2\vec{a}=2\hat{i}-4\hat{j}+4\hat{k}$

$3\vec{b}=6\hat{i}+12\hat{j}-3\hat{k}$

$\therefore\quad 2\vec{a}-3\vec{b}=-4\hat{i}-16\hat{j}+7\hat{k}$

$\therefore\quad |2\vec{a}-3\vec{b}| = \sqrt{(-4)^2+(-16)^2+(7)^2} = \sqrt{321}$

3. Here, $\vec{a}+\vec{b}=4\hat{i}+3\hat{j}-2\hat{k}$

$\therefore\quad |\vec{a}+\vec{b}| = \sqrt{4^2+3^2+(-2)^2} = \sqrt{29}$

$\therefore$ unit vector in the direction of $\vec{a}+\vec{b}$

$$=\frac{\vec{a}\ +\ \vec{b}}{|\vec{a}\ +\ \vec{b}|} = \frac{1}{\sqrt{29}}\ (4\hat{\imath}+3\hat{\jmath}-\ 2\hat{k})$$

4. For any vector $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k},\ a,b$ and c are direction ratios and $\frac{\vec{a}}{|\vec{r}|},\frac{\vec{b}}{|\vec{r}|}$ and $\frac{\vec{c}}{|\vec{r}|}$ are the direction cosines.

Here, $r=2\hat{i}+\hat{j}-\hat{k},$ so the direction ratios are $a=2,\ \ b=1,\ \ c=-1$

Also, $|\vec{r}| = \sqrt{2^2+1^2+(-1)^2}\ \ = \sqrt{6}$

$$\therefore\ l=\frac{2}{\sqrt{6}}, m=\frac{1}{\sqrt{6}}, n=\frac{-1}{\sqrt{6}}$$

5. Let $\vec{a}=2\hat{i}+3\hat{j}=-\hat{k}$ and $\vec{b}=4\hat{i}+6\hat{j}-2\hat{k}=-2\hat{k}$

Thus clearly $\vec{b}=2(2\hat{i}+3\hat{j}-\hat{k})=2\vec{a}$

is a scalar multiple of $\vec{a}$.

Hence, $\vec{a}$ and $\vec{b}$ are collinear vectors.

1. Here, $\overrightarrow{PQ} = (-1-1)\,\hat{\imath} + (1-2)\,\hat{\imath} + (2-3)\,\hat{k}$
$= -2\ \hat{\imath} - \hat{\jmath} - \hat{k}$

$\therefore |\overrightarrow{PQ}| = \sqrt{(-2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$

$\therefore$ unit vector in the direction of $\overrightarrow{PQ}$

$$= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{\sqrt{6}}\,(-2\,\hat{\imath} - \hat{\jmath} - \hat{k}\,)$$

2. (i) $\overrightarrow{OR} = \dfrac{2\overrightarrow{OQ} + \overrightarrow{OP}}{2+1} = \dfrac{2\left(\vec{a} + \vec{b}\right) + \left(3\vec{a} + 2\vec{b}\right)}{3} = \dfrac{5\vec{a}}{3}$

(ii) $\overrightarrow{OR} = \dfrac{2\overrightarrow{OQ} - \overrightarrow{OP}}{2-1} = \dfrac{2\left(\vec{a} + \vec{b}\right) - \left(3\vec{a} - 2\vec{b}\right)}{3} = 3\vec{b} - \vec{a}.$

3. Let A of Δ ABC be considered as the origin of vectors (Figure 18)

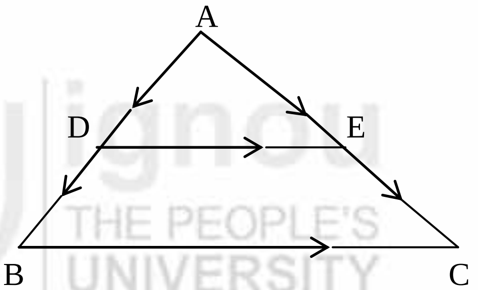


Figure 18

Let D and E be the mid-points of sides AB, CA respectively. Then

$\overrightarrow{AD} = \dfrac{1}{2}\ \overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AE}} = \dfrac{1}{2}\ \overrightarrow{\mathrm{AC}}$

Now, $\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE}$

$= -\overrightarrow{AD} + \overrightarrow{AE}$

$= -\dfrac{1}{2}\,\overrightarrow{AB} + \dfrac{1}{2}\overrightarrow{AC}$

$= \dfrac{1}{2}\,\overrightarrow{BA} + \dfrac{1}{2}\overrightarrow{AC}$

$= \dfrac{1}{2}\,\overrightarrow{BA} + \dfrac{1}{2}\overrightarrow{AC}$

$= \dfrac{1}{2}\,\overrightarrow{BC}$

$\Rightarrow\ \overrightarrow{DE} \parallel \overrightarrow{BC}$ and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\,|\overrightarrow{BC}|$

4. ABCD be the quadrilateral and O be the point of intersection of AC and BD (Figure 19) choose O as the origin of vectors.

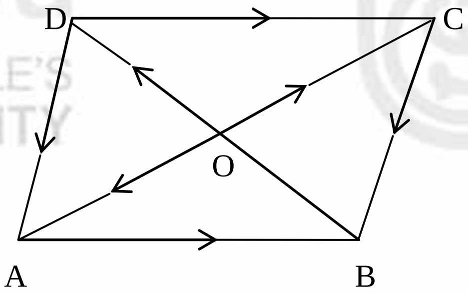


Figure 19

O is the mid-point of AC and BD

$\therefore \quad \overrightarrow{OP} = \dfrac{\overrightarrow{OA}+\overrightarrow{OD}}{2}$

i.e., $\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OD}}=\vec{\mathrm{O}}$ and $\overrightarrow{\mathrm{OC}}+\overrightarrow{\mathrm{OA}}=\vec{\mathrm{O}}$

i.e., iff $\overrightarrow{OB} - \overrightarrow{OC} = \overrightarrow{OA}-\overrightarrow{OD}$

i.e., if $\overrightarrow{CB} = \overrightarrow{DA}$

also equivalently $\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OA}}=\overrightarrow{\mathrm{OC}}-\overrightarrow{\mathrm{OD}}$

i.e., $\overrightarrow{\mathrm{AB}}=\overrightarrow{DC}$

5. P is the mid-point of AC

$\therefore \quad \overrightarrow{OP} = \dfrac{\overrightarrow{OA}+\overrightarrow{OD}}{2}$

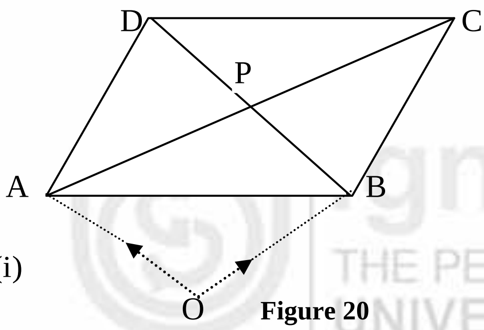


Figure 20

or $2\overrightarrow{\mathrm{OP}}=\overrightarrow{OA}+\overrightarrow{OC}$.....................(i)

Again, P is also mid-point of BD

$\therefore \quad \overrightarrow{OP} = \dfrac{1}{2}\ \overrightarrow{OB}+\ \overrightarrow{\mathrm{OC}}$ ....(ii)

or $2\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{OD}$

Adding (i) and (ii) we get

or $4\overrightarrow{\mathrm{OP}}=\overrightarrow{OA}+\overrightarrow{OC}+\overrightarrow{OB}+\overrightarrow{OD}$

or $4\overrightarrow{\mathrm{OP}}=\overrightarrow{OA}+\overrightarrow{OC}+\overrightarrow{OB}+\overrightarrow{OD}$

6. Here, $\overrightarrow{AB} = (2-6)\,\hat{\imath} + (-3+7)\,\hat{\jmath} + (1+1)\,\hat{k}$

$= -4\hat{\imath} + 4\,\hat{\jmath} + 2\hat{k}$

and $\overrightarrow{AC} = (4-6)\,\hat{\imath} + (-5+7)\hat{\jmath} + (0+1)\,\hat{k}$

$= -\ 2\,\hat{\imath} + 2\hat{\jmath} + \hat{k}$

clearly, $\overrightarrow{AB} = 2\,\overrightarrow{AC}$

So, $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ are collinear vectors. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are also coinitial, therefore A, B and C are collinear points.

1.6 SUMMARY

In this unit, we discuss mathematical concept of vector. In **section 1.2**, first of all, the concept, as distinct from that of a scalar, is defined. Then concepts of position vector, direction cosines of a vector, coinitial vectors, collinear vectors, coplanar vectors, are defined and explained. In **section 1.3**, method of expressing a vector in 3-dimensional space in terms of standard unit vectors is discussed. In **section 1.4**, first, method of finding a vector joining two points is discussed. Then, section formula for finding position vector of the point which divides the vector joining two given points, is illustrated.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.5**.

UNIT 2 VECTORS – 2

Structure

2.0 INTRODUCTION

In the previous unit, we discussed vectors and scalars. We learnt how to add and subtract two vectors, and how to multiply a vector by a scalar. In this unit, we shall discuss multiplication of vectors. There are two ways of defining product of vectors. We can multiply two vectors to get a scalar or a vector. The former is called scalar product or dot product of vectors and the latter is called vector product or cross product of vectors. We shall learn many applications of dot product and cross product of vectors. We shall use dot product to find angle between two vectors. Two vectors are perpendicular if their dot product is zero. Dot product helps in finding projection of a vector onto another vector. The cross product of two vectors is a vector perpendicular to both the vectors.

If cross product of two vectors is zero then the two vectors are parallel (or collinear). Cross product of vectors is also used in finding area of a triangle or a parallelogram. Using the two kinds of products, we can also find product of three vectors. Many of these products will not be defined. In this unit, we shall discuss the two valid triple products, namely, the scalar triple product and the vector triple product.

2.1 OBJECTIVES

After studying this unit, you should be able to:

- define scalar product or dot product of vectors;
- find angle between two vectors;
- find projection of a vector on another vector;

- define cross product or vector product of vectors;
- use cross product to find area of a parallelogram vector product of vectors;
- define scalar triple product and vector triple product of vectors.

2.2 SCALAR PRODUCT OF TWO VECTORS

Definition : The scalar product or the dot product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$ is defined by

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\,cos\,\theta$$

where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Also the scalar product of any vector with the zero vector is, by definition, the scalar zero. It is clear from the definition that the dot product $\vec{a}.\vec{b}$ is a scalar quantity.

Sign of the Scalar Product

If $\vec{a}$ and $\vec{b}$ are two non zero vectors, then the scalar product

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

is positive, negative or zero, according as the angle θ, between the vectors is acute, obtuse or right. In fact,

θ is acute $\Rightarrow$ $\cos\theta > 0$ $\Rightarrow$ $\vec{a}.\vec{b} > 0$

θ is right $\Rightarrow$ $\cos\theta = 0$ $\Rightarrow$ $\vec{a}.\vec{b} = 0$

θ is obtuse $\Rightarrow$ $\cos\theta < 0$ $\Rightarrow$ $\vec{a}.\vec{b} < 0$

Also, note that if and $\vec{a}$ and $\vec{b}$ are non zero vectors then $\vec{a}.\vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal) to each other.

If $\vec{a}$. is any vector, then the dot product $\vec{a}.\vec{a}$., of $\vec{a}$ with itself, is given by
$\vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos 0 = |\vec{a}|^2$

Thus, the length $|\vec{a}|$ of any vector $\vec{a}$ is the non negative square root $\sqrt{\vec{a}.\vec{a}}$, i.e., of the scalar product $\vec{a}\,\vec{a}$, i.e.,

$|\vec{a}| = \sqrt{\vec{a}.\vec{a}}$,

Angle between two Vectors

If θ is an angle between two non zero vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$\Rightarrow \qquad \cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$

or $\qquad \cos\theta = \dfrac{\vec{a}.\vec{b}}{\sqrt{\vec{b}.\vec{b}}\sqrt{\vec{a}\vec{a}}}$

So, the angle θ between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Properties of Scalar Product

1. Scalar product is cumulative, i.e., $\vec{a}.\,\vec{b} = \vec{b}.\vec{a}$ for every pair of vectors $\vec{a}$ and $\vec{b}$.
2. $\vec{a}.\,(-\vec{b}) = -(\vec{a}.\,\vec{b})$ and $(-\vec{a}).\,(-\vec{b}) = \vec{a}.\,\vec{b}$ for every pair of vectors $\vec{a}$ and $\vec{b}$.
3. $(\lambda_1\,\vec{a})\,.\,(\lambda_2\vec{b}) = (\lambda_1\,\lambda_2)\,(\vec{a}.\vec{b})$ where $\vec{a}$ and $\vec{b}$ are vectors and λ_1, λ_2 are scalars
4. (Distributivity) $\vec{a}.(\,\vec{b} + \vec{c}) = \vec{a}.\,\vec{b} + \vec{a}.\,\vec{c}$

The following identities can be easily proved using above properties.

(i) $\qquad (\vec{a} + \vec{b})\,.\,(\vec{a} - \vec{b}) \qquad = a^2 - b^2$ (here, $a^2 = \vec{a}.\vec{a} = |\vec{a}|^2$)

(ii) $\qquad (\vec{a} + \vec{b})^2 \qquad = a^2 + 2\vec{a}.\vec{b} + b^2$

(iii) $\qquad (\vec{a} - \vec{b})^2 \qquad = a^2 - 2\vec{a}.\vec{b} + b^2$

If $\hat{\imath}, \hat{\imath}$ and $\hat{k}$ are mutually perpendicular unit vectors, then

$\hat{\imath}.\,\hat{\imath} = \hat{\jmath}.\,\hat{\jmath} = \hat{k}.\,\hat{k} = 1$ and

$\hat{\imath}.\,\hat{\jmath}. = \hat{\jmath}.\,\hat{k} = \hat{k}.\hat{\imath} = 0 \qquad (1)$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ be two vectors in component form, then their scalar product is given by

$= a_1\hat{i}\,.\,(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{i}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$ (using distributivity)

$= a_2b_3(\hat{j}.\hat{k}) + a_3b_1(\hat{k}.\hat{i}) + a_3b_2(\hat{k}.\hat{j}) + a_3b_3(\hat{k}.\hat{k})$ (using properties)

$= a_1\,b_1 + a_2\,b_2 + a_3\,b_3$ (using (1))

Thus $\vec{a}.\,\vec{b} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$.

Projection of a Vectors on another Vector

Let $\vec{a} = \overrightarrow{OQ}$ and $\vec{b} = \overrightarrow{OB}$ be two vectors

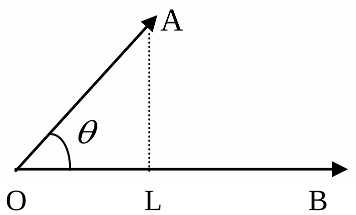


Figure 1

Drop a perpendicular form A on OB as shown in Figure 1. The projection of $\vec{a}$ on $\vec{b}$ is the vector $\overrightarrow{OL}$. If θ is the angle between $\vec{a}$ and $\vec{b}$, then projection of $\vec{a}$ and $\vec{b}$ has length $|\overrightarrow{OA}| \cos\theta$ and direction along unit vector $\vec{b}$. Thus, projection of $\vec{a}$ on $\vec{b} = (|\vec{a}| \cos\theta)\, \vec{b}$

$$= \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\hat{b} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

The scalar component of project of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\,.\vec{b}}{|\vec{b}|}$. Similarly, the scalar component of projection of $\vec{b}$ on $\vec{a}$ $\dfrac{\vec{a}\,.\vec{b}}{|\vec{a}|}$

Example 1: Find the angle θ between the following pair of vectors.

(a) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(b) $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ $\qquad \vec{b} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$

Solution: (a)

$$\vec{a}\,.\,\vec{b} = (2\hat{\imath} + 3\hat{k}).(\hat{\imath} + 4\hat{\jmath} - \hat{k})$$

$$= 2.1 + 0.4 + 3\,(-1)$$

$$= 2 - 3 = -1$$

$$|\vec{a}\,|^2 = \vec{a}\,.\,\vec{a} = (2\hat{\imath} + 3\hat{k}).(2\hat{\imath} + 3\hat{k})$$

$$= 2.2 + 0.0 + 3.3$$

$$= 13$$

$$\therefore \quad |a| = \sqrt{13}$$

and $$|\vec{b}|^2 = \vec{b}.\vec{b} = (\hat{\imath} + 4\hat{\jmath} - 3\hat{k}).(\hat{\imath} + 4\hat{\jmath} - 3\hat{k})$$

$$= 1.1 + 4.4 + (-3).(-3)$$

$$= 26$$

$$\therefore \quad |a| = \sqrt{26}$$

$$\therefore \quad \cos\theta = \frac{\vec{a}\,.\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{-1}{\sqrt{13}\sqrt{26}} = \frac{-1}{13\sqrt{2}}$$

$$\therefore \quad \theta = cos^{-1}\left(\frac{-1}{13\sqrt{2}}\right)$$

(b) Here, $\vec{a}\,.\,\vec{b} = (\hat{\imath} + 2\hat{\jmath} + 2\hat{k}).(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$

$$= 1.2 + 2.(-2) + 2.1$$

$$= 0$$

$$|\vec{a}\,| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|\vec{b}| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\therefore \quad \cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}\,||\vec{b}\,|} = \frac{0}{3.3} = 0$$

$$\therefore \quad \theta = \frac{\pi}{2}$$

.

Example 2 : Show that $|\vec{a}\,|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}\,|\vec{b} - |\vec{b}|\vec{a}$, for any two non zero vectors $\vec{a}$ and $\vec{b}$.

Solution : We know that two vectors are perpendicular if their scalar product is zero.

$$(|\vec{a}\,|\vec{b} + |\vec{b}|\vec{a}\,)\,.\,(|\vec{a}\,|\vec{b} - |\vec{b}|\vec{a}\,)$$

$$= |\vec{a}\,|\vec{b}.(|\vec{a}\,|\vec{b} - |\vec{b}|\vec{a}\,) + |\overrightarrow{AB}|\vec{a}\,.\,(|\vec{a}\,|\vec{b} - |\vec{b}|\vec{a}\,)$$

(using distributivity)

$$= |\vec{a}\,|^2 \quad (\vec{b}.\vec{b}\,) - |\vec{a}\,|\,|\vec{b}|(\vec{b}.\vec{a}\,) + |\vec{b}|\,|\vec{a}\,|\,(\vec{a}.\vec{b}\,) - |\vec{b}|^2(\vec{a}.\vec{a})$$

$$= |\vec{a}\,|^2|\vec{a}\times\vec{b} = 0|^2 - |\vec{a}\,|\,|\vec{b}|(\vec{a}.\vec{b}\,) + |\vec{a}\,|\,|\vec{b}|(\vec{a}.\vec{b}\,) - \vec{b}\ \ |\vec{a}\,|^2 = 0$$

$$\left(\because \vec{a}.\vec{b} = \vec{b}.\vec{a},\ \vec{a}.\vec{a} = |\vec{a}\,|^2 \text{ and } \vec{b}.\vec{b} = \overrightarrow{|b|}^2\right)$$

so, the given vectors are perpendicular.

Example 3 : Find the scalar component of projection of the vector

$$\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k} \text{ on the vector } \vec{b} = 2\hat{i} - 2\hat{j} - \hat{k}.$$

Solution : Scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

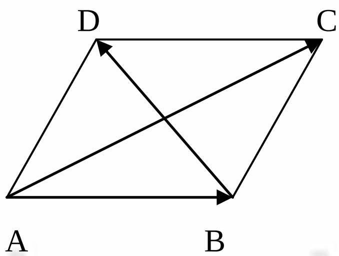
Here, $\vec{a}\,.\,\vec{b} = 2.2 + 3\,(-2) + 5(-1) = -7$

and $|\vec{b}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

$\therefore$ Scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{-7}{3}$

Example 4 : Show that the diagonals of a rhombus are at right angles.

Solution : Let A B C D be a rhombus (Figure 2)



(Figure 2)

ABCD being a rhombus, we have

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

$$\begin{aligned}
\therefore\ \overrightarrow{AC}.\overrightarrow{BD} &= (\overrightarrow{AB} + \overrightarrow{BC}).(\overrightarrow{AD} - \overrightarrow{AB}) \\
&= (\overrightarrow{AB} + \overrightarrow{AD}).(\overrightarrow{AD} - \overrightarrow{AB}) && (\because \overrightarrow{AC} = \overrightarrow{AD}) \\
&= \overrightarrow{AD}^2 - \overrightarrow{AB}^2 && (\because (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2) \\
&= \overrightarrow{AD}^2 - \overrightarrow{AD}^2 = 0 \\
&\Rightarrow \overrightarrow{AC} \perp \overrightarrow{BD} && (\because \overrightarrow{AB}^2 = \overrightarrow{AD}^2)
\end{aligned}$$

Example 5 : For any vectors $\vec{a}$ and $\vec{b}$, prove the triangle inequality

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

Solution : If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then the inequality holds trivially.
So let $|\vec{a}| \neq 0 \neq |\vec{b}|$. Then,

$$\begin{aligned}
|\vec{a} + \vec{b}|^2 &= (\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) \\
&= \vec{a}\vec{a} + \vec{a}\vec{b} + \vec{b}\vec{a} + \vec{b}\vec{b} \\
&= |\vec{a}|^2 + 2\vec{a}\vec{b} + |\vec{b}|^2 \quad (\because\ \vec{a}\vec{b} = \vec{a}\vec{b}) \\
&= |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2 \text{ where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b} \\
&\leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 \quad (\because \cos\theta \leq 1\ \forall\ \theta) \\
&= (|\vec{a}| + |\vec{b}|)^2
\end{aligned}$$

Hence $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

Remark : Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$, then $\vec{a} + \vec{b} = \overrightarrow{AC}$

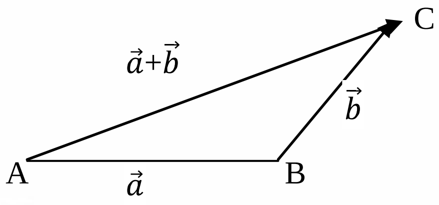


Figure 3

As shown in figure 3 inequality says that the sum of two sides of triangle is greater than the third side. If the equality holds in triangle inequality, i.e., $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$

Then $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

showing that the points A, B and C are collinear.

Check Your Progress – 1

1. If $\vec{a} = 5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} - 5\hat{k}$ then show that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

2. Find the angle between the vectors

(a) $\vec{a} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$ $\vec{b} = -4\hat{\imath} + 2\hat{k}$

(b) $\vec{a} = \hat{\imath} - \hat{\jmath} - 2\hat{k}$ $\vec{b} = -2\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

3. Find the vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$ and $\vec{b} = 7\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Also find the scalar component of projection of vector $\vec{b}$ on $\vec{a}$.

4. Prove the Cauchy – Schwarz Inequality

$|\vec{a}.\ \vec{b}| \leq |\vec{a}|\,|\vec{b}|$

5. Prove that $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

2.3 VECTOR PRODUCT (OR CROSS PRODUCT) OF TWO VECTORS

Definition : If $\vec{a}$ and $\vec{b}$ are two non zero and non parallel (or equivalently non collinear) vectors, then their vector product $\vec{a} \times \vec{b}$ is defined as

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ ($0 < \theta < \pi$) and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

If $\vec{a}$ and $\vec{b}$ are parallel (or collinear) i.e., when $\theta = 0$ or π, then we define the vector product of $\vec{a}$ and $\vec{b}$ to be the zero vector i.e., $\vec{a} \times \vec{b} = \vec{0}$. Also note that if either $\vec{a} = 0$ or $\vec{b} = 0$,then θ is not defined and we define $\vec{a} \times \vec{b} = 0$.

Properties of the Vector Product

1. $\vec{a} \times \vec{a} = \vec{0}$ since $\theta = 0$
2. Vector product is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$. $\vec{a}$,
   However, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
   We have $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system and $\vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\,|\sin\theta\,\hat{n}|$ where $\vec{b}$, $\vec{a}$ and $\hat{n}\ a$ right handed system. So the direction of $\hat{n}|$ is opposite to that of $\hat{n}$.

   Hence, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$
   $= -|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$,
   $= -\vec{b} \times \vec{a}$.
3. Let $\hat{\imath}, \hat{\jmath}, \hat{k}$ from a right–handed system of mutually perpendicular unit vectors in three dimensional space. Then

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$ and
$\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$
Also, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$, $\hat{\imath} \times \hat{k} = -\hat{\jmath}$

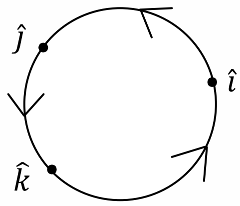


Figure 4

4. Two non zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if
$\vec{a} \times \vec{b} = \vec{0}$
5. Vector product is distributors over addition i.e., if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, then
(i) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
(ii) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$.
6. If λ is a scalar and $\vec{a}$ and $\vec{b}$ are vectors, then
$(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$

Vector Product in the component form

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and
$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$.
Then

$\vec{a} \times \vec{b} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$

$= a_1 b_1(\hat{\imath} \times \hat{\imath}) + a_1 b_2(\hat{\imath} \times \hat{\jmath}) + a_1 b_3(\hat{\imath} \times \hat{k}) + a_2 b_1(\hat{\jmath} \times \hat{\imath}) + a_2 b_3 (\hat{\jmath} \times \hat{k}) + a_2 b_3(\hat{\jmath} \times \hat{k}) + a_3 b_1(\hat{k} \times \hat{\imath}) + a_3 b_2(\hat{k} \times \hat{\jmath}) + a_3 b_3 (\hat{k} \times \hat{k})$

$= a_1 b_2\hat{k} + a_1 b_3(-\hat{\jmath}) + a_2 b_1(-\hat{k}) + a_2 b_3\hat{\imath} + a_3 b_1\hat{\jmath} + a_3 b_2(-\hat{\imath})$ (*using property* 3)

$= (a_2 b_3 - a_3 b_2)\,\hat{\imath} + (a_1 b_3 - a_3 b_1)\,\hat{\jmath} + (a_1 b_2 - a_2 b_1)\,\hat{k}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 6: Find $\vec{a} \times \vec{b}$ if $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Solution : We have

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$= (3 - 2)\,\hat{\imath} - (3 - 1)\,\hat{\jmath} + (2 - 1)\,\hat{k}$

$= \hat{\imath} - 2\hat{\jmath} + \hat{k}$

Example 7: Find a unit vector perpendicular to both the vectors

$$\vec{a} = 4\hat{i} + \hat{j} + 3\hat{k} \text{ and } \vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$$

Solution : A vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$= (2 - 3)\,\hat{\imath} - (-8 + 6)\,\hat{\jmath} + (4 - 2)\,\hat{k}$

$= -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{9} = 3.$

So the desired unit vector is

$$\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{a}|} = \frac{1}{3}(-\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$$

Example 8 : Prove the distributive law

$$\vec{a} \times (\vec{b}+\vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

using component form of vectors.

Solution : Let $\vec{a} = a_1\,\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\,\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and

$\vec{c} = c_1\,\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$.

So, $\vec{b} + \vec{c} = (b_1 + c_1)\,\hat{\imath} + (b_2 + c_2)\,\hat{\jmath} + (b_3 + c_3)\,\hat{k}$

Now,

$$\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

Example 9: Show that

$$|\vec{a} \times \vec{b}|^2 = (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2$$

Solution : $|\vec{a} \times \vec{b}|^2 = |\vec{a}||\vec{b}|\sin\theta$

$\therefore |\vec{a} \times \vec{b}|^2 = (|\vec{a}||\vec{b}|\sin\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}|^2|\vec{b}|^2\cos^2\theta)$

$= (\vec{a}\vec{a})^2(\vec{b}\vec{b}) - (\vec{a}.\vec{b})^2$

Example 10 : Find $|\vec{a} \times \vec{b}|$ if

$$|\vec{a}| = 10, \quad |\vec{b}| = 2, \quad \vec{a}.\vec{b} = 12.$$

Solution : Here $\vec{a}.\vec{b} = 12$

$$\therefore \text{Cos}\,\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{a}\vec{b}|} = \frac{12}{(10)(2)} = \frac{3}{5}$$

$$\therefore \text{Sin}\,\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|sin\theta$$

$$= 10 \times 2 \times \frac{4}{5} = 16$$

Area of Triangle

Let $\vec{a}$, $\vec{b}$ be two vectors and let θ be the angle between them (O < θ < π). Let O be the origin for the vectors as shown in figures (Fig. 5) below and let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

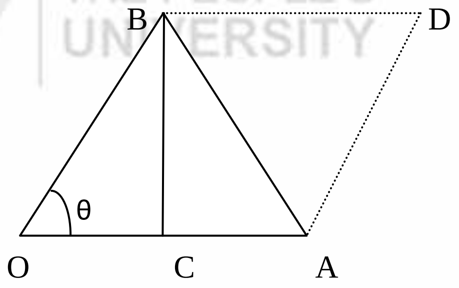


Figure 5

Draw BC⊥ OA.

Then BC = OB Sin θ = |b| sin θ

$$\therefore \text{ Area of } \Delta \text{ OAB } = \frac{1}{2}\,(\text{OA})(\text{BC})$$

$$= \frac{1}{2}|\vec{a}||\vec{b}| \sin\theta$$

$$= \frac{1}{2}|\vec{a} \times \vec{b}|$$

Thus, if $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given as $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Area of a Parallelogram

In above figure, if D is the fourth vertex of the parallelogram formed by O, B, A then its area is twice the area of the triangle OBA.

Hence, area of a parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides = $|\vec{a} \times \vec{b}|$.

Example 11: Find the area of Δ ABC with vertices A (1,3,2), B (2, –1,1) and C(–1, 2, 3).

Solution : We have

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (2-1)\,\hat{\imath} + (-1-3)\,\hat{\jmath} + (1-2)\,\hat{k} \\ &= \hat{\imath} - 4\hat{\jmath} - \hat{k}\end{aligned}$$

$$\begin{aligned}\text{and } \overrightarrow{BC} &= (-1-1)\,\hat{\imath} + (2-3)\,\hat{\jmath} + (3-2)\,\hat{k} \\ &= -2\,\hat{\imath} - \hat{\jmath} + \hat{k}\end{aligned}$$

$$\text{Vector Area of } \Delta \text{ ABC } = \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{BC})$$

$$= \frac{1}{2}\begin{vmatrix} \hat{\imath} & j & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,[-5\,\hat{\imath} + \hat{\jmath} - 9\hat{k}\,]$$

$$\therefore \text{ Area of } \Delta\, ABC = \frac{1}{2}\sqrt{5^2 + 1^2 + 9^2} = \frac{1}{2}\sqrt{107}$$

Example 12 : Show that $(\vec{a} - \vec{b})\times(\vec{a} + \vec{b}) = 2(\vec{a}\times\vec{b})$. Interpret the result geometrically.

Solution : $(\vec{a} - \vec{b})\times(\vec{a} + \vec{b})$

$= (\vec{a} - \vec{b})\times\vec{a} + (\vec{a} - \vec{b})\times\vec{b}$

$= \vec{a}\times\vec{a} - \vec{b}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{b}$

$= 0 + \vec{a}\times\vec{b} + \vec{a}\times\vec{b} + 0$

$= 2\,(\vec{a}\times\vec{b})$

Let ABCD be a parallelogram with $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = -\vec{b}$.

Then area of parallelogram = $\overrightarrow{AB}\times\overrightarrow{AD} = \vec{a}\times\vec{b}$

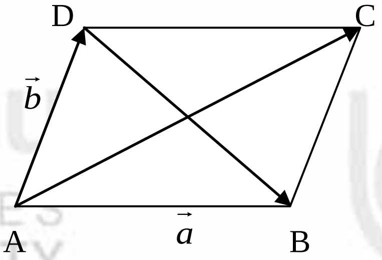


Figure 6

Also, diagonal $\overrightarrow{AC} = \vec{a} + \vec{b}$

and diagonal $\overrightarrow{DB} = \vec{a} - \vec{b}$

$\therefore\ (\vec{a} + \vec{b})\times(\vec{a} - \vec{b}) = \overrightarrow{AC}\times\overrightarrow{DB}$

= area of parallelogram formed by $\overrightarrow{AC}$ and $\overrightarrow{DB}$.

Thus, the above result shows that the area of a parallelogram formed by diagonals of a parallelogram is twice the area of the parallelogram.

Check Your Progress – 2

1. Find a unit vector perpendicular to each of the vector $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 4\,\hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + 2\,\hat{k}$

2. Show that
$$\vec{a}\times(\vec{b} + \vec{c}) + \vec{b}\times(\vec{c} + \vec{a}) + \vec{c}\times(\vec{a} + \vec{b}) = \vec{0}$$

3. For Vectors

   $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$

   Compute $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$.

   is $(\vec{a}\times\vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$.

4. If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a} + \vec{b} + \vec{c} = 0$, then prove that

   $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b}$.

5. Find the area of a parallelogram whose diagonals are

   $\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

2.4 TRIPLE PRODUCT OF VECTORS

Product of three vectors may or may not have a meaning. For example, $(\vec{a}.\vec{b}).\vec{c}$ has no meaning as $\vec{a}.\vec{b}$ is a scalar and dot product is defined only for vectors. Similarly, $(\vec{a}.\vec{b}) \times \vec{c}$ has no meaning. The products of the type $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a}.(\vec{b} \times \vec{c})$. are meaningful and called triple products. The former is a vector while the latter is a scalar.

Scalar Triple Product

Definition : Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors. The scalar product of $\vec{a}.(\vec{b} \times \vec{c})$ is called scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}, \vec{b}, \vec{c}]$. Thus $[\vec{a}, \vec{b}, \vec{c}] = \vec{a}.(\vec{b} \times \vec{c})$.

It is clear from the definition that $[\vec{a}, \vec{b}\ \vec{c}]$. Is a scalar quantity.

Geometrically, the scalar triple product gives the volume of a parallelepiped formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as adjacent sides.

Scalar Triple Product as a determinant

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

Then

$$\vec{a}.(\vec{b} \times \vec{c}) = \vec{a}.\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}).\left\{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\hat{k}\right\}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Thus, $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note : We can omit the brackets in $\vec{a}.(\vec{b} \times \vec{c})$ and just write $\vec{a}.\vec{b} \times \vec{c}$ because $(\vec{a}.\vec{b}) \times \vec{c}$ is meaningless.

Properties of Scalar Triple Product

1. $[\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b}] - [\vec{b},\vec{a},\vec{c}]$
   $= -[\vec{c},\vec{b},\vec{a}] = -[\vec{a},\vec{c},\vec{b}]$
   This is clear if we note the properties of a determinant as $[\vec{a},\vec{b},\vec{c}]$ can be expressed as a determinant.
2. In scalar triple product $\vec{a}.(\vec{b} \times \vec{c})$, the dot and cross can be interchanged. Indeed, $\vec{a}.(\vec{b} \times \vec{c}) = [\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b},]$
   $= \vec{c}.(\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}).\vec{c}.$
3. $= [p\vec{a}, q\vec{b}, r\vec{c}] = pqr[\vec{a},\vec{b},\vec{c}]$ where p, q and r are scalars. Again it is clear using properties of determinants.
4. If any two of $\vec{a}.\vec{b}$ and $\vec{c}$ are the same then $[\vec{a}\vec{b}\vec{c}] = 0$ For example, $[\vec{a},\vec{b},\vec{c}] = 0$

Coplanarity of three vectors

Theorem : Three vectors $\vec{a}.\vec{b}$ and $\vec{c}$ are coplanar if and only if $[\vec{a},\vec{b},\vec{c} = 0]$

Proof : First suppose that $\vec{a}.\vec{b}$ and $\vec{c}$ are coplanar . If $\vec{b}$ and $\vec{c}$ are parallel vectors, then $\vec{b} \times \vec{c} = 0$ and so $\vec{a}.(\vec{b} \times \vec{c}) = 0$. If $\vec{b}$ and $\vec{c}$ are not parallel, then $\vec{b} \times \vec{c}$ being perpendicular to plane containing $\vec{b}$ and $\vec{a}$, is also perpendicular to $\vec{a}$ because $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar.

$\therefore \quad \vec{a}(\vec{b} \times \vec{c}) = 0.$

Conversely, suppose that $[\vec{a},\vec{b},\vec{c}\,] = 0$. If $\vec{a}$ and $\vec{b} \times \vec{c}$ are both non zero, then $\vec{a}$ and $\vec{b} \times \vec{c}$ are perpendicular as their dot product is zero. But $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and hence $\vec{a},\vec{b}$ and $\vec{c}$ must lie in a plane, i.e., they are coplanar. If $\vec{a} = \vec{0}$ then $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar as zero vector is coplanar with any two vectors. If $\vec{b} \times \vec{c} = \vec{0}$, then $\vec{b} \times \vec{c}$ are parallel vectors and hence $\vec{a},\vec{b}$and $\vec{c}$ must be coplanar.

Note : Four points A, B , C and D are coplanar if the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

Example 13 : If $\vec{a} = 7\hat{\imath} + a_2 - 2\hat{\jmath} + 3\,\hat{k}$, $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 2\,\hat{k}$, and $\vec{c} = 2\hat{\imath} + 8\hat{\jmath}$ find $[\vec{a}, \vec{b}, \vec{c}]$

Solution : $[\vec{a}, \vec{b}, \vec{c}] = \vec{a}.\,(\vec{b} \times \vec{c})$

$$= \begin{vmatrix} 7 & -2 & 3 \\ 1 & -2 & 2 \\ 2 & 8 & 0 \end{vmatrix}$$

$= 7(0 - 16) + 2(0 - 4) + 3(8 + 4)$

$= -112 - 8 + 36$

$= -84$

Example 14 : Find the value of λ for which the vectors $\vec{a} = \hat{\imath} - 4\hat{\jmath} + \hat{k}$, $\vec{b} = \lambda\,\hat{\imath} - 2\hat{\jmath} + \hat{k}$, and $\vec{c} = 2\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$ are coplanar,

Solution : If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, we have $= \vec{a}.\,(\vec{b} \times \vec{c}) = 0$ i.e.,

$$\begin{vmatrix} 1 & -4 & 1 \\ \lambda & -2 & 1 \\ 2 & 3 & 3 \end{vmatrix} = 0$$

$\Rightarrow\ 1(-6 - 3) + 4(3\lambda - 2) + (3\lambda + 4) = 0$

$$\Rightarrow\ \lambda = \frac{13}{15}$$

Example 15 : If $\vec{a}, \vec{b}, \vec{c}$ are coplanar then prove that $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also coplanar.

Solution : Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar,

$\therefore\ [\vec{a}, \vec{b}, \vec{c}] = 0$

Now $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = (\vec{a} + \vec{b})\,.(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})$

$= (\vec{a} + \vec{b})\,.[(\vec{b} + \vec{c}) \times \vec{c} + (\vec{b} + \vec{c}) \times \vec{a}]$

$= (\vec{a} + \vec{b})\,.[\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}]$ (as $\vec{c} \times \vec{c} = \vec{0}$)

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{a}) + \vec{a}.(\vec{c} \times \vec{a}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.\,(\vec{b} \times \vec{a}) + \vec{b}.\,(\vec{c} \times \vec{a})$

$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{a}] + [\vec{a}, \vec{c}, \vec{a}] + [\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{a}] + [\vec{b}, \vec{c}, \vec{a}]$

$= 2\,[\vec{a}, \vec{b}, \vec{c}]$ (using property (4))

$= 0$

$\therefore\ \vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar

Vector Triple Product

Definition : Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ any three vectors. Then, the vectors $\vec{a} \times (\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}) \times \vec{c}$ are called vector triple products.

It is clear that, in general
$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

Note that $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$. And also $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. Thus $\vec{a} \times (\vec{b} \times \vec{c})$ lies in a plane containing the vectors $\vec{b}$ and $\vec{c}$, i.e., $\vec{a} \times (\vec{b} \times \vec{c})$, $\vec{b}$ and $\vec{c}$ are coplanar vectors.

We now show that for any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have
$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Let $\vec{a} = a_1\hat{\imath} + a_2\,\hat{\jmath} + a_3\hat{k}$
$\vec{b} = b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\hat{k}$
and $\vec{c} = c_1\hat{\imath} + c_2\,\hat{\jmath} + c_3\hat{k}$. Then

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\,\hat{\imath} + (b_3c_1 - b_1c_3)\hat{\jmath} + (b_1c_2 - b_2c_1)\,\hat{k}$$

$$\therefore\ \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$= [a_2\,(b_1c_2 - b_2c_1) - a_3\,(b_3c_1 - b_1c_3)\,]\hat{\imath} + [a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\,\hat{\jmath}$
$+ [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\,\hat{k}$

Also, $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$ (1)

$= (a_1c_1 + a_2c_2 + a_3c_3)\left(b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\hat{k}\right) - (a_1b_1 + a_2b_2 + a_3b_3)$
$(c_1\hat{\imath} + c_2\,\hat{\jmath} + c_3\hat{k})$

$= [b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3)\,]\,\hat{\imath}$
$+ [b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2\,(a_1b_1 + a_2b_2 + a_3b_3)\,]\,\hat{\jmath}$
$+ [b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3\,(a_1b_1 + a_2b_2 + a_3b_3)\,]\,\hat{k}$

$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3\,c_1 - b_1c_3)]\,\hat{\imath}$
$+ [a_3(b_2c_3 - b_3c_2) - a_1(b_1\,c_2 - b_2c_1)]\,j$
$+ [a_1(b_3c_1 + b_1c_3) - a_2(b_2\,c_3 - b_3c_2)]\,\hat{k}$ (2)

From (1) and (2), we have
$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c}) - (\vec{a}.\vec{b})\vec{c}$

It is also clear from this expression that $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector in the plane of $\vec{b}$ and $\vec{c}$.

Now $\vec{a} \times \vec{b} \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b})$

$$= -[(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}]$$

$$= (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\,\vec{a}$$

$$= (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\,\vec{a}$$

So, $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector in the plane of $\vec{b}$ and $\vec{a}$.

Theorem : Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors. Then $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ if and only if $\vec{a}$ and $\vec{c}$ are collinear.

Proof : First suppose $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

Now, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

and $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{c})\vec{a}$

So, $-(\vec{a}.\vec{b})\vec{c} = -(\vec{b}.\vec{c})\vec{a}$

$\Rightarrow \vec{c}$ and $\vec{a}$ are collinear vectors.

Conversely, suppose that $\vec{a}$ and $\vec{c}$ are collinear vectors.
Then there exist a scalar λ such that

$\vec{c} = \lambda\,\vec{a}$. Then

$-(\vec{a}.\vec{b})\vec{c} = -(\vec{a}.\vec{b})(\lambda\,\vec{a}) = -\lambda\,(\vec{a}.\vec{b})\vec{a}$

and $-(\vec{b}.\vec{c})\vec{a} = -(\vec{b}.\lambda\,\vec{a})\vec{a} = -\lambda\,(\vec{b}.\vec{a})\vec{a}$

$= -\lambda\,(\vec{a}.\vec{b})\vec{a}$

So, $-(\vec{a}.\vec{b})\vec{c} = -(\vec{b}.\vec{c})\vec{a}$

$\Rightarrow (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

i.e., $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

Example 16 : Show that

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$$

Solution : $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$

$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$

$= \vec{0}$ since dot product is commutative.

Example 17 : Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any four vectors. Then prove that

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

Solution : (i) Let $\vec{a} \times \vec{b} = \vec{r}$ Then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{r} \times (\vec{c} \times \vec{d})$$
$$= (\vec{r} \,.\, \vec{d})\vec{c} - (\vec{r}.\vec{c})\vec{d}$$
$$= (\vec{d}.\vec{r}\,)\vec{c} - (\vec{c}.\vec{r})\vec{d}$$
$$= [\vec{d}.\vec{a} \times \vec{b}]\vec{c} - [\vec{c}.\vec{a} \times \vec{b}]\vec{d}$$
$$= [\vec{d}, \vec{a}, \vec{b}]\vec{c} - [\vec{c}, \vec{a}, \vec{b}]\vec{d}$$
$$= [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$$

(ii) Let $\vec{c} \times \vec{d} = \vec{r}$. Then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{r}$$
$$= -\vec{r} \times (\vec{a}, \times \vec{b})$$
$$= -[(\vec{r}\,.\vec{b})\vec{a} - (\vec{r}.\vec{a})\vec{b}]$$
$$= (\vec{r}.\vec{a})\vec{b} - (\vec{r}.\vec{b})\vec{a}$$
$$= (\vec{a}.\vec{r})\vec{b} - (\vec{b}.\ \vec{r})\vec{a}$$
$$= (\vec{a}.(\vec{c} \times \vec{d})\,\vec{b} - (\vec{b}.(\vec{c} \times \vec{d})\vec{a}$$
$$= [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$$

Example 18 : Prove that

$$[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c}\,,\vec{c} \times \vec{a}\,] = [\vec{a}, \vec{b}, \vec{c}]^2$$

Solution : $[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}\,]$

$$= (\vec{a} \times \vec{b}).\,[(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})]$$
$$= (\vec{a} \times \vec{b})[(\vec{b} \times \vec{c}).\vec{a})\vec{c} - (\vec{b} \times \vec{c}).\vec{c})\vec{a}]$$
$$= (\vec{a} \times \vec{b}).\,[(\vec{b} \times \vec{c}).\,\vec{a}\,)\,\vec{c}\,)] \quad (\because (\vec{b} \times \vec{c}).\vec{c} = 0)$$
$$= [(\vec{b} \times \vec{c}).\vec{a})][(\vec{a} \times \vec{b}).\vec{c}] = [\vec{a}\,.(\vec{b} \times \vec{c})][\vec{a}.(\vec{b} \times \vec{c})]$$
$$= [\vec{a}, \vec{b}, \vec{c}]^2.$$

Example 19 : For vectors $\vec{a} = \hat{\imath} - 2\,\hat{\jmath} + \hat{k}$,
$\vec{b} = 2\,\hat{\imath} + \hat{\jmath} + \hat{k}$
and $\vec{c} = \hat{\imath} + 2\,\hat{\jmath} - \hat{k}$ verify that
$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Solution : We have

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$$

So, $\vec{a} \times (\vec{b} \times \vec{c})$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= -9\hat{\imath} - 6\hat{\jmath} - 3\hat{k} \qquad (1)$$

Also,

$\vec{a}.\vec{c} = (\hat{\imath} - 2\hat{\jmath} + \hat{k}).(\hat{\imath} + 2\hat{\jmath} + \hat{k})$

$= 1 - 4 - 1 = -4$

$\therefore (\vec{a}.\vec{c}).\vec{b} = -4 \quad (2\hat{\imath} + \hat{\jmath} + \hat{k}) = -8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$

and $\vec{a}.\vec{b} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}).(2\hat{\imath} + \hat{\jmath} + \hat{k}) = 1$

$\therefore (\vec{a}.\vec{b})\vec{c} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

Thus, $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -9\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$ (2)

Hence, from (1) and (2)

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Check Your Progress – 3

1. For vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$, $\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}, and\ \vec{c} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ find $[\vec{a}, \vec{b}, \vec{c}]$.
2. Find the volume of the parallelepiped whose edges are represented by

   $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}, \vec{c} = 3\hat{\imath} - \hat{\imath} + 2\hat{k}$
3. Show that the four points having position vectors

   $\hat{\imath} + \hat{\jmath} + 2\hat{k}$, $6\hat{\imath} + 11\hat{\jmath} + 2\hat{k}$, $\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$, $\hat{i} + \frac{1}{2}\hat{\jmath} + 4\hat{k}$, are coplanar.
4. Prove that

   $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$
5. For any vector $\vec{a}$ prove that

   $\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$
6. Prove that

   $\vec{a} \times [\vec{a} \times (\vec{a} \times \vec{b})] = (\vec{a}.\vec{a})(\vec{b} \times \vec{a})$

2.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. Here, $\vec{a} + \vec{b} = 6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$ and

   $\vec{a} - \vec{b} = 4\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

   So, $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}).(4\hat{\imath} - 4\hat{\jmath} + 2\hat{k})$

   $= 24 - 8 - 16 = 0$

   Hence $\vec{a} + \vec{b}$ and $\vec{a} + \vec{b}$ are perpendicular vectors.

2. (a) Here, $\vec{a} \cdot \vec{b} = 3(-4) + (-1).0 + 2.2$

$$= -12 + 4 = -8$$

$$|\vec{a}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(-4)^2 + 2^2} = \sqrt{20}$$

$$\therefore \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-8}{\sqrt{14}\sqrt{20}} = \frac{-4}{\sqrt{70}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right)$$

(b) Here, $\vec{a}.\vec{b} = 1(-2) + (-1)2 + (-2)4$

$$= -12$$

$$|\vec{a}| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{(-2)^2 + 2^2 + 4^2} = \sqrt{24}$$

$$\therefore \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-12}{\sqrt{6}\sqrt{24}} = -1$$

$$\Rightarrow \theta = \pi$$

3. Vector Projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

$$= \left(\frac{12}{54}\right)(7\hat{\imath} + \hat{\jmath} - 2\hat{k})$$

$$= \frac{14}{9}\hat{\imath} + \frac{2}{9}\hat{\jmath} - \frac{4}{9}\hat{k}$$

Scalar project of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{a}|} = \dfrac{12}{\sqrt{38}}$

4. The inequality holds trivially if $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c}) = \vec{0},\ \vec{b} = 0$

So, let $|\vec{a}| \neq 0$ or $|\vec{b}| \neq 0$

Now, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\therefore = \frac{|\vec{a}.\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \leq 1$$

Hence $|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$

5. $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}).(\vec{a} - \vec{b})$

$$= \vec{a}.\vec{a} - \vec{a}.\vec{b} - \vec{b}.\vec{a} - \vec{b}.\vec{b}$$

$$= |\vec{a}|^2 - 2\vec{a}.\vec{b} + |\vec{b}|^2$$

($\because$ dot product is commulative)

Check Your Progress – 2

1. Here $\vec{a}+\vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\vec{a}-\vec{b} = 3\hat{\jmath} - 6\hat{k}$

Let $\vec{c} = (\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})$. Then $\vec{c}$ is vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

$$\text{Now } \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -2 \\ 0 & 3 & -6 \end{vmatrix}$$

$$= (-6+6)\,\hat{\imath} + (-12+0)\hat{\jmath} + (6+0)\,\hat{k}$$

$$= 12\hat{\jmath} + 6\hat{k}$$

A unit vector in the direction of $\vec{c}$ is

$$\hat{c} = \frac{1}{|\vec{c}|}\,\vec{c} = \frac{1}{\sqrt{12^2+6^2}}(12\hat{\jmath} + 6\,\hat{k})$$

$$= \frac{1}{6\sqrt{5}}\,(12\hat{\jmath} + 6\,\hat{k})$$

$$= \frac{2}{\sqrt{5}}\hat{\jmath} + \frac{1}{\sqrt{5}}\hat{k}$$

So, $\vec{c}$ is a unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

2. $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$ (using distributivity)

$= \vec{a}\times\vec{b} - \vec{c}\times\vec{a} + \vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a} - \vec{b}\times\vec{c}$ $(\because \vec{b}\times\vec{a} = -\vec{a}\times\vec{b})$

$\vec{a}\times\vec{c} = -\vec{c}\times\vec{a}$ and $\vec{c}\times\vec{b} = -\vec{b}\times\vec{c} = 0$

3. $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -\hat{\imath} + \hat{\jmath} + 3\hat{k}$$

$$\therefore (\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} i & \hat{\jmath} & \hat{k} \\ -1 & 1 & 3 \\ 1 & 1 & -2 \end{vmatrix} = -5\,\hat{\imath} + \hat{\jmath} - 2\hat{k}$$

Also,

$$(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \hat{\imath} + 5\hat{\jmath} + 3\hat{k}$$

$$\text{So, } \vec{a}\times(\vec{b}\times\vec{c}) \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 1 & 5 & 3 \end{vmatrix} = -11\hat{\imath} - 2\,\hat{\jmath} + 7\hat{k}$$

Clearly,

$$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\,(\vec{b}\times\vec{c})$$

4. $(\vec{a}+\vec{b}+\vec{c}) = \vec{0}$

$\Rightarrow \vec{a}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{a}\times\vec{0} = \vec{0}$

i.e., $\vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c} = \vec{0}$

i.e., $\vec{a}\times\vec{b}+\vec{a}\times\vec{c} = \vec{0}$ $\quad (\because\ \vec{a}\times\vec{a} = \vec{0})$

i.e., $\vec{a}\times\vec{b} = -\vec{a}\times\vec{c}$

i.e., $\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ……………..(1)

Similarly,

$\vec{b}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{b}\times\vec{0} = \vec{0}$

i.e., $\vec{b}\times\vec{a}+\vec{b}\times\vec{c} = \vec{0}$

i.e., $\vec{b}\times\vec{c} = -\vec{b}\times\vec{a}$

i.e., $\vec{b}\times\vec{c} = \vec{a}\times\vec{b}$ ……………. (2)

From (1) and (2), we have

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

5. Let ABCD be the parallelogram

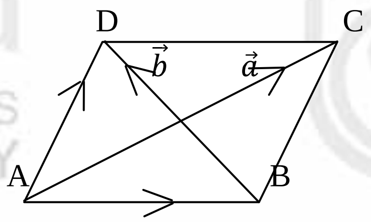


Figure 7

Now, $\overrightarrow{AB} = \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b}$

$= \frac{1}{2}(3\hat{\imath}+\hat{\jmath}-2\hat{k}) - \frac{1}{2}(\hat{\imath}-3\hat{\jmath}+4\hat{k})$

$= \hat{\imath}+2\hat{\jmath}-3\hat{k}$

and, $\overrightarrow{AD} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$

$= \frac{1}{2}(3\hat{\imath}+\hat{\jmath}-2\hat{k}) + \frac{1}{2}(\hat{\imath}-3\hat{\jmath}+4\hat{k})$

$= 2\hat{\imath}-\hat{\jmath}+\hat{k}$

So area of parallelogram ABCD

$= |\overrightarrow{AB}\times\overrightarrow{AD}|$

Now, $|\overrightarrow{AB} \times \overrightarrow{AD}| = \begin{vmatrix} \hat{\imath} & \hat{\imath} & \hat{k} \\ 1 & 2 & -3 \\ 2 & -1 & 1 \end{vmatrix}$

$= (2 – 3)\, \hat{\imath} – (1+ 6)\, \hat{\jmath} + (–1 – 4)\, \hat{k}$

$= –1\hat{\imath} –7\, \hat{\jmath} – 5\, \hat{k}$

$\therefore |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-1)^2 + (-7)^2 + (-5)^2}$

$= 5\sqrt{3}$

Hence, the area of parallelogram ABCD = $5\sqrt{3}$

Check Your Progress – 3

1. $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3\,\hat{\imath} + 5\hat{\jmath} - 7\,\hat{k}$

$\therefore \quad \vec{a}.(\vec{b} \times \vec{c}) = (2\hat{\imath} + \hat{\jmath} + 3\hat{k}).(3\hat{\imath} + 5\hat{\jmath}\, 3 - 7\hat{k})$

$= 6 + 5 – 21$

$= –10$

2. Volume of Parallelopiped = $|\vec{a}.(\vec{b} \times \vec{c})|$

Now, $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$= – 7$

$\therefore$ Volume $= |–7| = 7$

3. Let $\overrightarrow{OA} = \hat{\imath} - \hat{\jmath} + 2\,\hat{k}$

$\overrightarrow{OB} = 6\hat{\imath} + 11\,\hat{\jmath} + 2\,\hat{k}$

$\overrightarrow{OC} = \hat{\imath} + 2\hat{\jmath} + 6\,\hat{k}$

$\overrightarrow{OD} = \hat{\imath} + \frac{1}{2}\,\hat{\jmath} + 4\,\hat{k}$

So, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 5\hat{\imath} + 12\hat{\jmath}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -5\hat{\imath} - 9\hat{\jmath} + 4\,\hat{k}$

$\overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OC} = -\frac{3}{2}\hat{\jmath} - 2\,\hat{k}$

Now, $\overrightarrow{AB} \cdot (\overrightarrow{BC} \times \overrightarrow{CD}) = \begin{vmatrix} 5 & 12 & 0 \\ -5 & -9 & 4 \\ 0 & -\frac{3}{2} & -2 \end{vmatrix} = 0$

$\therefore \overline{AB}, \overline{BC}$ and $\overline{CD}$ are coplanar.

Hence A, B, C and D are coplanar.

4. Let $\vec{a} \times \vec{r} = \vec{r}$ Then

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \vec{r}.(\vec{c} \times \vec{d})$

$= (\vec{r} \times \vec{c}).\vec{d}$

$= ((\vec{a} \times \vec{b}) \times \vec{c}).\vec{d}$

$= [(\vec{c}.\vec{a})\,\vec{b} - (\vec{c}.\vec{a})\vec{a}].\vec{d}$

$= (\vec{c}.\vec{a})\,(\vec{b}.\vec{d}) - (\vec{c}.\vec{b})(\vec{a}.\vec{d})$

$= (\vec{a}.\vec{c})\,(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$

5. L.H.S. $= \hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k})$

$= [(\hat{\imath}.\hat{\imath})\vec{a} - (\hat{\imath}.\vec{a})\hat{i}] + [(\hat{\jmath}.\hat{\jmath})\vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath}] + [(\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}]$

$= \vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} + \vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath} + \vec{a} - (\hat{k}.\vec{a})\hat{k}$

$= 3\vec{a} - [(\hat{\imath}.\vec{a})\hat{i} + (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\vec{a})\hat{k}]$

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

So, $\hat{\imath}.\vec{a} = \hat{\imath}.(a_1\hat{\imath} - a_2\hat{\jmath} + a_3\hat{k}) = a_1$

Similarly, $\hat{\jmath}.\vec{a} = a_2$, $\hat{k}.\vec{a} = a_3$

$\therefore$ L.H.S. $= 3\vec{a} - (a_1\hat{\imath} - a_2\hat{\jmath} + a_3\hat{k})$

$= 3\vec{a} - \vec{a} = 2\vec{a}$

6. $\vec{a} \times (\vec{a} \times (\vec{a} \times \vec{b}))$

$= \vec{a} \times [(\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}]$

$= (\vec{a}.\vec{b})\,(\vec{a} \times \vec{a}) - (\vec{a}.\vec{a})\,(\vec{a} \times \vec{b})$

$= -(\vec{a}.\vec{a})\,(\vec{a}.\vec{b}) \quad (\because \vec{a} \times \vec{a} = \vec{0})$

$= (\vec{a}.\vec{a})\,(\vec{b} \times \vec{a}) \qquad (\because -\vec{a} \times \vec{b} = \vec{b} \times \vec{a})$

2.6 SUMMARY

This unit discusses various operations on vectors. The binary operation of scalar product is discussed in **section 2.2**. In the next section, the binary operation of vector product (also, called cross product) is illustrated. Finally in **section 2.4**, the ternary operation of tripe product of vectors is explained.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.5**.

UNIT 3 THREE-DIMENSIONAL GEOMETRY – 1

Structure

3.0 INTRODUCTION

Let us recall the concepts of two dimensional geometry. A point in plane is represented by an order pair of real numbers by a two-dimensional Cartesian coordinate system. In this unit, we shall see that a point in space is uniquely determined by an order triple of real numbers by a three dimensional Cartesian coordinate system. Once we have established a one to one correspondence between points in space and ordered triple of real numbers, we can extend the concept of distance between two points in space. We shall study distance formula for finding distance between two given points in space.

Also recall that we introduced the concepts of direction cosines and direction ratios of a vector. Infact, the same concept are valid for a directed line also. Infact, the concept of slope of a line in two dimensional plane is extended by direction cosines and direction ratios of a line in three dimensional plane.

We shall also study various forms of equation of a straight line in space. A straight line in space is uniquely determined if we know a point on the line and direction of the line or if we know two points on the line. Thus, we shall obtain equation of a straight line with a given point and parallel to a given direction, and equation of a straight line passing, through two given points. These equations are obtained both in vector and Cartesian forms.

Also recall that two lines in a plane either intersect or are parallel (or are coincident). But in space, we may have two non-intersecting and non-parallel lines. Such lines are called skew lines. In the last section, we shall introduce the concept of distance between two lines and find formula for calculating the shortest distance between two skew lines.

3.1 OBJECTIVES

After completing this unit, will be able to :

- find the distance between two points in space;
- find the direction cosines and direction ratios of a line passing through two points;
- find the equation of a straight line passing through a given point and parallel to a given vector in vector form and in Cartesian form;
- find the vector and Cartesian equations of a line passing through two given points;
- define the terms skew lines and coplanar lines; and
- find the distance between two skew lines.

3.2 THREE DIMENSIONAL SPACE

Let us recall that a point in plane is uniquely determined by an ordered pair of real numbers through a two dimensional Cartesian coordinate system. Similarly, there is one to one correspondence between points in space and ordered triplets of real numbers through a three dimensional Cartesian coordinate system, by fixing a point 0 as origin 0 and three mutually perpendicular lines through O as x-axis, y-axis and z-axis. The three axes are taken in such a way that they form a right-handed system. With any point P, we associate a triple of real number (x,y,z) in the following manner :

We drop a perpendicular from P to the xy–plane meeting it at M and take PM = z. From M drop perpendiculars on x-axis and y-axis meeting them at A and B respectively. Take MB = x and MA = y

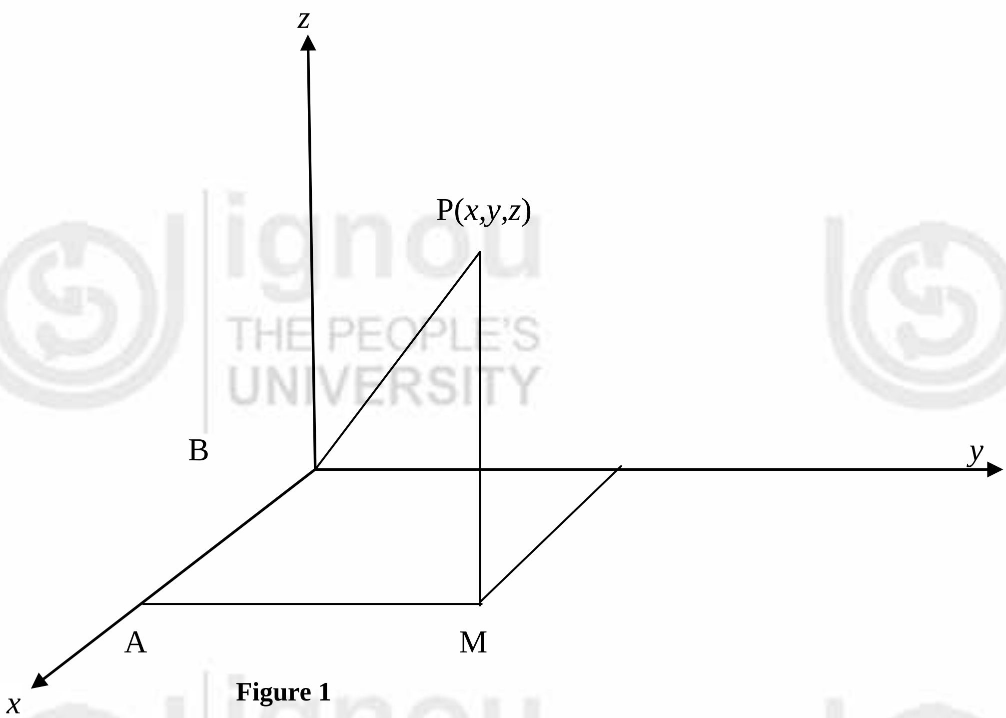


Figure 1

Also, given a triple (x,y,z) we can locate a point P in space uniquely whose Cartesian coordinates are (x,y,z)

Distance between two Points

Recall from the Unit 'Vectors – I' that if P is any point with coordinate (x,y,z) and position vector $\vec{r}$ then

$$\overrightarrow{OP} = \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

The distance OP of any point P(x,y,z) from the origin is given by
OP = $|\overrightarrow{OP}| = |\vec{r}\,| = \sqrt{x^2 + y^2 + z^2}$

Similarly, if P (x_1, y_1, z_1) and $Q(x_2, y_2, z_2)$ are two points with postion $\vec{r}_1$ vectors $\vec{r}_2$

Then

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r}_2 - \vec{r}_1$

$= (x_2\,\hat{\imath} + y_2\,\hat{\jmath} + z_2\,\hat{k}) - (x_1\,\hat{\imath} + y_1\,\hat{\jmath} + z_1\,\hat{k})$

$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\,\hat{\jmath} + (z_2 - z_1)\,\hat{k}$

and PQ = $|\overrightarrow{PQ}| = |\vec{r}_2 - \vec{r}_1|$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

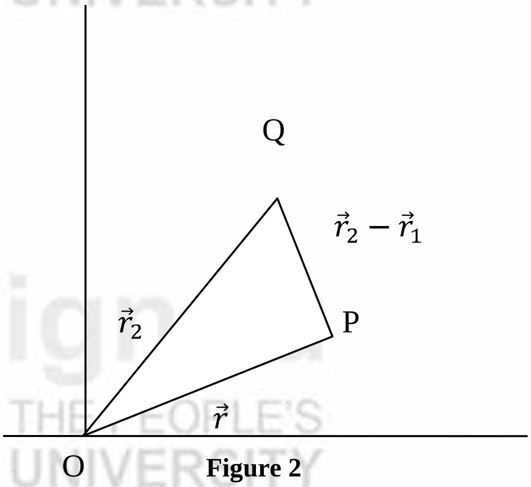


Figure 2

Thus, the distance between two points P (x_1, y_1, z_1) and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_{2,} - y_1)^2 + (z_2 - z_1)^2}$$

We have studies section formula (in Unit – 1 on Vectors) to find position vector of a point R which divides the line segment joining the points P and Q with position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively. If $\vec{r}_1$ is the position vector of R, then

$$\vec{r}_1 = \frac{m\,\vec{r}_2 + n\,\vec{r}_1}{m+n}$$

If P has coordinates P $(x_1, \ y_1, z_1)$ and Q has coordiantes $Q(x_2, \ y_2, z_2)$ and R has coordinates (x,y,z), then

$$x = \frac{m\,x_2 + n\,x_1}{m+n}$$

$$y = \frac{m\,y_2 + n\,y_1}{m+n}$$

$$z = \frac{m\,z_2 + n\,z_1}{m+n}$$

If we put $m = n = 1$, we get the coordinate of mid-point of PQ given as

$$x = \frac{x_2 + \ x_1}{2}$$

$$y = \frac{y_2 + \ y_1}{2}$$

$$z = \frac{z_2 + z_1}{2}$$

Example 1 : Find the distance between the points $P(1, –1, 0)$ and Q (2, 3, –1)

Solution : $PQ = \sqrt{(2-1)^2 + (3+1)^2 + (-1-0)^2}$

$$= \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$$

Direction Cosines and Direction Ratios

If α, β *and* γ are the angles which a non-zero vector $\vec{r}$ makes with positive x–axis, y–axis and z– axis respectively, then cos α, cos β and cos γ are called the direction cosines of $\vec{r}_1$ we write

$l = cos\alpha, m = cos\beta$ *and* $n = cos\gamma$

Also, $l^2 + \ m^2 + \ \ n^2 = 1$

If a, b, c are numbers proportional to l, m and n respectively, then a, b, c are called the direction ratios of $\vec{r}_1$

$$= \frac{a}{l} = \frac{b}{m} = \frac{c}{n}$$

with $l^2 + m^2 + n^2 = 1$.

Which gives

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

If $\vec{V} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$ is any vector, its direction ratios are *a*, *b*, *c* and its direction cosines are

$$l = \frac{a}{|\vec{V}|}$$

$$m = \frac{b}{|\vec{V}|}$$

$$n = \frac{c}{|\vec{V}|}$$

Consider the line joining the points P $(x_{1,}\ y_{1,}z_{1,})$ and $Q(x_{2,}\ y_{2,}z_{2,})$ whose position vectors

are $\vec{r}_1$ and $\vec{r}_2$

Now $\overrightarrow{AB} = \vec{r}_1 - \vec{r}_1$

$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\,\hat{\jmath} + (z_2 - z_1)\hat{k})$

Therefore, direction ratios of AB are $x_1 - x_2, y_2 - y_1$ and $z_2 - z_{1.}$ And the direction cosines of *AB* are

$$l = \frac{x_2 - x_1}{|\vec{r}\ |}$$

$$m = \frac{y_2 - y_1}{|\vec{r}\ |}$$

$$n = \frac{z_2 - z_1}{|\vec{r}\ |}$$

Where $\vec{r} = \vec{r}_1 - \vec{r}_1$

Thus, the direction cosines of a line joining points A$(x_{1,}\ y_{1,}z_{1,})$ and $B(x_{2,}\ y_{2,}z_{2,})$ are given by

$$\frac{x_2 - x_1}{AB}$$

$$\frac{y_2 - y_1}{AB}$$

$$\frac{z_2 - z_1}{AB}$$

Where *AB* is the distance between *A* and *B*

Example 2 : Find the direction cosines of a line which makes equal angles with the axes.

Solution : Let *l*, *m*, *n* be the direction cosines of the given line. Then

$$l^2 + m^2 + n^2 = 1$$

Since the given line makes equal angles with the axes, therefore, we have $l = m = n$

So, $l^2 + l^2 + l^2 = 1$

$or \quad 1 = \pm \frac{1}{\sqrt{3}}$

Thus $l = m = n \pm \frac{1}{\sqrt{3}}$

Example 3: Find the direction cosines of the line passing through the two points (1,2,3) and (–1,1,0)

Solution : We know that the direction cosines of the line passing through two points A$(x_{1,}\ y_{1,}z_{1,}) and\ B(x_{2,}\ y_{2,}z_{2,})$ are given by

$$l = \frac{x_2 - x_1}{AB}$$

$$m = \frac{y_2 - y_1}{AB}$$

$$n = \frac{z_2 - z_1}{AB}$$

$AB = \sqrt{(x_2 - x_2)^2 + (y_{2,} - y_2)^2 + (z_2 - z_1)^2}$

Here A is (1,2,3) and B is (–1,1,0)

So, *AB*,

where $AB = \sqrt{(-1-1)^2 + (1-2)^2 + (0-3)^2}$

$= \sqrt{4+1+9} = \sqrt{14}$

Thus, the direction cosines of line joining *A* and *B* are

$$l = \frac{-2}{\sqrt{14}}$$

$$m = \frac{-1}{\sqrt{14}}$$

$$n = \frac{-3}{\sqrt{14}}$$

Check Your Progress 1

1. Prove by finding distances that the three points (–2,3,5), (1,2,3) and (7,0,–1) are collinear.
2. Show that the points (0,7,10) (–1,6,6) and (–4,9,6) from an isosceles right angled triangle.
3. The direction ratios of a line are 1, –2, –2. What are its direction cosines ?
4. A line makes angles of 45° and 60° with the positive axes of *x* and *y* respectively. What angle does it make with the positive axis of *x* ?
5. Find the direction cosines of the line passing through the two points (–2, 4, 5) and (1,2,3).

3.3 EQUATION OF A STRAIGHT LINE IN SPACE

A line in space is completely determined once we know one of its points and its direction. We shall use vectors to measure direction and find equations of straight line in space.

(a) **Equation of a straight line passing through a fixed point A and parallel to the vector $\vec{b}$.**

Let $\vec{a}$ be the position vector of the given point A and let $\vec{r}$ be the position vector of any point P on the given line.

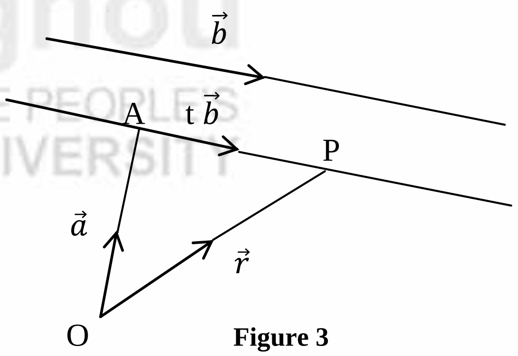


Figure 3

We have

$$\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + \overrightarrow{AP}$$

Since $\overrightarrow{AP}$ is parallel to $\vec{b}$, we must have $\overrightarrow{AP}$ = t $\vec{b}$ for some scalar *t*.

$\therefore \quad \vec{r} = \vec{a} + t\vec{b}$

Thus, each point *P* on the line has position vector $\vec{a} + t\vec{b}$ for some scalar *t*. Conversely, for each value of the scalar *t*, $\vec{a} + t\vec{b}$ is the position vector of a point of the line.

Hence, the vector equation of the line is

$$\vec{r} = \vec{a} + t\vec{b} \qquad (1)$$

Where t is a parameter.

Let us now write (1) in Cartesian form.

Let the coordinates of point A be (x_1, y_1, z_1)

So that $\vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$

Suppose the line has direction ratios a, b and c, then $\vec{b} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Substituting in (1), we get

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + t(a\hat{\imath} + b\hat{\jmath} + c\hat{k})$$

Equating the coefficients of $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ we get

$$x = x_1 + ta, \quad y = y_1 + tb \ \text{ and } z = z_1 + tc \qquad (2)$$

These are the parametric equations of the line.

Eliminating the parameter *t*, from (2), we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad (3)$$

This the Cartesian equation of a straight line.

Cartesian Equation of a straight line passing through the point (x_1, y_1, z_1) and having direction cosines *l, m, n* is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad (4)$$

Equations (1), (2), (3) and (4) are different forms of equations of a straight line passing through a given point and parallel to a given direction.

(b) Equation of a straight line passing through two given points

Let us now find equation of a straight line passing through two distinct points A (x_1, y_1, z_1) B (x_2, y_2, z_2). Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B respectively.

Since the points A and B lie on the line therefore $\overrightarrow{AB}$ is parallel to the line.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \vec{b} - \vec{a}$

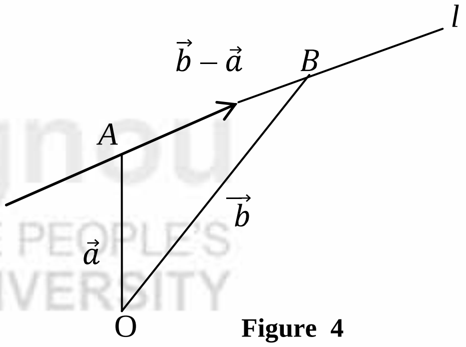


Figure 4

Thus, the vector equation of line passing through A and parallel to $\overrightarrow{AB}$ is

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) \qquad (5)$$

Equation (5) is the vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$.

Equation (5) can also be written as

$$\vec{r} = (1-t)\,\vec{a} + t\,\vec{b},\ t \in \mathrm{R} \qquad (6)$$

Let us now derive the Cartesian form the vector equation (5)

We have

$\vec{a} = x_1\,\hat{\imath} + y_1\hat{\jmath} + z_1\,\hat{k}$

$\vec{b} = x_2\,\hat{\imath} + y_2\hat{\jmath} + z_2\,\hat{k}$

Let $\vec{r} = x_1\,\hat{\imath} + y_1\hat{\jmath} + z_1\,\hat{k}$

Substituting these values in (5), we get

$x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k} = x_1\,\hat{\imath} + y_1\hat{\jmath} + z_1\,\hat{k} + t\,[(x_2 - x_1)\,\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$

Equating the coefficients of $\hat{\imath}, \hat{\jmath}, \hat{k}$ we get

$x = x_1 + t\,(x_2 - x_1)$
$y = y_1 + t\,(y_1 - y_1)$
$z = z_1 + t\,(z_2 - z_1)$

Eliminating t, we obtain

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad (7)$$

Equation (7) is the Cartesian form of equation of a straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Example 5 : Find the equations of the line (both Vector and Cartesian) passing through the point (1, –1,–2) and parallel to the vector $3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$.

Solution : We have

$\vec{a} = \hat{\imath} - \hat{\jmath} - 2\hat{k}$ and

$\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$

Therefore, the vector equation of the line is

$\vec{r} = (\hat{\imath} - \hat{\jmath} - 2\hat{k}) + t(3\hat{\imath} - 2\hat{\jmath} + 5\hat{k})$

Now $\vec{r}$ is the position vector of any point P (x,y,z) on the line.
So, $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$(\hat{\imath} - \hat{\jmath} - 2\hat{k}) + t(3\hat{\imath} - 2\hat{\jmath} + 5\hat{k})$

$= (1 + 3t)\,\hat{\imath} - (1 + 2t)\,\hat{\jmath} + (-2 + 5t)\,\hat{k}$

Thus $x = 1 + 3t$, $y = -1 - 2t$, $z = -2 + 5t$

Eliminating t, we get

$$\frac{x-1}{3} = \frac{y+1}{-2} = \frac{z+2}{5}$$

Example 6 : Find the Vector and Cartesian equation of the line passing through the points (–2, 0, 3) and (3,5, –2)

Solution : We have

$\vec{a} = -2\hat{\imath} + 3\hat{k}$ and

$\vec{b} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$

So, $\vec{b} - \vec{a} = 5\hat{\imath} + 5\hat{\jmath} - 5\hat{k}$

The vector equation of a line passing through two points with position vectors is given by

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

$\therefore\ \vec{r} = (-2\hat{\imath} + 3\hat{k})$ and $t(5\hat{\imath} + 5\hat{\jmath} - 5\hat{k})$

If $\vec{r}$ is the position vector of the point P (x, y, z) then we have

$x\hat{\imath} + y\,\hat{\jmath} + z\hat{k} = (-2\hat{\imath} + 3\hat{k}) + t\,(5\hat{\imath} + 5\,\hat{\jmath} - 5\hat{k})$

$= (-2 + 5t) + 5\,t\,\hat{j} + (3 - 5t)\hat{k}$

So, $x = -2 + 5t$, $y = 5t$, $z = 3-5t$

Eliminating t, we get

$$\frac{x+2}{5} = \frac{y}{5} = \frac{z-3}{-5}$$

Example 7 : The Cartesian equation of a line is

$$\frac{x+3}{4} = \frac{y-2}{5} = \frac{z+5}{1}$$

Find the Vector equation of the line.

Solution : Comparing the given equation with the standard form

$$\frac{x-x_{1,}}{a} = \frac{y-y_{1,}}{b} = \frac{z-z_{1,}}{c}$$

We get

$x_{1,} = -3$, $y_{1,} = 2$ $\quad z_{1,} = -5$ $a = 4, b = 5, c = 1$

Thus, the required line passes through the point $(-3, 2, -5)$ and is parallel to the vector $4\hat{\imath} + 5y\,\hat{\jmath} + \hat{k}$.

Thus, the equation of line in vector form is

$\vec{r} = (-3\hat{\imath} + 5\,\hat{\jmath} - 6\hat{k}) + t\,(4\hat{\imath} + 5\,\hat{\jmath} + \hat{k})$

Angle between two lines

Consider the two lines with vector equations

$\vec{r} = \vec{a} + t\,\vec{b}$ (8)

$\vec{r} = \overrightarrow{a'} + t\,\vec{b}'$ (9)

The angle θ between these lines is defined as the angle between the directions of $\vec{a}$ and $\vec{b}$ Also, we know that the angle between the vectors $\vec{b}$ and $\vec{b}'$ is given by

$$cos\theta = \left|\frac{\vec{b}\vec{b}'}{|\vec{b}||\vec{b}'|}\right|. \qquad (10)$$

Thus, (10) gives the angle between the lines (8) and (9). If the equation of two lines are given in Cartesian form

$$\frac{x - x_{1,}}{a_{1,}} = \frac{y - y_{1,}}{b_{1,}} = \frac{z - z_{1,}}{c_{1,}} \text{ and}$$

$$\frac{x - x_{2,}}{a_{2,}} = \frac{y - y_{2,}}{b_{2,}} = \frac{z - z_{2,}}{c_{2,}}$$

then the angle θ between them is given by

$$\text{Cos } \theta = \pm \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + a_2{}^2 + a_2{}^2}} \qquad (11)$$

as $\sin^2 \theta = 1 - \cos^2 \theta$, therefore, we also have

$$\sin \theta = \pm \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + a_2{}^2 + a_2{}^2}} \qquad (12)$$

Thus, the angle θ between two lines whose direction ratios are a_1, b_1, c_1 and a_2, b_2, c_2 is given by (11) or (12).

If instead of direction ratios, we take direction cosines l, m, n and l_2, m_2, n_2 of the two lines, then (10) and (11) can be written as

$$\cos \theta = \text{ and } \quad l_1 l_2 +, m_1 m_2 + n_1 n_2$$

and $\sin \theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2}$

It is clear from these relations that two lines are perpendicular to each other if and only if

$$l_1 l_2 +, m_1 m_2 + n_1 n_2 = 0$$

and parallel to each other if and only if

$$l_1 l_2 +, m_1 m_2 + n_1 n_2 = 0$$

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

Similarly, two lines with direction ratios a_1, $b_{1,} c_1$ and a_2, $b_{2,} c_2$ are

(i) perpendicular iff $a_1, a_{2+}, b_{2,} b_{1+} c_2 c_2$ and

(ii) parallel iff

$$\frac{a_1}{a_1} = \frac{b_1}{b_1} = \frac{c_1}{c_1}$$

Example 8: Find the angle between the lines

$$\vec{r} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k} + t(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$$
$$\vec{r} = 3\hat{\imath} - 5\hat{k} + s(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})$$

Solution : Here $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})$

If θ is the angle between the two lines, then

$$cos\,\theta = \left|\frac{\vec{b}\vec{b}'}{|\vec{b}||\vec{b}'|}\right|$$

$$\left|\frac{(\hat{\imath} - 2\hat{\jmath} + 2\hat{k}).(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})}{\sqrt{1+4+4}\,\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21}$$

Hence, $\theta = Cos^{-1}\left(\frac{19}{21}\right)$

Example 9 : Find the angle between the pair of lines

$$\frac{x-5}{2} = \frac{y-3}{1} = \frac{z-1}{-3} \text{ and}$$

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z+5}{-1}$$

Solution : The direction ratios of the line

$$\frac{x-5}{2} = \frac{y-3}{1} = \frac{z-1}{-3}$$

are 2, 1, and –3. similarly, the direction ratios of the line

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z+5}{-1} \text{ are } 3, 2, \text{and} - 1$$

Therefore, the angle θ between them is given by

$$\cos\theta = \left|\frac{2.3+1.2+(-3)(-1)}{\sqrt{2^2+1^2+(-3)^2}\sqrt{3^2+2^2+(-1)^2}}\right|$$

$$= \frac{11}{\sqrt{14}\,\sqrt{14}} = \frac{11}{14}$$

Hence, $\theta = cos^{-1}\left(\frac{11}{14}\right)$

Example 10: Find the equation of the line passing through the point (–1,3,–2) and perpendicular to the two lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}.$$

Solution : Let the equation of the given line be

$$\frac{x+2}{l} = \frac{y-3}{m} = \frac{z+2}{n}$$

Since (i) is perpendicular to both the lines therefore,

$l + 2m + 3n = 0$ ……………………(ii)

$-3l + 2m + 5n = 0$ …………………(iii)

Solving (ii) and (iii) we get

$$\frac{l}{10-6} = \frac{m}{-9-5} = \frac{n}{2+6}$$

Or

$$\frac{l}{4} = \frac{m}{-14} = \frac{n}{8}$$

$$\frac{l}{4} = \frac{m}{-7} = \frac{n}{8}$$

$\therefore$ The required equation is

$$\frac{x+2}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

Check Your Progress 2

1 Find the Vector equation of the line passing through the point (3,1,4) and parallel to the vector $-\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Also find the Cartesian equation of the line.

2 Find the vector and Cartesian equation of the line passing through (1,0,–4) and is parallel to the line

$$\frac{x+1}{3} = \frac{z+2}{4} = \frac{z-2}{2}$$

3. Find the vector equation for the line through the points (3,4,–7) and (1, –1,6) Also find the Cartesian equation.

4. Find the angle between the following pairs of lines.

(i) $\quad \frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ *and* $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$

(ii) $\quad \vec{r} = 3\hat{\imath} + 2\hat{\jmath} - 4\hat{k} + t(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$

$\quad \vec{r} = 5\hat{\jmath} - 2\hat{k} + S(\widehat{4\imath} - \hat{\jmath} + 8\hat{k})$

5. Find k so that the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \; and \; \frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$$

are at right angles.

3.4 SHORTEST DISTANCE BETWEEN TWO LINES

Two lines in space may either lie in the same plane or in different planes. In the former case, such lines are called coplanar lines. Clearly line which do not lie in the same planes are called non coplanar lines. We know that two lines in plane are either intersecting or parallel. But in space, two non coplanar lines may neither intersect, nor be parallel to each other . Such lines are called skew lines.

Definition : Two non coplanar liens are called skew lines if they are neither parallel nor intersecting.

The shortest distance between two lines is the join of a point in one line with one point on the other so that the length of the segment so obtained is the smallest.

Clearly, the shortest distance between two intersecting lines is zero and shortest distance between two parallel lines is the distance by which the two lines are separated. For skew lines, the direction of shortest distance is perpendicular to both the lines.

Let us now find an expression for the shortest distance between two skew lines.

Consider two skew liens l_1 and l_2 (Fig. 5) with vector equation

$$\vec{r} = \vec{a} + t\vec{b}$$
$$\text{and } \vec{r} = \vec{c} + p\,\vec{d}$$

Let A and C be points on $\vec{r}$ and $\vec{r}$ $respectively.$

Let A and C be points on l_1 and l_2 with position vectors $\vec{a}$ and $\vec{c}$ respectively.

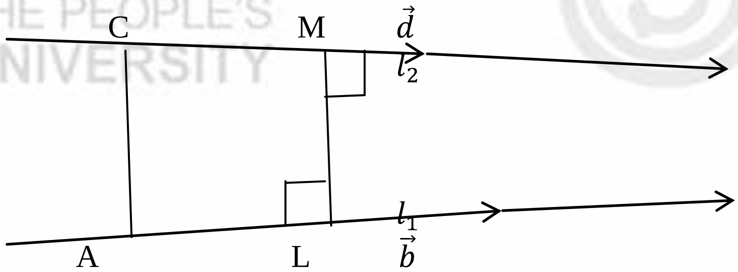


Figure 5

The line LM of shortest distance is perpendicular to both l_1 and l_2 therefore is parallel to $\vec{b} \times \vec{d}$.

The length of LM is the project of $\overrightarrow{AC}$ on $\overrightarrow{LM}$

So, LM = AC |Cos θ |, when θ is the angle between $\overrightarrow{AC}$ on $\overrightarrow{LM}$

Now a unit vector $\hat{n}$ along by $\overrightarrow{LM}$ *is given by*

$$\hat{n} = \frac{\vec{b} \times \vec{d}}{|\vec{b} \times \vec{d}\,|}$$

$$\text{So}, \overrightarrow{LM} = \text{LM}\,\hat{n} = \text{LM}\left(\frac{\vec{b} \times \vec{d}}{|\vec{b} \times \vec{d}\,|}\right)$$

$$\text{Also } Cos\,\theta \quad \frac{\overrightarrow{AC}\,.\,\overrightarrow{LM}}{(AC)(LM)} = \frac{(\text{LM})[\,\overrightarrow{AC}.(\vec{b} \times \vec{d}\,)}{(AC)(LM)|\vec{b} \times \vec{d}\,|}$$

$$= \frac{\overrightarrow{AC}\,.\,(\vec{b} \times \vec{d}\,)}{(AC)|\vec{b} \times \vec{d}\,|}$$

$$= \frac{(\overrightarrow{c} - \vec{a}\,)\,(\vec{b} \times \vec{d}\,)}{(AC)|\vec{b} \times \vec{d}\,|}$$

Hence, LM = AC |Cos θ|

$$= \frac{(\overrightarrow{c} - \vec{a}\,).(\vec{b} \times \vec{d}\,)}{|\vec{b} \times \vec{d}\,|} \qquad \ldots \text{(i)}$$

$$= \frac{(\overrightarrow{c}\,\vec{b}\,\vec{d}\,)\,(\vec{a}\,\vec{b}\,\vec{d}\,)}{|\vec{b} \times \vec{d}\,|}$$

Remarks :

1. Two lines will intersect if and only if the shortest distance between them is zero i.e. LM = 0

   or $(\vec{c}-\overrightarrow{a})\,(\vec{b}-\overrightarrow{d}) = 0$

2. If two lines are parallel and are given by $\overrightarrow{r} = \overrightarrow{a} + t\,\overrightarrow{b}$ and $\overrightarrow{r} = \overrightarrow{c} + p\,\overrightarrow{b}$, then they are coplanar. Then the distance between these two lines is given by

$$d = \left|\frac{\overrightarrow{b} \times (\overrightarrow{c} - \overrightarrow{a})}{|\,\overrightarrow{b}|}\right|$$

   This is because d = LM = AC |cos $\frac{\pi}{2} - \theta$ | (see. Fig. 6) so that

$$\text{d} = \text{AC}\,|\sin\theta\,|\,\vec{c} - \vec{a}|\left|\frac{\overrightarrow{b} \times (\overrightarrow{c} - \overrightarrow{a})}{|\,\overrightarrow{b}\,| \times |\overrightarrow{c} - \overrightarrow{a}|}\right|$$

$$= \left| \frac{\vec{b} \times (\vec{c} - \vec{a})}{|\vec{b}|} \right|$$

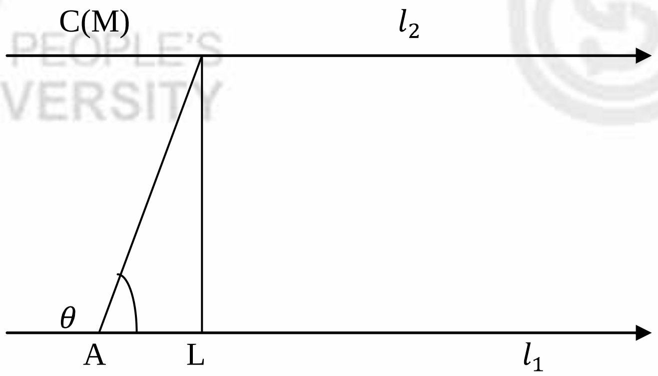


Figure 6

Cartesian form the distance between skew lines

Let the equation of two skew lines l_1 *and* l_2

$$l_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$\text{and } l_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

The corresponding vector from equation of l_1 and l_2 respectively are

$\vec{r} = (x_1\hat{\imath} + y_1\ \hat{\jmath} + z_1\hat{k}) + \text{t} (a_1\hat{\imath} + b_1\ \hat{\jmath} + c_1\hat{k})$

and $\vec{r} = (x_2\hat{\imath} + y_2\ \hat{\jmath} + z_2\hat{k}) + t(a_2\hat{\imath} + b_2\ \hat{\jmath} + c_2\hat{k})$.
Comparing these equations with $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$ and substituting the values in (i), we see that the distance between l_1 and l_2 is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}$$

Example 11 : Find the shortest distance between the lines

$$\vec{r} = (3\hat{\imath} + 4\ \hat{\jmath} - 2\hat{k}) + \text{t} (-\hat{\imath} + 2\ \hat{\jmath} + \hat{k})$$

$$\text{and } \vec{r} = (\hat{\imath} - 7\ \hat{\jmath} + 2\hat{k}) + (\hat{\imath} + 3\ \hat{\jmath} - 2\hat{k})$$

Solution : Comparing with equations $\vec{r} = \vec{a} + \text{t}\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$, we have

$\vec{a} = 3\hat{\imath} + 4\ \hat{\jmath} - 2\hat{k}$

$\vec{b} = -\hat{\imath} + 2\ \hat{\jmath} + \hat{k}$

$\vec{c} = \hat{\imath} - 7\ \hat{\jmath} + 2\hat{k}$

$\vec{d} = \hat{\imath} + 3\ \hat{\jmath} - 2\hat{k}$

$\therefore\ \vec{a} - \vec{c} = 2\ \hat{\imath} - 11\ \hat{\jmath}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}$

$= \hat{\imath} + 3\ \hat{\jmath} - 5\hat{k}$

So $|\vec{b} \times \vec{d}| = \sqrt{1 + 9 + 25} = \sqrt{35}$

Thus, Shortest distance

$$= \left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

$$= \left|\frac{(-2\hat{\imath} - 11\ \hat{\jmath}).(\hat{\imath} + 3\ \hat{\jmath} - 5\hat{k})}{\sqrt{35}}\right|$$

$$= \frac{35}{\sqrt{35}} = \sqrt{35}$$

Example 12 : Find the shortest distance between the lines whose vector equations are :

$$\vec{r} = (1 + \lambda)\hat{\imath} + (2 - \lambda)\ \hat{\jmath} + (1 + \lambda)\hat{k}$$

and $\vec{r} = 2(1 + \mu)\hat{\imath} + (1 - \mu)\ \hat{\jmath} + (-1 + 2\mu)\hat{k}$

Solution : The two equations are:

$$\vec{r} = (\hat{\imath} + 2\ \hat{\jmath} + \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$$

and $\vec{r} = (2\hat{\imath} - \hat{\jmath} - \hat{k}) + \mu\,(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

Here, we have

$\vec{a} = \hat{\imath} + 2\ \hat{\jmath} + \hat{k}$

$\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{c} = 2\ \hat{\imath} - \hat{\jmath} - \hat{k}$

$\vec{d} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$

So, $\vec{c} - \vec{a} = \hat{\imath} - 3\ \hat{\jmath} - 2\hat{k}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$$= -3\hat{\imath} + 3\,\hat{k}$$

So $|\vec{b} \times \vec{d}| = \sqrt{9+9} = 3\sqrt{2}$

Hence, the shortest distance between the two lines =

$$= \left|\frac{(\vec{b} \times \vec{d}).(\vec{c} - \vec{a})}{|\vec{b} \times \vec{d}|}\right|$$

$$= \left|\frac{(-3\hat{\imath} - 3\,\hat{k}).(\hat{\imath} - 3\,\hat{\jmath} - 2\hat{k})}{\sqrt[3]{2}}\right|$$

$$= \left|\frac{-3-6}{\sqrt[3]{2}}\right| = \frac{3}{\sqrt{2}} = \frac{\sqrt[3]{2}}{2}$$

Example 13 : Show that the lines

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z-3}{-5}$$

and $\dfrac{x-8}{7} = \dfrac{y-4}{1} = \dfrac{z-5}{3}$

intersect.

Solution : Given lines are :

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z-3}{-5}$$

and $\dfrac{x-5}{4} = \dfrac{y-7}{4} = \dfrac{z-5}{4}$

converting these equations into vectors, we get

$$\vec{r} = (\,5\hat{\imath} + 7\,\hat{\jmath} - 3\hat{k}\,) + t\,(\,4\hat{\imath} + 4\,\hat{\jmath} - 5\hat{k}\,)$$

and $\vec{r} = (8\,\hat{\imath} + 4\,\hat{\jmath} + 5\hat{k}\,) + s\,(7\,\hat{\imath} + \hat{\jmath} + 3\hat{k}\,)$

Comparing with equations $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ respectively, we have

$\vec{a} = 5\hat{\imath} + 7\,\hat{\jmath} - 3\hat{k}$

$\vec{b} = 4\hat{\imath} + 4\,\hat{\jmath} - 5\hat{k}$

$\vec{c} = 8\hat{\imath} + 4\,\hat{\jmath} + 5\hat{k}$

$\vec{d} = 7\,\hat{\imath} + \hat{\jmath} + 3\hat{k}$

$\therefore\ \vec{c} - \vec{a} = 3\,\hat{\imath} - 3\,\hat{\jmath} + 8\hat{k}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$

$= 17\hat{\imath} + 47\ \hat{\jmath} - 24\hat{k}$.

Now $(\vec{c} - \vec{a}\ ).(|\vec{b} \times \vec{d}) = (3\hat{\imath} - 3\ \hat{\jmath} + 8\hat{k}).(17\hat{\imath} - 47\ \hat{\jmath} - 24\hat{k}) = 51 + 141 - 192 = 0.$

Thus, the shortest distance between the two lines =

$$\left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right| = 0$$

Hence, the two lines intersect.

Check Your Progress – 3

1. Find the shortest distance between the two lines whose vector equations are
   $\vec{r} = \hat{\imath} - \ \hat{\jmath}\ + \mathrm{t}\ (\ 2\hat{\imath} + \hat{k}\ )$
   and $\vec{r} = (2\hat{\imath} - \ \hat{\jmath}\ ) + \mathrm{s}\ (\ \hat{\imath} + \hat{\jmath} - \hat{k}\ )$

2. Find the shortest distance between the two lines
   $\vec{r} = (1 - t)\hat{\imath} + (\mathrm{t} - 2)\ \hat{\jmath} + (3 - 2t)\hat{k}$
   and $\vec{r} = (s + 1)\hat{\imath} + (2\mathrm{s} - 1)\ \hat{\jmath} + (2s + 1)\hat{k}$

3. Find the distance between the lines
   $\dfrac{x-1}{2} = \dfrac{y+1}{2} = z$ and

   $\dfrac{x+1}{5} = \dfrac{y-2}{2}; z = 2$

4. Determine whether the following pair of lines interest
   $\vec{r} = (2\lambda + 1)\ \hat{\imath} - (\lambda + 1)\ \hat{\jmath} + (\lambda + 1)\ \hat{k}$

   $\vec{r} = (3\ \mu\ + 2)\ \hat{\imath} - (5\ \mu\ + 5)\ \hat{\jmath} + (2\ \mu\ - 1)\ \hat{k}$

3.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. Let A, B, C denote the points (–2,3,5), (1,2,3) and (7,0,–1) respectively

   Then AB = $\sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2}$

   $= \sqrt{9+1+4} = \sqrt{14}$

   BC = $\sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2}$

AB = $\sqrt{36+4+16} = \sqrt{56} = 3\sqrt{14}$

AC = $\sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2}$

$\sqrt{81+9+36} = \sqrt{126} \quad 3\sqrt{14}$

Therefore, AB + BC = AC

Hence, the three points are Colliner.

2. Let A (0,7,10), B (–1,6,6) and C(–4,9,6) denote the given points. Then

AB = $\sqrt{(-1+0)^2 + (6-7)^2 + (6-10)^2}$

AB = $\sqrt{1+1+16} = 3\sqrt{2}$

BC= $\sqrt{(-4+0)^2 + (9-6)^2 + (6-6)^2}$

= $\sqrt{9+9} = 3\sqrt{2}$

AC = $\sqrt{(-4-0)^2 + (9-7)^2 + (6-10)^2}$

AC = $\sqrt{16+4+16}$ = 6

Since AB = BC, therefore, the triangle $\Delta\, ABC\ is\ isosceles.$

Further, AB^2 + BC^2 = $(3\sqrt{2})^2$ +$(3\sqrt{2})^2 = 18 + 18 = 36$

Also $BC^2 = 36$

$\therefore\ AB^2 + BC^2 = AC^2$

Hence, $\Delta\, ABC$ is a right angled triangle.

3. Let *a* = 1, *b* = –2, *c* = –2

The direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{1+4+4}} = \frac{1}{9}$$

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{1+4+4}} = \frac{1}{9}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{-2}{\sqrt{1+4+4}} = \frac{-2}{9}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{1+4+4}} = \frac{-2}{9}$$

4. Let α, β, γ be the angles which the line makes with the x-axis, y-axis, y-axis and z-axis respectively.

and $\cos\beta = \cos 60^\circ = \frac{1}{2}$

Now, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\frac{1}{2} + \frac{1}{4} + cos^2 r = 1$$

$$\frac{3}{4} + cos^2 r = 1$$

$$cos^2 r = \frac{1}{4}$$

$$cos\, r = \pm\frac{1}{2}$$

Or $\cos^2\gamma = \frac{1}{4}$ i.e., $\cos^2\gamma = \pm\frac{1}{2}$

$\Rightarrow \gamma = 60^\circ$ or 120°

5. Let A(–2, 4, –5 and B(1, 2, 3) denote the given points. Then

AB = $\sqrt{(1+2)^2 + (2-4)^2 + (3+5)^2} = \sqrt{9+4+64} = \sqrt{77}$

The direction cosines of the line joining the points A and B are given by

$$l = \frac{x_2 - x_1}{AB}, m = \frac{y_2 - y_1}{AB}, and\ n = \frac{z_2 - z_1}{AB}$$

Hence, the direction cosines are

$$l = \frac{3}{\sqrt{77}}, m = \frac{-2}{\sqrt{77}}, n = \frac{8}{\sqrt{77}}$$

Check Your Progress – 2

1. We have

$\vec{a} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$ and

$\vec{b} = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$

Therefore, the vector equation of the line is

$\vec{r} = (3\hat{\imath} + \hat{\jmath} + 4\hat{k}) + t\,(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$

If $\vec{r}$ is the position vector of point (x, y, z), then

$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = (3\hat{\imath} + \hat{\jmath} + 4\hat{k}) + t(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$

= (3–t) $\hat{\imath}$ + (1–t) $\hat{\jmath}$ + (4–2t) $\hat{k}$

$\therefore$ x = 3 – t, y = 1 + t, z = 4–2t

Eliminating t, we get

$$\frac{x-3}{-1} = \frac{y-1}{1} = \frac{z-4}{-2}$$

which is the equation of line in Cartesian form.

2. A vector parallel to the line

$$\frac{x+2}{3} = \frac{y-4}{1} = \frac{z-2}{2}$$

is $3\hat{\imath} + \hat{\jmath} + 2\hat{k}$

Thus we have to find equation of a line passing through (1, 0, 4) & parallel to the vector $3\hat{\imath} + \hat{\jmath} + 4\hat{k}$

So, $\vec{a} = \hat{\imath} - 4\hat{k}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\therefore$ the vector eqatuon of required line is

$\vec{r} = (\hat{\imath} - 4\hat{k}) + t\,(3\hat{\imath} + \hat{\jmath} + 2\hat{k})$

Also, the Cartesian equation of the line is

$$\frac{x-1}{3} = \frac{y-0}{1} = \frac{z+4}{2}$$

3. Let A (3, 4,–7) and B(1, –1, 6) denote the given points. The direction ratios of AB are 1–3, –1–4, 6+7 or –2, –5, 13

$\therefore \quad \vec{b} = -2\hat{\imath} - 5\hat{\jmath} + 13\hat{k}$

Also as A (3, 4, –7) lies on the line,

$\therefore \quad \vec{a} = \overrightarrow{OA} = 3\hat{\imath} + 4\hat{\jmath} - 7\hat{k}$

Hence, the vector equation is

$\vec{r} = \vec{a} + t\,\vec{b}$

i.e., $\vec{r} = (3\hat{\imath} + 4\hat{\jmath} - 7\hat{k}) + t\,(-2\hat{\imath} - 5\hat{\jmath} + 13\hat{k})$

Also, Cartesian equation of a line passing through the points $(x_1\ y_1\ , z_1)$ $(\ x_2\ , x_2, x_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

So, the required Cartesian equation is

$$\frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z-13}{-7}$$

4(i) The given liens are

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \quad \ldots.(1)$$

$$\text{and } \frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \quad \ldots.(2)$$

Therefore, the direction ratios of line (1) are 1, 0, –1 and the direction ratios of line (2) are 3,4,5

If θ is the angle between the two lines, then

Cos θ =

$$\text{Cos}\theta = \left|\frac{1.3 + 0.4 + (-1)5}{\sqrt{1^2 + 0^2 + 1^2}\sqrt{3^2 + 4^2 + 5^2}}\right|$$

$$= \left|\frac{3-5}{\sqrt{2}\sqrt{50}}\right| = \left|\frac{-2}{\sqrt{100}}\right| = \left|\frac{-2}{10}\right| = \left|\frac{-1}{5}\right| = \frac{1}{5}$$

Hence $\theta = \cos^{-1}\left(\frac{1}{5}\right)$

(i) Here, $\vec{b} = \hat{\imath} - 2\,\hat{\jmath} + 2\hat{k}$ and $\vec{b'} = 4\hat{\imath} + \hat{\jmath} + 8\hat{k}$
If θ is the angle between the two lines,

$$\text{Then Cos}\theta = \left|\frac{\vec{b}.\vec{b}'}{|\vec{b}||\vec{b}'|}\right|$$

$$= \left|\frac{(\hat{\imath} - 2\,\hat{\jmath} + 2\hat{k}\,)(4\hat{\imath} + \hat{\jmath} + 8\hat{k})}{\sqrt{1^2 + (-2)^2 + 2^2}\sqrt{4^2 + 1^2 + 8^2}}\right|$$

$$= \left|\frac{4 - 2 + 16}{\sqrt{9}\sqrt{81}}\right| = \frac{18}{27} = \frac{2}{3}$$

Hence $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

5. The given lines are

$$= \frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \quad \ldots(1)$$

$$\text{and } \frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5} \quad \ldots(2)$$

The direction ratios of line (1) are –3, 2*k*, 2 and direction ratios of line (2) are 3*k*, 1, –5. Since the lines (1) and (2) are at right angles, therefore

$(-3)(3k) + (2k)(1) + 2(-5) = 0$

or $-9k + 2k - 10 = 0$

or $-7k = 10$

$$or \quad k = \frac{10}{7}$$

Check Your Progress – 3

1. Here

$\vec{a} = \hat{\imath} - \hat{\jmath}$

$\vec{b} = 2\hat{\imath} + \hat{k}$

$\vec{c} = 2\hat{\imath} - \hat{\jmath}$ and

$\vec{d} = \hat{\imath} + \hat{\jmath} - \hat{k}$

$\therefore\ \vec{c} - \vec{a} = (2\,\hat{\imath} - \hat{\jmath}) + (\,\hat{\imath} - \hat{\jmath}) = \hat{\imath}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= -\hat{\imath} + 3\,\hat{\jmath} + 2\hat{k} =$

$\therefore |\vec{b} \times \vec{d}\,| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

The shortest distance between the two lines is

$$= \left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

$$\left|\frac{\hat{\imath}(-\hat{\imath} + 3\,\hat{\jmath} + 2\hat{k})}{\sqrt{14}}\right| = \left|\frac{-1}{\sqrt{14}}\right| = \frac{-1}{\sqrt{14}}$$

2. The given equation can be written as

$\vec{r} = (\,\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}\,) + t\,(\,-\hat{\imath} + \hat{\jmath} - 2\hat{k}\,)$ and

$\vec{r} = (\,\hat{\imath} - \hat{\jmath} - \hat{k}\,) + s\,(\,\hat{\imath} + 2\,\hat{\jmath} - 2\hat{k}\,)$

Here, $\vec{a} = \hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}$

$\vec{b} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\vec{c} = \hat{\imath} - \hat{\jmath} - \hat{k}$

$\vec{d} = \hat{\imath} + 2\,\hat{\jmath} - 2\hat{k}$

$\vec{c} - \vec{a} = (\hat{\imath} - \hat{\jmath} - \hat{k}) - (\,\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k})$

$= \hat{\jmath} - 4\hat{k}$

Also, $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

$= 2\hat{\imath} - 4\,\hat{\jmath} - 3\hat{k}$

$\therefore \ |\vec{b} \times \vec{d}\,| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

The shortest distance between the two lines is

$$= \left|\frac{(\vec{b} \times \vec{d}).(\vec{c} - \vec{a})}{|\vec{b} \times \vec{d}|}\right| = \left|\frac{(2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}).(\hat{\jmath} - 4\hat{k})}{\sqrt{29}}\right| = \left|\frac{-4+12}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

3. The given lines are

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1} \qquad \ldots (1)$$

and $\dfrac{x+1}{5} = \dfrac{y-2}{1} = \dfrac{z-2}{0}$

Converting these equations into vector form, we have

$\vec{r} = (\hat{\imath} - \hat{\jmath}) + t\,(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$

and $\vec{r} = (-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) + s\,(5\hat{\imath} + \hat{\jmath})$

Comparing these equation with $\vec{r} = \vec{a} + t\vec{b}$

and $\vec{r} = \vec{c} + s\,\vec{d}$, we have

$\vec{a} = (\hat{\imath} - \hat{\jmath})$

$\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

$\vec{c} = -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\vec{b} = 5\hat{\imath} + \hat{\jmath}$

Now, $\vec{c} - \vec{a} = (-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) - (\hat{\imath} - \hat{\jmath}) = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$
and

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 1 \\ 5 & 1 & 0 \end{vmatrix}$$

$= -\hat{\imath} + 5\hat{\jmath} - 13\hat{k}$

$\therefore |\vec{b} \times \vec{d}\,| = \sqrt{(-1)^2 + (5)^2 + (-13)^2} = \sqrt{1+25+169} = \sqrt{195}$

$$\text{Shortest distance} = \left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

$$= \left|\frac{(-\hat{2\imath} + 3\hat{\jmath} + 2\hat{k}).(-\hat{\imath} + 5\hat{\jmath} - 13\hat{k})}{\sqrt{195}}\right| = \left|\frac{2+15-26}{\sqrt{14}}\right| = \frac{9}{\sqrt{195}}$$

4. The given lines are

$\vec{r} = \hat{\imath} - \hat{\jmath} + \hat{k} + (2\hat{\imath} - \hat{\jmath} + \hat{k})$ and

$\vec{r} = 2\hat{\imath} - 5\hat{\jmath} - \hat{k} + \mu(3\hat{\imath} - 5\hat{\jmath} + 2\hat{k})$

Here, $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{c} = 2\hat{\imath} - 5\hat{\jmath} + \hat{k}$

$\vec{d} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$

Now, $\vec{c} - \vec{a} = -\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$

$= 3\hat{\imath} - \hat{\jmath} - 7\hat{k}$

$\therefore |\vec{b} \times \vec{d}| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{59}$

$$\text{Shortest Distance} = \left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

$$= \left|\frac{(\hat{\imath} - 4\hat{\jmath} - 2\hat{k}).(3\hat{\imath} - \hat{\jmath} - 7\hat{k})}{\sqrt{59}}\right| = \left|\frac{3+4+14}{\sqrt{59}}\right| = \frac{21}{\sqrt{59}}$$

3.6 SUMMARY

The unit, as the title suggests, is about three-dimensional geometry. In **section 3.2**, first, the concept of three dimensional space is illustrated. Then formula for distance between two points in three-dimensional space, is derived. Then, the concepts of direction cosines and direction ratios are explained. In **section 3.3**, formulae for finding equations of a straight line are derived when (i) a point of the line and a vector parallel to the required line are given and when (ii) a pair of points is given. In **section 3.4**, formula for finding shortest distance between pair of straight lines in three-dimensional space, is first derived and then is used in solving problems.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.5**.

UNIT 4 LINEAR PROGRAMMING

Structure

4.0 INTRODUCTION

We first make the idea clear through an illustration.

Suppose a furniture company makes chairs and tables only. Each chair gives a profit of ₹ 20 whereas each table gives a profit of ₹30. Both products are processed by three machines M_1, M_2 and M_3. Each chair requires 3 hrs 5 hrs and 2 hrs on M_1, M_2 and M_3. respectively, whereas the corresponding figures for each table are 3, 2, 6. The machine M_1 can work for 36 hrs per week, whereas M_2 and M_3 can work for 50 hrs and 60 hrs, respectively. How many chairs and tables should be manufactured per week to maximise the profit?

We begin by assuming that x chairs and y tables, be manufactured per week. The profit of the company will be ₹ $(20x + 30y)$ per week. Since the objective of the company is to maximise its profit, we have to find out the maximum possible value of $P = 20x + 30y$. We call this as the objective function.

To manufacture x chairs and y tables, the company will require $(3x + 3y)$ hrs on machine M_1 . But the total time available on machine M_1 is 36 hrs. Therefore, we have a constraint $3x + 3y \leq 36$.

Similarly, we have the constraint $5x + 2y \leq 50$ for the machine M_2 and the constraint $2x + 6y \leq 60$ for the machine M_3.

Also since it is not possible for the company to produce negative number of chairs and tables, we must have $x \geq 0$ and $y \geq 0$. The above problem can now be written in the following format :

Maximise

$$P = 20x + 30y$$

subject to

$$3x + 3y \le 36$$
$$5x + 2y \le 50$$
$$2x + 6y \le 60$$

and $x \ge 0,\ y \ge 0$

We now plot the region bounded by constraints in Fig. 1 The shaded region is called the **feasible region** or the **solution space** as the coordinators of any point lying in this region always satisfy the constraints. Students are encouraged to verify this by taking points (2,2),(2,4) (4,2) which lies in the feasible region.

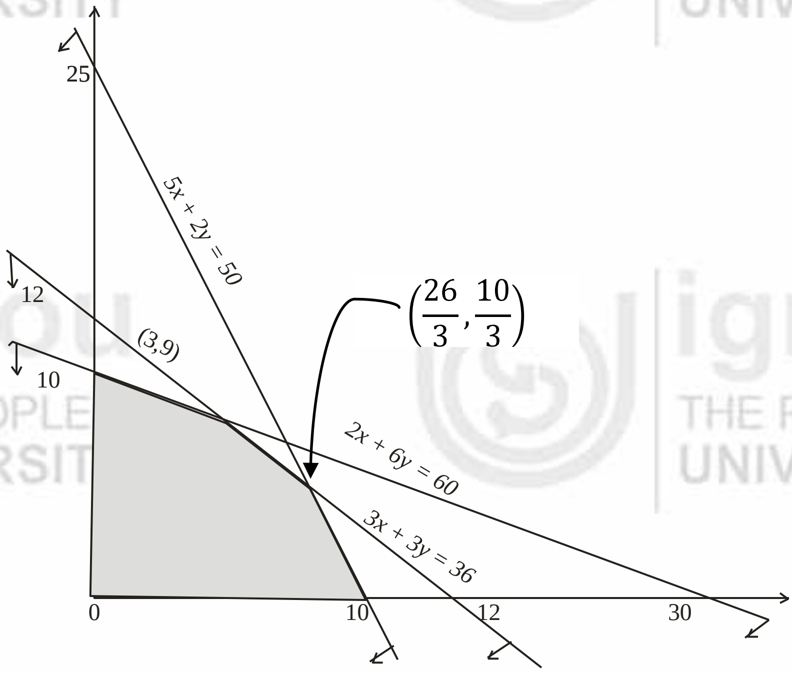


Figure 1

We redraw the feasible region without shading to clarify another concept (see Figure 2).

For any particular value of P, we can draw in the objective function as a straight line with slope $\left(\frac{-2}{3}\right)$ This is because $P = 20x + 30y$ is a straight line, which can be written as

$$y = \left(\frac{-2}{3}\right)x + \frac{P}{30}$$

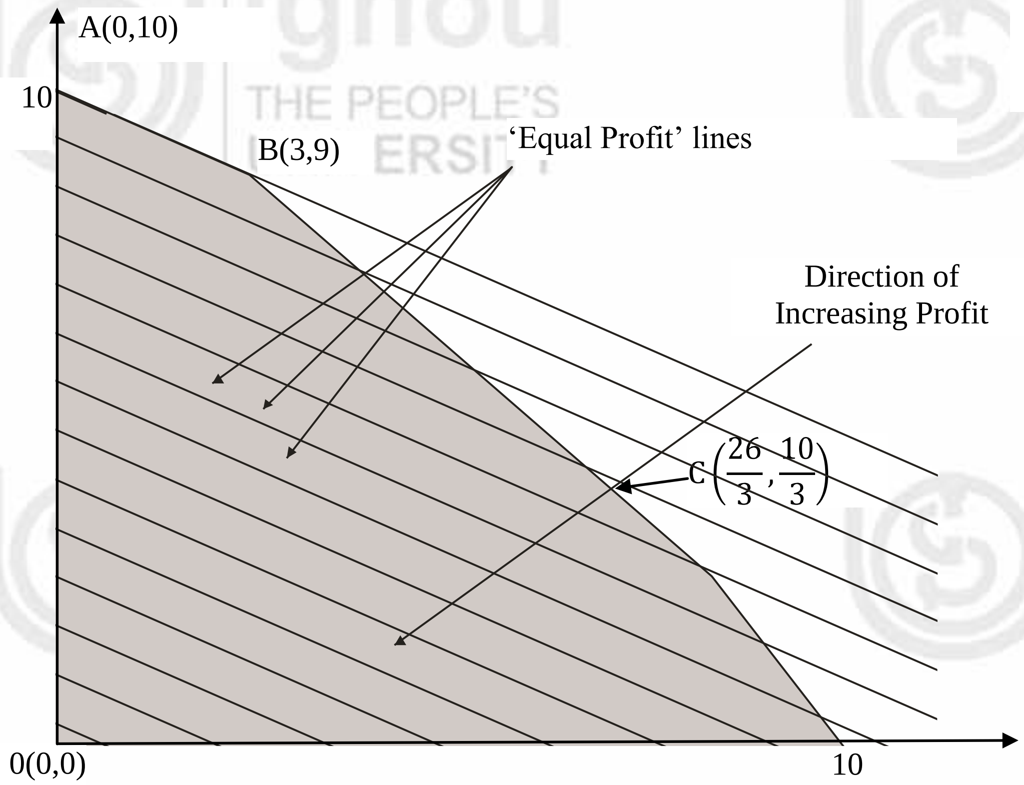


Figure 2

and this generates a family of parallel lines with slope $\left(\frac{-2}{3}\right)$ but with different intercepts on the axes. On any particular line, the different combinations of x and y (chairs and tables) all yield the same profit (P).

However, as P increases so does intercept $\frac{P}{30}$ (the $y-$intercept), so the lines with higher y-intercepts yield higher profits.

The problem is therefore to maximise the y-intercept, while at the same time remaining within constraints (or, feasible region). The part of the profit line which fall within the feasible region have been heavily drawn. Profit lines drawn farthest away from the origin (0, 0) yield the highest profits. Therefore, the highest profit yielded within the feasible region is at point B(3, 9). Therefore, the maximum profit is given by ₹ (20× 3 + 30× 9) = ₹ 330.

We are now ready for the definition of linear programming – the technique of solving the problem such as above.

What is Linear Programming

Linear because the equations and relationships introduced are linear. Note that all the constraints and the objective functions are linear. *Programming* is used in the sense of method, rather than in the computing sense.

In fact, linear programming is a technique for specifying how to use limited resources or capacities of a business to obtain a particular objective, such as least cost, highest margin or least time, when those resources have alternative uses.

4.1 OBJECTIVES

After studying this unit, you should be able to:

- define the terms-objective function, constraints, feasible region, feasible solution, optimal solution and linear programming;
- draw feasible region and use it to obtain optimal solution;
- tell when there are more than one optimal solutions; and
- know when the problem has no optimal solution.

4.2 LINEAR PROGRAMMING

We begin by listing some definitions.

Definitions

Objective Functions: If $a_1, a_2, . . . , a_n$ are constants and x_1 $x_2, ..., x_n$ are variables, then the linear function $Z = a_1x_1 + a_2x_2 + ... + a_nx_n$ which is to be maximised or minimised is called objective function*.

Constraints: These are the restrictions to be satisfied by the variables x_1, x_2 ..., x_n. These are usually expressed as inequations and equations.
Non-negative Restrictions: The values of the variables x_1 x_2, ..., x_n involved in the linear programming problem (LPP) are greater than or equal to zero (This is so because most of the variable represent some economic or physical variable.)

Feasible Region: The common region determined by all the constraints of an LPP is called the feasible region of the LPP.

Feasible Solution: Every point that lies in the feasible region is called a feasible solution. Note that each point in the feasible region satisfies all the constraints for the LPP.

Optimal Solution: A feasible solution that maximises or minimizes the objective function is called an optimal solution of the LPP.

* In this unit we shall work with just two variables.

Definition

Convex Region

A region ***R*** in the coordinate plane is said to be convex if whenever we take two points A and *B* in the region **R**, the segment joining *A* and *B* lies completely in ***R*.**

The student may observe that a feasible region for a linear programming is a convex region.

One of the properties of the convex region is that maximum and minimum values of a function defined on convex regions occur at the corner points only. Since all feasible regions are convex regions, maximum and minimum values of optimal functions occur at the corner points of the feasible region.

The region of Figure 3 is convex but that of Figure 4is not convex.

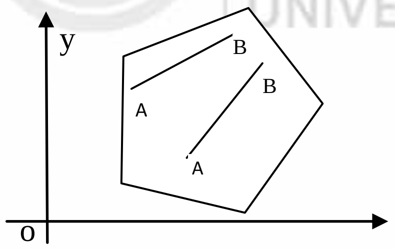


Figure 3 : Convex Region

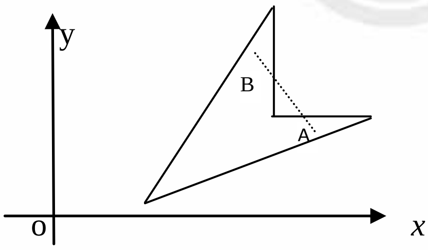


Figure 4 : Not Convex

4.3 TECHNIQUES OF SOLVING LINEAR PROGRAMMING PROBLEM

There are two techniques of solving an L.P.P. (Linear Programming Problem) by graphical method. These are

(i) Corner point method, and

(ii) Iso profit or Iso-cost method.

The following procedure lists the coner point method.

Corner Point Method*

Step 1 Plot the feasible region.

Step 2 Find the coordinates of the cornor points of the feasible region.

Step 3 Calculate the value of the objective function at each of the corner points of the feasible region.

Step 4 Pick up the maximum (or minimum) value of the objective function from amongst the points in step 3.

* The method explained in Example 1 is the iso-profit method. You are advised to use corner method unless you are specifically asked to do the problem by the iso-profit or iso-cost method.

Example 1 Find the maximum value of $5x + 2y$ subject to the constraints

$$-2x - 3y \leq -6$$
$$x - 2y \leq 2$$
$$6x + 4y \leq 24$$
$$-3x + 2y \leq 3$$
$$x \geq 0,\ y \geq 0.$$

Solution

Let us denote $5x + 2y$ by P.

Note that $-2x - 3y \leq -6$ can be written as $2x + 3y \geq 6$.

We can write the given LPP in the following format :

Maximise

$$P = 5x + 2y$$

subject to

$$2x + 3y \geq 6 \qquad \text{(I)}$$
$$x - 2y \leq 2 \qquad \text{(II)}$$
$$6x + 4y \leq 24 \qquad \text{(III)}$$
$$-3x + 2y \leq 3 \qquad \text{(IV)}$$

and $\quad x \geq 0,\ y \geq 0$

We plot the feasible region bounded by the given constraints in Figure 5

The lines I and II intersect in A (18/7, 2/7)

The lines II and III intersect in B (7/2, 3/4)

The lines III and IV intersect in C (3/2, 15/4)

The lines IV and I intersect in D (3/13, 24/13)

Let us evaluate P at A, B, C and D

P(A) = 5 (18/7) + (2/7) = 94/7   P(B) = 5 (7/2) + 2(3/4) =19

P(C) = 5 (3/2) + 2(15/4) = 15   P(D) = 5 (3/13) + 2(24/13)= 63/13

Thus, maximum value of P is 19 and its occurs at $x = 7/2$, $y = 3/4$.

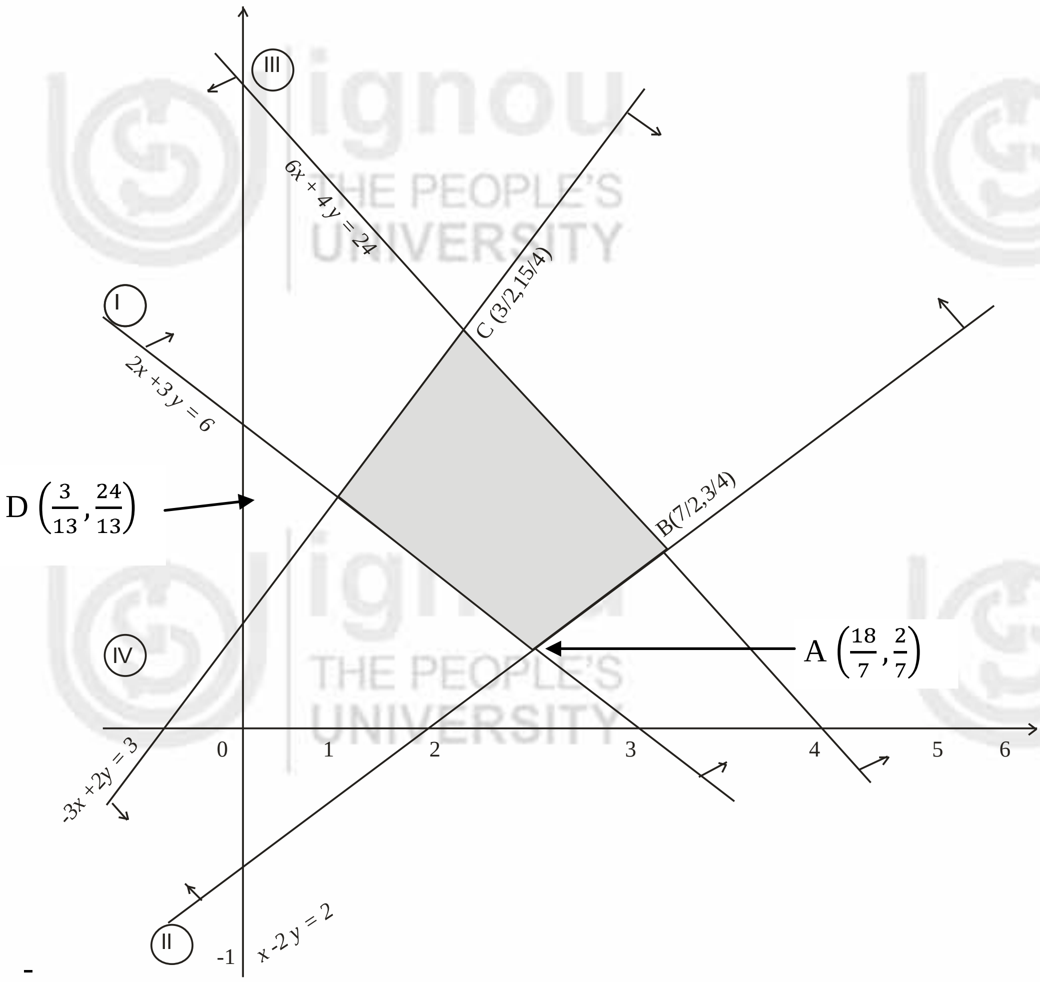


Figure 5

Example 2 Find the maximum value of $2x + y$ subject to the constraints

$$x + 3y \geq 6$$
$$x - 3y \leq 3$$
$$3x + 4y \leq 24$$
$$-3x + 2y \leq 6$$
$$5x + y \geq 5$$
$$x, y \geq 0$$

Solution

We write the given question in the following format :

Maximise

$$P = 2x + y$$

subject to

$x + 3y \geq 6$	(I)
$x - 3y \leq 3$	(II)
$3x + 4y \leq 24$	(III)
$-3x + 2y \leq 6$	(IV)
$5x + y \geq 5$	(V)
and $x \geq 0, y \geq 0$	(VI)

The feasible region is sketched in Figure 6

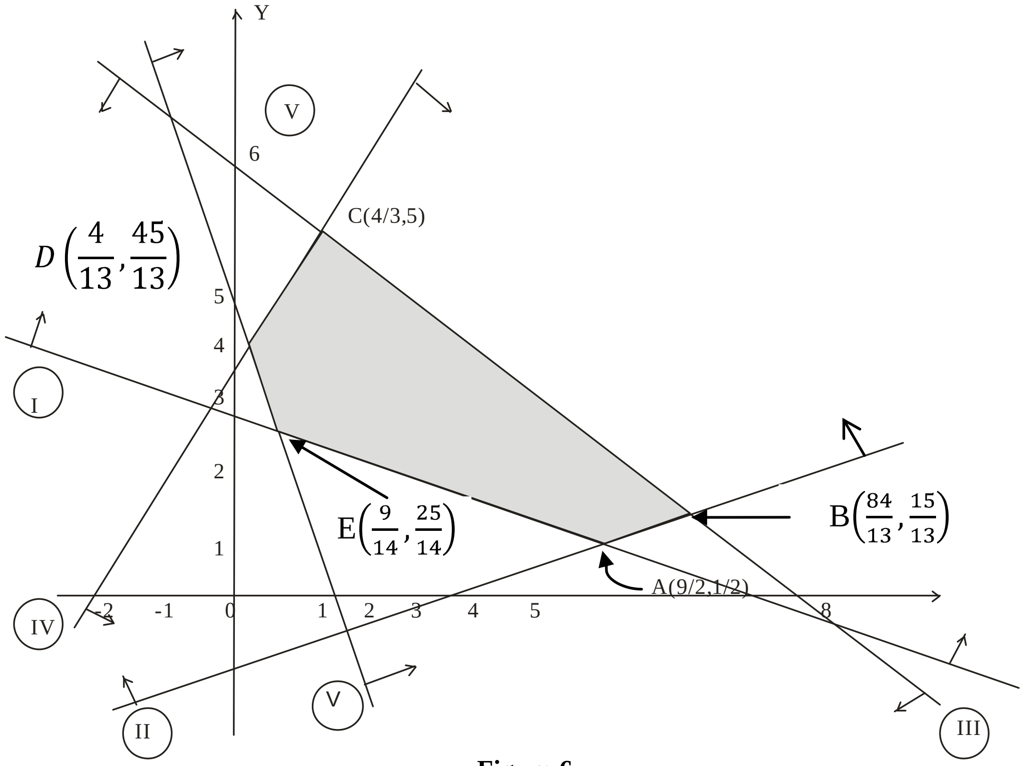


Figure 6

The lines I and II intersect in A(9/2, 1/2)

The lines II and III intersect in B(84/13, 15/13)

The lines III and IV intersect in *C(4/3, 5)*

The lines IV and V intersect in D(4/13, 45/13)

The lines V and I intersect in E(9/14, 25/14).

Let us evaluate *P* at the points A, B, *C, D* and E as follows:

$P(A) = 2(9/2) + 1/2 = 19/2$

$P(B) = 2(84/13) + 15/13 = 183/13$

$P(C) = 2(4/3) + 5 = 23/3$

$P(D) = 2(4/13) + 45/13 = 53/13$

$P(E) = 2(9/14) + 25/14 = 43/14$

The maximum value of P is 183/13 which occurs at $x = 84/13$ and $y = 15/13$.

Example 3: An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by first class. Determine how many of each type of tickets must be sold in order to maximise the profit for the airline? What is the maximum profit?

Solution

Let number of first class tickets sold be x and the number of economy class tickets sold be y.

As the airline makes a profit of ₹400 on first calss and ₹300 on economy class, the profit of the airlines is $400x + 300y$

As the aeroplane can carry maximum of 200 passengers, we must have $x + y \leq 200$. As the airline reserves at least 20 tickets for the first class, $x \geq 20$.

Next, as the number of passengers preferring economy class is at least four times the number of passengers preferring the first class, we must have $y \geq 4x$.
Also, $x \geq 0, y \geq 0$.

Therefore the LPP is

Maximixe		
	$P = 400x + 300y$	[objective function]
subject to	$x + y \leq 200,$	[capacity function]
	$y \geq 20$	[first class function]
	$y \geq 4x$	[preference constraint]
	$x \geq 0, y \geq 0$	[non-negativity]

We plot the feasible region in Figure 7

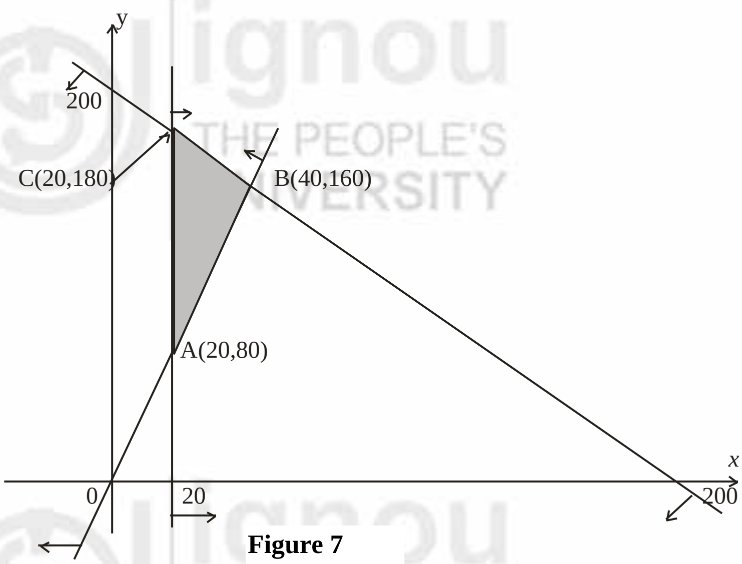


Figure 7

We now calculate the profit at the corner points of the feasible region.

P(A) = P(20,80) = (400)(20) + (300)(80) = 8000 + 24000 = 32000

P(B) = P(20,160) = (400)(40) + (300)(160) = 64000

P(C) = P(20,180) = (400)(20) + (300)(180) = 62000

Hence, profit of the airlines maximum at B i.e., when 40 tickets of first class and 160 tickets of ecoomy class are sold. Also, maximum profit is ₹ 64,000.

Example 4: Suriti wants to invest at most ₹ 12000 in Savings Certificate and National Savings Bonds. She has to invest at least ₹ 2000 in Savings Certificate and at least ₹ 4000 in National Savings Bonds. If the rate of interest in Saving Certificate is 8% per annum and the rate of interest on National Saving Bond is 10% per annum, how much money should she invest to earn maximum yearly income ? Find also the maximum yearly income ?

Solution

Suppose Suriti invests ₹ x in saving certificate and ₹ y in National Savings Bonds.

As she has just ₹ 12000 to invest, we must have $x + y \leq 12000$.

Also, as she has to invest at least ₹ 2000 in savings certificate $x \geq 2000$.

Next, as she must invest at least Rs. 4000 in National Savings Certificate $y \geq 4000$. Yearly income from saving certificate = ₹ $\frac{8x}{100}$ = $0.08x$ and from

National Savings Bonds = ₹ $\frac{10y}{100}$ = Rs. $0.1y$

$\therefore$ Her total income is ₹ P where

$P = 0.08x + 0.1y$

Thus, the linear programming problem is

Maximise

$$P = \frac{8x}{100} + \frac{10y}{100}$$

subject to

$x + y \leq 12000$	[Total Money Constraint]
$x \geq 2000$	[Savings Certificate Constraint]
$y \geq 4000$	[National Savings Bonds Constraint]
$x \geq 0, y \geq 0$	[Non-negativity Constraint]

However, note that the constraints $x \geq 0$, $y \geq 0$, are redundant in view of $x \geq 2000$ and $y \geq 4000$.

We draw the feasible region in Figure 8

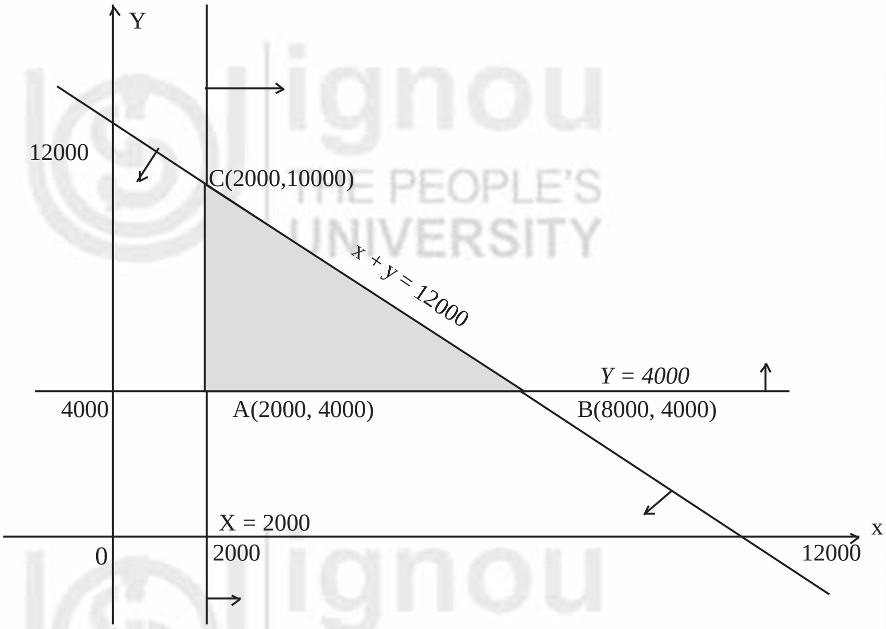


Figure 8

We now calculate the profit at the corner points of the feasible region.

We have

P(A) = P(2000,4000) = (0.08) (2000) + (0.1)(4000)
= 160 +400 = 560

P(B) = P(8000,4000) = (0.08) (2000) + (0.1)(4000)
= 640 +400 = 1040

P(C) = P(2000,10000) = (0.08) (2000) + (0.1)(10000)
= 160 +1000 =1160.

Thus, she must invest ₹ 2000 in Savings certificate and ₹ 10000 in National Savings Bonds in order to earn maximum income.

Example 5 If a young man rides his motor cycle at 25 km per hour, he has to spend ₹ 2 per km on petrol; if he rides it at a faster speed of 40 km per hour, the petrol cost increases to ₹ 5 per km. He wishes to spend at most ₹ 100 on petrol and wishes to find what is maximum distance he can travel within one hour. Express this as a linear programming problem and then solve it.

Solution

Let x km be the distance travlled at the rate of 25 km/h and y km be the distance travelled at the rate of 40 km/h. Then the total distance covered by the young man is D = $(x + y)$ km.

The money spent in travelling x km (at the rate of 25 km/h) is $2x$ and the money spent in travelling y km (at the rate of 40 km/h) is $5y$. Thus, total money spent during the journey is ₹ $(2x + 5y)$. Since the young man wishes to spend at most Rs. 100 on the journey, we must have $2x + 5y \leq 100$.

Also the time taken to travel x km is $\frac{x}{25}h$ and the time taken to travel y km is $\frac{y}{40}h$. Since the young man has just one hour, we must have

$$\frac{x}{25}+\frac{x}{40} \le 1.$$

Also, note that $x \ge 0,\ y \ge 0$.

The mathematical formulation of the linear programming problem is

Maximise

$$D = x + y$$

subject to

$$2x + 5y \le 100$$

$$\frac{x}{25}+\frac{x}{40} \le 1$$

and $x \ge 0,\ y \ge 0.$

The feasible region is sketched in Figure 9.

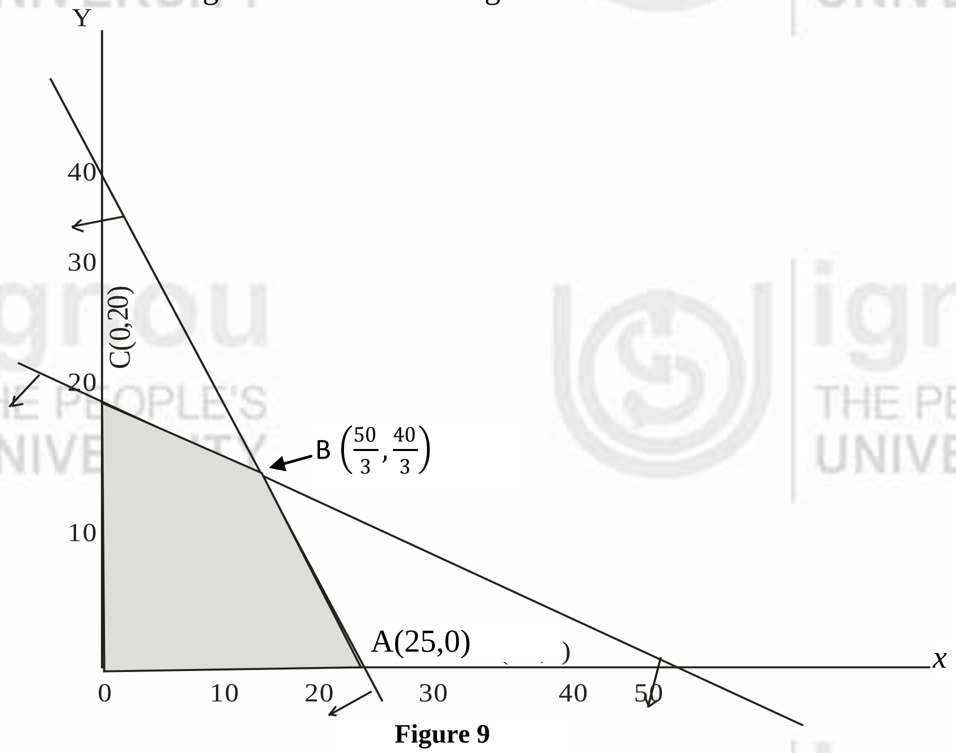


Figure 9

The corner points of feasible region are O(0,0), A(25,0), B$\left(\frac{50}{3},\frac{40}{3}\right)$ and C(0,20)

Let us evaluate D at these points

D(O) = 0 + 0 = 0

D(A) = 25 + 0 = 25

$$D(B) = \frac{50}{3}+\frac{40}{3} = 30$$

D(C) = 0 + 20 = 20

Thus, the maximum values of D is 30 which occurs at x = 50/3, y = 40/3

When do we have Multiple Solutions ?

Whenever the objective function isoprofit (iso-cost) line is parallel to one of the constraints we have multiple optimal solutions to the linear programming problem. We illustrate this in the following example.

Example 6 The *xyz* company manufactures two products A and B. They are processed on the same machine. A takes 10 mintues per item and B takes 2 minutes per items on the machine. The machine can run for a maximum of 35 hours in a week. Product A requires 1 kg and product B 0.5 kg of the raw material per item, the supply of which is 600 kg per week. Note more than 800 items of product B are required per week. If the product A costs ₹ 5 per items and can be sold for ₹ 10 and Product B costs ₹ 6 per items and can be sold for ₹ 8 per item. Determine how many items per week be produced for A and B in order to maximize the profit.

Solution

We summarise the information given in the question as follows :

	Product A (x)	**Product B(y)**	**Constraint**
Machine	10 (min. per item)	2 (min. per item)	$\leq 35 \times 60 = 2100$
Material	1 (kg per item)	0.5 (kg per item)	≤ 600
Number		for the Proudct B	≤ 800
Profit	₹ (10 –5) = ₹ 5 per item	₹ (8–6) = ₹ 2 per item	Maximise P

Suppose x items of **A** and y items of **B** are be manufactured. The given problem can be written as follows.

Maximise

$$P = 5x + 2y$$

subject to

$$10x + 2y \leq 2100 \quad \text{(Machine constraint)}$$
$$x + 0.5y \leq 600 \quad \text{(Material constraint)}$$
$$y \leq 800 \quad \text{(Restriction on B)}$$
$$x \geq 0, y \geq 0 \quad \text{(Non-negativity)}$$

The feasible region has been shaded in Figure 10.

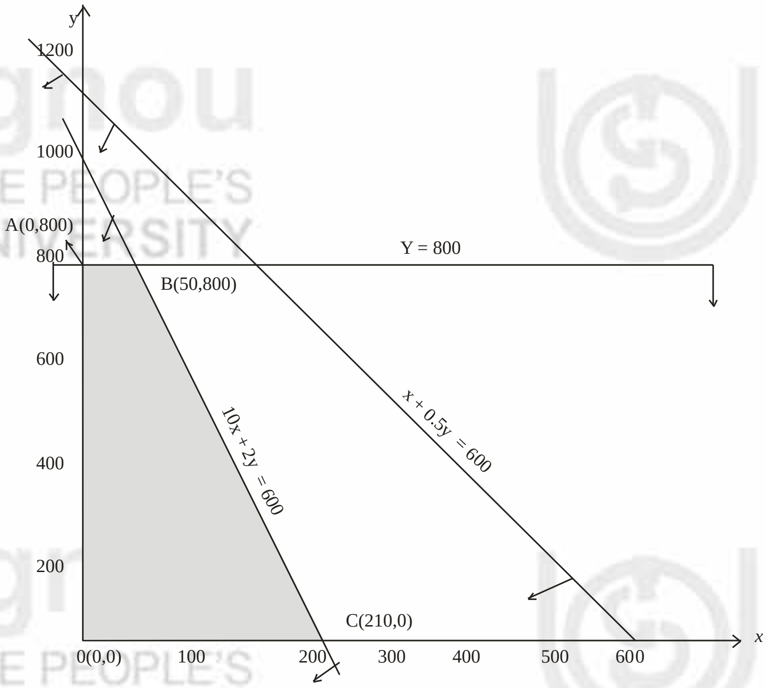


Figure 10

We now calculate the value of P at the corner points of the feasible region

P(A) = P(0,800) = 1600

P(B) = P(50,800) = 1850

P(C) = P(210,0) = 1050

P(O) = P(0,0) = 0

Maximum profit is ₹ 1850 for x = 50 and y = 800

Redundant Constraints

In the above example, the constraitn $x + 0.5, y \leq 600$ does not affect the feasible region. Such a constraints is called as redundant constraint. Redundant constraints are unnecessary in the formulation and solution of the problem, because they do not affect the feasible region.

Example 7 The manager of an oil refinery wants to decide on the optimal mix of two possible blending processes 1 and 2, of which the inputs and outputs per product runs as follows :

Process	**Crude *A***	**Crude *B***	**Gasoline *X***	**Gasoline *Y***
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of crudes **A** and **B** are 200 untis and 150 units, respectively. At least 100 units of gasoline *X* and 80 untis of *Y* are required. The profit per production run from processes 1 and 2 ₹ 300 and Rs. 400 respectively. Formulate the above as linear programming and solve it by graphical method.

Solution

Let process 1 be run x times and process 2 be y times. The mathematical formulation of the above linear programming problem is given by

Maximise

$$P = 300\,x + 400\,y$$

subject to

$5x + 4y \leq 200$ (constraint on Crude A)

$3x + 5y \leq 200$ (constraint on Crude B)

$5x + 4y \geq 100$ (constraint on gasoline X)

$8x + 4y \geq 80$ (constraint on gasoline Y)

$x \geq 0, y \geq 0$ (non-negativity)

The feasible region has been shaded in Figure 11.

We have

$P(\mathrm{A}) = P(0,30) = 12000$

$$P(B) = P\left(\frac{400}{13}, \frac{150}{13}\right) = \frac{180000}{13} = 13486\ \frac{2}{13}$$

$P(\mathrm{C}) = P(40,0) = 12,000$

$P(\mathrm{D}) = P(25,0) = 7500$

$P(\mathrm{E}) = P(0,200) = 8000$

Thus, the profit is maximum when $x = \frac{400}{13}, y = \frac{150}{13}$. Hence, 30 runs of process 1 and 11 runs of process 2 will give maximum profit.

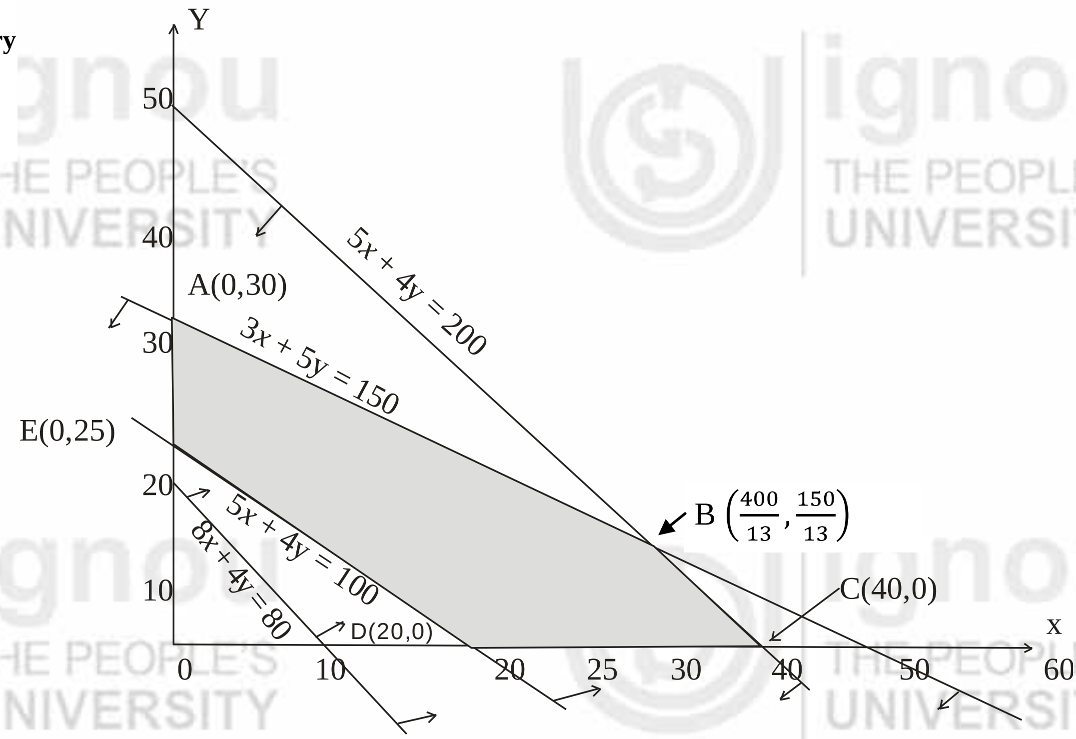


Figure 11

Remark : The constraint $8x + 4y \geq 80$ does not affect the feasible region, that is, the constraint $8x + 4y \geq 80$ is a redundant constraint.

No Feasible Solution

In case the solution space or the feasible region is empty, that is, there is no point which satisfies all the constraints, we say that the linear programming problem has no feasible solution.

Illustration : We illustrate this in the following linear programming problem.

Maximise

$$P = 2x + 5y$$

subject to

$$x + 2y \leq 10$$
$$x \geq 12$$

and $\quad x \geq 0,\ y \geq 0$

we draw the feasible region in Fig 12.

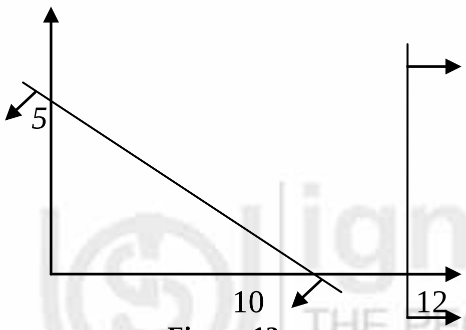


Figure 12

The direction of arrows indicate that the feasible region is empty. Hence, the given linear programming problem has no feasible solution.

Unbounded solution

Sometimes the feasible region is unbounded. In such cases, the optimal solution may not exist, because the value of the objective function goes on increasing in the unbounded region.

Illustration Let us look at the following illustration.

Maximise

$$P = 7x + 5y$$

subject to

$$2x + 5y \geq 10$$
$$x \geq 4$$
$$y \geq 3$$
$$x \geq 0,\ y \geq 0$$

has no bounded solution.

We draw the feasible region in Fig. 13

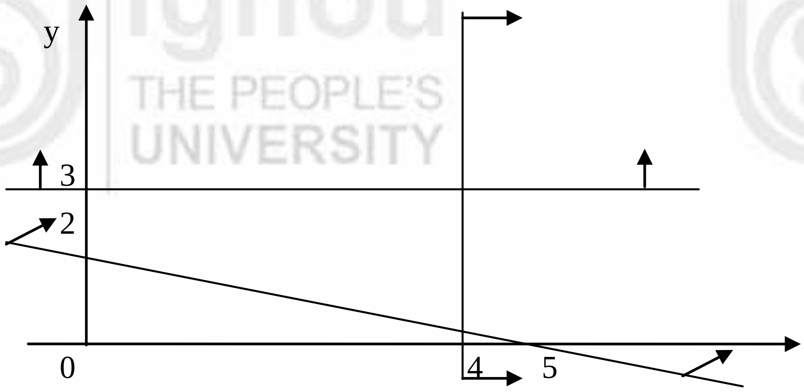


Figure 13

The constraint $2x + 5y \geq 10$ is a redundant constraint.

The feasible region is unbounded. Note that the linear programming problem has no bounded solution.

Chcek Your Progress – 1

1. Best Gift Packs company manufactures two types of gift packs, type A and type B. Type A requires 5 minutes each for cutting and 10 minutes assembling it. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are at most 200 minutes available for cutting and at most 4 hours available for assembling. The profit is ₹ 50 each for type A and ₹ 25 each for type B. How many gift packs of each type should the company manufacture in order to maximise the profit ?

2. A manufacturer makes two types of furniture, chairs and tables. Both the products are processed on three machines A_1, A_2 and A_3. Machine A_1 requires 3 hrs for a chair and 3 hrs for a table, machine A_2 requires 5 hrs for a chair and 2 hrs for a table and machine A_3 requires 2 hrs a chair and 6 hrs for a table. Maximum time available on machine A_1, A_2, A_3 is 36 hrs, 50 hrs and 60 hrs respectively. Profits are ₹ 20 per chair and ₹ 30 per table. Formulate the above as a linear programming problem to maximise the profit and solve it.

3. A manufacturer wishes to produce two types of steel trunks. He has two machines A and B. For completing, the first type of trunk, he requires 3 hrs on machine A and 2 hrs on machine B whereas the second type of trunk requires 3 hrs on machine A and 3 hrs on machine B. Machines A and B can work at the most for 18 hrs and 14 hrs per day respectively. He earns a profit of ₹30 and ₹ 40 per trunk of first type and second type respectively. How many trunks of each type must he make each day to make maximum profit? What is his maximum profit?
4. A new businessman wants to make plastic buckets. There are two types of available plastic bucket making machines. One type of machine makes 120 buckets a day, occupies 20 square metres and is operated by 5 men. The corresponding data for second type of machine is 80 buckets, 24 square metres and 3 men. The available resources with the businessman are 200 sq. metres and 40 men. How many machines of each type the manufacturer should buy, so as to maximise the number of buckets?
5. A producer has 20 and 10 units of labour and capital respectively which he can use to produce two kinds of goods X and Y. To produce one unit of goods X, 2 units of capital and 1 unit of labour is required. To produce one unit of goods Y, 3 units of labour and 1 unit of capital is required. If X and Y are priced at ₹ 80 and ₹ 100 per unit respectively, how should the producer use his resources to maximize the total revenue? Solve the problem graphically.
6. A firm has available two kinds of fruit juices – pineapple and orange juice. These are mixed and the two types of mixtures are obtained which are sold as soft drinks A and B. One tin of A needs 4 kgs of pineapple juice and 1 kg of orange juice. One tin of B needs 2 kgs of pineapple juice and 3 kgs of orange juice. The firm has available only 46 kgs of pineapple juice and 24 kgs of orange juice. Each tin of A and B sold at a profit of ₹ 4 and ₹ 2 respectively. How many tins of A and B should the firm produce to maximise profit?

4.4 COST MINIMISATION

We illustrate the concept by the following example.

Example 7 *:* A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure, certain nutrient constituents, it is necessary to buy two products (call them A and B*)* in addition. The contents of the various products, per unit, in nutrient constituents *(e.g.*, vitamins, proteins, etc.) is given in the following table:

Nutrients	Nutrient content in product A	B	Minimum amount of nutrient
M_1	36	6	108
M_2	3	12	36
M_3	20	10	100

The last column of the above table gives the minimum amounts of nutrient constituients M_1, M_2, M_3 which must given to the pigs. If products A and B cost ₹ 20 and ₹ 40 per unit respectively, how much each of these two products should be bought, so that the total cost is minimised?

Solution

Let x units of A and y units B be purchased. Our goal is to minimise the total cost

$$C = 20x + 40y$$

By using x units of A and y units of B, we shall get $36x + 6y$ units of M_1 Since we need at least 108 units of M_1, we must have

$$36x + 6y \geq 108$$

Similarly for M_2 we have $3x + 12y \geq 36$ and for M_3 we have $20x + 10y \geq 100$. Also, we cannot use negative numbers of x and y. Thus, our problem is

Minimise

$$C = 20x + 40y$$

subject to

$$36x + 6y \geq 108$$

$$3x + 12y \geq 36$$

$$20x + 10y \geq 100$$

$$x \geq 0, \; y \geq 0$$

We now draw the constraints on the same graph to obtain the feasible region. Since $x \geq 0$, $y \geq 0$, we shall restrict ourself only to the first quadrant. See Figure 14. We obtain point B by solving $36x + 6y = 108$ and $20x + 10y = 100$ and point C by solving $3x + 12y = 36$ and $20x + 10y = 100$. The feasible region has been shaded.

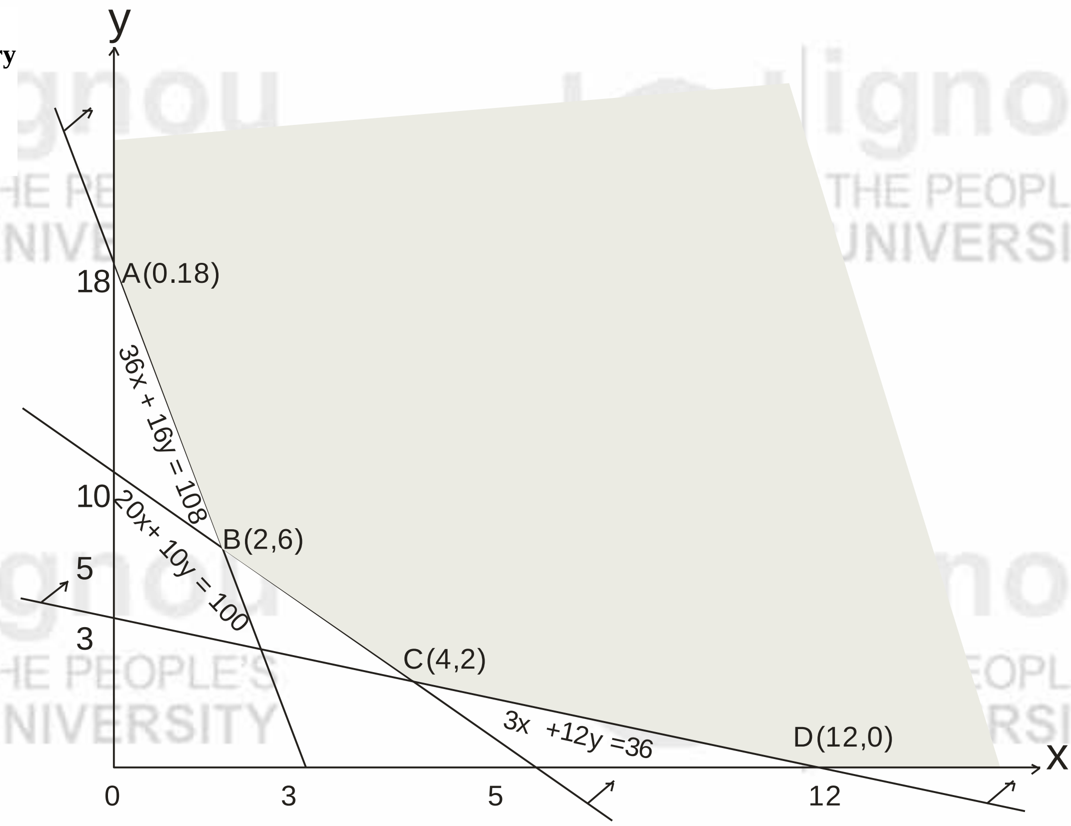


Figure14

We next draw a family of straight lines

$$y = -\frac{1}{2}x - \frac{C}{40}$$

with varying values of **C**. See Figure 15

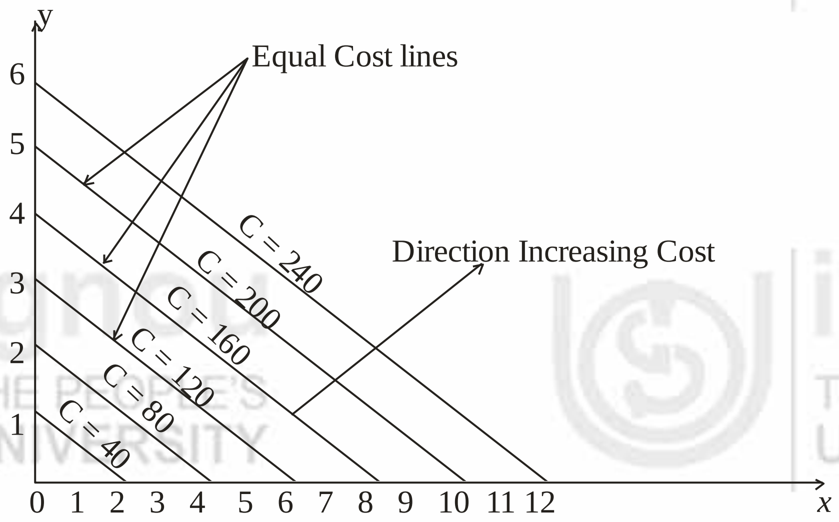


Figure 15

It is clear from here that in order to have least possible cost, we should take a cost line which intersects the feasible region and is as near to the origin as possible. Therefore we should consider only the corner points of the feasible region as possible candidates for least cost.

$C(A) = C(0, 18) = 720$

$C(B) = C(2, 6) = 20\times 2 + 40\times 6 = 280$

$C(C)=C(4,2)=20\times4+40\times2= 160$

$C(D) = C(12, 0) = 240$

Minimum cost is obtained at **C**, that is, for $x = 4$, $y= 2$. Minimum possible cost is ₹160.

Remark: The procedure for finding least cost is the same as that for the maximization of profit.

Example 8 A diet for a sick person must contain at least 1400 units of vitamins, 50 units of minerals and 1400 of calories. Two foods A and B are available at a cost of ₹ 4 and ₹ 3 per unit, respectively. If one unit of A contains 200 units of vitamins, one unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, two units of minerals and 40 calories. Find what combination of food be used to have least cost?

Solution

Let x units of food A and y units of food B be used to give the *sick* person the least quantities of vitamins, minerals and calories.

The total cost of the food is $C = 4x + 3y$. We wish to minimise C.

By consuming x units of A and y units of B, the sick person will get $200x + 100y$ units of vitamins. Since the least quantity of vitamin required is 1400, we must have $200x + 100y \geq 1400$

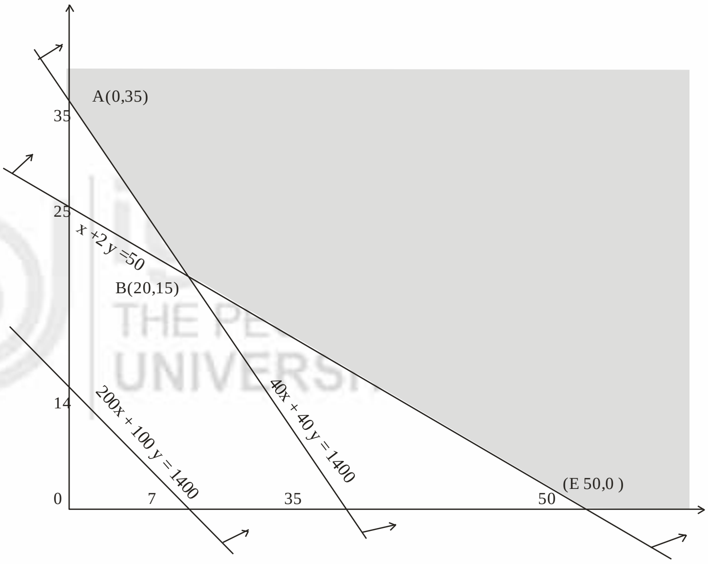


Figure16

Similarly, we must have

Minerals : $x + 2y \geq 50$

Calories : $40x + 40y \geq 1400$

Also, since x and y cannot be negative, we must have $x \geq 0$, $y \geq 0$

Thus, the linear programming problem is

Minimise

$$C = 4x + 3y$$

subject to

$$200x + 100y \geq 1400$$

$$x + 2y \geq 50$$

$$40x + 40y \geq 1400$$

and $x \geq 0, y \geq 0$

We draw the feasible region of the above linear programming problem in fig. 16. Note that constraint $200x + 100y \geq 1400$ is a redundant constraint. The other two intersect in (20,15).

The corner points of the feasible regions are A(0, 35), B(20,15) and E(50,0). We find the value of C at each of these corner points.

C(A) = 4(0) + 3(35) = 105
C(B) = 4(20) + 3(15) = 125
C(E) = 4(50) + 3(0) = 200

Thus, least cost is occurs at $x = 0$, $y = 35$.

Example 9 Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values for rice are 0.05 g and 0.5 g, respectively. Wheat costs ₹2 per kg and rice ₹ 8. The minimum daily requirements of protein and carbohydrates for an average child are 50 g and 200 g, respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of protein and carbohydrates at minimum cost? (The protein and carbohydrate values given here are fictitious and may be quite different from the actual values.)

Solution

Let x kg of wheat and y kg of rice be given to the child to give him at least the minimum requirements of protein and carbohydrates. Then cost of the food is $(2x + 8y) = C$ (say).

Since one gram wheat contains 0.1 g of proteins, x kg of wheat will contain $(1000x)(0.1) = 100x$ grams of protein.

Similarly, y kg of rice contain $(100y)(0.05) = 50y$ grams of protein. Thus, x kg of wheat and y kg of rice will contain $(100x + 50y)$ gms of protein.

As the minimum requirement of protein is 50 g, we must have

$$100x + 50y \geq 50.$$

Similarly, for carbohydrate, we must have

$$(1000x)\,(0.25) + (1000y)\,(0.5) \geq 200$$

$$\text{or} \quad 250x + 500y \geq 200$$

Also, x and y must be non-negative, that is, $x \geq 0$, $y \geq 0$.

Thus, the linear programming problem is

Minimise

$$C = 2x + 8y$$

subject to

$$100x + 50y \geq 50$$

$$250x + 500y \geq 200$$

and $x \geq 0,\ y \geq 0.$

We draw the feaible region of its problem in Figure 17

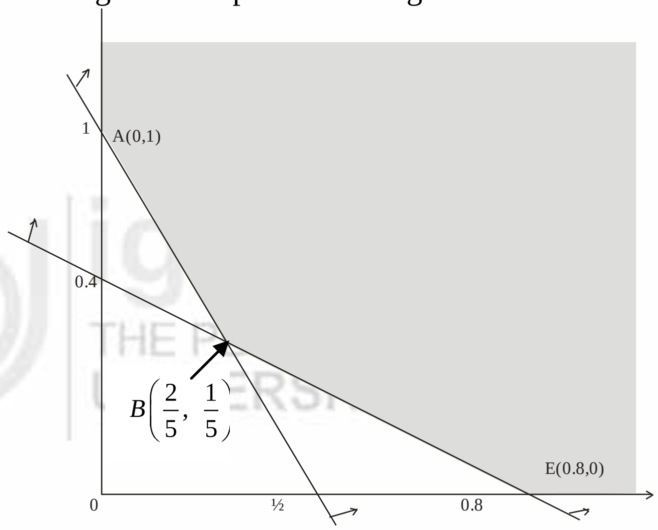


Figure17

The corner points of the feasible region are $A(0,1)$, $B(2/5,1/5)$ and $E(0.8,0)$.
Let us evaluate C at the conrer points of the feasible region.

$C\,(A) = 2(0) + 8(1) = 8$

$$C(B) = 2\left(\frac{2}{5}\right) = 8\left(\frac{1}{5}\right) = \frac{12}{5} = 2.4$$

$C\,(E) = 2(0.8) + 8(0) = 1.6$

Thus, the cost is least when $x = 0.8$, $y = 0$. The least cost is Rs. 1.60.

Example 10 An animal feed manufacturer produces a compound mixture from two materials A and B. A costs ₹2 per kilograms and B, ₹ 4 per kilogram. Material A is supplied in packs of 25 kilograms and material B in packs of 50 kilograms. A batch of at least 1,00,000 kilograms of the mixture is to be produced with the specification that at least 40,000 kilograms of material A, should be used in the manufacture, which ensures the minimum guaranteed content of the ingredient.

How many packs of A and *B* should be purchased in order to minimise cost?

Solution

Let x packs of *A* and y packs of *B* be purchased in order to meet the requirement. Our goal is to minimise the total cost. The total cost is

$$C = 2 \times 25x + 4 \times 50y = 50x + 200y$$

The constraints are

$$25x + 50y \geq 100000$$
$$50x \geq 40000$$

Therefore, the linear programming problem is

Minimise

$$C = 50x + 200y$$

subject to

$$25x + 50y \geq 100000$$
$$50x \geq 40000$$
$$x \geq 0, \; y \geq 0$$

We can rewrite the problem as

Minimise

$$C = 50x + 200y$$

subject to

$$x + 2y \geq 4000$$
$$x \geq 800$$
$$x \geq 0, \; y \geq 0$$

The feasible region of the above linear programming problem is given in Figure 17

Let us evaluate C at the corner points of the feasible region.

$$C(A) = 50 \times 800 + 50 \times 1600$$
$$= 40000 + 80000 = 120000$$
$$C(B) = 50 \times 4000 + 50 \times 0 = 200000$$

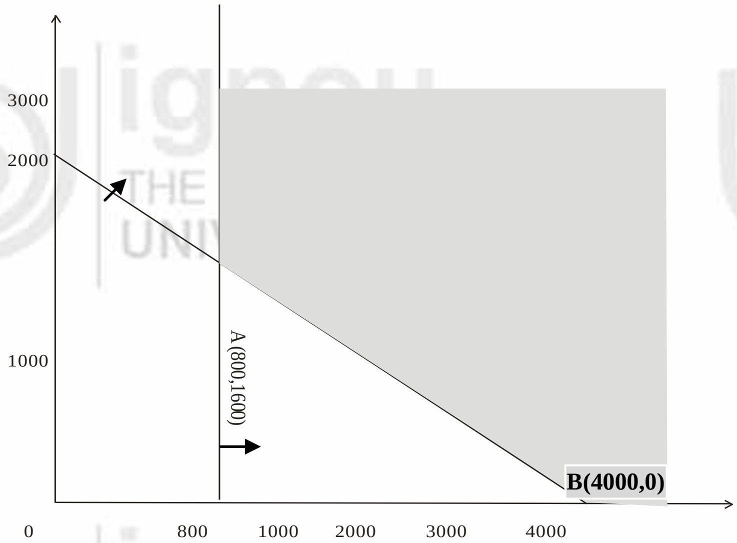


Figure 18

Therefore, cost is minimum when $x = 800, y = 1600$. The minimum possible cost is Rs. 1,20,000.

Check Your Progress – 2

1. Two tailors, A and B, earn ₹ 150 and ₹ 200 per day respectively. A can stitch 6 shirts and 4 pants while B can stitch 10 shirts and 4 pants per day. How many days shall each work if it is desired to produce (at least) 60 shirts and 32 pants at a minimum labour cost ? Also calculate the least cost.
2. A dietician mixes together two kinds of food in such a way that the mixture contains at least 6 units of vitamin A, 7 units of vitamin B, 11 units of vitamin C and 9 units of vitamin D. The vitamin contents of 1 unit food X and 1 unit of food Y are given below:

	Vitamin A	Vitamin B	Vitamin C	Vitamin D
Food X	1	1	1	2
Food Y	2	1	3	1

One unit of food X costs ₹ 5, whereas one unit of good Y cost ₹ 8. Find the least cost the mixture which will produce the dersired diet. Is there any redundant constraint ?

3. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 of calories. Two foods, A and B, are available at a cost of ₹ 4 and ₹ 3 per unit respectively. If one unit of A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamin, 2 units of minerals and 40 calories, find what combination of foods should be used to have the least cost ? Also calculate the least cost.

4. A tailor needs at least 40 large buttons and 60 small buttons. In the market, buttons are available in boxes or cards. A box contains 6 large and two small buttons and a card contains 2 large and 4 small buttons. If the cost of a box is 30 paise and card is 20 paise. Find how many boxes and cards should he buy as to minimise the expenditure ?

4.5 ANSWERS TO CHECK YOUR PROGRESS

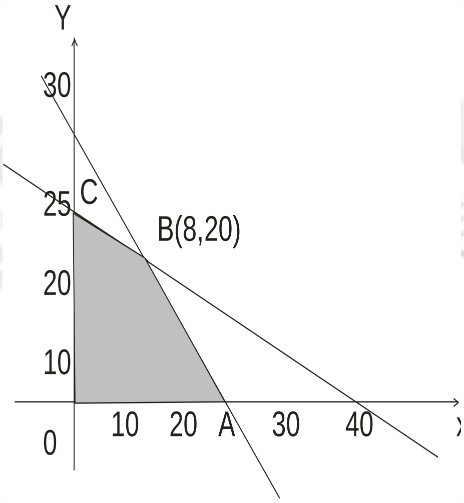


Figure 19

1. Type A : x, Type B : y then LPP is

   Maximize

   $P = 50x + 25y$

   subject to

   $5x + 8y \le 200$ [Cutting constraint]

   $10x + 8y \le 240$ [Assembly constraint]

   $x \ge 0$, $y \ge 0$ [Non- negativity]

   P(A) = 1200

   P (B) = 900

   P (C) = 625

   P(0) = 0

   Thus, profit is maximum when $x = 24$, $y = 0$

   Maximum profit = Rs. 1200

2. Chairs : x, Tables : y, then LPP is

   Maximize

   $P = 20x + 30y$

   subject to

   $3x + 3y \le 36$ [Machine A_1 constraint]

   $5x + 2y \le 50$ [Machine A_2 constraint]

   $2x + 6y \le 60$ [Machine A_3 constraint]

   $x \ge 0, y \le 0$ [Non-negativity]

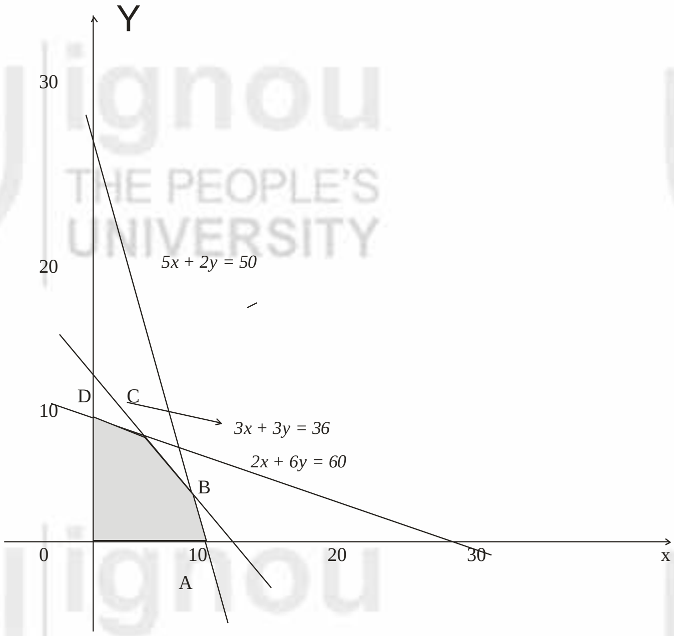


Figure 20

P (A) = 20 (10) + 30(0) = 200

P (B) $= 20\left(\frac{26}{3}\right) + 30\left(\frac{10}{3}\right) = \frac{820}{3}$

P (C) = 20 (3) + 30(9) = 330

P (D) = 20 (10) + 30(0) = 200

P(0) = 20(0) + 30(0) = 0

Thus, profit is maximum

when $x = 3$, $y = 9$.

Maximum Profit = Rs. 330

3. First type trunks : x Second type trunks : y

   The LPP is

   Maximise $P = 30x + 40y$,

   subject to

   $3x + 3y \leq 18$ [Machine A constraint]

   $2x + 3y \leq 14$ [Machine B constraint]

   $x \geq 0, y \geq 0$ [non-negativity]

   $P(A) = 180$

   $P(B) = 200$

   $P(C) = 560/3$

   P(O) = 0

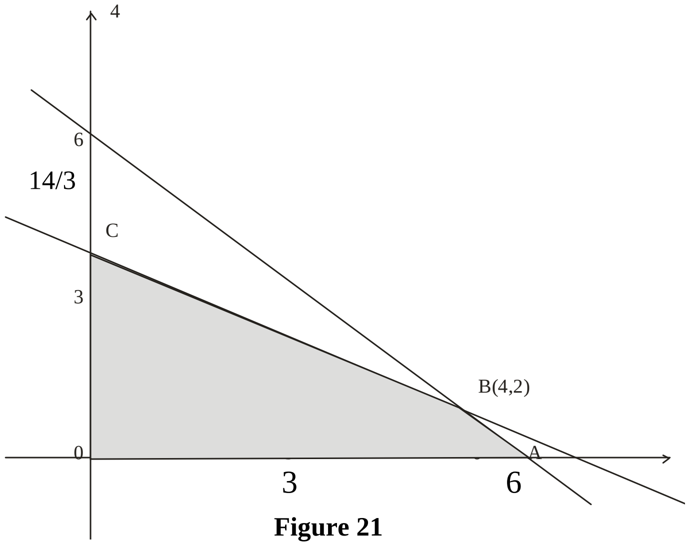


Figure 21

Thus, profit is maximum when $x = 4$, $y = 2$ and maximum profit is Rs. 200.

4. We write the information in the question in tabular form as follows :

	First type of machine (x)	First type of machine (x)	Constraint
Area	20 (sq. m per machine)	24 (sq. m per machine)	≤ 200
Labour	5 (per machine)	3 (per machine)	≤ 40
Buckets	120 (per machine)	80 (per machine)	Maximise N

Let x machines of the firs x pe and y machines of the second type be purchased.

We have to

Maxmise

$$N = 120\,x + 80y$$

subject to

$$20\,x + 24\,y \leq 200 \qquad \text{(Area constraint)}$$

$$5x + 3y \leq 40 \qquad \text{(Labour constraint)}$$

$$x \geq 0,\ y \geq 0 \qquad \text{(Non-negativity)}$$

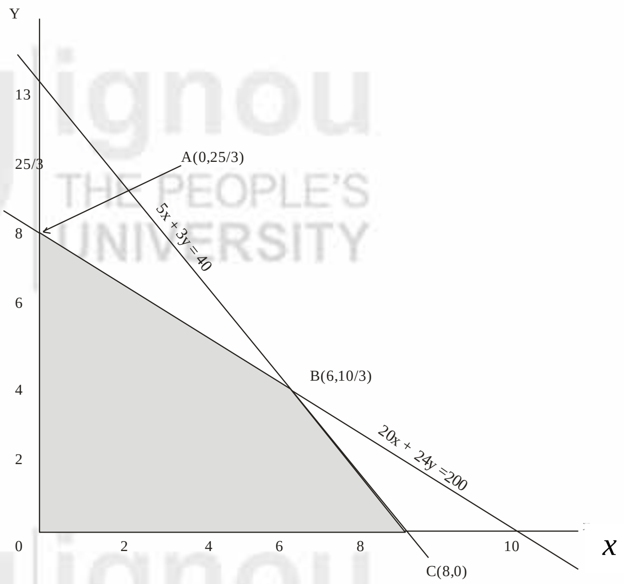


Figure 22

The feasible region for the above linear programming problem has been shaded in the figure.

We find the value of N at the cornor points of the feasible region. We have

$$N\,(A) = N\left(0, \frac{25}{3}\right) = \left(\frac{2{,}000}{3}\right) = 666\frac{2}{3}$$

$$\text{N (B)} = \text{N}\left(6, \frac{10}{3}\right) = \frac{2960}{3} = 986\frac{2}{3}$$

N (C) = N(8,0) = 960

N (O) = N(0,0) = 0

Thus, the value of N is maximum when $x = 6, y = 10/3$. As y cannot be in fraction, we take $x = 6$, $y = 3$.

5. We summarise the information given in the question in tabular form as follows:

	Good X	Good Y	Constraint
Capital	2	1	10
Labour	1	3	20
Revenue	80	100	Maximize R

Let x units of X and y units of Y be produced. The above problem can be written as

Maxmize

$$R = 80x + 100y$$

subject to

$2x + y \leq 10$ (Capital constraint)

$x + 3y \leq 20$ (Labour constraint)

$x \geq 0, y \geq 0$ (Non-negativity)

We shade the feasible region in following figure.

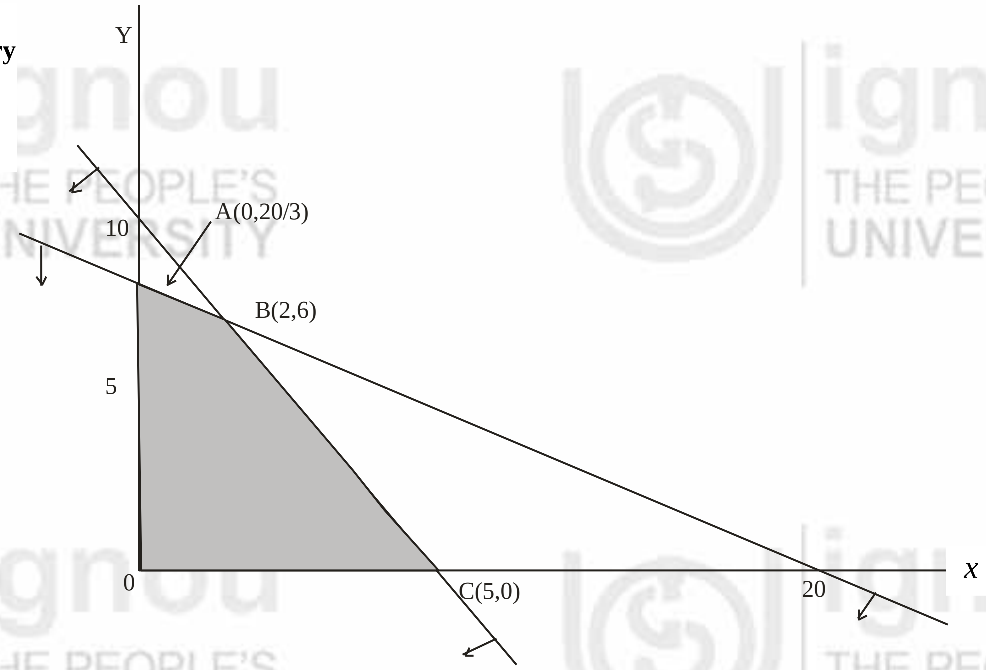


Figure 23

We now check the revenue at corner points of the feasible region.

R (A) = 80(0) + 100(20/3)= 2000/3 = $666\frac{2}{3}$

R(B) = 80(2) + 100(6) = 760

R(C) = 80(5) + 100(0) = 400

R(0) = 80(0) + 100(0) = 0

This shows that the revenue is maximum when x = 2 and y = 6

i.e. when 2 units of x and 6 units y are produced and the maximum revenue is Rs. 760.

6. We summarise the information given in the above question in the following table.

	Drink A	Drink B	Constraint
Pineapple juice	4(kg per tin)	2(kg per tin)	$\leq$ 46
Orange juice	1(kg per tin)	3(kg per tin)	$\leq$ 24
Profit	4	2	Maximize P

Let x tins of drink A and y tins of drink B be filled up. The above problem can be written as

Maximise

$$P = 4x + 2y$$

Subject to

$4x + 2y \leq 46$ (pineabpple juice constraint)

$x + 3y \leq 24$ (organge juice constraint)

$x \geq 0, y \geq 0$ (non-negativity)

The feasible region has been shaded. See the following figure.

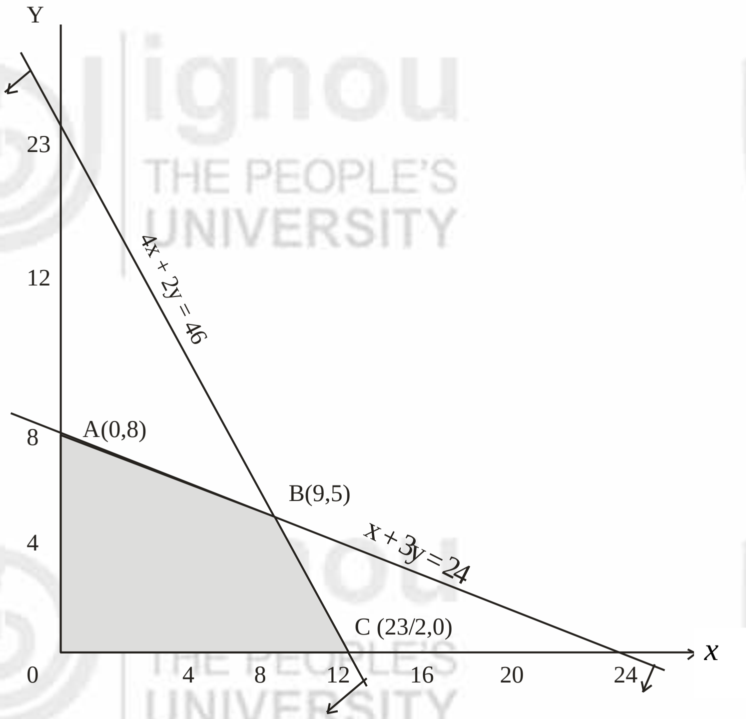


Figure 24

We have

P(A) = 4 (0) + 2(8) = 16

P(B) = 4(9) + 2(5) = 46

$$P(C) = 4\left(\frac{23}{2}\right) + 2(0) = 46$$

P(0) = 0

We have maximum profit at *B* and *C*. In this case, we say that we have multiple solutions. In fact, each point on the segment *BC* gives a profit of ₹ 46. This is because the segment *BC* is one of the isoprofit lines.

Check Your Progress 2

1. Suppose tailor A works for x days and tailor *B* work for y days.
   The LPP is

   Minimise
   $$C = 150x + 200y$$
   subject to

   $6x + 10y \geq 60$ [Shirts constraints]

   $4x + 4y \geq 32$ [Pants constraint]

   $x \geq 0, y \geq 0$ [Non–negativity]

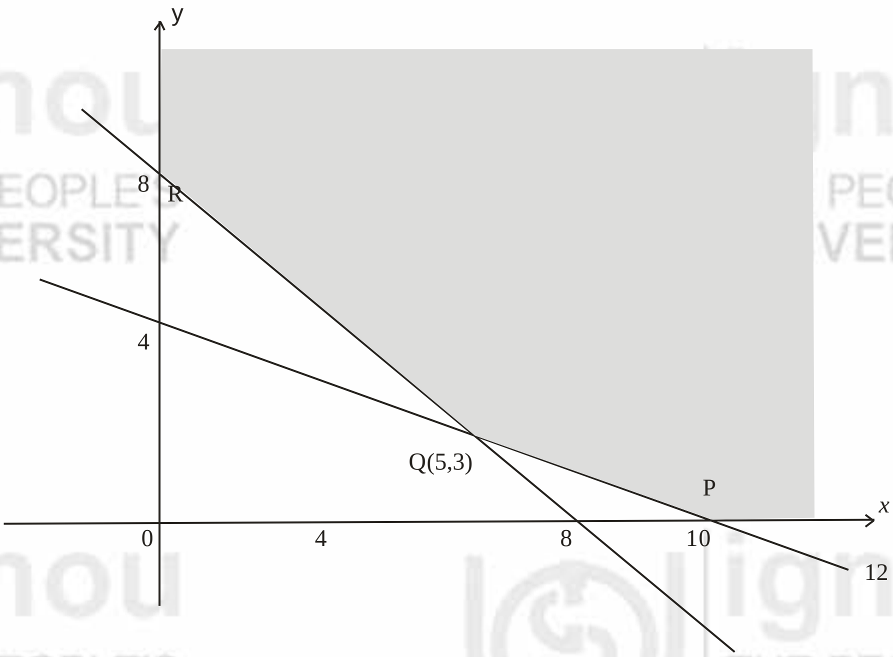


Figure 25

C (P) = 1500

C (Q) = 1350

C(R) = 1600

Thus, cost is least when x =5 and y = 3

2. Food X : x units and Food Y : y units LPP is

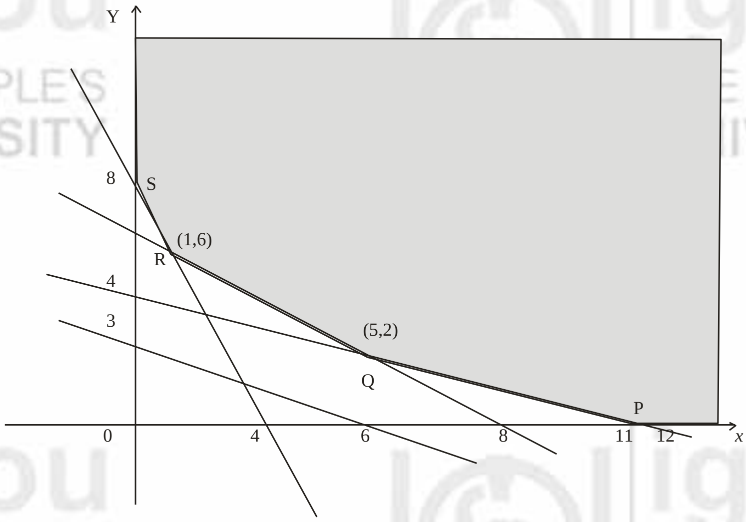


Figure 26

Minimise

$$C = 5x + 8y$$

subject to

$$x + 2y \geq 6$$

$$x + y \geq 7$$

$$x + 3y \geq 11$$

$$2x + y \geq 8$$

$$x \geq 0, y \geq 0$$

Now, C(P) = 55, C (Q) = 41

C(R) = 30, C(S) = 64

Thus, C is least when $x = 5$, y = 2

least Cost is Rs. 41

The constraint $x + 2y \geq 6$ is a redundant constraint.

3. Let x units of food A and y units of food B be used. The LPP is

Minimise

$$C = 4x + 3y$$

subject to

$200x + 100y \geq 400$	(vitamins constraints)
$x + 2\,y \geq 50$	(minerals constraints)
$40x + 40y \geq 1400$	(calories constraints)
$x \geq 0, y \geq 0$	(non-negativity)

We have

C(P) = 200, C (Q) = 125, C (R) = 110,

C(S) = 120

Thus, C is least when $x = 5$, $y = 30$.

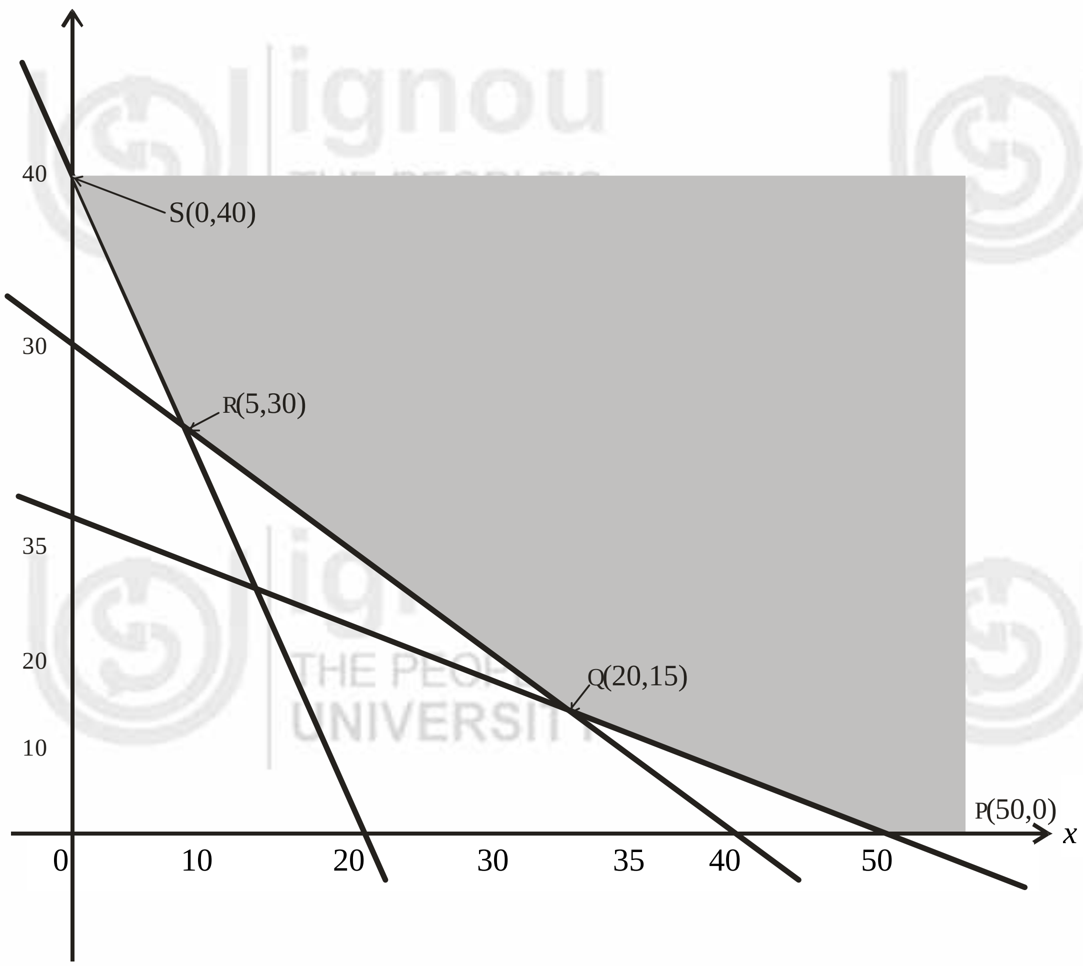


Figure 27

4. Let x boxes and y cards be purchased.

Cost of x boxes is $30x$ paise and cost of y cards is $20y$ paise.

$\therefore$ Cost incurred by the tailor (in paise) is $30x + 20y$

Number of large buttons obtained from x boses and y cards is $2x + 4y$.

According to given condition.

$$6x + 2y \geq 40$$

Number of small buttons obtained from x boxes and y cards is $2x + 4y$.

According to the given condition

$$2x + 4y \geq 60$$

Also, $x \geq 0, y \geq 0$

Thus, the linear programming problem is

Minimize

$C = 30\,x + 20y$ [objective function]

subject to

$6x + 2y \geq 40$ [large button constraint]

$2\,x + 4y \geq 60$ [small button constraint]

$x \geq 0, y \geq 0$ [non-negativity]

We draw the feasible in the following figure.

We now calculate the cost at the corner points of the feasible region.

C(A) = C(0,20) = 30(0) + (20)(20) = 400

C(B) = C(5/2, 55/2) = (30) (5/2) = (20) (55/2) = 75 + 550 = 525

C(D) = C (30,0) = (30)(30) + (20)(0) = 900

Figure 28

Thus, the least cost occurs when the tailor purchases just 20 cards and the least cost is 400.

4.7 SUMMARY

The unit is about the mathematical discipline of linear programming. In **section 4.0**, a number of relevant concepts including that of objective function, feasible region/solution space are introduced. Then the nomenclature 'linear programming' is explained. In **section 4.2** the above concepts alongwith some other relevant concepts are (formally) defined. **Section 4.3** explains the two graphical methods for solving linear programming problems (L.P.P.). viz. (i) corner point method (ii) iso-profit and iso-cost method. The methods are explained through a number of examples. **Section 4.4** discusses methods of cost minimisation in context of linear programming problems.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 4.5**.