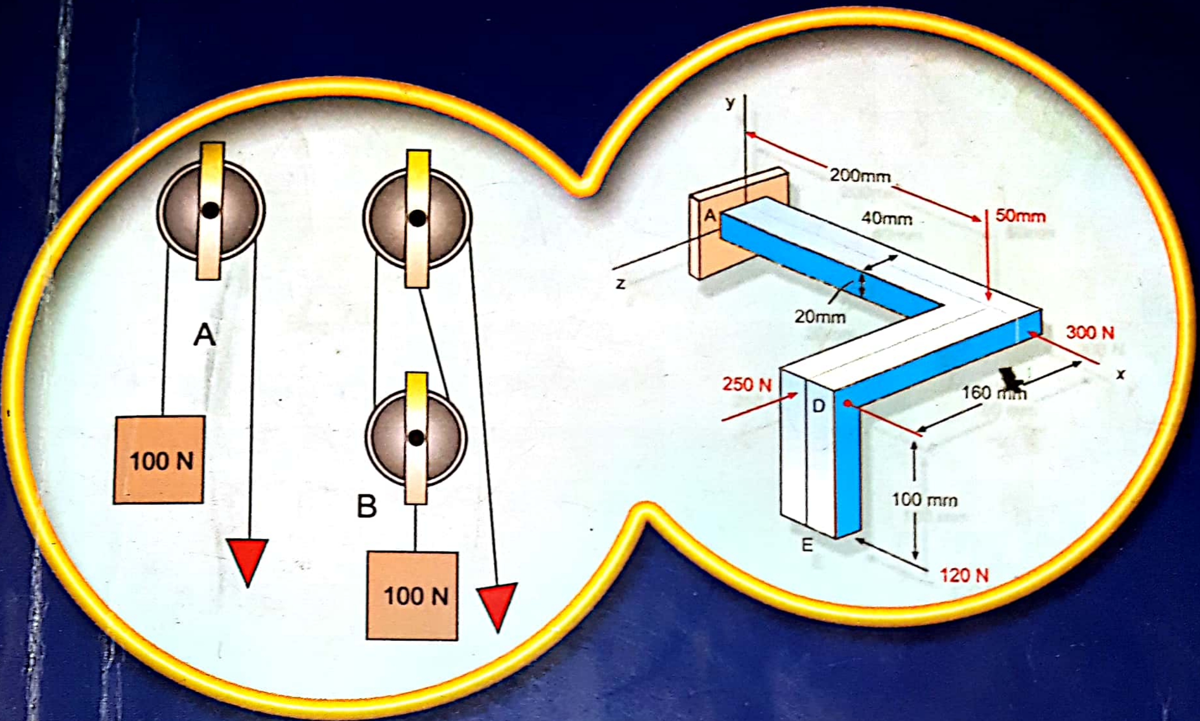
For first year students of BCE, BIE, BME, BAME, BCT, BEX, BEL, BGE, B. Agri & B.Arch



APPLIED MECHANICS

FOR ENGINEERS

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Contents

Chapter 1
Applied Mechanics : Introduction

1.1 Definition

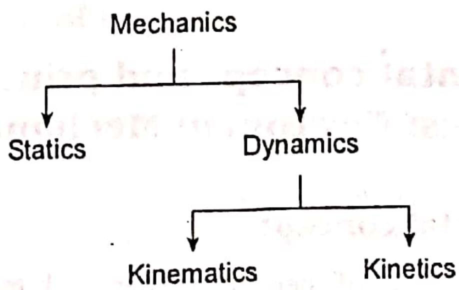
Mechanics is a basic subject that deals with the response of particles and rigid bodies to mechanical disturbance. Mechanical disturbances in our context means forces, moments, displacements, couple. It essentially deals with the study of bodies at rest or in motion when subjected to external mechanical disturbances. This subject can be subdivided into three branches:

a) Rigid body mechanics

b) Deformable body mechanics

c) Fluid mechanics

In this course we will study rigid body mechanics as it is a basics for the study of fluid mechanics and mechanics of deformable body. Rigid body mechanics is further divided into statics and dynamics; the former dealing with bodies in rest, and later with bodies in motion. Dynamics in further divided into kinematics and kinetics. Kinematics is used to relate displacement, velocity, acceleration and time without reference to the cause of motion. In kinetics, both the motion and its causes are considered. Hence kinetics is science of relation between force mass and motion of body.



1.1.1 Scope of applied mechanics

Mechanics is the foundation of most engineering sciences and is prerequisite to their study. The purpose of mechanics is to explain and predict natural and physical phenomena and thus forms the foundations for engineering application. It is essential to have a knowledge of applied mechanics for the design and analysis of many types of structural members, mechanical components and electrical devices encountered in engineering.

1.2 Concept of Rigid and Deformed bodies

1.2.1 Concept of rigid body

A body has a definite shape and consists of numbers of particles. A particle is an object that has infinitely small volume (size that can be neglected) but has a mass. Hence the rigid body can be considered as a combination of large number of particles in which the distance between any two particles remains constant (i.e. size and shape of body do not change) when it is acted upon by a force system. This model is important because the material properties of any body that is assumed to be rigid will not have to be considered when studying the effects of forces acting on body. In most of the case, the actual deformations occurring in structures, machines and mechanism and the like are usually small, and the rigid body assumption is suitable for analysis.

1.2.2 Deformable body

If a body changes its shape and size when it is acted upon by an external force, then it is called deformable body. An elastic body undergoes deformation but regains its original shape after the removal of external forces.

1.3 Fundamental concept and principle of mechanics: Newtonian Mechanics

1.3.1 Fundamental concept

Although the study of mechanics goes back to the time of Aristotle (384–322 B.C.) and Archimedes (287–212 B.C.), one has to wait Newton (1642–1727 A.D.) to find a satisfactory formulation of its fundamental principles. While its limitation have now been recognized, Newtonian mechanics still remains the basis of today's engineering science. After the Newton's laws were postulated, important techniques for their application were developed by Evler, D'Alembert, Lagrange and others. Their validity remains unchallenged until Einstein formulated his theory of relativity.

The basic concepts used in mechanics are space, time, mass and force. These concept cannot be truly defined rather they should be accepted on the basis of our intuitition and experience. The concept of space is associated with the notation of the position of point P. The position of P can be defined by three length measured from certain reference called origin, in the three direction. These length are called coordinates of P. Time is a measure of succession of events. The concept of mass is used to characterize and compare bodies on the basis of experiment. A force represents the action of one body on another. Force is characterized by its point of application, its magnitude, and its direction. In Newtonian mechanics, space, time and mass are absolute concepts, independent of each other. (This is not true in relativistic mechanics, by Einstein, where the time of an event depends upon its position and where the mass of a body varies with its velocity.)

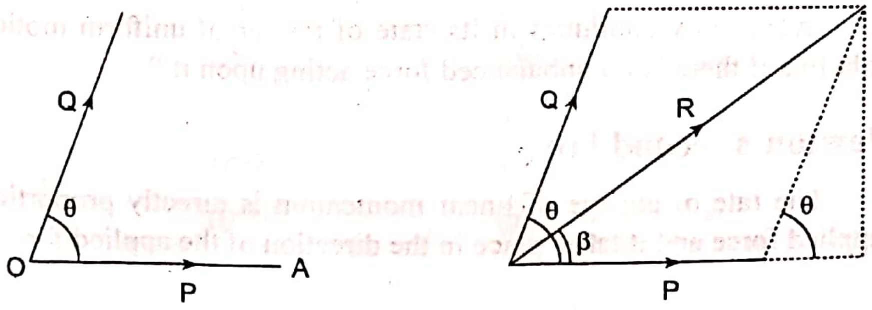
1.3.2 Principle of Mechanics: Newtonian

The study of elementary mechanics rests on six fundamental principles based on experimental evidence.

1. The parallelogram law of forces
2. The principle of transmissibility of force
3. Newton's first law of motion
4. Newton's Second law of motion
5. Newton's third law of motion
6. Newton's law of gravitation

1. Parallelogram law of forces

The parallelogram law is used to determine the resultant of two forces acting at a point in a plane and inclined to each other at an angle. It states that "If two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point."



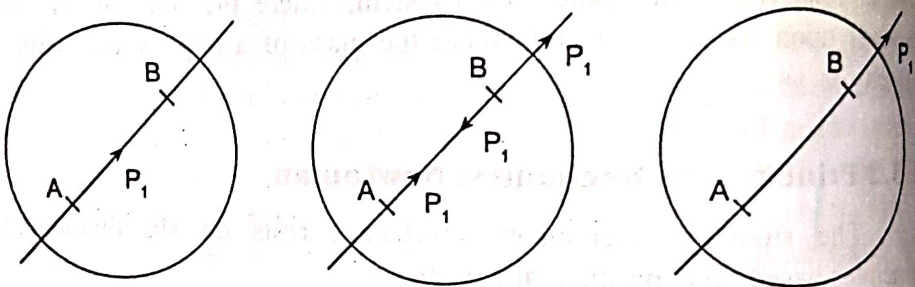
$$\therefore \vec{R} = \vec{P} + \vec{Q}$$

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta}$$

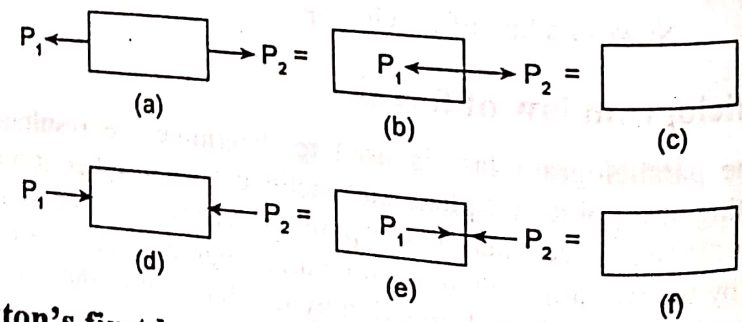
2. Principle of transmissibility of force

When the point of application of force acting on a body is shifted to any other point on the line of action of the force without changing its direction, there occurs no change in the equilibrium of the body.



Limitation of transmissibility of force

It may be used to determine the condition of motion or equilibrium of rigid bodies but it should be avoided or atleast used with care, to compute the external forces acting on these bodies and in determining internal forces and deformations.



3. Newton's first law

"Every body continues in its state of rest or of uniform motion in a straight line if there is no unbalanced force acting upon it."

4. Newton's second law

"The rate of change of linear momentum is directly proportional to the applied force and it takes place in the direction of the applied force."

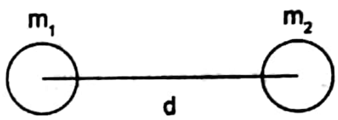
5. Newton's third law

"To every action, there is equal and opposite reaction."

6. Newton's law of gravitation

"Everybody in the universe attracts every other body with a force directly proportional to the product of their mass and inversely proportional to the square of the distance separating them."

$$F = \frac{G\, m_1 m_2}{r^2}\ ;\ G = 6.67 \times 10^{-11}\ Nm^2/kg^2$$



where G is constant.

1.4 System of unit

Based on the units used to represent the three fundamental quantities namely mass (m), length (l) and thme (t) the commonly used system of units are

a) FPS system — Foot, pound, second
b) CGS system — centimeter, gram, second
c) MKS system — meter, kilogram, second

1.4.1 SI prefixes

When a numerical quantity is either very large or very small, the units used to define its size may be modified by using a prefix.

Exponential form	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-3}	mili	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

1.5 Review of vector algebra

1.5.1 Dot product

If $\vec{A}$ and $\vec{B}$ are two vectors and θ is the angle between them, then the dot product is defined as

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta = AB\cos\theta$$

The result of the dot product of two vectors is scalar. So it is also called scalar product.

1.5.1.1 Properties of scalar product

a) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

b) $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

c) $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$

s) $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$

If $\vec{A} \cdot \vec{B} = 0$ and A and B $\neq 0$; then A and B are orthogonal (perpendicular) vector.

1.5.2 Cross product

If $\vec{A}$ and $\vec{B}$ are two vectors and θ is the angle between them, then the cross product is defined as

$$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta\, \hat{n}$$

The result of the cross product of two vector is again a vector, so it is also called vector product.

1.5.2.1 Properties of vector product

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

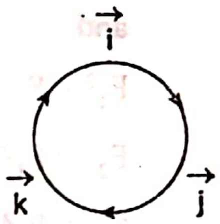
$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j}$$



If $\quad \vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$

$\qquad \vec{B} = B_1 \vec{i} + B_2 \vec{j} + B_3 \vec{k}$

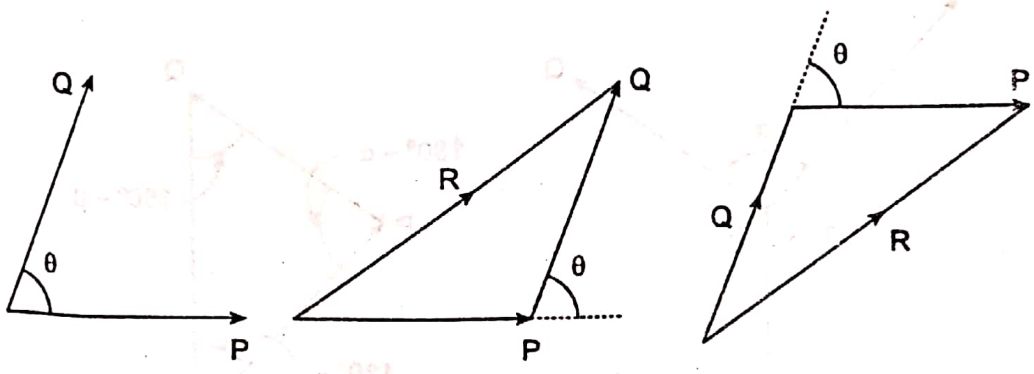
then,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

1.5.3 Triangle law of forces

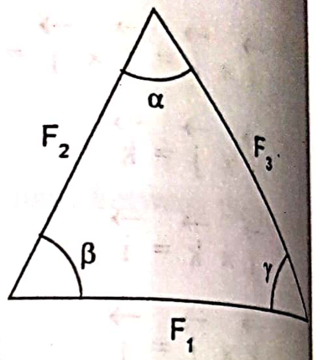
"If two forces acting simultaneously on a body are represented by the sides of a triangle taken in order, their resultant is represented by the third side of triangle taken in the opposite order."

$$\vec{R} = \vec{P} + \vec{Q}$$



The following trigonometric relations can be applied while working out solution by the triangle law.

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\gamma} = \frac{F_3}{\sin\beta}$$



and

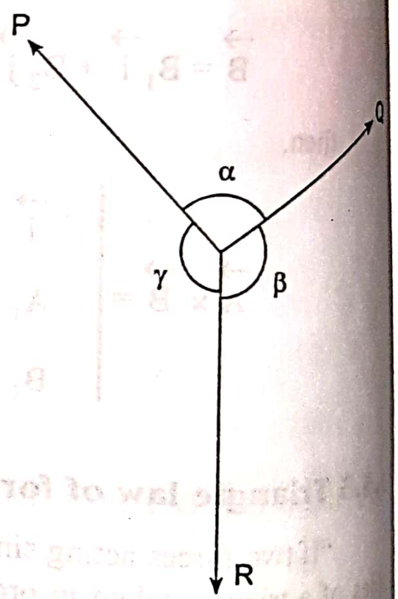
$$F_1^2 = F_2^2 + F_3^2 - 2F_2F_3 \cos\alpha$$

$$F_2^2 = F_1^2 + F_3^2 - 2F_1F_3 \cos\gamma$$

$$F_3^2 = F_1^2 + F_2^2 - 2F_1F_2 \cos\beta$$

1.5.4 Lami's theorem

"If a body is in equilibrium under the action of three forces, then each force is proportional to the sine of the angle between the other two forces."



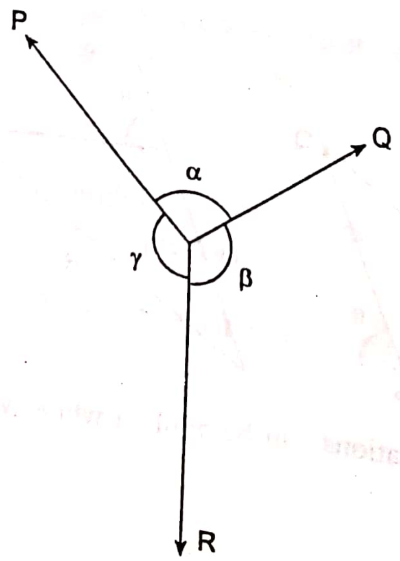
Mathematically,

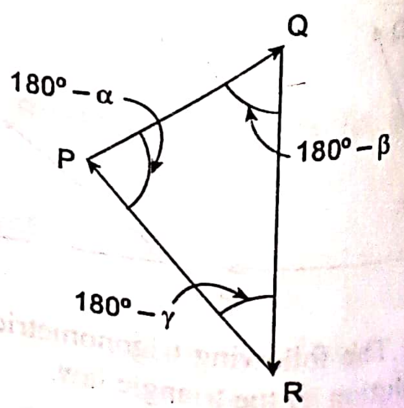
$$\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha}$$

All the forces must be outgoing from common point.

Proof:

Let P, Q, R be the three concurrent forces in equilibrium as shown in figure (a). Since the forces are vector, we can move them to form a triangle as shown in figure (b).





Applying sine rule, we get

$$\frac{P}{\sin(180-\beta)}=\frac{Q}{\sin(180-\gamma)}=\frac{R}{\sin(180-\alpha)}$$

$$\frac{P}{\sin\beta}=\frac{Q}{\sin\gamma}=\frac{R}{\sin\alpha}$$ Hence proved.

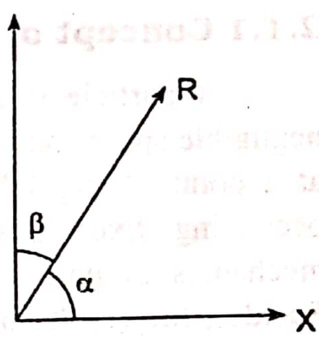
1.5.5 Resolution of forces

Finding out the component of a given forces in two given direction (normally perpendicular) is called resolution.

Consider angle α:

$$R_x = R\cos\alpha$$

$$R_y = R\sin\alpha$$



Consider angle β:

$$R_x = R\sin\beta$$

$$R_y = R\cos\beta$$

Note: If X and Y are not perpendicular, then the case will be different. It will be discussed in chapter 3.

Chapter 2

Basic Concept in Statics and Static Equilibrium

2.1 Concept of particles and free body diagram

2.1.1 Concept of particles

A particle is an object that has infinitely small volume (occupies negligible space) but has a mass which can be considered to be concentrated at a point. A rigid body is a combination of large number of particles occupying fixed positions with respect to each other. The study of mechanics of particle is obviously a prerequisite to that of rigid body. Besides, the results obtained for a particle can be used directly in a large number of problems dealing with the conditions of rest or motion of actual bodies.

When the distances travelled by the body are much more larger than the size of body itself, then we can approximate that body as particle. Sometimes heavenly bodies like earth is treated as particle, although size is very large, why? It is because the perimeter of earth is very small in comparison to the distance travelled by earth in revolving around the sun.

If the size is comparable to the distance travelled, then we treat them as a finite body.

2.1.2 Free body Diagram (FBD)

The force analysis of a structure is made in a simplified way by considering the equilibrium of a portion of the structure. For that, the portion is drawn separately showing applied forces, self weight and reactions at the point of contact with other bodies. The resulting diagram is known as Free Body Diagram (FBD). So, FBD is a sketch of body (space diagram) drawn in such a way that it shows all the reaction forces, applied forces and moments on the body.

2.1.2.1 Guidelines for drawing FBD

i) FBD should have no external supports or connections.

ii) The self weight ($W = mg$) should be indicated with vertical downward arrow.

iii) The reactions from the supports and connections should be indicated.

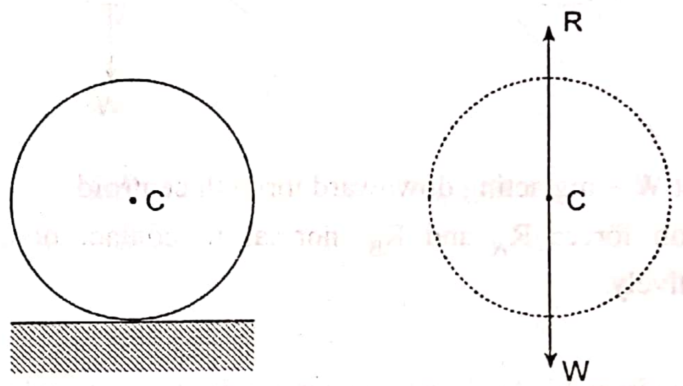
iv) The uncut member force should not be shown in FBD.

v) Tension in rope or string is directed towards support.

vi) The adopted coordinate system and sense of unknown force should be shown in FBD.

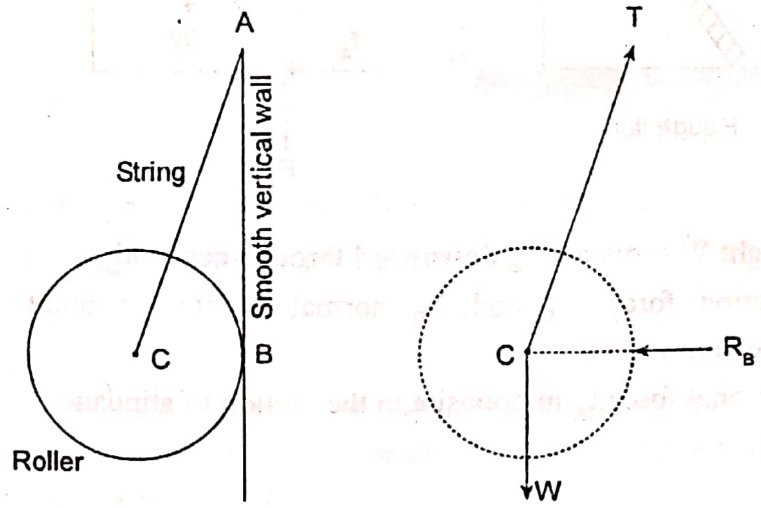
2.1.2.2 Few Examples of FBD

a) A sphere resting on a frictionless plane surface



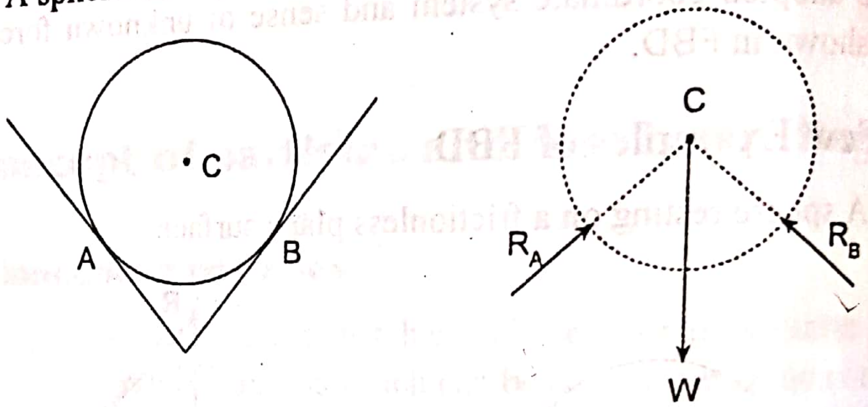
- Force $W = mg$ acting downward through centroid.
- Reaction R at the point of contact with surface. This R acts upward normal to the surface.

b) A circular roller (of weight W) hangs by a string and rest against a smooth (frictionless) vertical wall



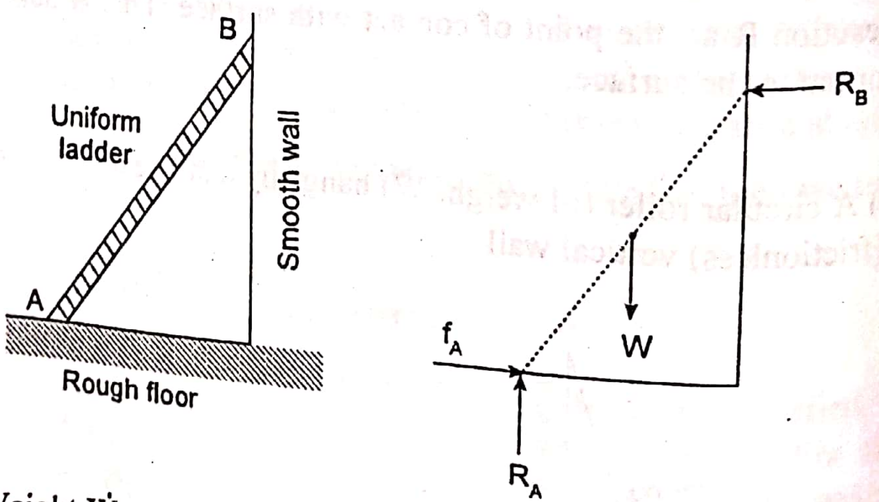
- Force $W = mg$ acting downward through centroid.
- Wall reaction R_B at point of contact. normal to wall along BC.
- Tension T towards support A along CA.

c) A sphere resting in a v-shaped groove.



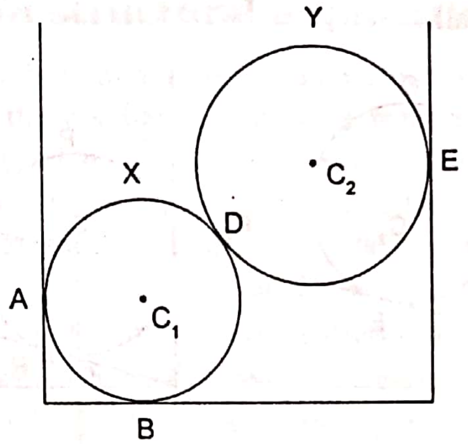
- Weight $W = mg$ acting downward through centroid.
- Reaction forces R_A and R_B, normal to contact points A and B respectively.

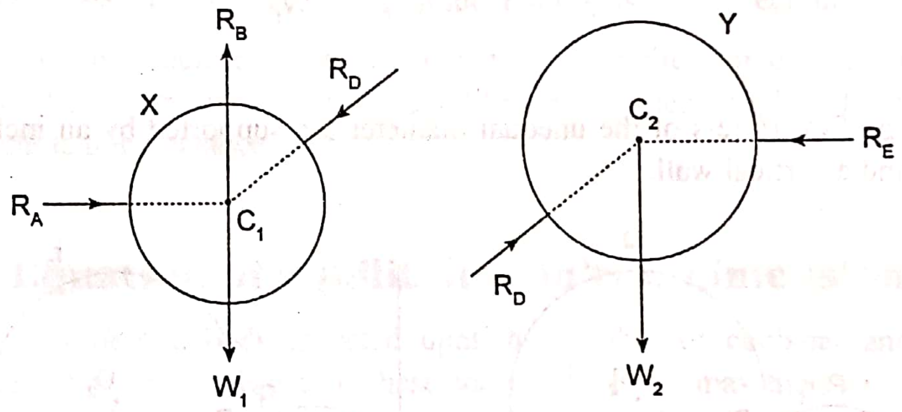
d) A uniform ladder leans against a smooth wall and rest on rough floor.

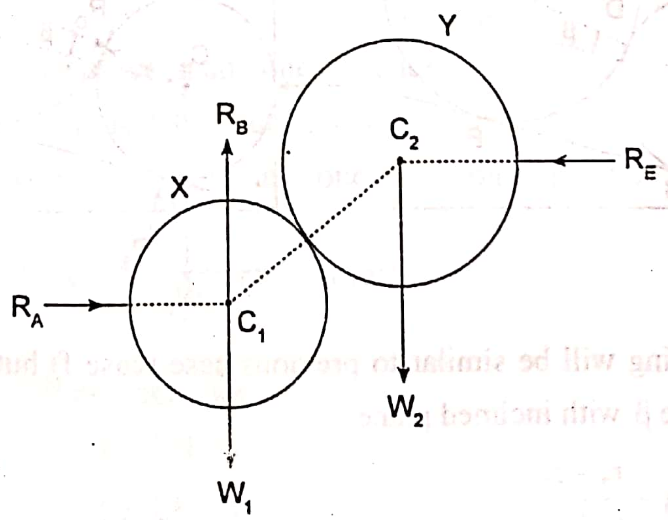


- Weight $W = mg$ acting downward through centroid
- Reaction force R_A and R_B normal to contact point A and B respectively.
- Frictional force f_A in opposite to the motion of slippage.

e) Two spheres X and Y placed in a vessel.



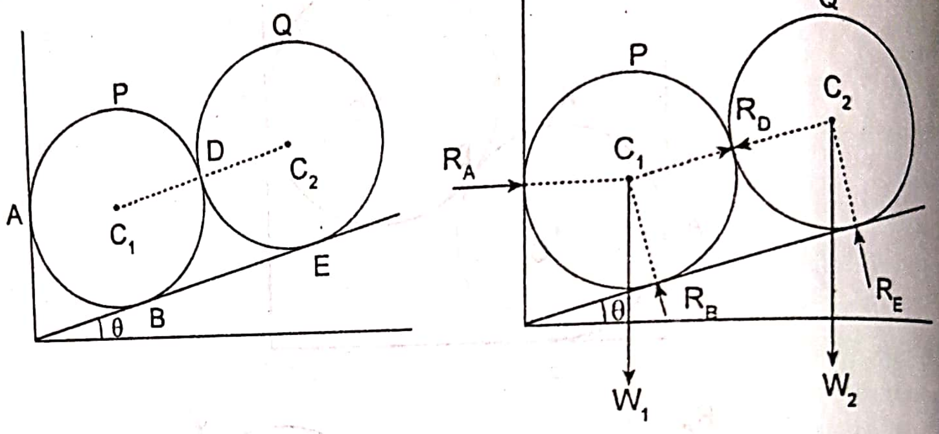




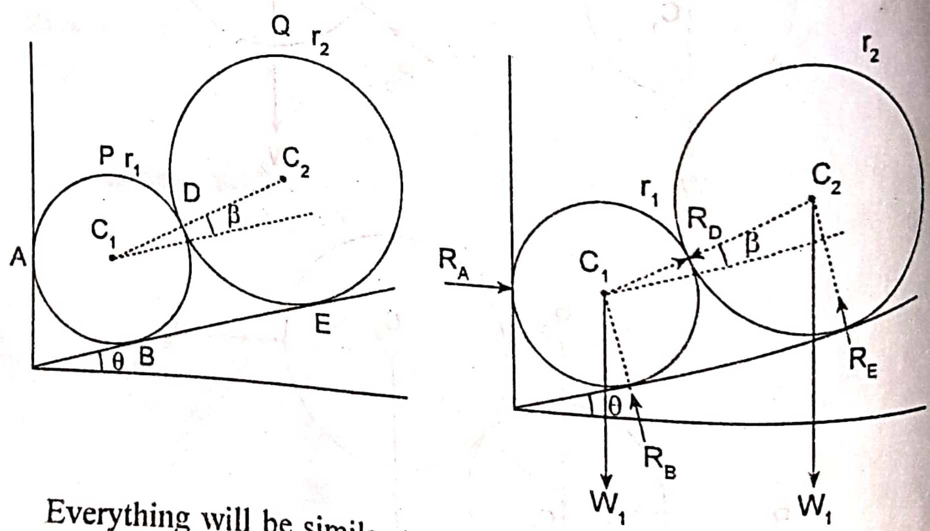
- Weight w_1 and w_2 vertically downward through e.g.
- R_A, R_B and R_E normal to contact surface.
- R_D on both sphere at contact point O along C_1C_2 in opposite direction.

Note: Reaction of sphere X on sphere Y and vice-versa at contact point D are equal in magnitude, opposite in direction and are collinear. So, they do not appear in the FBD.

f) Two rollers of the same diameter are supported by an inclined plane and a vertical wall.



g) Two rollers of the unequal diameter are supported by an inclined plane and a vertical wall.



Everything will be similar to previous case (case f) but the line c_1c_2 will make angle β with inclined plane.

$$\text{So, } \sin\beta = \frac{r_2 - r_1}{r_2 + r_1}$$

Here W_1 makes angle θ and W_2 makes angle θ with R_B and R_E respectively. Here θ is the angle of inclined plane.

2.2 Physical meaning of equilibrium and its essence in structural application

When the numbers of forces act on a rigid body, we have various methods of finding the resultant force. This resultant force will produce same effect as produced by number of forces. In some cases, it is quite possible for a resultant to be zero. In such case, the net effect of the given force is zero, and the particle is said to be in equilibrium. Such a set of forces, where resultant is zero, are called equilibrium forces. Hence the resultant refers to the single force which produces the same effect as is done by the combined effect of several forces. The force, which brings the set of forces in equilibrium is called an equilibrant. As a fact, the equilibrant is equal to the resultant force in magnitude, but opposite in direction.

In any structure, there will be number of applied force, moments and couple. For a structure to be in equilibrium, all these applied forces and moment is to be balanced.

2.3 Equation of equilibrium in two dimensions

Consider a body is acted upon by number of coplanar and non-concurrent forces. Because of these forces, the body may have one of the following states:

a) A body may move in any one direction

b) The body may rotate about itself without moving

c) The body may move in any one direction and rotate about itself at same time

d) The body may be completely at rest

Case a:

The equilibrium equations will be

i) $\Sigma F_x = 0$ i.e. $\Sigma H = 0$

ii) $\Sigma F_y = 0$ i.e. $\Sigma V = 0$

Case b:

The equilibrium equation will be

$$\Sigma \vec{M} = 0$$

Case c:

The equilibrium equations will be

i) $\Sigma F_x = 0$

ii) $\Sigma F_y = 0$

iii) $\Sigma M = 0$

Case d:

It is in equilibrium condition and following three equations of equilibrium are already satisfied.

i) $\Sigma F_x = 0$

ii) $\Sigma F_y = 0$

iii) $\Sigma M = 0$

So, it can be concluded that for a system to be in equilibrium

$$\Sigma \vec{F} = \vec{R} = 0$$

$$\Sigma \vec{M} = 0$$

The equilibrium equations for 3-D can be

$$\Sigma \vec{F} = 0$$

$$\Sigma \vec{M} = 0$$

which can be further expanded as

$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$

$\Sigma M_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$

where

ΣF_x = algebraic sum of forces in x-direction

ΣF_y = algebraic sum of forces in y-direction

ΣF_z = algebraic sum of forces in z-direction

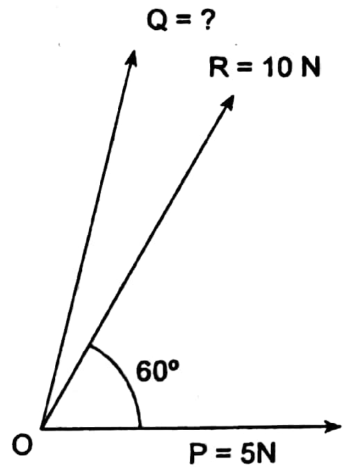
ΣM_x = algebraic sum of moment about y-z plane

ΣM_y = algebraic sum of moment about x-z plane

ΣM_z = algebraic sum of moment about x-y plane

Worked Out Examples

1. The resultant of two forces is 10 N and it is inclined at 60° to one of the forces whose magnitude is 5 N. Determine the magnitude and direction of the other force.



Solution:

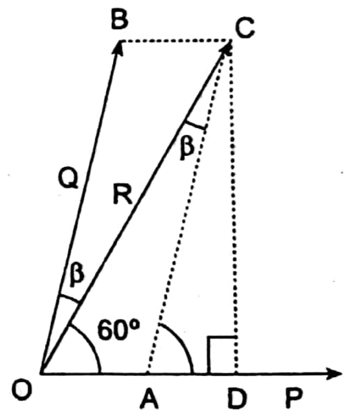
Let $P = 5$

$R = 10\,N$

$Q =$ unknown force

Let the angle between R and Q be β.

From the figure; ΔOAC



$$\frac{P}{\sin\beta} = \frac{Q}{\sin 60} = \frac{R}{\sin(180-(60+\beta))}$$

$$\therefore \quad \frac{Q}{\sin 60} = \frac{R}{\sin(60+\beta)} = \frac{P}{\sin\beta}$$

$$\frac{R}{\sin(60+\beta)} = \frac{P}{\sin\beta}$$

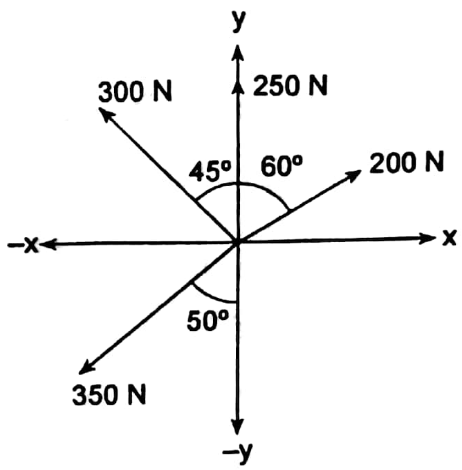
$$\Rightarrow \quad \frac{10}{\sin(60+\beta)} = \frac{5}{\sin\beta}$$

solving,

$$\beta = 30°$$

$$\therefore \quad Q = \frac{P}{\sin 30} \times \sin 60° = 5\sqrt{3}\,N = 8.66\,N$$

2. Determine the magnitude and direction of the resultant



$$\vec{R} = \Sigma\vec{F_x} + \Sigma\vec{F_y}$$

ΣF_x ($\rightarrow$ +ve)

$$= 200\sin 60 - 300\sin 45° - 350\sin 50$$

$$= -307.04\,N$$

$$= 307.04\,N\,(\leftarrow)$$

ΣF_y ($\uparrow$ +ve)

$= 200 \cos 60 + 250 + 300 \cos 45 - 350 \cos 50$

$= 337.156$ N ($\uparrow$)

$\therefore \quad \vec{R} = \Sigma \vec{F}_x + \Sigma \vec{F}_y$

$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

$= \sqrt{(307.04)^2 + (337.15)^2}$

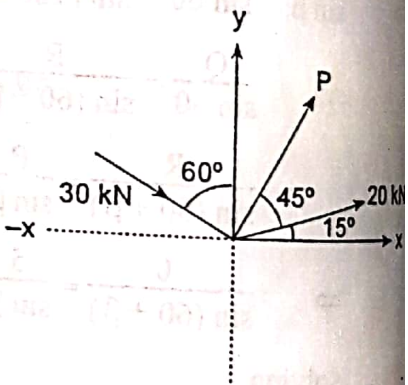
$= 456.01$ N

$$\tan \alpha = \frac{\Sigma F_y}{\Sigma F_x} = \frac{337.15}{307.04}$$

$\therefore \quad \alpha = 47.67°$

$R = 456.01; \ \alpha = 47.67°$

3. Determine the value of P such that the resultant is horizontal. Also find the value of equlibrant.



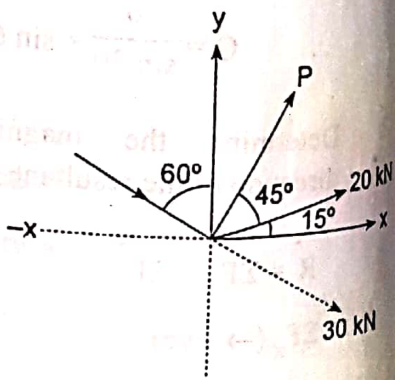
Solution:

Here 30 kN is directed towards the origin. To resolve the force it must be directed away from origin. So, produce 30 kN on the same line of action.

ΣF_x ($\rightarrow$ +ve)

$= 20 \cos 15 + P \cos 60 + 30 \sin 60$

$= 45.3 + 0.5P$



ΣF_y($\uparrow$ +ve)

$= 20 \sin 15 + P \sin 60 - 30 \sin 60$

$= -9.823 + \frac{\sqrt{3}}{2} P$

Since the resultant is horizontal

$\Sigma F_y = 0$

$\therefore \quad \frac{\sqrt{3}}{2} P = 9.823$

$\therefore \quad P = 11.34$ kN

$$\vec{R} = \Sigma \vec{F}_x$$

$$= 45.3 + 0.5 \times 11.34$$

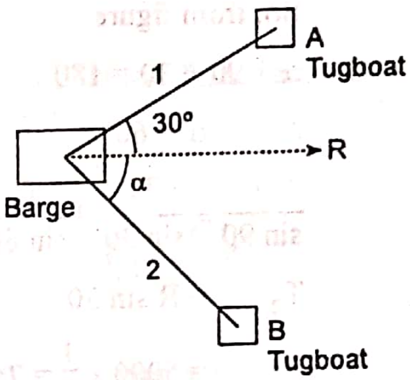
$$R = 50.97 \text{ kN } (\rightarrow)$$

$\therefore$ value of equlibrant $= -R = 50.97$ kN ($\leftarrow$)

4. A barge is pulled by two tugboats. If the resultant of the forces exerted by the tugboat is a 5000 N force directed along the axis of barge, determine

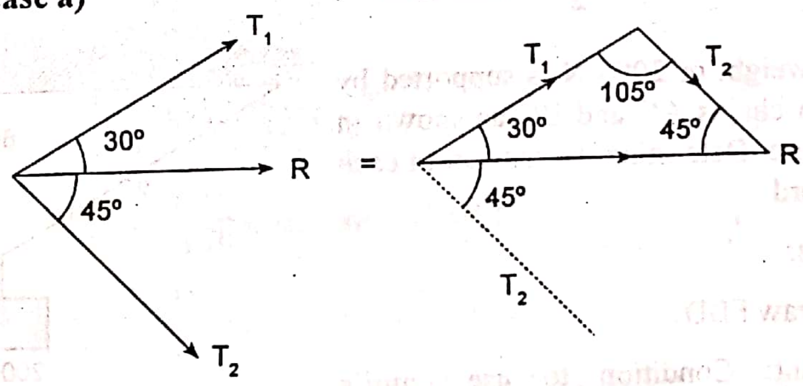
   a) the tension in each of the rope knowing $\alpha = 45°$

   b) the valve of α for which the tension in rope 2 is minimum and minimum T_2.



Solution:

Case a)



$\alpha = 45°$

Draw FBD and triangle for forces

Using sine law

$$\frac{T_1}{\sin 45} = \frac{T_2}{\sin 30} = \frac{R}{\sin 105}$$

$$T_1 = \frac{R}{\sin 105} \times \sin 45 = \frac{5000}{\sin 105} \times \sin 45$$

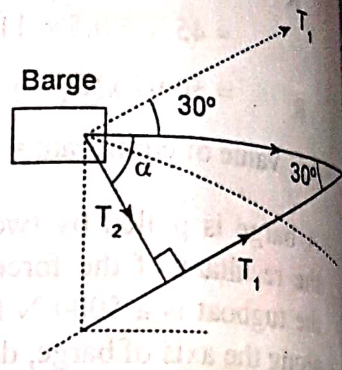
$$= 3660.25 \text{ N}$$

$$T_2 = \frac{R}{\sin 105} \times \sin 30 = \frac{5000}{\sin 105} \times \sin 30$$

$$= 2588.19 \text{ N}$$

Case b:

We know that the perpendicular distance is the shortest distance. So, we draw a perpendicular line on T_1 to find minimum T_2.



So, from figure

$\alpha + 90 + 30 = 180$

$\therefore \quad \alpha = 60°$

$$\frac{R}{\sin 90} = \frac{T_2}{\sin 30} = \frac{T_1}{\sin 60}$$

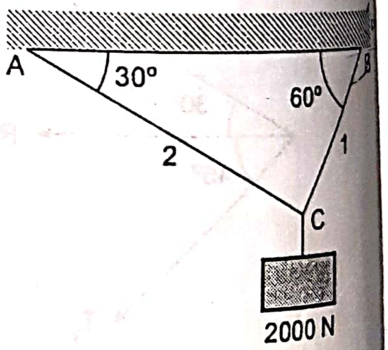
$T_2 \quad = R \sin 30$

$\quad = 5000 \times \frac{1}{2} = 2500$ N (minimum)

$T_1 \quad = R \sin 60$

$\quad = 5000 \times \frac{\sqrt{3}}{2} = 4330.12$ N

5. A weight of 2000 N is supported by two chains AC and BC as shown in figure. Determine the tension in each chord.



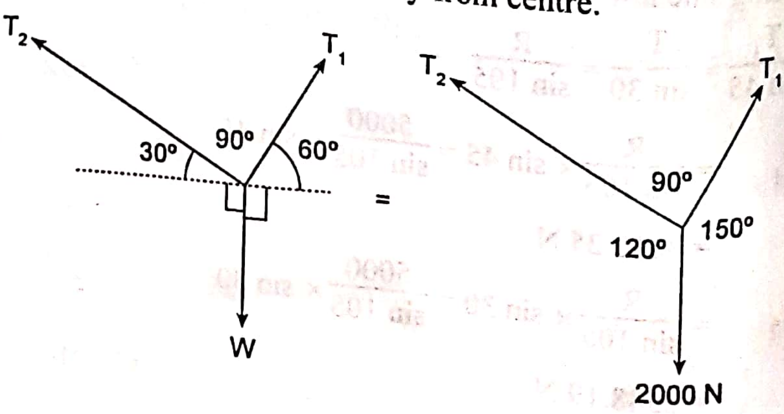
Solution:

Draw FBD.

Hint: Condition to use Lami's theorem

a) At most 3 forces

b) All forces directed away from centre.



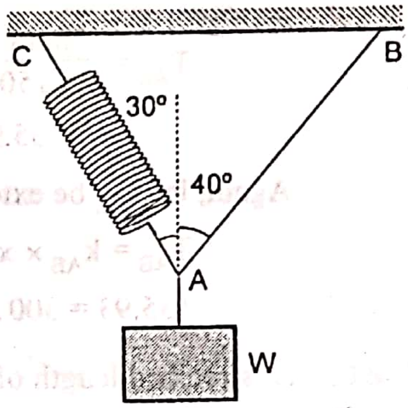
Given FBD satisfy the condition to use Lami's theorem.

So, $$\frac{T_1}{\sin 120} = \frac{T_2}{\sin 150} = \frac{2000}{\sin 90}$$

$\therefore$ $T_1 = 2000 \sin 120 = 1732$ N

$T_2 = 200 \sin 150 = 1000$ N

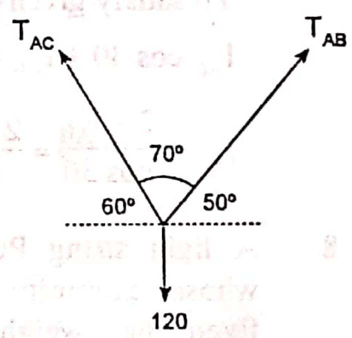
6. A container of weight W is subjected from ring A to which cable AB and spring AC are attached. The constant of spring is 100 N/m and its unstretched length is 3 m. Determine the tension in the cable when a) W = 120 N b) W = 160 N



Solution:

Case a)

W = 120 N



Using Lami's theorem

$$\frac{T_{AC}}{\sin 140} = \frac{T_{AB}}{\sin 150} = \frac{W}{\sin 70}$$

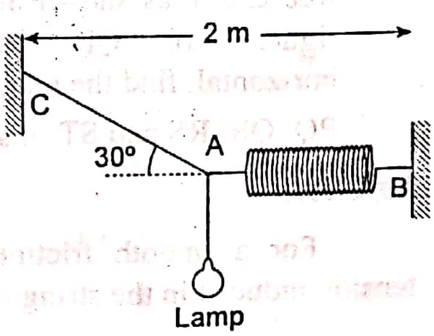
$\therefore$ $$T_{AC} = \frac{120}{\sin 70} \times \sin 140 = 82.08 \text{ N}$$

$$T_{AB} = \frac{120}{\sin 70} \times \sin 150 = 63.85 \text{ N}$$

Similarly change W = 160 N and get new value of T_{AB} = 85.13 N.

Note: Spring constant and unstretched length are superfluous data (here). Students can be asked to calculate the extension on the spring.

7. Determine the required length of cord AC so that 8 kg lamp can be suspended in the position shown. The underformed length of spring AB is 0.4 m and the spring has stiffness of 300 N/m.

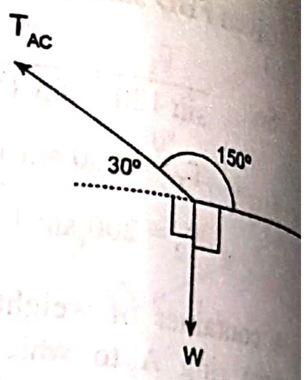


Solution:

Draw FBD

Using Lami's theorem

$$\frac{T_{AC}}{\sin 90} = \frac{T_{AB}}{\sin 120} = \frac{W}{\sin 150}$$



$$T_{AC} = \frac{78.48}{\sin 150} \times \sin 90$$

$$= 156.96 \text{ N}$$

$$T_{AB} = \frac{78.48}{\sin 150} \times \sin 120$$

$$= 135.93 \text{ N}$$

Again, let x_{AB} be extension of spring AB

$$T_{AB} = k_{AB} \times x_{AB} \qquad [F = kx]$$

$$135.93 = 300 \times x_{AB}$$

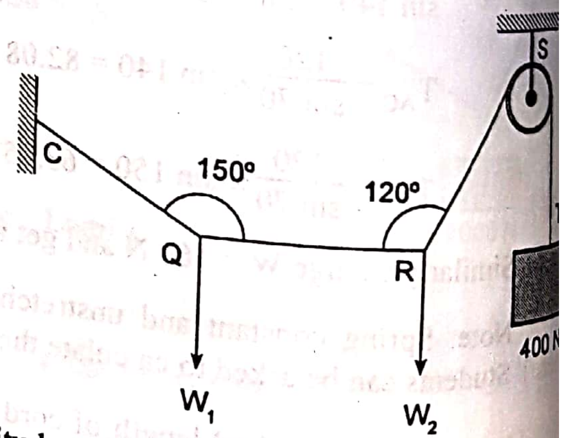
$\therefore$ stretched length of AB $= l_{AB} = 0.453 + 0.4 = 0.853$ m

To satisfy given condition

$$l_{AC} \cos 30 + l_{AB} = 2$$

$$l_{AC} = \frac{2 - l_{AB}}{\cos 30} = \frac{2 - 0.853}{\cos 30} = 1.324 \text{ m}$$

8. A light string PQRST whose extremity P is fixed, has weight W_1 and W_2 attached to it at Q and R. It passes round a small smooth peg at S carrying a weight of 400 N at the free end T as shown in figure. If QR is horizontal, find the magnitude of W_1 and W_2 and tension in the portion PQ, QR, RS and ST of string.



Solution:

For a smooth frictionless peg, pulley and continuous string, the tension induced in the string is same throughout.

$\therefore$ tension at ST $= T_{ST} = 400$ N

tension at RS $= T_{RS} = 400$ N

At point R:

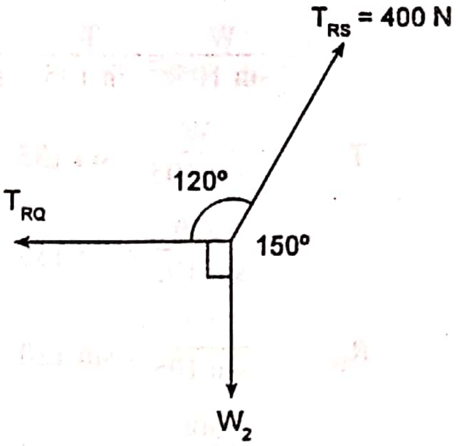
$$\frac{W_2}{\sin 120} = \frac{T_{RS}}{\sin 90} = \frac{T_{RQ}}{\sin 150}$$

$$\therefore\ W_2 = \frac{400}{\sin 90} \times \sin 120$$

$$= 346.41\ \text{N}$$

$$T_{RQ} = \frac{400}{\sin 90} \times \sin 150$$

$$= 200\ \text{N}$$



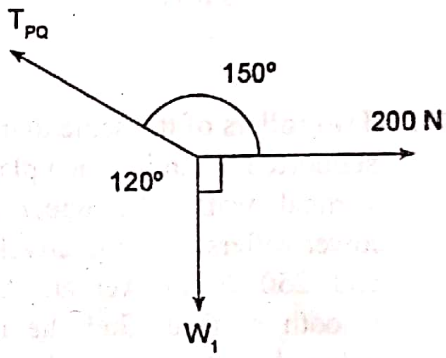
At point Q

$$\frac{W_1}{\sin 150} = \frac{200}{\sin 120} = \frac{T_{PQ}}{\sin 90}$$

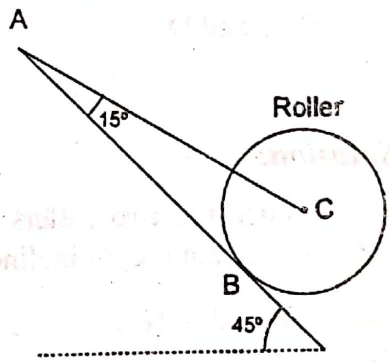
$$\therefore\ W_1 = \frac{200}{\sin 120} \times \sin 150$$

$$= 115.47\ \text{N}$$

$$T_{PQ} = 230.94\ \text{N}$$

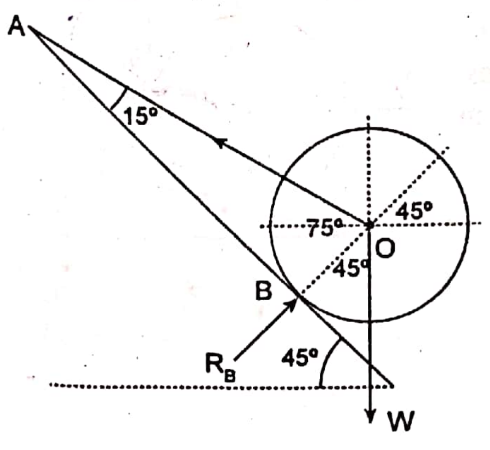


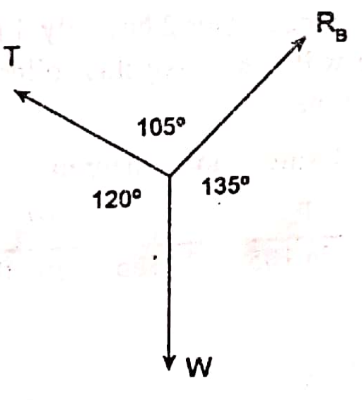
9. A roller of weight 500 N rests on a smooth inclined plane and is kept free from rolling down by a string as shown in figure. Workout the tension in the string and reaction at contact point B.



Solution:

Draw the FBD and use the trigonometry and equation of equilibrium.





Using Lami's theorem

$$\frac{W}{\sin 105} = \frac{T}{\sin 135} = \frac{R_B}{\sin 120}$$

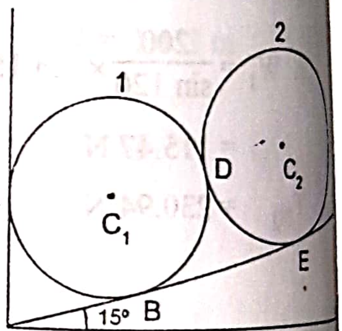
$$T = \frac{W}{\sin 105} \times \sin 135$$

$$= \frac{500}{\sin 105} \times \sin 135 = 366.02 \text{ N}$$

$$R_B = \frac{W}{\sin 105} \times \sin 120$$

$$= \frac{500}{\sin 105} \times \sin 120 = 448.28 \text{ N}$$

10. Two rollers of the same diameter are supported by an inclined plane and a vertical wall. The upper and the lower rollers are respectively 200 N and 250 N in weight. Assuming smooth surface, find the reactions induced at the points of support A, B, C and D.



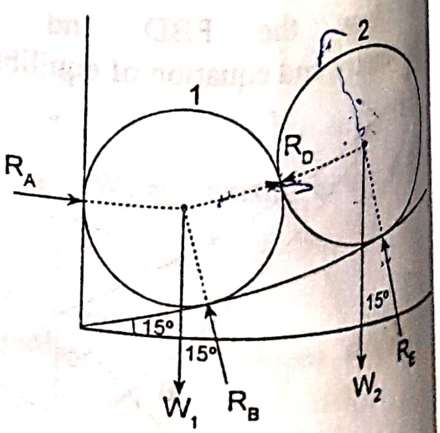
Solution:

Since the two rollers are of same diameter, line joining the centre C_1C_2 will be parallel to inclined plane.

$W_1 = 250$ N

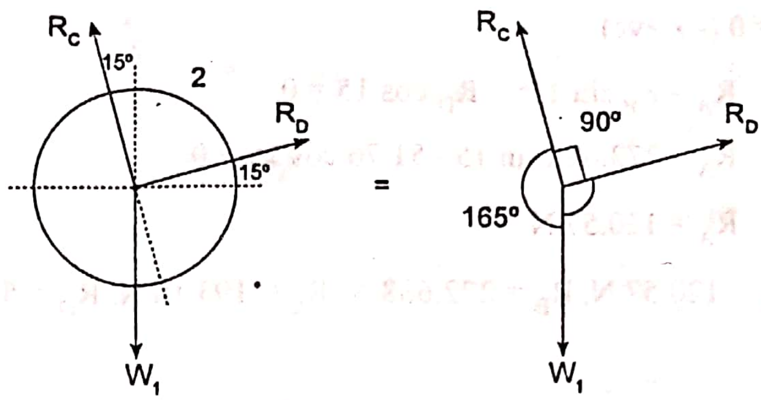
$W_2 = 200$ N

Since roller 2 has only 3 forces, so we will first solve this roller using Lami's theorem.



Using Lami's theorem

$$\frac{R_C}{\sin 105} = \frac{R_D}{\sin 165} = \frac{W_2}{\sin 90}$$



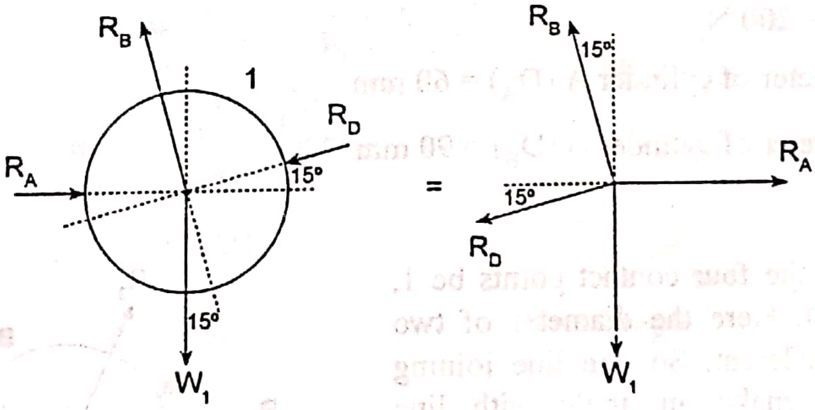
$$R_C = \frac{W_2}{\sin 90} \times \sin 105 = \frac{200}{\sin 90} \times \sin 105$$

$$= 193.18 \text{ N}$$

$$R_D = \frac{W_2}{\sin 90} \times \sin 165 = \frac{200}{\sin 90} \times \sin 165$$

$$= 51.76 \text{ N}$$

Now for further calculation, draw FBD of roller 1.



Here in the FBD of roller 2, the number of force system is greater than 3 (i.e. 4), so, Lami's theorem cannot be used. So, we go for resolution of force and use equation of equilibrium

i.e. $\Sigma F_x = 0 \quad \Sigma F_y = 0$

Now, from FBD

i) $\Sigma F_y = 0$ (↑ +ve)

or, $R_B \cos 15 - R_D \sin 15 - W_1 = 0$

or, $R_B \cos 15 - 51.76 \sin 15 - 250 = 0$

∴ $R_B = 272.688$ N

ii) $\Sigma F_x = 0\ (\rightarrow +ve)$

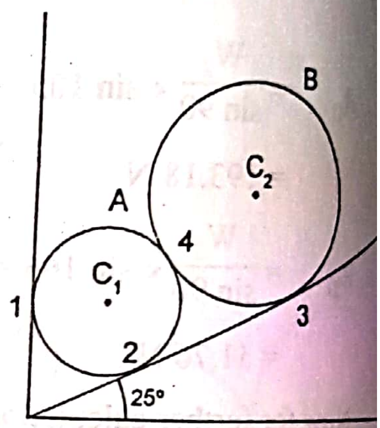
or, $R_A - R_B \sin 15° - R_D \cos 15 = 0$

or, $R_A - 272.688 \sin 15 - 51.76 \cos 15 = 0$

$\therefore$ $R_A = 120.57$ N

$\therefore R_A = 120.57$ N, $R_B = 272.688$ N, $R_C = 193.18$ N, $R_D = 51.76$ N

11. The cylinder A and B rest in an inclined plane which makes an angle of 25° with horizontal as shown in figure below. Determine reaction at contact points. Take



weight of cylinder A (W_A)

= 100 N

weight of cylinder B (W_B)

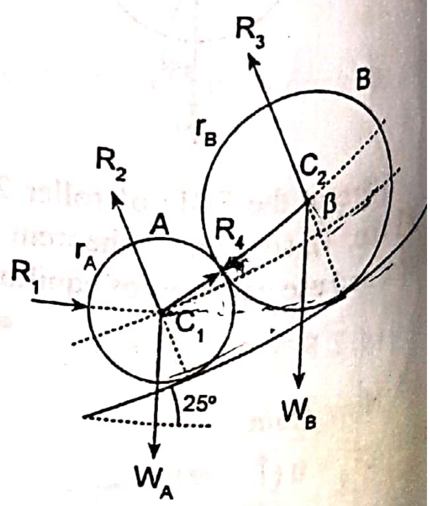
= 200 N

diameter of cylinder A (D_A) = 60 mm

diameter of cylinder B (D_B) = 90 mm

Solution:

Let the four contact points be 1, 2, 3 and 4. Here the diameter of two roller is different. So, the line joining C_1C_2 will make an angle with line parallel to inclined plane (say β).



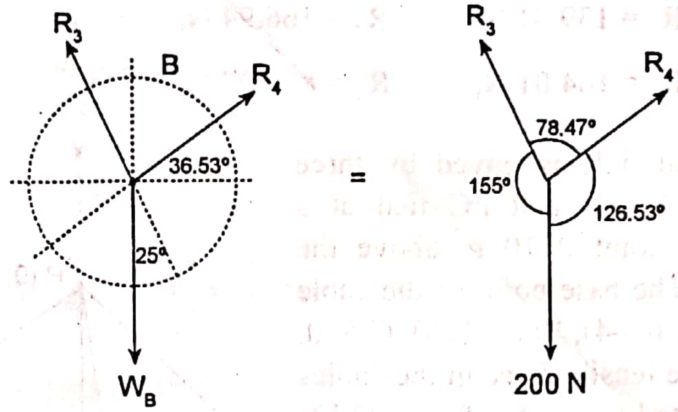
Calculation of angle β

$$\sin\beta = \frac{r_B - r_A}{r_B + r_A}$$

$$= \frac{45 - 30}{45 + 30} = \frac{15}{75} = \frac{1}{5}$$

$\therefore$ $\beta = 11.53°$

First we will solve cylinder B as it includes 3 forces; using Lami's theorem

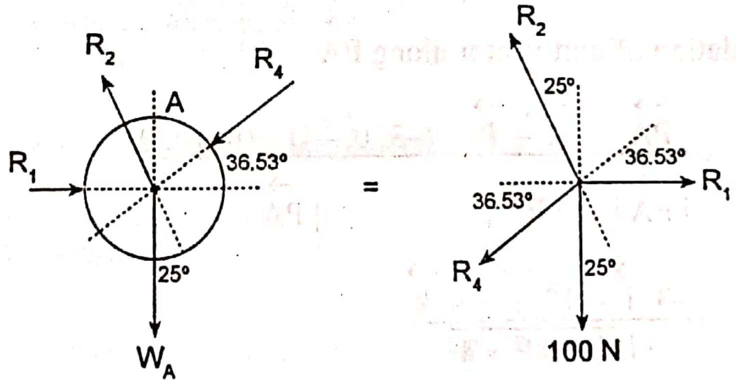


$$\frac{R_3}{\sin 126.53} = \frac{R_4}{\sin 155} = \frac{200}{\sin 78.47}$$

$$\therefore \quad R_3 = \frac{200}{\sin 78.47} \times \sin 126.53 = 164.01 \text{ N}$$

$$R_4 = \frac{200}{\sin 78.47} \times \sin 155 = 86.1972 \text{ N}$$

For further calculation, draw FBD of sphere A.



$\Sigma F_y = 0$ (↑ +ve)

$R_2 \cos 25° - R_4 \sin 36.53 - W_A = 0$

$R_2 \cos 25 - 86.19 \sin 36.53 - 100 = 0$

$\therefore \quad R_2 = 166.94$ N

$\Sigma F_x = 0$ (→ +ve)

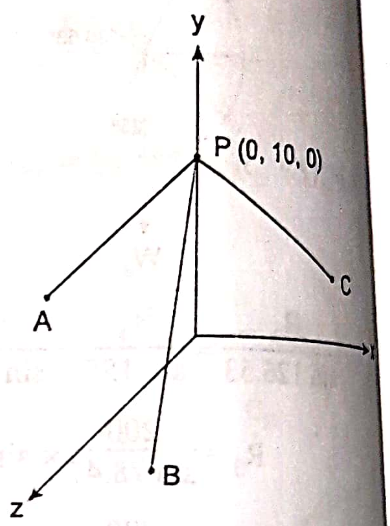
$R_1 - R_2 \sin 25 - R_4 \cos 36.53 = 0$

$R_1 - 166.94 \sin 25 - 86.19 \cos 36.53 = 0$

$\therefore \quad R_1 = 139.81$ N

$$\therefore \quad R_1 = 139.81\ \text{N}, \qquad R_2 = 166.94\ \text{N},$$
$$R_3 = 164.01\ \text{N}, \qquad R_4 = 86.197\ \text{N}$$

12. A vertical pole is guyed by three cables PA, PB and PC tied at a common point P 10 m above the ground. The base point of the cable are A(–3, 0, –4), B(1, –1, 5), C(5, 0, –1). If the tensile force in the cables are adjusted to be 15, 18 and 20 kN respectively. Make calculation for the force on the pole at P.



Solution:

The steps in the solution of this kind is to find out the direction of different vector say $\overrightarrow{PA}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$. For this we will first calculate the unit vector and multiply it by magnitude.

Calculation of unit vector along $\overrightarrow{PA}$

$$\hat{n}_1 = \frac{\overrightarrow{PA}}{|\overrightarrow{PA}|} = \frac{\vec{A} - \vec{P}}{|\overrightarrow{PA}|} = \frac{(-3, 0, -4) - (0, 10, 0)}{|\overrightarrow{PA}|}$$

$$= \frac{-3\vec{i} - 10\vec{j} - 4\vec{k}}{\sqrt{3^2 + 10^2 + 4^2}}$$

$$= -0.268\vec{i} - 0.894\vec{j} - 0.3577\vec{k}$$

$\therefore$ Force along $\overrightarrow{PA} = \vec{F}_1 = \hat{n}_1 . 15$

$$= 15(-0.268\vec{i} - 0.894\vec{j} - 0.3577\vec{k})$$

$$\vec{F}_1 = -4.02\vec{i} - 13.41\vec{j} - 5.36\vec{k}$$

along PB,

$$\hat{n}_2 = \frac{\overrightarrow{PB}}{|\overrightarrow{PB}|} = \frac{\vec{B} - \vec{P}}{|\overrightarrow{PB}|} = \frac{(1, -1, 5) - (0, 10, 0)}{|\overrightarrow{PB}|}$$

$$= \frac{\vec{i} - 11\vec{j} + 5\vec{k}}{\sqrt{1^2 + 11^2 + 5^2}}$$

$$\hat{n}_2 = 0.0824\vec{i} - 0.9072\vec{j} + 0.4123\vec{k}$$

Force along $\vec{PB} = \vec{F}_2 = \hat{n}_2.18$

$$= 18(0.0824\vec{i} - 0.9072\vec{j} + 0.4123\vec{k})$$

$$\vec{F}_2 = 1.48\vec{i} - 16.34\vec{j} + 7.42\vec{k}$$

along PC,

$$\hat{n}_3 = \frac{\vec{PC}}{|\vec{PC}|} = \frac{\vec{C} - \vec{P}}{|\vec{PC}|} = \frac{(5, 0, -1) - (0, 10, 0)}{|\vec{PC}|}$$

$$= \frac{5\vec{i} - 10\vec{j} - \vec{k}}{\sqrt{5^2 + 10^2 + 1^2}}$$

$$\hat{n}_3 = 0.4454\vec{i} - 0.8908\vec{j} - 0.08908\vec{k}$$

Force along $\vec{PC} = \vec{F}_3 = \hat{n}_3.20$

$$= 20(0.4454\vec{i} - 0.8908\vec{j} - 0.08908\vec{k})$$

$$\vec{F}_3 = 8.91\vec{i} - 17.8\vec{j} - 1.78\vec{k}$$

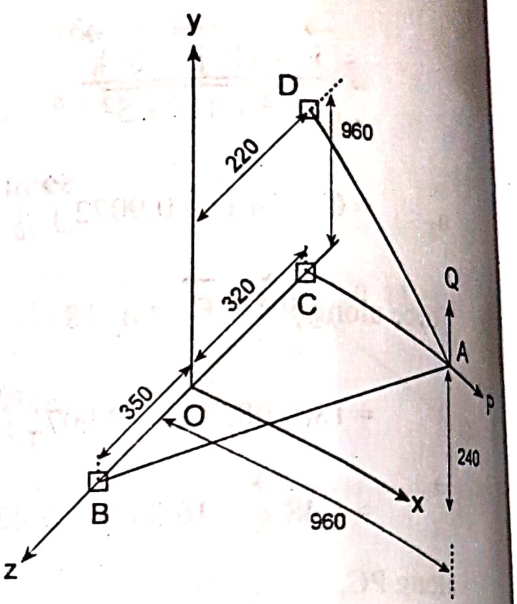
$\therefore$ Force on pole P is

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$= -4.02\vec{i} - 13.41\vec{j} - 5.36\vec{k} + 1.48\vec{i} - 16.34\vec{j} + 7.42\vec{k}$$
$$+ 8.91\vec{i} - 17.8\vec{j} - 1.78\vec{k}$$

$$\therefore \vec{R} = 6.37\vec{i} - 47.54\vec{j} + 0.28\vec{k}$$

13. Three cables are connected at A, where the force P and Q are applied as shown. Knowing that P = 1200 N, determine the range of valves of Q for which cable AD is taut.



Solution:

First, find the number of forces $\vec{AB}$, $\vec{AC}$, $\vec{AD}$, $\vec{P}$, $\vec{Q}$.

Second is to find 3D coordinates of A, B, C and D.

A (0.96, 0.24, 0) [look value parallel to axes]

B (0, 0, 0.35)

C (0, 0, –0.32)

D (0, 0.96, –0.22)

$\vec{P}$ is parallel to x-axis; $P\vec{i}$

$\vec{Q}$ is parallel to y-axis; $Q\vec{j}$

Now, find the force $\vec{AB}$, $\vec{AC}$, $\vec{AD}$, $\vec{P}$ and $\vec{Q}$

Unit vector along AB

$$\hat{AB} = \hat{n}_1 = \frac{\vec{AB}}{|\vec{AB}|} = \frac{\vec{B} - \vec{A}}{|\vec{AB}|}$$

$$= -0.914\,\vec{i} - 0.25\,\vec{j} + 0.33\,\vec{k}$$

Unit vector along AC

$$\hat{AC} = \hat{n}_2 = \frac{\vec{AC}}{|\vec{AC}|} = \frac{\vec{C} - \vec{A}}{|\vec{AC}|}$$

$$= -0.923\,\vec{i} - 0.23\,\vec{j} - 0.3\,\vec{k}$$

Unit vector along AD

$$\hat{AD} = \hat{n}_B = \frac{\vec{AD}}{|\vec{AD}|} = \frac{\vec{D} - \vec{A}}{|\vec{AD}|}$$

$$= -0.78\,\vec{i} + 0.59\,\vec{j} - 0.18\,\vec{k}$$

$$\vec{P} = P\,\vec{i}$$

$$\vec{Q} = Q\,\vec{j}$$

Although $Q = 0$, AD cable remains taut. This is the minimum range of Q. To find the maximum range, we first assume the condition of unstretching AD i.e. $T_{AD} = 0$.

For equilibrium at A;

$$\vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AD} + \vec{P} + \vec{Q} = 0$$

$$\text{if } \vec{T}_{AD} = 0$$

$$\vec{T}_{AB} + \vec{T}_{AC} + \vec{P} + \vec{Q} = 0$$

$$T_{AB}(\hat{AB}) + T_{AC}(\hat{AC}) + 1200\,\vec{i} + Q\,\vec{j} = 0$$

$$T_{AB}(-0.914\,\vec{i} - 0.25\,\vec{j} + 0.33\,\vec{k}) + T_{AC}(-0.923\,\vec{i} - 0.23\,\vec{j} - 0.3\,\vec{k}) + 1200\,\vec{i} + Q\,\vec{j} = 0$$

Equating the coefficient

$-0.914T_{AB} - 0.923T_{AC} + 1200 = 0$ ......(i)

$-0.25T_{AB} - 0.23T_{AC} + Q = 0$ ......(ii)

$0.33T_{AB} - 0.3T_{AC} = 0$ ......(iii)

Solving we get,

$T_{AB} = 621.98$ N $\quad T_{AC} = 684.185$ N

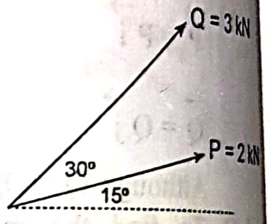
$Q = 312.859$

$\therefore$ Range of Q is $0 < Q < 312$ N.s

Practice Questions

1. The magnitude of two forces is such that when acting at right angle produce a resultant force of $\sqrt{20}$ and when acting at 60° produce a resultant equal to $\sqrt{28}$. Workout the two forces.

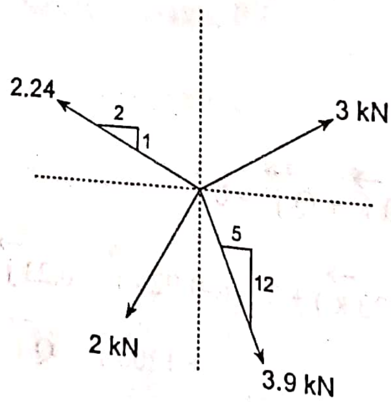
2. Use triangle law to find the resultant. Also verify using parallelogram law. Verify it graphically.



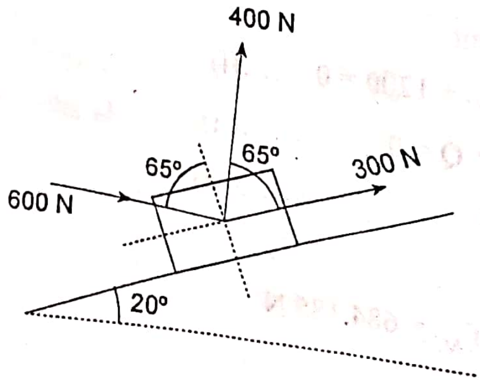
3. The resultant of two forces P and Q acting at a point is R. The resultant R gets doubled when Q is either doubled or its direction is reversed. Show that $P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$.

4. Determine the resultant

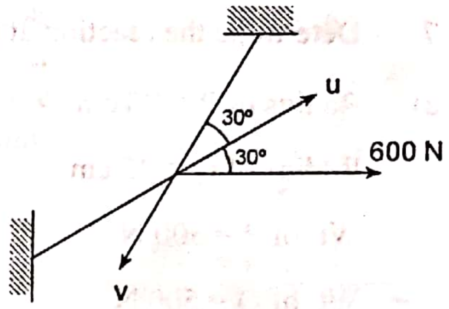
a)



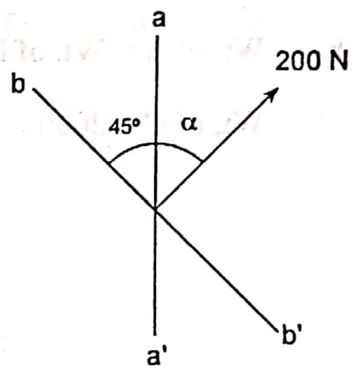
b)



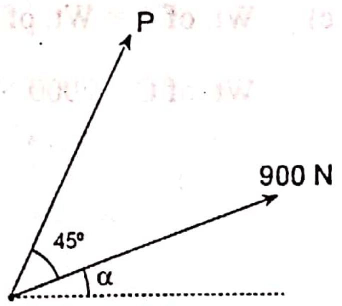
5. a) Resolve horizontal 600 N force along u and v axes.



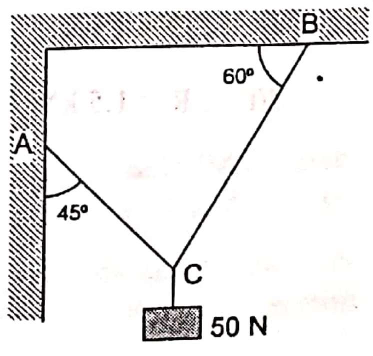
b) A 200 N force is to be resolved into components along lines a–a′ and b–b′. If the component along a–a′ is to be 150 N, determine the value of α. Also, find the corresponding value of component along b–b′.



c) Knowing that magnitude of P is 600 N, determine a) angle α if the resultant R of the two forces is to be vertical; b) the corresponding magnitude of R.



6. Determine the tension in string.



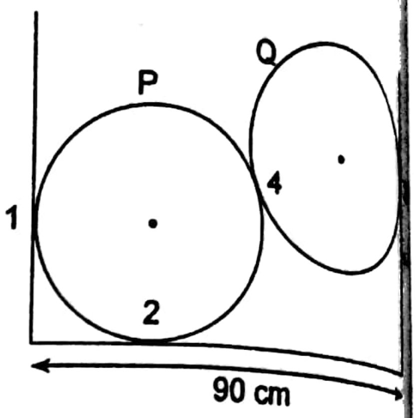
7. **Determine the reaction at all contact points.**

a) Radius of P = 25 cm

Radius of Q = 25 cm

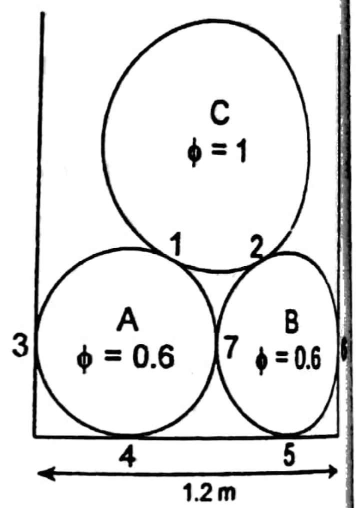
Wt. of P = 500 N

Wt. of Q = 500 N



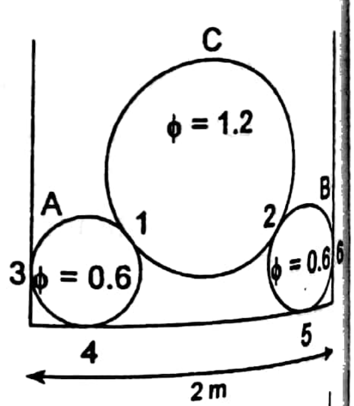
b) Wt. of A = Wt. of B = 1000 N

Wt. of C = 2000 N



c) Wt. of A = Wt. of B = 1000 N

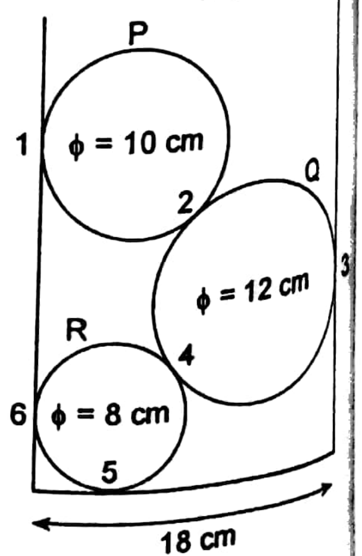
Wt. of C = 2000 N



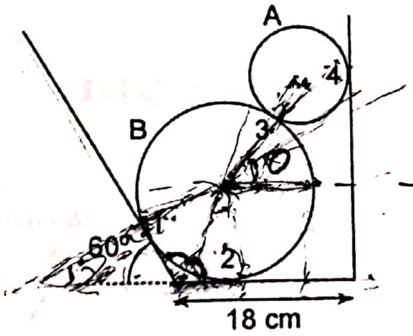
d) Wt. of P = 2 kN

Wt. of Q = 4 kN

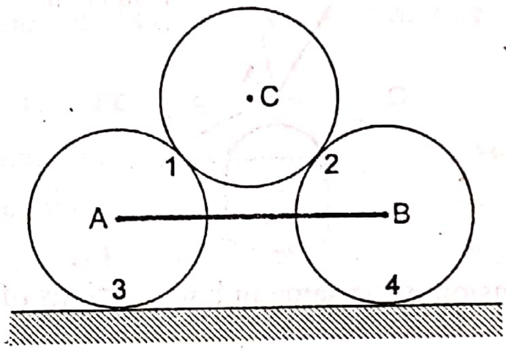
Wt. of R = 1.5 kN



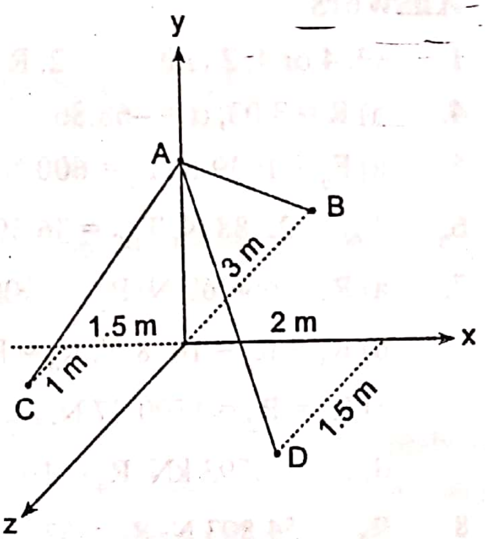
8. Two cylinders A and B rest in a channel as shown in figure. The cylinder A has diameter of 10 cm and weighs 200 N whereas the cylinder B has diameter of 18 cm and weighs 500 N. Determine the contact forces.



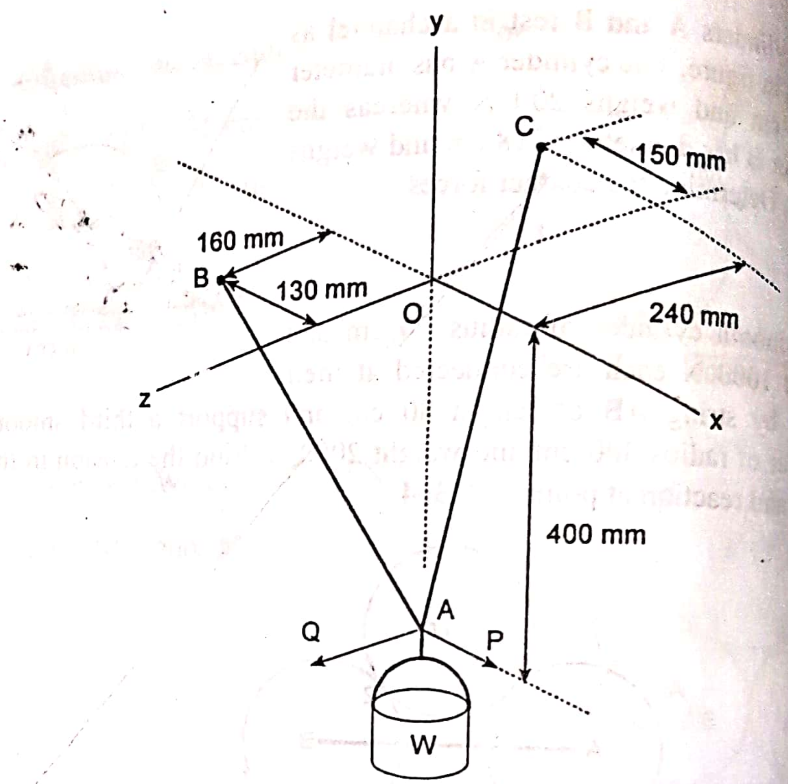
9. Two smooth cylinder of radius 30 cm and weight 1000 N each are connected at their centre by string AB of length 80 cm and support a third smooth cylinder of radius 300 cm and weight 2000 N. Find the tension in the string and reaction at point 1, 2, 3, 4.



10. In a system shown a 5 m long beam is held in vertical position AO by three guy wire AB, AC and AD. If a tension equivalent to 600 N is induced in AD and the resultant force at A is to be vertical. Calculate tension in wire AC and AB.



11. A container of weight W is suspended from ring A. Cable BAC passes through the ring and is attached to fixed supports at B and C. Two forces $P = P\vec{i}$ and $Q = Q\vec{k}$ are applied to the ring to maintain the container in the position shown. Knowing that W = 376 N, determine P and Q.



(Hint: The tension is the same in both portions of cable BAC.)

Answers

1. (2, 4 or 4, 2 unit 2. $R = 4.836$, $\alpha = 33.06$
4. a) $R = 3.03$, $\alpha = -68.86$ b) $R = 1019$ N, $\alpha = 26.01$
5. a) $F_u = 1039$ N; $F_v = 600$ N b) $\alpha = 103°$, 276 N c) 72.2°, 1391 N
6. $T_{AC} = 25.88$ N, $T_{BC} = 36.59$ N
7. a) $R_1 = 666.67$ N, $R_2 = 1000$ N, $R_3 = 666.67$ N, $R_4 = 833.3$ N
   b) $R_1 = R_2 = 1078$ N, $R_4 = R_5 = 2000$ N
   c) $R_1 = R_2 = 1590.87$ N, $R_3 = R_6 = 1237.38$ N, $R_4 = R_5 = 1999.87$ N
   d) $R_2 = 2.593$ kN, $R_4 = 10$ kN, $R_6 = 8$ kN
8. $R_1 = 154.893$ N, $R_2 = 622.55$ N, $R_3 = 240.82$ N, $R_4 = 134.14$ N
9. $R_1 = 1341.2$ N, $R_2 = 1341.2$ N, $R_3 = R_4 = 2000$ N, $T = 893.95$ N
10. $T_{AB} = 591.66$ N, $T_{AC} = 761.24$ N
11. $P = 131.2$ N, $Q = 29.6$ N

Force ac

3.1 Differen

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concentrated loac
force (weight) are

Characteristi

- Force hav
- Force hav
- Force is
  action.

3.1.1 Point f

A finite

3.1.2 Body f

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gravitational fo

3.1.3 Surfac

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Chapter 3

Force acting on particles and rigid body

3.1 Different types of force

Force is defined as an external agent which change or tends to change the speed, direction or shape of system. Different types of forces are concentrated load (point force), surface traction (pressure load) and body force (weight) are applied in different shapes structure.

Characteristics of force

- Force have magnitude and direction, so it is a vector quantity.
- Force have point of application.
- Force is transmissible vector i.e. it can be moved along its line of action.

3.1.1 Point force

A finite force which is assumed to act through point.

3.1.2 Body force

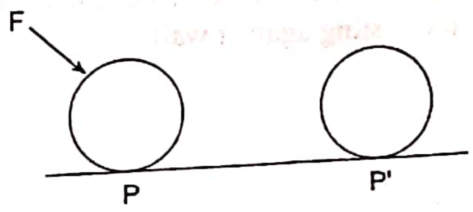
A force which acts on each element of body is called body force. e.g. gravitational force, inertia force, electromagnetic force etc .

3.1.3 Surface force

The force which acts on the surface or area elements of the body is called surface forcc. When the area considered lies on the actual boundary of the body, the surface force distribution is termed as surface traction.

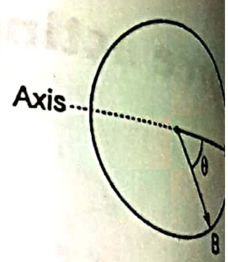
3.1.4 Translation force

The force which moves or tends to move a body from one point to another point is called translation force.



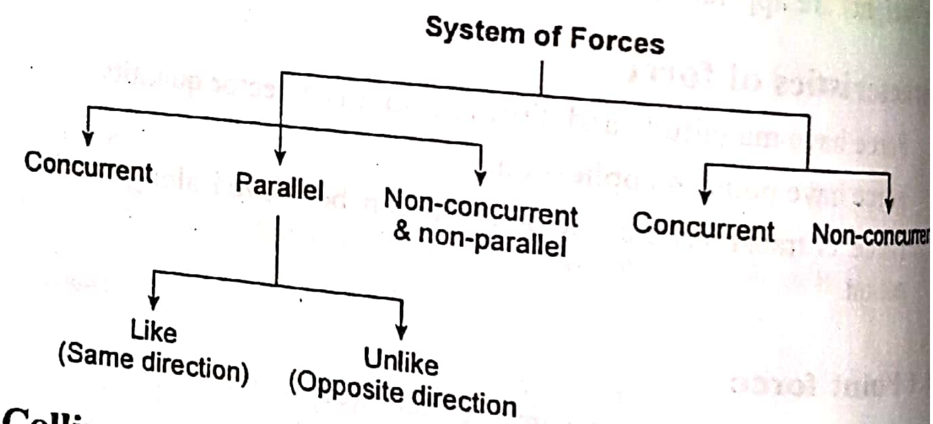
3.1.5 Rotational force

The force which rotates or tends to rotate a body around a central axis is called rotational force. e.g. turning of bolt by wrench.



3.2 System of forces

When several forces of different magnitude and direction act up body, they constitute a force system considering the plane in which f are applied and depending upon the position of line of action, force ma classified as follows:

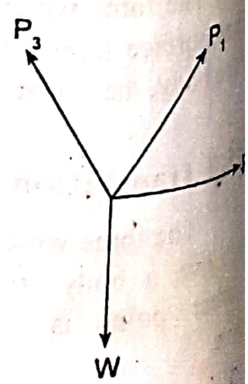


- **Collinear force**

The line of action of all forces lie along the same straight line. force on rope.

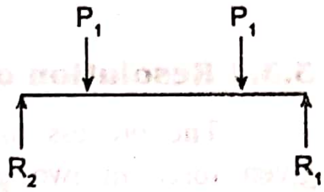
- **Coplanar concurrent force**

All forces lie in the same plane having different directions but their lines of action act at a point, called point of concurrency. e.g. forces on rod resting against wall.



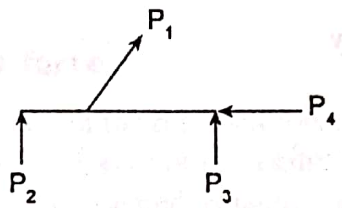
• Coplanar parallel force

The line of action of all forces are parallel to each other and lie in single plane. e.g. system of vertical load on beam.



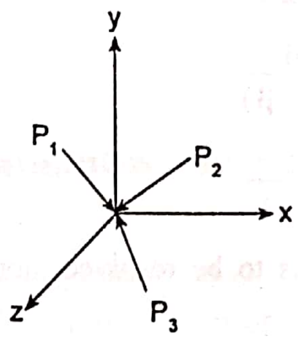
• Coplanar non concurrent and non parallel forces

All the forces lie in the same plane but their line of action do not pass through a single point and force are not parallel. e.g. forces on a ladder resting against wall and person climbing.



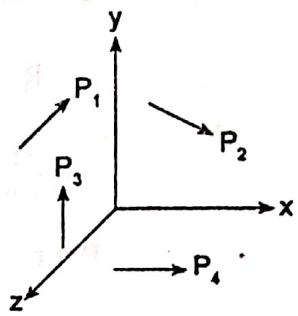
• Non-coplanar concurrent forces

All forces do not lie in the same plane but their line of action passes through single point. e.g. forces on a tripod carrying a camera.



• Non-coplanar non concurrent forces

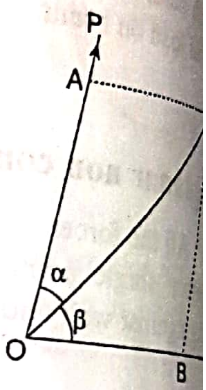
All forces do not lie in the same plane, and their lines of action do not meet at a single point. e.g. forces acting on a moving bus.



3.3 Resolution and composition of force

3.3.1 Resolution of forces

The process to find the components of given force in two given directions is called resolution. These component forces will produce same effect on the body as the given single force.



Let us find the component of force R in the direction making angles α and β with its line of action. Let the resolved force be P and Q as shown in figure.

From trigonometry

$\angle OCB = \angle AOC = \alpha$

$\angle OBC = 180 - (\alpha + \beta)$

Applying sin rule to ΔOBC

$$\frac{OB}{\sin\alpha} = \frac{BC}{\sin\beta} = \frac{OC}{\sin(180 - \alpha + \beta)}$$

$$\frac{Q}{\sin\alpha} = \frac{P}{\sin\beta} = \frac{R}{\sin(\alpha + \beta)}$$

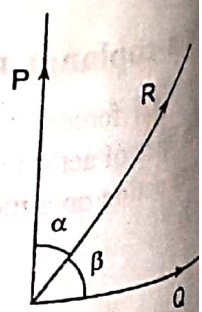
$$\therefore \quad P = R\frac{\sin\beta}{\sin(\alpha+\beta)}$$

$$Q = R\frac{\sin\alpha}{\sin(\alpha+\beta)}$$

When the force R is to be resolved along perpendicular directi then

$$\alpha + \beta = 90°$$

$$\therefore \quad \alpha = 90° - \beta$$



$$\therefore \quad P = R\frac{\sin\beta}{\sin(\alpha+\beta)}$$

$$= R\frac{\sin\beta}{\sin 90}$$

$$\therefore \quad P = r\sin\beta$$

Also, $Q = R\dfrac{\sin\alpha}{\sin(\alpha+\beta)}$

$$= R\frac{\sin\alpha}{\sin 90}$$

$$= R\sin\alpha$$

$$= R\sin(90-\beta)$$

$$= R\cos\beta$$

$\therefore$ for perpendicular axes

$Q = R\cos\beta$ and $P = R\sin\beta$

or $Q = R\sin\alpha$ and $P = R\cos\alpha$

(Note the difference for different reference angle.)

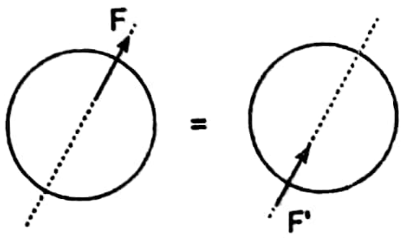
3.3.2 Composition of force

The process of finding out the resultant force, out of number of given forces, is called composition of forces or sometimes compounding of forces. The resultant of forces can be calculated using following methods

1. Parallelogram law
2. Triangle law
3. Method of resolution

(The numerical method and examples has been discussed in chapter two.)

3.4 Principle of transmissibility and equivalent force



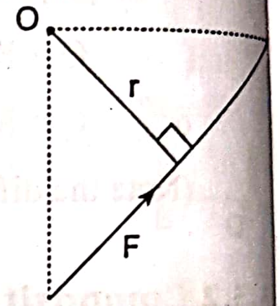
The principle of transmissibility states that the conditions of equilibrium or motion of rigid body will remain unchanged if a force F acting at a given point of the rigid body is replaced by a force F of same magnitude and same direction, but acting at different point, provided that the two forces have the same line of action as in figure. The two forces F and F' have the same effect on the rigid body and are said to be equivalent.

3.5 Moments and couples

3.5.1 Moments of force

A coplanar non concurrent force system consists of a set of forces that lie in the same plane but the line of action of all the forces do not meet at a single point.

Moment of force about a point is defined as the turning tendency of a force about that point. It is measured by the product of force and the perpendicular distance of the line of action of force from that point.



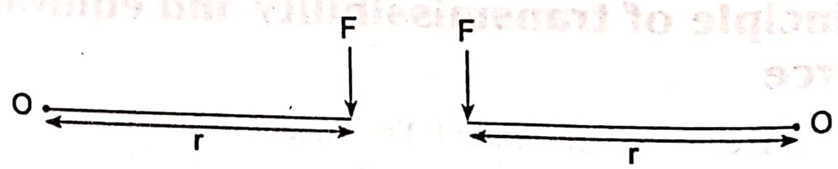
Moment of force F about O

$$\vec{M}_O = \vec{r}_{F/O} \times \vec{F}$$

Here $\vec{r}_{F/O}$ is the distance of point of application of force with respect to point O, about which moment is to be calculated.

Scalarly, $M_O = r \times F$

The point O is called moment centre and the distance r is called moment arm. The tendency of turning of the body due to the moment of force may be clockwise or anticlockwise. The corresponding moments are referred as clockwise moment and anticlockwise moment.



For a 3D case:

$$M_O = \vec{r} \times \vec{F}$$

$$M_O = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

Notes:

- Moment of force about a point

$$M_O = \vec{r} \times \vec{F}$$

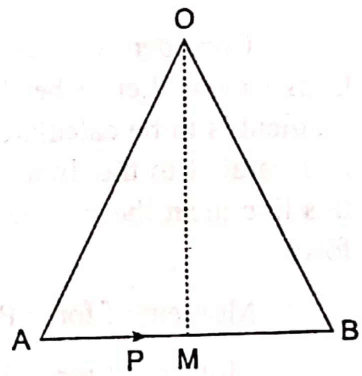
(Be careful to select $\vec{r}$)

- Moment of force about an axis $\vec{i}$, $\vec{j}$ and $\vec{k}$ is

$$\vec{M} \cdot \vec{i}\ , \vec{M} \cdot \vec{j}\ , \vec{M} \cdot \vec{k}$$

In general the moment of $\vec{F}$ about any axis AB will be $\vec{M} \cdot \hat{n}$ where M is the moment of force $\vec{F}$ about any point on the axis AB and $\hat{n}$ is the unit vector along AB.

3.5.1.1 Graphical representation of moment



Let a force $\vec{P}$ be represented by vector AB and O be the point about which the moment is to be determined.

Moment of force P about O

$$M_P = P \times OM$$

$$= AB \times OM$$

$$= 2 \times \frac{1}{2} \times AB \times OM$$

$$= 2 \times \left(\frac{1}{2} \times AB \times OM\right)$$

$$M_P = 2 \times \text{Area of triangle AOB}$$

Hence the moment of force about any point is geometrically equal to twice the area of the triangle whose base is the line that represent the force and whose vertex is the point about which the moment is required to calculate.

3.5.1.2 Principle of moments

A body acted upon by a number of coplanar forces will be in equilibrium, if the algebraic sum of the moments of all forces about a point in the same plane is zero.

i.e. $\Sigma \text{moment} = 0$

$\Sigma M = 0$

i.e. clockwise moment = anticlockwise moment

Note: For a body to be in equilibrium

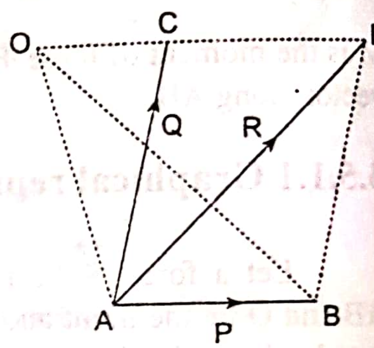
$\Sigma F_x = 0$

$\Sigma F_y = 0$

$\Sigma F_z = 0$

3.5.1.3 Varignon's Theorem

It states that "Moment of a resultant of two forces, about a point lying in the plane of the forces, is equal to the algebraic sum of moments of these forces about the same point."



Consider two concurrent forces P and Q as shown. Let O be the point about which moment is to be calculated. Through O draw a line parallel to the direction of force P and let this line meet the line of action of force Q at C. Suppose R be the resultant force.

Moment of force P about O = $2 \times \Delta AOB$

Moment of force Q about O = $2 \times \Delta AOC$

Moment of force R about O = $2 \times \Delta AOD$

From geometry:

$\Delta AOD = \Delta AOC + \Delta ACD$

$\Delta AOD = \Delta AOC + \Delta ABD$ (why?)

$\Delta AOD = \Delta AOC + \Delta AOB$ (why?)

$\therefore \quad \Delta AOD = \Delta AOC + \Delta AOB$

$$\frac{M_R}{2} = \frac{M_Q}{2} = \frac{M_P}{2}$$

$\therefore \quad M_R = M_Q + M_P$

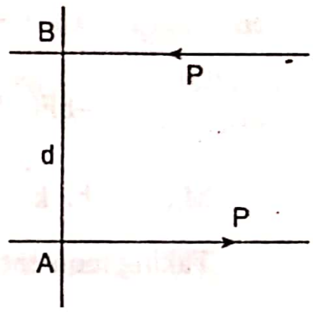
this principle can be extended for number of forces "moment of resultant of number of forces about a point lying in the plane of force is equal to the algebraic sum of the moment of these forces about the same point.

$$\vec{r} \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \vec{r} \times \vec{F}_3 + \ldots$$

$$\vec{r} \times \vec{R} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \vec{r} \times \vec{F}_3 + \ldots$$

3.5.2 Moment of a couple

Two parallel forces equal in magnitude but opposite in direction separated by a finite distance are said to form a couple.



Moment of couple = $P \times d$

Sign convention

- Clockwise moment is negative
- Anticlockwise moment is positive.

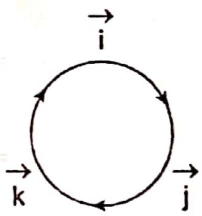
If we go for vector method

$$\text{Moment of couple} = \vec{r}_{A/B} \times P\vec{i}$$

$$= -d\vec{j} \times P\vec{i}$$

$$= -dP(-\vec{k})$$

$$= dP\vec{k}$$



(why $\vec{r}_{A/B}$ but not $\vec{r}_{B/A}$? Discuss in class)

3.5.2.1 Characteristic of couple

a) algebric sum of the forces constituting the couple is zero.

b) couple cannot be balanced by single force.

c) couple is a free vector.

3.5.2.2 Couple as a free vector

The only effect of a couple is to produce a moment, It is possible to represent a couple with a vector, the couple vector, which is equal to the moment of the couple. The couple vector is a free vector and will be represented by a special symbol to distinguish it from force vector.

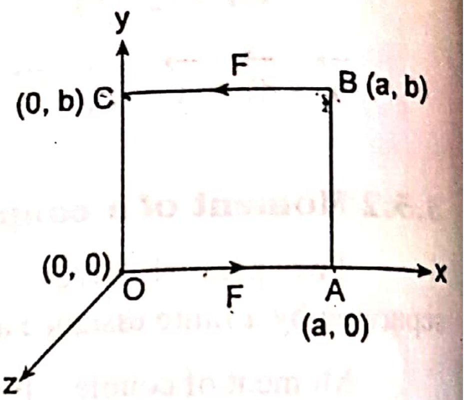
Proof:

To prove couple is a free vector, we need to show moment of couple about any point will be same.

$r_{BC} = r_C - r_B$

$r_{OC} = r_C - r_O$

Taking moment about O:



$$M_O = \vec{r} \times \vec{F}$$

$$= \vec{r}_{C/O} \times -F\vec{i}$$

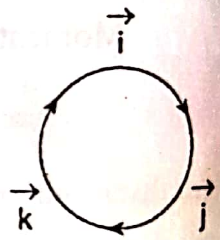
$$= [(0, b) - (0, 0)] \times -F\vec{i}$$

$$= b\vec{j} \times -F\vec{i}$$

$$= -bF(-\vec{k})$$

$$M_O = bF\vec{k}$$

Taking moment about A



$$M_A = \vec{r} \times \vec{F}$$

$$= \vec{r}_{C/A} \times -F\vec{i}$$

$$= [(0, b) - (a, 0)] \times -F\vec{i}$$

$$= (-a\vec{i} + b\vec{j}) \times -F\vec{i}$$

$$= aF\vec{i} \times \vec{i} - bF(-\vec{k})$$

$$= 0 + bF\vec{k}$$

$$M_A = bF\vec{k}$$

Taking moment about B

$\vec{r}_{AB}$

$r_A - r_B$

$r_{BA} = r_A - r_B$

$$M_B = \vec{r} \times \vec{F}$$

$$= \vec{r}_{A/B} \times F\vec{i}$$

$$= [(a, 0) - (a, b)] \times F\vec{i}$$

$$= -b\vec{j} \times F\vec{i}$$

$$= -bF(-\vec{k})$$

$$M_B = bF\vec{k}$$

Since $M_O = M_A = M_B$, it is proved that couple is a free vector.

3.5.3 Resultant of coplanar, non concurrent force system

Different steps which are to be followed to determine the resultant of coplanar, non-concurrent force system in magnitude, direction and position are listed below:

Steps:

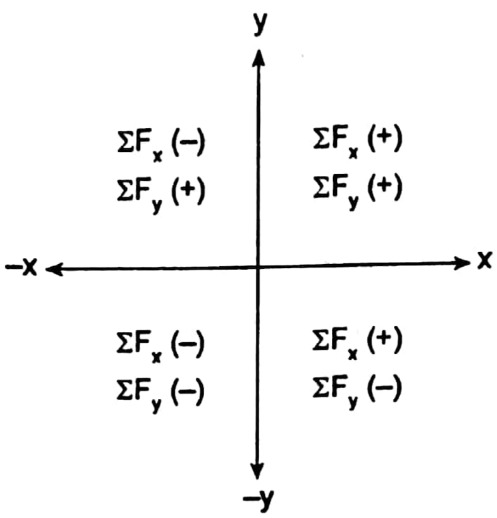
1. Resolve the given forces horizontally (ΣF_x) and vertically (ΣF_y).
2. Determine magnitude of resultant

   $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

3. Find the direction of resultant force with respect to x-axis using

   $$\tan\alpha = \frac{\Sigma F_y}{\Sigma F_x} \Rightarrow \alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x}$$

   Depending upon the sign of ΣF_x and ΣF_y decide in which quadrant the resultant lies.



4. Obtain the algebric sum of moments of all forces about any point say O.

5. Mark the position of resultant such that it produces the sa direction of moment about point O.
6. Apply varignon's theorem of moment to find the exact position resultant.

$\Sigma M_O = R \times d$

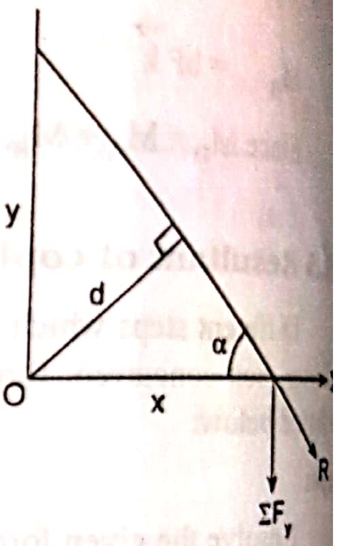
also, $\Sigma M_O = \Sigma F_y \times x$

also, $\sin\alpha = \dfrac{d}{x}$

$\therefore \quad d = x \sin\alpha$

also, $\tan\alpha = \dfrac{y}{x}$

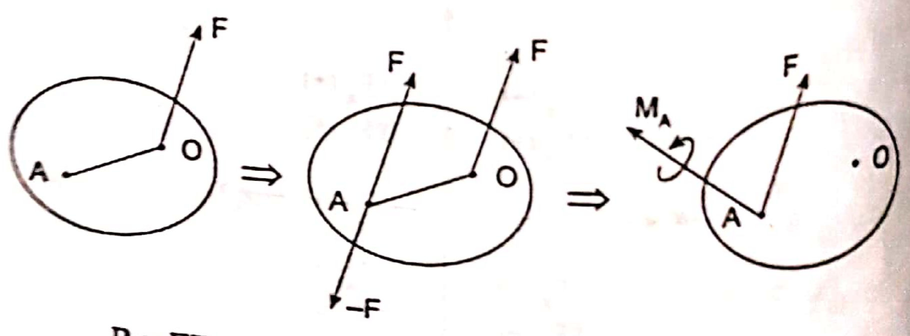
$y = x \tan\alpha$



3.6 Resolution of force into forces and a couple

3.6.1 Reducing a force system to a force and a couple at a given point A

The force is the resultant R of the system and is obtained by addi the various forces; the moment of the couple is the moment resultant of th system and is obtained by adding the moments about A of the vario forces.



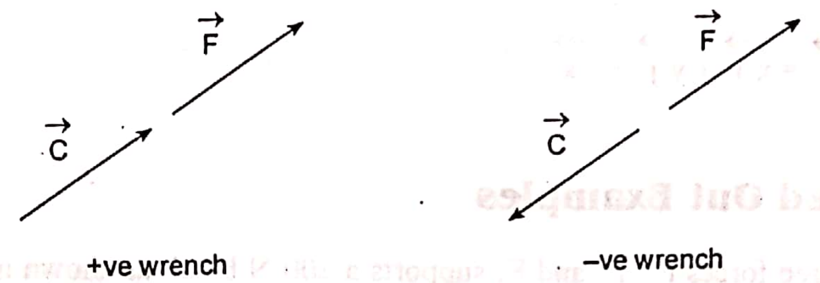
$R = \Sigma F$

$M_A^R = \Sigma r \times F$

where the position vector r is drawn from A to any point on the line of action of F.

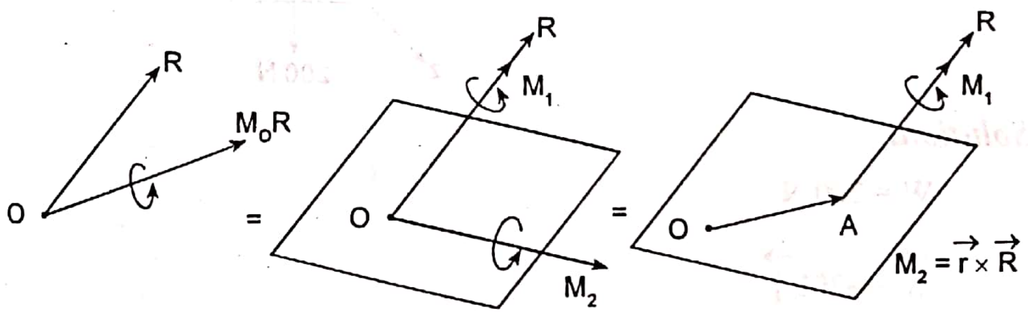
3.6.2 Reduction of a system of forces to a wrench

A wrench consists of a force and a couple with the same line of action. If the directions are same, it is called positive wrench and if the directions are opposite, it is called negative wrench.



A zero wrench is a condition in which both forces and couple are zero.

If a given system is comprised of forces which are not concurrent, coplanar or parallel, the equivalent force-couple system at a point O will consist of a force R and a couple vector M_O^R which are not mutually perpendicular [to check use dot product]. In this case the system cannot be reduced to a single force but can be reduced to a wrench – the combination of a force R and a couple vector M_1 directed along a common line of action called axis of wrench. The ratio $p = \frac{M_1}{R}$ is called pitch of wrench.



Steps:

1. Reduce the given system to an equivalent force couple system $[\vec{R}, \vec{M}_O^R]$ typically located at origin O.
2. Determine the pitch p

$$p = \frac{M_1}{R} = \frac{\vec{R} \cdot \vec{M}_O^R}{R^2}$$

Couple vector $\vec{M}_1 = p.\vec{R}$

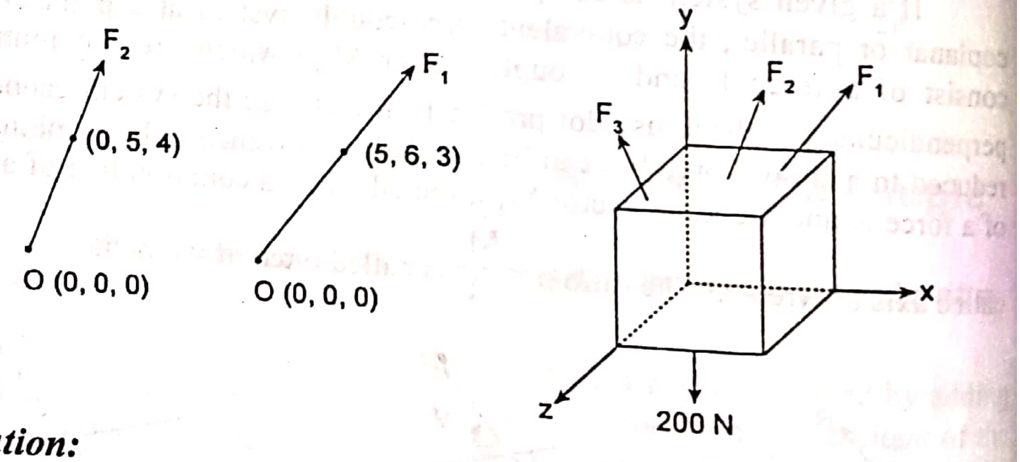
3. Express that the moment about O of the wrench is equal to the moment resultant M_O^R of the force-couple system at O.

$$\vec{M}_1 + \vec{r} \times \vec{R} = \vec{M}_O^R$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Worked Out Examples

1. Three forces F_1, F_2 and F_3 supports a 200 N block as shown in figure. If the magnitude of F_1 and F_2 are 60 N and 50 N respectively. Find the magnitude and direction of F_3 for equilibrium.



Solution:

$$W = 200 \text{ N}$$

$$\vec{W} = -200\vec{j}$$

$$\text{Unit vector along } F_1 = \hat{n}_1 = \frac{\vec{F}_1}{|\vec{F}_1|}$$

$$= \frac{5\vec{i} + 6\vec{j} + 3\vec{k}}{\sqrt{5^2 + 6^2 + 3^2}} = \frac{5\vec{i} + 6\vec{j} + 3\vec{k}}{8.3667}$$

$$\hat{F}_1 = \hat{n}_1 = 0.597\vec{i} + 0.717\vec{j} + 0.358\vec{k}$$

Unit vector along $F_2 = \hat{n}_2 = \dfrac{\vec{F_2}}{|\vec{F_2}|}$

$$= \frac{0\vec{i} + 5\vec{j} + 4\vec{k}}{\sqrt{0^2 + 5^2 + 4^2}} = \frac{5\vec{j} + 4\vec{k}}{6.403}$$

$$\hat{F}_2 = n_2 = 0.78\vec{j} + 0.624\vec{k}$$

$$\vec{F_1} = 60.\hat{F}_1$$

$$= 60(0.597\vec{i} + 0.717\vec{j} + 0.358\vec{k})$$

$$\vec{F_1} = 35.82\vec{i} + 43.02\vec{j} + 21.48\vec{k}$$

$$\vec{F_2} = 50.\hat{F}_2$$

$$= 50.(0\vec{i} + 0.78\vec{j} + 0.624\vec{k})$$

$$\vec{F_2} = 0\vec{i} + 39\vec{j} + 31.2\vec{k}$$

For equilibrium,

$$\Sigma\vec{F} = 0$$

$$\vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{W} = 0$$

$$35.82\vec{i} + 43.02\vec{j} + 21.48\vec{k} + 39\vec{j} + 31.2\vec{k} + \vec{F_3} - 200\vec{j} = 0$$

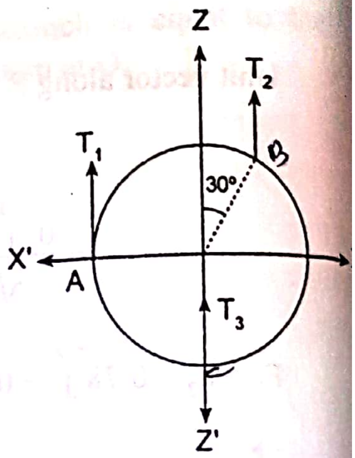
or, $35.82\vec{i} - 117.98\vec{j} + 52.68\vec{k} + \vec{F_3} = 0$

$$\therefore \vec{F_3} = -35.82\vec{i} + 117.98\vec{j} - 52.68\vec{k}$$

$$|\vec{F_3}| = \sqrt{(-35.82)^2 + (117.98)^2 + (-52.68)^2}$$

$$= 134\text{ N}$$

2. **A homogeneous plate of mass 30 kg is supported by three vertical wire as shown in figure. The plate is 60 cm in diameter. Determine the tension in wire.**



Solution:

The plate rest on XZ plane.

A (–0.3, 0, 0)

B (r sin 30, 0, r cos 30)

or, B(0.3 × 0.5, 0, 0.3 × 0.866)

or, B(0.15, 0, 0.26)

C (0, 0, –0.3)

∴ A(–0.3, 0, 0), B(0.15, 0, 0.26), C(0, 0, –0.3)

For equilibrium

$\Sigma F = 0$ and $\Sigma M = 0$

$$\Sigma \vec{F} = 0$$

$$T_1 \vec{j} + T_2 \vec{j} + T_3 \vec{j} - w \vec{j} = 0$$

$$T_1 + T_2 + T_3 = 30 \times 9.81$$

$$T_1 + T_2 + T_3 = 294.3 \text{ ......(i)}$$

$$\Sigma \vec{M}_O = 0$$

$$\vec{r}_{A/O} \times T_1 + \vec{r}_{B/O} \times T_2 + \vec{r}_{C/O} \times T_3 = 0$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -0.3 & 0 & 0 \\ 0 & T_1 & 0 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.15 & 0 & 0.26 \\ 0 & T_2 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & -0.3 \\ 0 & T_3 & 0 \end{vmatrix} = 0$$

$$-0.3T_1\vec{k} - 0.26T_2\vec{i} + 0.3T_3\vec{i} + 0.15T_2\vec{k} = 0$$

$$(0.3T_3 - 0.26T_2)\vec{i} + (0.15T_2 - 0.3T_1)\vec{k} = 0$$

Equating coefficient of like vectors

$$0.3T_3 - 0.26T_2 = 0 \Rightarrow T_3 = 0.8667T_2 \quad \text{......(ii)}$$

$$0.15T_2 - 0.3T_1 = 0 \Rightarrow T_1 = 0.5T_2 \quad \text{......(iii)}$$

From eq. (i), (ii) and (iii)

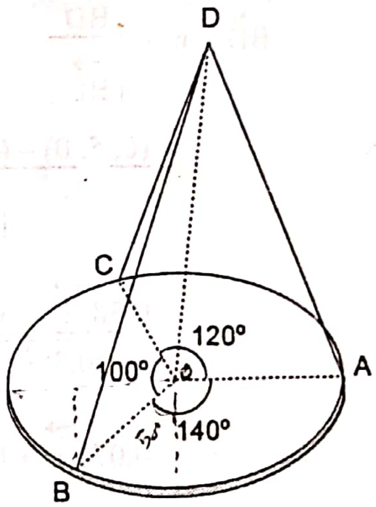
$$0.5T_2 + T_2 + 0.8667T_2 = 294.3$$

$$\therefore T_2 = \frac{294.3}{2.3667} = 124.35 \text{ N}$$

$$T_1 = 62.175 \text{ N}$$

$$T_3 = 107.77 \text{ N}$$

3. A homogeneous circular plate of mass 50 kg is supported by three wires. The angular distance between the points of attachment on the circumference of the plate with respect to centre of the plate makes an angle of 100° degree while other two angular distance are 120° and 140° as shown in figure. The three wire are attached to a single point on the ceiling which is 5 m vertically above the centroid of the plate. The plate has diameter of 1 m. Calculate the force developed in each wire.



Solution:

A (0.5, 0, 0)

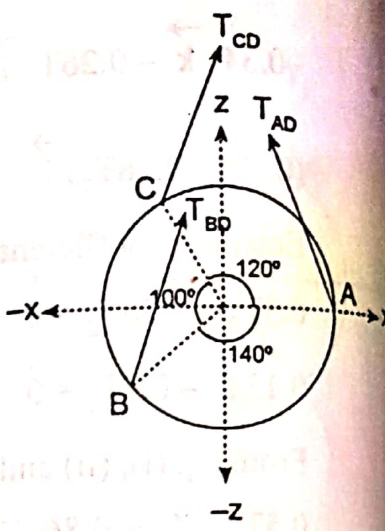
B(–0.5 sin 50, 0, –0.5 cos 50) = (–0.383, 0, –0.321)

C(–0.5 cos 60, 0, 0.5 sin 60) = (–0.25, 0, 0.433)

D (0, 5, 0)

$\therefore$ A(0.5, 0, 0), B(–0.383, 0, –0.321),

C(–0.25, 0, 0.433), D(0, 5, 0)



Unit vector along AD

$$\hat{AD} = \hat{n}_1 = \frac{\vec{AD}}{|\vec{AD}|} = \frac{\vec{D} - \vec{A}}{|\vec{AD}|}$$

$$= \frac{(0, 5, 0) - (0.5, 0, 0)}{\sqrt{(-0.5)^2 + 5^2}}$$

$$= \frac{-0.5\ \vec{i} + 5\ \vec{j}}{5.024}$$

$$= -0.1\ \vec{i} + 0.99\ \vec{j}$$

$$\vec{T}_{AD} = T_{AD}(-0.1\ \vec{i} + 0.99\ \vec{j}\ )$$

Unit vector along BD

$$\hat{BD} = \hat{n}_2 = \frac{\vec{BD}}{|\vec{BD}|}$$

$$= \frac{(0, 5, 0) - (-0.38, 0, -0.32)}{|\vec{BD}|}$$

$$= \frac{0.38\ \vec{i} + 5\ \vec{j} + 0.32\ \vec{k}}{\sqrt{0.38^2 + 5^2 + 0.32^2}} = \frac{0.38\ \vec{i} + 5\ \vec{j} + 0.32}{5.024}$$

$$= 0.076\ \vec{i} + 0.99\ \vec{j} + 0.064\ \vec{k}$$

$$\therefore\ \vec{T}_{BD} = T_{BD}(0.076\ \vec{i} + 0.99\ \vec{j} + 0.064\ \vec{k}\ )$$

Unit vector along CD

$$\hat{CD} = \hat{n}_3 = \frac{\vec{CD}}{|\vec{CD}|}$$

$$= \frac{(0, 5, 0) - (-0.25, 0, 0.433)}{|\vec{CD}|}$$

$$= \frac{0.25\vec{i} + 5\vec{j} - 0.433\vec{k}}{\sqrt{0.25^2 + 5^2 + 0.433^2}} = \frac{0.25\vec{i} + 5\vec{j} - 0.433\vec{k}}{5.024}$$

$$= 0.05\vec{i} + 0.99\vec{j} - 0.086\vec{k}$$

$$\therefore \vec{T}_{CD} = T_{CD}(0.05\vec{i} + 0.99\vec{j} - 0.086\vec{k})$$

Now for equilibrium

$$\Sigma\vec{F} = 0$$

$$\vec{T}_{AD} + \vec{T}_{BD} + \vec{T}_{CD} + \vec{W} = 0$$

$$T_{AD}(-0.1\vec{i} + \vec{j}) + T_{BD}(0.076\vec{i} + \vec{j} + 0.064\vec{k})$$
$$+ T_{CD}(0.05\vec{i} + \vec{j} - 0.086\vec{k}) - 490.5\vec{j} = 0$$

Equating the coefficient of like vectors

$-0.1T_{AD} + 0.076T_{BD} + 0.05T_{CD} = 0$ ......(i)

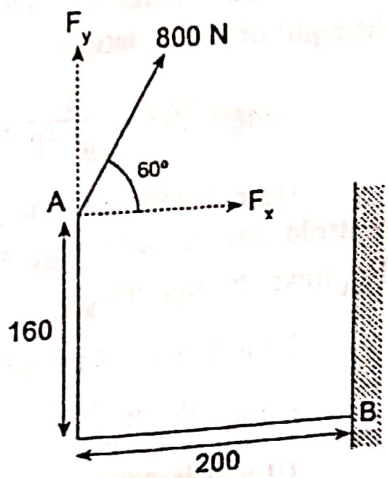
$T_{AD} + T_{BD} + T_{CD} = 490.5$ ......(ii)

$0.064T_{BD} - 0.086T_{CD} = 0$ ......(iii)

Solving,

$T_{AD} = 193$ N, $T_{BD} = 170.5$ N, $T_{CD} = 126.9$ N

4. A force of 800 N acts on a bracket as shown. Determine the moment of the force about B.



Solution:

You can use both scalar and vector method.

Method 1: Scalar method

Resolve the force 800 N to x and y axes.

$$\therefore \quad F_x = 800 \cos 60 = 400 \text{ N}$$

$$F_y = 800 \sin 60 = 692.82 \text{ N}$$

$$\Sigma M_B \; +ve \circlearrowleft = -F_x\,160 - F_y\,200$$

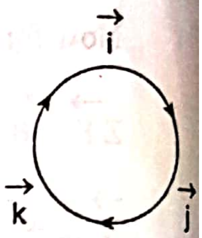
$$= -400 \times 0.1 - 692.82 \times 0.2$$

$$= -202.56 \text{ Nm}$$

$\therefore$ moment about B is 202.56 Nm clockwise.

Method 2: Vector method

A(–0.2, 0.16), B(0, 0)



$$\vec{M}_B = \vec{r} \times \vec{F}$$

$$= \vec{r}_{A/B} \times (800 \cos 60\,\vec{i} + 800 \sin 60\,\vec{i})$$

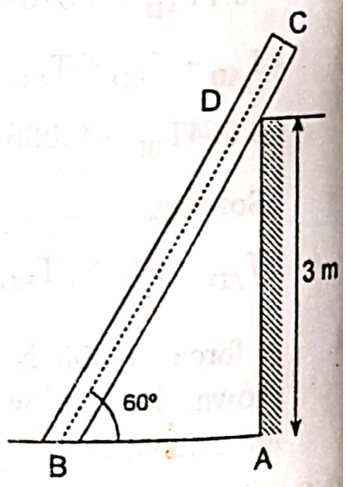
$$= (-0.2\,\vec{i} + 0.16\,\vec{j}) \times 400\,\vec{i} + 692.82\,\vec{j}$$

$$= (-0.2 \times 692.82)k - (0.16 \times 400)k$$

$$= -202.56 \text{ Nm } \vec{k}$$

[Here –ve sign indicates clockwise moment.]

5. Determine the tension in the cable AB which holds a post BC of 4 m length from sliding. The post has a mass of 10 kg. Assume all surface are smooth.



Solution:

Draw a FBD, calculate length BD and use the equilibrium equation.

$$\text{Length BD} = \frac{3}{\sin 60} = 2\sqrt{3}$$

Here weight (w = mg) lies downward from centroid and reaction R_{BV} and R_D perpendicular to contact point, tension T_{BA} towards support.

Length BG = 2 m (why?); where G is the point c.g.

$$\Sigma M_B = 0 \; +ve \circlearrowright$$

(It is better to use moment equation at a point where maximum force act.)

$$mg \times 2\cos 60 - R_D\, 2\sqrt{3} = 0$$

$$98.1 - R_D . 2\sqrt{3} = 0$$

$$\therefore\ R_D = 28.31\ N$$

$$\Sigma F_y = 0 \ +ve$$

$$R_{BV} - mg + R_D \sin 30 = 0$$

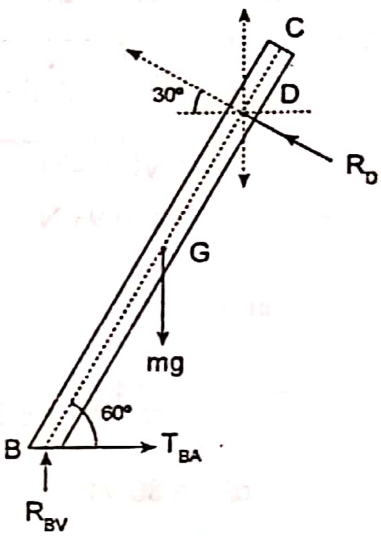
(not useful for this question.)

$$\Sigma F_x = 0 \ +ve$$

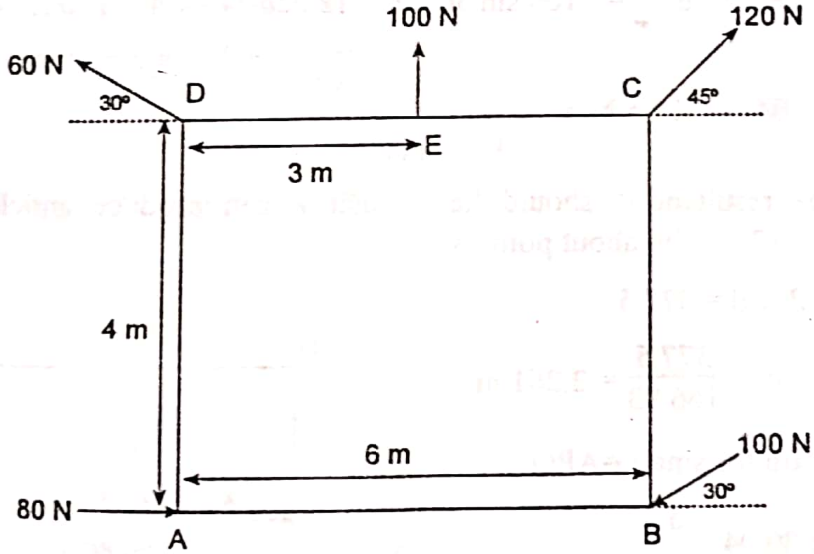
$$T_{BA} - R_D \cos 30 = 0$$

$$T_{BA} = R_D \cos 30$$

$$= 28.31 . \frac{\sqrt{3}}{2} = 24.51\ N$$



6. A plate measuring 6 m × 5 m is acted upon by a set of forces in its plane as shown in figure. Determine the magnitude direction and position of resultant force.



Solution:

$$+ve \rightarrow \Sigma F_x = 120 \cos 45 - 60 \cos 30 + 80 - 100 \cos 30$$

$$= 84.85 - 51.96 + 80 - 86.6$$

$$= 26.29\ N\ (\rightarrow)$$

$$+ve\uparrow\ \Sigma F_y = 100 + 60 \sin 30 + 120 \sin 45 - 100 \sin 30$$

$$= 100 + 30 + 84.85 - 50$$

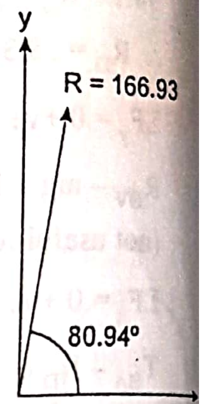
$$= 164.85\ N\ (\uparrow)$$

$$\vec{R} = \Sigma\vec{F_x} + \Sigma\vec{F_y}$$

$$= \sqrt{(\Sigma\vec{F_x})^2 + (\Sigma\vec{F_y})^2}$$

$$= \sqrt{(26.29)^2 + (164.85)^2}$$

$$= 166.93 \text{ N}$$



$$\tan\alpha = \frac{\Sigma F_y}{\Sigma F_x}$$

$$= \frac{164.85}{26.29} = 6.27$$

$$\therefore \alpha = 80.94°$$

Since both ΣF_x and ΣF_y are positive, resultant lies in first quadrant.

To find the position, we have to calculate moment at any poi (say A).

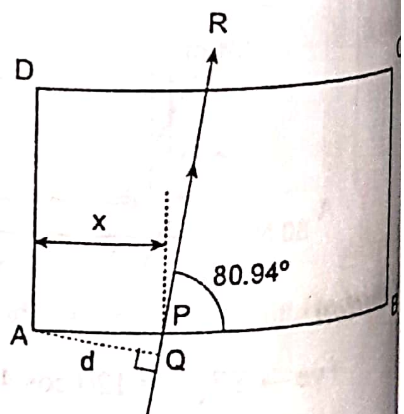
$$\therefore \Sigma M_A \text{ +ve} = -100 \sin 30 \times 6 - 120 \cos 45 \times 4 + 120 \sin 45 \times 6 + 100 \times 3 + 60 \cos 30 \times 4$$

$$\therefore \Sigma M_A = 377.5 \text{ Nm}$$

The resultant R should lie so that it can produce anticlockwis moment of 377.5 Nm about point A.

$$\therefore R \times d = 377.5$$

$$\therefore d = \frac{377.5}{166.93} = 2.261 \text{ m}$$



Again for small ΔAPQ

$$\sin 80.94 = \frac{d}{x}$$

$$\therefore x = \frac{d}{\sin 80.94}$$

$$= \frac{2.261}{\sin 80.94}$$

$$= 2.29 \text{ m}$$

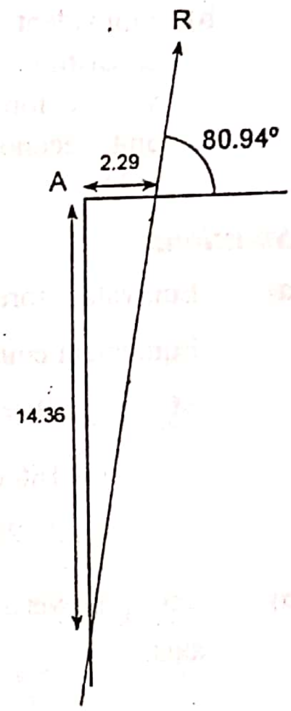
Alternative vector approach

Let a resultant force R be $\vec{r'} = x\vec{i} + y\vec{j}$ distance away from point A. Then, from varignon's theorem

$$\vec{r'} \times \vec{R} = \Sigma \vec{r} \times \vec{F}\ 0$$

or, $(x\vec{i} + y\vec{j}) \times (26.29\vec{i} + 164.85\vec{j})$

$$= 6\vec{i} \times (-100 \sin 30\vec{j}) + (6\vec{i} + 4\vec{j}) \times (120 \cos 45\vec{i} + 120 \sin 45\vec{j}) + 3\vec{i} \times 100\vec{j} + 4\vec{j} \times (-60 \cos 30\vec{i})$$



or, $164.85x\vec{k} - 26.29y\vec{k}$

$$= -300\vec{k} + 509.116\vec{k} - 339.41\vec{k} + 300\vec{k} + 207.84\vec{k}$$

or, $164.85x - 26.29y = 377.5$

or, $\dfrac{x}{\frac{377.5}{164.85}} - \dfrac{y}{\frac{377.5}{26.29}} = 1$

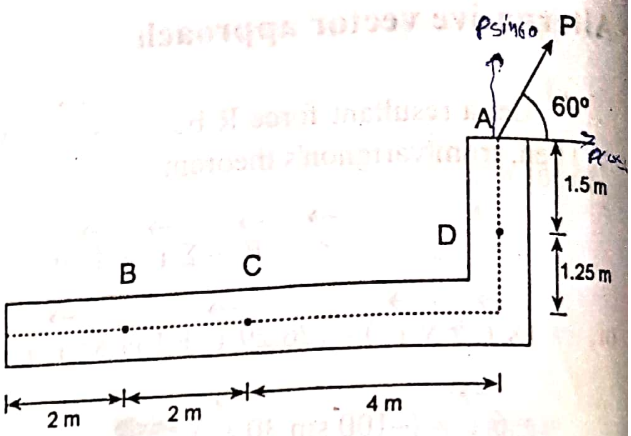
or, $\dfrac{x}{2.29} + \dfrac{y}{-14.36} = 1$

So, A as a origin

x-intercept = 2.29 m

y-intercept = –14.36 m

7. A 160 N force P is applied at point A of the structural member. Replace P with
   a) Equivalent force couple at C
   b) Equivalent system consisting of a vertical force at B and second force at D.



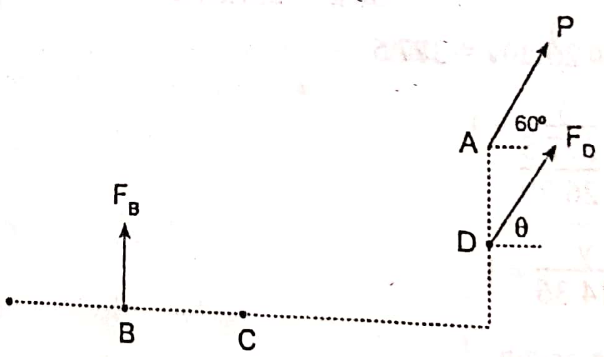
Solution:

a) Equivalent force at C = P = 160 N $\angle 60°$

Equivalent couple is moment of this force P about C

$$M_C = -P\cos 60 \times (1.5 + 1.25) + P\sin 60 \times 4$$
$$= -160\cos 60 \times 2.75 + 160\sin 60 \times 4$$
$$= 334.25 \text{ Nm}$$

b) Let F_B be vertical force at B and F_D be force at D at angle θ with x-axis.



Using varignon's theorem

Moment of P about D = moment of F_B about D + moment of F_D about D

$$\therefore\ P\cos 60 \times 1.5 = F_B \times 6 + 0$$

$$\frac{160\cos 60 \times 1.5}{6} = F_B$$

$$\therefore\ F_B = 20 \text{ N } (\uparrow)$$

Also, equating vertical component

or, $P \sin 60 = F_B + F_D \sin \theta$

or, $160 \sin 60 = 20 + F_D \sin \theta$

$\therefore$ $F_D \sin \theta = 118.56$ ......(i)

Also, equating horizontal component

or, $P \cos 60 = F_D \cos \theta$

or, $160 \times \cos 60 = F_D \cos \theta$

$\therefore$ $F_D \cos \theta = 80$ ......(ii)

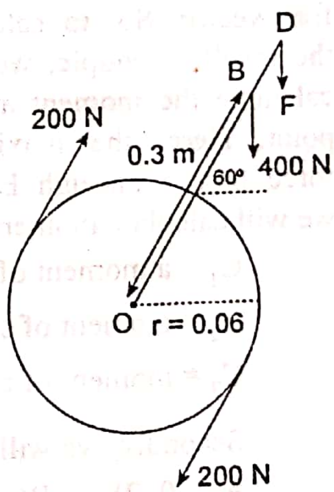
From (i) and (ii)

$\tan \theta = 1.482$

$\therefore \theta = 56°$

$\therefore F_D = \dfrac{80}{\cos 56} = 143.03$ N

8. Replace the couple and force shown in figure by an equivalent single force applied to lever.



Solution:

Here 200 N forces forms a couple.

So, couple $C_O = 200 \times 0.12$

$= 24$ Nm

$\therefore$ $C_O = -24$ Nm

Note this C_O is a free vector.

Moment of force 400 N at O

$M_O = 400 \times OB \cos 60$

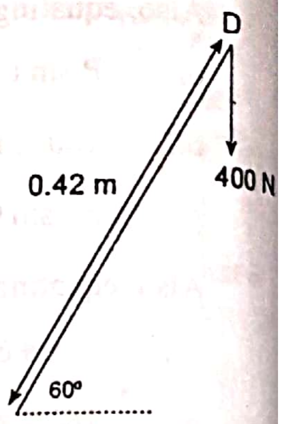
$= 400 \times 0.15$

$= -60$ Nm

$\therefore$ Total moment $= (-60 - 24)$ Nm

$= -84$ Nm about O

Now the equivalent single force also produce the same moment about O. Let it be vertical through D at distance d from O. Then,



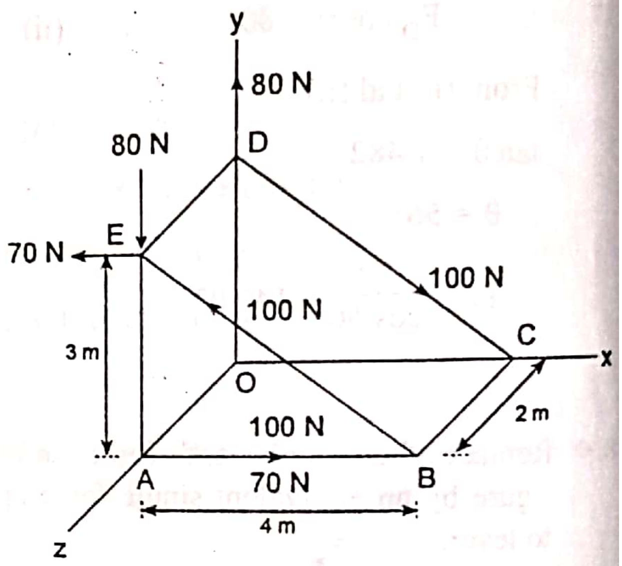
$F \times d \cos 60 = 84$

$400 \times d \cos 60 = 84$

$\therefore\ d = 0.42$ m

So, the equivalent force is F = 400 N at d = 0.42

9. Three pairs of couples are acted on the triangular block as shown in figure below. Determine the single resultant couple.



Solution:

We know couple is a free vector. So, to calculate the resultant couple, we will calculate the moment at any point. Here, the maximum force passes through E. So, we will calculate moment at E. Let

C_1 = a moment of couple at E due to 70 N force

C_2 = moment of couple at E due to 80 N force

C_3 = moment of couple at E due to 100 N force

Secondly, we will write the coordinates

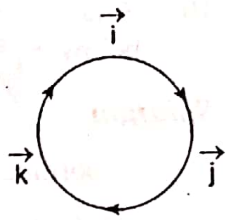
A(0, 0, 2), B(4, 0, 2), C(4, 0, 0) D(0, 3, 0) E(0, 3, 2)

Here the direction of 100 N is unknown. So, find the unit vector along DC.

$$\text{Unit vector of DC} = \hat{DC} = \frac{\vec{DC}}{|\vec{DC}|} = \frac{\vec{C} - \vec{D}}{|\vec{DC}|}$$

$$= \frac{(4, 0, 0) - (0, 3, 0)}{|\vec{DC}|}$$

$$= \frac{4\vec{i} - 3\vec{j}}{\sqrt{4^2 + 3^2}}$$



$$= \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j} = 0.8\vec{i} - 0.6\vec{j}$$

$$\therefore \vec{C_1} = \vec{r} \times \vec{F}$$

$$= \vec{r}_{A/E} \times 70\vec{i}$$

$$= [(0, 0, 2) - (0, 3, 2)] \times 70\vec{i}$$

$$= -3\vec{j} \times 70\vec{i} = 210\vec{k}$$

$$\vec{C_2} = \vec{r}_{D/E} \times \vec{F}$$

$$= (0, 3, 0) - (0, 3, 2) \times 80\vec{j}$$

$$= -2\vec{k} \times 80\vec{j}$$

$$= 160\vec{i}$$

$$\vec{C_3} = \vec{r}_{D/E} \times \vec{F}$$

$$= -2\vec{k} \times 100(0.8\vec{i} - 0.6\vec{j})$$

$$= -2\vec{k} \times (80\vec{i} - 60\vec{j})$$

$$= -160\vec{j} - 120\vec{i}$$

$\therefore$ Single resultant couple

$$\vec{C} = \vec{C_1} + \vec{C_2} + \vec{C_3}$$

$$= 210\vec{k} + 160\vec{i} - 160\vec{j} - 120\vec{i}$$

$$= 40\vec{i} - 160\vec{j} + 210\vec{k} \text{ Nm}$$

magnitude = 267.02 Nm

22. Find the resultant of force couple system at point A as shown in figure below. F_1 = 100 kN, F_2 = 300 kN.

Solution:

Coordinates

O (0, 0, 0)

A (0, 0, 3)

C (5, 0, 3)

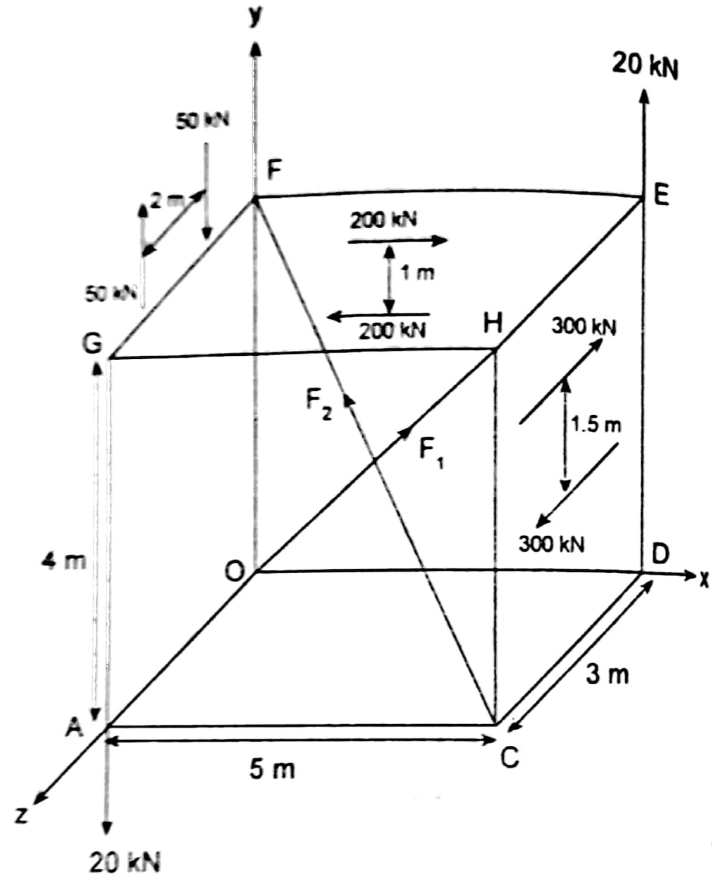
D (5, 0, 0)

E (5, 4, 0)

F (0, 4, 0)

G (0, 4, 3)

H (5, 4, 3)



Let

C_1 = moment of couple at A due to 50 N forces

C_2 = moment of couple at A due to 200 kN force

C_3 = moment of couple at A due to 300 kN force

Unit vector along $F_1 = \hat{F}_1 = \dfrac{\overrightarrow{OH}}{|\overrightarrow{OH}|}$

$$= \frac{5\vec{i} + 4\vec{j} + 3\vec{k}}{\sqrt{5^2 + 4^2 + 3^2}}$$

$$= 0.707\vec{i} + 0.565\vec{j} + 0.424\vec{k}$$

$\therefore \vec{F}_1 = 100.\hat{F}_1$

$= 70.7\vec{i} + 56.5\vec{j} + 42.4\vec{k}$

Unit vector along $F_2 = \hat{F}_2 = \dfrac{\vec{CF}}{|\vec{CF}|} = \dfrac{\vec{F} - \vec{C}}{|\vec{CF}|}$

$= \dfrac{(0, 4, 0) - (5, 0, 3)}{|\vec{CF}|}$

$= \dfrac{-5\vec{i} + 4\vec{j} - 3\vec{k}}{\sqrt{5^2 + 4^2 + 3^2}}$

$= -0.707\vec{i} + 0.565\vec{j} - 0.424\vec{k}$

$\therefore \vec{F}_2 = 300.\hat{F}_2$

$\therefore \vec{F}_2 = 300(-0.707\vec{i} + 0.565\vec{j} - 0.424\vec{k})$

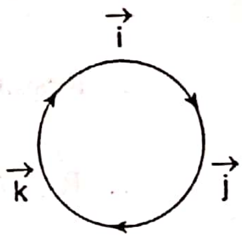
$= -212.\vec{i} + 169.68\vec{j} - 127.27\vec{k}$

Moment due to F_1 about A is

$\vec{M}_1 = \vec{r} \times \vec{F}$

$= \vec{r}_{O/A} \times \vec{F}_1$

$= -3\vec{k} \times (70.7\vec{i} + 56.\vec{j} + 42.4\vec{k})$



$= -212.1\vec{j} + 169.68\vec{i}$

$= 169.68\vec{i} - 212.1\vec{j}$

Moment due to F_2 about A is

$\vec{M}_2 = \vec{r} \times \vec{F}$

$= \vec{r}_{C/A} \times \vec{F}_2$

$= 5\vec{i} \times (-212.1\vec{i} + 169.68\vec{j} - 127.27\vec{k})$

$$= 636.39\vec{j} + 848.4\vec{k}$$

Moment due to force 20 kN at E about A

$$\vec{M}_3 = \vec{r} \times \vec{F}$$

$$= \vec{r}_{E/A} \times 20\vec{j}$$

$$= [(5, 4, 0) - (0, 0, 3)] \times 20\vec{j}$$

$$= (5\vec{i} + 4\vec{j} - 3\vec{k}) \times 20\vec{j}$$

$$= 100\vec{k} + 60\vec{i}$$

$$= 60\vec{i} + 100\vec{k}$$

$$\vec{C}_1 = -(50 \times 2)\vec{i} = -100\vec{i}$$

$$\vec{C}_2 = -(200 \times 1)\vec{k} = -200\vec{k}$$

$$\vec{C}_3 = -(300 \times 1.5)\vec{i} = -450\vec{i}$$

∴ Resultant force = $\vec{R} = \vec{F}_1 + \vec{F}_2$

$$\therefore \vec{R} = 70.7\vec{i} + 56.5\vec{j} + 42.4\vec{k} - 212.1\vec{i} + 160.68\vec{j} - 127.27\vec{k}$$

$$= -141.4\vec{i} + 226.1\vec{j} - 84.87\vec{k}$$

$$\vec{M}_R = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \vec{C}_1 + \vec{C}_2 + \vec{C}_3$$

$$= 169.68\vec{i} - 212.1\vec{j} + 636.39\vec{j} + 848.4\vec{k} + 60\vec{i} + 100\vec{k} - 100\vec{i} - 200\vec{k} - 450\vec{i}$$

$\therefore \vec{M}_R = -320.32\vec{i} + 424.29\vec{j} + 748.4\vec{k}$ kNm about A.

11. Replace the two wrenche in figure by a single wrench. Indicate its line xz plane.

Solution:

First find resultant o moment at origin.

$$\vec{R} = 50\vec{i} + 40\vec{j}$$

$$\vec{M}_O^R = \vec{C}_1 + \vec{C}_2 + \vec{r}$$

$$= -100\vec{j} + 8$$

$$= -100\vec{j} + 8$$

$$= 80\vec{i} + 200$$

$$\text{pitch } p = \frac{\vec{R} \cdot \vec{M}_o^R}{R^2}$$

$$= \frac{(50\vec{i} + 4}{(}$$

$$= \frac{4000 + 80}{4100}$$

$$= \frac{12000}{4100}$$

$\therefore \quad p = 2.926$

∴ couple vector M

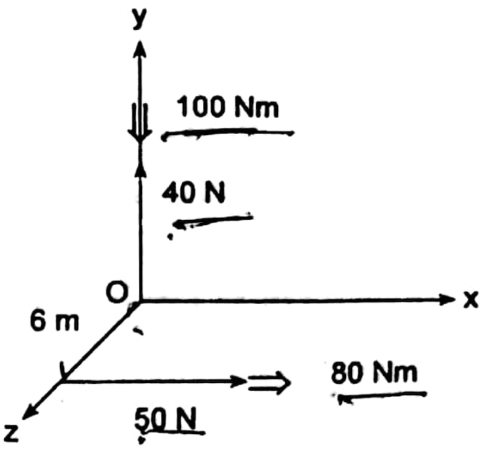
$$= 2.926\ (5$$

$$= 146.34\ \vec{i}$$

To find line of act

$$\vec{M}_1 + \vec{r} \times \vec{R} = M$$

11. Replace the two wrenches as shown in figure by a single equivalent wrench. Indicate its line of action in xz plane.



Solution:

First find resultant of force and moment at origin.

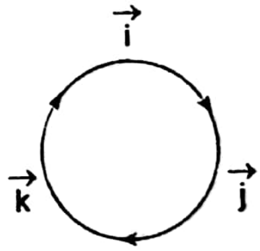
$$\vec{R} = 50\,\vec{i} + 40\,\vec{j}$$

$$\vec{M_O}^R = \vec{C_1} + \vec{C_2} + \vec{r} \times \vec{F}$$

$$= -100\,\vec{j} + 80\,\vec{i} + (6\,\vec{k} \times 50\,\vec{i})$$

$$= -100\,\vec{j} + 80\,\vec{i} + 300\,\vec{j}$$

$$= 80\,\vec{i} + 200\,\vec{j}$$



$$\text{pitch } p = \frac{\vec{R} \cdot \vec{M_o}^R}{R^2}$$

$$= \frac{(50\,\vec{i} + 40\,\vec{j}) \cdot (80\,\vec{i} + 200\,\vec{j})}{\left(\sqrt{(50^2 + 40^2)}\right)^2}$$

$$= \frac{4000 + 8000}{4100}$$

$$= \frac{12000}{4100}$$

$$\therefore \quad p = 2.926$$

$$\therefore \text{ couple vector } \vec{M_1} = p.\vec{R}$$

$$= 2.926\,(50\,\vec{i} + 40\,\vec{j})$$

$$= 146.34\,\vec{i} + 117.04\,\vec{j}$$

To find line of action

$$\vec{M_1} + \vec{r} \times \vec{R} = \vec{M_O}^R$$

$$\vec{M}_O^R = 146.34\vec{i} + 117.04\vec{j} + (x\vec{i} + y\vec{j} + z\vec{k}) \times (50\vec{i} + 40\vec{j})$$

$$= 80\vec{i} + 200\vec{j}$$

for xz plane y = 0

$$\therefore\ 146.34\vec{i} + 117.04\vec{j} + (x\vec{i} + z\vec{k}) \times (50\vec{i} + 40\vec{j})$$

$$= 80\vec{i} + 200\vec{j}$$

$$40x.\vec{k} + 50z\vec{j} - 40z\vec{i} = -66.34\vec{i} + 82.96\vec{j}$$

Equating the coefficient

$40x = 0$

$\therefore\ x = 0$

$z = 1.65$

$$\therefore\ \vec{r} = 1.65\vec{k}$$

$$\vec{M}_1 = 146.34\vec{i} + 117.04\vec{j}\ \text{Nm}$$

12. Compute a wrench out of a force $\vec{F} = 5\vec{i} + 7\vec{j} + 18\vec{k}$ N acting (4, 4, 3) M and a couple $\vec{C} = 18\vec{i} + 7\vec{j} + 5\vec{k}$ Nm.

Solution:

$$\vec{R} = 5\vec{i} + 7\vec{j} + 18\vec{k}$$

$$\vec{M}_O^R = \vec{r} \times \vec{F} + \vec{C}$$

$$= (4\vec{i} + 4\vec{j} + 3\vec{k}) \times (5\vec{i} + 4\vec{j} + 18\vec{k}) + (18\vec{i} + 7\vec{j} + 5\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 3 \\ 5 & 7 & 18 \end{vmatrix} + (18\vec{i} + 7\vec{j} + 5\vec{k})$$

$$=(72-21)\vec{i} - (72-$$

$$= 51\vec{i} - 57\vec{j} + 8\vec{k}$$

$$= 60\vec{i} - 50\vec{j} + 13$$

$$\therefore\ \text{pitch } p = \frac{\vec{R}\,.\,M_O^R}{R^2}$$

$$= \frac{(5\vec{i} + 7\vec{j} + 18}{(5}$$

$$= \frac{345 + (-350) + 2}{398}$$

$$= 0.575$$

$$\therefore\ \vec{M}_1 = p.\vec{R} = 0.575(5\vec{i}$$

$$= 2.875\vec{i} + 4.02\vec{j}$$

13. Force $\vec{F} = (3\vec{i} - 6\vec{j} +$

Replace this force with a passes through (2, 5, 10) m

Solution:

Let A(6, 3, 2) and B(2, 5,

Introduce two forces F at

New value of force at B

= 3

You can calculate couple

Couple C = $\vec{r}_{A/B} \times \vec{F}$

$= (A - B) \times (3\vec{i}$

$$=(72-21)\vec{i}-(72-15)\vec{j}+(28-20)\vec{k}+18\vec{i}+7\vec{j}+5\vec{k}$$

$$=51\vec{i}-57\vec{j}+8\vec{k}+18\vec{i}+7\vec{j}+5\vec{k}$$

$$=60\vec{i}-50\vec{j}+13\vec{k}$$

$$\therefore \text{ pitch } p=\frac{\vec{R}\cdot M_O^R}{R^2}$$

$$=\frac{(5\vec{i}+7\vec{j}+18\vec{k})\cdot(69\vec{i}-50\vec{j}+13\vec{k})}{(5^2+7^2+18^2)}$$

$$=\frac{345+(-350)+234}{398}$$

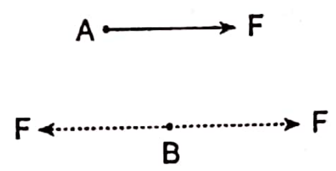
$$=0.575$$

$$\therefore \vec{M}_1=p.\vec{R}=0.575(5\vec{i}+7\vec{j}+18\vec{k})$$

$$=2.875\vec{i}+4.02\vec{j}+10.35\vec{k}\text{ Nm}$$

13. Force $\vec{F}=(3\vec{i}-6\vec{j}+4\vec{k})$ N passes through point (6, 3, 2) m. Replace this force with an equivalent system, where the force $\vec{F}$ passes through (2, 5, 10) m.

Solution:



Let A(6, 3, 2) and B(2, 5, 10)

Introduce two forces F at B as shown.

New value of force at B = $\vec{F}$

$$=3\vec{i}-6\vec{j}+4\vec{k}$$

You can calculate couple at A or B.

Couple $C=\vec{r}_{A/B}\times\vec{F}$

$$=(A-B)\times(3\vec{i}-6\vec{j}+4\vec{k})$$

$$= (4\vec{i} - 2\vec{j} - 3\vec{k}) \times (3\vec{i} - 6\vec{j} + 4\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & -8 \\ 3 & -6 & 4 \end{vmatrix}$$

$$= (-8 - 48)\vec{i} - (16 + 24)\vec{j} + (-24 + 6)\vec{k}$$

Couple $C = -56\vec{i} - 40\vec{j} - 18\vec{k}$

$\therefore$ Equivalent system at B is

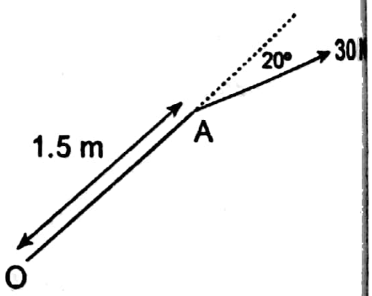
$\vec{F} = 3\vec{i} - 6\vec{j} + 4\vec{k}$ N

$\vec{C} = -56\vec{i} - 40\vec{j} - 18\vec{k}$ Nm

Check: couple $C = \vec{r}_{B/A} \times -\vec{F}$ (why –ve for $\vec{F}$?)

Practice Questions

1. A 30 N force acts on the end of the 1.5 m lever as shown. Determine the moment of force about O



[Ans. 15.4 Nm]

2. Forces $\vec{F} = (3\vec{i} - 6\vec{j} + 4\vec{k})$ N goes through point (6, 3, 2) m

Replace this force with an equivalent system, where the force $\vec{F}$ goes through point (2, –5, 10) m.

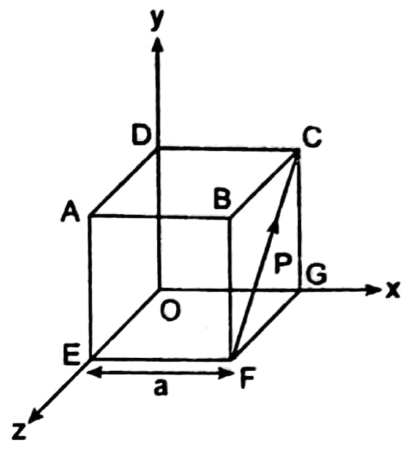
[Ans. $F = 3\vec{i} - 6\vec{j} + 4\vec{k}$; $C = -16\vec{i} - 40\vec{j} - 48\vec{k}$]

3. A cube of side a is acted upon by a force P as shown. Determine the moment of P

a) about A

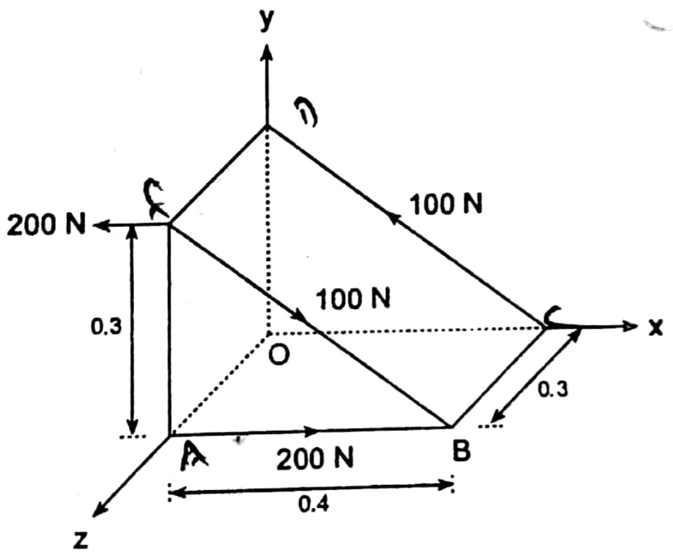
b) about edge AB

c) about diagonal AG of cube



[Ans. a) $\frac{aP}{\sqrt{2}}(\vec{i} + \vec{j} + \vec{k})$; b) $\frac{aP}{\sqrt{2}}$; c) $-\frac{aP}{\sqrt{6}}$]

4. Replace the two couples of a triangular block as shown in figure by a single resultant couple.

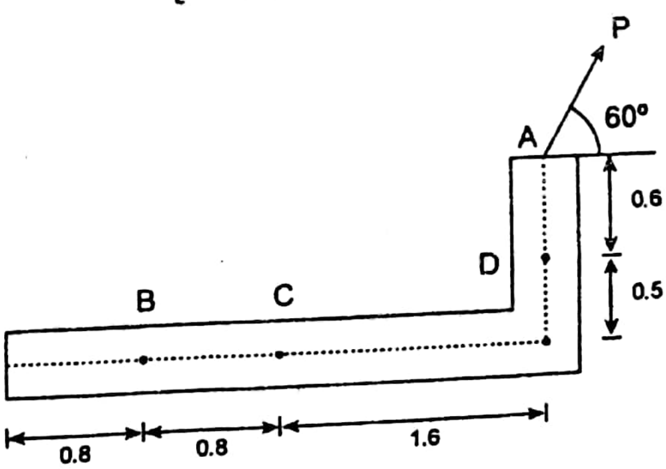


[Ans. $(18\vec{i} + 24\vec{j} + 60\vec{k})$ Nm]

5. A 700 N force P is applied at point A of structural member. Replace P with

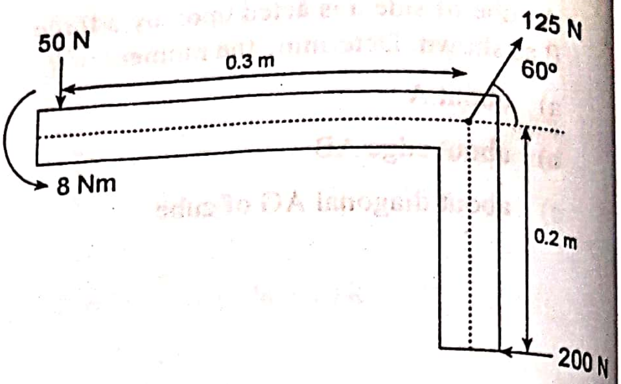
a) An equivalent force couple at point C.

b) An equivalent system consisting of a vertical force at B and second force at D.



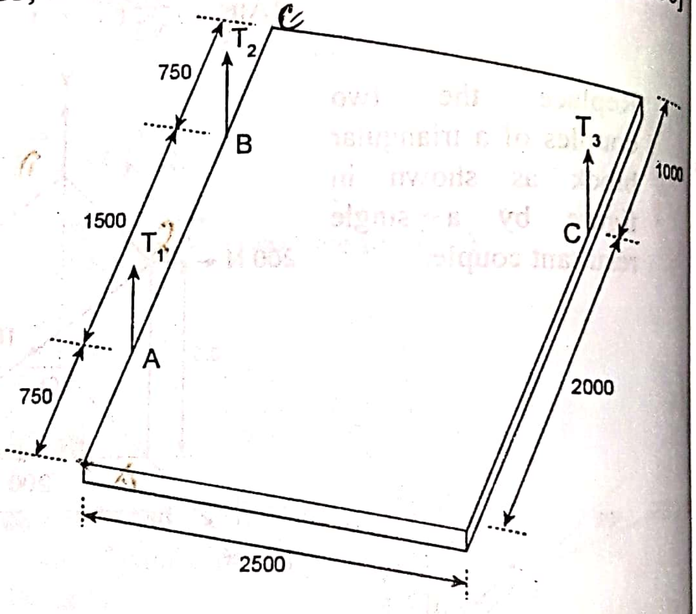
[Ans. a) 700 N, 585 Nm; b) 87.5 N, 626 N]

6. An angle bracket has been subjected to three forces and a couple as shown in figure. Determine the resultant of this system of forces. Locate the point of position of resultant.



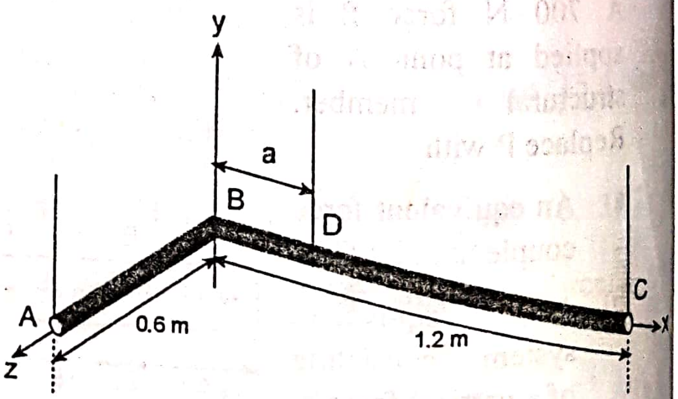
[Ans. R = 149.33, α = 157.44, x = 0.292 m, y = 0.124 m, d = 0.1138]

7. Three vertical wires as shown in figure support a plate of 50 kg. Determine the tension in each wire.



[Ans. T_1 = 20.83 kg, T_2 = 4.17 kg, T_3 = 25 kg]

8. Two steel pipe AB and BC each having a weight per unit length of 30 N/m are welded together at B and supported by three wires. Knowing a = 0.4 m, determine the tension in each wire.



[Ans. T_A = 9 N, T_D = 40.5 N, T_C = 4.5 N]

9. If two forces of same magnitude 25 kN acts at point A and B as shown in figure. Force A passes through C and B through D.

a) Determine equivalent force couple system at O.

b) Find equivalent wrench, pitch and axis of wrench.

[Ans. a) 41.5

b) 13.7

10. Four forces act on a 700 × 375 mm plate as shown

a) Find the resultant of the force

b) Find the position of resultant.

[Ans. $(519.81\,\vec{i} + 1199.2\,\vec{j})$ N; 25

11. Determine the resultant of the four acting tangentially to a circle of r m as shown. What will be the loca the resultant with respect to centre circle.

[Ans. F

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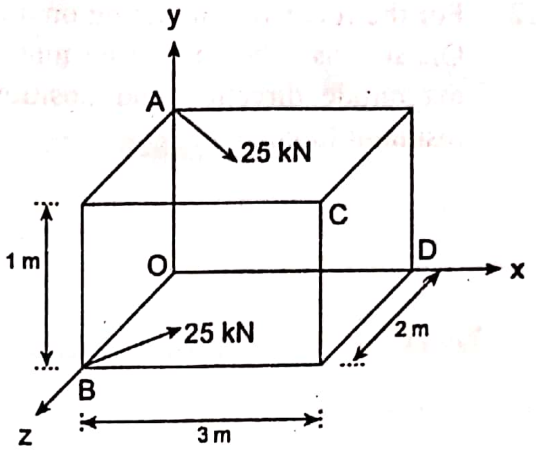
9. If two forces of same magnitude 25 kN acts at point A and B as shown in figure. Force A passes through C and B through D.

   a) Determine equivalent force couple system at O.

   b) Find equivalent wrench, pitch and axis of wrench.



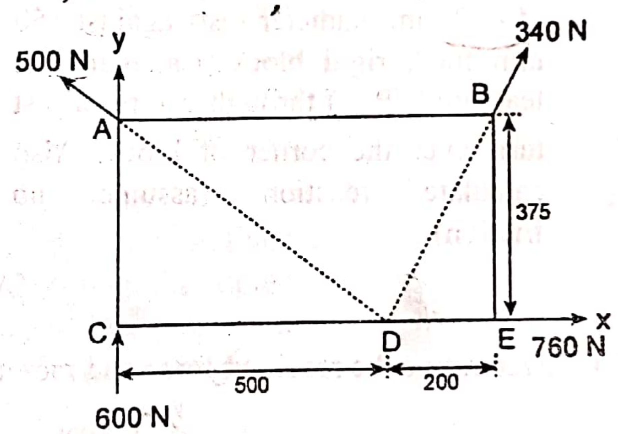
[Ans. a) $41.5\vec{i}$, $13.75\vec{i} + 41.5\vec{j} - 20.75\vec{k}$;

b) $13.75\vec{i}$, 0.33 m, x = 0, y = 0.5, z = 1]

10. Four forces act on a 700 × 375 mm plate as shown

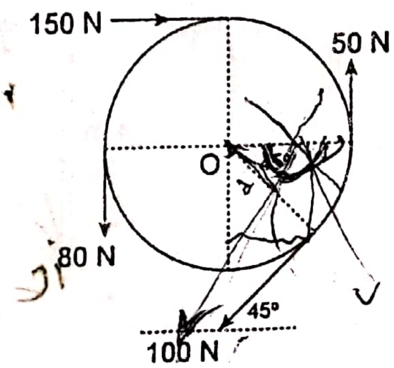
    a) Find the resultant of the force

    b) Find the position of resultant.



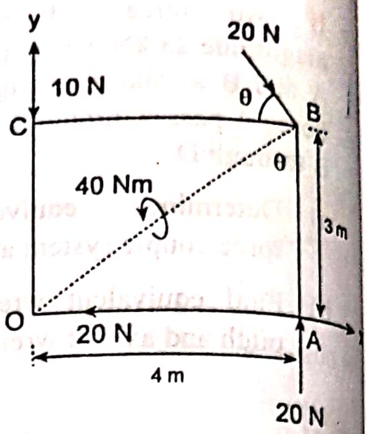
[Ans. $(519.81\vec{i} + 1199.2\vec{j})$ N; 250 mm from point C of edge CE and 412.76 mm from A of edge AB]

11. Determine the resultant of the four forces acting tangentially to a circle of radius 3 m as shown. What will be the location of the resultant with respect to centre of the circle.



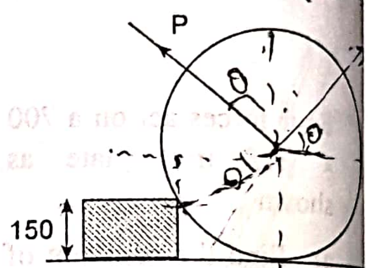
[Ans. R = 128.18 N, α = 51.8, d = 2.795 m]

12. For the force system acting on a body OABC as shown. Determine the magnitude direction and position of resultant force.



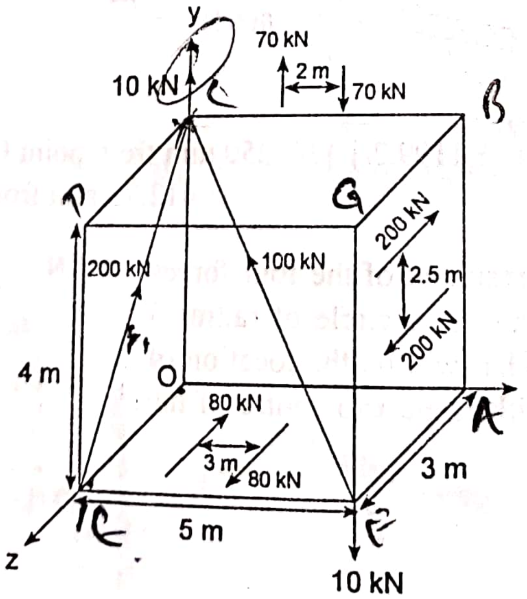
[Ans. R = 10 N α = 216.87, d = 2 m, x = 3.33 m, y = 2.5 m

13. A uniform wheel weighing 30 kN and of 600 mm diameter rests against 150 mm thick rigid block. Calculate the least pull (P_{min}) through centre to just turn over the corner of block. Also calculate reaction (assume no friction).



[Ans. P_{min} = 25.981, R_{xn} = 15kN

14. Determine the resultant force and moment about point O.



[Ans: $\vec{R} = -70.72\vec{i} + 216.57\vec{j} - 162.43\vec{k}$ kN

$\vec{M}_o = -119.71\vec{i} - 240\vec{j} + 92.85\vec{k}$ kNm

Chap

Centre of Gravit

Moment

4.1 Concept of centre o

A body comprises of several So weight is the force of attractio weight is proportional to mass of the can be considered as parallel force Thus centre of gravity is defined weight of the body is assumed to ac

Centre of mass is the point mass of body is assumed to act. It d of gravity only when the gravitat uniform and non-parallel.

The centroid or centre of the point where the whole area assumed to be concentrated. It is a of gravity when a body has area bu

4.1.1 Centroid of Lines, ar

a) Centroid of lines

Let us consider a homoge or curved wire of length L in the crossectional area be A and dl length.

Then, differential volume

Total volume V = AL

Using principle of momen

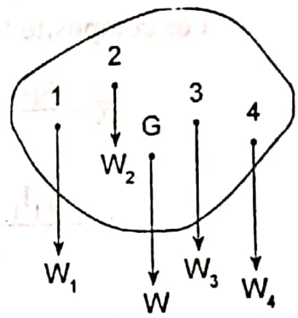
$\overline{X}.V = \int x\, dv$

Chapter 4

Centre of Gravity, Centroid and Moment of Inertia

4.1 Concept of centre of gravity and centroid

A body comprises of several parts and its every part possess weight. So weight is the force of attraction between a body and the earth. The weight is proportional to mass of the body. The weight of all parts of a body can be considered as parallel forces directed towards the centre of earth. Thus centre of gravity is defined as the point through which the whole weight of the body is assumed to act. It is denoted by c.g. or G.



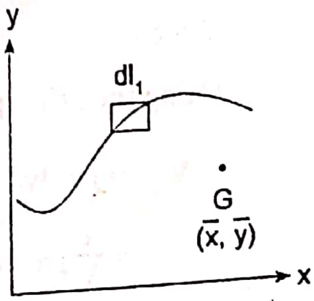
Centre of mass is the point where the whole mass of body is assumed to act. It differs from centre of gravity only when the gravitational field is not uniform and non-parallel.

The centroid or centre of area is defined as the point where the whole area of the figure is assumed to be concentrated. It is analogous to centre of gravity when a body has area but not weight.

4.1.1 Centroid of Lines, areas and volume

a) Centroid of lines

Let us consider a homogeneous slender rod or curved wire of length L in the xy plane. Let the crossectional area be A and dl be the differential length.



Then, differential volume dv = Adl

Total volume V = AL

Using principle of moments

$\bar{X}.V = \int x\, dv$

$$\bar{X} = \frac{1}{V}\int x\, dv$$

$$= \frac{1}{A.L}\int x\, Adl$$

$$= \frac{1}{L}\int x\, dl$$

$$\bar{Y}v = \int y\, dv$$

$$\bar{Y} = \frac{1}{V}\int y\, dv$$

$$= \frac{1}{A.L}\int y\, Adl$$

$$\bar{Y} = \frac{1}{L}\int y\, dl$$

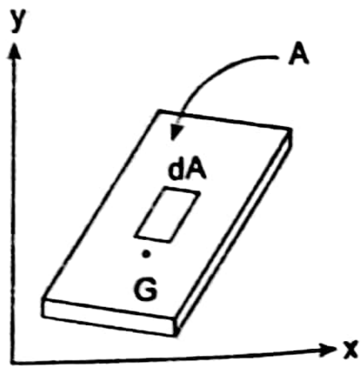
For composite line

$$\bar{X} = \frac{x_1l_1 + x_2l_2 + x_3l_3 + \ldots\ldots}{l_1 + l_2 + l_3 + \ldots\ldots}$$

$$\bar{Y} = \frac{y_1l_1 + y_2l_2 + y_3l_3 + \ldots\ldots}{l_1 + l_2 + l_3 + \ldots\ldots}$$

b) Centroid of area

Consider a homogeneous thin plate of uniform thickness. Let dA be the differential area of the plate in the xy plane so that the differential volume is given by



$$dV = tdA$$

$$V = t.A$$

Using the principle of moment

$$\bar{X}V = \int x\, dv$$

$$= \frac{1}{V}\int x\, dv$$

$$= \frac{1}{t.A}\int x\, t\, dA$$

$$= \frac{1}{A}\int x\, dA$$

$$\overline{Y}\,V = \int y\, dv$$

$$\overline{Y} = \frac{1}{V}\int y\, dv$$

$$= \frac{1}{t.A}\int y.\, t\, dA$$

$$= \frac{1}{A}\int y\, dA$$

For composite area

$$\overline{X} = \frac{A_1X_1 + A_2X_2 + A_3X_3 + \ldots}{A_1 + A_2 + A_3 + \ldots}$$

$$\overline{Y} = \frac{A_1Y_1 + A_2Y_2 + A_3Y_3 + \ldots}{A_1 + A_2 + A_3 + \ldots}$$

c) Centroid of volume

$$\overline{X} = \frac{1}{V}\int x\, dv$$

$$\overline{Y} = \frac{1}{V}\int y\, dv$$

$$\overline{Z} = \frac{1}{V}\int z\, dv$$

4.1.2 First moment of an area

The first moment of element of an area about any axis is the product of area of element and the perpendicular distance between the element and axis.

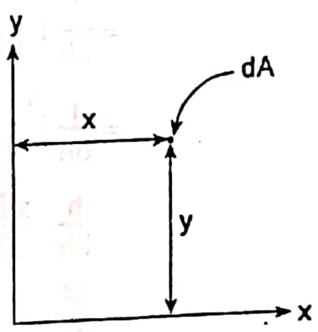
about x-axis: $d\,M_x = y\, dA$

about y-axis: $dM_y = x\, dA$

for a finite area

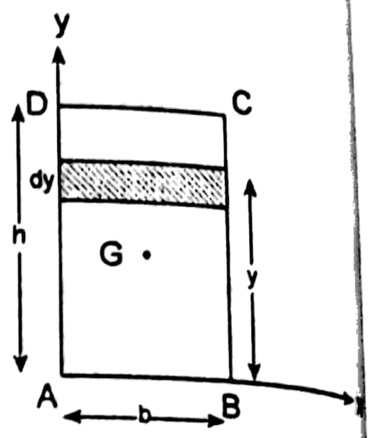
$M_x = \overline{Y}\,.\,A$ or $\int y\, dA$

$M_y = \overline{X}\,.\,A$ or $\int x\, dA$



4.1.3 Determination of centroid of area of rectangle

Consider a rectangle ABCD of breadth b and height h. Consider strip of thickness dy located at a distance of y as shown.



Elemental area $dA = b\,dy$

Moment about x-axis $dM_x = y\,dA$

$= by\,dy$

Area of rectangle $A = b \times h$

Let $\overline{Y}$ be the distance of centroid from x-axis.

Then from moment principle:

$$A \times \overline{y} = \int dM_x$$

$$bh \times \overline{y} = \int_0^h by\,dy = b\frac{y^2}{2}\Bigg|_0^h = \frac{b}{2}.h^2\frac{1}{bh}$$

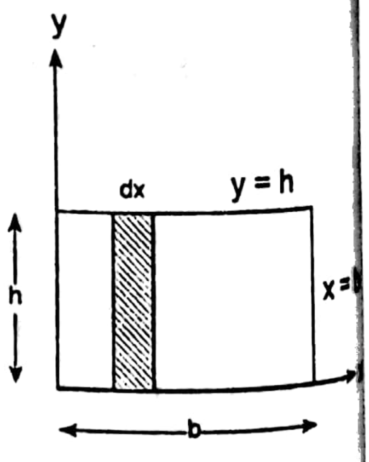
$$\therefore \ \overline{y} = \frac{h}{2}$$

Similarly, $\overline{x} = \frac{b}{2}$

$$\therefore \ (\overline{x}, \overline{y}) = \left(\frac{b}{2}, \frac{h}{2}\right)$$

Using integration approach:

Consider vertical strip:



$$\overline{x} = \frac{1}{A}\int x_{el}\,dA$$

$$= \frac{1}{bh}\int x.y\,dx$$

$$= \frac{1}{bh}\int_0^b x.h\,dx$$

$$= \frac{h}{bh}.\frac{x^2}{2}\Bigg|_0^b$$

$$= \frac{b^2}{2b}$$

$$= \frac{b}{2}$$

$$\bar{y} = \frac{1}{A}\int y_{el}\, dA$$

$$= \frac{1}{bh}\int \frac{y}{2} . y\, dx$$

$$= \frac{1}{2bh} . \int_0^b y^2\, dx$$

$$= \frac{1}{2bh}\int_0^b h^2\, dx$$

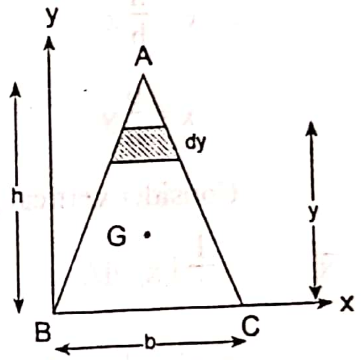
$$= \frac{h^2}{2bh} . b$$

$$= \frac{h}{2}$$

$$\therefore (\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{h}{2}\right)$$

4.1.4 Determination of centroid of area of triangle

Consider a triangle ABC with width b and height h. Consider an elemental strip of width l thickness dy and located at distance y from the base BC as shown.



Elemental area $dA = l\, dy$

Moment about x-axis $dM_x = y\, dA$

Area of triangle $A = \frac{1}{2} \times b \times h$

Let $\bar{y}$ be the distance of centroidal from x-axis; then by moment principle

$$A \times \bar{y} = \int d\, Mx$$

$$\frac{1}{2} bh \times \bar{y} = \int_0^h y\, dA$$

$$\frac{1}{2} \times bh \times \bar{y} = \int_0^h y . l\, dy$$

$$= \int_0^h b\left(1 - \frac{y}{h}\right) . y\, dy$$

$$= b\left[\int_0^h y\,dy - \int_0^h \frac{y^2}{h}\,dy\right]$$

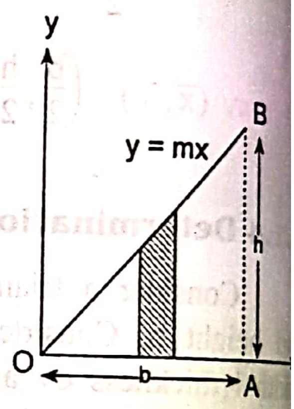
$$= b\left[\frac{h^2}{2} - \frac{h^3}{3h}\right]$$

$$\therefore\ \bar{y} = \frac{h}{3}$$

Centroid of triangle is $\frac{1}{3}$ of height from base and $\frac{2}{3}$ of height fr vertex.

Integration approach:

Consider a right angled triangle passing throug origin.



$$y = mx$$

$$h = m.b$$

$$\therefore\ m = \frac{h}{b}$$

$$\therefore\ y = \frac{h}{b}x$$

$$x = \frac{b}{h}y$$

Consider vertical strip:

$$\bar{x} = \frac{1}{A}\int x_{el}\,dA \qquad\qquad \bar{y} = \frac{1}{A}\int y_{el}\,dA$$

$$= \frac{1}{\left(\frac{bh}{2}\right)}\int x.y\,dx \qquad\qquad = \frac{2}{bh}\int \frac{y}{2}\cdot y\,dx$$

$$= \frac{2}{bh}\int_0^b x.\frac{h}{b}x\,dx \qquad\qquad = \frac{1}{bh}\int_0^b \frac{h^2}{b^2}\cdot x^2\,dx$$

$$= \frac{2h}{b^2h}\int_0^b x^2\,dx \qquad\qquad = \frac{h^2}{b^3h}\cdot \left.\frac{x^3}{3}\right|_0^b$$

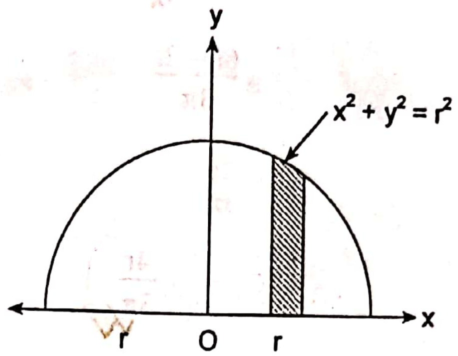
$$= \frac{2}{b^2}\cdot\frac{b^3}{3} \qquad\qquad = \frac{h}{b^3}\cdot\frac{b^3}{3}$$

$$= \frac{2b}{3} \qquad\qquad = \frac{h}{3}$$

$= \frac{2}{3}$ from vertex O $\qquad\qquad = \frac{1}{3}$ from base OA

4.1.5 Determination of centroid of semi-circle

Consider a semi-circle as shown in figure. Draw a small vertical stripes as shown.



$$A = \frac{\pi r^2}{2}$$

$$\bar{x} = \frac{1}{A}\int x_{el}\, dA$$

$$= \frac{2}{\pi r^2}\int_{-r}^{r} x.y\, dx$$

$$= \frac{2}{\pi r^2}\int_{-r}^{r} x.\sqrt{r^2 - x^2}\, dx$$

Let $x = r\sin\theta$

$dx = r\cos\theta\, d\theta$

$x \rightarrow -r; \quad \theta \rightarrow -\frac{\pi}{2}$

$x \rightarrow r; \quad \theta \rightarrow \frac{\pi}{2}$

$$\bar{x} = \frac{2}{\pi r^2}\int_{-\pi/2}^{\pi/2} r\sin\theta\ .\ r\cos\theta\ .\ r\cos\theta\, d\theta$$

$$= \frac{2r}{\pi}\int_{-\pi/2}^{\pi/2} \sin\theta\ .\ \cos^2\theta\, d\theta$$

$= 0$ [It is because of symetricity about y-axis]

$$\bar{y} = \frac{1}{A}\int y_{el}\, dA$$

$$= \frac{2}{\pi r^2}\int_{-r}^{r} \frac{y}{2}.y\, dx$$

$$= \frac{1}{\pi r^2}\int_{-r}^{r} y^2\, dx$$

$$= \frac{1}{\pi r^2}\int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \frac{1}{\pi r^2}\left[\int_{-r}^{r} r^2\, dx - \int_{-r}^{r} x^2\, dx\right]$$

$$= \frac{1}{\pi r^2}\left[r^2 . x\Big|_{-r}^{r}\right] - \frac{1}{\pi r^2} . \frac{x^3}{3}\Big|_{-r}^{r}$$

$$= \frac{1}{\pi r^2} \cdot 2r^3 - \frac{1}{3\pi r^2} \cdot 2r^3$$

$$= \frac{2r}{\pi} - \frac{2r}{3\pi}$$

$$= \frac{6r - 2r}{3\pi}$$

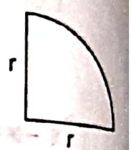
$$\overline{y} = \frac{4r}{3\pi}$$

$$\therefore \ (\overline{x}, \overline{y}) = \left(0, \frac{4r}{3\pi}\right)$$

Centroid of common shapes of lines

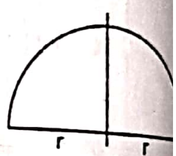
a) Quarter-circular arc

$\overline{x} = \frac{2r}{\pi}$ $\overline{y} = \frac{2r}{\pi}$ Length $= \frac{\pi r}{2}$



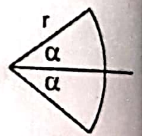
b) Semi-circular arc

$\overline{x} = 0$ $\overline{y} = \frac{2r}{\pi}$ Length $= \pi r$



c) Arc of circle

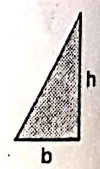
$\overline{x} = \frac{r \sin \alpha}{\alpha}$ $\overline{y} = 0$ Length $= 2\alpha r$



Centroid of common shapes of area

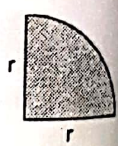
a) Triangular area

$\overline{y} = \frac{h}{3}$ Area $= \frac{bh}{2}$



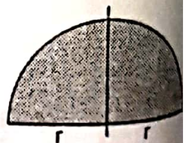
b) Quarter-circular area

$\overline{x} = \frac{4r}{3\pi}$ $\overline{y} = \frac{4r}{3\pi}$ Area $= \frac{\pi r^2}{4}$



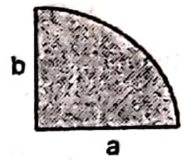
c) Semicircular area

$\overline{x} = 0$ $\overline{y} = \frac{4r}{3\pi}$ Area $= \frac{\pi r^2}{2}$



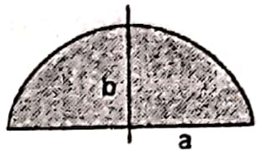
d) Quarter elliptical area

$\bar{x} = \dfrac{4a}{3\pi}$ $\quad\quad$ $\bar{y} = \dfrac{4b}{3\pi}$ $\quad\quad$ $\text{Area} = \dfrac{\pi ab}{4}$



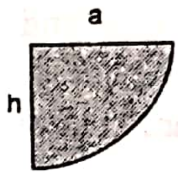
e) Semi elliptical area

$\bar{x} = 0$ $\quad\quad$ $\bar{y} = \dfrac{4b}{3\pi}$ $\quad\quad$ $\text{Area} = \dfrac{\pi ab}{4}$



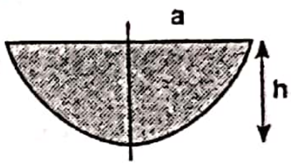
f) Semiparabolic area

$\bar{x} = \dfrac{3a}{8}$ $\quad\quad$ $\bar{y} = \dfrac{3h}{5}$ $\quad\quad$ $\text{Area} = \dfrac{2ah}{3}$



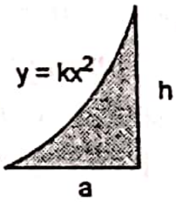
g) Parabolic area

$\bar{x} = 0$ $\quad\quad$ $\bar{y} = \dfrac{3h}{5}$ $\quad\quad$ $\text{Area} = \dfrac{4ah}{3}$



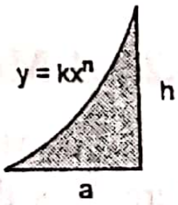
h) Parabolic spandrel

$\bar{x} = \dfrac{3a}{4}$ $\quad\quad$ $\bar{y} = \dfrac{3h}{10}$ $\quad\quad$ $\text{Area} = \dfrac{ah}{3}$



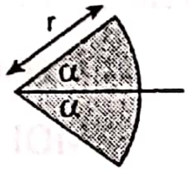
i) General spandrel

$\bar{x} = \dfrac{n+1}{n+2}a$ $\quad\quad$ $\bar{y} = \dfrac{n+1}{4n+2}h$ $\quad\quad$ $\text{Area} = \dfrac{ah}{n+1}$

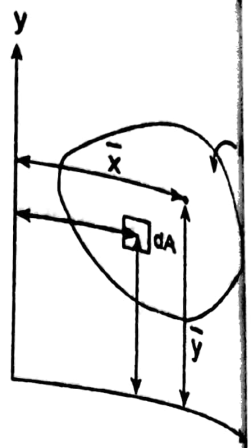


j) Circular sector

$\bar{x} = \dfrac{2r\sin\alpha}{3\alpha}$ $\quad\quad$ $\bar{y} = 0$ $\quad\quad$ $\text{Area} = \alpha r^2$



Consider the area A with centroid at G. The moment of inertia of any plane area A is the second moment of all the small area dA. Comprising the area A about any axis in the plane of area A.



Second moment of area about y-axis

$$I_{yy} = \Sigma(dA\,.\,x)\,x$$

where (dA.x) is the first moment of area dA about y-axis and (dA.x)x is the moment of the first moment of area A, called second moment, about same y-axis.

Similarly, $I_{xx} = \Sigma(dA.y)\,y$

In integration form:

$I_{xx} = \int y^2\,dA$ $\qquad$ $I_{yy} = \int x^2\,dA$

Inertia refers to the property of a body by virtue of which body resi any change in its state of rest or of uniform motion. Moment of area essentially an measure of resistance to bending and is applied while dealin with the deflection or deformation of number in bending. Its unit is four power of length mm^4, cm^4, m^4 etc.

4.2.1.1 Moment of inertia of rectangular lamina

Consider a rectangular lamina of width b and height h as show Here G is the centroid. Consider an elemental area dA at a distance of from base OA.

Elemental area dA = b dy

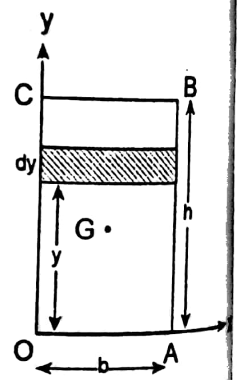
MOI about OA = I_{xx}

$$I_{xx} = \int y^2\,dA$$

$$= \int_0^h y^2\,b\,dy$$

$$= b\left.\frac{y^3}{3}\right|_0^h$$

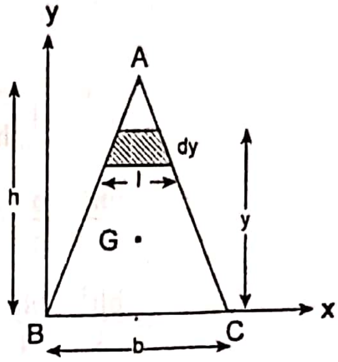
$$\therefore\; I_{xx} = \frac{bh^3}{3}$$



Similarly, $I_{yy} = \dfrac{hb^3}{3}$ [simply change the orientation.]

4.2.1.2 Moment of inertia of triangular lamina

Consider a triangular lamina of base b and height h as shown. The triangle may be considered to consist of a number of infinitely small elemental component parallel to the base BC. Let one such elemental component at a distance y from base BC and thickness dy.



Let l be the length of element

$$\frac{l}{b} = \frac{h-y}{h}$$

$$\therefore\ l = b\left(1-\frac{y}{h}\right)$$

elemental area $dA = l\ dy$

$$= b\left(1-\frac{y}{h}\right)dy$$

Moment of inertia (MOI) about BC = I_{xx}

$$I_{xx} = \int y^2\, dA$$

$$= \int_0^h y^2\, b\left(1-\frac{y}{h}\right)dy$$

$$= b\int_0^h y^2\, dy - \frac{b}{h}\int_0^h y^3\, dy$$

$$= b\,\frac{h^3}{3} - \frac{bh^4}{4h}$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$= \frac{bh^3}{12}$$

Integration approach:

Consider a triangle as shown in figure.

$$y = mx$$

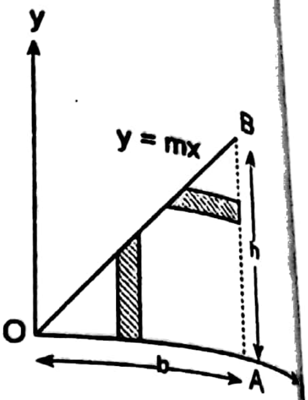
$$h = mb$$

$$m = \frac{h}{b}$$

$$\therefore\ y = \frac{h}{b}x \qquad x = \frac{b}{h}y$$

Centr

Consider a horizontal strip



$$I_{xx} = \int y^2\, dA$$

$$= \int_0^h y^2 (b - x)\, dy$$

$$= \int_0^h by^2\, dy - \int_0^h xy^2\, dy$$

$$= b\frac{h^3}{3} - \int_0^h \frac{b}{h} y\, y^2\, dy$$

$$= \frac{bh^3}{3} - \frac{b}{h} \cdot \frac{y^4}{4}\Big|_0^h$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$= \frac{bh^3}{12} \text{ unit}$$

= MOI about base OA

Alternative approach

To calculate I_{yy}, consider vertical strip

$$I_{yy} = \int x^2\, dA$$

$$= \int_0^b x^2\, y\, dx$$

$$= \int_0^b x^2 \frac{h}{b} x\, dx$$

$$= \frac{h}{b} \int_0^b x^3\, dx$$

$$= \frac{h}{b} \cdot \frac{b^4}{4}$$

$$= \frac{hb^3}{4}$$

= MOI about vertex at O.

4.2.1.3 Moment of inertia of circular lamina

The circular lamina may be considered as consisting of elemental concentric rings. Consider one such elemental ring at a radius r and having thickness dr

$\frac{6h^3}{3}$

Polar MOI of elemental ring = $dA \times r^2$

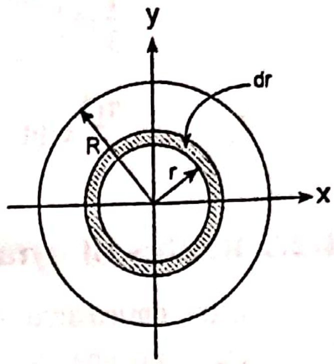
$= 2\pi r\, dr \times r^2$

$dI_p = 2\pi r^3\, dr$

$\therefore I_p = \int_0^R 2\pi r^3\, dr$

$= 2\pi \frac{R^4}{4}$

$= \frac{\pi R^4}{2}$



if diameter $D = 2R$, $I_p = \frac{\pi D^4}{32}$

$\therefore I_{xx} = I_{yy} = \frac{I_p}{2} = \frac{\pi D^4}{64} = \frac{\pi R^4}{4}$

Note polar MOI = $I_{xx} + I_{yy}$

4.2.1.4 Moment of inertia of semi-circle

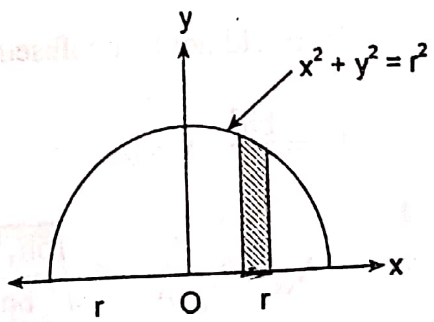
Calculate MOI of semi-circle about base.

$dI_{xx} = \frac{1}{3} y^3\, dx$

$I_{xx} = \int d\, I_{xx}$

$= \int \frac{1}{3} y^3\, dx$

$= \frac{1}{3} \int_{-r}^{r} (r^2 - x^2)^{3/2}\, dx$



$x = r \sin\theta$

$dx = r \cos\theta\, d\theta$

$x \to -1, \qquad r \to -\frac{\pi}{2}$

$x \to 1, r \to \frac{\pi}{2}$

$I_{xx} = \frac{1}{3} \int_{-\pi/2}^{\pi/2} (r^2 - r^2 \sin^2\theta)^{3/2} \cdot r \cos\theta\, d\theta$

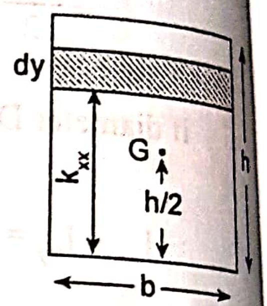
$$= \frac{1}{3} r^4 \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$I_{xx} = \frac{\pi r^4}{8} \text{ unit}$$

4.2.2 Radius of gyration

If the entire area (or mass) of a lamina is concentrated at a point such that there is no change in the moment of inertia about a given axis, then distance of that point from the given axis is called radius of gyration.

Mathematically,



$I = AK^2$

I = Moment of inertia

A = Area of body

K = Radius of gyration

$$\therefore \ k = \sqrt{\frac{I}{A}}$$

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}\ ; \ k_{yy} = \sqrt{\frac{I_{yy}}{A}}$$

$$\text{polar radius of gyration} = k_{zz} = \sqrt{\frac{I_{xx} + I_{yy}}{A}}$$

It should not be confused with centroid of area, ordinate $\overline{y} = \frac{h}{2}$. Here;

$$I_{xx} = \frac{bh^3}{3}, \ A = bh$$

$$\therefore \ k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{bh_3/3}{bh}}$$

$$\therefore \ k_{xx} = \frac{h}{\sqrt{3}}$$

Centroid $\overline{y}$ depends upon the first moment abut radius of gyration k_{xx} depends upon second moment of area.

4.3 Use of parallel axis theorem and perpendicular axis theorem

4.3.1 Parallel axis theorem

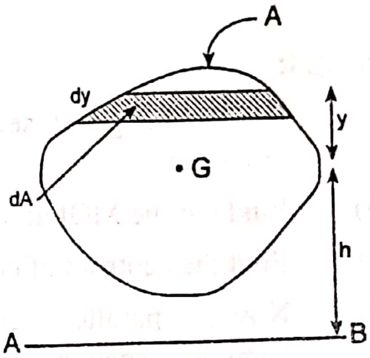
This is also called transfer theorem. It is used to transfer the MOI about one axis to other axis parallel to each other. It states "If the moment of inertia of a plane about an centroidal axis is represented by I_G; the moment of inertia about any other axis AB, parallel to centroidal axis, and at a distance of h from centroid is given by

$$I_{AB} = I_G + Ah^2$$

A is area of lamina.

Proof:

Consider a strip with elemental area dA at a distance of y from centroidal axis. Then, MOI of strip about an axis through C.G. is $y^2\,dA$



MOI of whole area about C.G. is

$$I_G = \int y^2\,dA$$

MOI of whole area about AB is

$$I_{AB} = \int (y + h)^2\,dA$$

$$= \int (y^2\,dA + 2hy\,dA + h^2\,dA)$$

$$\therefore\ I_{AB} = \int y^2\,dA + 2h \int h\,dA + h^2 \int dA$$

$$= I_G + 2h \int y\,dA + Ah^2$$

Here $\int y\,dA$ is the first moment of area about c.g. and is equal to $\bar{y}.A$, where $\bar{y}$ is the distance between the section and the axis passing through c.g. when whole area is considered, this $\bar{y} = 0$

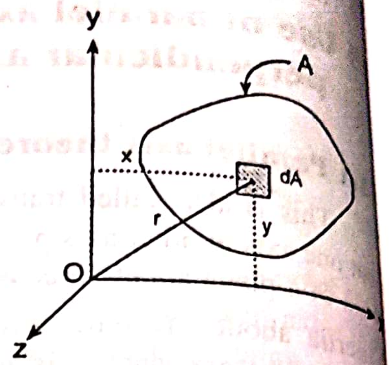
$$\therefore\ \int y\,dA = 0$$

$$\therefore\ I_{AB} = I_G + Ah^2$$

4.3.2 Perpendicular axis theorem

If I_{xx} and I_{yy} be the moment of inertia of a plane section about two perpendicular axes meeting at O, the moment of inertia I_{zz} about the axis z, perpendicular to both previous axes is given by

$I_{zz} = I_{xx} + I_{yy}$



Consider a small lamina of area dA at x and y distance from perpendicular axes y and x respectively.

From geometry; $r^2 = x^2 + y^2$

$$I_{zz} = \int r^2\, dA$$

$$= \int (x^2 + y^2)\, dA$$

$$= \int x^2\, dA + \int y^2\, dA$$

$$\therefore\ I_{zz} = I_{xx} + I_{yy}$$

4.4 Moment of inertia of composite section

Steps:

a) Divide the given section into simple area i.e. rectangle, triangle, circle etc.

b) Find out the MOI about their own centre of gravity.

c) Find the centroid of composite section.

d) Now use parallel axes theorem to transfer to new centroidal axes (of composite section) and about new required axes (if any).

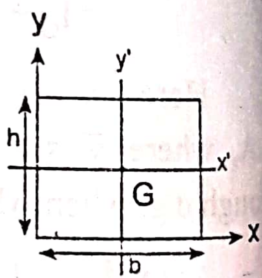
Moments of inertia of common geometric shapes

a) Rectangle

$$I_{x'} = \frac{1}{12} bh^3; \qquad I_{y'} = \frac{1}{12} b^3h;$$

$$I_x = \frac{1}{3} bh^3; \qquad I_y = \frac{1}{3} b^3h,$$

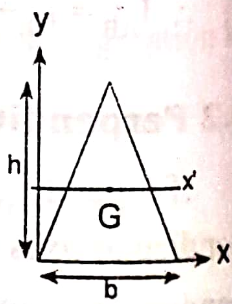
$$J_c = \frac{1}{12} bh(b^2 + h^2)$$



b) Triangle

$$I_{x'} = \frac{1}{36} bh^3;$$

$$I_x = \frac{1}{12} bh^3$$



c) Circle

$I_x = I_y = \frac{1}{4}\pi r^4$

$J_O = \frac{1}{2}\pi r^4$

Semicircle

$I_x = I_y = \frac{1}{8}\pi r$

$J_O = \frac{1}{4}\pi r^4$

Quarter circle

$I_x = I_y = \frac{1}{16}$

$J_O = \frac{1}{8}\pi r^4$

Ellipse

$I_x = \frac{1}{4}\pi ab$

$I_y = \frac{1}{4}\pi a^3$

$J_O = \frac{1}{4}\pi a$

Worked O

1. Locate th

Solution:

Locate

dθ at origin. L

Length

coord

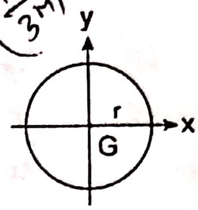
$\overline{X}$

Scanned by CamScanner

c) Circle

$$I_x = I_y = \frac{1}{4}\pi r^{-4};$$

$$J_O = \frac{1}{2}\pi r^4$$

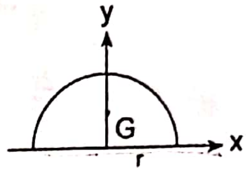


$I_x = I_G + \frac{\pi r^2}{4} \times \left(\frac{4r}{3\pi}\right)^2$

Semicircle

$$I_x = I_y = \frac{1}{8}\pi r^{-4};$$

$$J_O = \frac{1}{4}\pi r^{-4}$$



$I_{GxGx} = \frac{\pi r^4}{8}$

$\frac{\pi r^4}{16} - \frac{\pi \times 16 r^4}{4 \times 9\pi^2}$

$\frac{\pi r^4}{16} - \frac{4r^4}{9\pi}$

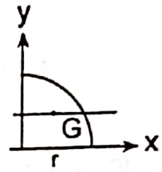
$\left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$

$0.055 r^4$

Quarter circle

$$I_x = I_y = \frac{1}{16}\pi r^{-4};$$

$$J_O = \frac{1}{8}\pi r^4$$

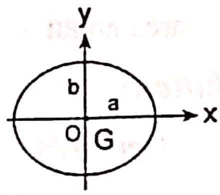


Ellipse

$$I_x = \frac{1}{4}\pi ab^3;$$

$$I_y = \frac{1}{4}\pi a^3 b;$$

$$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$$

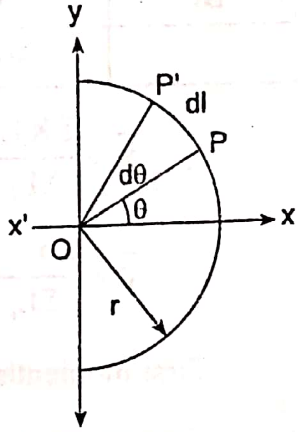


Worked Out Examples

1. Locate the centroid of line of semi-circular arc.

Solution:

Locate the small arc PP′ which subtends angle $d\theta$ at origin. Let P is at an angle θ with OB.



Length of arc PP′ = $dl = r d\theta$

coordinates of P is $(r\cos\theta, r\sin\theta)$

$$\bar{X} = \frac{1}{l}\int x\, dl$$

$$= \frac{1}{\int dl}\int x\, dl$$

$$= \frac{1}{\int_{-\pi/2}^{\pi/2} r\, d\theta} \int_{-\pi/2}^{\pi/2} (r \cos\theta) \,.\, r\, d\theta$$

$$= \frac{1}{\pi r} r^2 \sin\theta \Big|_{-\pi/2}^{\pi/2}$$

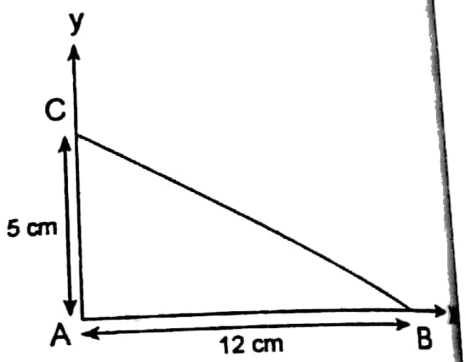
$$= \frac{1}{\pi r} 2r^2$$

$$= \frac{2r}{\pi}$$

Since figure is symmetrical about x-axis, $\overline{Y} = 0$

$$\therefore\ (\overline{X}, \overline{Y} = \frac{2r}{\pi}, 0)$$

2. Locate the centroid of the composite line. Also find the first moment of area about X and Y axes.



Solution:

Here, AB = 12 cm

AC = 5 cm

$BC = \sqrt{5^2 + 12^2} = 13$ cm

Segment	Length L_i (cm)	$\overline{X}_i$ cm	$\overline{Y}_i$ cm	$\overline{X}_i L_i$	$\overline{Y}_i L_i$
AB	12	6	0	72	0
AC	5	0	2.5	0	12.5
BC	13	6	2.5	78	32.5
	30			150	45

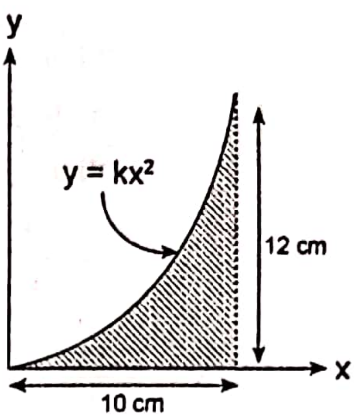
$$\therefore\quad \overline{X} = \frac{\Sigma X_i L_i}{\Sigma L_i} = \frac{150}{30} = 5 \text{ cm}$$

$$\overline{Y} = \frac{\Sigma Y_i L_i}{\Sigma L_i} = \frac{45}{30} = 1.5 \text{ cm}$$

First moment about x-axis $M_x = \Sigma Y_i L_i = 45$ cm^2

First moment about y-axis $M_y = \Sigma X_i L_i = 150$ cm^2

3. Determine by direct integration the location of the centroid of shaded area.



Solution:

Steps:

i) Determine constant k.

ii) Calculate small elemental rectangle and integrate it.

Given curve

$$y = kx^2$$

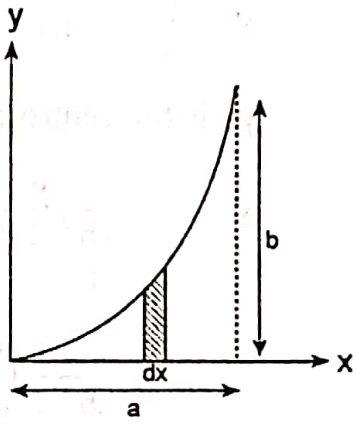
At origin, $0 = k.0^2$ not useful.

Again, $b = ka^2$

$$\therefore \quad k = \frac{b}{a^2}$$

So, $$y = \frac{b}{a^2}x^2$$

$$x = \frac{a}{\sqrt{b}}.\sqrt{y}$$



Draw a elemental rectangle parallel to y-axis having width of dx. The height of elemental rectangle will be y.

Elemental area $dA = y\ dx$

$$\therefore A = \int dA = \int_0^a y\ dx$$

$$= \int_0^a \frac{b}{a^2}.x^2\ dx$$

$$= \frac{b}{a^2}\cdot\frac{a^3}{3}$$

$$\therefore A = \frac{ab}{3}$$

Now, $\overline{X} = \frac{1}{A} x_{el}\ dA$

$$= \frac{3}{ab}\int_0^a x \cdot \frac{b}{a^2}x^2\,dx$$

$$= \frac{3}{ab}\cdot\frac{b}{a^2}\int_0^a x^3\,dx$$

$$= \frac{3}{a^3}\cdot\frac{a^4}{4}$$

$$\therefore\ \overline{X} = \frac{3a}{4}$$

Again,

$$\overline{Y} = \frac{1}{A}\int y_{el}\,dA$$

y_{el} is the centroid of elemental rectangle.

$$\overline{Y} = \frac{1}{\frac{ab}{3}}\int \frac{y}{2}\cdot y\,dx \quad \left(\text{Here } y_{el} = \frac{y}{2} \text{ why?}\right)$$

$$= \frac{3\times 1}{ab\,.\,2}\int_0^a y^2\,dx$$

$$= \frac{3}{2ab}\int_0^a \frac{b^2}{a^4}\,.\,x^4\,dx$$

$$= \frac{3}{2ab}\cdot\frac{b^2}{a^4}\cdot\frac{a^5}{5}$$

$$\therefore\ \overline{Y} = \frac{3b}{10}$$

$$\therefore\ \overline{X} = \frac{3a}{4},\ \overline{Y} = \frac{3b}{10}$$

if a = 10 cm, b = 12 cm

$\overline{X} = 7.5$ cm $\qquad \overline{Y} = 3.6$ cm

Alternative method:

Till now we take vertical elemental rectangle, but we can take horizontal rectangle also.

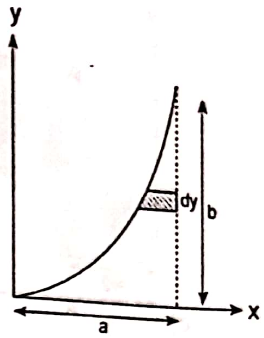
Be careful while calculating the length of elemental rectangle.

Elemental area dA

$\therefore\ dA = (a - x)\,dy$ (but not x dy, why?)

$\therefore\ A = \int dA = \int_0^b (a - x)\,dy$

$= \frac{ab}{3}$



$\overline{X} = \frac{1}{A}\int x_{el}\,dA$

$= \frac{1}{A}\int_0^b \left(x + \frac{a-x}{2}\right).(a - x)\,dy$ (why?)

$= \frac{3a}{4}$

Again,

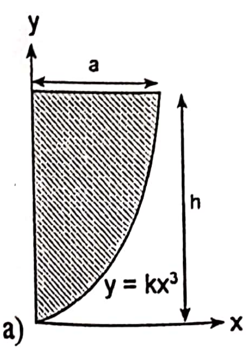
$\overline{Y} = \frac{1}{A}\int y_{el}\,dA$

$= \frac{1}{A}\int_0^b y.(a - x)\,dy$ (why?)

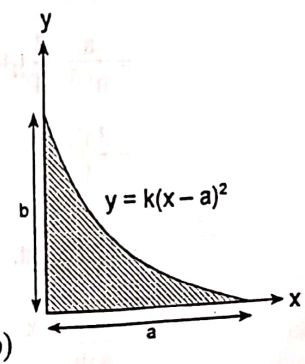
$= \frac{3b}{10}$

So, proper selection of elemental rectangle is required for fast and easy solution.

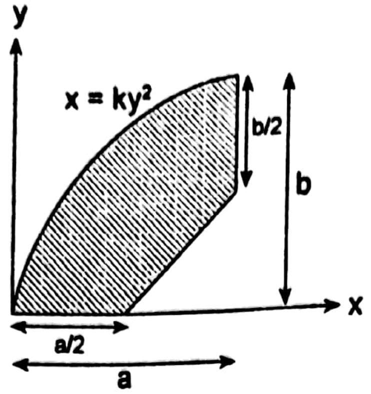
4. Locate the centroid of shaded area:



a)



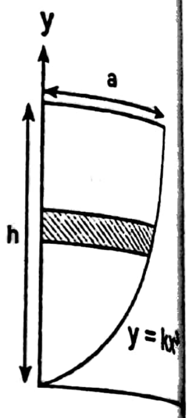
b)



c)

Solution:

a) Take a horizontal strip of thickness dy.



$y = kx^3$

$h = ka^3$

$\therefore\ k = \dfrac{h}{a^3}$

$\therefore\ y = \dfrac{h}{a^3} . x^3$

$x = \dfrac{a.y^{1/3}}{h^{1/3}}$

Small area $dA = x\,dy$

$$\therefore\ A = \int dA$$

$$= \int_0^h x\,dy$$

$$= \int_0^h \frac{a.y^{1/3}}{h^{1/3}} = \frac{a}{h^{1/3}} \cdot \left.\frac{y^{4/3}}{4/3}\right|_0^h$$

$$= \frac{a}{h^{1/3}} \cdot \frac{3}{4} h^{4/3}$$

$$= \frac{3ah}{4}$$

$$\overline{X} = \frac{1}{A}\int x_{el}\,dA$$

$$= \frac{4}{3ah}\int_0^h \frac{x}{2} \cdot x\,dy$$

$$= \frac{2}{3ah}\int_0^h x^2\,dy$$

$$= \frac{2}{3ah} \int_0^h \frac{a^2\, y^{2/3}}{h^{2/3}}\, dy$$

$$= \frac{2a}{3h^{5/3}} \int_0^h y^{2/3}\, dy$$

$$= \frac{2a}{3h^{5/3}} \cdot \frac{y^{5/3}}{5/3} \Bigg|_0^h$$

$$= \frac{2a}{3} \times \frac{3}{5}$$

$$\therefore\ \overline{X} = \frac{2a}{5}$$

$$\overline{Y} = \frac{1}{A} \int y_{el}\, dA$$

$$= \frac{4}{3ah} \int_0^h y \,.\, x\, dy$$

$$= \frac{4}{3ah} \int_0^h y \cdot \frac{a}{h^{1/3}} \,.\, y^{1/3}\, dy$$

$$= \frac{4}{3h^{4/3}} \cdot \int_0^h y^{4/3}\, dy$$

$$= \frac{4}{3} \cdot \frac{1}{h^{4/3}} \cdot \frac{y^{7/3}}{7/3} \Bigg|_0^h$$

$$= \frac{4}{7} \,.\, h$$

$$\therefore\ \overline{Y} = \frac{4h}{7}$$

b) Take vertical strip

$y = k(x - a)^2$

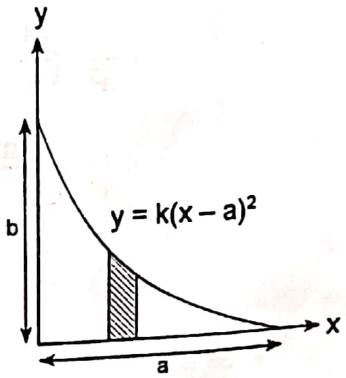
$x \rightarrow a,\ y \rightarrow 0$ at point A.

$O = k.0$ not useful.

$x \rightarrow 0,\ y \rightarrow b$ at point B

$b = k(0 - a)^2$

$b = k.a^2$



$k = \frac{b}{a^2}$

$\therefore\ y = \frac{b}{a^2}.(x-a)^2$

$dA = y\,dx$

$\therefore$ Area $A = \int_0^a y\,dx$

$= \frac{b}{a^2}\int_0^a (x-a)^2\,dx$

$= \frac{b}{a^2}\int_0^a (x^2 - 2ax + a^2)\,dx$

$= \frac{b}{a^2}.[\frac{a^3}{3} - 2a.\frac{a^2}{2} + a^3]$

$= \frac{b}{a^2}.\frac{a^3}{3}$

$\therefore\ A = \frac{ab}{3}$ unit

$\overline{X} \quad = \frac{1}{A}\int x_{el}\,dA$

$= \frac{3}{ab}\int_0^a xy\,dx$

$= \frac{3}{ab}\int_0^a x.\frac{b}{a^2}(x-a)^2\,dx$

$= \frac{3}{ab}\int_0^a \frac{b}{a^2}.x(x^2 - 2ax + a^2)\,dx$

$= \frac{3}{ab}\int_0^a (x^3 - 2ax^2 + a^2x)\,dx$

$= \frac{3}{a^3}.[\frac{a^4}{4} - 2a.\frac{a^3}{3} + \frac{a^4}{2}]$

$= \frac{3}{a^3}.\left[\frac{3a^4 - 8a^4 + 6a^4}{12}\right]$

$= \frac{3}{a^3}.\frac{a^4}{12}$

$\therefore\ \overline{X} \quad = \frac{a}{4}$

$$\overline{Y} = \frac{1}{A}\int y_{el}\, dA$$

$$= \frac{3}{ab}\int_0^a \frac{y}{2} \cdot y\, dx$$

$$= \frac{3}{2ab}\int_0^a y^2\, dx$$

$$= \frac{3}{2ab}\int_0^a k^2(x-a)^4\, dx$$

$$= \frac{3}{2ab} \cdot \frac{b^2}{a^4}\int_0^a (x-a)^4\, dx$$

$$= \frac{3b}{2a^4}\int_0^a (x-a)^4\, dx$$

Let $x - a = z$

$$dx = dz$$

$x \to 0;\qquad z \to -a$

$x \to a;\qquad z \to 0$

$$\therefore\ \overline{Y} = \frac{3b}{2a^4}\int_{-a}^{0} z^4\, dz$$

$$= \frac{3b}{2a^4} \cdot \left.\frac{z^4}{5}\right|_{-a}^{0}$$

$$= \frac{3b}{10a^4} \cdot a^4$$

$$= \frac{3b}{10}$$

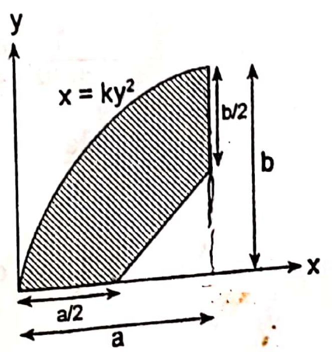
c) Required area = Area of curve – Area of triangle

Let area of curve = A_1

area of triangle = A_2

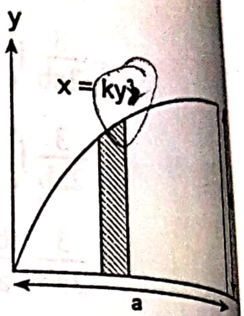
Centroid of curve $(\overline{X}_1, \overline{Y}_1)$

Centroid of triangle = $(\overline{X}_2, \overline{Y}_2)$



Then, $\overline{X} = \dfrac{A_1\overline{X}_1 - A_2\overline{X}_2}{A_1 - A_2}$; $\overline{Y} = \dfrac{A_1\overline{Y}_1 - A_2\overline{Y}_2}{A_1 - A_2}$

Now consider the curve only



$x = ky^2$

$a = kb^2$

$\therefore\ k = \dfrac{a}{b^2}$

$\therefore\ x = \dfrac{a}{b^2} \cdot y^2$

$y = \dfrac{b}{\sqrt{a}}\sqrt{x}$

$dA = \int y\,dx$

$A_1 \quad = \int_0^a \dfrac{b}{\sqrt{a}} \cdot \sqrt{x}\,dx$

$= \dfrac{b}{\sqrt{a}} \cdot \int_0^a x^{1/2}\,dx \quad \Rightarrow \quad A_1 = \dfrac{b}{\sqrt{a}} \cdot \dfrac{a^{3/2}}{3/2} = \dfrac{2ab}{3}$ unit

$\therefore\ A_1 = \dfrac{2ab}{3}$ unit

$\overline{X}_1 \quad = \dfrac{1}{A}\int x_{el} \cdot dA$

$= \dfrac{3}{2ab}\int_0^a x.y\,dx$

$= \dfrac{3}{2ab} \cdot \dfrac{b}{\sqrt{a}}\int_0^a x \cdot x^{1/2}\,dx$

$= \dfrac{3}{2a^{3/2}} \cdot \int_0^a x^{3/2}\,dx$

$= \dfrac{3}{2a^{3/2}} \cdot \dfrac{a^{5/2}}{5/2}$

$\therefore\ \overline{X}_1 = \dfrac{3a}{5}$

$\overline{Y}_1 \quad = \dfrac{1}{A}\int y_{el}\,dA$

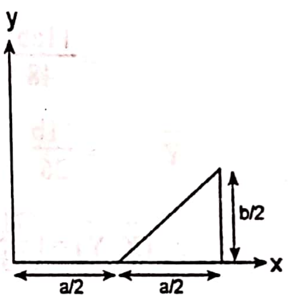
$$= \frac{3}{2ab}\int_0^a \frac{y}{2} . y \, dx$$

$$= \frac{3}{4ab}\int_0^a \frac{b^2}{a} . x \, dx$$

$$= \frac{3b}{4a^2} . \frac{x^2}{2}\Big|_0^a$$

$$\therefore \ \bar{Y}_1 = \frac{3b}{8}$$

Consider triangle only.



$$\text{Area } A_2 = \frac{1}{2} \times \frac{a}{2} \times \frac{b}{2} = \frac{ab}{8}$$

$$\bar{X}_2 = \frac{a}{2} + \frac{2}{3} . \frac{a}{2} = \frac{a}{3} + \frac{a}{2}$$

$$\bar{Y}_2 = \frac{1}{3} . \frac{b}{2} = \frac{b}{6}$$

$\therefore$ Area required $= A_1 - A_2$

$$= \frac{2ab}{3} - \frac{ab}{8} = \frac{13ab}{24}$$

$$\bar{X} = \frac{A_1X_1 - A_2X_2}{A_1 - A_2}$$

$$= \frac{\frac{2ab}{3} \times \frac{3a}{5} - \frac{ab}{8} \times \left(\frac{a}{3} + \frac{1}{2}\right)}{\frac{13ab}{24}}$$

$$= \frac{\frac{2a^2b}{5} - \frac{ab}{8} \times \frac{5a}{6}}{\frac{13ab}{24}}$$

$$= \frac{96a^2b - 25a^2b}{48 \times 5} \times \frac{24}{13ab}$$

$$= \frac{71a^2b}{130ab}$$

$$\therefore \ \bar{X} = \frac{71a}{130}$$

Scanned by CamScanner

$$\overline{Y} = \frac{A_1Y_1 - A_2Y_2}{A_1 - A_2}$$

$$= \frac{\frac{2ab}{3} \times \frac{3b}{8} - \frac{ab}{8} \times \frac{b}{6}}{\frac{13ab}{24}}$$

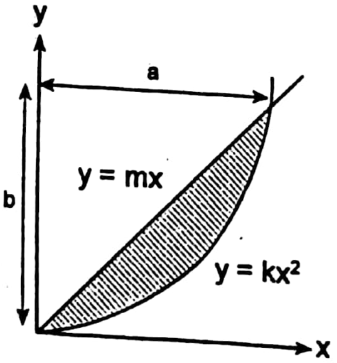
$$= \frac{\frac{ab^2}{4} - \frac{ab^2}{48}}{\frac{13ab}{24}}$$

$$= \frac{11ab^2}{48} \times \frac{24}{13ab}$$

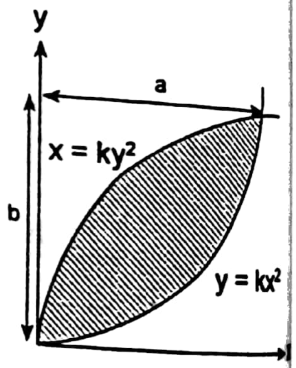
$$\overline{Y} = \frac{11b}{26}$$

$$\therefore\ (\overline{X}, \overline{Y}) = \left(\frac{71a}{130}, \frac{11b}{26}\right)$$

5. Locate the centroid of shaded area



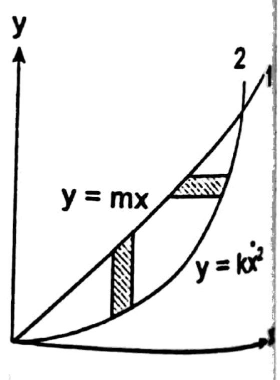
a)



b)

Solution:

a) For the intersection of curve, the solving method is easier when we draw both horizontal and vertical strip.



To calculate $\overline{X}$, take vertical strip.

To calculate $\overline{Y}$, take horizontal strip.

Let $y_1 = mx_1, \quad y_2 = kx_2^2$

$b = mab = k.a^2$

$\therefore \quad m = \frac{b}{a} \qquad k = \frac{b}{a^2}$

or, $\quad y_1 = \frac{b}{a} x_1 \qquad x_1 = \frac{a}{b} y_1$

$y_2 = \frac{b}{a^2} {x_2}^2 \qquad x_2 = \frac{a}{\sqrt{b}} \cdot \sqrt{y_2}$

For the calculation of area you can take either strip.

Taking vertical strip

$dA = (y_1 - y_2)\, dx$

$$\therefore A = \int dA = \int_0^a (y_1 - y_2)\, dx$$

$$= \int_0^a \left(\frac{b}{a} x_1 - \frac{b}{a^2} {x_2}^2\right) dx$$

For vertical strip $x_1 = x_2 = x$ (say)

$$\therefore A = \int_0^a \left(\frac{b}{a} x - \frac{b}{a^2} x^2\right) dx$$

$$= \int_0^a \frac{b}{a} x\, dx - \int_0^a \frac{b}{a^2} x^2\, dx$$

$$= \frac{b}{a} \cdot \frac{a^2}{2} - \frac{b}{a^2} \cdot \frac{a^3}{3}$$

$$= \frac{ab}{2} - \frac{ab}{3}$$

$$\therefore A = \frac{ab}{6} \text{ unit}$$

Calculation of $\overline{X}$; consider vertical strip

$$\overline{X} = \frac{1}{A} \int x_{el}\, dA$$

$$= \frac{6}{ab} \int_0^a x.(y_1 - y_2)\, dx$$

$$= \frac{6}{ab} \int_0^a x \left[\frac{b}{a} x_1 - \frac{b}{a^2} {x_2}^2\right] dx$$

$$= \frac{6}{ab} \int_0^a \left(\frac{b}{a} x^2 - \frac{b}{a^2} x^3\right) dx$$

$$= \frac{6}{ab}\left[\frac{b}{a}\cdot\frac{a^3}{3} - \frac{b}{a^2}\cdot\frac{a^4}{4}\right]$$

$$= \frac{6}{ab}\cdot\left(\frac{a^2b}{3} - \frac{a^2b}{4}\right)$$

$$\therefore\ \overline{X} = \frac{a}{2}$$

Calculation of $\overline{Y}$, consider horizontal strip

$$\overline{Y} = \frac{1}{A}\int y_{el}\, dA$$

$$= \frac{6}{ab}\int_0^a y(x_2 - x_1)\, dy$$

$$= \frac{6}{ab}\int_0^b y\left(\frac{a}{\sqrt{b}}\cdot\sqrt{y_2} - \frac{a}{b}y_1\right)dy$$

$$= \frac{6}{ab}\int_0^b \left(\frac{a}{\sqrt{b}}\cdot y^{3/2} - \frac{a}{b}y^2\right)dy$$

$$= \frac{6}{ab}\left[\frac{a}{\sqrt{b}}\cdot\frac{b^{5/2}}{5/2} - \frac{a}{b}\cdot\frac{b^3}{3}\right]$$

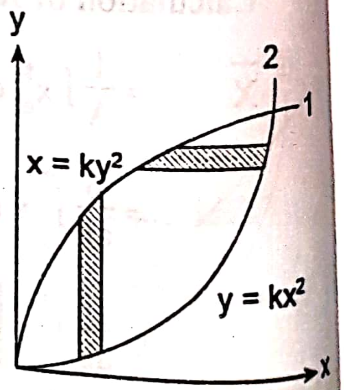
$$= \frac{6}{ab}\left[\frac{2a}{5}\cdot b^2 - \frac{ab^2}{3}\right]$$

$$\therefore\ \overline{Y} = \frac{2b}{5}$$

$$\therefore\ (\overline{X}, \overline{Y}) = \left(\frac{a}{2}, \frac{2b}{5}\right)$$

b) Here

$x_1 = ky_1^2$	$y_2 = kx_2^2$
$a = k.b^2$	$b = k.a^2$
$\therefore\ k = \frac{a}{b^2}$	$k = \frac{b}{a^2}$
$x_1 = \frac{a}{b^2}.y_1^2$	$y_2 = \frac{b}{a^2}.x^2$
$y_1 = \frac{b}{\sqrt{a}}.\sqrt{x_1}$	$x_2 = \frac{a}{\sqrt{b}}.\sqrt{y_2}$



$$A = \int dA = \int_0^a (y_1 - y_2)\, dx$$

$$= \int_0^a \frac{b}{\sqrt{a}} \sqrt{x_1} - \frac{b}{a^2} x_2^2$$

Here for vertical strip, $x_1 = x_2 = x$

$$A = \int_0^a \left(\frac{b}{\sqrt{a}} \sqrt{x} - \frac{b}{a^2} x^2\right) dx$$

$$= \frac{b}{\sqrt{a}} \cdot \frac{a^{3/2}}{3/2} - \frac{b}{a^2} \cdot \frac{a^3}{3}$$

$$= \frac{2b}{3} .a - \frac{ab}{3}$$

$$= \frac{ab}{3} \text{ unit}$$

Calculation of $\overline{X}$, consider vertical strip

$$\overline{X} = \frac{1}{A} \int x_{el}\, dA$$

$$= \frac{1}{A} \int_0^a x(y_1 - y_2)\, dx$$

$$= \frac{1}{\frac{ab}{3}} \int_0^a x\left[\frac{b}{\sqrt{a}} \sqrt{x} - \frac{b}{a^2} x^2\right] dx$$

$$= \frac{1}{\frac{ab}{3}} \int_0^a \left(\frac{b}{\sqrt{a}} x^{3/2} - \frac{b}{a^2} x^3\right) dx$$

$$= \frac{1}{\frac{ab}{3}} \left[\frac{b}{\sqrt{a}} \cdot \frac{a^{5/2}}{5/2} - \frac{b}{a^2} \cdot \frac{a^4}{4}\right]$$

$$= \frac{3}{ab} \left[\frac{2b}{5} \cdot a^2 - \frac{a^2 b}{4}\right]$$

$$= \frac{3}{ab} \cdot \frac{8a^2b - 5a^2b}{20}$$

$$\therefore \overline{X} = \frac{9a}{20}$$

Calculation of $\overline{Y}$, consider horizontal strip

$$\overline{Y} = \frac{1}{A}\int y_{el}\, dA$$

$$= \frac{3}{ab}\int_0^b y\,(x_2 - x_1)\, dy$$

$$= \frac{3}{ab}\int_0^b y\left(\frac{a}{\sqrt{b}}\sqrt{y_2} - \frac{a}{b^2}\, y_1^{\,2}\right) dy$$

$$= \frac{3}{ab}\int_0^b \left(\frac{a}{\sqrt{b}}\, y^{3/2} - \frac{a}{b^2}\, y^3\right) dy$$

$$= \frac{3}{ab}\left[\frac{a}{\sqrt{b}} \cdot \frac{b^{5/2}}{5/2} - \frac{a}{b^2} \cdot \frac{b^4}{4}\right]$$

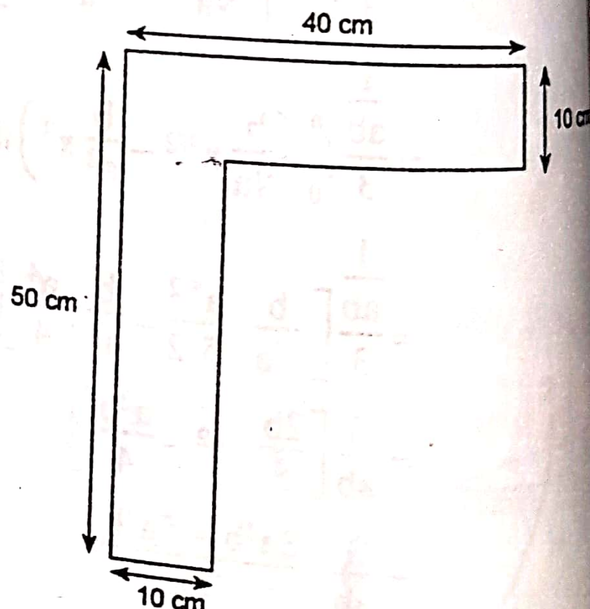
$$= \frac{3}{ab}\left[\frac{2a}{b}\, b^2 - \frac{ab^2}{4}\right]$$

$$= \frac{3}{ab} \cdot \frac{3ab^2}{20}$$

$$\therefore \overline{Y} = \frac{9b}{20}$$

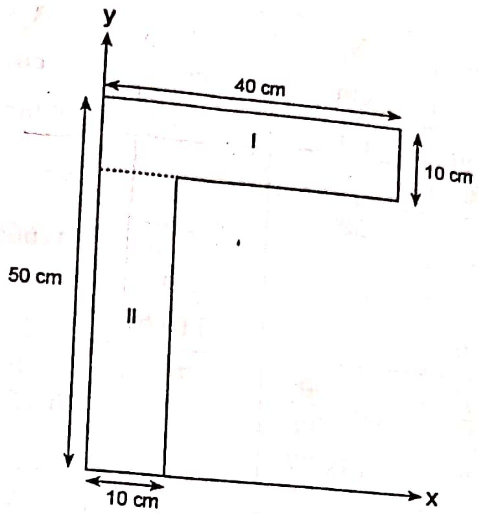
$$\therefore (\overline{X}, \overline{Y}) = \left(\frac{9a}{20}, \frac{9b}{20}\right)$$

6. Locate the centroid of composite area.



Solution:

Here the axis is not given, so we have to assume axes. Also, the required section is sum of two rectangle.

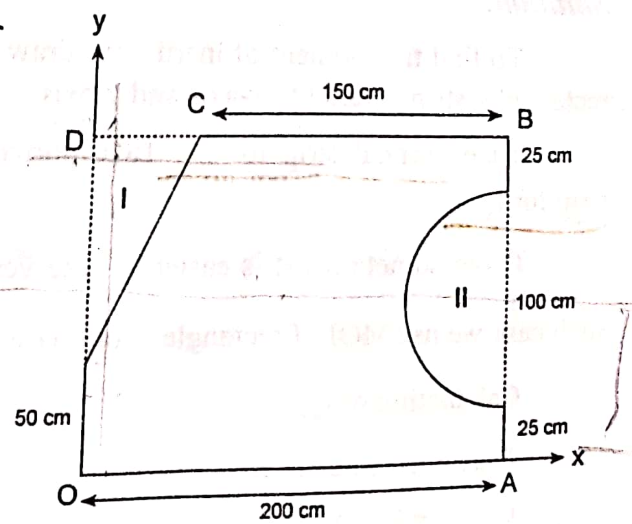


segment	Area (A_i) cm^2	X_i cm	Y_i cm	A_iX_i	A_iY_i
Rect (i)	400	20	45	8000	18000
Rect (ii)	400	5	20	2000	8000
	800			1000	26000

$$\bar{X} = \frac{\Sigma A_i X_i}{\Sigma A_i} = \frac{10000}{800} = 12.5 \text{ cm}$$

$$\bar{Y} = \frac{\Sigma A_i Y_i}{\Sigma A_i} = \frac{26000}{800} = 32.5 \text{ cm}$$

7. Locate the centroid of the given composite area.



Solution:

Required area

= Rect. OABD

– Fig (i)

– fig (ii)

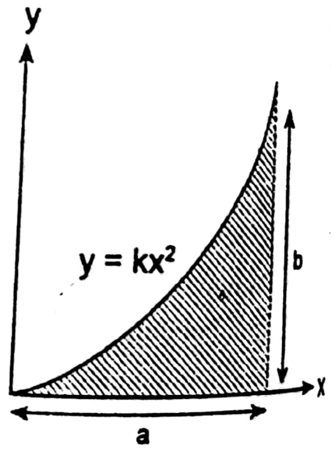
Segment	Area (A_i) cm^2	X_i cm	Y_i cm	A_iX_i cm^3	A_iY_i cm^3
Rect OABD	$200 \times 150 = 30000$	100	75	3000000	22500(0)
Fig. (i)	$-\frac{1}{2} \times 50 \times 100 = -2500$	$\frac{50}{3}$	$50 + \frac{100}{3} \times 2 = 116.67$	(−) 41666.67	(−) 29167(5)
Fig. (ii)	$-\pi \times \frac{50^2}{2} = -3925$	$200 - \frac{4r}{3\pi} = 178.77$	75	(−) 701672.25	(−) 294375
	$\Sigma = 23575$			$\Sigma =$ 2256661	$\Sigma =$ 166395(

(−) area for not required area

$$\therefore \quad \bar{X} = \frac{\Sigma A_i X_i}{\Sigma A_i} = \frac{225661}{23575} = 95.72 \text{ cm}$$

$$\bar{Y} = \frac{\Sigma A_i Y_i}{\Sigma A_i} = \frac{1663950}{23575} = 70.58 \text{ cm}$$

8. Calculate the MOI of the shaded area by direct integration about axes.



Solution:

To find the moment of inertia, we draw two rectangular strip parallel to x-axis and y-axis.

Take vertical strip for I_{yy}. Take horizontal strip for I_{xx}.

(Note sometimes it is easier to take vertical strip to calculate I_{xx}. In such case we use MOI of rectangle about base is $\frac{bh^3}{3}$ and integrate it.)

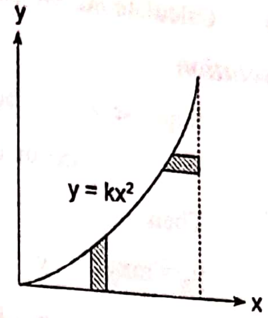
Calculation of I_{xx}:

Take horizontal strip

$$I_{xx} = \int y^2 \, dA$$

$$= \int y^2 (a - x) \, dy$$

$$= \int_0^b y^2 a\, dy - \int_0^b y^2 x\, dy$$



$$= \frac{ab^3}{3} - \int_0^b y^2 \frac{a}{\sqrt{b}} \sqrt{y}\, dy$$

$$= \frac{ab^3}{3} - \frac{a}{\sqrt{b}} \cdot \frac{b^{7/2}}{7/2}$$

$$= \frac{ab^3}{3} - \frac{2}{7} ab^3$$

$$= \frac{ab^3}{21} \text{ unit}$$

Alternative:

Take vertical strip

$$I_{xx} = \int_0^a \frac{y^3}{3} dx$$

$$= \frac{1}{3} \int_0^a \frac{b^3}{a^6} x^6\, dx$$

$$= \frac{1}{3} \frac{b^3}{a^6} \frac{a^7}{7}$$

$$I_{xx} = \frac{ab^3}{21} \text{ unit}$$

Calculation of I_{yy}:

Take vertical strip

$$I_{yy} = \int x^2\, dA$$

$$= \int_0^a x^2 . y\, dx$$

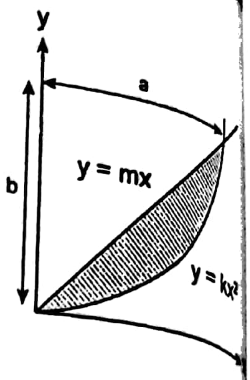
$$= \int_0^a x^2 \frac{b}{a^2} x^2\, dx$$

$$= \frac{b}{a^2} \int_0^a x^4\, dx$$

$$= \frac{b}{a^2} \frac{a^5}{5}$$

$$I_{yy} = \frac{a^3 b}{5} \text{ unit}$$

9. **Calculate the MOI of given area.**



Solution:

Suppose $y = mx$ be eq (ii)

$y = kx^2$ be eq (i)

Then,

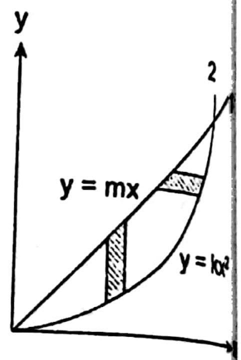
$y_2 = mx_2 \qquad y_1 = kx_1^2$

$b = m.a \qquad b = ka^2$

$\therefore\ m = \frac{b}{a} \qquad \therefore\ k = \frac{b}{a^2}$

$y_2 = \frac{b}{a} x_2 \qquad y_1 = \frac{b}{a^2} x_1^2$

$x_2 = \frac{a}{b} y_2 \qquad y_2 = \frac{a}{\sqrt{b}} \sqrt{y_1}$



Calculation of I_{xx}:

Take horizontal strip

$$I_{xx} = \int y^2\, dA$$

$$= \int_0^b y^2 (x_1 - x_2)\, dy$$

$$= \int_0^b y^2 \left(\frac{a}{\sqrt{b}} \sqrt{y_1} - \frac{a}{b} y_2\right) dy$$

$$= \int_0^b \left(\frac{a}{\sqrt{b}} y^{5/2} - \frac{a}{b} y^3\right) dy$$

$$= \frac{a}{\sqrt{b}} \cdot \frac{b^{7/2}}{7/2} - \frac{a}{b} \cdot \frac{b^4}{4}$$

$$= \frac{2a}{7} . b^3 - \frac{ab^3}{4}$$

$$\therefore\ I_{xx} = \frac{ab^3}{28} \text{ unit}$$

(It is not recommended to use vertical strip to calculate I_{xx} here why?)

Calculation of I_{yy}:

Take vertical strip

$$I_{yy} = \int x^2\, dA$$

$$= \int_0^a x^2 (y_2 - y_1)\, dx$$

$$= \int_0^a x^2 \left[\frac{b}{a} x_2 - \frac{b}{a_2} x_1^2\right] dx$$

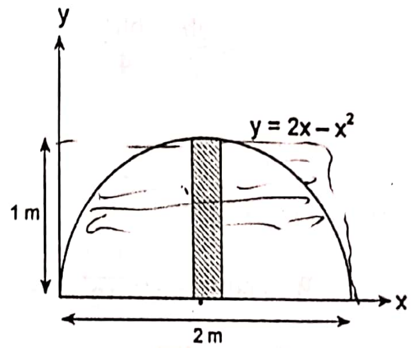
$$= \int_0^a \left(\frac{b}{a} x^3 - \frac{b}{a^2} x^4\right) dx$$

$$= \frac{b}{a} \cdot \frac{a^4}{4} - \frac{b}{a^2} \cdot \frac{a^5}{5}$$

$$= \frac{a^3 b}{4} - \frac{a^3 b}{5}$$

$$\therefore\ I_{yy} = \frac{a^3 b}{20} \text{ unit}$$

10. Calculate the MOI about x-axis.



Solution:

Here it is recommended to use vertical strip to calculate I_{xx}.

$$I_{xx} = \frac{bh^3}{3} \qquad \text{[for rectangle]}$$

$$\therefore\ I_{xx} = \int_0^2 \frac{1}{3} y^3\, dx$$

$$= \frac{1}{3}\int_0^2 (2x - x^2)^3\, dx$$

$$= \frac{1}{3}\int_0^2 (8x^3 - x^6 - 3(2x)^2x^2 + 3.2x.x^4)\, dx$$

$$= \frac{1}{3}\int_0^2 (8x^3 - x^6 - 12x^4 + 6x^5)\, dx$$

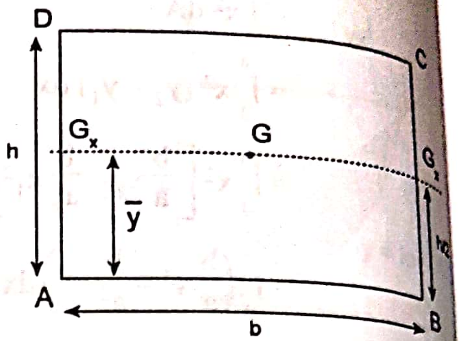
$$= \frac{1}{3}\left[\frac{8x^4}{4} - \frac{x^7}{7} - 12\frac{x^5}{5} + \frac{6x^6}{6}\right]_0^2$$

$$= \frac{1}{3}\left[2 \times 16 - \frac{128}{7} - \frac{384}{5} + 64\right]$$

$$= \frac{1}{3}[0.914]$$

$$I_{xx} = 0.3047 \text{ m}^4$$

11. Calculate the moment of inertia of rectangle and triangle through centroid.



Solution:

a) for rectangle:

We know, $I_{AB} = \dfrac{bh^3}{3}$

Using parallel axis theorem;

$$I_{AB} = I_{GXGX} + A(\bar{y})^2$$

$$\frac{bh^3}{3} = I_{GXGX} + bh\left(\frac{h}{2}\right)^2$$

$$I_{GXGX} = \frac{bh^3}{3} - bh\frac{h^2}{4}$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$\therefore\ I_{GXGX} = \frac{bh^3}{12}$$

$\dfrac{bh^3}{12} + b \times h \times \dfrac{h^2}{4}$

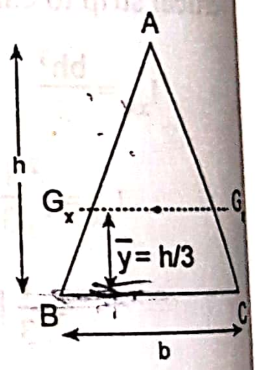
$I_x = I_G + A \times h^2$

$\dfrac{bh^3}{12} + \dfrac{bh^3}{4}$

$\dfrac{4bh^3}{3 \times 12}$

b) For triangle:

We know, $I_{BC} = \dfrac{bh^3}{12}$



Using parallel axis

$$I_{BC} = I_{GXGX} + A\,(\bar{y})^2$$

$$\frac{bh^3}{12} = I_{GXGX} + \frac{1}{2} \times b \times h\left(\frac{h}{3}\right)^2$$

$$\therefore\ I_{GXGX} = \frac{bh^3}{12} - \frac{1}{2}\cdot\frac{bh^3}{9}$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$= \frac{bh^3}{36} \text{ unit}$$

12. Calculate the MOI of semicircular section about centroidal axes.

Solution:

Here, MOI of semicircular about the base AC is equal to the half of MOI of circular section about AC.

$\therefore$ MOI$_{AC} = \frac{1}{2} \times \frac{\pi r}{4}$

The $\bar{y} = \frac{4r}{3\pi}$

[Distance

Using parallel ax

$I_{AC} = I_{GXGX} + A.$

$\frac{\pi r^4}{8} = I_{GXGX} + \frac{\pi r}{2}$

$\frac{\pi r^4}{8} = I_{GXGX} + \frac{\pi}{}$

$I_{GXGX} = \frac{\pi r^4}{8} - \frac{8}{9}$

$\therefore$ $I_{GXGX} = 0.1$

MOI about ce

13. Find the MOI a

Solution:

Here the M directly calculated calculate MOI abo MOI of curve abou of triangle about sa

Let $y = kx^2$

$y = $

$\therefore$ $y_1 =$

This equati

$b = k.0 + c$

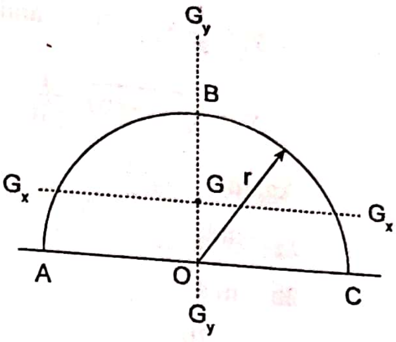
$\therefore$ $c = b$

Again, 2b

$b = ka^2$

$\therefore$ $k = \frac{b}{a^2}$

$$\therefore \text{ MOI}_{AC} = \frac{1}{2} \times \frac{\pi r^4}{4} = \frac{\pi r^4}{8}$$



The $\overline{y} = \dfrac{4r}{3\pi}$

[Distance of G from AC]

Using parallel axes theorem

$$I_{AC} = I_{GXGX} + A.h^2$$

$$\frac{\pi r^4}{8} = I_{GXGX} + \frac{\pi r^2}{2} \cdot \left(\frac{4r}{3\pi}\right)^2$$

$$\frac{\pi r^4}{8} = I_{GXGX} + \frac{\pi r^2}{2} \cdot \frac{16 r^2}{9\pi^2}$$

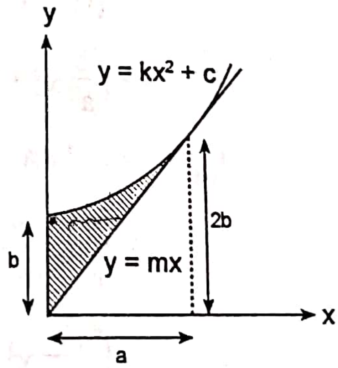
$$I_{GXGX} = \frac{\pi r^4}{8} - \frac{8 r^4}{9\pi}$$

$$\therefore \ I_{GXGX} = 0.11 r^4$$

MOI about centroidal y-axis is $I_{GYGY} = \dfrac{\pi r^4}{8} = 0.393 r^4$

13. Find the MOI about axes.

Solution:



Here the MOI about y axis can be directly calculated using vertical strip. To calculate MOI about x axis, we first calculate MOI of curve about x axis and subtract the MOI of triangle about same x axis.

Let $y = kx^2 + c$ ......(i)

$y = mx$ ......(ii)

$\therefore \quad y_1 = k x_1^2 + c$

This equation passes through (0, b) and (a, 2b)

$b = k.0 + c$

$\therefore \ c = b$

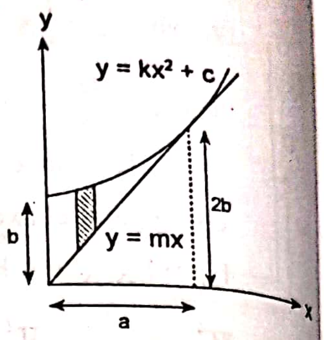
Again, $2b = k.a^2 + b$

$b = ka^2$

$\therefore \ k = \dfrac{b}{a^2}$

$\therefore\ y_1 = \frac{b}{a^2} x_1{}^2 + b$ and

$x_1 = \sqrt{(y_1 - b)} \cdot \frac{a}{\sqrt{b}}$



Again,

$y_2 = mx_2$

$2b = m.a$

$\therefore\ m = \frac{2b}{a}$

$\therefore\ y_2 = \frac{2b}{a} x_2$ and $x_2 = \frac{a}{2b} y_2$

Calculation of I_{yy}

Consider a vertical strip

$$dA = (y_1 - y_2)\,dx$$

$$= \left\{\left(\frac{b}{a^2} x_1{}^2 + b\right) - \left(\frac{2b}{a} x_2\right)\right\} dx$$

$$= \left(\frac{b}{a^2} x^2 + b - \frac{2b}{a} x\right) dx$$

$$I_{yy} = \int x^2\, dA$$

$$= \int_0^a x^2 \left(\frac{b}{a^2} x^2 + b - \frac{2b}{a} x\right) dx$$

$$= \int_0^a \left(\frac{b}{a^2} x^4 + bx^2 - \frac{2b}{a} x^3\right) dx$$

$$= \frac{b}{a^2} \cdot \frac{a^5}{5} + b.\frac{a^3}{3} - \frac{2b}{a} \cdot \frac{a^4}{4}$$

$$= \frac{a^3 b}{5} + \frac{a^3 b}{3} - \frac{a^3 b}{2}$$

$$= \frac{6a^3 b + 10a^3 b - 15a^3 b}{30}$$

$$\therefore\ I_{yy} = \frac{a^3 b}{30} \text{ unit}$$

Calculation of I_{xx}

Required MOI = MOI of curve
– MOI of tr

$$I_{xx} = \int \frac{1}{3} y^3\, dx - \text{MOI of } \Delta$$

$$= \int_0^a \frac{1}{3}\left(\frac{b}{a^2} x^2 + b\right)^3 dx - \ldots$$

[ΔMOI = ...

$$= \frac{1}{3} \int_0^a \left(\frac{b^3}{a^6} x^6 + b^3 + \frac{3b^2\ldots}{a^4}\ldots\right.$$

$$= \frac{1}{3}\left[\frac{b^3 a^7}{a^6 .7} + b^3 a + \frac{3b^2\ldots}{a^4}\ldots\right.$$

$$= \frac{1}{3}\left[\frac{ab^3}{7} + ab^3 + \frac{3ab^3}{5}\ldots\right.$$

$$= \frac{1}{3} \frac{5ab^3 + 70ab^3 + 21\ldots}{35}$$

$$= \frac{96ab^3}{105} - \frac{2ab^3}{3}$$

$$\therefore\ I_{xx} = \frac{26ab^3}{105} \text{ unit}$$

14. Find the moment of inertia ... axis of the shaded plane ... integration method.

Solution:

Setps:

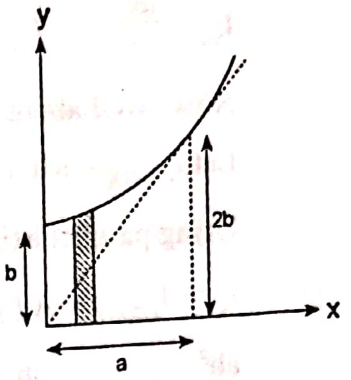
1. Calculate the centroid of a...
2. Calculate MOI about axes...
3. Transfer the MOI abo... centroidal axes.

In previous questions, we...

$A = \frac{ab}{3};\quad \overline{X} = \frac{3a}{4};$

Calculation of I_{xx}

Required MOI = MOI of curve
 – MOI of triangle



$$I_{xx} = \int \frac{1}{3} y^3\, dx - \text{MOI of } \Delta$$

$$= \int_0^a \frac{1}{3}\left(\frac{b}{a^2}x^2 + b\right)^3 dx - \frac{a(2b)^3}{12}$$

$$[\Delta \text{MOI} = \frac{bh^3}{12}]$$

$$= \frac{1}{3}\int_0^a \left(\frac{b^3}{a^6}x^6 + b^3 + \frac{3b^2x^4}{a^4} . b + 3\frac{b}{a^2}x^2b^2\right) dx - \frac{8ab^3}{12}$$

$$= \frac{1}{3}\left[\frac{b^3a^7}{a^6.7} + b^3a + \frac{3b^2}{a^4}\frac{a^5}{5} b + \frac{3b^3}{a^2}\frac{a^3}{3}\right] - \frac{2}{3}ab^3$$

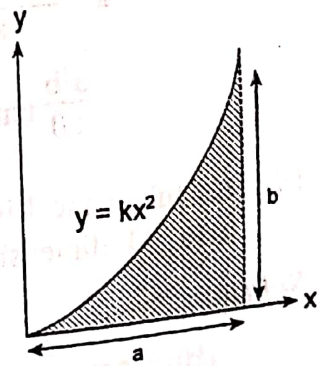
$$= \frac{1}{3}\left[\frac{ab^3}{7} + ab^3 + \frac{3ab^3}{5} + ab^3\right] - \frac{2}{3}ab^3$$

$$= \frac{1}{3}\frac{5ab^3 + 70ab^3 + 21ab^3}{35} - \frac{2}{3}ab^3$$

$$= \frac{96ab^3}{105} - \frac{2ab^3}{3}$$

$$\therefore\ I_{xx} = \frac{26ab^3}{105} \text{ unit}$$

14. Find the moment of inertia about centroidal axis of the shaded plane area by using integration method.



Solution:

Setps:

1. Calculate the centroid of area.
2. Calculate MOI about axes
3. Transfer the MOI about axes to this centroidal axes.

In previous questions, we have calculated

$$A = \frac{ab}{3}; \qquad \overline{X} = \frac{3a}{4}; \qquad \overline{Y} = \frac{3b}{10}$$

$$I_{xx} = \frac{ab^3}{21}; \qquad I_{yy} = \frac{a^3b}{5}$$

Now, MOI about centroidal x axis

Let I_{GXGX} = MOI about centroidal x axis

Using parallel axis theorem:

$$I_{xx} = I_{GXGX} + A.(\overline{y})^2$$

$$\frac{ab^3}{21} = I_{GXGX} + \frac{ab}{3}\left(\frac{3b}{10}\right)^2$$

$$\therefore I_{GXGX} = \frac{ab^3}{21} - \frac{3ab^3}{100}$$

$$= \frac{37ab^3}{2100} \text{ unit}$$

MOI about centroidal y axis:

Let I_{GYGY} be the MOI about centroidal y-axis.

Then,

$$I_{yy} = I_{GYGY} + A(\overline{x})^2$$

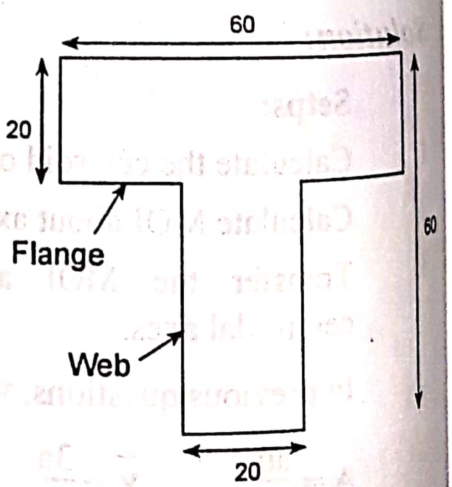
$$\frac{a^3b}{5} = I_{GXGX} + \frac{ab}{3}\cdot\left(\frac{3a}{4}\right)^2$$

$$I_{GYGY} = \frac{a^3b}{5} - \frac{3a^3b}{16}$$

$$= \frac{16a^3b - 15a^3b}{80}$$

$$= \frac{a^3b}{80} \text{ unit}$$

15. Calculate the MOI about centroidal axes. All dimension in cm.



Solution:

[Horizontal rectangle is called flange and vertical rectangle is called web]

Here the axis is not given. So, we should first give the axes. If possible draw a axis to divide the figure symmetrically.

Required figure = Rect. I + Rect. II

20

Since the figure is

Calculation of $\overline{Y}$

Segment
Rectangle I
Rectangle II

$$\therefore \overline{Y} = \frac{\Sigma A_i y_i}{\Sigma A_i}$$

$$= \frac{76000}{2000}$$

$\therefore$ G (0, 38)

MOI about cent

$$I_{GXGX} = \left(\frac{bh^3}{12}\right) +$$

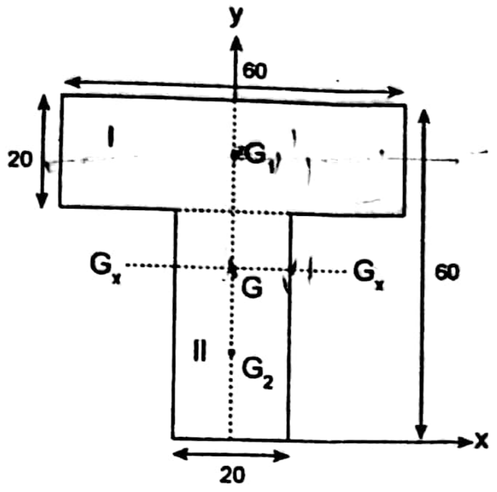
$$= \frac{60 \times 2}{12}$$

$= 21280$

$\therefore I_{GXGX} = 578$

MOI about cen

Since the area



Since the figure is symmetric about y-axis; $\overline{X} = 0$.

Calculation of $\overline{Y}$

Segment	Area (A_i) cm^2	$\overline{y}_i$ cm	A_iY_i cm^3
Rectangle I	$60 \times 20 = 1200$	$40 + 10 = 50$	60000
Rectangle II	$40 \times 20 = 800$	20	16000
	2000		76000

$$\therefore \overline{Y} = \frac{\Sigma A_i y_i}{\Sigma A_i}$$

$$= \frac{76000}{2000} = 38 \text{ cm}$$

$\therefore$ G (0, 38)

MOI about centroidal x-axis

$$I_{GXGX} = \left(\frac{bh^3}{12} + Ah^2\right)_I + \left(\frac{bh^3}{12} + Ah^2\right)_{II}$$

$$= \frac{60 \times 20^3}{12} + 60 \times 20 \times 12^2 + \frac{20 \times 40^3}{12} + 20 \times 40 \times 18^2$$

$$= 212800 + 365866$$

$\therefore I_{GXGX} = 578666 \text{ cm}^4$

MOI about centroidal y-axis

Since the area is symmetrical about y-axis

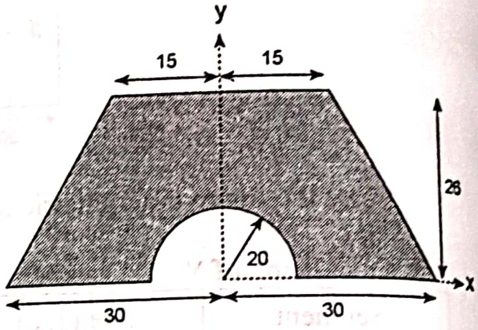
$$I_{GYGY} = \left(\frac{bh^3}{12}\right)_I + \left(\frac{bh^3}{12}\right)_{II}$$

$$= \frac{20 \times 60^3}{12} + \frac{40 \times 20^3}{12}$$

$$= 360000 + 26666$$

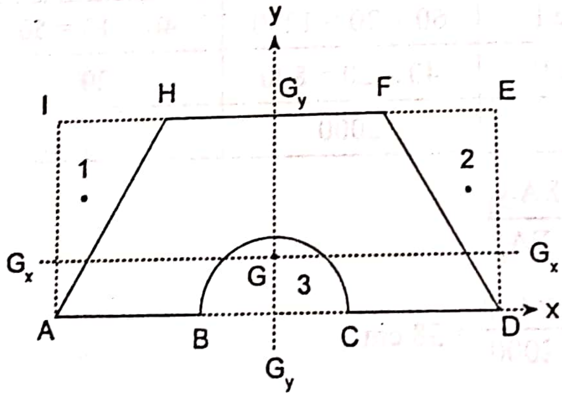
$$\therefore\ I_{GYGY} = 386666\ \text{cm}^4$$

16. Calculate the moment of inertia of the given shaded area about its centroidal axes. (All dimensions in cm)



Solution:

Required figure
= Rect. ADEI
– fig (1) – fig (2)
– fig (3)



The required figure is symmetric about y-axis.

$\therefore\ \overline{x} = 0$

For calculation of $\overline{y}$

Figure	Area (A_i) cm²	y_i (cm)	A_iy_i (cm³)
Rec. ADEI	$60 \times 26 = 1560$	13	20280
Tri (1)	$-\frac{1}{2} \times 15 \times 26$ $= -195$	$\frac{2}{3} \times 26 = 17.33$	(–)3380
Tri (2)	– 195	17.33	(–)3380

Semi-circle (3)	$-\pi\frac{20^2}{2}$ $= -628.31$	
	$\Sigma = 541.69$	

$$\therefore\ \overline{y} = \frac{\Sigma A_iy_i}{\Sigma A_i} = \frac{8186.74}{541.69} = 15.12$$

$$I_{GXGX} = \frac{60 \times 26^3}{12} + 60 \times 26 \times ($$

$$-2 \times \left\{\frac{15 \times 26^3}{36}\right.$$

$$-\{(0.11$$

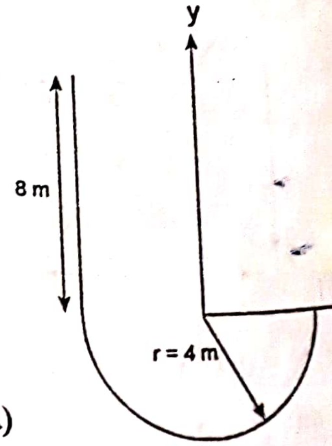
$$= 58027.6\ \text{cm}^4$$

$$I_{GYGY} = \frac{26 \times 60^3}{12} - 2 \times \left\{\frac{26 \times}{3}\right.$$

$$= 156543.14\ \text{cm}^4$$

Practice Questions

1. Locate the centroid of comp



a)

Semi-circle (3)	$-\pi \frac{20^2}{2}$ $= -628.31$	$\frac{4r}{3\pi} = 8.48$	(−) 5333.26
	$\Sigma = 541.69$		$\Sigma = 8186.74$

$$\therefore\ \bar{y} = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{8186.74}{541.69} = 15.12 \text{ cm}$$

$$I_{GXGX} = \frac{60 \times 26^3}{12} + 60 \times 26 \times (15.12 - 13)^2$$
$$- 2 \times \left\{ \frac{15 \times 26^3}{36} + \left(\frac{1}{2} \times 15 \times 26\right)(17.33 - 15.12)^2 \right\}$$
$$- \{(0.11 \times 20^4) + 628.31 \times (15.12 - 8.488)^2\}$$

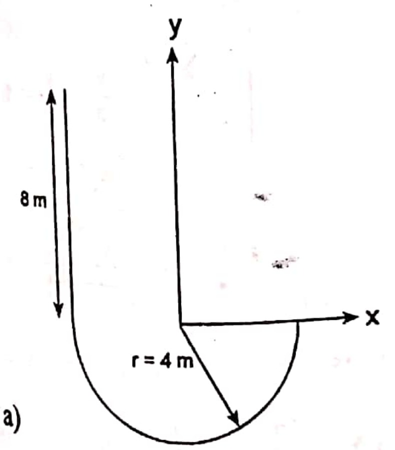
$$= 58027.6 \text{ cm}^4$$

$$I_{GYGY} = \frac{26 \times 60^3}{12} - 2 \times \left\{ \frac{26 \times 15^3}{36} + 195 \times (30 - 5)^2 \right\} - \frac{\pi \times 20^4}{8}$$

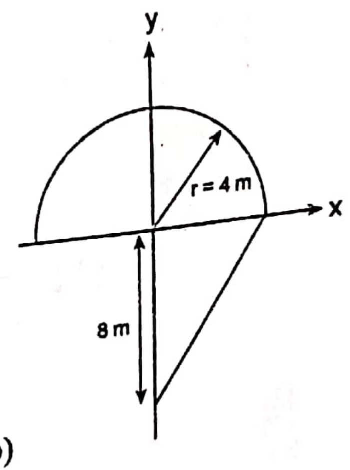
$$= 156543.14 \text{ cm}^4$$

Practice Questions

1. Locate the centroid of composite lines

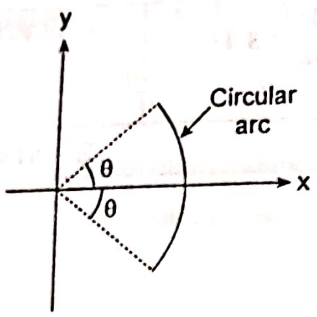


a)

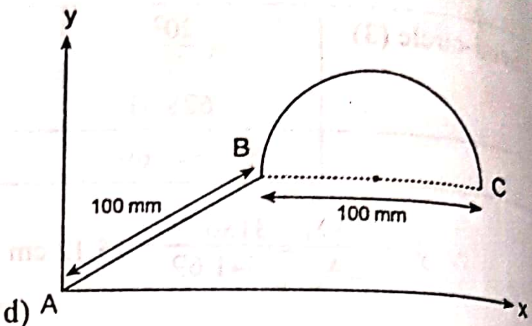


b)

c)



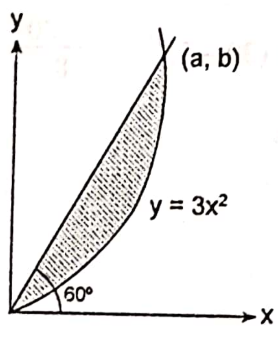
d)



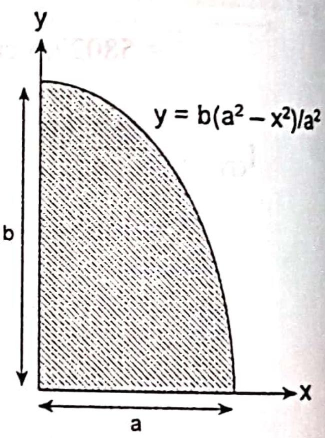
[Ans. a) $\overline{X}, \overline{Y} = -1.57, 0$; b) $\overline{X}, \overline{Y} = 0.569, 1.395$;
c) $\overline{X} = \dfrac{r \sin\theta}{\theta}$; d) $\overline{X} = 100.3$ mm, $\overline{Y} = 59.72$ mm]

2. Locate the centroid of the shaded area

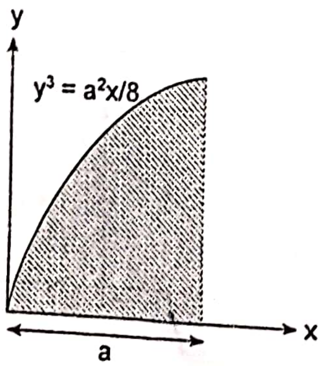
a)



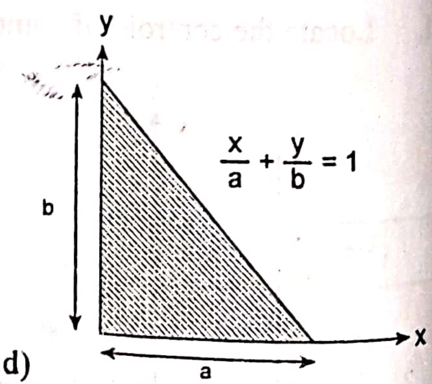
b)



c)



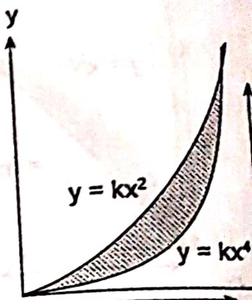
d)



[a) $\overline{X} = 0.289, \overline{Y} = 0.4$; b) $\overline{X} = \dfrac{3a}{8}, \overline{Y} = \dfrac{2b}{5}$;
c) $\overline{X} = \dfrac{4a}{7}, \overline{Y} = \dfrac{a}{5}$; d) $\overline{X} = \dfrac{a}{3}, \overline{Y} = \dfrac{b}{3}$]

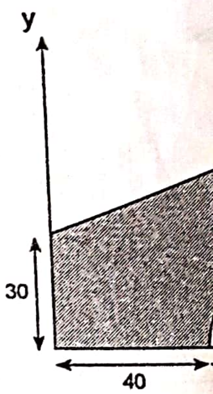
3. Locate the centroid of

a)



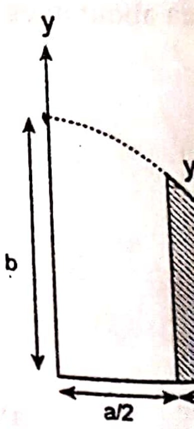
4. Locate the centroid

a)



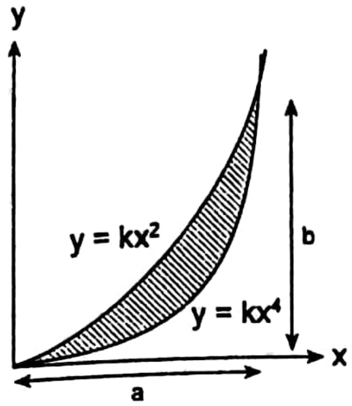
5. Find the moment

a)

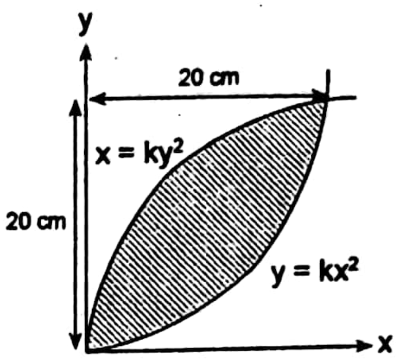


Scanned by CamScanner

3. Locate the centroid of shaded area:



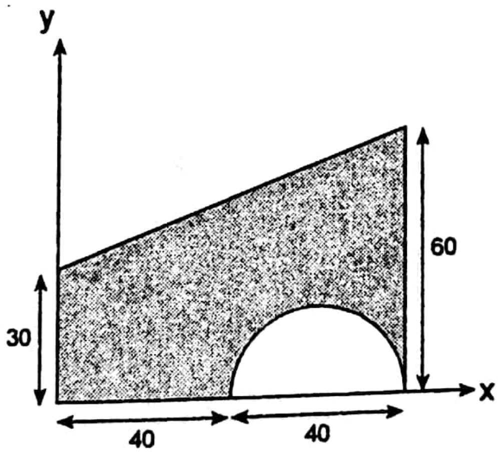
a)



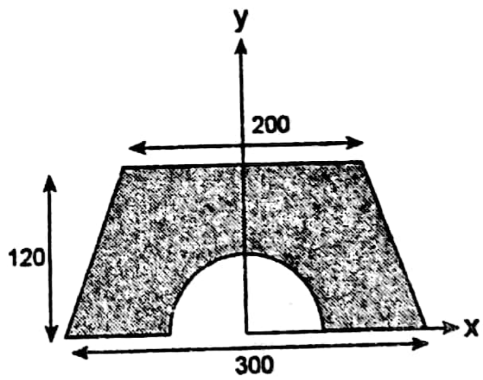
b)

$$[a)\ \overline{X} = \frac{5a}{8},\ \overline{Y} = \frac{b}{3};\ b)\ \overline{X} = 9 \text{ cm},\ \overline{Y} = 9 \text{ cm}]$$

4. Locate the centroid of composite area:



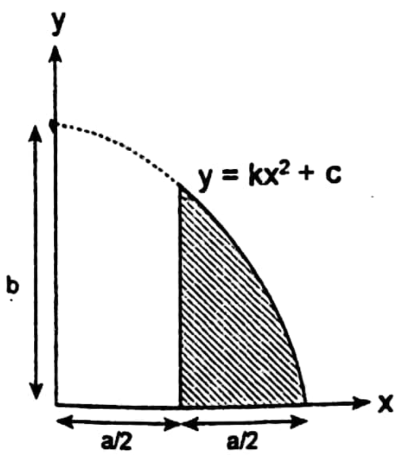
a)



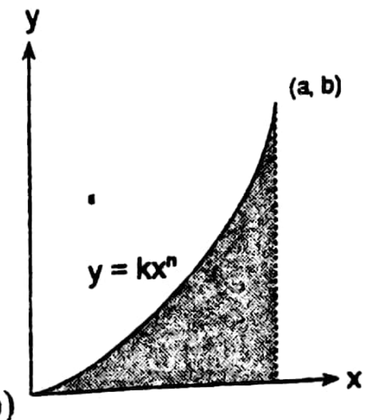
b)

$$[a)\ \overline{X} = 41.1 \text{ cm},\ \overline{Y} = 26.5 \text{ cm};\ b)\ \overline{X} = 0,\ \overline{Y} = 69.1 \text{ cm}]$$

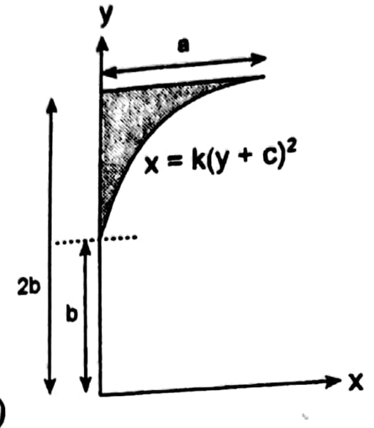
5. Find the moment of inertia about x-y axes.



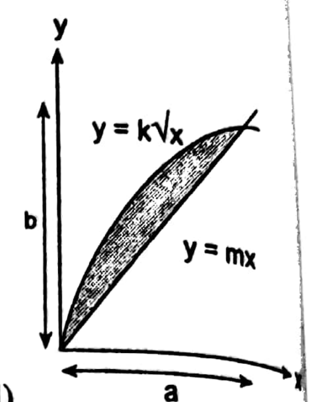
a)



b)



c)



d)

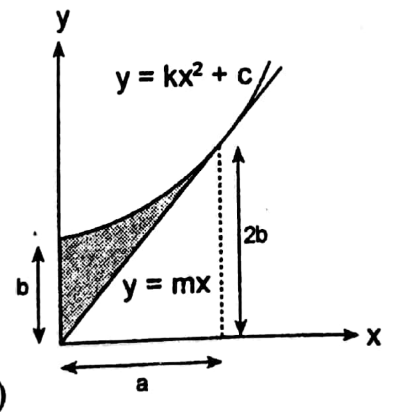
[Ans.: a) $I_{xx} = 0.0430ab^3$ unit, $I_{yy} = \frac{47}{60}a^3b$ unit;

b) $I_{xx} = \frac{ab^3}{3(3n+1)}$ unit, $I_{yy} = \frac{a^3b}{n+3}$ unit;

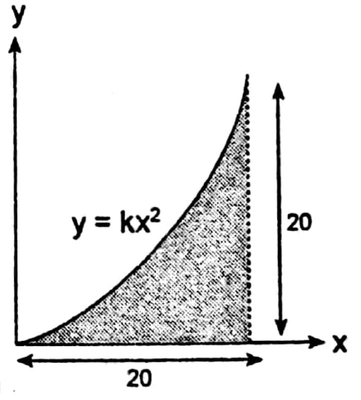
c) $I_{xx} = \frac{31}{30}ab^3$ unit, $I_{yy} = \frac{1}{21}a^3b$ unit; d) $I_{xx} = \frac{ab^3}{20}$ unit, $I_{yy} = \frac{a^3b}{28}$ unit]

8.

Find the moment of inertia about centroidal axes:



a)



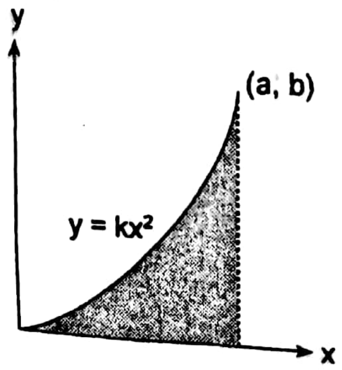
b)

[Ans.: a) $I_{GXGX} = \frac{6ab^3}{175}$ unit, $I_{GYGY} = \frac{a^3b}{80}$ unit;

b) $I_{GXGX} = 59200\ cm^4$, $I_{GYGY} = 2000\ cm^4$]

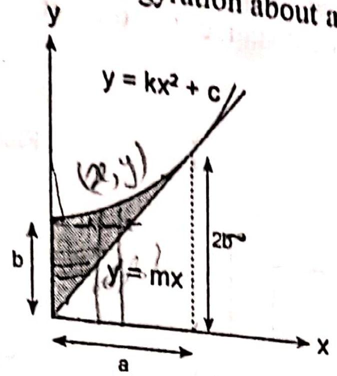
9.

Determine the radius of gyration of the shaded area about axes:



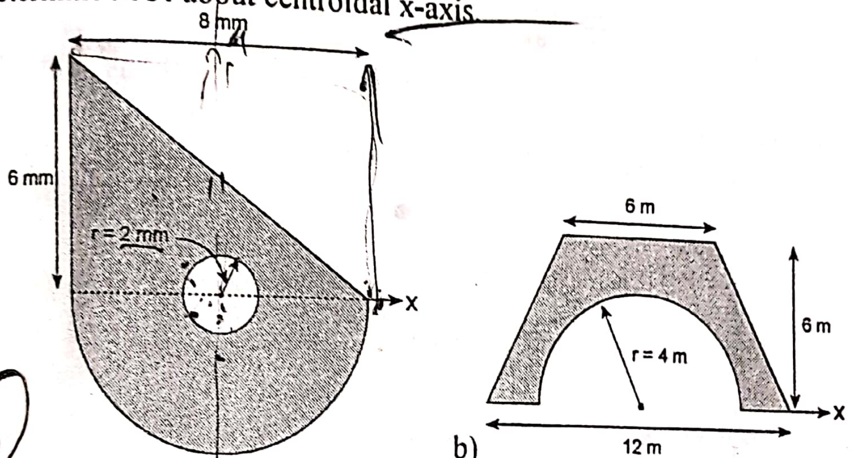
[Ans.: $K_{xx} = \frac{b}{\sqrt{7}}$ unit, $K_{yy} = \frac{a\sqrt{3}}{\sqrt{5}}$ unit]

8. Determine the polar radius of gyration about axes



[Ans.: $K_{zz} = \sqrt{\dfrac{7a^2 + 52b^2}{70}}$]

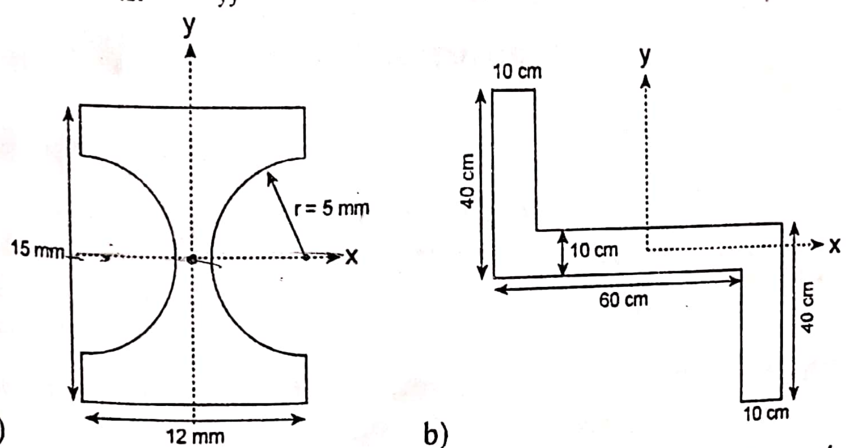
9. Determine MOI about centroidal x-axis.



a) b)

[Ans.: a) $I_{GXGX} = 231.15$ mm^4; b) $I_{GXGX} = 83.76$ m^4]

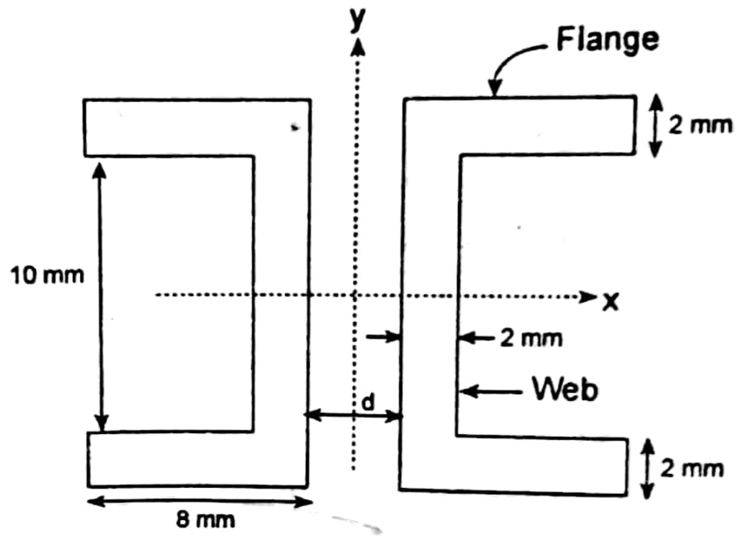
10. Determine I_{xx} and I_{yy} of centroidal axes:



a) b)

[Ans.: a) $I_{XX} = 2884.13$ mm^4, $I_{YY} = 839.74$ mm^4;
$I_{xx} = 2.9 \times 10^5$ cm^4, $I_{yy} = 5.6 \times 10^5$ cm^4]

11. Two equal channels are kept as shown at a distance d between them. Calculate the value of distance d if the centroidal moments of inertia I_{xx} and I_{yy} are equal.



[Ans.: d = 3.28 mm]

Chapter 5
Friction

5.1 Introduction

When a body slides over another body, a force is exerted at a surface of contact by the stationary body on a moving body. This resisting force is called the force of friction and acts in the direction opposite to the direction of motion.

Friction is quite undesirable and needs to be mitigated in some machines and process like power screw, bearing, gear, fluid flow etc. where the presence of friction would cause loss of power, wearing and tearing of equipment etc. However, the working of many device such as frictional brake and clutches, belt and rope drive, holding and fastening of device depends upon friction. Friction in both a liability and a necessity and is often referred as a necessary evil.

5.1.1 Characteristic of friction

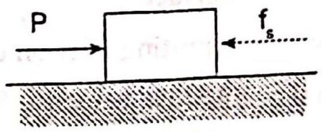
a) It always acts in a direction opposite to that in which motion is intended.

b) It exists as along as the tractive force acts. So, it is passive force.

c) It is self-adjusting force i.e. only that much comes to play as is just sufficient to prevent motion.

5.2 Static and Dynamic frictions

5.2.1 Static friction

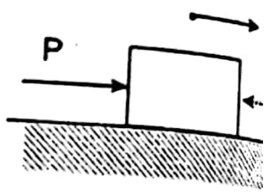


The friction force that develops between mating surface when subjected to external force but there is no relative motion between them is called static friction.

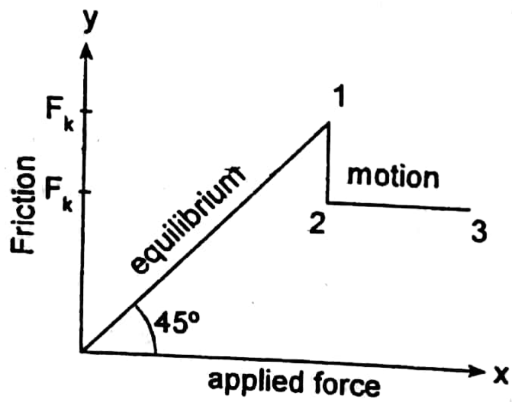
If force P acts towards right, then f_s equal to P acts in opposite direction to prevent motion. Then the force f_s is static friction.

5.2.2 Dynamic friction

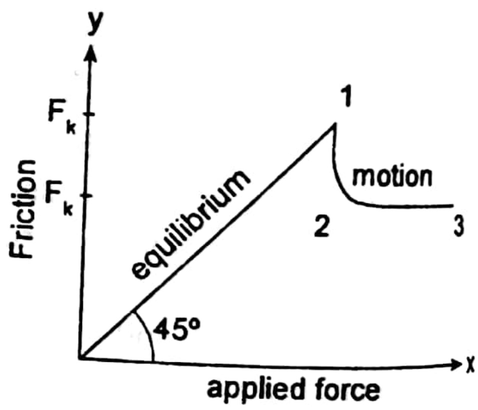
The frictional force that develops between mating surfaces when subjected to external forces and there is relative motion between them is called dynamic friction. It is also called kinetic friction.



If force P acts towards right, then f_k less than P acts in opp direction and the body move with acceleration a.



Theoretically



Theoretically

5.3 Laws of solid friction (static or dynamic)

Based on the experiment, following laws of friction have been state

i) Friction acts tangential to the surface in contact and is in a directi opposite to that in which motion is to take place.

ii) Frictional force is maximum at the instant of impending motion.

iii) The magnitude of limiting friction bears a constant ratio to th normal reaction between the mating surface. This ratio drops to slightly lower value when motion starts.

iv) Limiting friction is independent of the area and shape of contac surface.

v) Limiting friction depends upon the nature of the surface in contact.

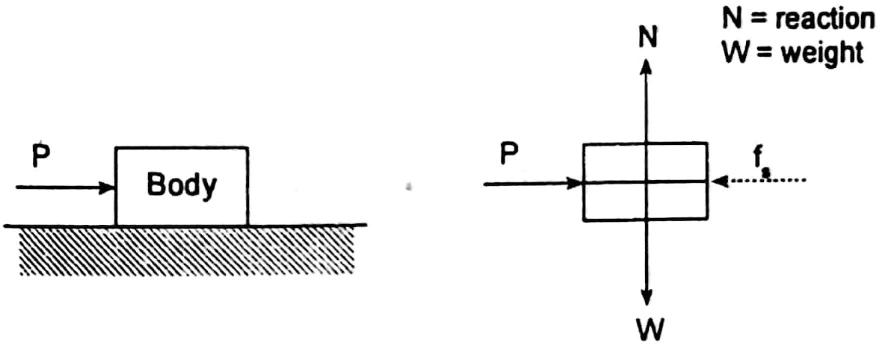
vi) At low velocity, friction is independent of the velocity. But at higher speed, there will be slight reduction in friction.

5.4 Terms related to friction

a) Coefficient of friction

The frictional force is proportional to normal reaction i.e. $F \propto R$. The ratio $\frac{F}{R}$ is called coefficient of friction. So, coefficient of friction is defined as the ratio of frictional force to normal reaction.

When the system is in state of impending motion, the frictional force has maximum value f_s. If μ_s is the coefficient of static friction, then $\mu_s = \frac{F_s}{R}$



For impending motion

$P - f_s = 0$

$P - \mu_s.N = 0$

$\therefore\ \mu_s = \frac{P}{N}$ ......(i)

When motion starts, the maximum force of friction falls to some lower value, called kinetic friction F_k. If μ_k is the coefficient of kinetic friction, then

$$\mu_k = \frac{f_k}{R}$$

for motion

$P - f_k = m.a$ where m = mass and a = acceleration

$P - \mu_k.N = m.a$

$\frac{P - ma}{N} = \mu_k$ ......(ii)

Since $a > 0$, from eq. (i) and (ii) $\mu_s > \mu_k$. So, the coefficient of static friction is always greater than kinetic friction.

b) Limiting friction

The maximum value of frictional force which acts on the body when it just starts to slide over another body is called limiting friction.

Mathematically, $F_{\mu m} = \mu_s.N$

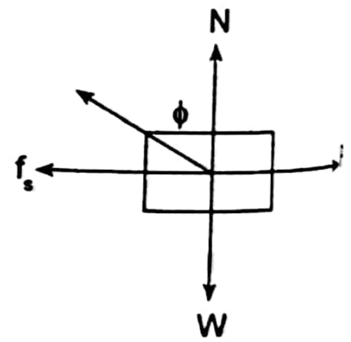
where N is normal reaction.

c) Impending motion

When the applied force on a body becomes equal to maximum frictional force i.e. limiting friction acting on it, the body just comes to motion. This statge of motion is called impending motion.

d) Angle of friction

It is defined as the angle which the resultant of normal reaction and limiting frictional force makes with the normal reaction.



N = Normal reaction

f_s = Limiting friction

R = Resultant of N and f_s

$$= \sqrt{N^2 + f_s^2}$$

$\therefore\ \tan\phi = \frac{f_s}{N} = \mu_s$ where ϕ is angle of friction.

e) Angle of repose

The angle α of the inclined plane at which the block resting on it is about to slide down the plane is called angle of repose.

At equilibrium

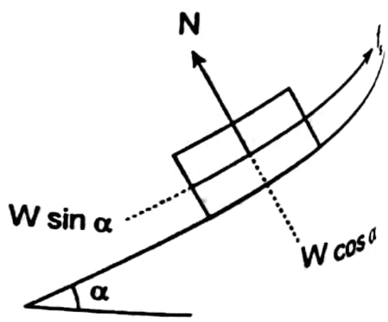
$f_s = W \sin\alpha$

$\mu_s N = W \sin\alpha$

$\mu_s.W\cos\alpha = W\sin\alpha$

$$\mu_s = \frac{W\sin\alpha}{W\cos\alpha}$$

$\therefore\ \tan\alpha = \mu_s$

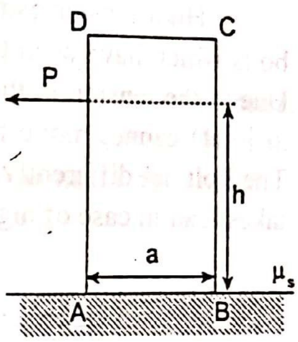


In previous, angle of friction $\tan\phi = \mu_s$.

So, angle of friction = angle of repose.

5.5 Condition of Tipping and sliding of a block

Here tipping refers to the turning of block ABCD due to the formation of couple. The condition for no tipping is $h = \frac{a}{2\mu_s}$

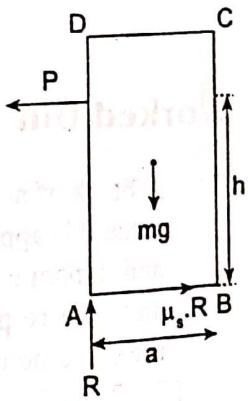


or, $h < \frac{a}{2\mu_s}$ i.e. $h \le \frac{a}{2\mu_s}$

h is point of application of force P above the base of block AB.

Proof:

At extreme point of tipping i.e. at just tipping condition, the block will be in equilibrium and reaction R will be at point A.



So, at equilibrium

i) $\Sigma M_A = 0$ +ve

$$mg\frac{a}{2} - P.h = 0$$

$$P.h = mg\frac{a}{2}$$

ii) $\Sigma F_x = 0 \rightarrow +ve$

$$\mu_s.R - P = 0$$

$$\therefore P = \mu_s.R$$

iii) $\Sigma F_y = 0 \uparrow +ve$

$$R - mg = 0$$

$$R = mg$$

Combining all three equations above

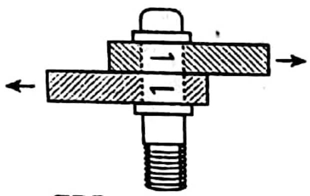
$$\mu_s.R.h = R.\frac{a}{2}$$

$$h = \frac{a}{2\mu_s}$$

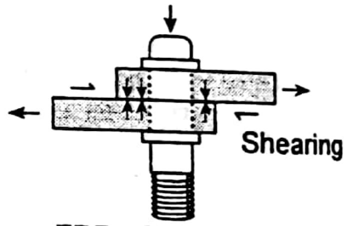
So, for no tipping $h \leq \frac{a}{2\mu_s}$. Proved.

5.6 High tension friction grip bolts

High tension (strength) friction grip bolt are high strength structural bolts which have been tightened such as to induce tension in the bolt shank. Due to the tension in the bolt, the interface between the plies (steel members in joint) cannot move relative to each other because of friction resistance. The bolt act differently than normal bolts or rivets. Friction along interface takes load in case of high tension friction grip bolt subject to shear.



FBD ordinary bolt



FBD of HTFG bolt

Worked Out Example

1. A block of mass 20 kg is rest on a horizontal floor. To slide the block force P is applied along the horizontal. The coefficient of static friction and dynamic friction between the block and floor are stated to be 0.3 and 0.25 respectively. For the each value of P given below, work out the corresponding value of frictional force F and complete the table.

Applied force P(N)	15	30	45	58.86	60	75
Frictional force F (N)						

Solution:

Weight of block $w = mg = 20 \times 9.81 = 196.2$ N

Normal reaction $R = w = 196.2$ N

Limiting friction $F_{lim} = \mu_s.R = 0.3 \times 196.2 = 58.86$ N

The block will remain at rest till the applied force is less than 58.86 N.

When applied force is greater than F_{lim}, block will move and frictional force reduces to $\mu_k R = 0.25 \times 196.2\ N = 49.05\ N$.

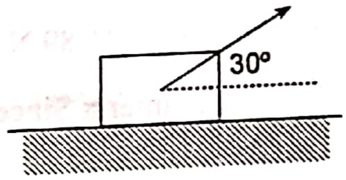
Applied force P(N)	15	30	45	58.86	60	75
Frictional force F (N)	15	30	45	58.86	49.05	49.05

2. A wooden block of weight 100 N rests on a horizontal plane. Determine the force required to just a) pull it b) push it. Take coefficient of friction $\mu = 0.3$ for all surface. Comment on the result.

Solution:

Case a)

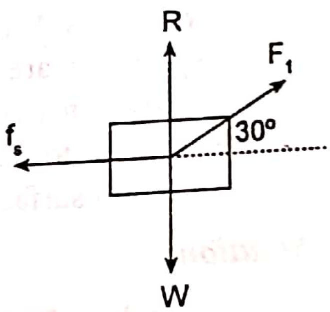
Let F_1 be the force required to pull.



$\Sigma F_x = 0\ (\rightarrow +ve)$

$F_1 \cos 30 - f_s = 0$

$F_1.\cos 30 - \mu_s.R = 0$ ......(i)



$\Sigma F_y = 0 \uparrow +ve$

$F_1 \sin 30 + R - W = 0$

$F_1 \sin 30 + R - 100 = 0$

$0.5F_1 + R = 100$

$\therefore R = 100 - 0.5F_1$ ......(ii)

From (i) and (ii)

$$F_1 \frac{\sqrt{3}}{2} - 0.3(100 - 0.5F_1) = 0$$

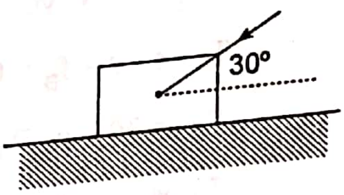
$0.866F_1 - 30 + 0.15F_1 = 0$

$1.016F_1 = 30$

$\therefore F_1 = 29.53\ N$

Case b)

Let F_2 be the force required to push.



For just motion

$\Sigma F_y = 0 \uparrow +ve$

$R - W - F_2 \sin 30 = 0$

$R - 100 - 0.5F_2 = 0$

$\therefore\ R = 100 + 0.5F_2$

$\Sigma F_x = 0 \rightarrow +ve$

$f_s - F_2 \cos 30 = 0$

$\mu_s R - 0.866F_2 = 0$

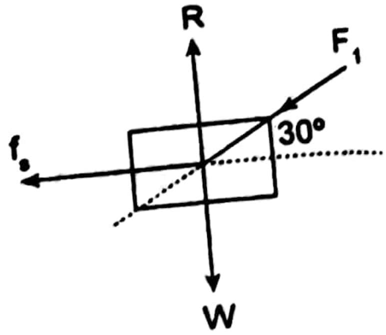
$0.3(100 + 0.5F_2) - 0.866F_2 = 0$

$30 + 0.15F_2 - 0.866F_2 = 0$

$30 + (-)0.716F_2 = 0$

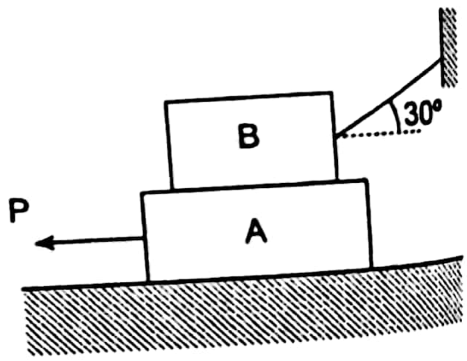
$0.716F_2 = 30$

$\therefore\ F_2 = 41.89\ N$



Comment: Since $F_2 > F_1$; it is easier to pull the block than to push it.

3. Two blocks A and B of 40 N and 20 N respectively are in equilibrium position as shown in figure. Calculate the force P required to move block A. Take $\mu = 0.3$ for all surface.



Solution:

First draw FBD of block A and B.

Consider block B

$\Sigma F_y = 0 \uparrow +ve$

$R_B + T \sin 30 - W_B = 0$

$R_B + 0.5T = 20$

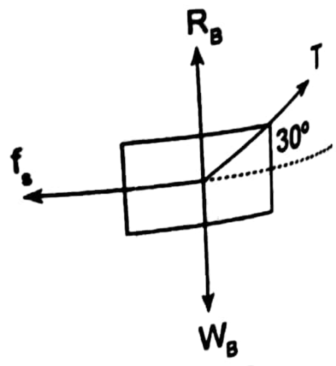
$\therefore\ R_B = 20 - 0.5T$

$\Sigma F_x = 0 \rightarrow +ve$

$T \cos 30 - f_B = 0$

$T \cos 30 - \mu_s . R_B = 0$

$T \cos 30 - 0.3(20 - 0.5T) = 0$



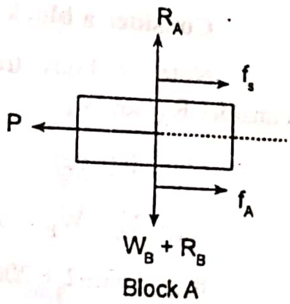
Block B

$$0.866T - 6 + 0.15T = 0$$

$$\therefore\ T = 5.905\ N$$

$$\therefore\ R_B = 20 - 0.5 \times 5.905 = 17.047N$$

Consider block A



$$\Sigma F_y = 0 \uparrow +ve$$

$$R_A - W_A - R_B = 0$$

$$\therefore\ R_A = 40 + 17.04 = 57.04\ N$$

$$\Sigma F_x = 0 \rightarrow +ve$$

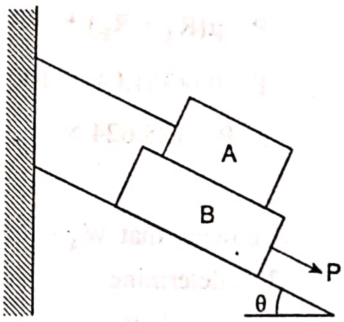
$$f_A + f_B - P = 0$$

$$\mu_s(R_A + R_B) - P = 0$$

$$0.3(57.04 + 17.047) = P$$

$$\therefore\ P = 22.22\ N$$

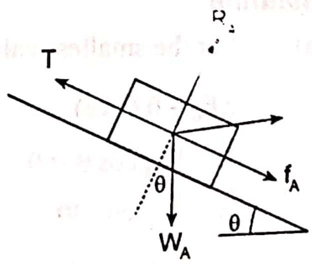
4. Block A in the figure weighs 120 N and block B weigh 200 N. The cord is parallel to inclined. If the $\mu = 0.6$, $\theta = 30°$, what force P applied to B acting down and parallel to incline will start motion. Find the value of tension in cord.



Solution:

Consider a block A

For a inclined plane, we use x-axis as axis parallel to incline plane and y-axis as axis perpendicular to incline plane.



$$\Sigma F_y = 0\ (+ve)$$

$$R_A - W_A \cos\theta = 0$$

$$R_A - 120 \cos 30 = 0$$

$$R_A - 103.92 = 0$$

$$\therefore\ R_A = 103.92\ N$$

$$\Sigma F_x = 0\ (+ve)$$

$$f_A - T + W_A \sin\theta = 0$$

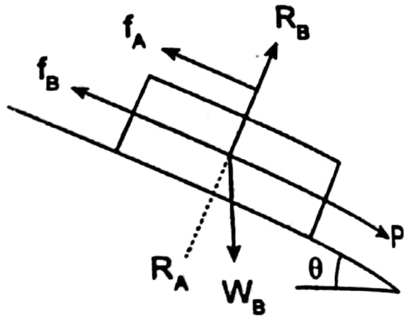
$\mu.R_A - T + 120 \sin 30 = 0$

$0.6 \times 103.92 - T + 60 = 0$

$\therefore\ T = 122.35\ N$

Consider a block B

Note: A body transfer reaction not the weight. So, we have to consider R_A not W_A.



$\Sigma F_y = 0$ (+ve)

$R_B - R_A - W_B \cos \theta = 0$

$R_B = 103.92 + 200 \cos 30$

$= 277.12\ N$

Also, $\Sigma F_x = 0$ (+ve)

$P - f_A - f_B + w_B \sin \theta = 0$

$P - \mu(R_A + R_B) + 200 \sin 30 = 0$

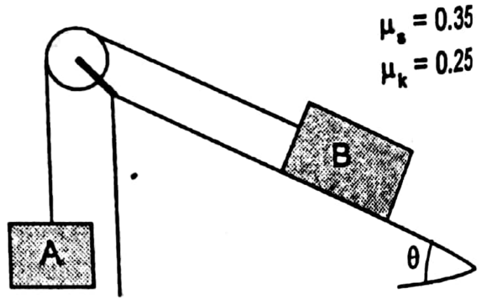
$P - 0.6(381.04) + 100 = 0$

$\therefore\ P = 128.624\ N$

5. Knowing that W_A = 25 kg and θ = 30°, determine

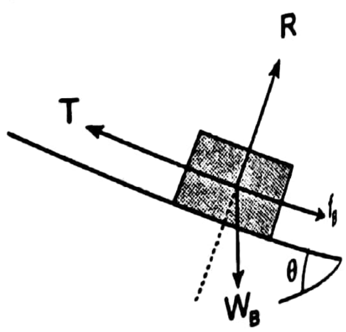
a) the smallest value.

b) the largest value of W_B for which system is in equilibrium.



Solution:

a) For the smallest value of W_B, it will move upward.



$\Sigma F_y = 0$ (+ve)

$R - W_B \cos \theta = 0$

$R = W_B \cos 30$

$\Sigma F_x = 0$ (+ve)

$f_B - T + W_B \sin \theta = 0$

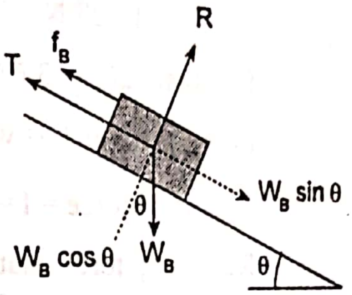
$\mu_s R - 25 + W_B \sin 30 = 0 \quad [T = W_A]$

$$0.35 \times W_B \cos 30 + W_B \sin 30 = 25$$

$$W_B (0.303 + 0.5) = 25$$

$$\therefore\ W_{B(min)} = 31.13 \text{ kg}$$

b) For the largest value of W_B, it will move downward.



$$\Sigma F_y = 0\ (+ve)$$

$$R - W_B \cos\theta = 0$$

$$R = W_B \cos 30$$

$$\Sigma F_x = 0\ (+ve)$$

$$W_B \sin 30 - T - f_B = 0$$

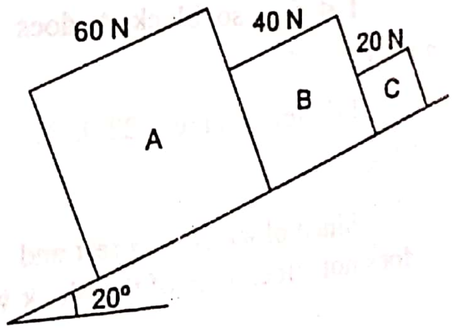
$$W_B \times 0.5 - 25 - \mu_s.R = 0$$

$$W_B \times 0.5 - 25 - 0.35(0.866\ W_B) = 0$$

$$\therefore\ W_B = \frac{25}{0.1968}$$

$$\therefore\ W_{B(max)} = 127 \text{ kg}$$

6. Three blocks are placed on a 20° incline in contact with each other. Determine which of the block will move and friction force on each block.



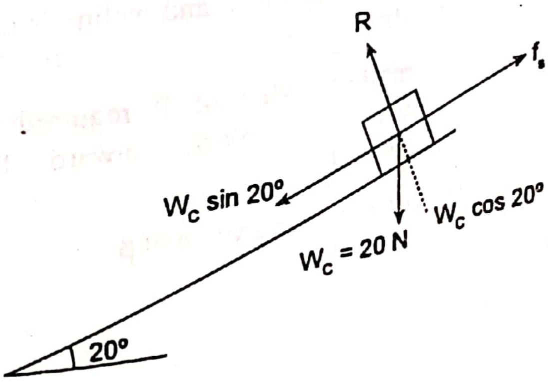
for A and C:

$\mu_s = 0.5$ and $\mu_k = 0.4$

for B: $\mu_s = 0.3$ and $\mu_k = 0.2$

Solution:

Block C:



$$f_{lim} = \mu_s.R$$

$$= 0.5 \times 20 \cos 20°$$

$$= 9.396 \text{ N}$$

$$f = 20 \sin 20$$

$$= 6.8404 \text{ N}$$

$f < f_{lim}$; so block C does not move.

Frictional force = f = 6.8404 N

Block B:

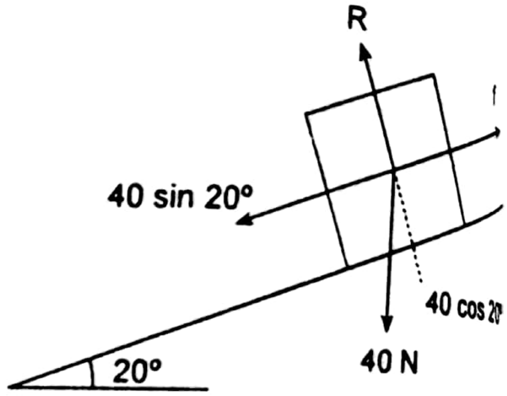
$f_{lim} = \mu_s.R = 0.3 \times 40 \cos 20$

$= 11.27$ N

$f = 40 \sin 20 = 13.68$

$f > f_{lim}$; so block B will move.

Frictional force = f = 11.27 N



Resulting force that push the block A = 13.68 – 11.27 = 2.4 N

Block A:

$f_{lim} = \mu_s.R = 0.5 \times 60 \cos 20 = 28.19$ N

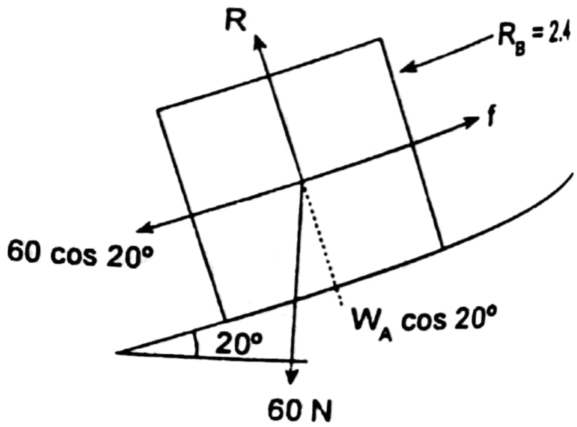
$f - R_B - 60 \sin 20 = 0$

$f - 2.4 - 20.52 = 0$

$\therefore\ f = 22.92$

$f < f_{lim}$; so block A does not move.

Frictional force = 22.92 N

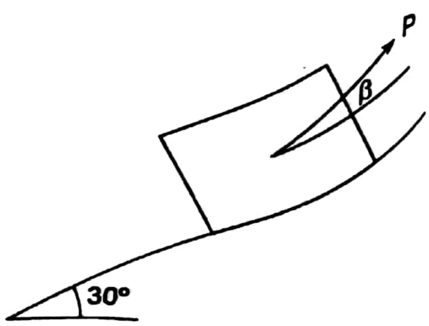


Since block A is in rest and it does not slide, none of the block will slide.

7. Knowing that the coefficient of friction between 25 kg block and incline is μ = 0.25; determine

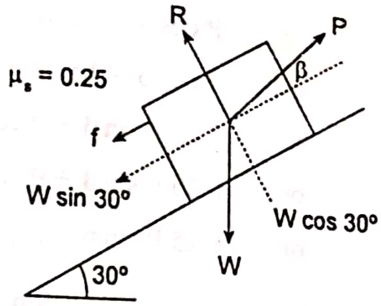
   a) smallest value of P required to move the block upward the inclined.

   b) corresponding value of β.



Solution:

Method one: Drawing method



Replace frictional force f and normal reaction R by S where

$S^2 = f^2 + R^2$

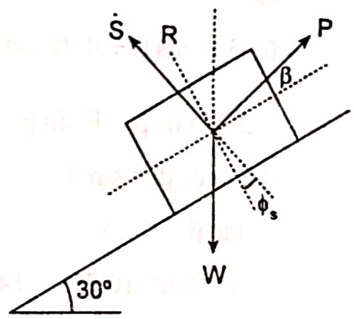
For just motion:

$\tan \phi_s = \mu_s$

$\tan \phi_s = 0.25$

$\therefore \phi_s = 14.036°$

For minimum P; P is perpendicular to S and B must equal to ϕ_s.



Angle between S and W is $30 + \phi_s$ = 44.036°

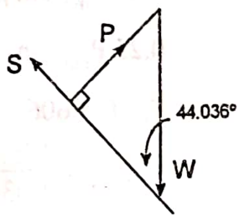
$\therefore \sin 44.036 = \dfrac{P}{W}$

$\therefore P = P_{min} = W \sin 44.036$

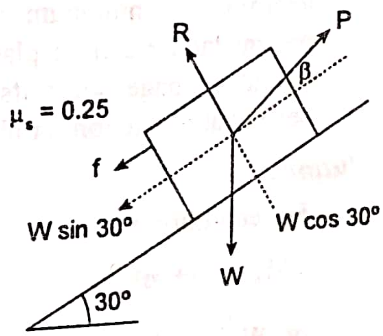
$= 25 \times 9.81 \times \sin 44.036$

$= 170.97$ N

$\beta = 14.036°$



Alternative method:



$\Sigma F_x = 0$ (+ve)

$P \cos \beta - W \sin 30 - f = 0$

$P \cos \beta - W \sin 30 - \mu_s . R = 0$

$P \cos \beta - 0.5\, W - 0.25\, R = 0$ ...(i)

$\Sigma F_y = 0 \uparrow$ (+ve)

$P \sin \beta + R - W \cos 30° = 0$

$P \sin \beta + R - 0.866\, W = 0$

$\therefore R = 0.866\, W - P \sin \beta$ ......(ii)

From eq. (i) and (ii)

or, $P \cos\beta - 0.5\ W - 0.25(0.866\ W - P \sin\beta) = 0$

or, $P \cos\beta - 0.5\ W - 0.2165\ W + 0.25\ P \sin\beta = 0$

or, $0.25\ P \sin\beta + P \cos\beta - 0.7165\ W = 0$

or, $0.25\ P \sin\beta + P \cos\beta - 0.7165 \times 25 \times 9.81 = 0$

or, $0.25\ P \sin\beta + P \cos\beta - 175.72 = 0$

For minimum value of P; $\frac{dP}{d\beta} = 0$

$$\frac{d}{d\beta}(0.25\ P \sin\beta + P \cos\beta - 175.72) = 0$$

$$0.25\ P \cos\beta + 0.25 \sin\beta \frac{dP}{d\beta} - P \sin\beta + \cos\beta \frac{dP}{d\beta} = 0$$

$$0.25\ P \cos\beta - P \sin\beta = 0$$

$$0.25 \cos\beta = \sin\beta$$

$\therefore\ \tan\beta = 0.25$

$\therefore\ \beta = \tan^{-1}(0.25) = 14.036°$

Again,

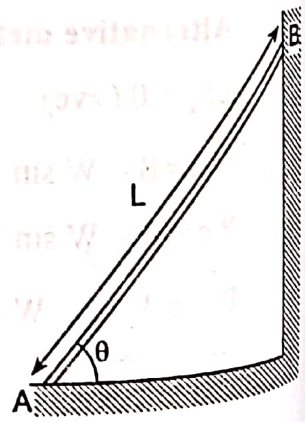
$$0.25\ P_{min} \sin\beta + P_{min} \cos\beta - 175.72 = 0$$

$$0.25\ P_{min} \sin 14.036 + P_{min} \cos 14.036 - 175.72 = 0$$

$$P_{min}(0.0606 + 0.9701) = 175.72$$

$$\therefore\ P_{min} = \frac{175.72}{1.0307} = 170.48\ N$$

8. Determine the minimum angle θ at which a uniform ladder can be placed against a wall without slippage under its own weight. The coefficient of friction for all surfaces is 0.2.



Solution:

For equilibrium:

$\Sigma M_A = 0$ (+ve)

or, $W\frac{L}{2}\cos\theta - R_B\ L \sin\theta - f_B\ L \cos\theta = 0$

or $\frac{W}{2}\cos\theta - R_B \sin\theta - \mu\ R_B \cos\theta = 0$ ......(i)

$\Sigma F_y = 0\ (\uparrow +ve)$

$R_A - W + f_B = 0$

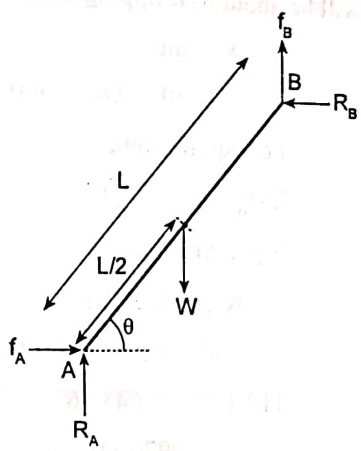
$R_A - W + \mu\, R_B = 0$

$R_A + \mu R_B - W = 0 \quad ......(ii)$

$\Sigma F_x = 0\ (\rightarrow +ve)$

$f_A - R_B = 0$

$\mu . R_A = R_B \quad ......(iii)$



From eq. (ii) and (iii)

$R_A + \mu.\mu R_A = W$

$R_A(1 + 0.04) = W$

$\therefore\ W = 1.04\ R_A \quad ......(iv)$

From eq. (i)

$\frac{W}{2} \cos\theta - R_B \sin\theta - \mu\, R_B \cos\theta = 0$

$\frac{1.04\ R_A}{2} \cos\theta - 0.2\ R_A \sin\theta - 0.4\ R_A \cos\theta = 0$

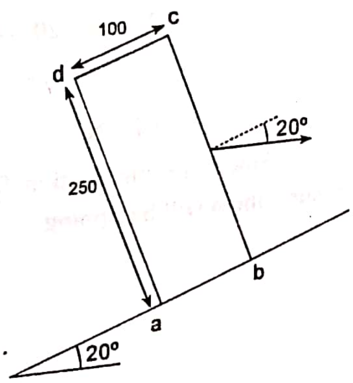
$0.52 \cos\theta - 0.2 \sin\theta - 0.04 \cos\theta = 0$

$0.48 \cos\theta = 0.2 \sin\theta$

$\therefore\ \tan\theta = \frac{0.48}{0.2} = 2.4$

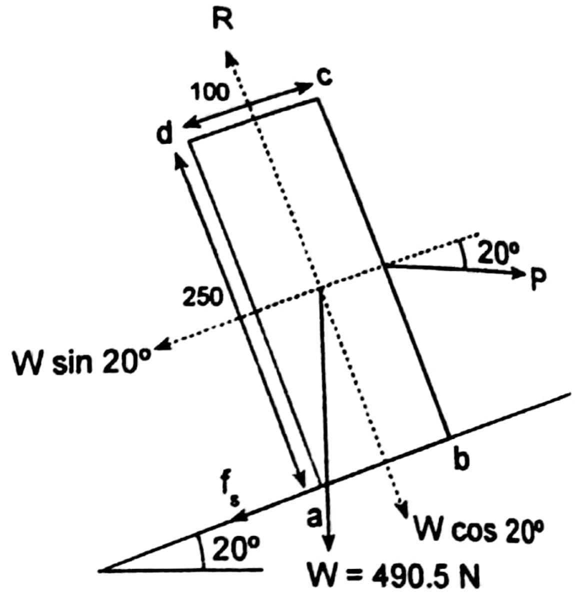
$\therefore\ \theta = \tan^{-1} 2.4 = 67.38°$

9. The homogeneous block has a mass of 50 kg and the coefficient of friction between the block and plane is 0.4. If the force P is increased gradually until the motion ensures, will the block slide or tip and for what value of P.



Solution:

Let the block be represented by abcd. It is visible that the tipping of block will be about b (if tipping condition).



$W = mg$

$= 50 \times 9.81 = 490.5$ N

For equilibrium;

$\Sigma M_b = 0$ (+ve)

$P \cos 20 \times 125$

$- W \cos 20 \times 50$

$- W \sin 20 \times 125 = 0$

$117.47P - 23045.96$

$- 20970.11 = 0$

$117.47\,P = 44016.07$

$\therefore\ P = 374.7$ N

$\Sigma F_y = 0$ (+ve)$\uparrow$

$R - W \cos 20 - P \sin 20 = 0$

$R = \sin 20 + W \cos 20$

$= 374.7 \sin 20 + 490.5 \cos 20$

$= 589.08$ N

$\therefore$ Limiting friction $F_{lim} = \mu R$

$= 0.4 \times 589.08 = 235.63$ N

$\Sigma F_x = 0$ (+ve)$\rightarrow$

$P \cos 20 - 2 \sin 20 - f_s = 0$

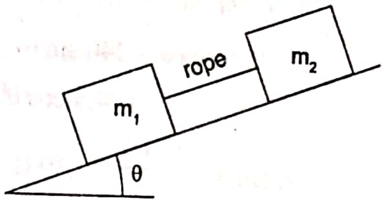
$\therefore\ f_s = 374.7 \cos 20 - 490.5 \sin 20$

$= 352.1 - 167.7$

$= 184.4$ N

Since limiting friction $f_{lim} > f_s$, there will not be sliding. If motion ensues, there will be tipping.

10. Two masses $m_1 = 240$ N and $m_2 = 150$ N are tied together by a rope parallel to the incline plane surface as shown in figure. If μ_s for m_1 is 0.3 and that for m_2 is 0.55. Find:

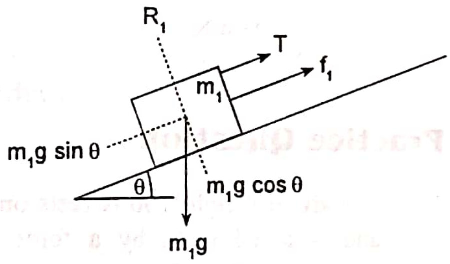


a) the value of θ for which masses will just start slipping

b) tension in the rope.

Solution:

The tension in the rope will be same for both block. Let it be T. During just slipping, the block will tends to move downward.



Consider mass m_1

$\Sigma F_y = 0$ (+ve)↑

$R_1 - m_1 g \cos\theta = 0$

$\therefore\ R_1 = m_1\, g \cos\theta$ N

$\Sigma F_x = 0$ (+ve)→

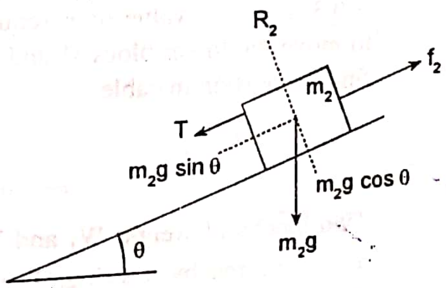
$T + f_1 - m_1\, g \sin\theta = 0$

$T + \mu_1 . R_1 - m_1 g \sin\theta = 0$

$T + 0.3\, m_1 g \cos\theta - m_1 g \sin\theta = 0$

$\therefore\ T = m_1 g\,(\sin\theta - 0.3 \cos\theta)$

$= 240(\sin\theta - 0.3\cos\theta)$ ......(i)



Consider mass m_2

$\Sigma F_y = 0$ (+ve)↑

$R_2 - m_2\, g \cos\theta = 0$

$R_2 = m_2\, g \cos\theta = 150 \cos\theta$

$\Sigma F_x = 0$ (+ve)→

$f_2 - T - m_2 g \sin\theta = 0$

$\mu_2 R_2 - 240\,(\sin\theta - 0.3\cos\theta) - 150 \sin\theta = 0$

or, $0.55(150 \cos \theta) - 140 \sin \theta + 72 \cos \theta - 150 \sin \theta = 0$

or, $82.5 \cos \theta - 240 \sin \theta + 72 \cos \theta - 150 \sin \theta = 0$

or, $390 \sin \theta = 154.5 \cos \theta$

or, $\tan \theta = \dfrac{154.5}{390} = 0.3961$

$\therefore \quad \theta = \tan^{-1} 0.3961 = 21.62°$

Again,

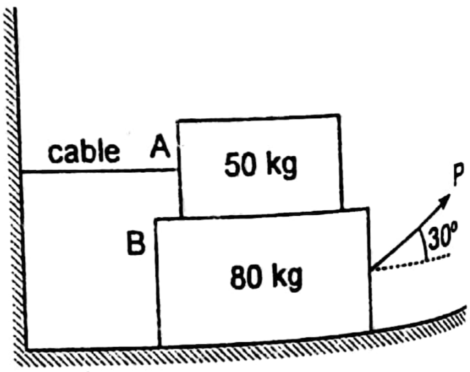
$$\begin{aligned} T &= 240 (\sin \theta - 0.3 \cos \theta) \\ &= 240 (0.368 - 0.2788) \\ &= 240 \times 0.0891 \\ &= 21.4 \text{ N} \end{aligned}$$

Practice Question

1. A body of weight 100 N rests on a rough horizontal surface ($\mu = 0.3$) and is acted upon by a force applied at an angle of 30° to the horizontal. What force is required to just cause the body to slide over the surface? Also, determine the inclination and magnitude of minimum force required to set the block into impending motion.

(Ans.: 29.53 N, 16.7°, 28.73 N)

2. Two blocks A and B weighing 50 kg and 80 kg are in equilibrium in the position as shown. The coefficient of friction for all surface = 0.3. Find the value of P required to move the lower block B and also find the tension in cable.



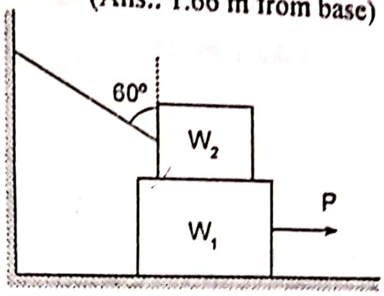
(Ans.: P = 521.4 N, T = 147.15 N)

3. Two blocks of weight W_1 and W_2 rest on a rough inclined plane and are connected by a string. The coefficient of friction between these blocks and the plane is μ_1 and μ_2 respectively. Show that for the impending motion, the inclination is given by $\tan \theta = \dfrac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$

4. A uniform ladder of weight 250 N and length 5 m is placed against a vertical wall in a position where its inclination to horizontal is 60°. A man weighing 800 N climbs the ladder. If the coefficient of friction for all surfaces = 0.2; at what position along a ladder will he induce slippage?

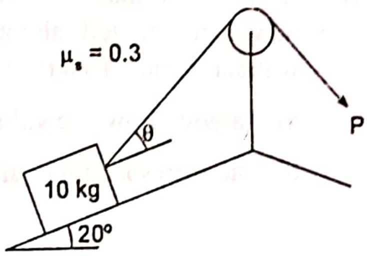
(Ans.: 1.66 m from base)

5. A block of weight W_1 = 1800 N rest on a horizontal surface and supports on the top of it another block of weight W_2 = 1000 N as shown in figure. The coefficient of friction for all surface is 0.4. Find the magnitude of P to slide lower block.



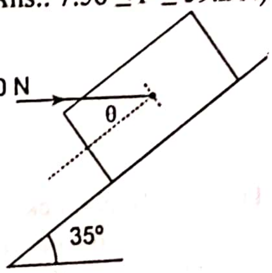
(Ans.: P = 1370 N)

6. Knowing that θ = 25°, determine the range of values of P for which equilibrium is maintained.



(Ans.: $7.56 \le P \le 59.2$ N)

7. Determine the smallest value of θ for which the block of 20 kg is in equilibrium. Take $\mu_s = 0.3$.

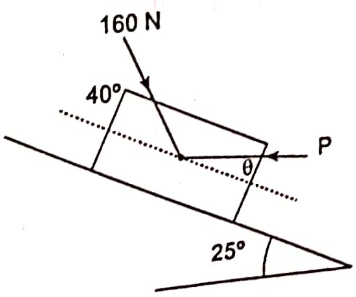


(Ans.: θ = 28.9°)

8. Neglecting the mass of block, determine

a) the smallest value of P for which the block is in equilibrium.

b) the corresponding value of θ. Take $\mu_s = 0.35$.



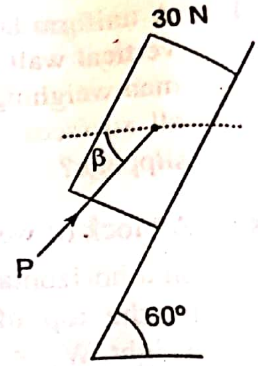
(Ans.: P_{min} = 81.7 N, θ = 19.29°)

9. For the 30 N block, determine

a) the smallest value of P required to maintain equilibrium.

b) the corresponding value of β.

Take $\mu = 0.25$.

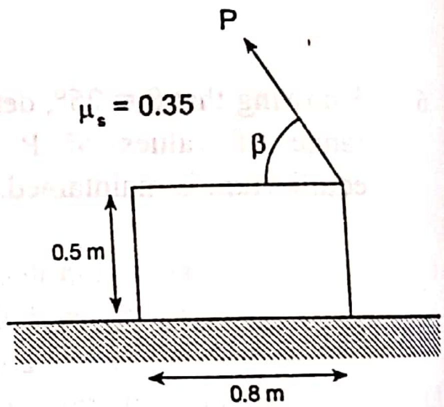


(Ans.: $P_{min} = 21.6$ N, $\beta = 46°$)

10. A block of mass 40 kg must be moved to the left along the floor without tipping. Determine

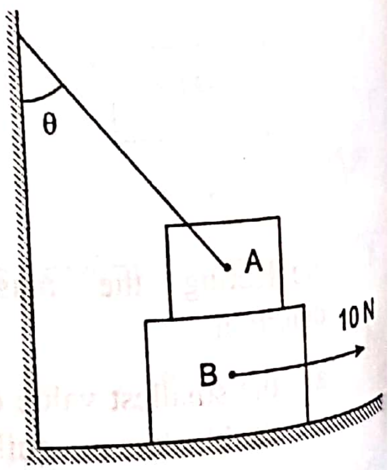
a) largest allowable value of β.

b) the corresponding value of P.



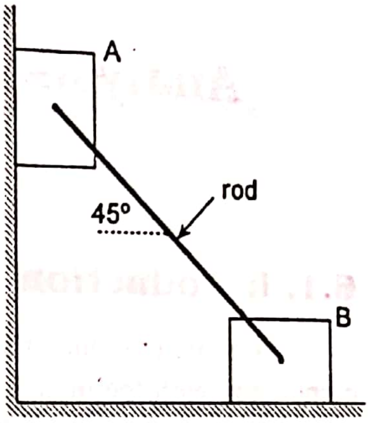
(Ans.: $\beta = 58.1°$, $P = 166.4$ N)

11. The 16 N block A is attached to link AC and rests on the 24 N block B. Determine the value of θ for which motion of block B is impending. Take $\mu = 0.2$ for all.



(Ans.: $\theta = 40.9°$)

12. Two identical blocks A and B are connected by a rod and rest as shown in figure. The sliding motion of two blocks impends when rod makes an angle of 45° with horizontal. Calculate the coefficient of friction assuming it to be same for both surface.



(Ans.: $\mu = 0.414$)

Chapter 6
Analysis of Beam and Frame

6.1. Introduction

A structural member designed to support loads applied at vario point along with the member is known as beam. In most cases, the loads a perpendicular to the axis of beam and will cause only shear and bending i the beam. When the load are not at right angle to the beam, they will al produce axial force in the beam.

Frames are simply the combination of the network of horizon beam and vertical column. They are designed to carry the loads and transf it to the support.

6.2 Objective of structural design

The structural design should have following main objectives:

a) Strength: To resist safely the stresses induced by the loads i various section.

b) Stability: To prevent overturning, sliding or buckling of structu under the action of load.

c) Serviceability: To ensure satisfactory performance under load.

6.3 Discrete and continuum

Various assumption are made during the application of principles mechanics to practical problems on the basis of assumption, a structure m be:

a) Continnum: In this idealization, the elements are assumed to distributed continuously throughout the surface. The solutions of th structures are represented by partial differential equations and are mo complex.

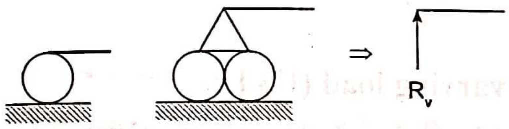
b) Discrete (Skeletal): In this idealization, the elements are assume to be composed of finite number of small element. Depending upon th resources and time availability, we divide the structure into number of small compound i.e. in basic 1-D element (line element), 2-D element (basic triangle element) and 3-D element (cube or parallelepiped). Larger the number of discrete elements, more accurate will be the result. This assumption makes the analysis easier.

6.4 Types of support

Basically there are three types of supports. These are based on natures of independent displacement called degree of freedom.

a) Roller support

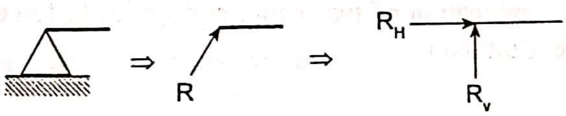
It gives rise to one force which is perpendicular to the plane supporting the plane.



No. of reaction = 1

b) Hinge support

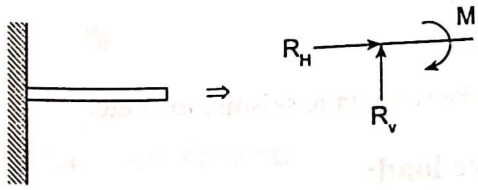
It gives rise to one force reaction whose direction is unknown. It can be resolved into two forces along x and y axes.



No. of reaction = 2

c) Fixed support

It gives rise to one force reaction (having two components) and one reaction moment.



No. of reaction = 3

6.5 Classification of loads

Different types of loads are:

a) Point load:

It is assumed to act at a point. It may be vertical, horizontal or inclined. Unit: kg, N, kN.

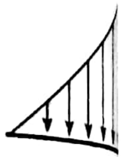
b) Uniformly distributed load (UDL):

Load is distributed along the structure uniformly. It is in rectangular form. Unit: kg/m, kH/m.

Total load is calculated as
W = base × height = Area of rectangle

c) Uniformly varying load (UVL):

Load which varies with uniform intensity. Intensity varies with respect to length. Unit: kg/m, kN/m



Total load $=\frac{1}{2}\times$ base × height = Area of triangle

d) Couple:

t is the combination of two equal and opposite forces separated certain distance. Unit: Nm.

e) Moment:

It is the product of force and distance. Unit: Nm

f) Static and Dynamic load:

Static load are gradually applied load. They do not give ris vibration whereas the dynamic loads give rise to load and vibration as we

g) Imposed load:

Extra load like wind, rain, seismic load etc.

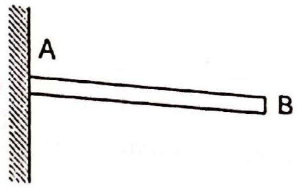
h) Dead and live load:

Dead load is the self-load or load of structure itself whereas the load are movable load.

6.6 Classification of beams

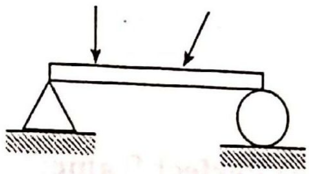
The beams are classified as:

a) Cantilever beam:



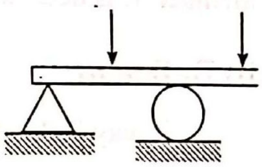
A cantilever is a beam whose one end is fixed and the other end is free. AB is the length of cantilever.

b) Simply supported beam:



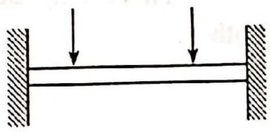
A beam which rests as one hinge support and other roller support.

c) Over handing beam:



An overhanging beam is one one or both the ends projects beyond the support.

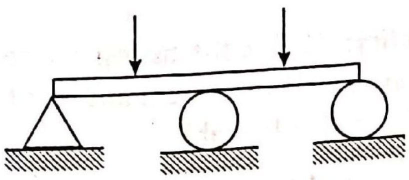
d) Fixed beam:



A beam whose both ends are rigidly fixed or built into its supporting walls and columns.

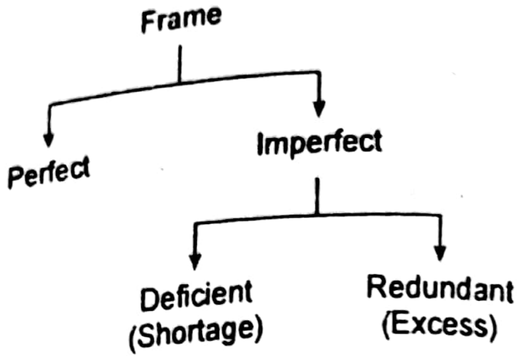
e) Continuous beam:

This have more than two supports.



6.7 Classification of frame

A frame is the network of beams and columns joined together to carry load and transfer to the support. If the joint is rigid, the frame is rigid jointed frame and if the joints are hinge connected, it is called pin jo... frame.



a) Perfect frame:

There is sufficient number of support connections and structu... member. It is determinate and stable both statically and geometrically.

b) Deficient:

It may lack either structural member or support connection or both.

c) Redundant:

There may be either excess of support connections or member... both.

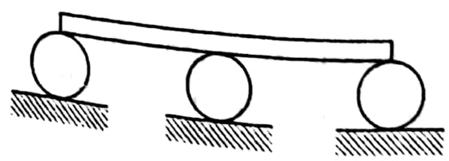
6.8 Concept of beams and frames

6.8.1 Stability

The structure is stable if and only if it is statically and geometrical... stable.

a) Statical stability: This is the necessary condition for the stabili... If unknown quantities are equal to the equations of static equilibrium, th... the structure is said to be statically stable.

b) Geometrical stability: This is the sufficient condition for th... stability. There must be at least one end fixed or combination of one hin... and one roller.



It is determinate and statically stable but geometrically unstable since it cannot resist inclined load.

6.8.2 Static determinacy and indeterminacy

A structure is said to be statically determinate if all the internal member forces and reactions can be determined using the equation of static equilibrium.

For beam:

External indeterminacy = $r - (3 + c)$

Internal indeterminacy = 0

Total degree of static indeterminacy = $r - (3 + c) + 0$

$= r - (3 + c)$

where r = no. of support reaction and c = no. of equations due to special condition (hinge).

For frame:

External indeterminacy = $r - (3 + c)$

Internal indeterminacy = $3 \times$ total no. of cuts required to have open configuration

Directly, you can use

Total degree of static indeterminacy = $(3m + r) - (3j + c)$

where j = no. of joints.

To get internal indeterminacy you can subtract external indeterminacy from total.

6.8.3 Kinematic determinacy and indeterminacy

It is related with the joint displacements. It is the total number of freedom that are allowed for the joints in a structure.

For beam:

We can find degree of kinematic indeterminacy by counting the freedom at joints.

For frame:

Degree of kinematic indeterminacy (DKI) = $3j - (m + r)$

6.9 Calculation of axial force, shear force and bending moment and drawing AFD, SFD and BMD

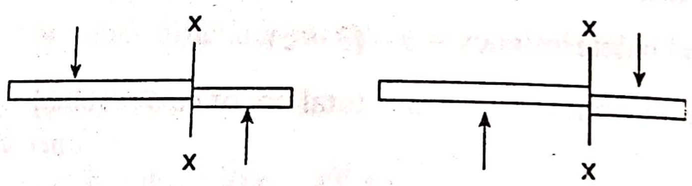
6.9.1 Axial force (AF)

At any cross-section of member it is the algebraic sum of all forces acting parallel to the longitudinal axis on either side of the section. A tensile is positive while compressive is negative.

Tensile (+ve) Compressive (–ve)

6.9.2 Shear force (SF)

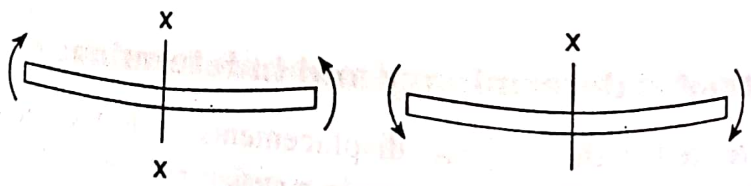
It is the algebraic sum of all forces acting transverse to the member on either side of the section.



–ve shear +ve shear

6.9.3 Bending moment (BM)

It is the algebraic sum of the moments of all forces on either side of the section.



+ve BM
Sagging condition

–ve BM
Hogging condition

The beam in hogging condition is subjected to negative bending moment and one in a sagging condition is subjected to positive bending moment.

6.9.4 Axial force, shear force and bending moment diagrams (AFD, SFD and BMD)

The graphical representation of axial force, shear force and bending moment which are introduced in the structural members are called AFD, SFD and BMD. Values of axial force, shear force and bending moment are plotted along y-axis corresponding to the length of the member along x-axis.

6.9.5 Important properties of SF and BM:

BM is zero at simple support.

a) Under UDL, SFD varies linearly but BMD varies parabolically.

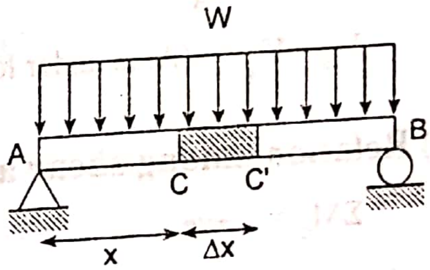
b) Maximum BM occurs at point of zero shear.

c) The point of zero BM is known as point of contraflexure or inflexion point or point of virtual hinge.

d) Abrupt change in loading cause change in slope of SFD and BMD.

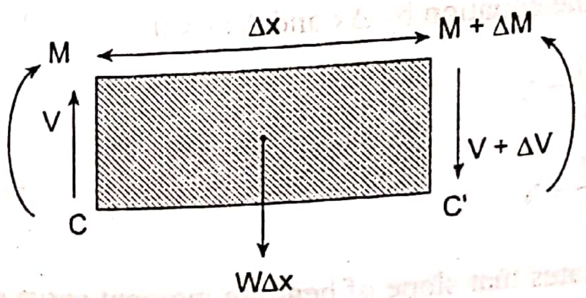
e) Concentrated load produce vertical lines in SFD and BMD.

6.9.6 Relation among load, shear and bending moment.



Consider a simply supported beam AB carrying distributed W per unit length. Let C and C′ be the two points on the beam at a distance of Δx from each other. The shear and BM at C will be denoted by V and M and is assumed positive. Therefore, the shear and BM at C′ will be denoted by $v + \Delta v$ and $M + \Delta M$ respectively.

Let us detach the portion of beam CC′ and redraw the free body diagram FBD.



The force exerted on the free body include a load of magnitude $W\Delta x$ and internal forces.

a) Relation among load and shear:

$$\Sigma F_y = 0 \uparrow ve$$

$$V - (V + \Delta V) - W\Delta x = 0$$

$$\Delta V = -W\Delta x$$

Dividing both side by Δx and let $\Delta x \to 0$;

$$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = -W$$

$$\frac{dv}{dx} = -W$$

indicates for a beam loaded as shown. Slope of shear curve is negative, absolute value of slope at any point equals to load per unit length at that point.

Integrating

$$\int_A^B dv = -\int_{x_A}^{x_B} W\,dx$$

$$V_B - V_A = -(\text{Area under load curve})$$

b) Relation among shear and BM

$$\Sigma M_{c'} = 0 + ve$$

$$M + \Delta M - M - V\Delta x + W\Delta x\frac{\Delta x}{2} = 0$$

$$\Delta M = V\Delta x - \frac{1}{2}W(\Delta x)^2$$

Dividing the equation by Δx and $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} = V$$

$$\frac{dM}{dx} = V$$

This indicates that slope of bending moment curve equals to value of shear. The equation shows that BM is maximum when shear force is zero. This property tells where the beam is likely to fall under bending.

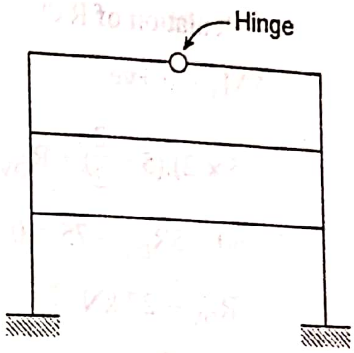
Integrating,

$$\int_A^B dM = \int_{x_A}^{x_B} V\,dx$$

$M_B - M_A$ = area under shear curve

Worked Out Examples

1. Calculate external, internal and total statical indeterminacy of given frame.



Solution:

Here, m = 9, j = 8, r = 6 (fixed support), c = 1 (hinge)

External indeterminacy

$= r - (3 + c)$

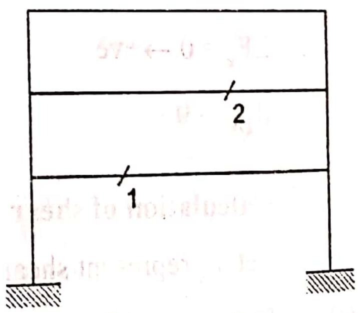
$= 6 - (3 + 1) = 2$

Total indeterminacy $= (3m + r) - (3j + c)$

$= (3 \times 9 + 6) - (3 \times 8 + 1) = 8$

$\therefore$ Internal indeterminacy $= 8 - 2 = 6$

Or, Directly you can calculate total internal indeterminacy

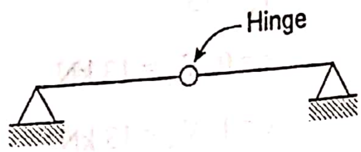


= 3 × no. of cuts for open configuation

$= 3 \times 2$

$= 6$

2. Calculate external indeterminacy of given beam.



Solution:

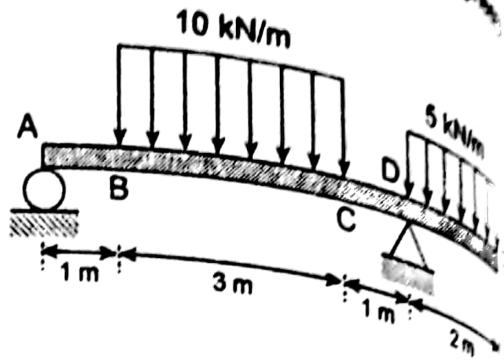
Here, r = 4 (for hinge support)

c = 1 (hinge point)

External indeterminacy $= r - (3 + c)$

$= 4 - (3 + 1) = 4 - 4 = 0$

3. Draw SFD, BMD of the given beam loaded as shown.



Solution:

Draw FBD:

$R_{AV} \rightarrow Rx^n$ at A

$R_{DH} \rightarrow Rx^n$ at D horizontal

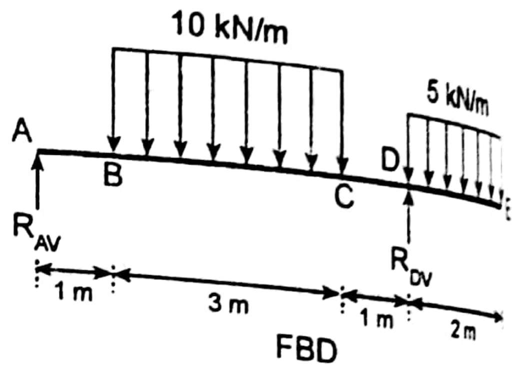
$R_{DV} \rightarrow Rx^n$ at D vertical

Calculation of Rx^n

$\Sigma M_A = 0$ +ve

$$-(5 \times 2).(5 + \frac{2}{2}) + R_{DV} \times 5 - (10 \times 3)\,(1 + \frac{3}{2}) = 0$$

$-60 + 5R_{DV} - 75 = 0$



$\therefore\ R_{DV} = 27$ kN (↑)

$\Sigma F_y = 0$ ↑+ve

$R_{AV} - 30 + R_{DV} - 10 = 0$

$R_{AV} - 40 + 27 = 0$

$R_{AV} = 13$ kN (↑)

$\Sigma F_x = 0$ →+ve

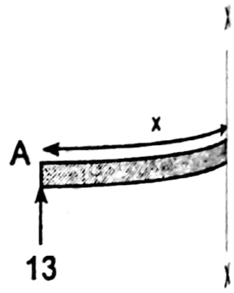
$R_{DH} = 0$

Calculation of shear force and BM

Let V_x represent shear force and M_x represent moment at section.

(i) For span AB $[0 \leq x \leq 1\text{m}]$

$V_x = 13$



$x = 0;\ V_A = 13$ kN

$x = 1;\ V_B = 13$ kN

$M_x = 13x$

$x = 0;\ M_A = 0$

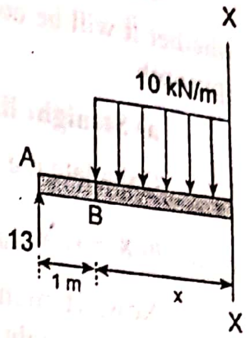
$x = 1;\ M_B = 13$ kNm

(ii) For Span BC [$0 \le x \le 3$m]

$V_x = 13 - 10x$

$x = 0$; $V_B = 13$ kN

$x = 3$; $V_C = -17$ kN



Here values of shear force varies from +ve to -ve; so check for zero-shear

$V_x = 13 - 10x$

$0 = 13 - 10x$

$\therefore$ $x = 1.3$ m

It is 1.3 m from B towards C.

$M_x = 13(1 + x) - (10x)\frac{x}{2}$

$= 13 + 13x - 5x^2$

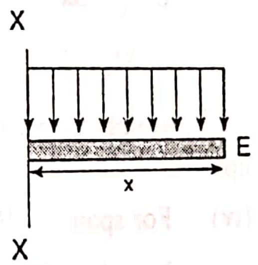
$x = 0$; $M_B = 13$ kNm

$x = 3$; $M_C = 13 + 13 \times 3 - 5 \times 9$

$= 7$ kNm

$x = 1.3$; $M_{max} = 21.45$ kNm

For span ED [$0 \le x \le 2$m]



For our simplicity, we have solved few span parts from left to right and will solve few span from right to left. But you can also solve either totally from left to right or right to left. The important thing is sign convention. Be sure you have used proper sign convention as stated.

(iii) For span ED [$0 \le x \le 2$m]

$V_x = 5x$

$x = 0$; $V_E = 0$

$x = 2$; $V_D = 10$ kN

$M_x = -5x\frac{x}{2} = -2.5x^2$

$x = 0$; $M_E = 0$

$x = 2$; $M_D = -10$ kNm

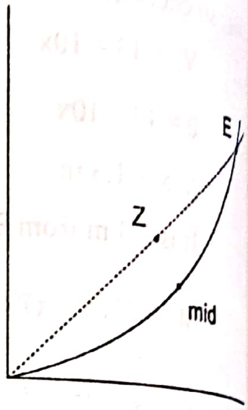
Here $M_x = -2.5x^2$ represent the curve. So to plot we have to check whether it will be concave upward or concave downward. We will have two approach.

a) Straight line

Calculate the moment M_x at middle of span.

at $x = 1$; $M_{mid} = -2.5 \times 1 = -2.5$ kN = point Z (say)

Now, If moments value three point E, Z and D forms straight line



$$M_Z = \frac{M_D + M_E}{2}$$

$$= \frac{0 - 10}{2} = -5 \text{ kNm}$$

$M_Z > M_{mid}$; so the curve is concave upward.

b) Calculus approach

Since we have taken span from right to left, –ve sign is to be neglected for equation.

$M_x = 2.5x^2$

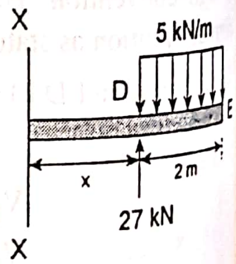
$dM_x = 5x$

$d^2 M_x = 5$

which is positive and gives the minimum value and is concave upward.

(iv) For span DC $[0 \le x \le 1\text{m}]$

$V_x = 10 - 27 = -17$



$x = 0$; $V_D = -17$ kN

$x = 1$; $V_C = -17$ kN

$M_x = -10(1 + x) + 27x$

$= -10 - 10x + 27x$

$= 17x - 10$

$x = 0$; $M_D = -10$ kNm

$x = 1$; $M_C = 7$ kNm

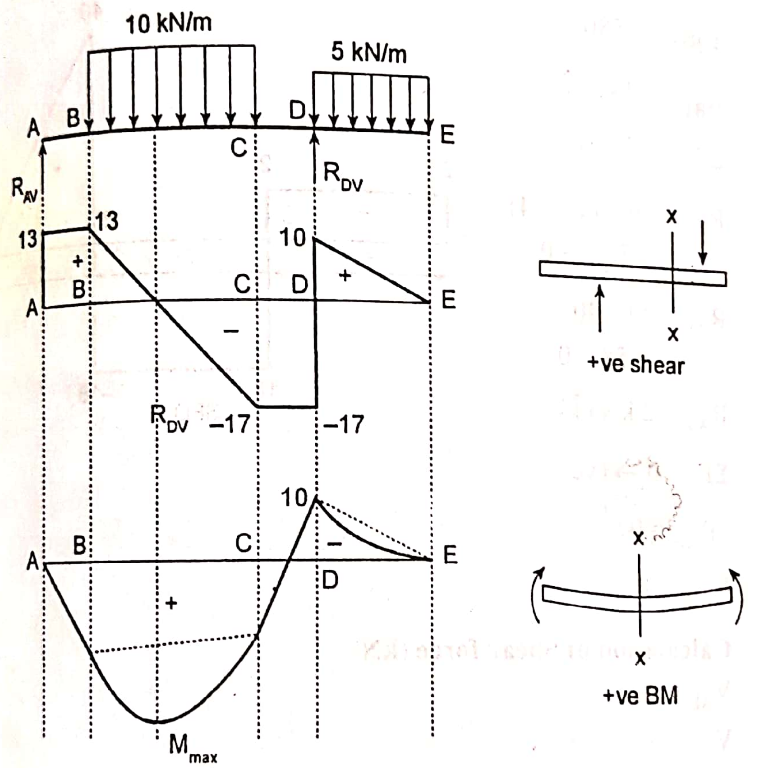
Here values of BM varies from –ve to +ve, so check for zero BM.

$M_x = 17x - 10$

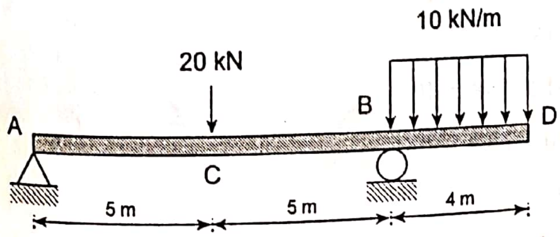
$0 = 17x - 10$

$x = \frac{10}{17} = 0.59$

It is 0.59 m from D towards C. This point is known as point of contraflexure.



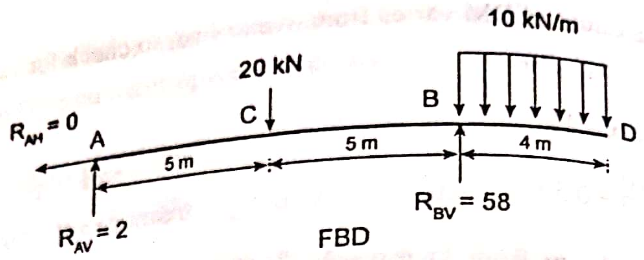
4. Draw SFD and BMD for the given beam.



Solution:

$\Sigma M_A = 0$ +ve

$20 \times 5 - R_{BV} \times 10 + 10 \times 4 \times (10 + 2) = 0$



$100 - 10R_{BV} + 480 = 0$

$10R_{BV} = 580$

$R_{BV} = 58$ kN($\uparrow$)

$\Sigma F_y = 0 \uparrow$+ve

$R_{AV} - 20 - (10 \times 4) + R_{BV} = 0$

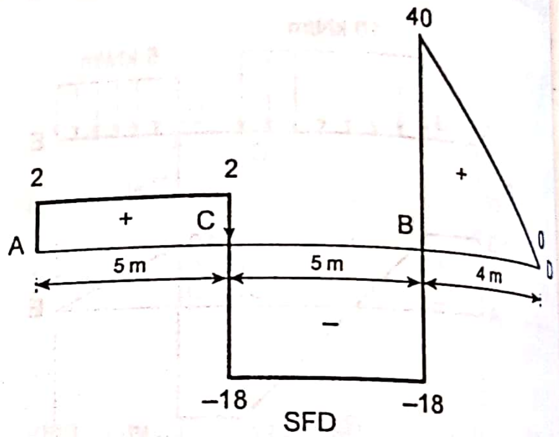
$R_{AV} - 20 - 40 + 58 = 0$

$R_{AV} = 2$ kN ($\uparrow$)

$\Sigma F_x = 0 \rightarrow$+ve

$-R_{AH} = 0$

$R_{AH} = 0$



Calculation of Shear force (kN)

$V_{AL} = 0$

$V_{AR} = 2$

$V_{CL} = 2$

$V_{CR} = 2 - 20 = -18$

$V_{BL} = -18$

$V_{BR} = -18 + 58 = 40$

$V_{DL} = 40 - 10 \times 4 = 0$

$V_{DR} = 0$

Calculation of BM: (kNm)

$M_{AL} = 0$

$M_{AR} = 0$

$M_{CL} = 0 + 10 = 10$ (Add (+) area under V-curve)

$M_{CR} = 10$

$M_{BL} = 10 - 90 = -8$

$M_{BR} = -80$

$M_{DL} = -80 + \frac{1}{2} \times 4$

$M_{DR} = 0$

Let P be the mid of

But from give

condition

$M_{PL} = M_{BR}$

$+\frac{1}{2}(20 + 40$

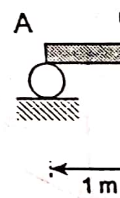
$= -80 + 60$

$= -20$ kNm

So, it suggest lo

value.

5. Draw SFD and B



Solution:

Calculation of

$\Sigma M_A = 0$ +ve

$\left(\frac{1}{2} \times 3 \times 20\right).$

$30 \times 3 + 15 \times$

$\therefore R_{BV} = 27.$

$M_{CR} = 10$

$M_{BL} = 10 - 90 = -80$ (Subtract (–) area)

$M_{BR} = -80$

$M_{DL} = -80 + \frac{1}{2} \times 4 \times 40 = 0$

$M_{DR} = 0$

Let P be the mid of BD. If it is straight line $M_{PL} = -40$ kNm

But from given condition

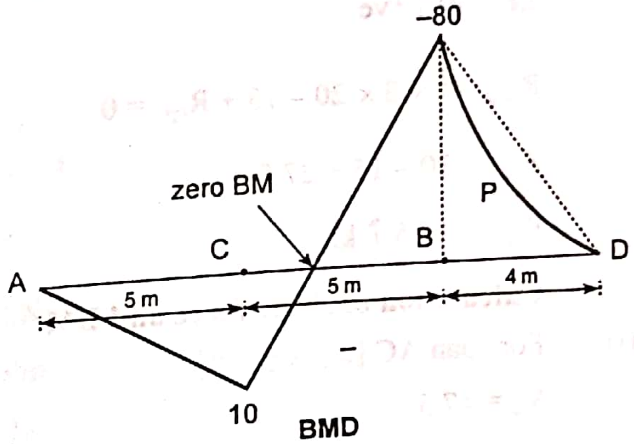
$M_{PL} = M_{BR}$

$+\frac{1}{2}(20 + 40).2$

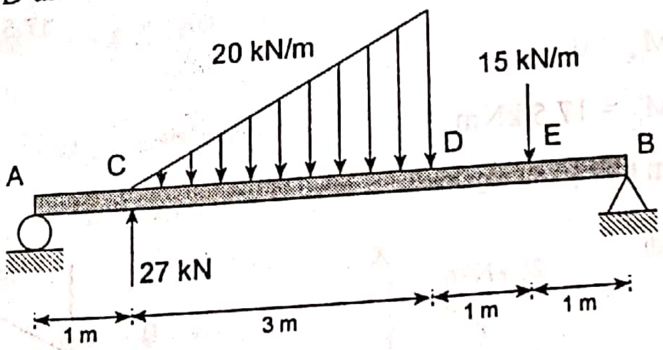
$= -80 + 60$

$= -20$ kNm

So, it suggest lower value.



5. Draw SFD and BMD of the beam loaded as follows:



Solution:

Calculation of reaction:

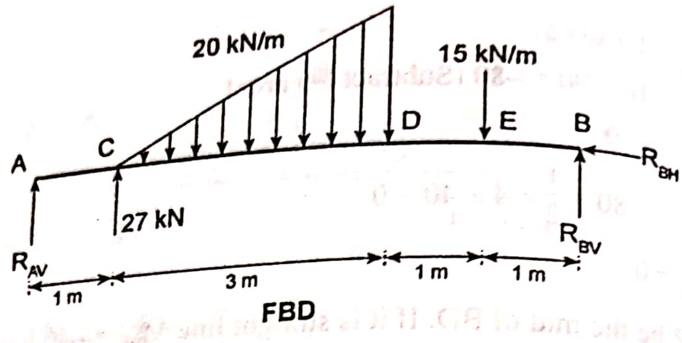
$\Sigma M_A = 0$ +ve

$$\left(\frac{1}{2} \times 3 \times 20\right).\left(1 + \frac{2}{3}.3\right) + 15 \times 5 - R_{BV} \times 6 = 0$$

$$30 \times 3 + 15 \times 5 - 6\,R_{BV} = 0$$

$$\therefore R_{BV} = 27.5\ (\uparrow)\ \text{kN}$$

Scanned by CamScanner



$\Sigma F_y = 0 \uparrow +ve$

$R_{AV} - \frac{1}{2} \times 3 \times 20 - 15 + R_{BV} = 0$

$R_{AV} - 30 - 15 + 27.5 = 0$

$R_{AV} = 17.5 \uparrow$ kN

Calculation of shear force and BM

(i) For span AC $[0 \leq x \leq 1\text{m}]$

$V_x = 17.5$

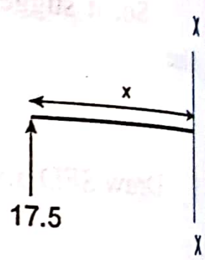
$x = 0; V_A = 17.5$ kN

$x = 1; V_B = 17.5$ kN

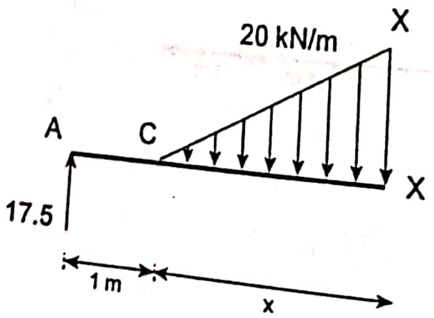
$M_x = 17.5x$

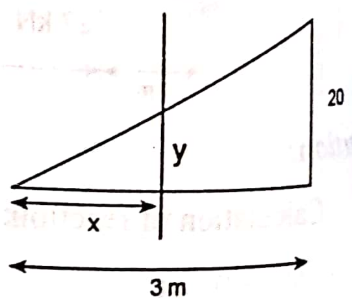
$x = 0; M_A = 0$

$x = 1; M_C = 17.5$ kNm



(ii) For span CD $[0 \leq x \leq 3\text{m}]$





$V_x = 17.5 - \frac{1}{2} \times 20 \frac{x}{3}$

$= 17.5 - \frac{10}{3} x^2$

$x = 0;$ $V_C = 17.5$ kN

$x = 3;$ $V_D = -12.5$ kN

Values of shear varies from +ve to –ve, so check for zero shear.

$0 = 17.5 - \frac{10}{3}x^2$

$\frac{10}{3}x^2 = 17.5$

$x = 2.29$ m

It is from C to D.

$M_x = 17.5\,(1 + x) - \frac{1}{2} \times \frac{20x}{3}\frac{x}{3}$

$M_x = 17.5 + 17.5x - \frac{10}{9}x^3$

$x = 0;$ $M_C = 17.5$ kNm

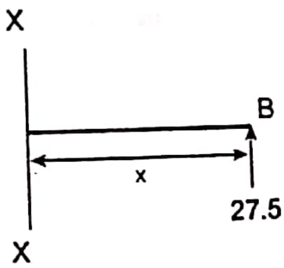
$x = 3;$ $M_D = 40$ kNm

$x = 2.29,$ $M_{max} = 44.23$

$x = 1.5;$ $M_{mid} = 40$ kN

$\frac{M_C + M_D}{2} = 27.75$ kNm

(iii) For span BE $[0 \le x \le 1\text{m}]$



$V_x = -27.5$

$x = 0,$ $V_B = -27.5$ kN

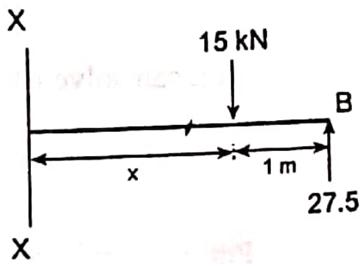
$x = 1,$ $V_E = -27.5$ kN

$M_x = 27.5x$

$x = 0,$ $M_B = 0$

$x = 1,$ $M_E = 27.5$ kNm

(iv) For span ED $[0 \le x \le 1\text{m}]$



$V_x = -27.5 + 15 = -12.5$ kN

$x = 0,$ $V_E = -12.5$ kN

$x = 1,$ $V_D = -12.5$ kN

$M_x = 27.5(1 + x) - 15x$

$= 27.5 + 12.5\,x$

$x = 0, \qquad M_E = 27.5 \text{ kNm}$

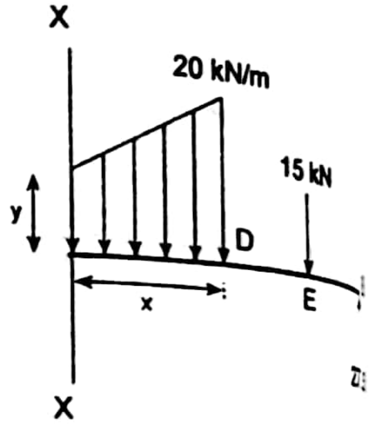
$x = 1, \qquad M_D = 40 \text{ kNm}$

Alternative method of span DC

Sometimes the triangular load has to be solved by making trapezoid

For span DC $[0 \le x \le 3\text{m}]$

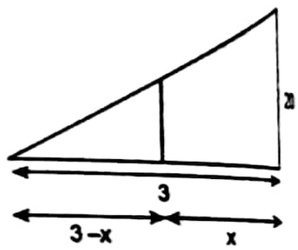
Before shear force calculation the height of trapezium at section X–X, let say it is y.



$$\frac{20}{y} = \frac{3}{3-x}$$

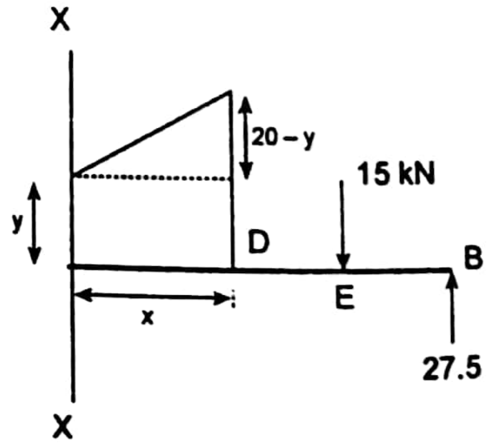
$$y = \left(\frac{3-x}{3}\right)20$$

Now the trapezoid can be divided into triangle and rectangle as shown.



$$V_x = -27.5 + 15 + xy + \frac{1}{2}\, x(20 - y)$$

$$V_x = -12.5 + x\left(\frac{3-x}{3} \cdot 20\right) + \frac{1}{2}\, x\left(20 - \frac{3-x}{3} \cdot 20\right)$$



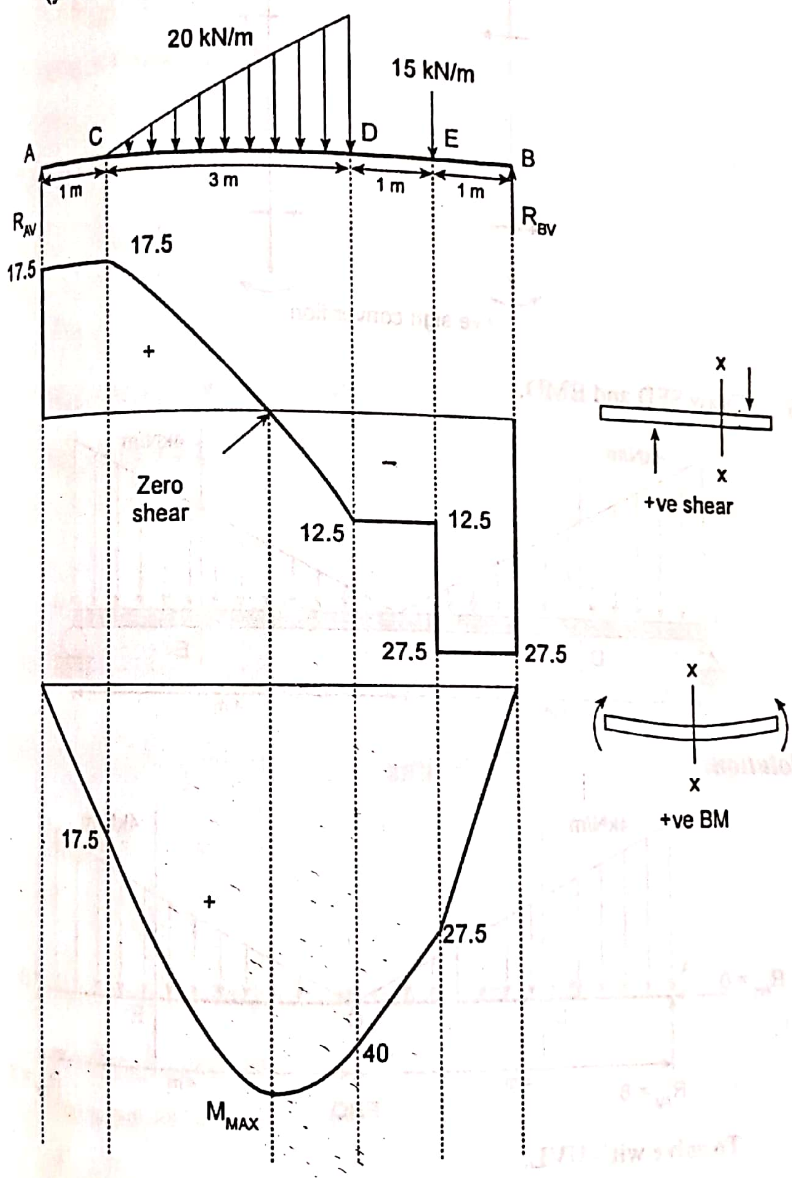
(you can solve it)

$$M_x = 27.5(2 + x) - 15(1 + x) - \frac{1}{2}\, x(20 - y)\frac{2x}{3} - xy\frac{x}{2}$$

Put $y = \frac{3-x}{3} \cdot 20$

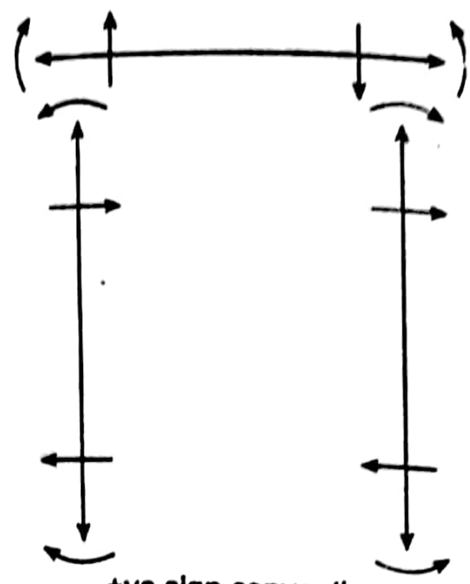
$$M_x = \frac{10}{9}x^3 - 10x^2 + 12.5x + 40$$

(you can solve now.)

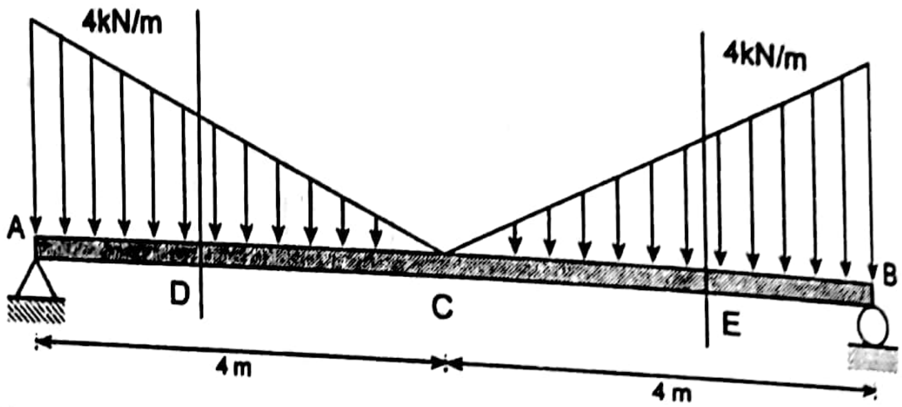


Analysis of frame

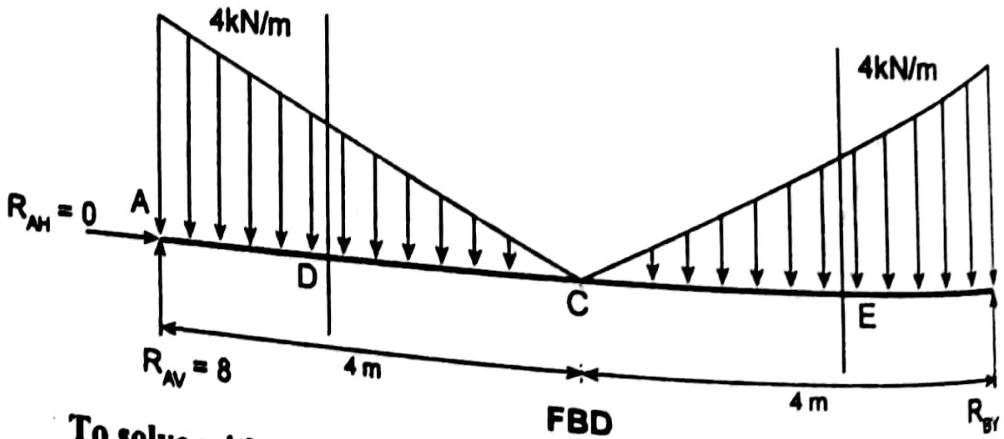
Assume the individual elements of frame should be in equilibrium if whole frame is in equilibrium.



6. Draw SFD and BMD.



Solution:



To solve with UVL:

Draw SFD and BMD.

$\Sigma M_A = 0$ +ve

$$\frac{1}{2} \times 4 \times 4 \times \left(\frac{1}{3} \cdot 4\right) + \frac{1}{2} \times 4 \times 4 \times \left(4 + \frac{2}{3} \cdot 4\right) - R_{BV} \times 8 = 0$$

$\therefore\ R_{BV} = 8$ kN↑

$\Sigma F_y = 0 \uparrow +ve$

$R_{AV} + R_{BV} = 16$

$\therefore R_{AV} = 8 \text{ kN}$

Shear Calculation:

$V_{AL} = 0$

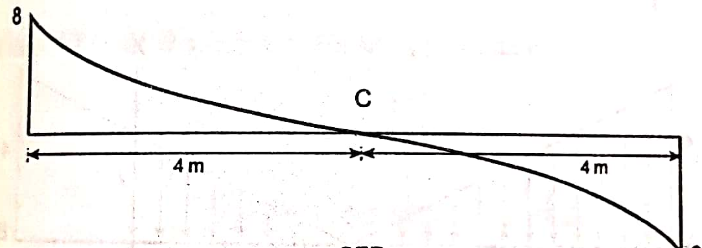
$V_{AR} = 8 \text{ kN}$

$V_{CL} = 8 - \frac{1}{2} \times 4 \times 4 = 0$

$V_{CR} = 0$

$V_{BL} = 0 - \frac{1}{2} \times 4 \times 4 = -8 \text{ kN}$

$V_{BR} = -8 + 8 = 0$



SFD

$V_{DL} = 8 - \frac{1}{2}(4 + 2).2$ (D is mid-point of AC)

$= 2$

$V_{EL} = 0 - \frac{1}{2} \times 2 \times 2 = -2$ (E is mid-point of CB)

Bending Calculation

For span AC ($0 \le x \le 4$ m)

$$\frac{4}{y} = \frac{4}{4-x} \Rightarrow y = 4 - x$$

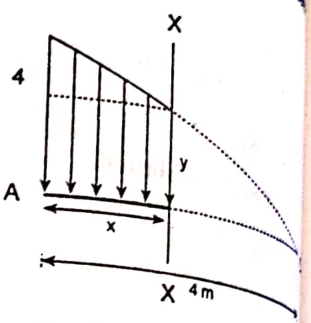
$$M_x = 8x - xy \cdot \frac{x}{2} - \frac{1}{2} x(4-y) \frac{2}{3} x$$

$$= 8x - \frac{x^2}{2}(4-x) - \frac{x^3}{3}(x)$$

$$= 8x - 2x^2 + \frac{x^3}{2} - \frac{x^3}{3}$$

$$= \frac{48 - 12x^2 + 3x^3 - 2x^3}{6}$$

$$= \frac{x^3 - 12x^2 + 48x}{6}$$



$x = 0;\quad M_A = 0$

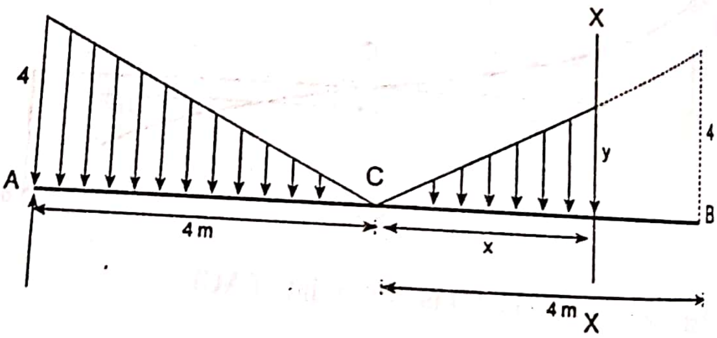
$x = 2;\quad M_{mid} = 9.33$ kNm

$x = 4;\quad M_C = 10.667$

for straight line $M_{mid} = \frac{0 + 10.667}{2} = 5.33$ kNm

For span CB:

$\frac{y}{4} = \frac{x}{4} \Rightarrow \quad y = x$



$$M_x = 8 \times (4 + x) - (\frac{1}{2} \times 4 \times 4).(x + \frac{2}{3} \times 4) - \frac{1}{2}.x.y.\frac{1}{3}x$$

$$= 32 + 8x - 8(x + \frac{8}{3}) - \frac{1}{6}x^3$$

$$= 32 + 8x - 8x - \frac{64}{3} - \frac{x^3}{6}$$

$$= 32 - \frac{64}{3} - \frac{x^3}{6}$$

$$= \frac{192 - 128 - x^3}{6}$$

$$= \frac{64 - x^3}{6}$$

$x = 0;\quad M_C$

$x = 4;\quad M_E$

$x = 2;\quad M_m$

for straight line M

Conclusion:

To calculate BM

other cases left-right m

7. Draw AFD, SFD

Solution:

Calculation of r

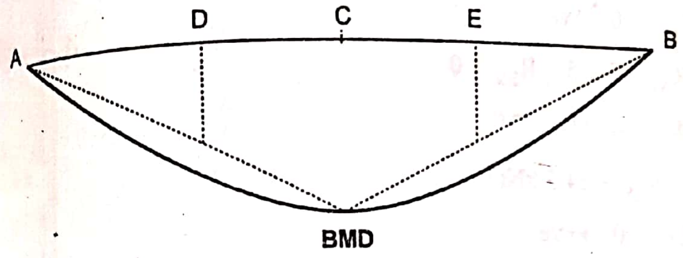
$\Sigma M_E = 0$ +ve

$x = 0;$ $M_C = 10.667$

$x = 4;$ $M_B = 0$

$x = 2;$ $M_{mid} = 9.33$

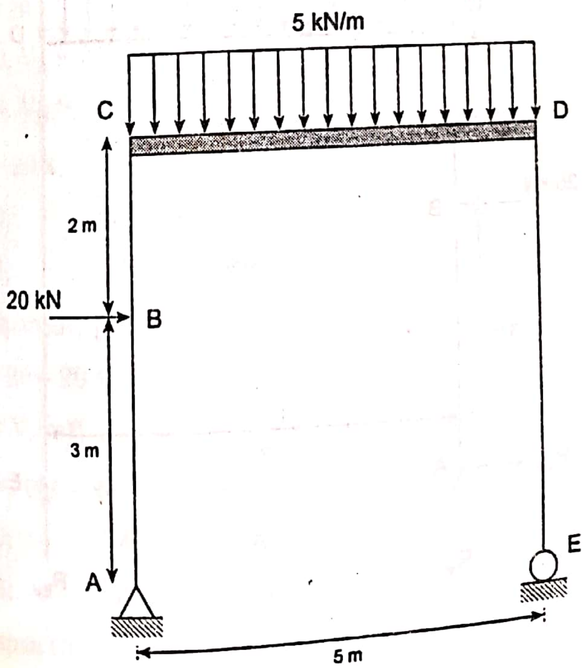
for straight line $M_{mid} = \dfrac{0 + 10.667}{2} = 5.33$ kNm



Conclusion:

To calculate BM for a triangular load use section method and for all other cases left-right method.

7. Draw AFD, SFD and BMD for the given frame.



Solution:

Calculation of reactions:

$\Sigma M_E = 0$ +ve

$$-R_{AV} \times 5 - 20 \times 3 + 25 \times \frac{5}{2} = 0$$

$$-5R_{AV} - 60 + 62.5 = 0$$

$$-5R_{AV} = -2.5$$

$$\therefore\ R_{AV} = 0.5\ \text{kN}(\uparrow)$$

$$\Sigma F_y = 0 \uparrow +\text{ve}$$

$$R_{AV} - 5 \times 5 + R_{EV} = 0$$

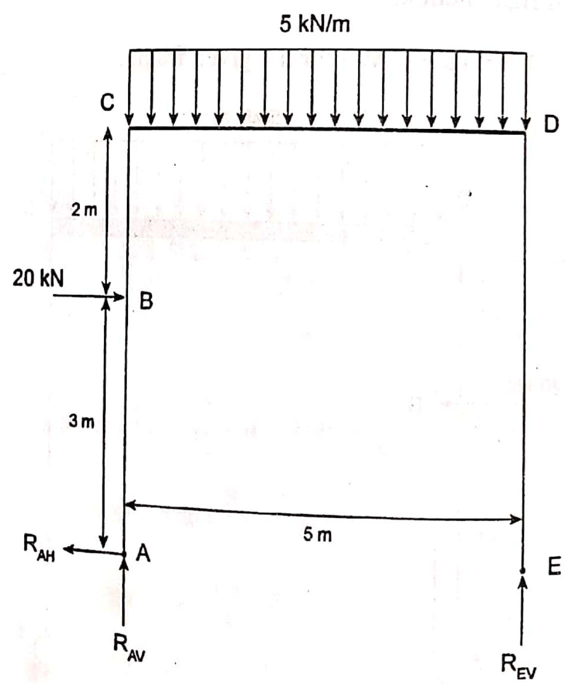
$$0.5 - 25 + R_{EV} = 0$$

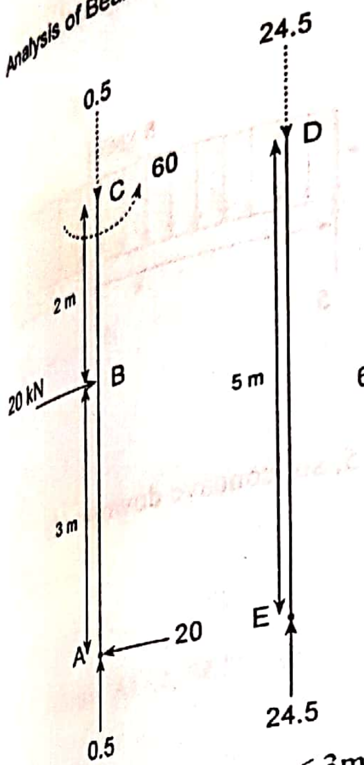
$$\therefore\ R_{EV} = 24.5\ \text{kN}(\uparrow)$$

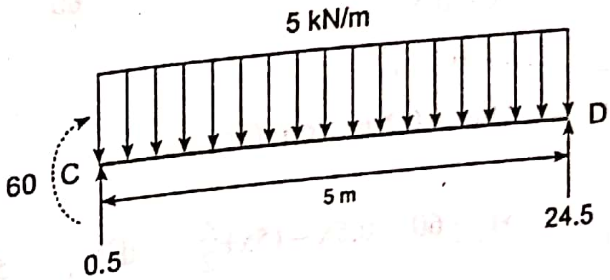
$$\Sigma F_x = 0 \rightarrow +\text{ve}$$

$$20 - R_{AH} = 0$$

$$\therefore\ R_{AH} = 20\ \text{kN}$$







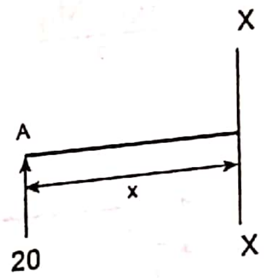
FBD

(i) For span AB $[0 \le x \le 3m]$

$V_x = 20$

$x = 0, V_A = 20$ kN

$x = 3; V_B = 20$ kN



$M_x = 20\,x$

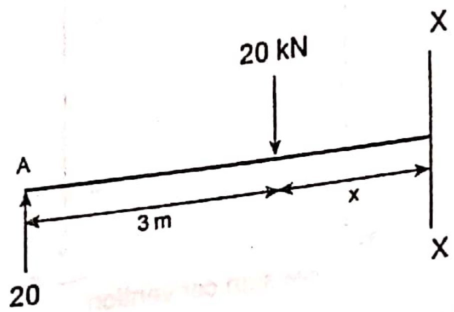
$x = 0; \quad M_A = 0$

$x = 3; \quad M_B = 60$ kNm

(ii) For span BC $[0 \le x \le 2m]$

$V_x = 20 - 20 = 0$

$V_B = V_C = 0$



$M_x = 20(3 + x) - 20x = 60$

$x = 0; \quad M_B = 60$ kNm

$x = 3; \quad M_B = 60$ kNm

(iii) For span CD $[0 \le x \le 5m]$

$V_x = 0.5 - 5x$

$x = 0; V_C = 0.5$ kN

$x = 5; V_D = -24.5$ kN

Here shear force varies from +ve to –ve; so check for zero shear

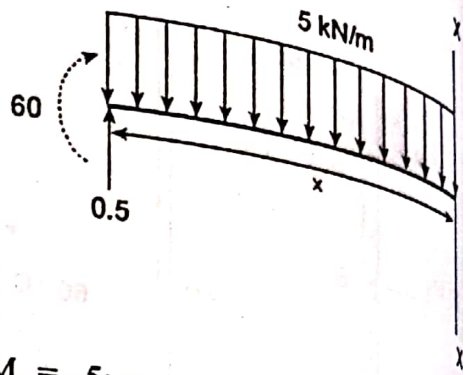
$V_x = 0.5 - 5x$

$0 = 0.5 - 5x$

$5x = 0.5$

$x = 0.1$ m

This is 0.1 m from C to D.



$M_x = 60 + 0.5x - (5x)\dfrac{x}{2}$ $\quad$ $d^2M_x = -5$; so, concave downward.

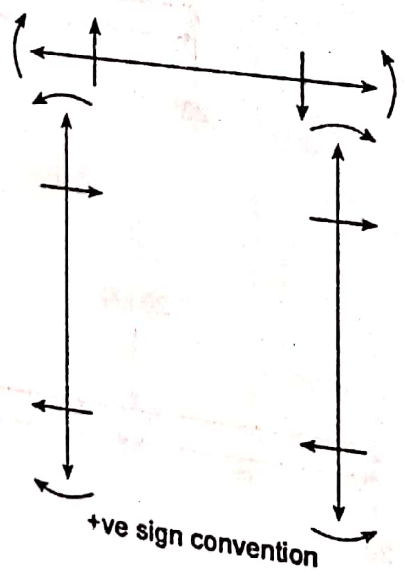
$= 60 + 0.5x - 2.5\,x^2$

$x = 0;$ $\quad$ $M_C = 60$ kNm

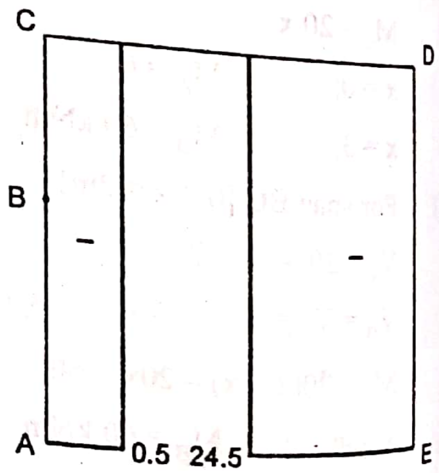
$x = 5;$ $\quad$ $M_D = 60 - 60 = 0$

$x = 2;$ $\quad$ $M_{mid} = 51$ kNm

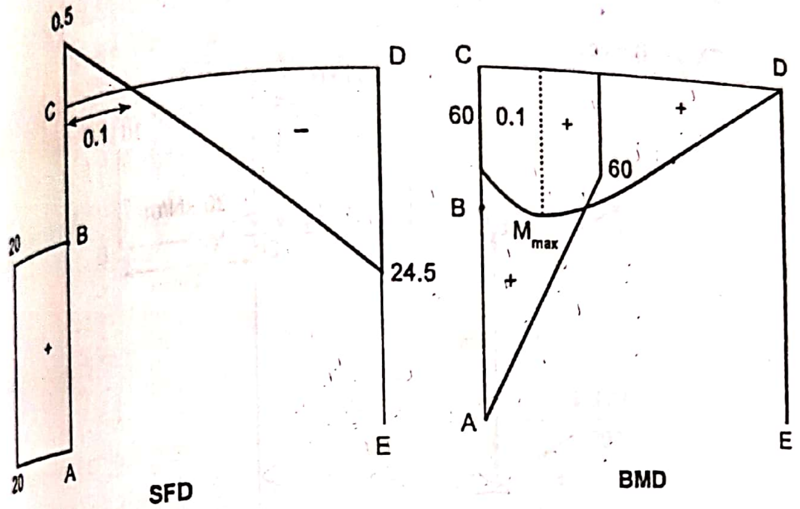
$\dfrac{M_C + M_D}{2} = 30$ kNm



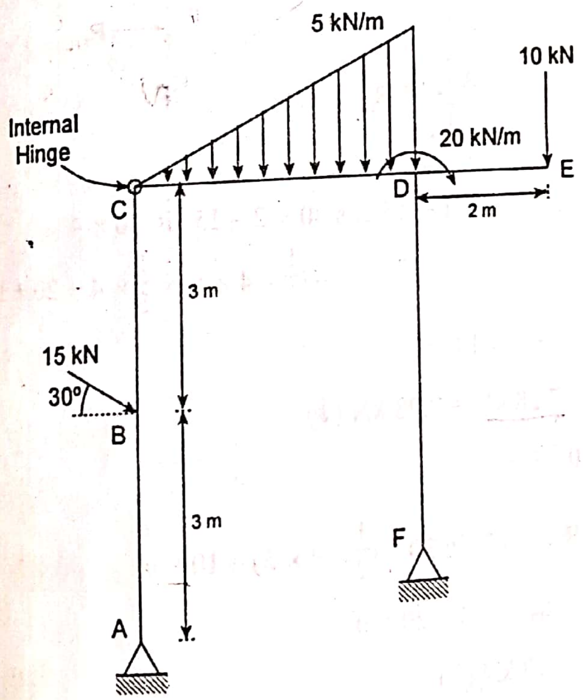
+ve sign convention



AFD



8. Draw AFD, SFD and BMD for the given frame.



Solution:

Calculation of rx^n

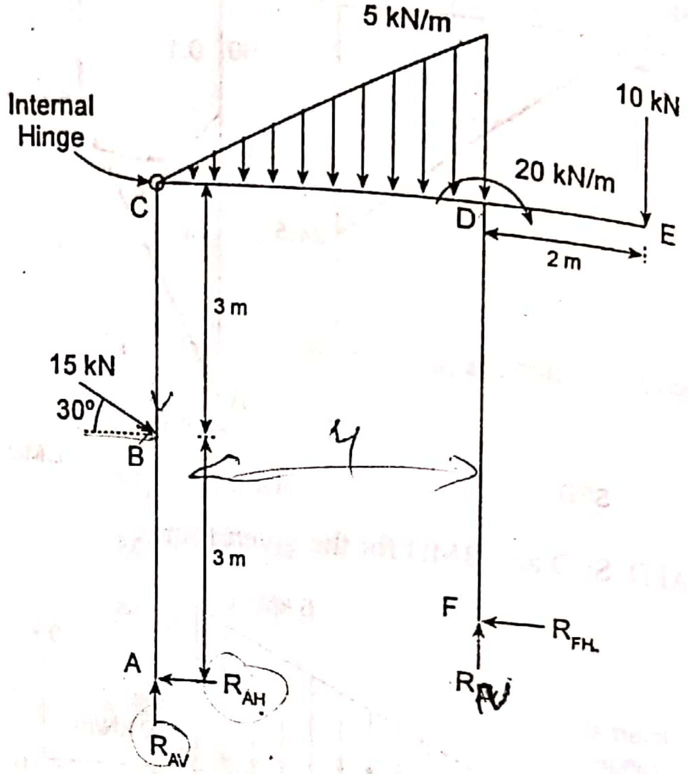
$\Sigma(M_C)$ left = 0 +ve

$R_{AH} \times 6 - 15 \cos 30 \times 3 = 0$

$6R_{AH} = 45 \cos 30$

$R_{AH} = 6.5$ kN (←)

$\Sigma M_F = 0$ +ve;



$$R_{AV} \times 4 + R_{AH} \times 1 + 15 \cos 30 \times 2 - 15 \sin 30 \times 4$$
$$- (\frac{1}{2} \times 4 \times 5) \times \frac{1}{3} \times 4 + 20 + 10 \times 2 = 0$$

$\therefore\ 4R_{AV} = -29.1426$

$R_{AV} = -7.28$ kN $= 7.28$ kN (↓)

$\Sigma F_y = 0$ ↑+ve

$$F_{AV} + R_{FV} - 15 \sin 30 - (\frac{1}{2} \times 4 \times 5) - 10 = 0$$

$$-7.28 + R_{FV} - 7.5 - 20 = 0$$

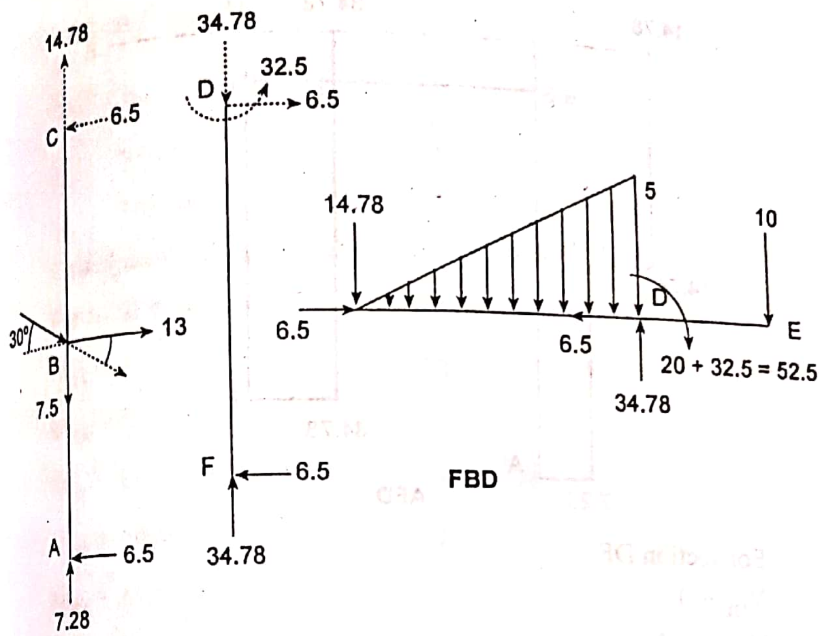
$R_{FV} = 34.78$ kN (↑)

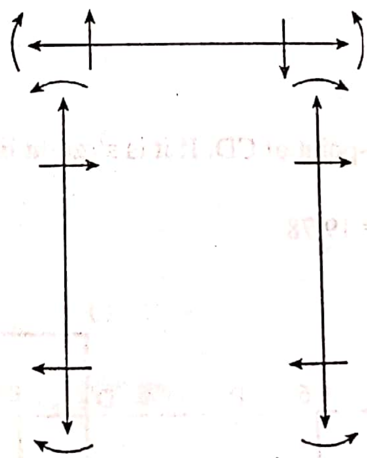
$\Sigma F_x = 0$ →+ve

$$-R_{AH} + 15 \cos 30 - R_{FH} = 0$$

$$-6.5 + 13 - R_{FH} = 0$$

$R_{FH} = 6.5$ kN (←)





+ve sign convention

Calculation of shear force (kN)

For section ABC:

$V_{AL} = 0$

$V_{AR} = 6.5$

$V_{BL} = 6.5$

$V_{BR} = 6.5 - 13 = -6.5$

$V_{CL} = -6.5$

$V_{CR} = 0$

For section CDE:

$V_{CL} = 0$

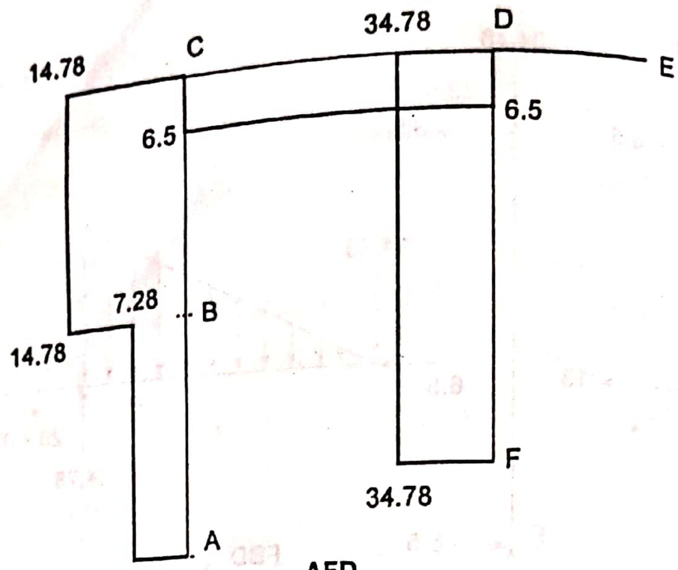
$V_{CR} = -14.78$

$V_{DL} = -14.78 - 10 = -24.78$

$V_{DR} = -24.78 + 34.78 = 10.0$

$V_{EL} = 10$

$V_{ER} = 10 - 10 = 0$



AFD

For section DF

$V_{DL} = 0$

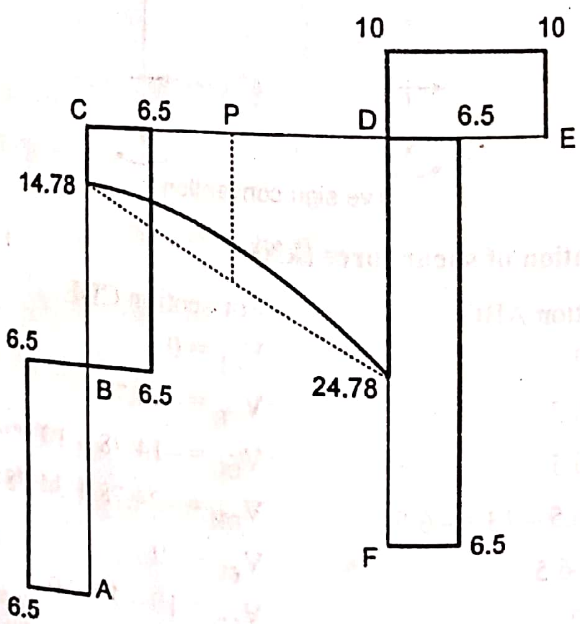
$V_{DR} = 6.5$

$V_{FL} = 6.5$

$V_{FR} = 6.5 - 6.5 = 0$

Let P be the mid-point of CD. If it is straight line

$$V_{PL} = \frac{V_C + V_D}{2} = 19.78$$



SFD

From condition:

$V_{PL} = V_{CR} + (-\frac{1}{2} \times 2 \times 2.5)$

$= -14.78 - 2.5$

$= 17.28$

Calculation of BM (kNm)

For section ABC:

$M_{AL} = 0$

$M_{AR} = 0$

$M_{BL} = 0 + 6.5 \times 3 = 19.5$

$M_{BR} = 19.5$

$M_{CL} = 19.5 - 6.5 \times 3 = 0$

$M_{CR} = 0$

For section DF

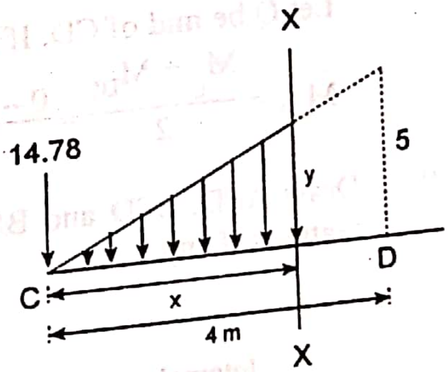
$M_{DL} = 0$

$M_{DR} = -32.5$

$M_{FL} = -32.5 + 6.5 \times 5 = 0$

$M_{FR} = 0$

For section CD:



$\frac{y}{5} = \frac{x}{4}$

$y = \frac{5x}{4} = 1.25x$

$M_x = -14.78x - \frac{1}{2} \times x \times y \times \frac{1}{3} \times x$

$= -14.78x - \frac{1}{6} x^2(1.25x)$

$= -14.78x - \frac{5}{2} x^3$

$x = 0;\ M_C = 0$

$x = 4$ m; $\quad M_{DL} = -72.45$ kNm

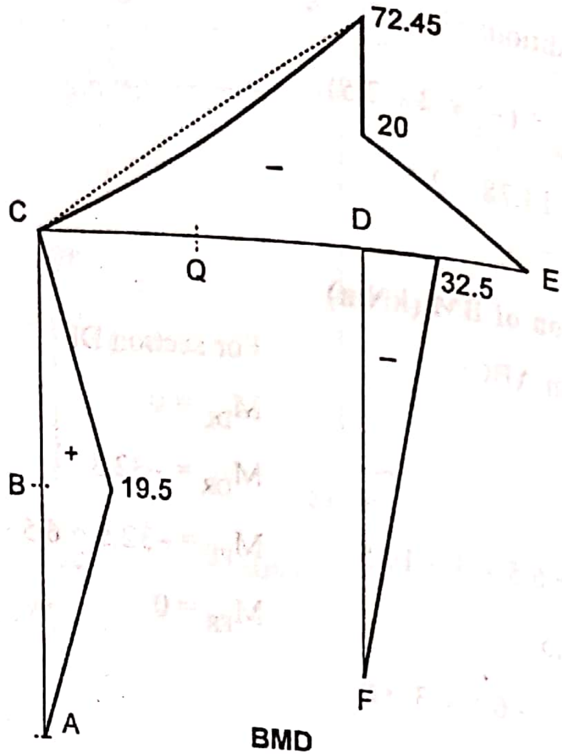
Let Q be mid of CD.

$x = 2$m; $\quad M_{\text{mid of CD}} = -32.22 = M_{Q2} = -31.22$

Again,

$M_{DR} = -72.45 + 52.5 = -19.95 \sim -20$

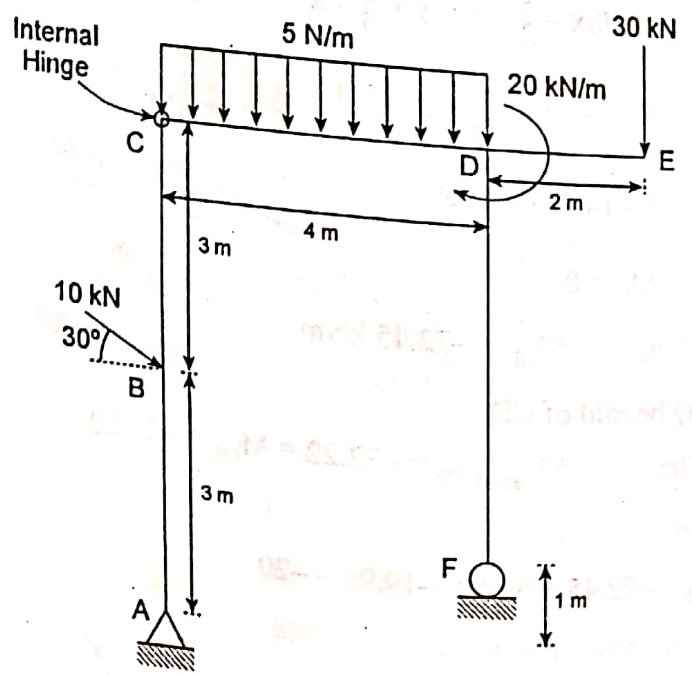
$M_{EL} = -20 + 10 \times 2 = 0$



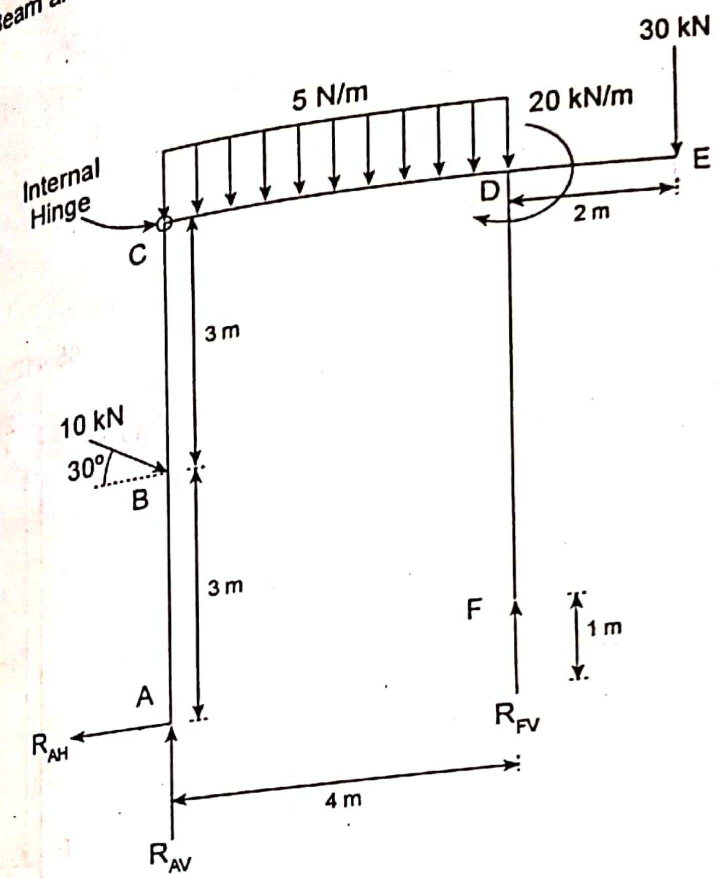
Let Q be mid of CD. If it is straight line

$$M_{Q2} = \frac{M_C + M_{DL}}{2} = \frac{0 - 72.45}{2} = -36.225$$

9. **Draw AFD, SFD and BMD of given frame. Indicate also the salient features if any.**



Solution:



Calculation of reaction

Taking left of internal hinge C

$\Sigma M_C = 0$ +ve

$-R_{AH} \times 6 + 10 \cos 30 \times 3 = 0$

$\therefore\ 6R_{AH} = 15\sqrt{3}$

$\therefore\ R_{AH} = 4.33$ kN $(\leftarrow)$

Taking $\Sigma M_F = 0$ +ve

$-R_{AV} \times 4 - 4.33 \times 1 - 10 \cos 30 \times 2 + 10 \sin 30 \times 4 + (5 \times 4).2 - 20 - 30 \times 2 = 0$

$-4R_{AV} = 41.65$

$\therefore\ R_{AV} = (-)\ 10.4125 = 10.4125\ (\downarrow)$

$\Sigma F_x = 0 \rightarrow$+ve

$-4.33 + 10 \cos 30 - R_{FH} = 0$

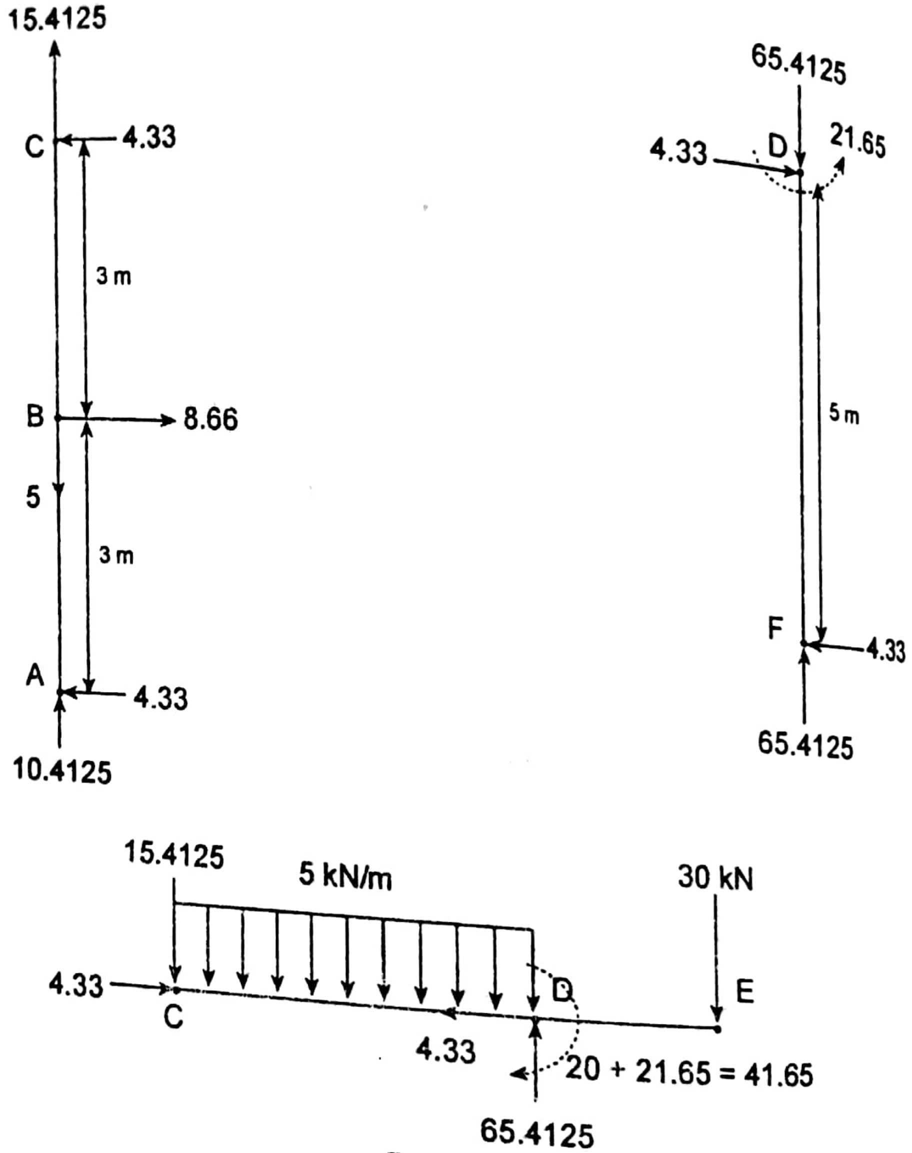
$\therefore\ R_{FH} = 8.66 - 4.33 = 4.33$ kN $(\leftarrow)$

$\Sigma F_y = 0 \uparrow +ve$

$R_{AV} - 10 \sin 30 - 5 \times 4 - 30 + R_{FV} = 0$

$-10.4125 - 5 - 20 - 30 + R_{FV} = 0$

$\therefore \; R_{FV} = 65.4125\ (\uparrow)\ \text{kN}$



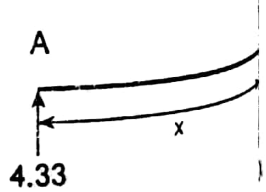
FBD

(i) For span AB $[0 \leq x \leq 3\text{m}]$

$V_x = 4.33$

$x = 0;$ $\quad V_A = 4.33\ \text{kN}$

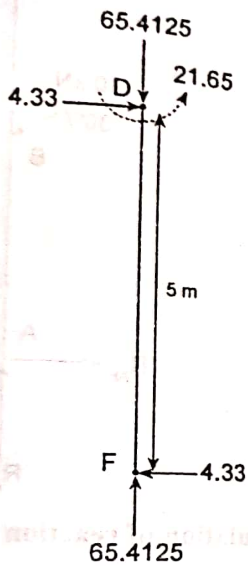
$x = 3;$ $\quad V_B = 4.33\ \text{kN}$

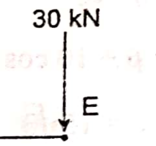


$M_x = 4.33x$

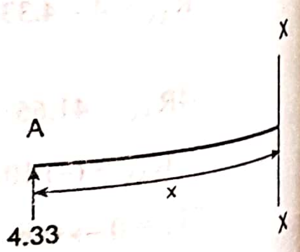
$x = 0;$ $\quad M_A = 0$

$x = 3;$ $\quad M_B = 13$





21.65 = 41.65



(ii) For span BC $[0 \le x \le 3\text{m}]$

$V_x = 4.33 - 8.66 = -4.33$ kN

$x = 0;$ $\quad V_B = -4.33$ kN

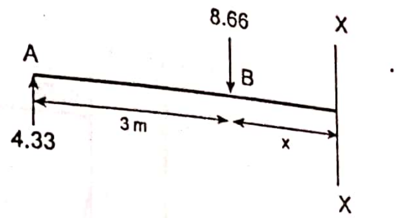
$x = 3;$ $\quad V_C = -4.33$ kN

$M_x = 4.33(3 + x) - 8.66x$

$= 13 - 4.33x$

$x = 0;$ $\quad M_B = 13$ kNm

$x = 3;$ $\quad M_C = 0$



(iii) For span FD $[0 \le x \le 5\text{m}]$

$V_x = 4.33$

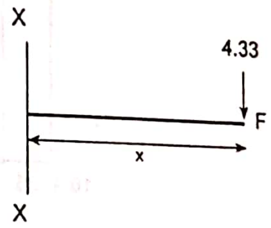
$x = 0;$ $\quad V_F = 4.33$ kN

$x = 5;$ $\quad V_D = 4.33$ kN

$M_x = -4.33x$

$x = 0;$ $\quad M_F = 0$

$x = 5;$ $\quad M_D = -21.65$ kN



(iv) For span ED $[0 \le x \le 2\text{m}]$

$V_x = 30$

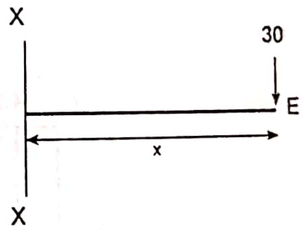
$x = 0;$ $\quad V_E = 30$ kN

$x = 2;$ $\quad V_D = 30$ kN

$M_x = -30x$

$x = 0;$ $\quad M_E = 0$

$x = 2;$ $\quad M_D = -60$ kNm

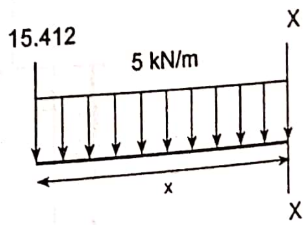


(v) For span CD $[0 \le x \le 4\text{m}]$

$V_x = -15.412 - 5x$

$x = 0;$ $\quad V_C = -15.412$ kN

$x = 4;$ $\quad V_D = -35.412$ kN



$M_x = -15.412\,x - 5x \cdot \dfrac{x}{2}$

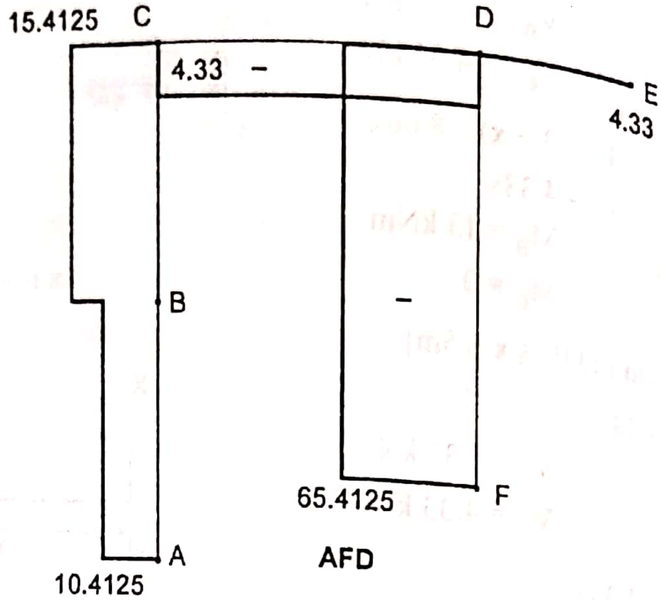
$d^2M_x = 5$; so, concave upward.

$= -15.412x - 2.5x^2$

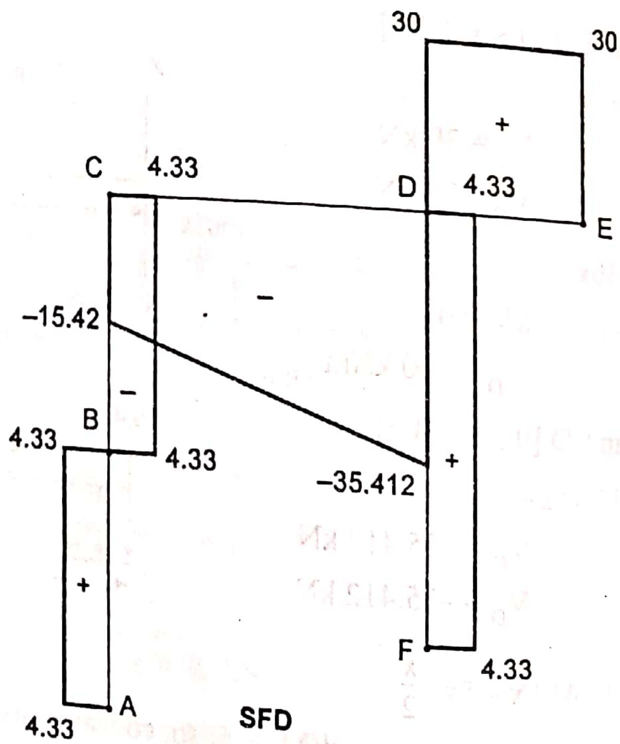
$x = 0;$ $\quad M_C = 0$

$x = 4;$ $\quad M_D = -101.648$ kNm

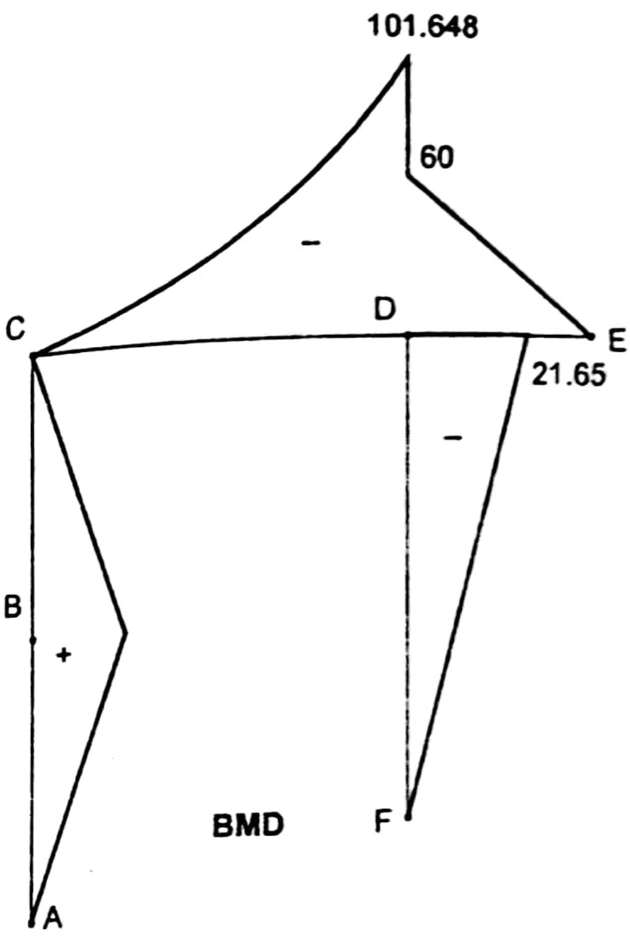
$$\frac{M_C + M_D}{2} = -50.824$$



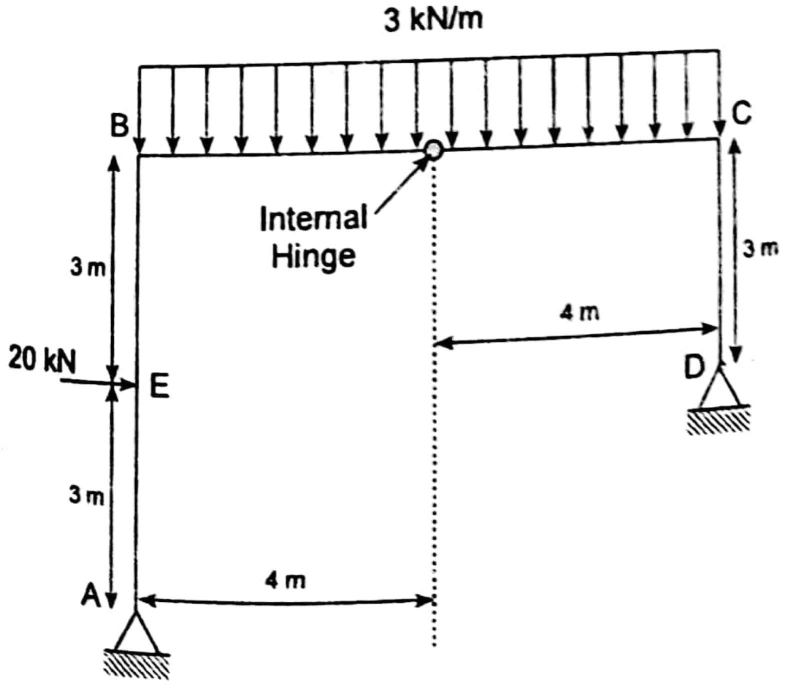
AFD



SFD



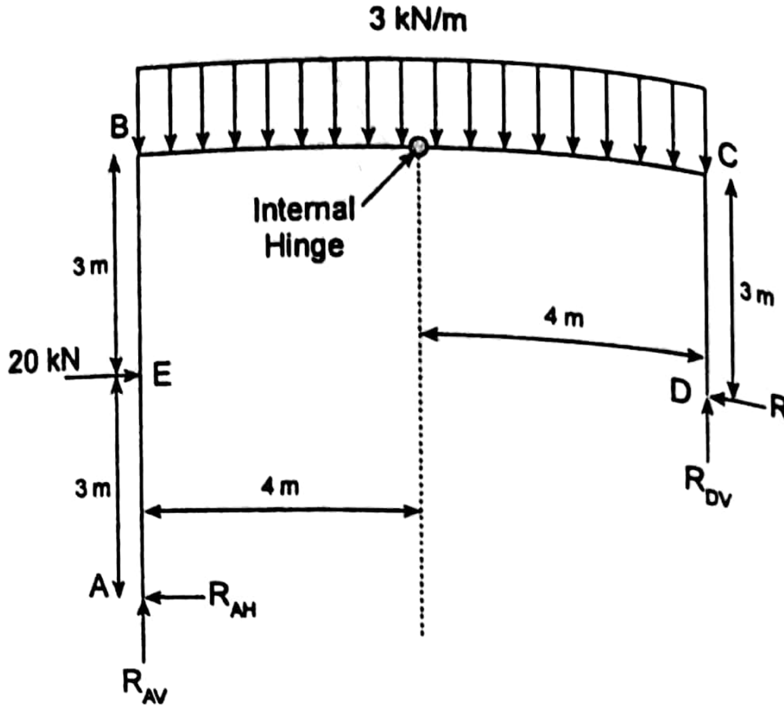
10. Draw AFD, SFD and BMD.



Solution:

$$\curvearrowleft (\Sigma M_F)_{left} = 0$$

$$R_{AV} \times 4 + R_{AH} \times 6 - 20 \times 3 - 3 \times 4 \times \frac{4}{2} = 0$$



$4R_{AV} + 6R_{AH} - 84 = 0$

$2R_{AV} + 3R_{AH} - 42 = 0$ ......(i)

$\Sigma M_D = 0$ +ve

$R_{AV} \times 8 + R_{AH} \times 3 - (3 \times 8) \times 4$

$8R_{AV} + 3R_{AH} - 96 = 0$ ......(ii)

Solving equation (i) and (ii)

$R_{AV} = 9$ kN, $R_{AH} = 8$ kN

Why we have taken $\Sigma M_D = 0$ but not $\Sigma M_A = 0$. It is because in equation (i) we have two variable R_{AV} and R_{AH}. So, to solve these values we need one more equation on these variable which we can get when we take $\Sigma M_D = 0$

$\Sigma F_y = 0 \uparrow$+ve

$R_{AV} - 3 \times 8 + R_{DV} = 0$

$\therefore\ 9 - 24 + R_{DV} = 0$

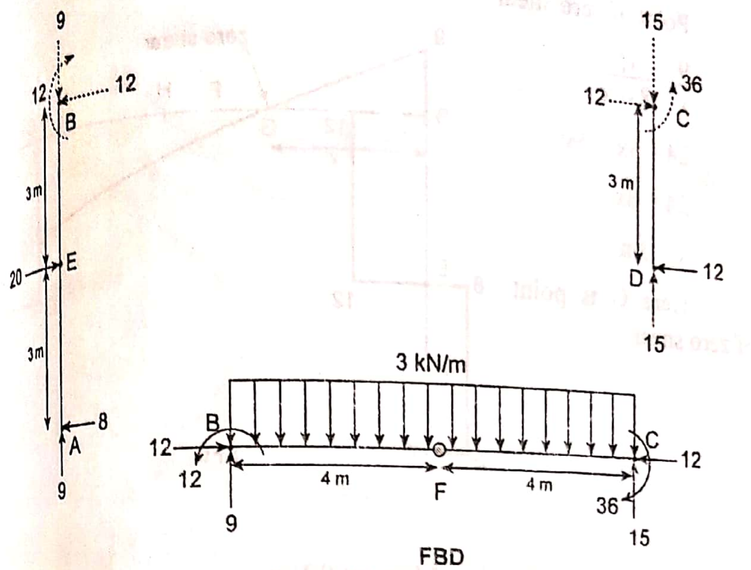
$\therefore\ R_{DV} = 15$ kN

$\Sigma F_x = 0 \rightarrow$+ve

$20 - R_{AH} - R_{DH} = 0$

$20 - 8 - R_{DH} = 0$

$\therefore\ R_{DH} = 12$ kN



Calculation of Shear Force (kN)

For section AEB:

$V_{AL} = 0$

$V_{AR} = 8$

$V_{EL} = 8$

$V_{ER} = 8 - 20 = -12$

$V_{BL} = -12$

$V_{BR} = -12 + 12 = 0$

For section CD:

$V_{CL} = 0$

$V_{CR} = 12$

$V_{DL} = 12$

$V_{DR} = 12 - 12 = 0$

For section BFC:

$V_{BL} = 0$

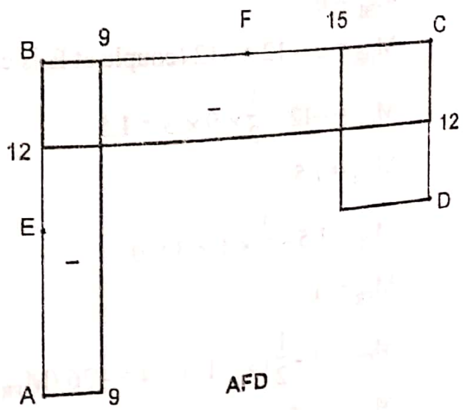
$V_{BR} = 9$

$V_{FL} = 9 - 12 = -3$

$V_{FR} = -3$

$V_{CL} = -3 - 12 = -15$

$V_{CR} = -15 + 15 = 0$



Point of zero shear

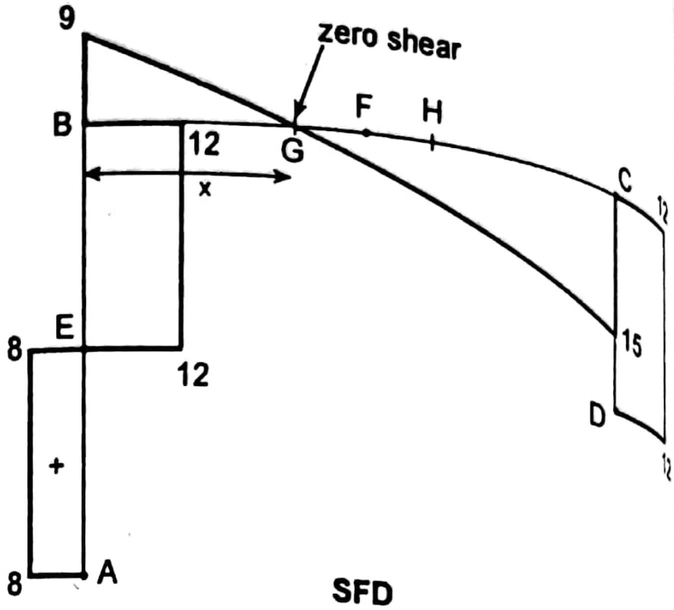
$\frac{9}{x} = \frac{15}{8-x}$

$24 - 3x = 5x$

$24 = 8x$

$x = 3m$

Here G is point of zero shear.



SFD

Calculation of Bending Moment (kNm)

For section AEB:

$M_{AL} = 0$

$M_{AR} = 0$

$M_{EL} = 0 + 24 = 24$ (Add positive area)

$M_{ER} = 24$

$M_{BL} = 24 - 36 = -12$ (Subtract negative area)

$M_{BR} = -12 + 12 = 0$

For section CD:

$M_{CL} = 0$

$M_{CR} = -36$

$M_{DL} = -36 + 36 = 0$

$M_{DR} = 0$

For section BFC:

$M_{BL} = 0$

$M_{BR} = 0 - 12 = -12$ (couple at B is counter-clockwise)

$M_{GL} = -12 + \frac{1}{2} \times 9 \times 3 = 1.5$

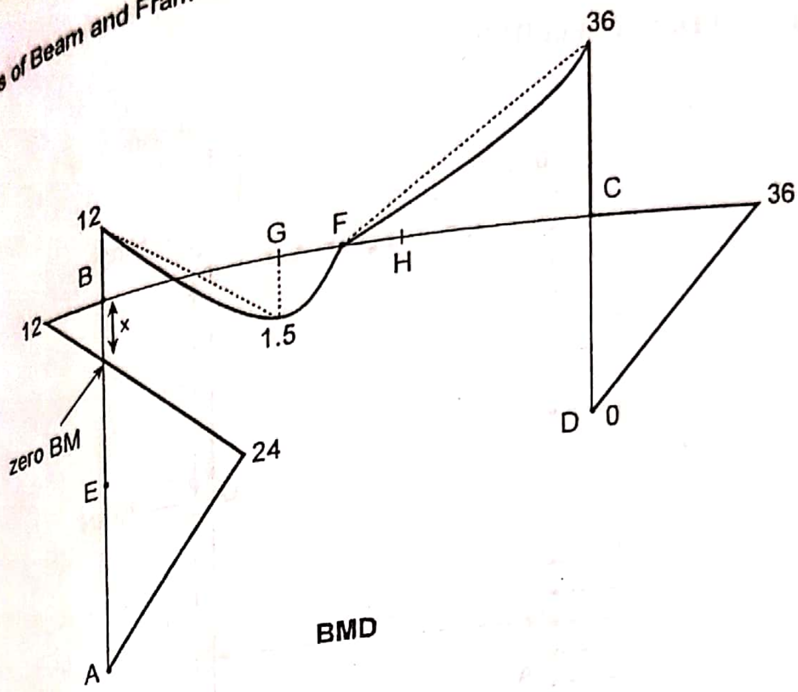
$M_{GR} = 1.5$

$M_{FL} = 1.5 - \frac{1}{2} \times 1 \times 3 = 0$

$M_{FR} = 0$

$M_{CL} = 0 - \frac{1}{2}(3 + 15) \times 4 = -36$ (M_{FR} – area of trapezoid)

$M_{CR} = -36 + 36 = 0$



BMD

$$\frac{12}{x} = \frac{24}{3-x}$$

$$3 - x = 2x$$

$$3 = 3x$$

$$x = 1$$

To plot between FHC

If it is straight line

$$M_{HL} = \frac{M_{FR} + M_{CL}}{2}$$

$$= \frac{0 - 36}{2}$$

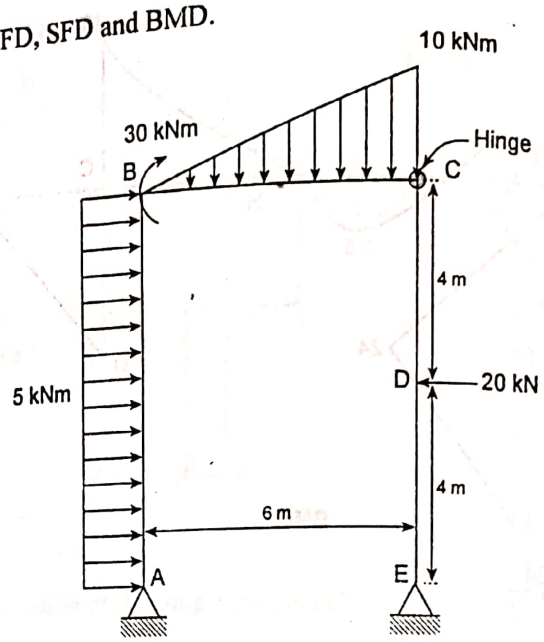
$$= -18$$

But from given condition,

$$M_{HL} = M_{FR} - \frac{1}{2} \times (3 + 9) \times 2$$

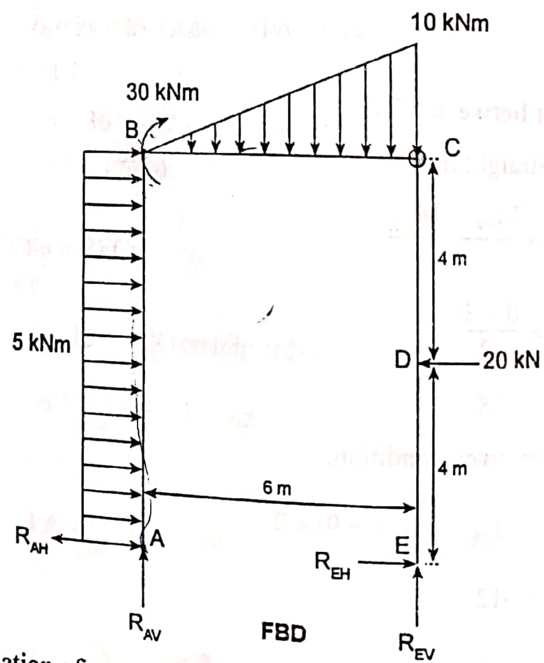
$$= -12$$

8. Draw AFD, SFD and BMD.



Solution:

Frame with UDL, UVL and Hinge



Calculation of rx^n

$\Sigma(M_C)_{right} = 0$ +ve

$$20 \times 4 - R_{EH} \times 8 = 0$$

$$80 = 8R_{EH}$$

$$\therefore\ R_{EH} = 10\ \text{kN}\ (\rightarrow)$$

$$\Sigma M_A = 0\ +\text{ve}$$

$$5 \times 8 \times 4 + 30 + \left(\frac{1}{2} \times 6\ \ 10\right)\frac{2}{3} \times 6$$

$$-20 \times 4 - R_{EV} \times 6 = 0$$

$$160 + 30 + 120 - 80 - 6R_{EV} = 0$$

$$6R_{EV} = 230$$

$$R_{EV} = 38.33\ \text{kN}\ (\uparrow)$$

$\Sigma F_x = 0\ (\rightarrow)+\text{ve}$

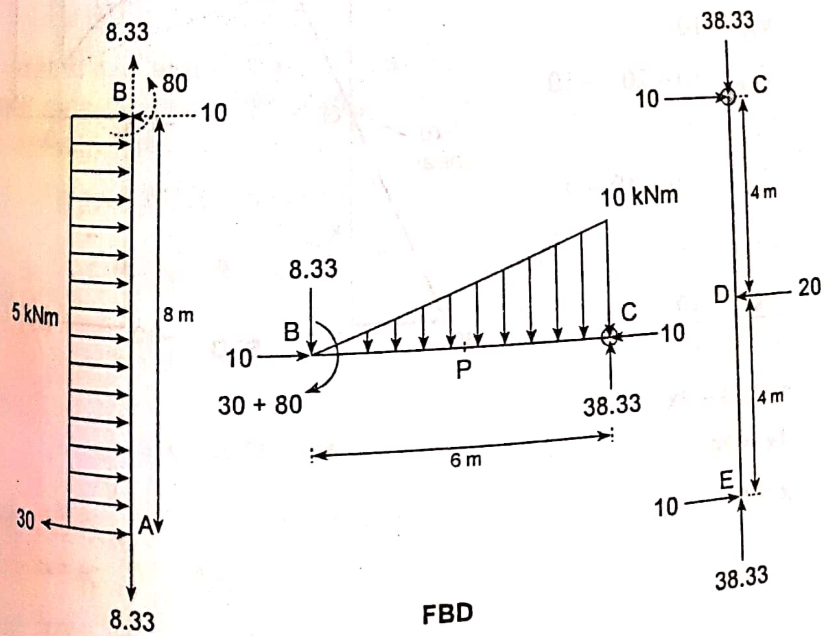
$-R_{AH} + 5 \times 8 - 20 + 10 = 0$

$R_{AH} = 30\ \text{kN}\ (\leftarrow)$

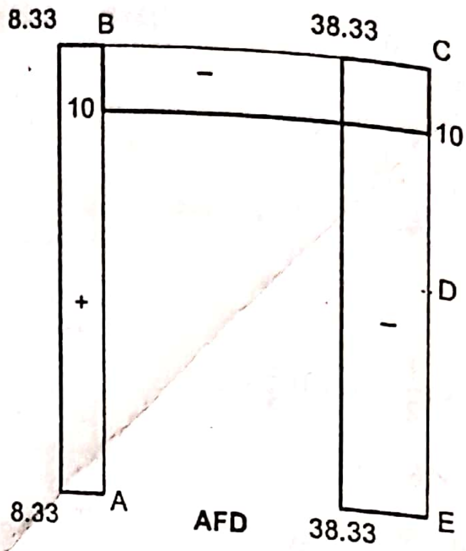
$\Sigma F_y = 0\ (\uparrow)+\text{ve}$

$R_{AV} - \left(\frac{1}{2} \times 6 \times 10\right) + R_{EV} = 0$

$R_{AV} = -8.33\ \text{kN}\ (\downarrow)$



FBD



Calculation SFD (kN):

For AB:

$V_{AL} = 0$

$V_{AR} = 30$

$V_{BL} = 30 - 40 = -10$

$V_{BR} = -10 + 10 = 0$

For BC:

$V_{BL} = 0$

$V_{BR} = -8.33$

$V_{CL} = -8.33 - 30 = -38.33$

$V_{CR} = -38.33 + 38.33 = 0$

For CDE:

$V_{CL} = 0$

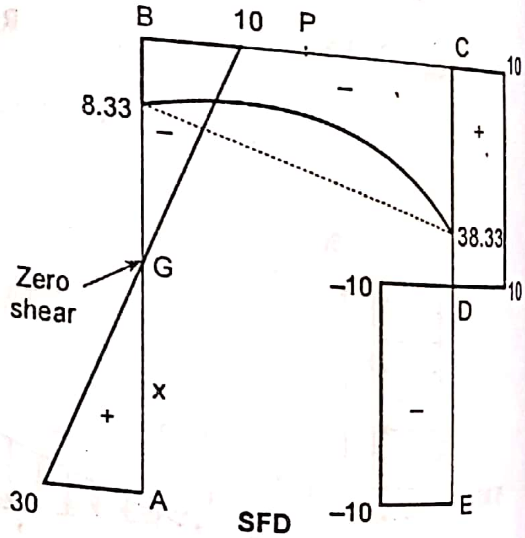
$V_{CR} = 10$

$V_{DL} = 10$

$V_{DR} = 10 - 20 = -10$

$V_{EL} = -10$

$V_{ER} = -10 + 10 = 0$

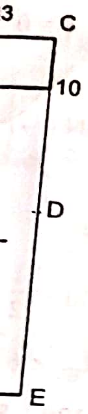


$$\frac{x}{8-x} = \frac{30}{10}$$

$x = 24 - 3x$

$4x = 24$

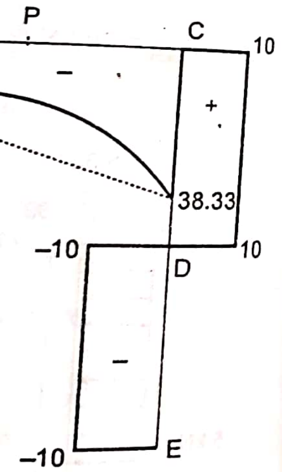
$x = 6$



33

$3 - 30 = -38.33$

$33 + 38.33 = 0$



Let P be mid of BC. If it is straight line:

$$V_{PL} = \frac{-(8.33 + 38.33)}{2} = -23.32$$

From given condition:

$$V_{PL} = V_{BR} - \frac{1}{2}\times\frac{6}{2}\times\frac{10}{2}$$

$$= -8.33 - 7.5$$

$$= -15.83 \text{ kN}$$

Calculation BM (kNm):

For AGB:

$M_{AL} = 0$

$M_{AR} = 0$

$M_{GL} = 0 + \frac{1}{2}\times 6\times 30 = 90$

$M_{GR} = 90$

$M_{BL} = 90 - \frac{1}{2}\times 2\times 10 = 80$

$M_{BR} = 80 - 80 = 0$

For CDE:

$M_{CL} = 0$

$M_{CR} = 0$

$M_{DL} = 0 + 10\times 4 = 40$

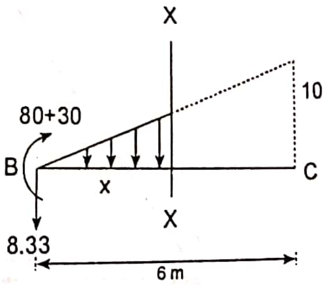
$M_{DR} = 40$

$M_{EL} = 40 - 40 = 0$

$M_{ER} = 0$

For BPE:

It is hard to calculate the exact area under curve. So better use section method for triangular load.



For span BC ($0 \le x \le 6$ m)

$$M_x = 110 - 8.33x - \frac{xy}{2}\,\frac{1}{3}x$$

$$= 110 - 8.33x - \frac{x^2}{2\,3}\cdot\frac{5x}{3}$$

$$= 110 - 8.33x - \frac{5x^3}{2\times 9}$$

$x = 0$; $\quad M_B = 110$ kNm

$x = 6$; $\quad M_C = 0$

$x = 3$; $M_{mid\,BC} = 77.51 \text{ kNm} = M_{ZL}$

If it is straight line, $M_{ZL} = \frac{0+110}{2} = 55$ kNm

Let α be mid-point of GB.

If straight line, $M_{QL} = \frac{90+80}{2} = 85$

From condition given,

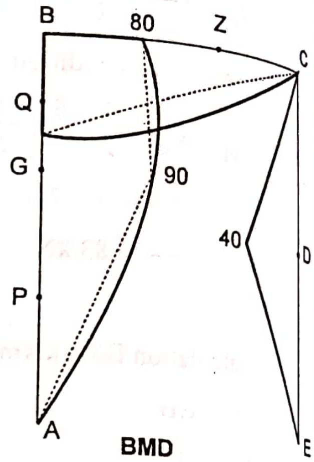
$$M_{QL} = M_{GR} - \frac{1}{2} \times 1 \times 5$$

$$= 90 - 2.5 = 87.5$$

Let P be the mid-point of AG.

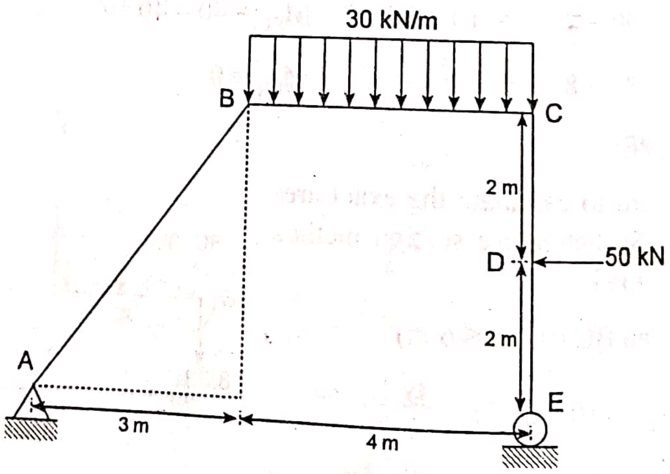
If it is straight line,

$$M_{PL} = \frac{90+0}{2} = 45$$



From condition: $M_{PL} = M_{AR} + \frac{1}{2} \times (15+30) \times 3 = 67.5$

9. Draw AFD, SFD and BMD.



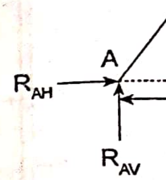
Solution:

Inclined frame:

Calculation of rx^n:

$\Sigma M_A = 0$ +ve

$$30 \times 4 \times (3 + \frac{4}{2}) - 50 \times 2 - R_{EV} \times 7 = 0$$



$R_{EV} = 71.428$

$\Sigma F_y = 0 \uparrow$+ve

$R_{AV} - 30 \times 4$

$R_{AV} = -71.42$

$\Sigma F_x = 0 \rightarrow$+v

$R_{AH} = 50$ kN

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\frac{4}{3} =$

$\cos\theta = \frac{3}{5}$;

Let two axis

H →

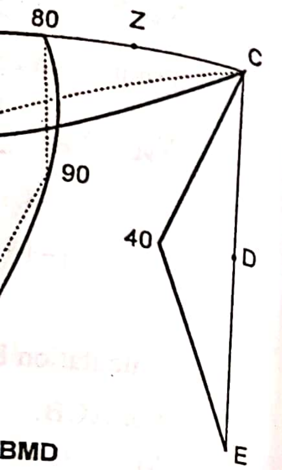
$N_1 = 48.572$

$H_1 = 48.572$

$N_2 = 50 \cos$

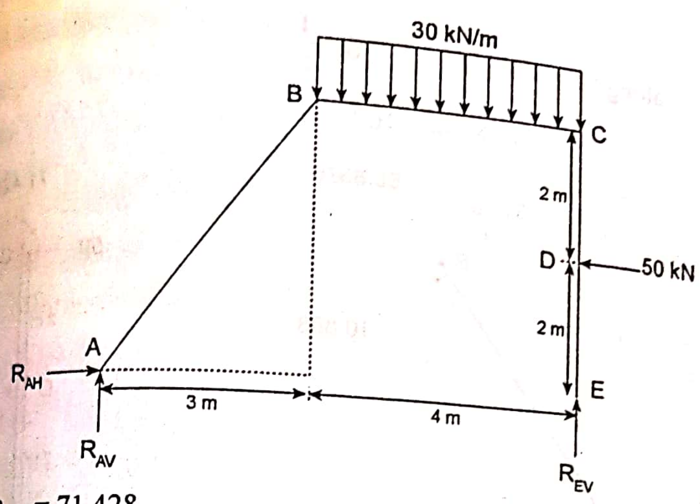
$H_2 = -50 \sin$

∴ along N



67.5

50 kN



$R_{EV} = 71.428$

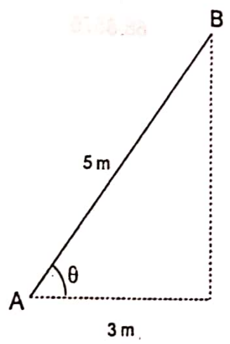
$\Sigma F_y = 0 \uparrow +ve$

$R_{AV} - 30 \times 4 + R_{EV} = 0$

$R_{AV} = -71.428 + 120 = 48.572$

$\Sigma F_x = 0 \rightarrow +ve$

$R_{AH} = 50 \text{ kN}$



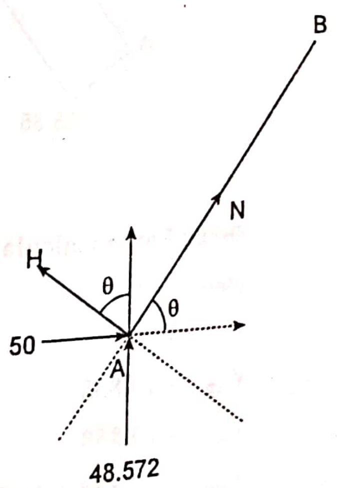
$$\tan\theta = \frac{4}{3}$$

$$\theta = \tan^{-1}\frac{4}{3} = 53.13$$

$$\cos\theta = \frac{3}{5}; \quad \sin\theta = \frac{4}{5}$$

Let two axis N → along AB

H → perpendicular to AB



$N_1 = 48.572 \sin\theta$

$H_1 = 48.572 \cos\theta$

$N_2 = 50 \cos\theta$

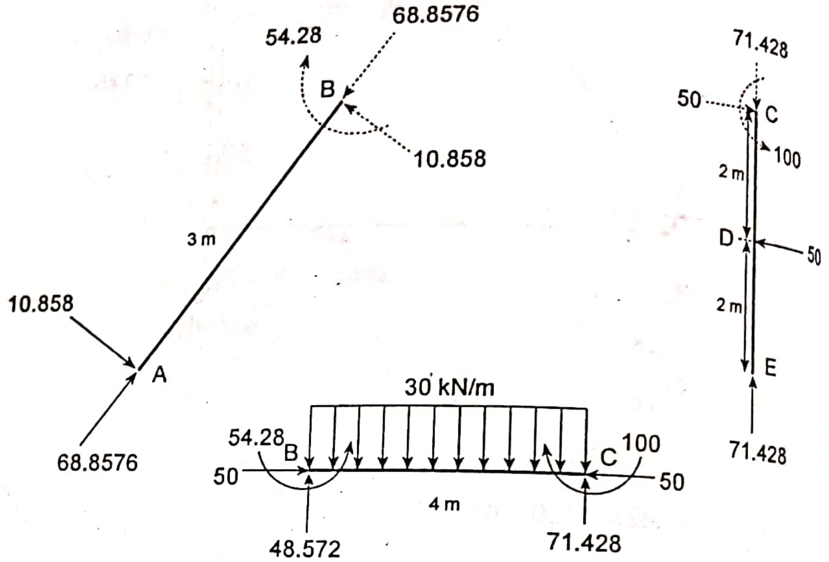
$H_2 = -50 \sin\theta$

$$\therefore \quad \text{along N} = 48.572 \times \frac{4}{5} + 50 \times \frac{3}{5}$$

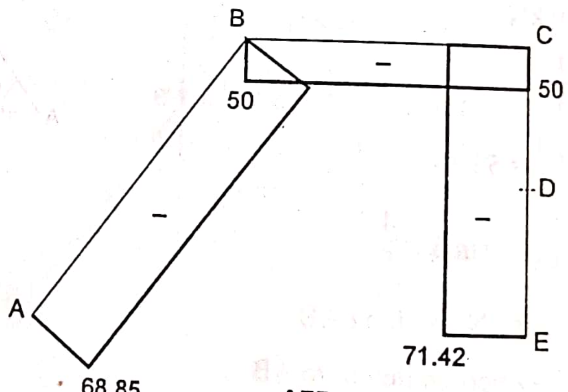
$$= 68.8576 \text{ kN}$$

$$\text{along H} = 48.572 \times \frac{3}{5} - 50 \times \frac{4}{5}$$

$$= -10.8568 = 10.8568$$



FBD



AFD

Shear Force Calculation:

Member AB

$V_{AL} = 0$

$V_{AR} = -10.858$

$V_{BL} = -10.858$

$V_{BR} = -10.858 + 10.858 = 0$

Analysis of Beam and Fra

Member BC

$V_{BL} = 0$

$V_{BR} = 48.572$

$V_{CL} = 48.572 -$

$V_{CR} = -71.428 -$

For member CE

$V_{CL} = 0$

$V_{CR} = 50$

$V_{DL} = 50$

$V_{DR} = 0$

$V_{EL} = 0$

$V_{ER} = 0$

For zero shear,

$\frac{x}{4-x} = \frac{48.57}{71.43}$

$\frac{x}{4-x} = 0.679$

$1.6792x = 2.71$

$\therefore\ x = 1.619$ m

Now you can p

For AB:

$M_{AL} = 0$

$M_{AR} = 0$

$M_{BL} = 0 - 54.2$

$M_{BR} = -54.28 -$

For BGC:

$M_{BL} = 0$

$M_{BR} = -54.28$

$M_{GL} = -54.28 -$

Member BC

$V_{BL} = 0$

$V_{BR} = 48.572$

$V_{CL} = 48.572 - 30 \times 4 = -71.428$ (+ve to –ve; check zero shear)

$V_{CR} = -71.428 + 71.428 = 0$

For member CE

$V_{CL} = 0$

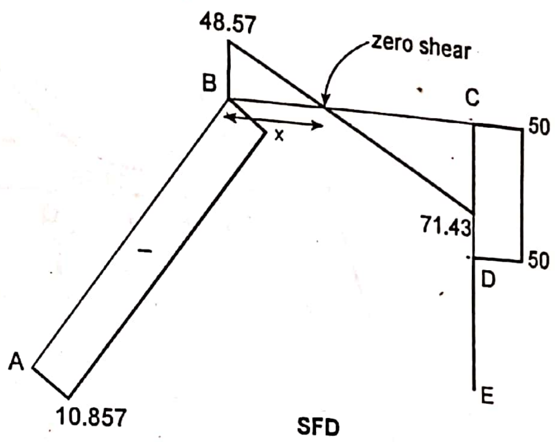
$V_{CR} = 50$

$V_{DL} = 50$

$V_{DR} = 0$

$V_{EL} = 0$

$V_{ER} = 0$



SFD

For zero shear,

$$\frac{x}{4-x} = \frac{48.57}{71.43}$$

$$\frac{x}{4-x} = 0.679$$

$1.6792x = 2.7198$

$\therefore$ x = 1.619 m [point of zero shear from B toards C]

Now you can proceed for BMD (kNm)

For AB:

$M_{AL} = 0$

$M_{AR} = 0$

$M_{BL} = 0 - 54.28 = -54.28$

$M_{BR} = -54.28 + 54.28 = 0$

For CDE:

$M_{CL} = 0$

$M_{CR} = 0 - 100 = -100$

$M_{DL} = -100 + 100 = 0$

$M_{DR} = 0$

$M_{EL} = 0$

$M_{ER} = 0$

For BGC:

$M_{BL} = 0$

$M_{BR} = -54.28$

$$M_{GL} = -54.28 + \frac{1}{2} \times 1.619 \times 48.57 = -14.96$$

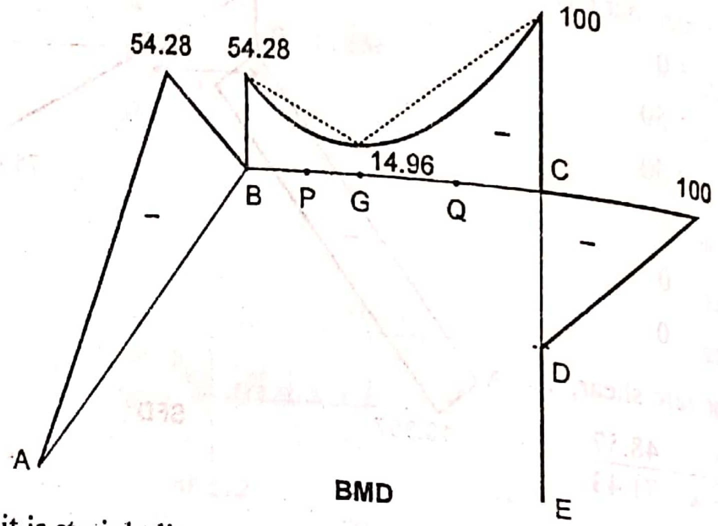
$M_{GR} = -14.96$

$M_{CL} = -14.96 - \frac{1}{2} \times (4 - 1.62) \times 71.73 = -100$

$M_{CR} = -100 + 100 = 0$

Let P and Q be mid-point of BG and GC respectively.

For BPG:



If it is straight line

$$M_{PL} = -34.62 \frac{(M_B + M_G)}{2}$$

But from condition

$$M_{PL} = M_{BR} + \frac{1}{2}(0.81) \times (48.57 + 24.285)$$

$$= -54.28 + 29.50$$

$$= -24.77 \text{ kNm}$$

For GQC:

If it is straight line

$$M_{QL} = -\left(\frac{100 + 14.96}{2}\right) = -57.48$$

But from condition

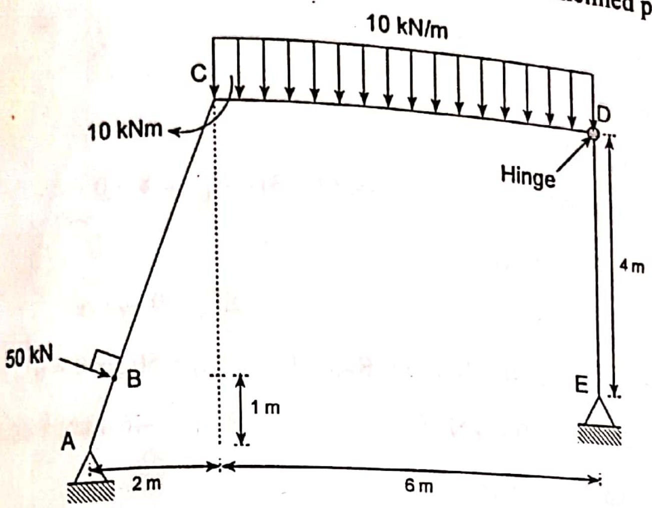
$$M_{GL} = M_{GR} - \frac{1}{2} \times \left(\frac{4 - 1.62}{2}\right)\left(\frac{71.43}{2}\right)$$

$$= -14.96 - 21.25$$

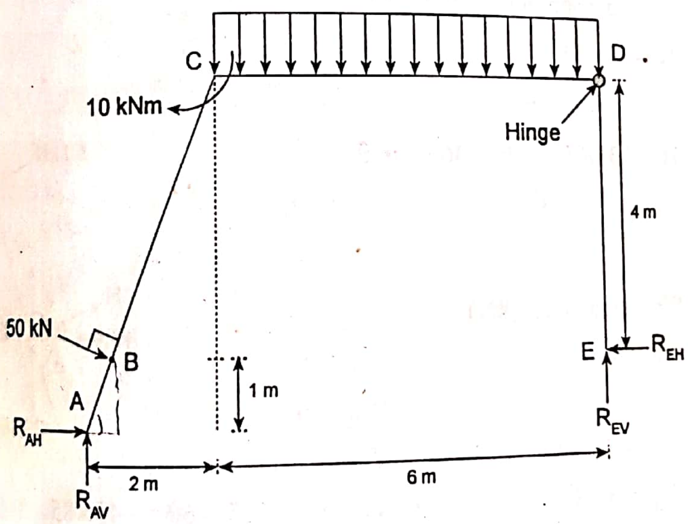
$$= -36.21 \text{ kNm}$$

Scanned by CamScanner

10. Draw AFD, SFD, BMD. (Inclined frame with load at inclined part)



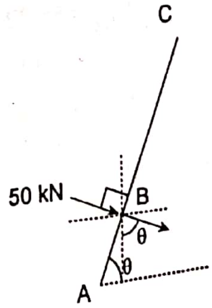
Solution:



$\tan\theta = \dfrac{5}{2}$

$\therefore\ \theta = \tan^{-1}\dfrac{5}{2} = 68.2$

$\sin\theta = 0.928$ $\qquad \cos\theta = 0.371$



$\therefore\ \sin\theta = \dfrac{1}{AB}$

$AB = \dfrac{1}{\sin\theta} = 1.077$

$\Sigma(M_D)_{right} = 0$

$R_{EH} \times 4 = 0$

$R_{EH} = 0$

$\Sigma M_A = 0$ +ve

$50 \times 1.077 + 10 + (10 \times 6) \times (2 + 3) - R_{EV} \times 8 = 0$

$R_{EV} = 45.48$ kN (↑)

$\Sigma F_y = 0$ ↑+ve

$R_{AV} - 50 \cos\theta - 10 \times 6 + R_{EV} = 0$

$\therefore R_{AV} = 33.065$ kN (↑)

$\Sigma F_x = 0$ →+ve

$R_{AH} + 50 \sin\theta = 0$

$R_{AH} = -46.4$ kN $= 46.4$ kN (←)

Consider two axes

N → along ABC

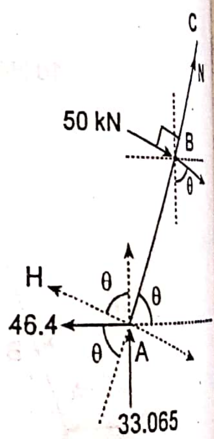
H → perpendicular to N

$N = 33.065 \sin\theta - 46.4 \cos\theta$

$= 13.469$

$H = 33.065 \cos\theta + 46.4 \sin\theta$

$= 55.32$



SF Calculation (kN)

For ABC:

$V_{AL} = 0$

$V_{AR} = 55.32$

$V_{BL} = 55.32$

$V_{BR} = 55.32 - 50 = 5.32$

$V_{CL} = 5.32$

$V_{CR} = 5.32 - 5.32 = 0$

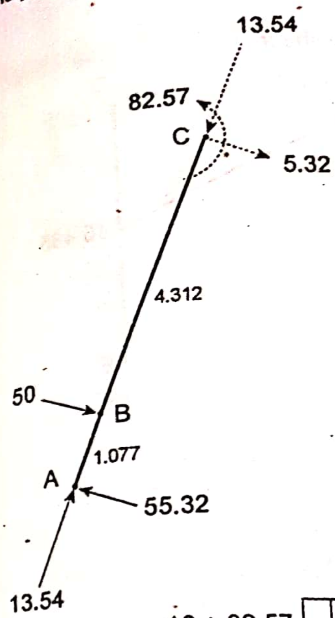
For CD

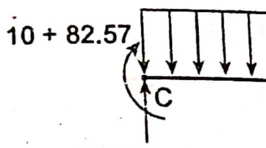
$V_{CL} = 0$

$V_{CR} = 14.515$

$V_{DL} = 14.515 - 60 = -45.485$

$V_{DR} = -45.485 + 45.485 = 0$





(33.065 – 50 cos θ
= 14.515

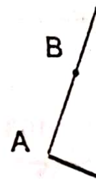
For DE

$V_{DL} = 0$

$V_{DF} = 0$

$V_{EL} = 0$

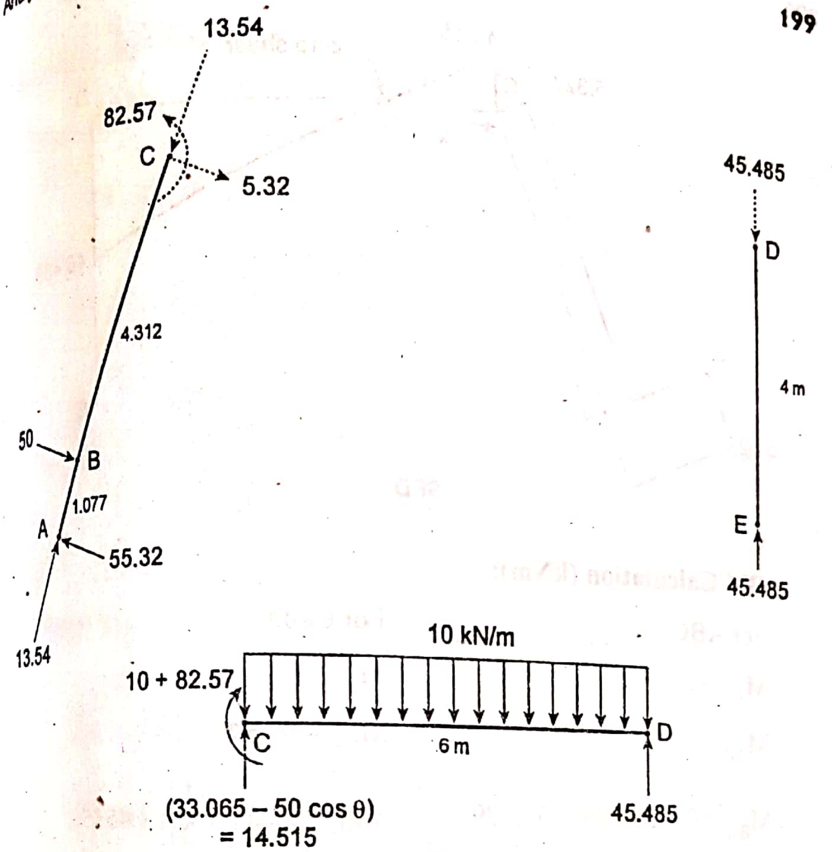
$V_{ER} = 0$



For zero shear

$$\frac{x}{14.515} = \frac{6 - x}{45.485}$$

$\therefore x = 1.4515$



FBD

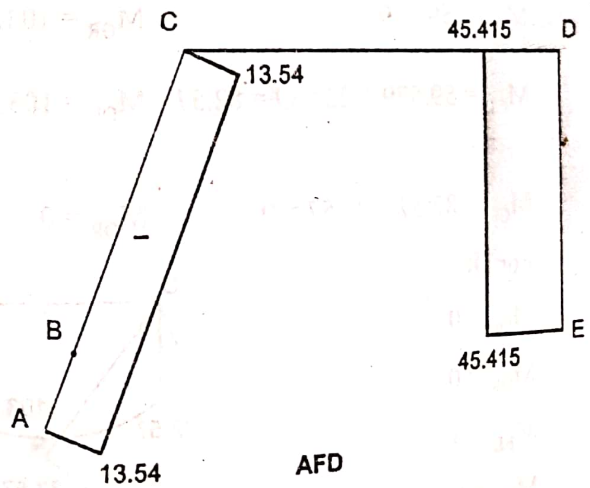
For DE

$V_{DL} = 0$

$V_{DF} = 0$

$V_{EL} = 0$

$V_{ER} = 0$

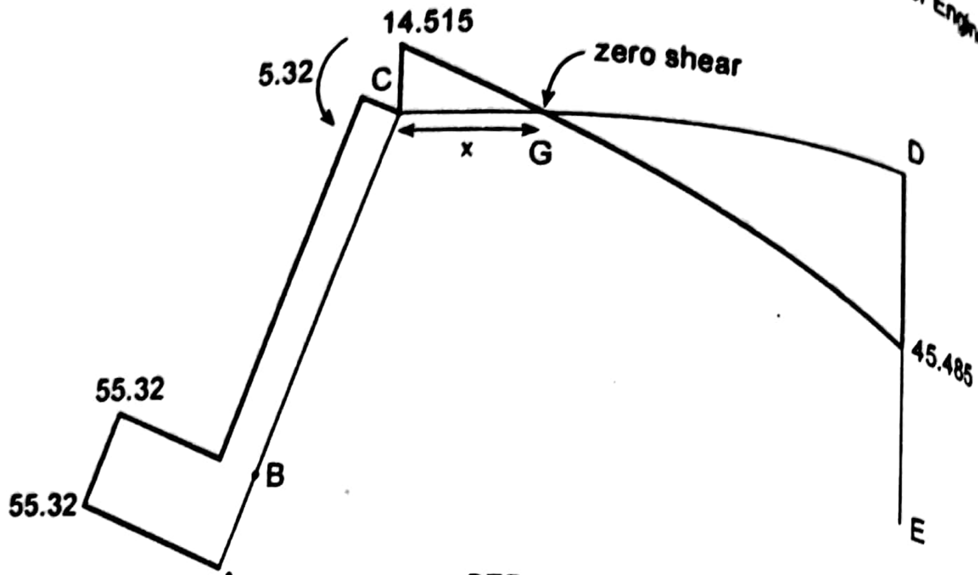


AFD

For zero shear

$$\frac{x}{14.515} = \frac{6 - x}{45.485}$$

$\therefore$ x = 1.4515



SFD

BM Calculation (kNm):

For ABC

$M_{AL} = 0$

$M_{AR} = 0$

$M_{BL} = 0 + 59.579 = 59.579$

$M_{BR} = 59.579$

$M_{CL} = 59.579 + 22.939 = 82.57$

$M_{CR} = 82.57 - 82.57 = 0$

For DE

$M_{DL} = 0$

$M_{DR} = 0$

$M_{EL} = 0$

$M_{ER} = 0$

For CGD

$M_{CL} = 0$

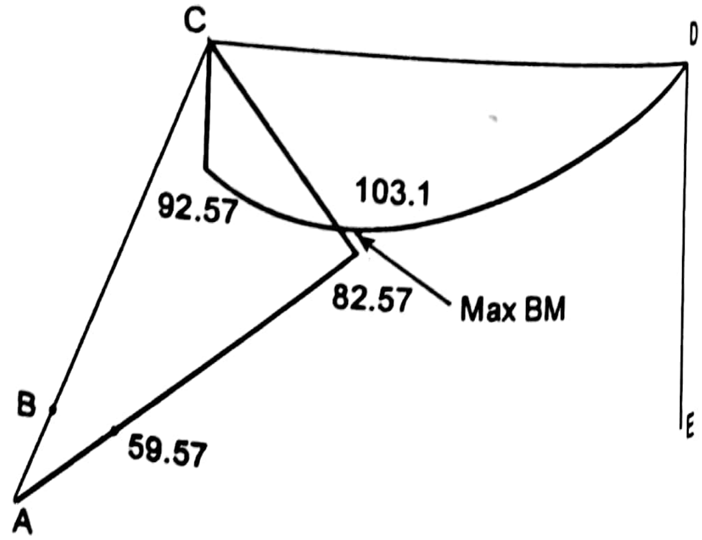
$M_{CR} = 92.57$

$M_{GL} = 92.57 + \frac{1}{2} \times 1.4515 \times 14.515 = 103.1$

$M_{GR} = 103.1$

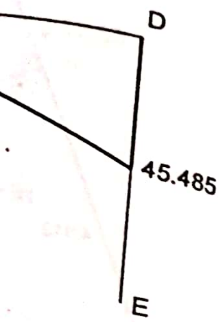
$M_{DL} = 103.1 - \frac{1}{2} \times (6 - 1.4515) \times 45.485 = 0$

$M_{DR} = 0$



BMD

d Mechanics for Engineers

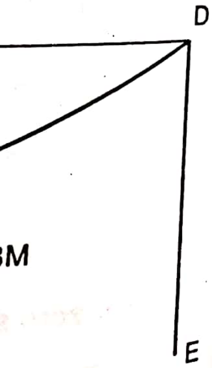


× 1.4515

14.515 = 103.1

(6 – 1.4515)

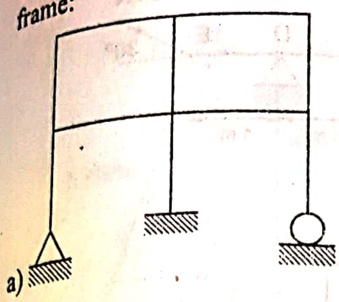
5.485 = 0

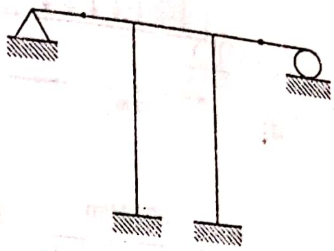


BM

Practice Question

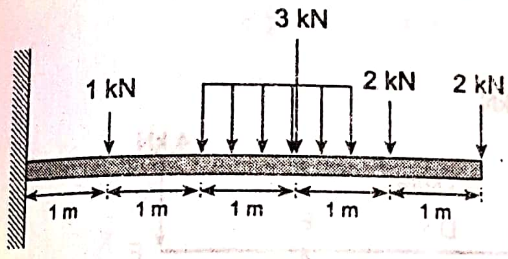
1. Calculate external, internal and total indeterminacy of following frame:

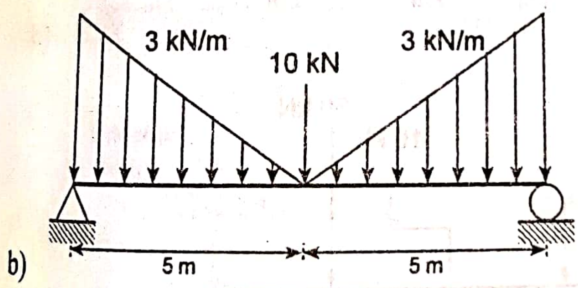


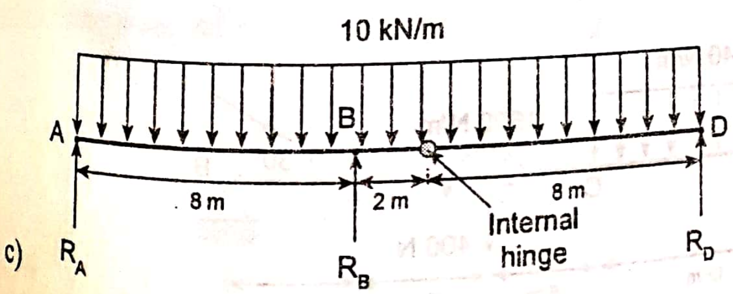


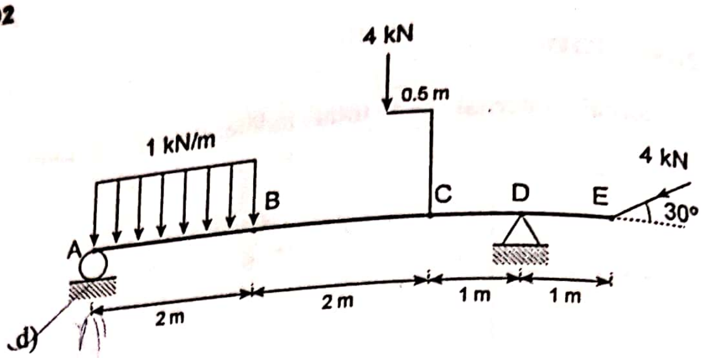
(Ans.: a) 3, 6, 9 b) 4, 0, 4)

2. Draw SFD and BMD for following beam.

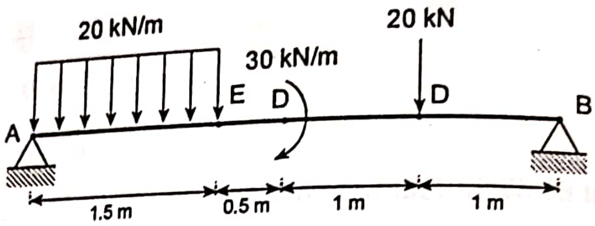




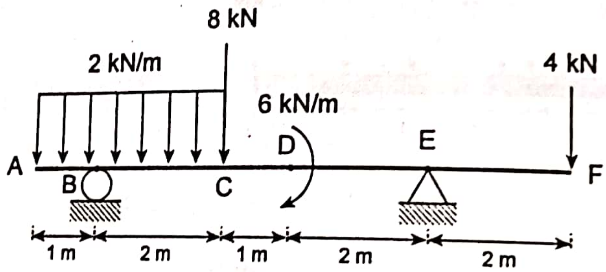




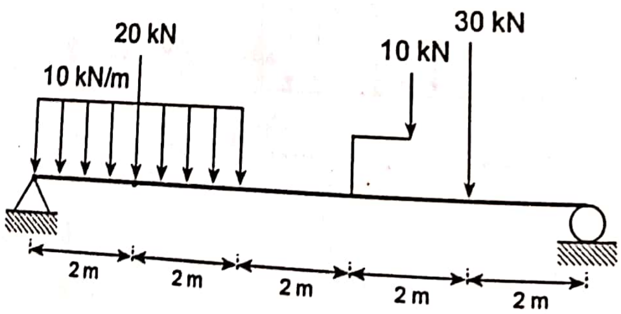
d)



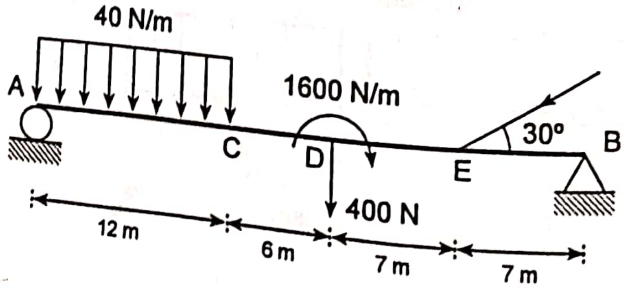
e)



f)

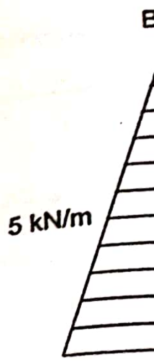


g)

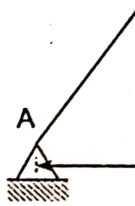


h)

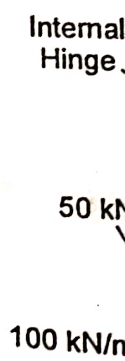
3. Draw AFD, SFD



a)

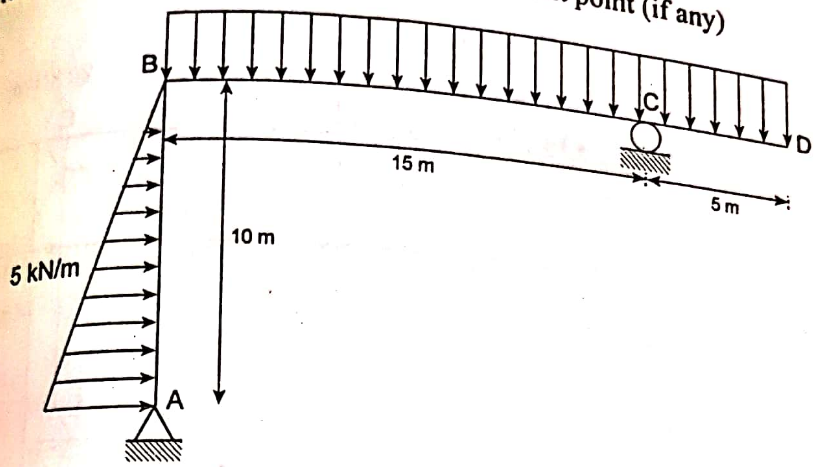


b)

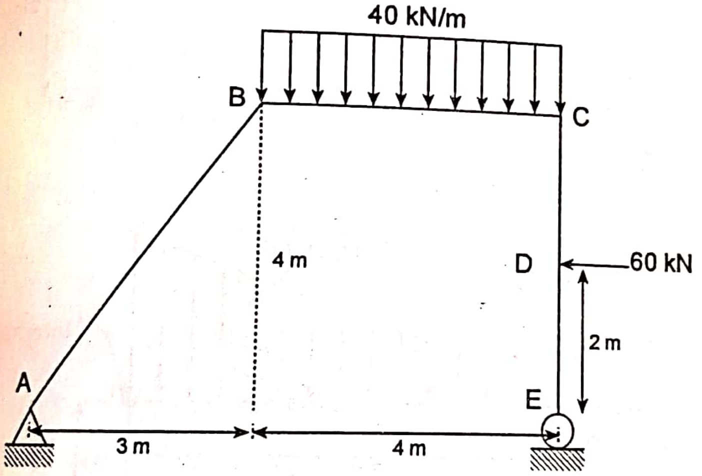


c)

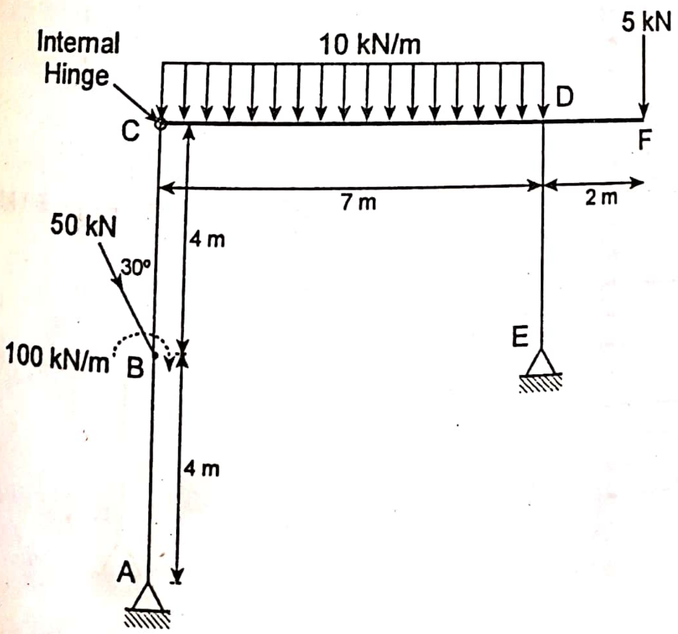
3. Draw AFD, SFD and BMD and show the salient point (if any)



a)

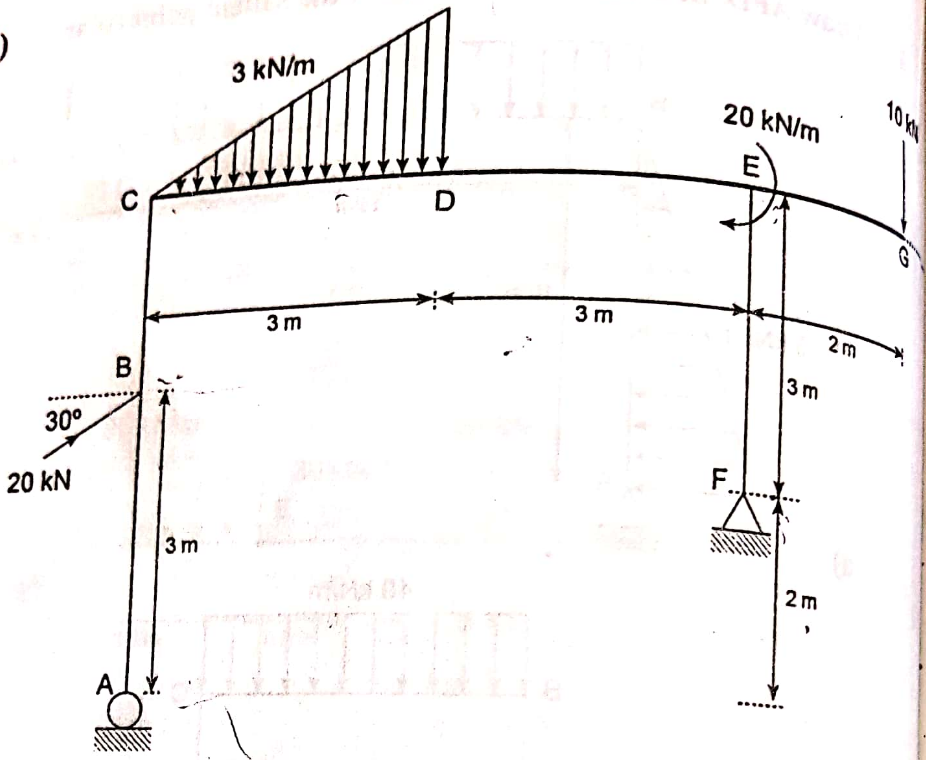


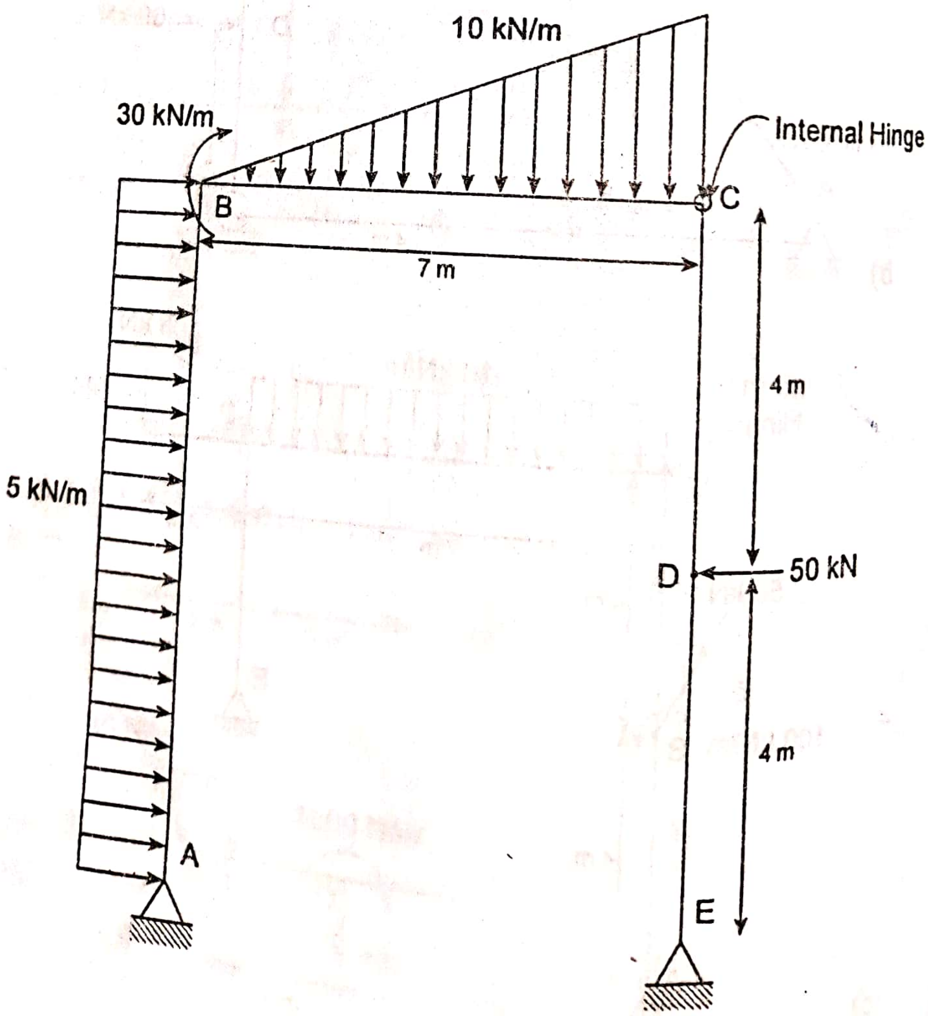
b)



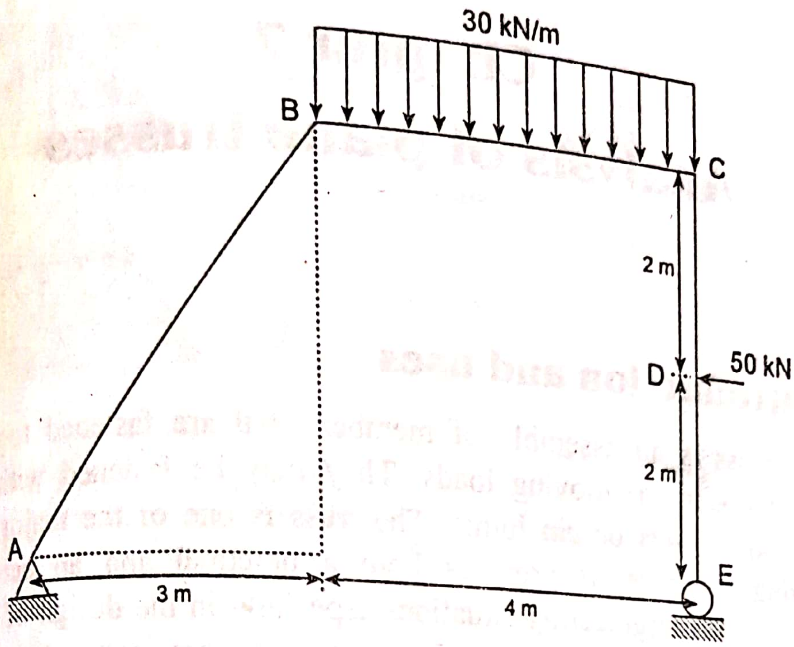
c)

d)

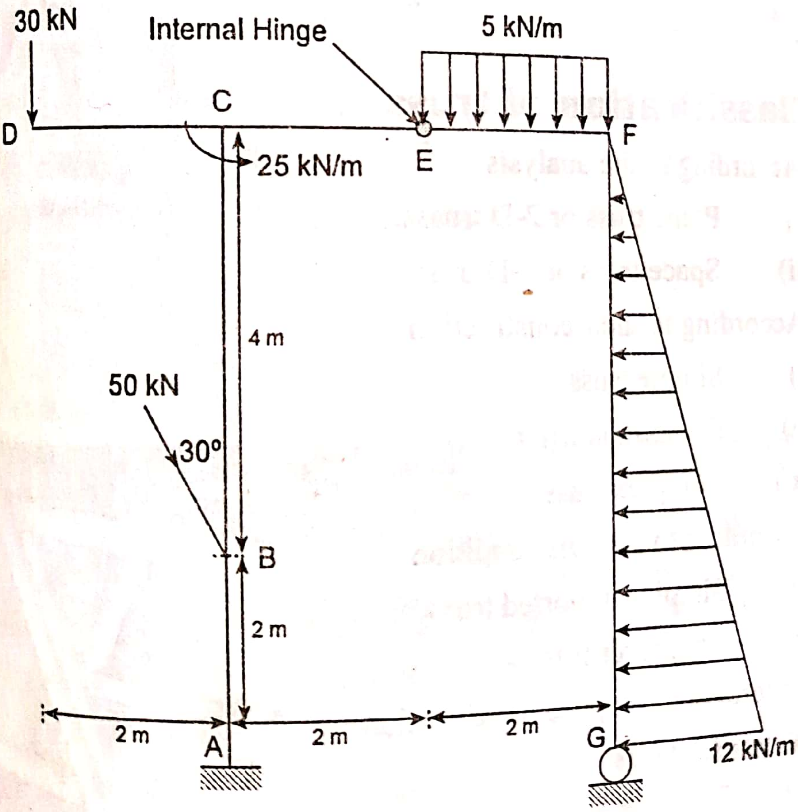

3 kN/m
20 kN/m
10 kN
C
D
E
G
3 m
3 m
2 m
B
30°
20 kN
3 m
3 m
F
2 m
A


10 kN/m
30 kN/m
Internal Hinge
B
C
7 m
4 m
5 kN/m
D
50 kN
4 m
A
E

e)


30 kN/m
B
C
2 m
D
50 kN
2 m
A
E
3 m
4 m

f)


30 kN
Internal Hinge
5 kN/m
C
D
E
F
25 kN/m
4 m
50 kN
30°
B
2 m
2 m
A
2 m
2 m
G
12 kN/m

g)

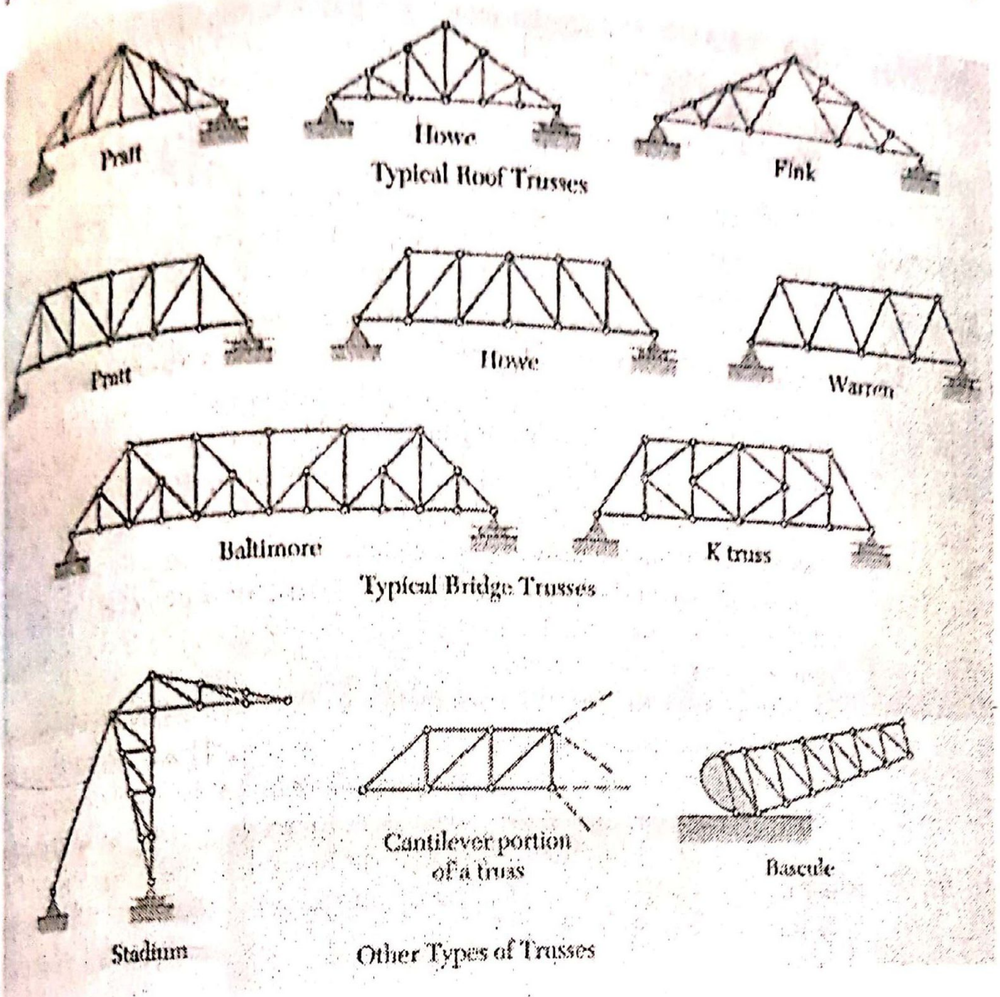
Chapter 7
Analysis of plane trusses

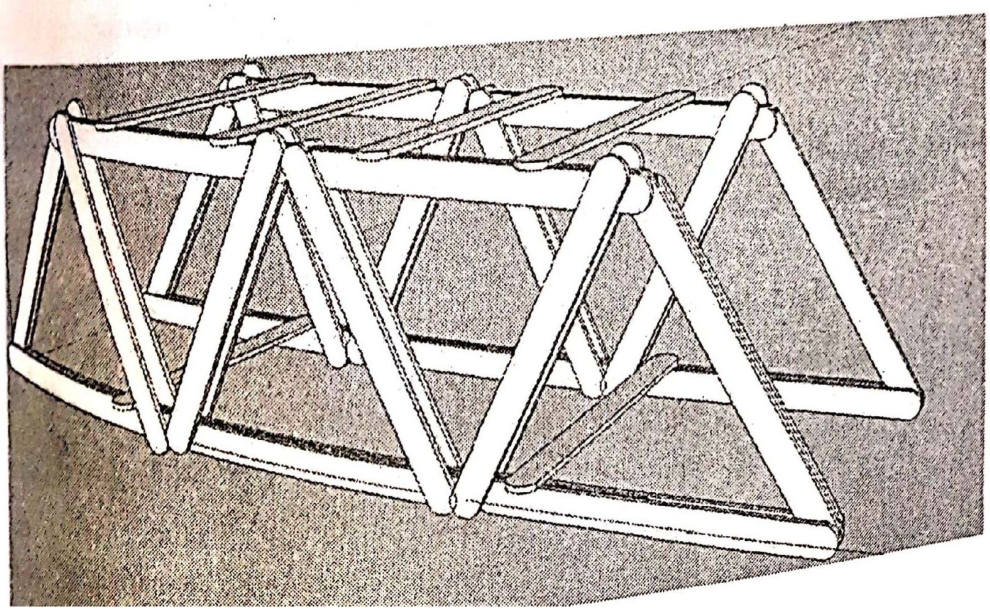
7.1 Introduction and uses

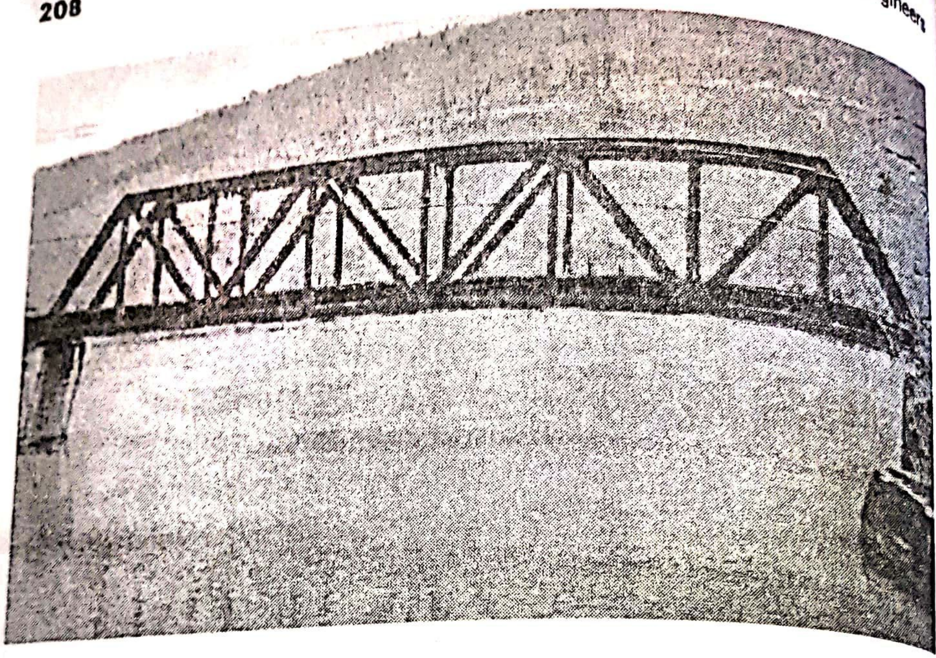
A truss is an assembly of members that are fastened together to support stationary or moving loads. They may be fastened with riveted joints, welded joints or pin joints. The truss is one of the major type of engineering structure. It provides both a practical and an economical solution to many engineering situations especially in the design of bridges, long span roof and buildings. Truss member are connected at their extremities only. In the system, the loads are transmitted through joints.

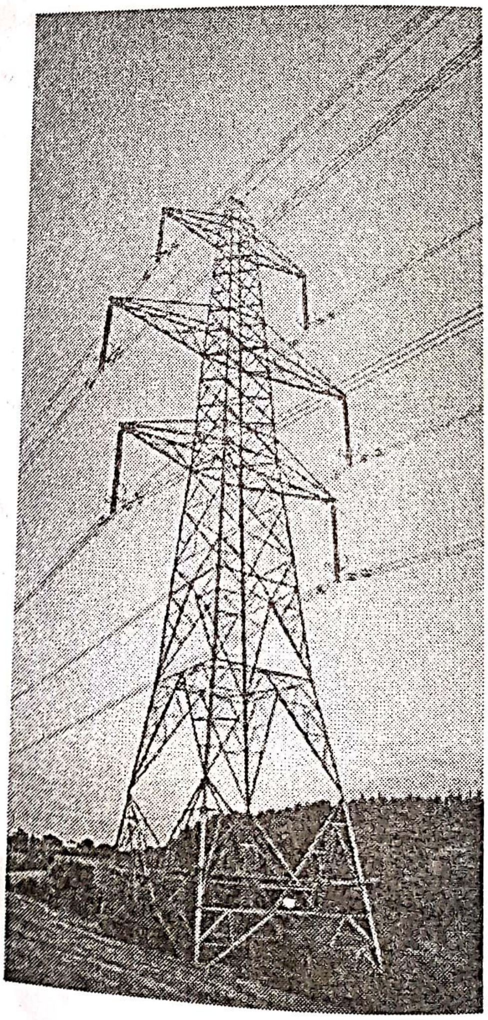
7.2 Classification of truss

a) According to the analysis
   i) Plane truss or 2-D truass
   ii) Space truss or 3-D truss
b) According to their construction
   i) Simple truss
   ii) Compound truss
   iii) Complex truss
c) According to support condition
   i) Simply supported truss
   ii) Cantilever truss
d) According to purpose of utilization
   i) Roof truss
   ii) Bridge truss
   iii) Tower truss etc.


Pratt
Howe
Fink
Typical Roof Trusses
Pratt
Howe
Warren
Baltimore
K truss
Typical Bridge Trusses
Stadium
Cantilever portion
of a truss
Bascule
Other Types of Trusses







7.3 Determinancy ... structure

The structure in wh... out by applying the eq... determinate, otherwise in... provides three equations ...

$\Sigma F_x = 0;$

If m be the numb... and j be the number of j... may arise.

a) m + r > 2j

The truss is said... solved with the help... indeterminate.

b) m + r = 2j

The truss is sa... statically determinate a...

c) m + r < 2j

The truss is sa... and is mechanism.

7.3.1 Degree of s...

A structure is ... and reactions can be ...

For plane trus...

a) If $m = 2j - r$

b) If $m > 2j - r$

c) If $m < 2j - r$

d) Total externa...

e) Total interna...

f) Total degree ...

7.3 Determinacy and indeterminancy of a structure

The structure in which the unknown reaction elements can be worked out by applying the equations of static equilibrium is called statically determinate, otherwise indeterminant structure. In case of plane truss, statics provides three equations which are useful to find the support reaction.

$$\Sigma F_x = 0; \qquad \Sigma F_y = 0; \qquad \Sigma M = 0$$

If m be the number of members, r be the number of support reactions and j be the number of joints in the truss, then the following three condition may arise.

a) $m + r > 2j$

The truss is said to be redundant or over rigid. These truss can be solved with the help of static equilibrium alone. So, it is statically indeterminate.

b) $m + r = 2j$

The truss is said to be efficient or perfect or rigid truss. Truss is statically determinate and stable.

c) $m + r < 2j$

The truss is said to be deficient or imperfect truss. Truss is unstable and is mechanism.

7.3.1 Degree of static indeterminacy

A structure is said to be statically determinate if all member forces and reactions can be evaluated using the equation of static equilibrium.

For plane truss

a) If $m = 2j - r$ Determinate

b) If $m > 2j - r$ Indeterminate

c) If $m < 2j - r$ Unstable

d) Total external indeterminacy $= r - 3$

e) Total internal indeterminacy $= m - (2j - 3)$

f) Total degree of static indeterminacy $= (r - 3) + m - (2j - 3)$

$$= m + r - 2j$$

7.3.2 Degree of kinematic indeterminacy (DKI)

It is related with the joint displacement. It is the total number of freedoms that are allowed for the joints in a structure.

- Roller support can resist only vertical translation but not the horizontal translation and rotation. So, degree of freedom is 2.
- For hinge support degree of freedom is 1.
- For fixed support degree of freedom is 0.

For truss

DKI = 2j – r

$$= 2 \times \text{no. of joints} - \text{no. of support reaction}$$

7.4 Idealization of a truss

Following assumptions are made while finding out the forces in the members of truss:

a) The members are connected by smooth frictionless pins.

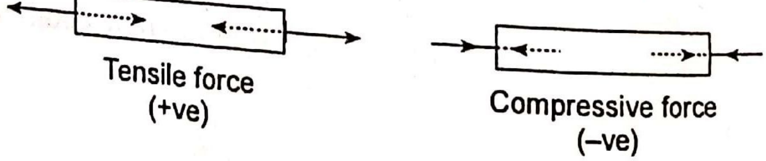
b) Self weight of member is neglected.

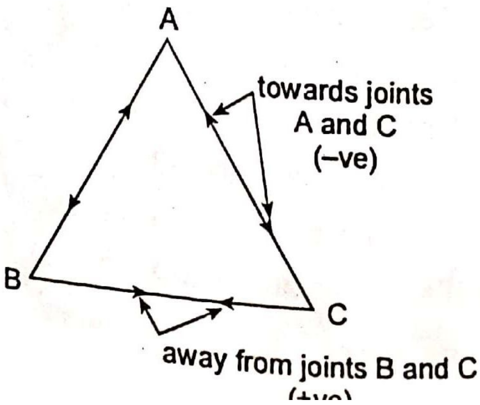
c) Loads are applied only at the joint.

d) Truss is statically determinate.

7.5 Nature of force

A tensile force in the member is indicated by arrows pointing away from the joints while a compressive force is towards the joints as shown.





During analysis, start by assuming either tensile (T) or compressive (C). After solving for a forces at a joint, a positive forces indicate correct assumption of the direction of forces. A negative forces means that the actual direction of force is opposite to the assumed.

7.6 Analysis of truss

Two methods are available:

a) Method of joints

b) Method of sections

7.6.1 Method of joints

The solution proceeds from one joint to the other. At every joint the forces are concurrent as the centre line of each member at a joint meet at a point. Equation of equilibrium at a joint is

$$\Sigma F_x = 0; \qquad \Sigma F_y = 0$$

Thus only two unknown forces can be determined. So we need to start with a joint with two unknown force at most.

7.6.2 Method of section

This method is more economical than the method of joints, if forces in only few members of the truss are desired. Divide the truss into two imaginary section by passing a cut (straight or crooked) through the truss such that the member in which force is to be determined is cut. To solve problem more than one cut may be required. Take one cut section of truss and use

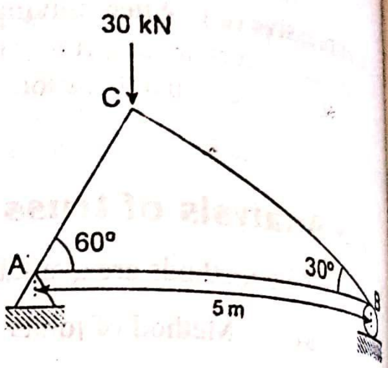
$$\Sigma M = 0; \qquad \Sigma F_y = 0; \qquad \Sigma F_x = 0$$

to solve the unknown forces.

The section has to be such that it does not cut more than three members in which the forces are to be determined.

Worked Out Examples

1. Find the forces in all the members of truss by joint methods.



Solution:

Draw FBD of given truss and use simple trigonometry to find length and angles.

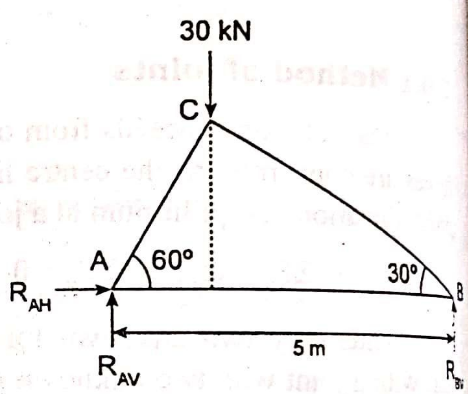
$AC = AB \cos 60 = 2.5$ m

Let us assume:

$R_{AV} \rightarrow Rx^n$ at A, (vertical)

$R_{AH} \rightarrow Rx^n$ at A, (horizontal)

$R_{BV} \rightarrow Rx^n$ at B, (vertical)



Calculation of Rx^n

Taking $\Sigma M_A = 0$ +ve

or, $30 \times AD - R_{BV} \times 5 = 0$

or, $30 \times 2.5 \cos 60 = 5 R_{BV}$

$\therefore\ 5 R_{BV} = 30 \times 1.25$

$\therefore\ R_{BV} = 7.5 \text{ kN} (\uparrow)$

$\Sigma F_y = 0 \uparrow +ve$

or, $R_{AV} - 30 + R_{BV} = 0$

or, $R_{AV} - 30 + 7.5 = 0$

or, $R_{AV} = 22.5 \text{ kN} (\uparrow)$

$\Sigma F_x = 0 \rightarrow +ve$

$R_{AH} = 0$

Joint A:

Unknown force is assumed to be tensile.

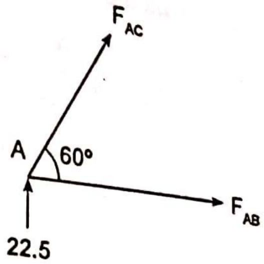
$\Sigma F_y = 0 \uparrow +ve$

$22.5 + F_{AC} \sin 60 = 0$

$$\therefore F_{AC} = -\frac{22.5}{\sin 60}$$

$$= -25.97 \text{ kN}$$

$$= 25.97 \text{ kN (C)}$$



–ve sign for AC indicate that the actual direction of force in the member AC is opposite to assumed direction.

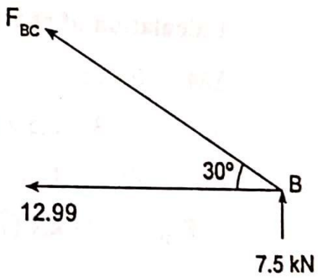
$\Sigma F_x = 0 \rightarrow +\text{ve}$

$F_{AB} + F_{AC} \cos 60° = 0$

$F_{AB} - 25.97 \cos 60 = 0$

$\therefore F_{AB} = 12.99$ kN (T)

Joint B:



$\Sigma F_y = 0 \uparrow +\text{ve}$

$7.5 + F_{BC} \sin 30 = 0$

$$7.5 + F_{BC}\frac{1}{2} = 0$$

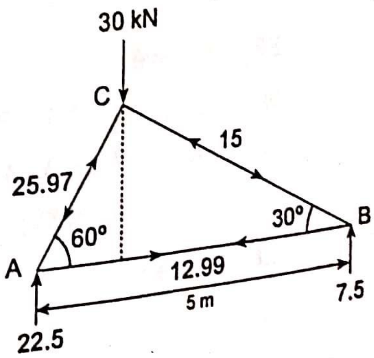
$\therefore F_{BC} = -15$ kN

$F_{BC} = 15$ kN (C)

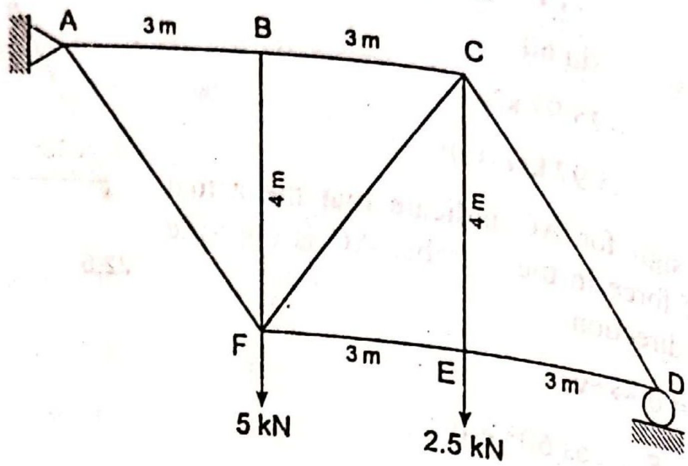
Member	Magnitude (kN)	Nature
AB	12.99	Tensile (T)
BC	15	Compressive (C)
CA	25.97	Compressive (C)

You can solve this truss without calculating the reaction value. It is because joint C has only two unknown. You can directly start from here.

(Try yourself)



2. **Find the forces in members of truss shown.**



Solution:

Draw FBD:

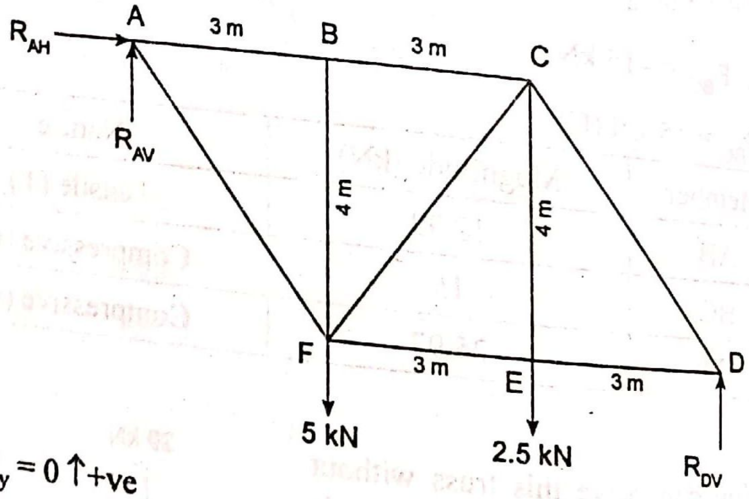
Calculation of reaction:

$\Sigma M_A = 0$ +ve

or, $5 \times 3 + 2.5 \times 6 - 9\,R_{DV} = 0$

or, $30 = 9\,R_{DV}$

$\therefore\ R_{DV} = 3.33$ kN (↑)



$\Sigma F_y = 0$ ↑+ve

or, $R_{AV} - 5 - 2.5 + R_{DV} = 0$

or, $R_{AV} - 7.5 + 3.33 = 0$

$\therefore\ R_{AV} = 4.167$ kN (↑)

$\Sigma F_x = -0$ →+ve

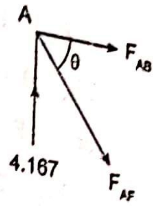
$R_{AH} = 0$

Joint A:

From trigonometry

$\tan\theta = \frac{4}{3}$

$\sin\theta = \frac{4}{5} = 0.8 \qquad \cos\theta = \frac{3}{5} = 0.6$



$\Sigma F_y = 0 \uparrow +ve;$

or, $\quad 4.167 - F_{AF}\sin\theta = 0$

$\therefore F_{AF} = 5.21$ kN (T)

$\Sigma F_x = 0 \rightarrow +ve$

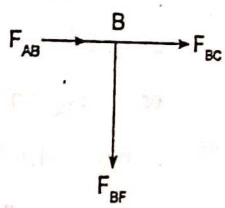
or, $\quad F_{AB} + F_{AF}\cos\theta = 0$

or, $\quad F_{AB} + 5.21 \times 0.6 = 0$

or, $\quad F_{AB} + 3.126 = 0$

$\therefore F_{AB} = -3.126$ kN $= 3.126$ kN (C)

Joint B:



$\Sigma F_y = 0 \uparrow +ve$

$F_{BF} = 0$ (zero force member)

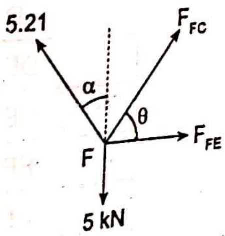
$\Sigma F_x = 0 \rightarrow +ve$

or, $\quad 3.126 + F_{BC} = 0$

$\therefore F_{BC} = -3.126$ kN $= 3.126$ kN (C)

Joint F

From trigonometry,



$\tan\alpha = \frac{3}{4}$

$\therefore \alpha = 36.87$

$\tan\theta = \frac{4}{3}$

$\therefore \theta = 53.13$

$\Sigma F_y = 0 \uparrow +ve$

or, $5.21 \cos 36.87 - 5 + F_{FC} \sin 53.13 = 0$

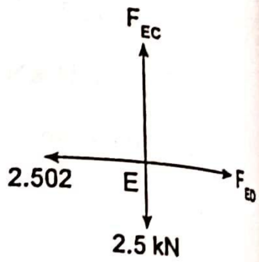
$\therefore F_{FC} = 1.04$ kN (T)

$\Sigma F_x = 0 \rightarrow +ve$

or, $F_{FE} + 1/04 \cos 53.13 - 5.21 \sin 36.87 = 0$

$\therefore F_{FE} = 2.502$ kN (T)

Joint E:



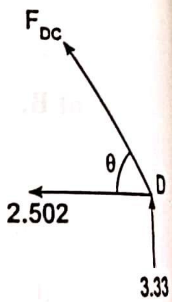
$\Sigma F_y = 0 \uparrow +ve$

$F_{EC} = 2.5$ kN (T)

$\Sigma F_x = 0 \rightarrow +ve$

$F_{ED} = 2.502$ kN (T)

Joint D:



$\tan\theta = \frac{4}{3}$

$\therefore \theta = 53.13$

$\Sigma F_y = 0 \uparrow +ve$

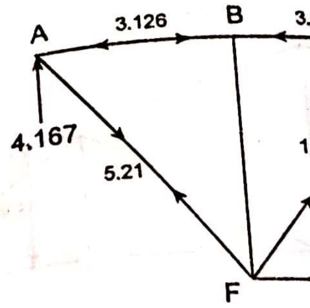
or, $F_{DC} \sin 53.13 + 3.33 = 0$

$\therefore F_{DC} = -4.167$ kN

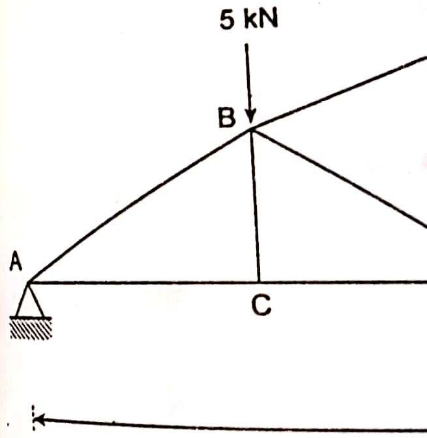
$= 4.167$ kN (C)

Member	Magnitude (kN)	Nature
AB	3.126	C
BC	3.126	C
CD	4.167	C
DE	2.5	T
CE	2.5	T
EF	2.502	T
CF	1.04	T
BF	0	–
AF	5.21	T

Graphical presentation



3. Determine the total degree ... given truss. Also determine ... given truss.



Solution:

Total external indeterm...

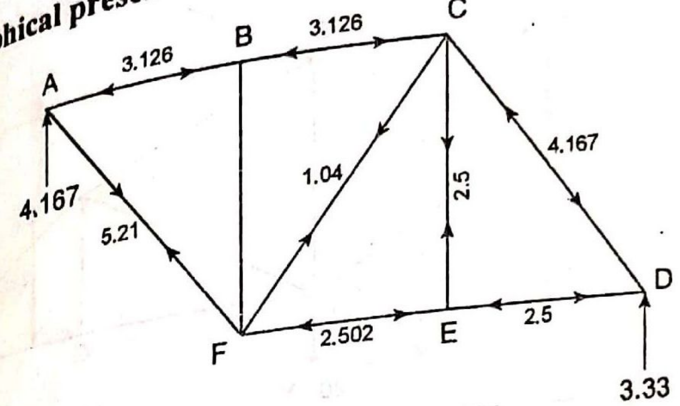
$= 3 - 3 = 0$

Total internal indeterm...

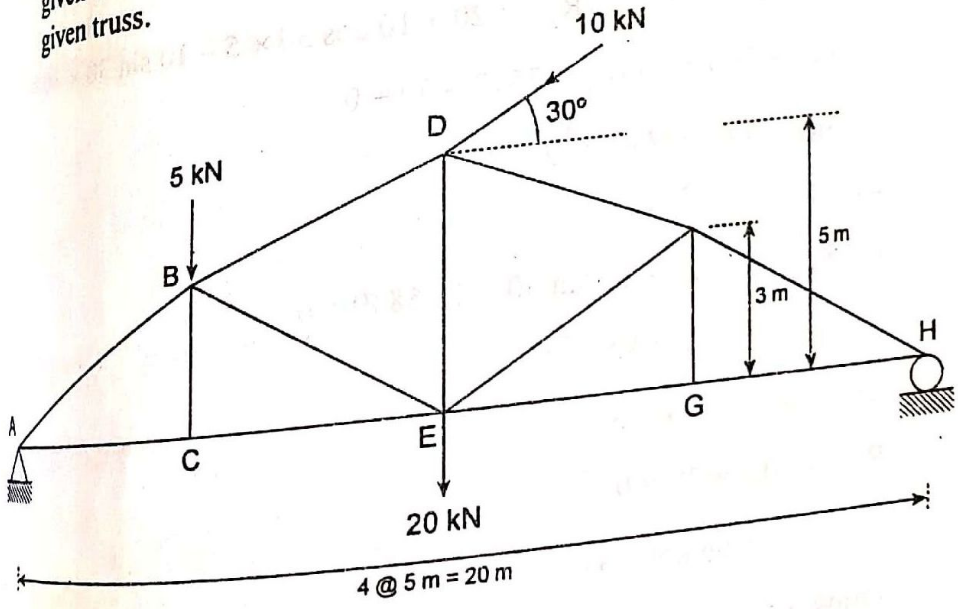
$= 13 - (2 \times 8 - ...$

$= 13 - 13 = 0$

Graphical presentation



3. Determine the total degree of internal, external indeterminacy of the given truss. Also determine the member force in CE, BE and BD for given truss.



Solution:

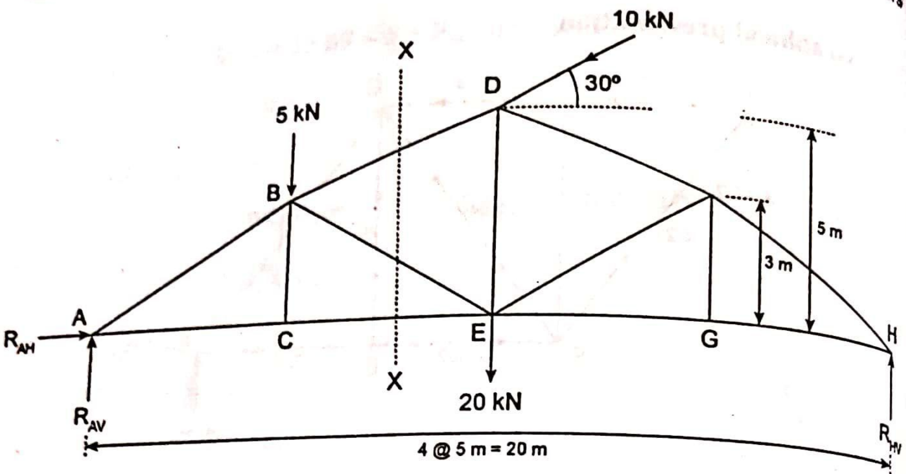
Total external indeterminacy $= r - 3$

$= 3 - 3 = 0$

Total internal indeterminacy $= m - (2j - 3)$

$= 13 - (2 \times 8 - 3)$

$= 13 - 13 = 0$



Calculation of rx^n

$\Sigma M_A = 0$ +ve ↶

or, $-5 \times 5 - 20 \times 10 + R_{HV} \times 20 + 10 \cos 30 \times 5 - 10 \sin 30 \times 10 = 0$

or, $-25 - 200 + 20R_{HV} + 25\sqrt{3} - 50 = 0$

$\therefore\ R_{HV} = 11.5849$ kN (↑)

$\Sigma F_y = 0$ ↑+ve

or, $R_{AV} - 5 - 20 - 10 \sin 30 + 11.5849 = 0$

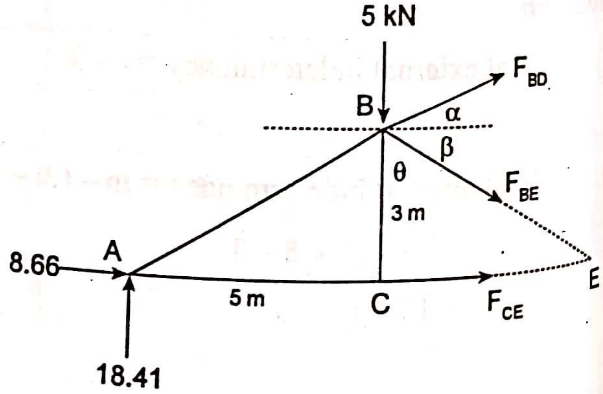
$\therefore\ R_{AV} = 18.4151$ kN (↑)

$\Sigma F_x = 0$ →+ve

$R_{AH} - 10 \cos 30 = 0$

$\therefore\ R_{AH} = 8.66$ kN (→)

Draw a section X–X as shown. Take the left part of section.



By solving,

$\alpha = 21.8°$

$\beta = 30.9°$

$\theta = 59.037°$

$\Sigma M_B = 0$ +ve ↶

or, $8.66 \times 3 - 18.41 \times + F_{CE} \times 3 = 0$

$\therefore\ F_{CE} = 22.03$ kN

$\Sigma M_E = 0$ +ve ↶

or, $-18.41 \times 10 + 5 \times 5$

or, $-F_{BD} \times 4.6423 = 1$

$\therefore\ F_{BD} = -34.282$ kN

$= 34.282$ kN (C

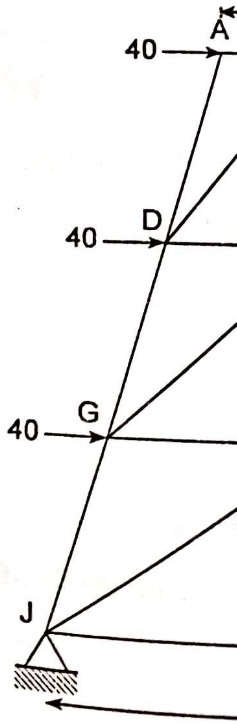
$\Sigma F_y = 0$ ↑+ve

or, $18.415 - 5 - F_{BE}$ si

or, $13.4151 - 0.51448$

$F_{BE} = 1.32816$ kN (T)

4. Determine the force in t



Solution:

Draw a section a–a

Use $\tan\theta = \dfrac{2.7}{0.8}$

$\therefore\ \theta = 73.5°$

$\therefore\ F_{CE} = 22.03$ kN

$\Sigma M_E = 0$ +ve

or, $-18.41 \times 10 + 5 \times 5 - F_{BD} \cos 21.8 \times 3 - F_{BD} \sin 21.8 \times 5 = 0$

or, $-F_{BD} \times 4.6423 = 159.15$

$\therefore\ F_{BD} = -34.282$ kN

$= 34.282$ kN (C)

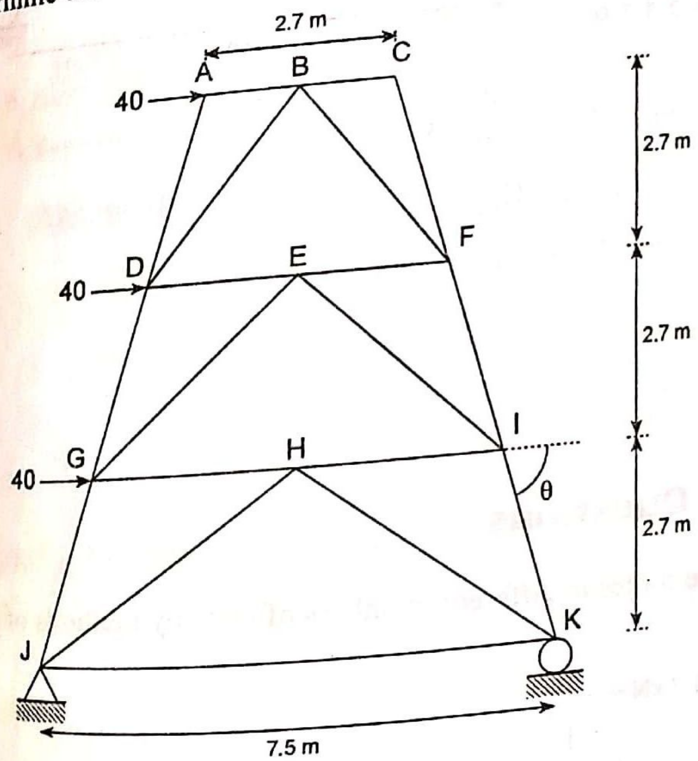
$\Sigma F_y = 0 \uparrow$+ve

or, $18.415 - 5 - F_{BE} \sin 30.96 - 34.28 \sin 21.81 = 0$

or, $13.4151 - 0.51448\ F_{BE} - 12.7317 = 0$

$F_{BE} = 1.32816$ kN (T)

4. Determine the force in the member IK for the truss shown.



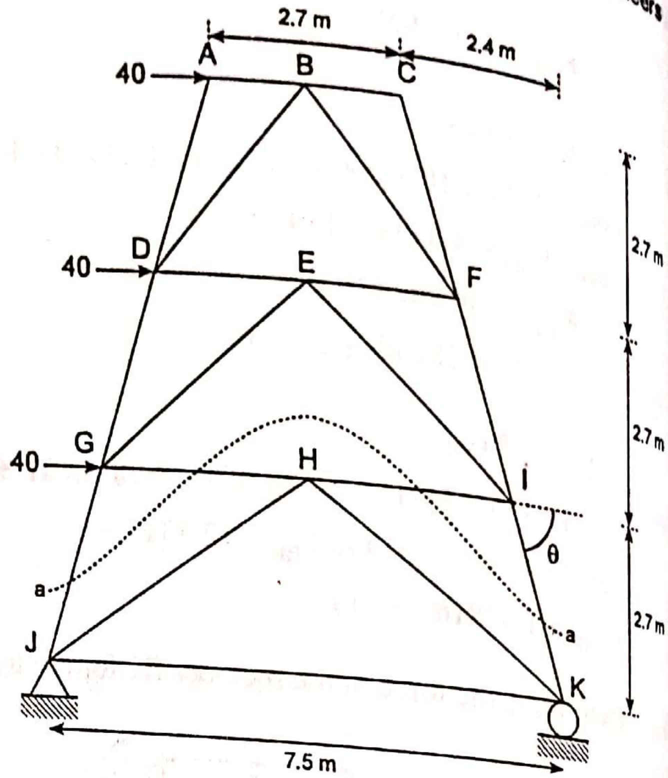
Solution:

Draw a section a–a

Use $\tan\theta = \dfrac{2.7}{0.8}$

$\therefore\ \theta = 73.5°$

Here we have select the section that cuts four unknown force (>3) because by visualization it is found that 3 unknown force passes through G.



GI = 7.5 – 2 × 0.8 = 5.9 m make in same line.

$\Sigma M_G = 0$ +ve

or, $F_{IK} \sin\theta \times 5.9$

$+ 40 \times 2.7$

$+ 40 \times 5.4 = 0$

$\therefore\ F_{IK} \sin 73.5 = (-)54.915$

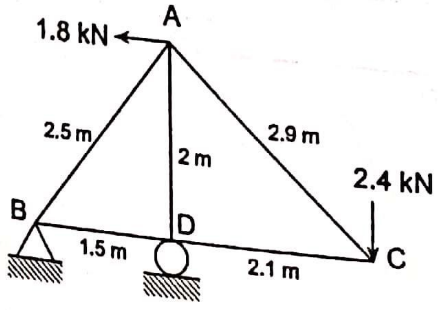
$\therefore\ F_{IK} = 57.27$ kN (C)

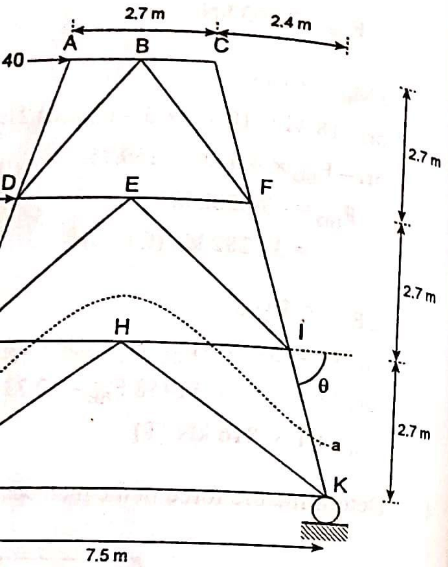
$F_{GJ} = 57.27$ kN (T)

Practice Questions

1. Find the forces in different members of truss by methods of joints.

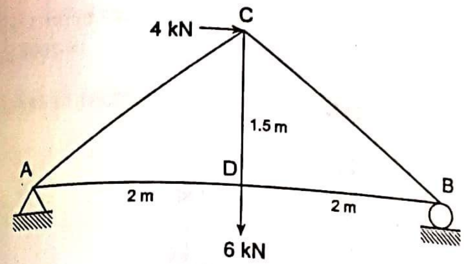
a)



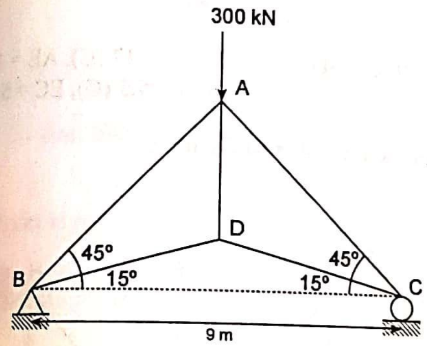


...ers of truss by methods of joints.

N



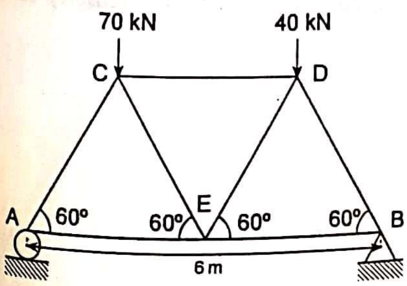
b)



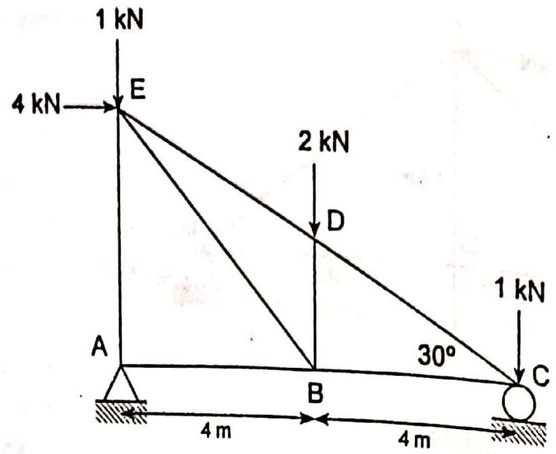
c)

(Ans. a) AB = 1.2 kN (T), BC = 2.52 kN (C), AD = 3.48 kN (T),
AC = 3.36 kN (C), CD = 2.52 kN (C)
b) AC = 2.5 (C), BC = 7.5 (C), AD = 6 (T),
DB = 6 (T), CD = 6 (T);
c) AB = 280 (C), AD = 100 (T), AC = 280 (C),
CD = 205 (T), DB = 205 (T))

2. Determine the member force using section methods.



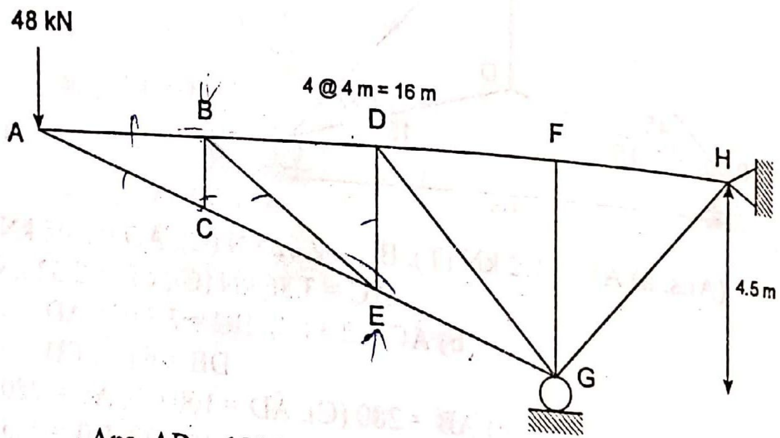
a)



b)

(Ans. a) AC = 29 (C), BD = 40 (C), CD = 17 (C), AE = 15 (T),
EB = 20 (T), ED = 5.5 (C), EC = 5.5 (T))

3. Determine the forces in each member of the truss.

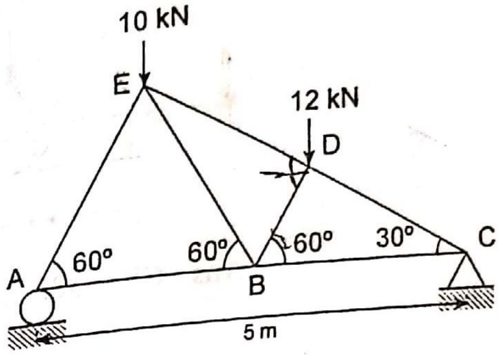


Ans. AB = 128 (T), AC = 136.7 (C), BD = DF = FH = 128 (T),
CE = EG = 136.7 (C), GH = 192.7 (C)

[Hint: for this type of cantilever truss, do not calculate the reaction. Start from the free end which has at most two unknown. The joint A has two unknown. So, start from here. Use trigonometry to calculate angle]

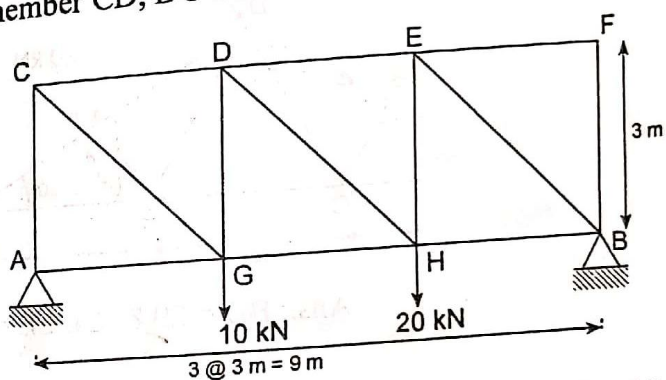
4. Determine the forces in the members of the truss loaded and supported as shown.

a) forces in member ED, BD and BC



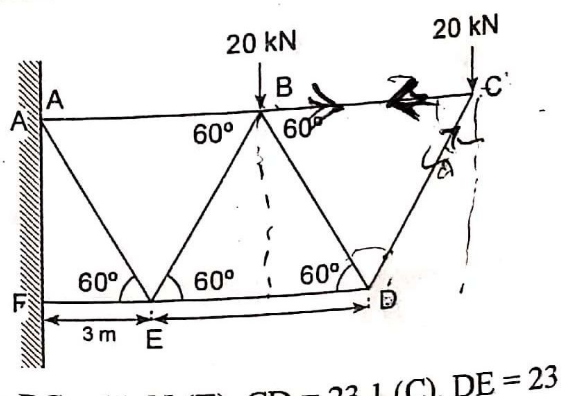
Ans. ED = 14 kN(CC), BD = 10.39 kN(CC), BC = 17.32 kN(CT)

b) forces in member CD, DG and GH



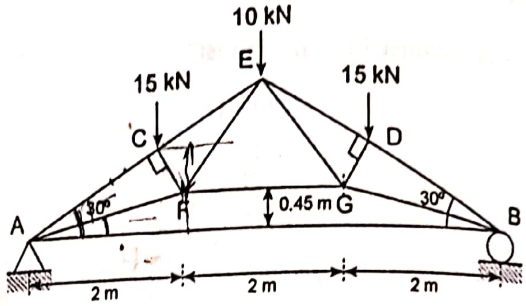
Ans.: CD = 13.33 kN (C), DG = 3.33 kN (C), GH = 13.33 kN (T)

c) forces in all members



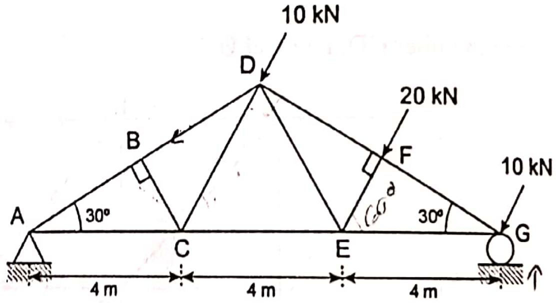
Ans.: AB = 46.2 (C), BC = 11.55 (T), CD = 23.1 (C), DE = 23.1 (C), EF = 69.3 (C), AE = 46.2 (T), BE = 46.2 (C), BD = 23.1 (T)

d) member CE and FG



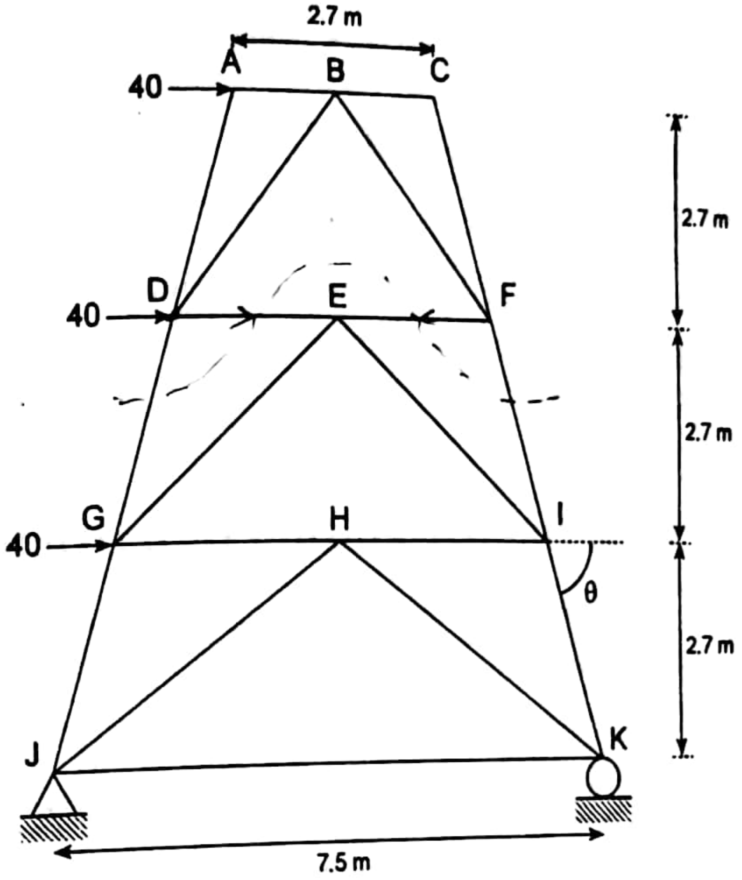
Ans.: CE = 38.5 kN (C), FG = 24.2 kN (T)

e) member BD, CE and CD.



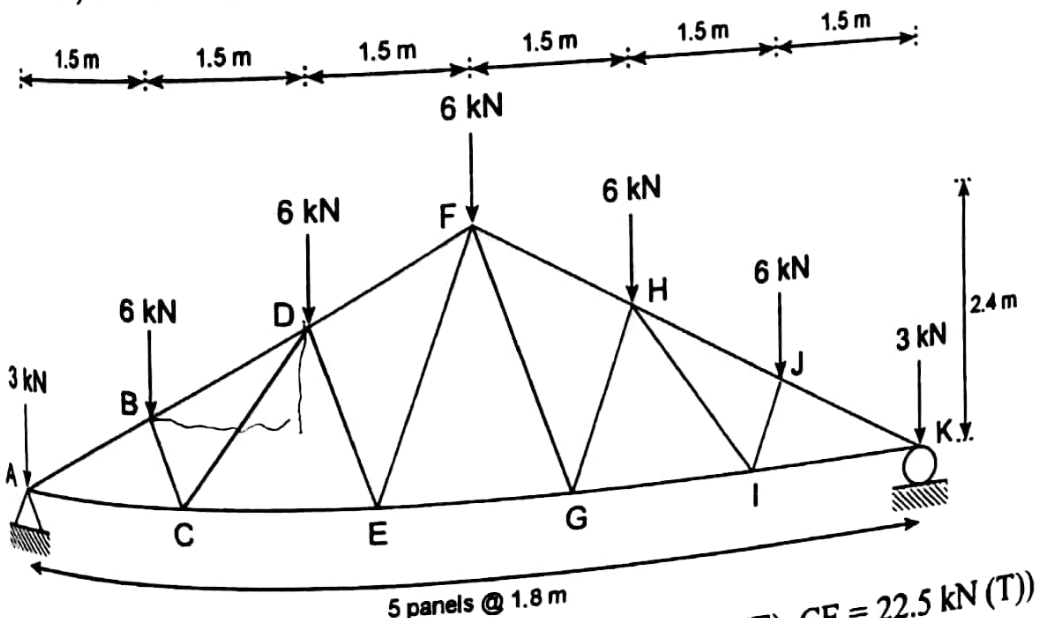
Ans.: BD = 29.8 (C), CE = 25.8 (T), CD = 0

5. Determine the force in member FI for the truss shown.



(Ans.: 26.2 kN(C))

6. A Fink roof truss is loaded as shown. Determine the force in members BD, CD and CE.



(Ans.: BD = 29.8 kN(C), CD = 6.25 kN(T), CE = 22.5 kN (T))

Chapter 8

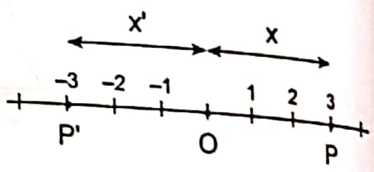
Kinematics of particle and rigid body

Kinematics is the study of geometry of motion. It is used to relate displacement velocity, acceleration and time, without reference to the cause of the motion.

8.1 Rectilinear motion of particle

When a particle moves along a straight line, the motion is said to be a rectilinear motion.

At any instant of time t, the particle will occupy a certain position on the straight line. To define the position P of the particle, we choose a fixed origin O on the straight line and a direction (say positive along x axis).

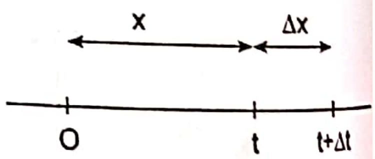


$$OP = x = +3\text{m}$$

$$OP' = x' = -3\text{m}$$

Here P and P' are called position coordinate and $\vec{OP}$ and $\vec{OP'}$ are called position vector.

When the position coordinate x of a particle is known for every value of time t, we say that the motion of particle is known. The timetable of the motion can be given in the form of an equation in x and t.



Then,

$$\text{Average velocity} = \frac{\Delta x}{\Delta t}$$

The instantaneous velocity v of the particle at the instant t is obtained from the average velocity by choosing shorter and shorter time intervals Δ and displacement Δx.

$$\text{Instantaneous velocity } v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$\therefore \quad v = \frac{dx}{dt}$

Similarly, $\quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

$v = \frac{dx}{dt}$

$\therefore \ dt = \frac{dx}{v}$

Again, $a = \frac{dv}{dt} = \frac{dv}{\frac{dx}{v}} = v\frac{dv}{dx}$

The relate the acceleration, velocity and position of particle.

...2 Determination of motion of particle and rigid body

The condition of motion will be specified by the type of acceleration ...t the particle possess. The acceleration of the particle may be expressed ...the function of one or more of the variable x, v and t. In order to ...ermine x in terms of t, it is necessary to perform two successive ...egration.

When acceleration is a given function of time i.e. $a = f(t)$

We know, $a = \frac{dv}{dt}$

or, $\quad dv = a\ dt$

$\therefore \quad dv = f(t)\ dt$

Integrating both sides

$\int dv = \int f(t)\ dt$

initially; $t = 0, v = v_0$

finally, $t = t, v = v$

$\int_{v_0}^{v} dv = \int_0^t f(t)\ dt$

$v - v_0 = \int_0^t f(t)\ dt$

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$v = v_0 + \int_0^t f(t)\,dt$ ......(i)

Again, $v = \dfrac{dx}{dt}$

$dx = v\,dt$

Integrating both side:

$\int dx = \int v\,dt$

at $t \to 0$; $\quad x \to x_0$

at $t \to t$; $\quad x \to x$

$\int_{x_0}^{x} dx = \int_0^t v\,dt$

or, $x - x_0 = \int_0^t [v_0 + \int_0^t f(t)\,dt]\,dt$

$\therefore\ x = x_0 + \int_0^t [v_0 + \int_0^t f(t)\,dt]\,dt$

Hence position x is obtained.

b) When acceleration is a given function of position i.e. a = f(x)

We know, $a = \dfrac{dv}{dt}$

Also, $a = v\dfrac{dv}{dx}$

or, $v\,dv = a\,dx$

or, $v\,dv = f(x)\,dx$

Integrating both side

$\int v\,dv = \int f(x)\,dx$

at $t \to 0$; $\quad x \to x_0$; $\quad v \to v_0$

at $t \to t$, $\quad x \to x$, $\quad v \to v$

$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} f(x)\,dx$

or, $\left.\dfrac{v^2}{2}\right|_{v_0}^{v} = \int_{x_0}^{x} f(x)\,dx$

or, $\dfrac{v^2}{2} - \dfrac{v_0^2}{2} = \int_{x_0}^{x} f(x)\,dx$

or, $v^2 = v_0^2 + 2\int_{x_0}^{x} f(x)$

or, $v = \Big[v_0^2 + 2\int_{x_0}^{x} f(x)$

Again, $v = \dfrac{dx}{dt}$

or, $dx = v\,dt$

or, $\int dx = \int v\,dt$

or, $\int_{x_0}^{x} dx = \int_0^t v\,dt$

or, $x - x_0 = \int_0^t v\,dt$

$\therefore\ x = x_0 + \int_0^t [v_0^2 + 2\int_{x_0}^{x} f$

Hence position x is obtained.

c) When acceleration is a give i.e. a = f(v)

We know, $a = \dfrac{dv}{dt}$

Also, $a = v\dfrac{dv}{dx}$

for, $a\,dx = v\,dv$

or, $f(v)\,dx = v\,dv$

or, $dx = \dfrac{v\,dv}{f(v)}$

at $t \to 0$; $\quad x \to x_0$, $\quad v$

at $t \to t$; $\quad x \to x$; $\quad v$

or, $$\left.\frac{v^2}{2}\right|_{v_0}^{v} = \int_{x_0}^{x} f(x)\,dx$$

or, $$\frac{v^2}{2} - \frac{v_0^2}{2} = \int_{x_0}^{x} f(x)\,dx$$

or, $$v^2 = v_0^2 + 2\int_{x_0}^{x} f(x)\,dx$$

or, $$v = \left[v_0^2 + 2\int_{x_0}^{x} f(x)\,dx\right]^{½}$$

Again, $v = \frac{dx}{dt}$

or, $dx = v\,dt$

or, $\int dx = \int v\,dt$

or, $$\int_{x_0}^{x} dx = \int_{0}^{t} v\,dt$$

or, $$x - x_0 = \int_{0}^{t} v\,dt$$

$$\therefore \quad x = x_0 + \int_{0}^{t} [v_0^2 + 2\int_{x_0}^{x} f(x)\,dx]^{1/2}\,dt$$

Hence position x is obtained.

c) When acceleration is a given function of velocity i.e. $a = f(v)$

We know, $a = \frac{dv}{dt}$

Also, $a = v\frac{dv}{dx}$

for, $a\,dx = v\,dv$

or, $f(v)\,dx = v\,dv$

or, $dx = \frac{v\,dv}{f(v)}$

at $t \to 0$; $x \to x_0$, $v \to v_0$

at $t \to t$; $x \to x$; $v \to v$

or, $\int dx = \int \frac{v\,dv}{f(v)}$

or, $\int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{v\,dv}{f(v)}$

or, $x - x_0 = \int_{v_0}^{v} \frac{v\,dv}{f(v)}$

$\therefore$ $x = x_0 + \int_{v_0}^{v} \frac{v\,dv}{f(v)}$

Hence position x is obtained.

d) Uniformly rectilinear motion

Uniform motion means covering equal distance over equal interv of time. In this case velocity is constant throughout the motion and hen acceleration a is zero for every value of t.

$\therefore$ $v = \frac{dx}{dt} = \text{constant} = u \text{ (say)}$

or, $\frac{dx}{dt} = u$

or, $dx = u\,dt$

or, $\int dx = \int u\,dt$

or, $x = ut + c_1$

at $t \to 0$; $x \to x_0$

$\therefore$ $c_1 = x_0$

$\therefore$ $x = x_0 + ut$

Hence the position of particle is obtained.

e) Uniformly accelerated rectilinear motion

In this motion acceleration a of the particle is constant.

$\therefore$ $\frac{dv}{dt} = \text{constant} = a \text{ (say)}$

or, $dv = a\,dt$

or, $\int dv = \int a\,dt$

or, $v = at + c_1$

at $t \to 0$, $\quad v \to v_0$

$\therefore c_1 = v_0$

$\therefore v = v_0 + at$ ......(i)

Also, $v = \frac{dx}{dt}$

or, $\frac{dx}{dt} = v_0 + at$

or, $dx = (v_0 + at)\, dt$

or, $\int dx = \int (v_0 + at)\, dt$

at $t \to 0$, $\quad x \to x_0$

or, $x = v_0 t + \frac{at^2}{2} + c_1$

$\therefore c_1 = x_0$

$\therefore x = x_0 + v_0 t + \frac{1}{2} at^2$ is required position.

Also, $a = v\frac{dv}{dx}$

or, $v\, dv = a\, dx$

or, $\int v\, dv = \int a\, dx$

or, $\frac{v^2}{2} = ax + c_1$

at $t \to 0$, $\quad v \to v_0, \quad x \to x_0$

or, $\frac{v_0^2}{2} = ax_0 + c_1$

$\therefore \quad c_1 = \frac{v_0^2}{2} - ax_0$

$\therefore \quad \frac{v^2}{2} = ax + \frac{v_0^2}{2} - ax_0$

or, $\frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$

$\therefore$

$v^2 = v_0^2 + 2a(x - x_0)$

f) Relative motion of two particles

Let position coordinate of A = x_A

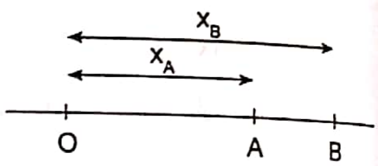
position coordinate of B = x_B

Relative position of B with respect to A is $x_{B/A}$ or x_{AB}

$$x_{B/A} = x_B - x_A$$

$$\therefore \quad x_B = x_A + x_{B/A}$$

Differentiating:



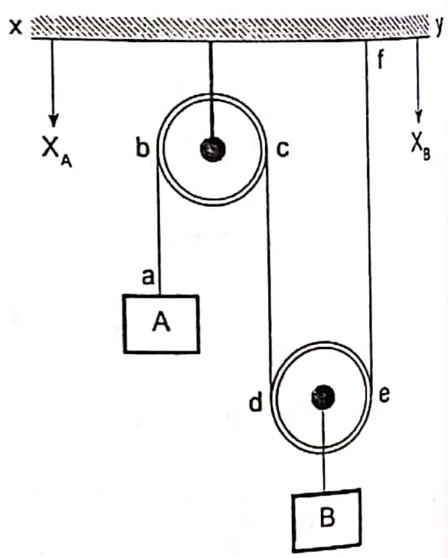
$$V_B = V_A + V_{B/A}$$

Again, differentiating

$$a_B = a_A + a_{B/A}$$

g) Dependent motion

The position of a particle sometimes depend upon the position of another or several other particles. The motion of such kind is called dependent motion.



Let the position of A be x_A and B be x_B with respect to reference XY.

From the figure; the position of block B depends upon the position of block A. Since the rope abcdef is of constant length.

$$ab + bc + cd + de + ef = \text{constant}$$

The length of portion of rope bc and de wrapped around the pulley remains fixed. So,

$$ab + cd + ef = \text{constant}$$

Since the length segment ab differ from x_A only by a constant and the length of segment cd and ef differ from x_B by a constant, we can write

$$x_A + x_B + x_B = \text{constant}$$

$$x_A + 2x_B = k \text{ ......(i)}$$

Differetiating

$v_A + 2V_B = 0$ ......(ii)

Again differentiating

$a_A + 2A_B = 0$ ......(iii)

8.3 Curvilinear motion

The motion of a particle along a curved path, other than a straight line is known as curvilinear motion. e.g. projectile motion, motion of satellite.

Position of particle on curved path at any instant is defined as position vector $\vec{r}$ where

$$\vec{r} = x\,\vec{i} + y\,\vec{j}$$

$$|\vec{r}| = \sqrt{x^2 + y^2}$$

velocity vector $\vec{v} = \dfrac{d\,\vec{r}}{dt} = \dfrac{d}{dt}(x\,\vec{i} + y\,\vec{j})$

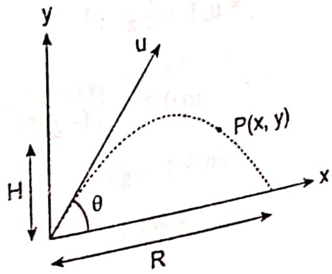
$$\vec{v} = \vec{i}\,\frac{dx}{dt} + \vec{j}\,\frac{dy}{dt}$$

$$\vec{v} = v_x\,\vec{i} + v_y\,\vec{j}$$

$$\vec{a} = a_x\,\vec{i} + a_y\,\vec{j}$$

8.3.1 Projectile motion

Let a body is projected with initial velocity u at an angle θ with ground as shown in figure.



Let H be the greatest height, R be the horizontal range and T be the time of flight of projectile.

Consider at instant of time t projectile reaches point P(x, y).

Then,

$u_y = u \sin\theta$

$u_x = u \cos\theta$ (remains constant throughout the motion. why?)

for point P

$x = u \cos\theta \times t$

$\therefore\ t = \dfrac{x}{u\cos\theta}$

Consider vertical motion;

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

for point P(x, y)

$y = u \sin\theta \,.\, t + \dfrac{1}{2}(-g)\, t^2$

or, $\quad y = u\sin\theta \dfrac{x}{u\cos\theta} - \dfrac{1}{2} \cdot g \cdot \dfrac{x^2}{u^2\cos^2\theta}$

or, $\quad y = x\tan\theta - \dfrac{1}{2}\dfrac{g}{u^2} x^2 \sec^2\theta$

$\therefore \quad y = x\tan\theta - \dfrac{1}{2}\dfrac{g}{u^2} x^2(1 + \tan^2\theta)$

This is the equation of parabola. Hence the projectile motion is parabolic in nature.

a) Calculation of time of flight (T)

During the total time of flight, the net height gain by projectile will be zero.

$s_y = u_y T + \dfrac{1}{2} a_y T^2$

$0 = u\sin\theta . T + \dfrac{1}{2}(-g)T^2$

$2u\sin\theta\, T = gT^2$

$\therefore\ T = \dfrac{2u\sin\theta}{g}$

b) Calculation of greatest height

At the maximum height of projectile the vertical component of velocity will be zero.

$$v_y^2 = v_y^2 + 2a_y s_y$$

$$0 = (u \sin\theta)^2 + 2(-g).H$$

$$u^2 \sin^2\theta = 2gh$$

$$\therefore H = \frac{u^2 \sin^2\theta}{2g}$$

For maximum height,

$$\sin^2\theta = 1 \text{ i.e. } \theta = 90^\circ$$

$$H_{max} = \frac{u^2}{2g} \text{ at } \theta = 90^\circ$$

c) Calculation of horizontal range

Since horizontal component of projectile velocity is constant.

$$R = u\cos\theta \times T$$

$$= u\cos\theta \times \frac{2u\sin\theta}{g}$$

$$= \frac{u^2}{g} 2\sin\theta\cos\theta$$

$$= \frac{u^2 \sin 2\theta}{g}$$

For maximum range

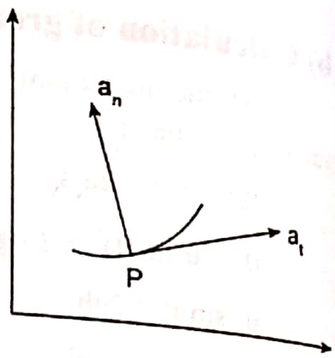
$$\sin 2\theta = 1 \text{ i.e. } \theta = 45^\circ$$

$$R_{max} = \frac{u^2}{g} \text{ at } \theta = 45^\circ$$

8.4 Tangential and normal component of acceleration

The tangential component of acceleration deflect the change in speed of particle while its normal component deflects change in direction of motion of particle. The acceleration of particle will be zero only if both of its component is zero.

The normal component is always directed towards the centre while tangential component is tangent to the curve. If $\hat{e}_t$ and $\hat{e}_n$ be the unit vector along tangential and normal component respectively. Then,



$$\vec{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{\delta}\hat{e}_n$$

$$\vec{a} = \vec{a}_t + \vec{a}_n$$

$a_t = \frac{dv}{dt}$; $a_n = \frac{v^2}{\delta}$ where δ is radius of curvature.

For a straight line motion i.e. rectilinear motion, radius of curvature $\delta = \infty$, $a_n = 0$.

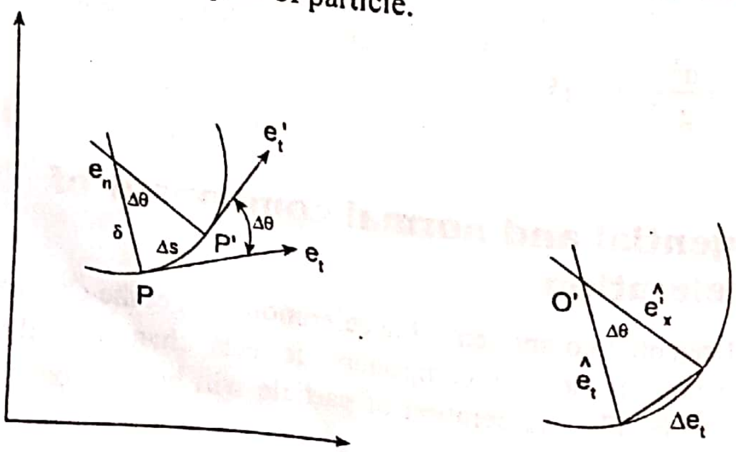
$\therefore \frac{dv}{dt} = a_t = a$ [for uniform circular motion $a_t = 0$; $a = a_n = \frac{v^2}{\delta}$]

Sometime δ, radius of curvature is calculated using

$$\delta = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

8.4.1 Derivation of tangential and normal component

The velocity of a particle is a vector tangent to the path of a particle but that the acceleration vector is in general, not tangent to the path. The acceleration vector may be resolved into component directed along the tangent and normal to the path of particle.



Consider $\hat{e}_t$ and $\hat{e}_n$ be the unit vectors along tangential and normal direction. Consider a particle a at point P having unit vector $\hat{e}_t$ tangent to the path and $\hat{e}_n$ perpendicular to $\hat{e}_t$. When P reaches to P′, $\hat{e}_t$ changes to $\hat{e}'_t$ and $\hat{e}_n$ changes to $\hat{e}_n'$ having radius of curvature δ.

Drawing both vectors from the same origin O, we define $\Delta e_t = e'_t - e_t$. Since e_t and e'_t are of unit length, their tips lie on the circle of radius 1. Denoting by Δθ the angle formed by e_t and e'_t.

Now, $\Delta s = PP' = \delta . \Delta\theta$ [use $\theta = \frac{l}{r}$]

$$\Delta\hat{e}_t = \hat{e}'_t - \hat{e}_t \cong \Delta\theta . \hat{e}_n$$

Now, $$\frac{d\theta}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{1}{\delta}$$

$$\frac{d\hat{e}_t}{d\theta} = \lim_{\Delta\theta \to 0} \frac{\Delta\hat{e}_t}{\Delta\theta} = \hat{e}_n$$

Now $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{e}_t)$$

$$= \frac{dv}{dt} \cdot \hat{e}_t + v\frac{d\hat{e}_t}{dt}$$

$$= \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{d\theta} \cdot \frac{d\theta}{ds} \cdot \frac{ds}{dt}$$

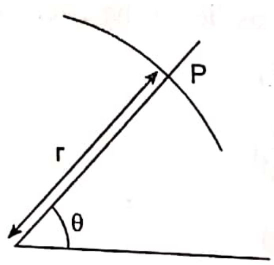
$$= \frac{dv}{dt}\hat{e}_t + v \, . \, \hat{e}_n \cdot \frac{1}{\delta} v$$

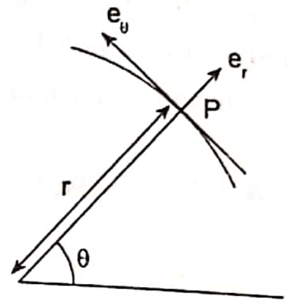
$$\vec{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{\delta}\hat{e}_n$$

$$= \vec{a}_t + \vec{a}_n$$

8.5 Radial and transverse component

In certain problems of plane motion the position of the particle P is defined by its polar coordinates r and θ as shown in figure. It is then convenient to resolve the velocity and acceleration of the particle into component parallel and perpendicular to the line OP respectively. These component are called radial and transverse component respectively.





The unit vector e_r defines the radial direction i.e. the direction in which P would move if r were increased and θ were kept constant.

The unit vector e_θ defines the transverse direction i.e. the direction in which P would move if θ were increased and r were kept constant.

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta}$$

$$\therefore \vec{v} = \vec{v}_r + \vec{v}_\theta = \dot{r}\, e_r + r\,\dot{\theta}\, e_\theta$$

$$a_r = \ddot{r} - r\,\dot{\theta}^2 \quad a_\theta = r\ddot{\theta} + 2\dot{r}\,\dot{\theta}$$

$$\vec{a} = \vec{a}_r + \vec{a}_\theta$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\, e_r + (r\ddot{\theta} + 2\dot{r}\,\dot{\theta})\, e_\theta$$

In case of particle moving along a circle of centre O, we have

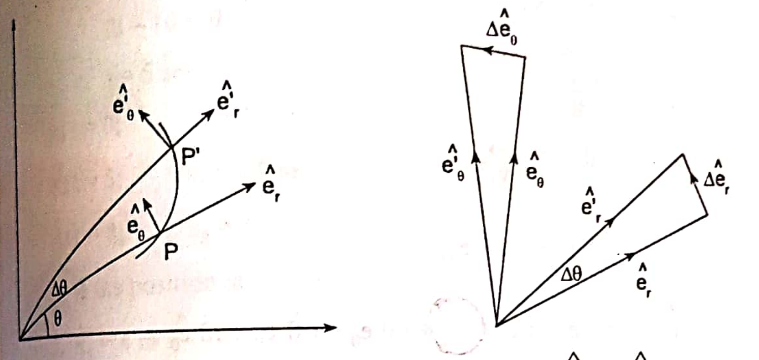
r = constant

$$\dot{r} = 0,\ \ddot{r} = 0$$

$$\therefore \vec{v} = r\dot{\theta}\, e_\theta;\ \vec{a} = -r\dot{\theta}^2\, e_r + r\ddot{\theta}\, e_\theta$$

8.5.1 Derivation of radial and transverse components

The position of the particle at P is defined by the coordinates r and θ as shown where r is length in metre and θ is angle in radian. The unit vectors in radial and transverse direction are denoted by $\hat{e}_r$ and $\hat{e}_\theta$ respectively.



As a particle moves from P to P′, the unit vector $\hat{e}_r$ and $\hat{e}_\theta$ changes to $\hat{e}'_r$ and $\hat{e}'_\theta$

$$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta} \cdot \frac{d\theta}{dt} = \hat{e}_\theta \cdot \dot{\theta}$$

Since $d\hat{e}_r$ is directed in the direction of $\hat{e}_\theta$

$$\dot{\hat{e}}_\theta = \frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta} \cdot \frac{d\theta}{dt} = -\hat{e}_r \cdot \dot{\theta}$$

Since $d\hat{e}_\theta$ is directed in the direction of $-\hat{e}_r$.

Now, $\vec{r} = r\hat{e}_r$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r)$$

$$= r\frac{d\hat{e}_r}{dt} + \hat{e}_r\frac{dr}{dt}$$

$$= r\dot{e}_r + e_r\dot{r}$$

$$= r\, e_\theta\, \dot{\theta} + \dot{r}\, e_r$$

$$= r\dot{\theta}\, e_\theta + \dot{r}\, e_r$$

$$\vec{v} = \dot{r}\, e_r + r\dot{\theta}\, e_\theta$$

$$\vec{v} = \vec{v}_r + \vec{v}_\theta$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d}{dt}[\vec{v}_r + \vec{v}_\theta]$$

$$= \frac{d}{dt}[\dot{r}\, e_r + r\dot{\theta}\, e_\theta]$$

$$\vec{a} = \ddot{r}\, e_r + \dot{r}\,\dot{e}_r + \dot{r}\, e_r + \dot{r}\dot{\theta}\, e_\theta + r\ddot{\theta}\, e_\theta + r\dot{\theta}\, \dot{e}_\theta$$

$$= \ddot{r}\, e_r + \dot{r}\dot{\theta}\, e_\theta + \dot{r}\dot{\theta}\, e_\theta + r\ddot{\theta}\, e_\theta + r\dot{\theta}\,(-)\dot{\theta}\, e_r$$

$$= (\ddot{r} - r\dot{\theta}^2)\, e_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\, e_\theta$$

$$= \vec{a}_r + \vec{a}_\theta$$

Quick Revision

$\vec{v}_r = \dot{r}\, e_r$; $\quad \vec{v}_\theta = r\dot{\theta}\, e_0$

$\vec{v} = \vec{v}_r + \vec{v}_\theta$

$\vec{a}_r = (\ddot{r} - r\dot{\theta}^2)\, e_r$; $\quad a_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta})\, e_\theta$

$\vec{a} = \vec{a}_r + \vec{a}_\theta$

Worked Out Example

1. The motion of particle is defined by the relation $x = t^2 - 10t + 20$, where x is expressed in meter and t in second. Determine
   a) when the velocity is zero?
   b) the position and the total distance travel when t = 8 seconds.

Solution:

$$x = t^2 - 10t + 20$$

$x = t^2 - 10t + 20,$
mine

$= 8$ seconds.

$v = \dfrac{dx}{dt} = 25 - 10$

If $v = 0$;

$2t - 10 = 0$

$\therefore \quad t = 5$ sec

At $t = 0$; $\quad x_0 = 20$ m

At $t = 5$; $\quad x_5 = -5$ m

At $t = 8$; $\quad x_8 = 4$ m

x_8 is the position at $t = 8$ sec.

$\therefore \; x_8 = 4$ m

Total distance travelled when $t = 8$ sec

$$s_8 = |x_5 - x_0| + |x_8 - x_5|$$
$$= |-5 - 20| + |4 - (-5)|$$
$$= 25 + 9 = 34 \text{ m}$$

2. The acceleration of a particle is directly proportional to the time t. At time $t = 0$ sec, the velocity of the particle $v = -16$ m/s. Knowing that both the velocity and the position coordinate are zero when $t = 4$ sec, write the equation of motion of particle.

Solution:

$a \propto t$

$a = kt$

Integrating

$\int a\,dt = \int kt\,dt$

or, $\quad v = k\dfrac{t^2}{2} + c_1$

At $t = 0$, $\quad v = -16$,

$\therefore \; -16 = c_1$

$\therefore \; v = \dfrac{kt^2}{2} - 16$

At $t = 4$, $\quad v = 0$

$$0 = k.\frac{16}{2} - 16$$

$\therefore\ k = 2$

$v = t^2 - 16$

Integrating with respect to time;

$$x = \frac{t^3}{3} - 16t + c_2$$

At t = 4 sec; x = 0

$$0 = \frac{64}{3} - 64 + c_2$$

$\therefore\ c_2 = 42.67$

$$\therefore\ x = \frac{t^3}{3} - 16t + 42.67$$

Equations of motion are

a) $x = \frac{t^3}{3} - 16t + 42.67$

b) $v = t^2 - 16$

c) $a = 2t$

3. The equation of motion of a particle is defined by the relation a = –0.4 v where a is the acceleration in m/s^2 and v is the velocity in m/s. Find the time for which velocity will be zero. It is known that velocity is 30 m/s at the instant of zero seconds.

Solution:

$a = -0.4\ v$

$$\frac{dv}{dt} = -0.4\ dt$$

Integrating

$\ln v = 0.4t + c_1$

At t = 0, v = 30

$\therefore\ \ln 30 = c_1$

$\therefore\ \ln v = -0.4t + \ln 30$

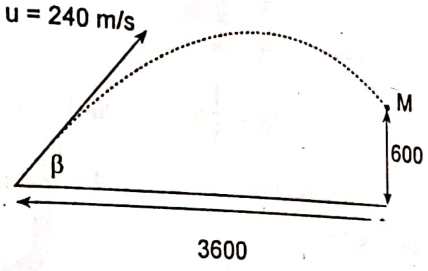
When v = 0

$\ln 0 = -0.4t + \ln 30$

or, $t = \infty$

$\therefore$ Time = infinity

4. A projectile is fired with an initial velocity of 240 m/s at a target M located 600 m above a gun G and at a horizontal distance of 3600 m. Neglecting air resistance, determine the value of firing angle β.



Solution:

For projectile motion

$$y = x \tan\beta - \frac{1}{2}\, g \frac{x^2}{u^2}(1 + \tan^2\beta)$$

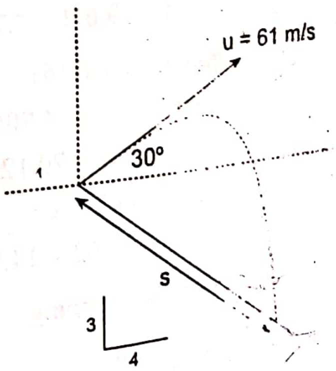
or, $$600 = 3600 \tan\beta - 4.905 \frac{(3600)^2}{(240)^2}(1 + \tan^2\beta)$$

or, $$600 = 3600 \tan\beta - 1103.6 - 1103.6 \tan^2\beta$$

or, $$1103.6 \tan^2\beta - 3600 \tan\beta + 1703.6 = 0$$

$\therefore$ $$\beta = 69.58 \text{ and } 29.85$$

5. A particle is projected at an angle of 30° with an initial velocity of 61 m/s as shown in figure. Find the slopping distance covered by the projectile.



Solution:

Initial velocity u = 61 m/s

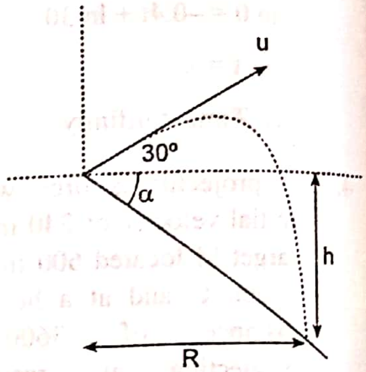
Projection angle $\theta = 30°$

Let α be the angle of inclination.

Then,

$$\tan\alpha = \frac{3}{4}$$

$$\alpha = 36.87°$$



Then,

$$\tan\alpha = \frac{h}{R}$$

$$\frac{3}{4} = \frac{h}{R}$$

$$R = \frac{4}{3}h \quad \text{and} \quad h = \frac{3}{4}R$$

Consider a vertical motion of projectile;

$$s_y = u_y.t + \frac{1}{2}a_y t^2$$

$$-h = u \sin\theta \cdot t - \frac{1}{2}gt^2$$

$$-h = 61 \sin 30\, t - \frac{1}{2} \times 9.81 \times t^2$$

$$-h = 30.5\, t - 4.905\, t^2 \quad ......(i)$$

Again,

$$R = u_x \times t$$

$$\frac{4h}{3} = u \cos\theta \times t$$

$$\frac{4h}{3} = 61 \cos 30 \times t$$

$$h = 39.62t \quad ......(ii)$$

From eq. (i) and (ii)

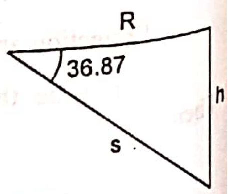
$$-39.62t = 30.5t - 4.905t^2$$

$$4.905t^2 = 70.12t$$

$$\therefore \quad t = 14.31 \text{ sec}$$

$$\therefore \quad h = 39.62 \times 14.31 = 566.97 \text{ m}$$

To find the slopping distance s

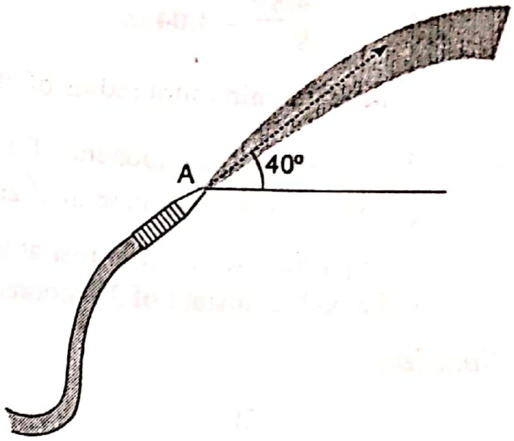


$$\sin 36.87 = \frac{h}{s}$$

$$\therefore s = \frac{h}{\sin 36.87}$$

$$= \frac{566.97}{0.6} = 944.94 \text{ m}$$

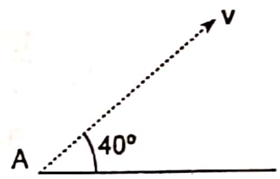
6. A nozzle discharges a stream of water as shown. It is determined that radius of curvature at A is $\delta_A = 9$ m. Determine the velocity v_A. The radius of curvature of trajectory at its maximum height.



Solution:

Case (a): At point A:

a_t will be along the v_A and a_n will be perpendicular to a_t.



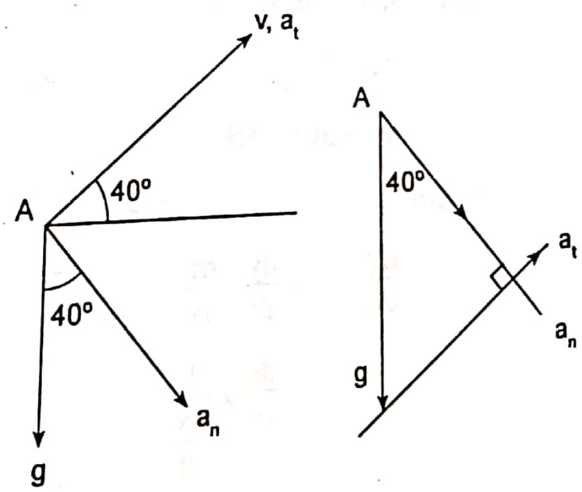
Now, again draw a triangle

$$\cos 40° = \frac{a_n}{g}$$

$$\therefore\ a_n = g\cos 40$$

$$= 9.81 \cos 40°$$

$$= 7.51\ \text{m/s}^2$$



Also, $a_n = \dfrac{v_A^{\ 2}}{\delta_A}$

$$7.51 = \frac{v_A^{\ 2}}{9}$$

$$\therefore\quad v_A = 8.22\ \text{m/s}$$

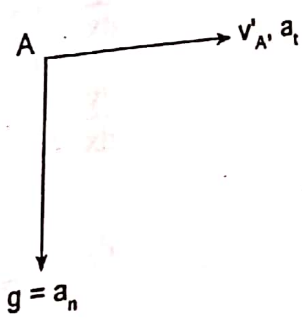
Case (b)

At maximum height

$v_A' = v_A \cos 40° = 6.2968$ m/s

$a_n = g$

$$\therefore\ A_n = \frac{v_A}{\delta}$$



$$\delta = \frac{(6.2968)^2}{9.81} = 4.04 \text{ m}$$

This is the minimum radius of curvature.

7. The rectangular component of acceleration for a particle are $a_x = 3t$, $a_y = 30 - 10t$ where a is in m/s^2 and t in seconds.

If the particle starts from rest at the origin, find the radius of curvature of the path at instant of 2 seconds.

Solution:

$$a_x = 3t \qquad v_x = \frac{3t^2}{2} + c_1$$

At $t = 0$, $v_x = 0$, $v_y = 0$

$\therefore \quad c_1 = 0$

$$\therefore \quad v_x = \frac{3t^2}{2}$$

Also, $a_y = 30 - 10t$

$$v_y = 30t - 5t^2 + c_2$$

At $t = 0$, $v_x = 0$, $v_y = 0$

$\therefore \quad c_2 = 0$

$\therefore \quad v_y = 30t - 5t^2$

Again,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

$$\frac{dy}{dx} = \frac{30t - 5t^2}{\frac{3t^2}{2}}$$

$$\frac{dy}{dx} = \frac{60t - 10t^2}{3t^2}$$

$$\frac{dy}{dx} = \frac{20}{t} - \frac{10}{3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{20}{t} - \frac{10}{3}\right]$$

$$= \frac{d}{dt}\left(\frac{20}{t} - \frac{10}{3}\right) \cdot \frac{dt}{dx}$$

$$= \frac{-20}{t^2} \cdot \frac{1}{v_x}$$

$$= \frac{-20}{t^2} \cdot \frac{2}{3t^2}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{40}{3t^4}$$

At t = 2 sec

$$\frac{dy}{dx} = \frac{20}{t} - \frac{10}{3}$$

$$= 10 - \frac{10}{3} = \frac{20}{3}$$

$$\frac{d^2y}{dx^2} = -\frac{40}{3t^4}$$

$$= -\frac{40}{3.2^4} = -\frac{40}{48}$$

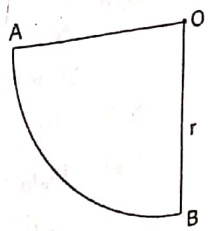
Radius of curvature

$$\delta = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{20}{3}\right)^2\right]^{3/2}}{\frac{-40}{48}}$$

$= -367.62$

$\therefore$ radius of curvature = 368 m.

8. An automobile enters a curved road at 30 km/hr and then leaves at 48 km/hr. The curved road is in the form of quarter of circle and has a length of 400 m. If the car travels at constant acceleration along the curve, calculate resultant acceleration at both ends of curve.



Solution:

$$P = \frac{2\pi r}{4}$$

$$400 = \frac{2\pi \times r}{4}$$

$\therefore$ r = 254.6 m

Initially, v_A = 30 km/hr = 8.33 m/s

Finally, v_B = 48 km/hr = 13.33 m/s

Again, $v_B^2 = v_A^2 + 2.a_t.s$

$(13.33)^2 = (8.33)^2 + 2 \times a_t \times 400$

$\therefore$ $a_t = 0.1353$ m/s²

At point A:

$$a_n = \frac{v_A^2}{\delta} = \frac{(8.33)^2}{254.6} = 0.2725 \text{ m/s}^2$$

At point B:

$$a_n = \frac{v_B^2}{\delta} = \frac{(13.33)^2}{254.6} = 0.6979 \text{ m/s}^2$$

Resultant acceleration at

a) Point A

$$(a_R)_A = (a_n)_A + a_t$$
$$= \sqrt{(a_n)_A^2 + at^2}$$
$$= \sqrt{(0.2725)^2 + (0.1353)^2}$$
$$= 0.304 \text{ m/s}$$

$$\tan\theta = \frac{at}{a_n}$$

$\therefore$ $\theta = 26.40°$

b) Point B:

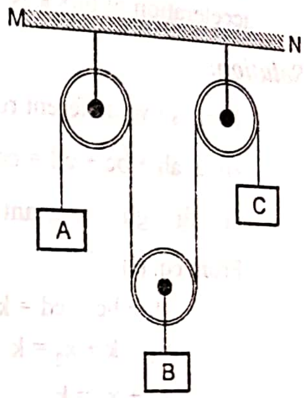
$$(a_R)_B = (a_n)_B + a_t$$
$$= \sqrt{(a_n)_B^2 + at^2}$$

$$= \sqrt{(0.6979)^2 + (0.1353)^2}$$
$$= 0.7108 \text{ m/s}^2$$

$$\tan\beta = \frac{at}{a_n}$$

$$\therefore \beta = 10.97°$$

9. For a pulley system as shown in figure, calculate the velocity and acceleration of the block C. If the velocity and accelerations of the blocks A and B are 4 m/s (↓), 2 m/s² (↑), 6 m/s (↑) and 4 m/s²(↑) respectively.



Solution:

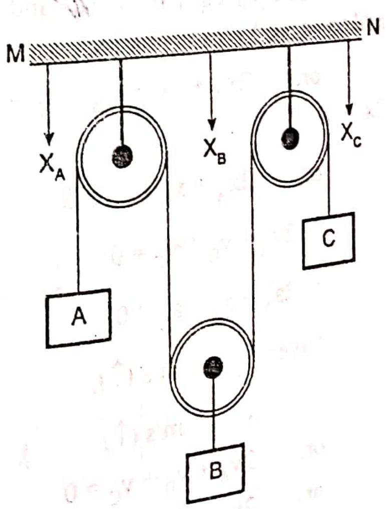
Let the position of block A, B and C be x_A, x_B, x_C from the reference MN.

Then, $x_A + x_B + x_C = k$

$$x_A + 2x_B + x_C = k$$

Differentiating

$$v_A + 2v_B + v_C = 0$$
$$a_A + 2a_B + a_C = 0$$



Given, $v_A = 4$ m/s (↓),

$v_B = 6$ m/s (↑)

$\therefore v_A + 2v_B + v_C = 0$

or, $-4 + 2 \times 6 + v_C = 0$

or, $-4 + 12 + v_C = 0$

$\therefore v_C = -8$ m/s $= 8$ m/s (↓)

Also, given $a_A = 2$ m/s² (↑),

$a_B = 4$ m/s² (↑)

or, $a_A + 2a_B + a_C = 0$

or, $2 + 2 \times 4 + a_C = 0$

or, $2 + 8 + a_C = 0$

$\therefore a_C = 10$ m/s² (↓)

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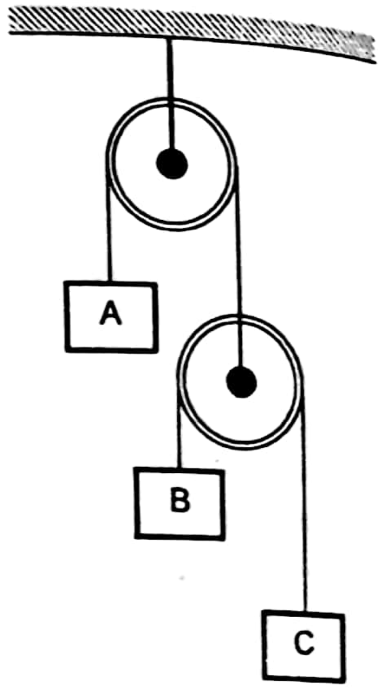
10. The connection of block A, B and C with pulley as shown in figure. If the block B has velocity 2 m/s (↑) and its acceleration 3 m/s² (↓) while block C has velocity and acceleration of 2 m/s (↑) and 4 m/s² (↓) respectively. Find the velocity and acceleration of block A.



Solution:

Here is two different rope.

Here, ab + bc + cd = constant = k ......(i)

ef + fg + gh = constant = k ......(ii)

From eq. (i)

$$ab + bc + cd = k$$

$$x_A + k + x_2 = k$$

$$x_A + x_2 = k \text{ ......(iii)}$$

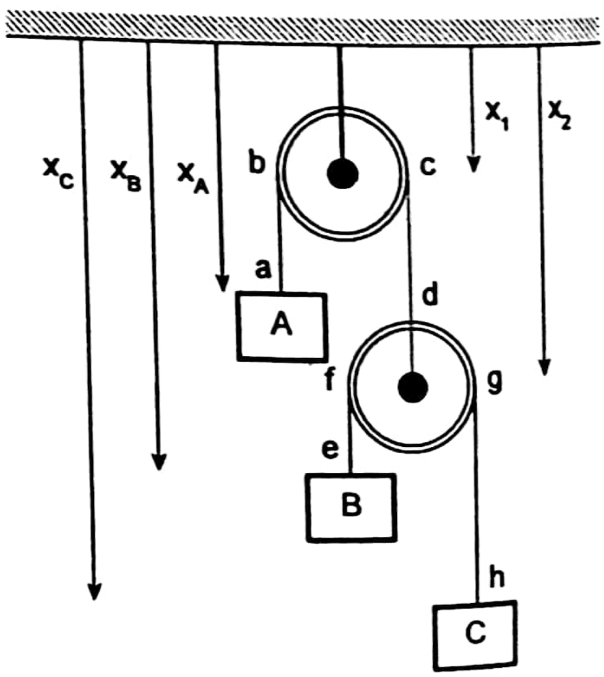
From eq. (ii)

$$ef + fg + gh = k$$

$$x_B - x_2 + k + x_C - x_2 = k$$

$$x_B + x_C - 2x_2 = k \text{ ......(iv)}$$

Multiply eq. (iii) by ~~(ii)~~ 2 and add to eq. (iv)



or, $2x_A + 2x_2 + x_B + x_C - 2x_2 = k$

or, $2x_A + x_B + x_C = k$

$\therefore\ 2v_A + v_B + v_C = 0$

$\therefore\ 2a_A + a_B + a_C = 0$

Given, $v_B = 2$ m/s (↑),

$v_C = 2$ m/s (↑)

or, $2v_A + v_B + v_C = 0$

or, $2v_A + 2 + 2 = 0$

or, $2v_A = -4$

$\therefore\ v_A = -2$ m/s = 2 m/s (↓)

Given, $a_B = 3\ m/s^2\ (\downarrow)$, $a_C = 4\ m/s^2\ (\downarrow)$

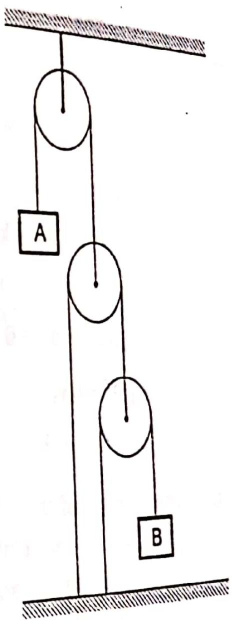
or, $2a_A + a_B + a_C = 0$

or, $2a_A - 3 - 3 = 0$

or, $2a_A = 6$

$\therefore\ a_A = 3\ m/s^2\ (\uparrow)$

11. Find the acceleration of body B if the acceleration of A is 4 m/s^2 ($\downarrow$) for the following connection.



Solution:

$ab + bc + cd = k$

$x_A + k + x_2 = k$

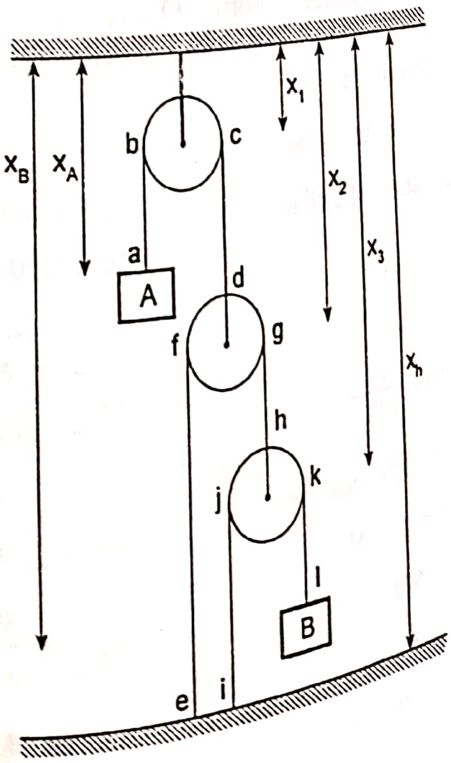
$x_A + x_2 = k$ ......(i)

$ef + fg + gh = k$

or, $x_h - x_2 + k + x_3 - x_2 = k$

or, $x_h - 2x_2 + x_3 = k$

or, $x_3 - 2x_2 = k$ ......(ii) $[x_h = k]$



$ij + jk + kl = \text{constant} = k$

or, $x_h - x_3 + k + x_B - x_3 = k$

or, $x_h - 2x_3 + x_B = k$

or, $x_B - 2x_3 = k$ ......(iii)

Multiply eq. (i) by 4, eq. (ii) by 2

$4x_A + 4x_2 = k$ ......(iv)

$2x_3 - 4x_2 = k$ ......(v)

$x_B - 2x_3 = k$ ......(vi) from eq. (iii)

Adding all (iv), (v) and (vi)

$4x_A + x_B = k$

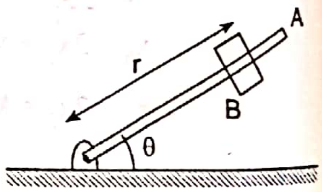
or, $4v_A + v_B = 0$

or, $4a_A + a_B = 0$

$\therefore$ $a_B = -4.a_A$

$= -4 \times 4 = -16$ m/s² (↓) = 16 m/s² (↑)

12. The rotation of the 0.9 m arm OA about O is defined by the relation $\theta = 0.15t^2$ where θ is expressed in radian and t in second. Collar B slides along the arm in such a way that its distance from O is $r = 0.9 - 0.12t^2$, r in meters and t in secs. After the arm OA has rotated through 30°, determine



a) total velocity of the collar

b) total acceleration of collar.

Solution:

Given, $\theta = 30°$

$$\theta = 30 \times \frac{\pi}{180} = 0.524^c$$

Also, $\theta = 0.15t^2$ (by question)

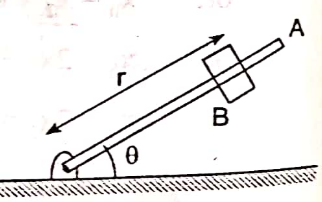
$0.524 = 0.15\,t^2$

or, $t^2 = 3.4906$

$\therefore$ $t = 1.868$ sec.

Now, $r = 0.9 - 0.12t^2$; $\theta = 0.15 \times (1.868)^2$

(iii)



the arm OA has rotated

$= 0.15 \times (1.868)^2$

At t = 1.868 sec

$$r = 0.9 - 0.12 \times (1.868)^2 \quad ; \quad \theta = 0.15 \times (1.868)^2$$

$$r = 0.481 \text{ m} \quad ; \quad \theta = 0.524^c$$

$$\dot{r} = -2 \times 0.12\, t \quad ; \quad \dot{\theta} = 0.3t$$

$$= -0.24 \times 1.868 \quad ; \quad = 0.3 \times 1.868$$

$$= -0.448 \text{ m/s} \qquad = 0.56 \text{ rad/sec}$$

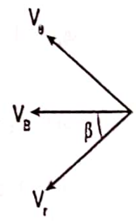
$$\ddot{r} = -0.24 \text{ m/s}^2 \qquad \ddot{\theta} = 0.3 \text{ rad/sec}^2$$

$$v_r = \dot{r} = -0.448 \text{ m/s}$$

$$v_\theta = r\dot{\theta} = 0.481 \times 0.56 = 0.270 \text{ m/s}$$

$$v_B = \sqrt{v_r^2 + v_\theta^2}$$

$$= \sqrt{(0.448)^2 + (0.270)^2} = 0.523 \text{ m/s}$$



$$\tan\beta = \frac{v_\theta}{v_r}$$

$$= \frac{0.270}{0.448}$$

$$\therefore \beta = 31.076°$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

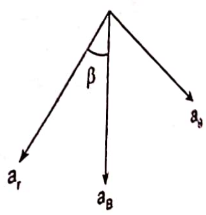
$$= -0.24 - 0.481 \times (0.561)^2$$

$$= -0.391 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$= 0.481 \times 0.3 + 2 \times (-0.449) \times 0.561$$

$$= -0.359 \text{ m/s}^2$$



$$\vec{a}_B = \vec{a}_r + \vec{a}_\theta$$

$$\therefore a_B = \sqrt{a_r^2 + a_\theta^2}$$

$$= \sqrt{(0.391)^2 + (0.359)^2}$$

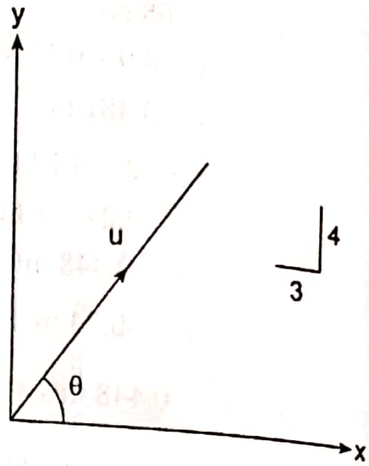
$$= 0.531 \text{ m/s}^2$$

$$\tan\beta = \frac{a_\theta}{a_r} = \frac{0.359}{0.391}$$

$$\therefore \beta = 42.55°$$

Acceleration of B with respect to OA is $a_{B/OA} = \ddot{r} = -0.240 \text{ m/s}^2$

13. A particle starting from origin is subjected to acceleration, such that $a_x = -3\ m/s^2$ and $a_y = -11\ m/s^2$. If the initial velocity is 80 m/s directed at a slope of 4 : 3 as shown. Compute the radius of curvature of path after 4 seconds. Also find the position at the end of 4 sec.

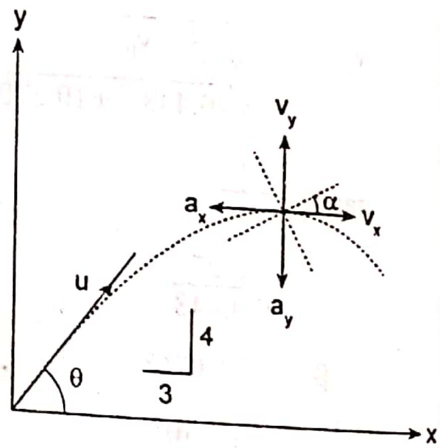


Solution:

$$\tan\theta = \frac{4}{3} = 1.333$$

So, $\sin\theta = \frac{4}{5} = 0.8$

$$\cos\theta = \frac{3}{5} = 0.6$$



At t = 0 sec

$(v_0)_x = 80\cos\theta = 48\ m/s$

$(v_0)_y = 80\sin\theta = 64\ m/s$

At t = 4 sec;

$(v_r)_x = (v_0)_x + a_x.t$

$= 48 - 3 \times 4 = 36\ m/s$

$(v_r)_y = (v_0)_y + a_y.t$

$= 64 - 11 \times 4 = 20\ m/s$

$|\vec{v}_r| = \sqrt{(v_r)_x^2 + (v_r)_y^2}$

$= \sqrt{36^2 + 20^2} = 41.18\ m/s$

$\tan\alpha = \frac{(v_4)_y}{(v_4)_x} = \frac{20}{36}$

$\therefore \alpha = 29.05°$

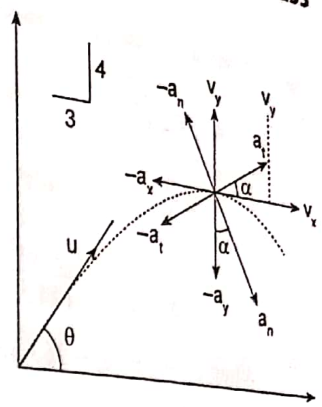
From figure $a_n = a_y \cos\alpha - a_x \sin\alpha$

$a_n = 11\cos 29.05 - 3\sin 29.05$

$= 8.1594\ m/s^2$

Also, $a_n = \dfrac{v^2}{\delta}$

$$\Rightarrow \delta = \frac{v^2}{a_n} = \frac{(41.18)^2}{8.1594} = 207.9 \text{ m}$$



$$x_4 = (v_0)_x.t + \frac{1}{2}.a_x.t^2$$

$$= 48 \times 4 + \frac{1}{2}.-3\ .\ 4^2$$

$$= 168$$

$$y_4 = (v_0)_y.t + \frac{1}{2}a_y t^2$$

$$= 64 \times 4 + \frac{1}{2}.-11\ .\ 4^2 = 168$$

$\therefore\ x_4, y_4 = 168, 168$

14. The motion of a particle is defined by the position vector $\vec{r} = 3t^2\vec{i} + 4t^3\vec{j} + 5t^4\vec{k}$ where $\vec{r}$ in meter and t in seconds. At the instant t = 4 sec, find the normal and tangential component of acceleration and the radius of curvature.

Solution:

$$\vec{r} = 3t^2\vec{i} + 4t^3\vec{j} + 5t^4\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 6t\vec{i} + 12t^2\vec{j} + 20t^3\vec{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 6\vec{i} + 24t\vec{j} + 60t^2\vec{k}$$

$$|\vec{v}| = [(6t)^2 + (12t^2)^2 + (20t^3)^2]^{1/2}$$

$$= [36t^2 + 144t^4 + 400t^6]^{1/2}$$

When t = 4 sec, $|V| = 1294.54$ m/s

$$|\vec{a}| = [6^2 + (24t)^2 + (60t^2)^2]^{1/2}$$

$$= [36 + 576t^2 + 3600t^4]^{1/2}$$

When t = 4 sec, $|a| = 964.81\ m/s^2$

$$a_t = \frac{d|v|}{dt}$$

$$= \frac{d}{dt}[(36t^2 + 144t^4 + 400t^6)^{1/2}]$$

$$= \frac{1}{2} \cdot \frac{1.(72t + 576t^3 + 2400t^5)}{(36t^2 + 144t^4 + 400t^6)^{1/2}}$$

When t = 4 sec, $a_t = 963.56\ m/s^2$

Also, $\vec{a} = \vec{a_t} + \vec{a_n}$

$$a_n = \sqrt{a^2 - a_t^2}$$

$$= \sqrt{(964.81)^2 - (963.56)^2}$$

$$= 49.1\ m/s^2$$

$$\therefore\ a_n = \frac{v^2}{\rho}$$

$$\therefore\ \rho = \frac{v^2}{a_n} = \frac{(1294.54)^2}{49.1} = 34131.03\ m$$

Radius of curvature = ρ = 34131.03 m

Practice Questions

1. The acceleration of a particle is defined by the relation $a = -2\ m/s^2$. Initially if velocity $v_0 = 8$ m/s and position $x_0 = 8$ m; determine the velocity, position and the total distance travelled at the instant of 6 sec.

   (Ans. –4 m/s; 20 m; 20 m)

2. The acceleration of a particle is defined by the relation a = 12x – 28 where a in m/s^2 and x in m. knowing that v = 8 m/s when x = 0; determine:

   a) the maximum value of x.

   b) the velocity when the particle has travelled a total distance of 3 m.

   (Ans. a) 2 m, b) –4.47 m/s)

the relation $a = -2\ m/s^2$.
$x_0 = 8$ m; determine the
led at the instant of 6 sec.
Ans. -4 m/s; 20 m; 20 m)

the relation $a = 12x - 28$
$v = 8$ m/s when $x = 0$;

d a total distance of 3 m.
Ans. a) 2 m, b) -4.47 m/s)

3. The acceleration of a particle is defined by the relation $a = kt^2$. Knowing that velocity is -32 m/s when time is zero and again velocity is $+32$ m/s when time is 4 sec. Write down the equation of motion knowing that the position of the particle is zero at the instant of 4 sec.

(Ans. $x = \frac{t^4}{4} - 32t + 64$; $v = t^3 - 32$, $a = 3t^2$)

4. The motion of particle is defined by the relation $x = 2t^3 - 12t^2 - 72t - 80$; x in m and t in sec. Determine the velocity, the acceleration and total distance travelled when $x = 0$.

(Ans. 288 m/s, 96 m/s^2, 944 m)

5. The acceleration of a particle in rectilinear motion is directly proportional to t^2. At $t = 0$, its velocity is -32 m/s. At $t = 2$ sec, the particle bounds to origin and its velocity becomes zero. Express the position x of the particle as a function of t.

(Ans. $x = t^4 - 32t + 48$)

6. A particle, moving in a straight line has an acceleration, $a = \sqrt{v}$, its displacement and velocity at time $t = 2$ sec, are $\frac{128}{3}$ m and 16 m/s. Find the displacement velocity and acceleration at time $t = 3$ sec.

(Ans. 60.75 m, 20.25 m/s, 4.5 m/s^2)

7. A projectile is fired from the edge of a 150 m cliff with an initial velocity of 180 m/s at an angle of 30° with the horizontal. Neglecting air resistance, find

a) the greatest elevation above the ground.

b) the horizontal distance from the gun to the point where the projectile strikes the ground.

c) the velocity with which it strikes ground.

(Ans. a) 563 m, b) 3103 m, c) 188 m/s, $\alpha = -34$)

8. A projectile is aimed at a marked on the horizontal plane through a point of projection and falls 10 m short when the angle of projection is 15° while overshoots the mark by 25 m when the inclination is 40°. Calculate the distance of target and required angle of projection. Neglect air resistance.

(Ans. $x = 46.09$ m, $\theta = 19.85°$)

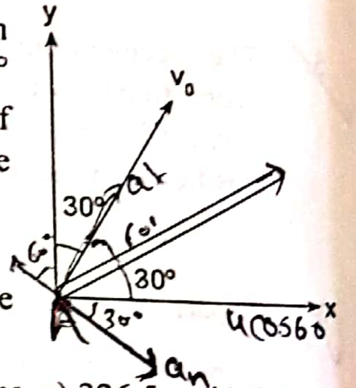
9. The velocity of a particle, at its greatest height is $\sqrt{\frac{2}{5}}$ times of its velocity at half of its greatest height. Show that the angle of projection is 60°.

10. A motorist is travelling on a curved section of highway of radius 750 m at the speed of 90 km/hr. The motorist suddenly applies the brake, causing the automobile to slow down at a constant rate. Knowing that after 8 sec, the speed has been reduced to 72 km/hr, determine the acceleration of the automobile immediately after brake has been applied. (Ans. 1.041 m/s², $\theta = 53.11°$)

11. A projectile is launched from point A with an initial velocity v_0 of 40 m/s at an angle of 30° with the vertical. Determine the radius of curvature of projectile described by the particle

a) At point A

b) At the point in the projectile where the velocity is parallel to inclined.



(Ans. a) 326.5 m, b) 62.8 m)

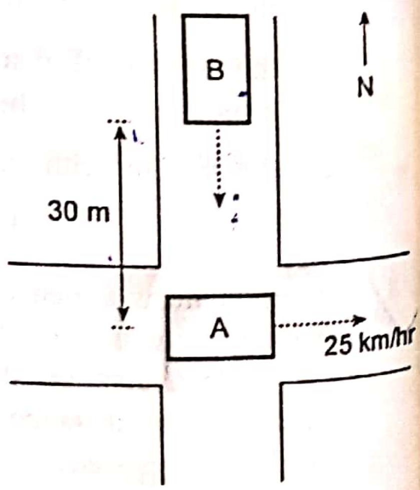
12. The motion of particle is given by the relation $v_x = 2\cos t$ and $v_y = \sin t$. It is known that initially both x and y coordinate are zero. Determine

a) Total acceleration at the instant of 2 sec.

b) The equation of path.

(Ans. a) 1.8678 m/s², $\theta = 192.99$, b) $x^2 + 4y^2 - 8y = 0$)

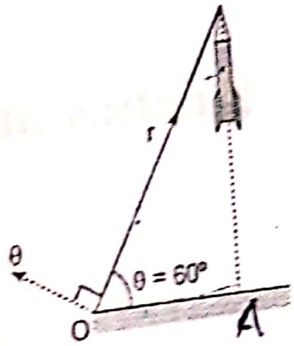
13. Automobile A is travelling east at the constant speed of 25 km/hr. As A crosses the intersection, auto B starts from rest 30 m north of intersection and moves south with a constant acceleration of 1.2 m/s². Determine the position, velocity and acceleration of B relative to A, five seconds after A crosses the intersection.



(Ans. $x_{B/A} = 37.821$, $v_{B/A} = 9.177$, $a_{B/A} = -1.2\,(\downarrow)$)

Scanned by CamScanner

A rocket is fired vertically and tracked by a radar station at O as shown in figure. When $\theta = 60^\circ$, it is found $r = 9000$ m, $\dot{\theta} = 0.02$ rad/sec, $\ddot{\theta} = 0.002$ rad/sec^2. Determine the velocity and acceleration of the rocket at the instant.



(Ans. $v = 360$ m/s $(\hat{\uparrow})$, $a = 60.94$ m/s^2 $(\hat{\uparrow})$)

5. The motion of particle is defined by the position vector $r = \frac{t^3}{4} + 3t$ and $\theta = 0.5t^2$ where r in meter and θ in radian. Determine the acceleration, velocity and radius of curvature at the instant $t = 4$ sec.

(Ans. $v = 113$ m/s, $a = 466.12$ m/s^2, $\rho = 27.41$ m)

$a_t =$

$a_\theta = \frac{v^2}{\rho}$

Chapter 9

Kinetics of Particles and Rigid Body

Kinetics is the study of relation existing between the forces acting on a body, the mass of body and the motion of body. It is used to predict the motion caused by given force or to determine the forces required to produce a given motion.

9.1 Newton's second law and momentum

9.1.1 Newton's second law

It states that "If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force."

$$F \propto a$$

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \text{constant}$$

Therefore when particle is subjected to several forces

$$\Sigma F = ma$$

$$\Sigma F_x = ma_x$$

$$\Sigma F_y = ma_y$$

9.1.2 Linear momentum

We know from Newton's second law

$$\Sigma \vec{F} = m\vec{a}$$

$$\Sigma \vec{F} = m\frac{d\vec{v}}{dt}$$

$$\Sigma \vec{F} = \frac{d}{dt}(m\vec{v})$$

The vector $m\vec{v}$ is called linear momentum or simply momentum. It has the same direction as the velocity and magnitude equal to product of mass and velocity. Denoting linear momentum by L

$$\vec{L} = m\vec{v}$$

Newton's law can be restated as "The resultant of the forces acting on a particle is equal to the rate of change of linear momentum of the particle."

$$\therefore \quad \Sigma\vec{F} = \dot{\vec{L}}$$

9.2 Equation of motion and dynamic equilibrium

9.2.1 Equation of motion

a) **Rectangular component**

$$\Sigma F_x = ma_x, \quad \Sigma F_y = ma_y, \quad \Sigma F_z = ma_z$$

b) **Tangential and normal components**

$$\Sigma F_t = ma_t \quad \Sigma F_n = ma_n \quad \Sigma F_t = m\frac{dv}{dt} \quad \Sigma F_n = m\frac{v^2}{\delta}$$

c) **Radial and transverse component**

$$\Sigma F_r = a_r \quad \Sigma F_\theta = a_\theta$$

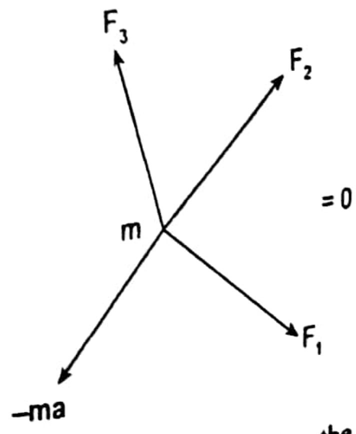
9.2.2 Dynamic equilibrium

From Newton's second law

$$\Sigma\vec{F} = m\vec{a}$$

or,

$$\Sigma\vec{F} - m\vec{a} = 0$$



The vector $-m\vec{a}$, of magnitude ma and direction opposite to that of acceleration is called inertia vector.

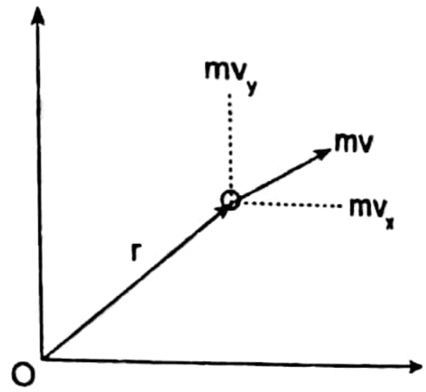
So, when the particle is considered to be in equilibrium under the given forces and inertia vector $-m\vec{a}$; it is said to be in dynamic equilibrium. The inertia vector measures the resistance that particle offer when we try to set them in motion or when we try to change the condition of their motion.

$$\Sigma F_x - ma_x = 0 \qquad \Sigma F_t - ma_t = 0$$
$$\Sigma F_y - ma_y = 0 \qquad \Sigma F_n - ma_n = 0$$
$$\Sigma F_z - mz_z = 0 \qquad \Sigma F_r - ar_r = 0$$
$$\Sigma F_\theta - a_\theta = 0$$

9.3 Angular momentum and rate of change

9.3.1 Angular momentum

The moment of the vector $m\vec{v}$ about a reference point say (O) is called angular momentum of the particle.



$$H_O = \vec{r} \times m\vec{v}$$

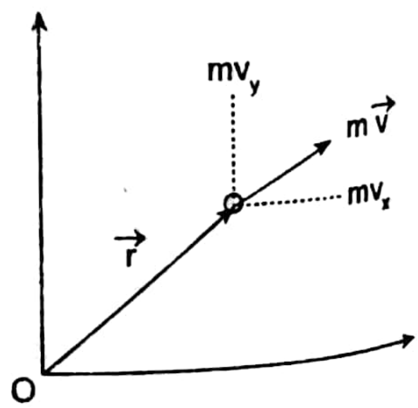
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$H_O = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ mv_x & mv_y & mz_z \end{vmatrix}$$

9.3.2 Rate of change of angular momentum

Consider a particle of mass m moving in the xy plane as shown in figure.

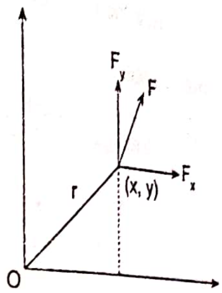


$$H_O = \vec{r} \times m\vec{v}$$
$$= m\,v_y.x - mv_x.y$$

$$\frac{d}{dt}(H_O) = \frac{d}{dt}(mv_y.x - mv_x.y)$$

$$\dot{H}_0 = m[\dot{x}\, v_y + x\dot{v}_y - \dot{v}_x\, y - \dot{y}\, v_x]$$

$$\begin{aligned} \dot{H}_0 &= [v_x . v_y + xa_y - ya_x - v_y v_x]m \\ &= m(xa_y - ya_x) \\ &= xma_y - yma_x \\ &= xF_y - yF_x \\ &= \text{moment of force about O} \end{aligned}$$



So, rate of change of angular momentum of a particle about any point at any instant is equal to the moment of force ($\vec{F}$) about that point.

9.4 Principle of impulse and momentum

Consider a particle of mass m acted upon by a force F, then from Newton's second law

$$\vec{F} = \frac{d}{dt} m\vec{v}$$

where $m\vec{v}$ is the linear momentum.

$$\vec{F}\, dt = d(m\vec{v})$$

Integrating both sides;

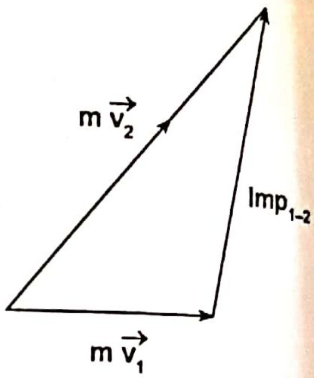
$$\int_{t_1}^{t_2} \vec{F}\, dt = \int_{v_1}^{v_2} d(m\vec{v})$$

$$\int_{t_1}^{t_2} \vec{F}\, dt = m\vec{v}_2 - m\vec{v}_1$$

$$\int_{t_1}^{t_2} \vec{F}\, dt = m\vec{v}_2 - m\vec{v}_1$$

$$\therefore \int_{t_1}^{t_2} \vec{F}\, dt + m\vec{v}_1 = m\vec{v}_2$$

Here the $\int_{t_1}^{t_2} \vec{F}\, dt$ is called linear impulse or simply impulse.



$$\therefore\ Imp_{1-2} = \int_{t_1}^{t_2} \vec{F}\, dt$$

$$= \vec{i} \int_{t_1}^{t_2} F_x dt + \vec{j} \int_{t_1}^{t_2} F_y dt + \vec{k} \int_{t_1}^{t_2} F_z dt$$

$$\therefore\ m\vec{v_1} + Imp_{1-2} = m\vec{v_2}$$

So, when a particle is acted upon by a force F during a given time interval, the final momentum $m\vec{v_2}$ of the particle can be obtained by adding vectorically its initial momentum and the impulse of the force $\vec{F}$ during the time interval considered.

When several forces acts on particle

$$m\vec{v_1} + \Sigma Imp_{1-2} = m\vec{v_2}$$

If problem involves two or more particles

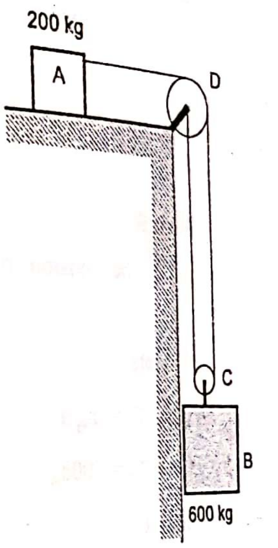
$$\Sigma m\vec{v_1} + \Sigma Imp_{1-2} = \Sigma m\vec{v_2}$$

If no external forces act

$$\Sigma m\vec{v_1} = \Sigma m\vec{v_2}$$

Worked Out Examples

1. The two blocks shown start from rest. The horizontal plane and the pulley are frictionless and the pulley is assumed to be of negligible mass. Determine the acceleration fo each block and the tension in each cord.



Solution:

In such type, we first find the relation between velocity and acceleration of each block and use Newton's second law.

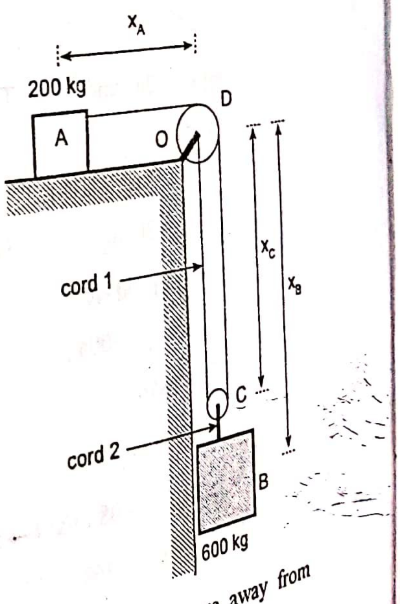
$x_A \rightarrow$ position of block A

$x_B \rightarrow$ poition of block B

$x_C \rightarrow$ position of block C

Here $x_C + k = x_B$

$\therefore x_B = x_C$



Here ADCO is a continuous cord.

$$AD + DC + CO = k$$

$$x_A + x_C + x_C = k$$

$$x_A + 2x_C = k$$

$$x_A + 2x_B = k$$

$$v_A = -2v_B \text{ and } a_A = -2a_B$$

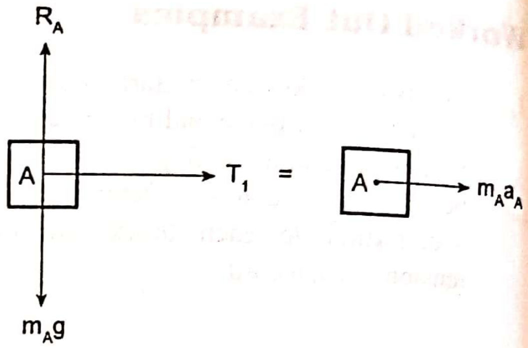
–ve sign shows that if block A moves towards reference pulley D, then block B will move away from reference pulley D and vice-versa.

Block A:

Let T_1 be tension on cord 1.

$\Sigma F_x = ma$

$T_1 = m_A.a_A = 200a_A$ ......(i)



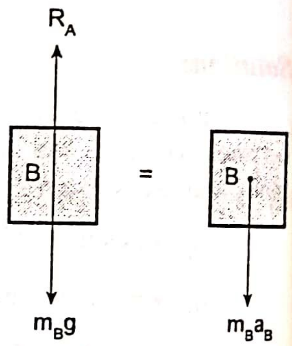
Block B

Let T_2 be tension on cord 2.

$\Sigma F_y = ma$

$m_B.g - T_2 = m_B.a_B$

$5886 - T_2 = 600a_B$ ......(ii)



Pulley C

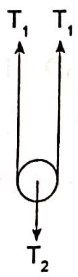
$T_1 + T_1 = T_2$

$2T_1 = -T_2$ ......(iii)

Using $a_A = 2a_B$ and $2T_1 = T_2$ in eq. (i)

or, $\dfrac{T_2}{2} = 200 \times 2a_B$

$\Rightarrow$ $T_2 = 800a_B$ ......(iv)



From eq. (ii) and (iv)

$5886 - 800a_B = 600a_B$

$1400a_B = 5886$

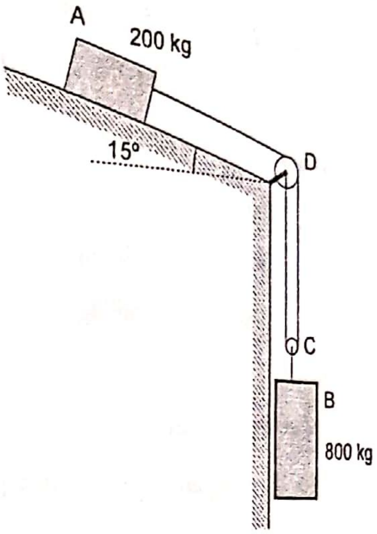
$\therefore$ $a_B = 4.204$ m/s² ($\downarrow$)

$\therefore$ $a_A = 2a_B = 8.408$ m/s² ($\rightarrow$)

$\therefore$ $T_2 = 800a_B = 3363.42$ N

$\therefore$ $T_1 = \dfrac{T_2}{2} = 1681.7$ N

2. Two blocks start from rest. The pulley are frictionless and having no mass. If u_k between block A and inclined plane is 0.4. Determine the acceleration of each block and tension in each cord.



Solution:

$x_A \rightarrow$ position of block A

$x_B \rightarrow$ position of block B

$x_C \rightarrow$ position of block C

$x_C + k = x_B$

$\therefore\ x_B = x_C$

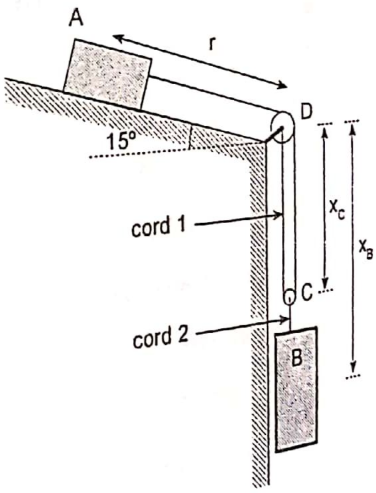
ADCO is a continuous cord.

$AD + DC + CO = k$

$x_A + 2x_C = k$

$x_A + 2x_B = k$

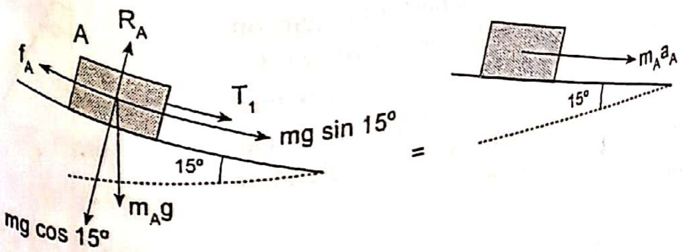
$v_A = -2v_B$ and $a_A = -2a_B$



Block A:

$\Sigma F_x = ma$

$$T_1 + m_A \cdot g \sin 15 - f_A = m_A a_A$$



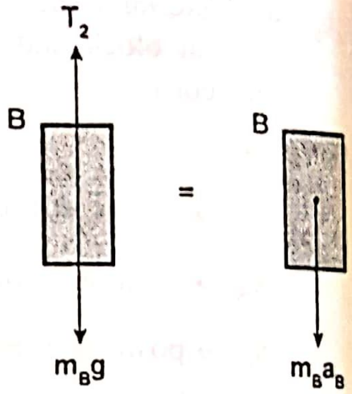
$$T_1 + 507.8 - \mu_k R_A = 200a_A$$

$$T_1 + 507.8 - 0.4\ m_A\ g \cos 15 = 200a_A$$

$T_1 + 507.8 - 758.05 = 200a_A$

$T_1 - 200a_A - 250.25 = 0$ ......(i)

Block B:



$\Sigma F_y = ma$

or, $m_B.g - T_2 = m_B.a_B$

or, $7848 - T_2 = 800a_B$ ......(ii)

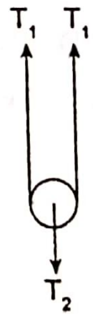
Pulley C

$T_2 = 2T_1$ ......(iii)

Use $a_A = 2a_B$ and $T_2 = 2T_1$ in eq. (i)

$\frac{T_2}{2} - 200(2a_B) - 250.25 = 0$

$T_2 - 800a_B - 500.5 = 0$ ......(iv)



From eq. (ii) and eq. (iv)

$T_2 - 800a_B - 500.5 = 0$

$-T_2 - 800a_B + 7848 = 0$

$-1600a_B = -7347.5$

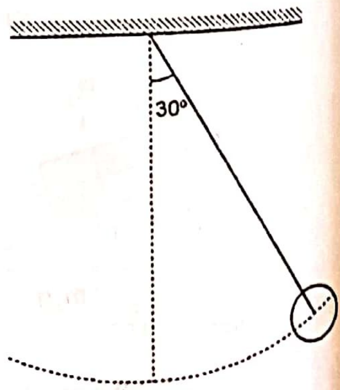
$\therefore\ a_B = 4.592\ m/s^2\ (\downarrow)$

$a_A = 9.184\ m/s^2$

$\therefore\ T_1 = 2087.125\ N$

$\therefore\ T_2 = 4174.25\ N$

3. The bob of 6 m pendulum describe an arc of a circle in a vertical plane. If the tension in the cord is 2 times the weight of bob for the position when the bob is displaced through an angle 30° from its mean position. Find the velocity and acceleration of bob for this situation.



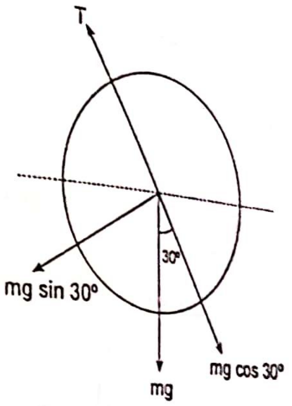
Solution:

Consider a tangential direction,

$mg \sin 30 = ma_t$

$g \sin 30 = a_t$

$\therefore a_t = 4.905 \text{ m/s}^2$



Normal component is directed towards centre

$T - mg \cos 30 = ma_n$

$2mg - mg \cos 30 = ma_n$

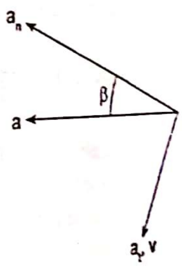
$2g - g \cos 30 = a_n$

$\therefore a_n = 11.12 \text{ m/s}^2$

$\therefore \vec{a} = \vec{a_t} + \vec{a_n}$

$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(4.905)^2 + (11.12)^2} = 12.15 \text{ m/s}^2$

$\tan \beta = \dfrac{a_t}{a_n} = \dfrac{4.905}{11.12}$



$\therefore \beta = 23.78°$

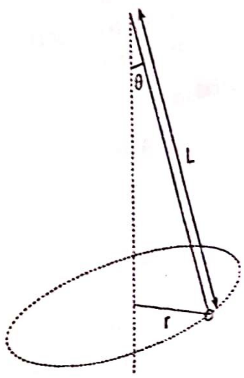
Again, $a_n = \dfrac{v^2}{\delta}$

$11.12 = \dfrac{v^2}{6}$

$v^2 = 66.745$

$\therefore v = 8.16$ (along a_t)

4. A 2 kg ball revolves in a horizontal circle as shown at a constant speed of 1.5 m/s knowing that L = 600 mm. Determine a) θ, b) tension.



Solution:

Radius $r = L \cos (90 - \theta)$

$= L \sin \theta$

$= 600 \sin \theta$

$\Sigma F_y = ma_y$

$T \cos \theta - W = 0$

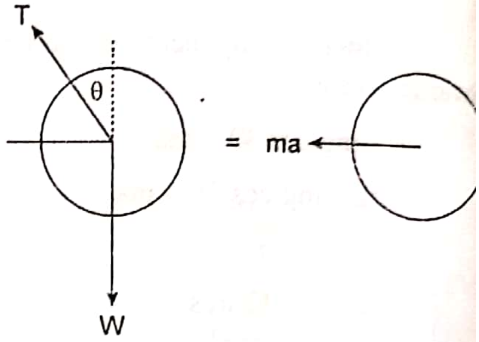
$T \cos \theta = 2g$ ......(i)

$\Sigma F_x = ma_x$

$T \sin \theta = ma$

$$\frac{2g}{\cos\theta}\sin\theta = 2a$$

$g\tan\theta = a$ ......(ii)



Also $a_n = \frac{v^2}{\delta}$

For uniform circular motion

$a_n = a$

$$\therefore\ g\tan\theta = \frac{v^2}{\left(\frac{600}{1000}\right)\sin\theta}$$

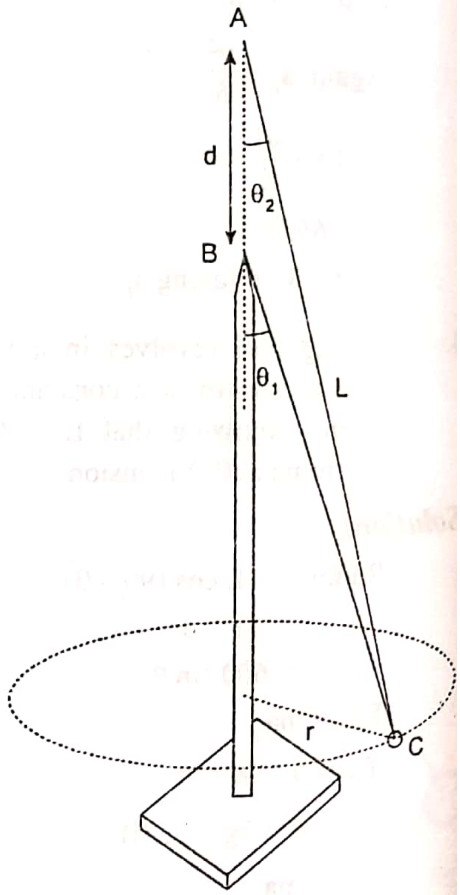
$$\tan\theta\sin\theta = \frac{(1.5)^2}{9.81\times 0.6}$$

$\tan\theta\sin\theta = 0.38226$

solving, $\theta = 34.21°$

$$\text{Tension } T = \frac{2g}{\cos\theta} = \frac{2\times 9.81}{\cos 34.21} = 23.72\text{ N}$$

5. A single wire ACB of length 2 m passes through a ring at C that is attached to a sphere which revolves at a constant speed v in the horizontal circle shown. Knowing that $\theta_1 = 60°$ and $\theta_2 = 30°$ and that the tension is the same in both portions of the wire, determine the speed v.



Solution:

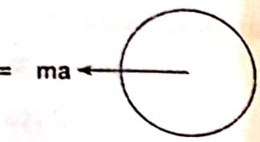
Let r be the radius of horizontal circle. The length of wire is

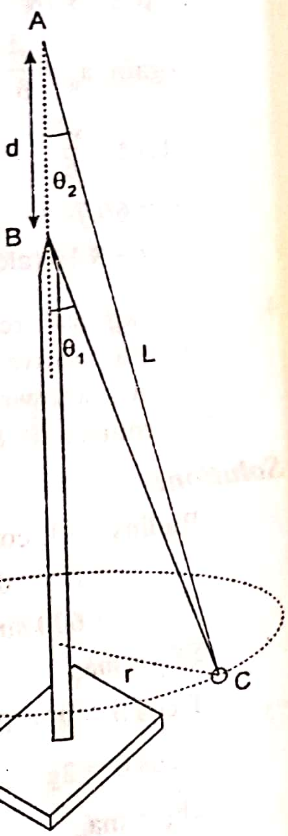
$$L = \frac{r}{\sin\theta_1} + \frac{r}{\sin\theta_2}$$

$$\therefore\ r = \frac{L\sin\theta_1\sin\theta_2}{\sin\theta_1 + \sin\theta_2}$$

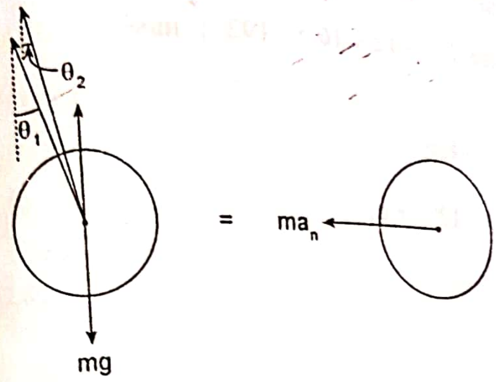
$\Sigma F_y = 0$

$$T\cos\theta_1 + T\cos\theta_2 - mg = 0$$





$$\therefore T = \frac{mg}{\cos\theta_1 + \cos\theta_2}$$



$$\Sigma F_x = ma$$

$$T\sin\theta_1 + T\sin\theta_2 = m\frac{v^2}{r}$$

$$mg\frac{\sin\theta_1 + \sin\theta_2}{\cos\theta_1 + \cos\theta_2} = mv^2\frac{\sin\theta_1 + \sin\theta_2}{2\sin\theta_1\sin\theta_2}$$

$$\frac{gL\sin\theta_1\sin\theta_2}{\cos\theta_1 + \cos\theta_2} = v^2$$

$$v^2 = 6.2193$$

$$\therefore v = 2.49 \text{ m/s}$$

6. The resultant external force acting on a 2 kg particle in space is $\vec{F} = (12t\,\vec{i} - 24t^2\,\vec{j} - 40t^3\,\vec{k})$ N. The particle is at origin and at rest initially. Determine y-component of acceleration, velocity and position at the instant of 4 sec.

Solution:

$$\vec{F} = 12t\,\vec{i} - 24t^2\,\vec{j} - 40t^3\,\vec{k} \text{ N}$$

$$\therefore F_x = 12t; \quad F_y = -24t^2; \quad F_z = -40t^3$$

Considering y-component

$$F_y = -24t^2$$

$$m\,a_y = -24t^2$$

$$a_y = -\frac{24}{2}t^2 = -12t^2$$

At t = 4 sec; $a_y = -12 \times 16 = -192\ \vec{j}\ m/s^2$

Again,

$$a_y = -12t^2$$

$$v_y = -12\frac{t^3}{3} + c_1$$

At t = 0; v = 0

$\therefore\ c_1 = 0$

$\therefore\ v_y = -4t^3$

At t = 4 sec; $v_y = -256\ \vec{j}\ m/s$

$v_y = -4t^3$

$$s_y = -4\frac{t^4}{4} + c_2 = -t^4 + c_2$$

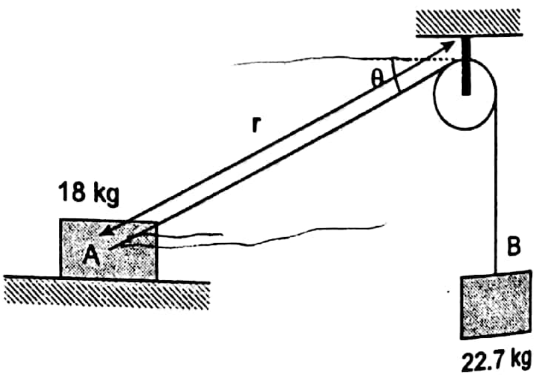
At t = 0, s = 0

$c_2 = 0$

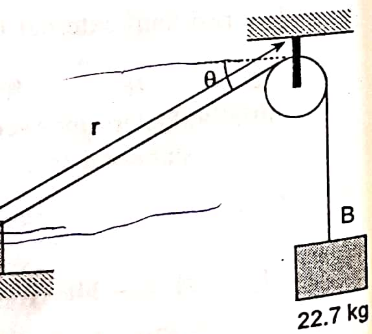
$s_y = -t^4$

At t = 4 sec; $s_y = -256\ \vec{j}\ m$

7. The two blocks are released from rest when r = 0.73 m and θ = 30°. Neglecting the mass of pulley and the effect of friction in the pulley and between block A and the horizontal surface. Determine



a) the initial tension in the cable.
b) the initial acceleration of block A
c) the initial acceleration of block B



Solution:

Let y_B be the position of block B from reference. Then for cable;

$$r + y_B = k$$

$$\dot{r} + v_B = 0$$

$$\ddot{r} + a_B = 0$$

$$\therefore \ddot{r} = -a_B$$

For block A

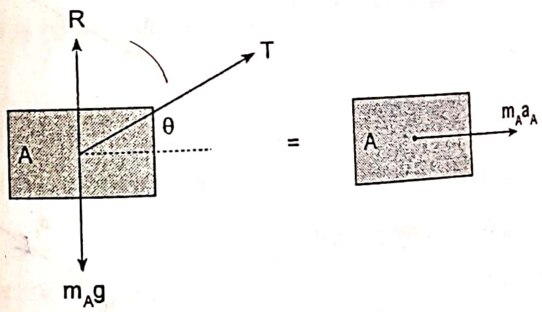
$$\Sigma F = ma$$

$$T\cos\theta = m_A . a_A$$

$$T = m_A . a_A \sec\theta$$

$$= 18\, a_A \sec 30$$

$$= 20.78\, a_A \quad \ldots\ldots(i)$$



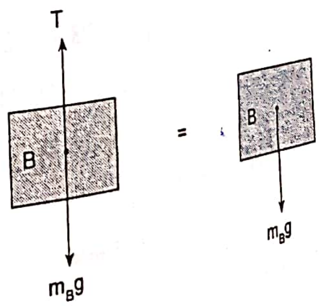
For block B

$$\Sigma F = ma$$

$$m_B.g - T = m_B.a_B$$

$$22.7 \times 9.81 - T = 22.7 \times a_B$$

$$222.68 - 20.78\, a_A = 22.7\, a_B \quad \text{(use eq. (i))}$$



$$\therefore 22.7\, a_B + 20.78\, a_A - 222.68 = 0 \quad \ldots\ldots(ii)$$

Now it is important to establish relation between a_A and a_B.

For initial case $a_B = a_A \cos\theta$

But for different case like if velocity of block A is given, the relation has to be calculated using radial and transverse component as below.

Proof for initial case

Positive direction for increase in radius and angle.

We know,

$$\cos\theta = \frac{-a_r}{a_A} \text{ [– sign w.r.t. A]}$$

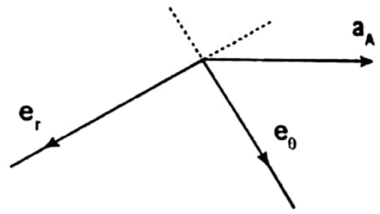
(a_A and a_r in same direction)

$\therefore\ a_r = -a_A \cos\theta$

Also,

$a_r = \ddot{r} - r\dot{\theta}^2$

initially $\dot{\theta} = 0$ [if velocity is given you have to calculate $\dot{\theta}$ using v_r, a_A and v_θ]

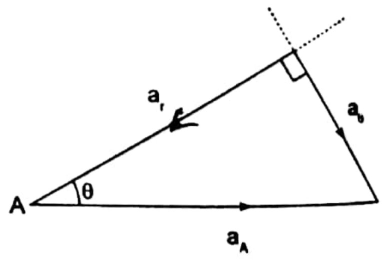


$\therefore\ a_r = \ddot{r}$

So, $\ddot{r} = -a_A \cos\theta$

$-a_B = -a_A \cos\theta$

$\therefore\ a_B = a_A \cos\theta$ hence proved.



From eq. (ii)

$$22.7\, a_B + 20.78\, a_A - 222.68 = 0$$

or, $22.7\, a_A \cos 30 + 20.78\, a_A - 222.68 = 0$

or, $19.658\, a_A + 20.78\, a_A - 222.68 = 0$

or, $40.438\, a_A = 222.68$

$\therefore$ $a_A = 5.5065\ m/s^2$

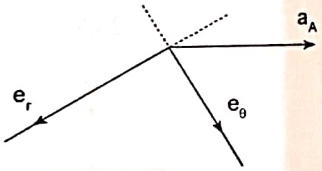
$\therefore$ $a_B = a_A \cos 30° = 4.768\ m/s^2$

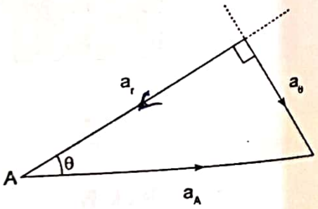
$\therefore$ $T = 20.78 a_A = 114.42\ N$

ty of block A is given, the relation
verse component as below.

dius and angle.

you





= 0

222.68 = 0

68 = 0

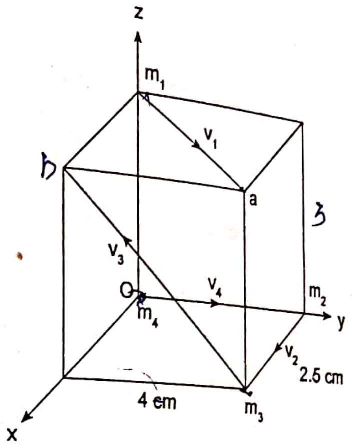
1) The system of particle at a time t is shown in the figure below

$v_1 = 7$ m/s $\quad m_1 = 0.5$ kg

$v_2 = 6$ m/s $\quad m_2 = 1.5$ kg

$v_3 = 5$ m/s $\quad m_3 = 1$ kg

$v_4 = 1.5$ m/s $\quad m_4 = 0.5$ kg



Determine:

a) total linear momentum of the system.

b) angular momentum of system about O.

c) angular momentum of system about a.

Solution:

Write the coordinate of all point and find the direction of v_1, v_2, v_3 and v_4.

$m_1(0, 0, 3)$ $\quad m_2(0, 4, 0)$ $\quad m_3(2.5, 4, 0)$ $\quad m_4(0, 0, 0)$

$O(0, 0, 0)$ $\quad a(2.5, 4, 3)$ $\quad b(2.5, 0, 3)$

$$\text{Unit vector along } v_1 = \hat{n}_1 = \frac{\overrightarrow{m_1 a}}{|\overrightarrow{m_1 a}|} = \frac{\vec{a} - \vec{m_1}}{|\overrightarrow{m_1 a}|}$$

$$= \frac{(2.5, 4, 3) - (0, 0, 3)}{|\overrightarrow{m_1 a}|}$$

$$= \frac{2.5\vec{i} + 4\vec{j}}{\sqrt{(2.5)^2 + 4^2}}$$

$$= \frac{2.5\vec{i} + 4\vec{j}}{4.71}$$

$$\hat{n}_1 = 0.529\vec{i} + 0.847\vec{j}$$

$$\therefore \vec{v_1} = \hat{n}_1 . v_1$$

$$= 7(0.529 \vec{i} + 0.847 \vec{j})$$

$$= 3.703 \vec{i} + 5.929 \vec{j}$$

Unit vector along $v_2 = \vec{i}$

$$\vec{v_2} = v_2 . \vec{i} = 6 \vec{i}$$

Unit vector along $v_3 = \hat{n}_3 = \dfrac{\vec{m_3 b}}{|\vec{m_3 b}|} = \dfrac{\vec{b} - \vec{m_3}}{|\vec{m_3 b}|}$

$$= \frac{(2.5, 0, 3) - (2.5, 4, 0)}{|\vec{m_3 b}|}$$

$$= \frac{-4 \vec{j} + 3 \vec{k}}{\sqrt{4^2 + 3^2}}$$

$$\hat{n}_3 = -0.8 \vec{j} + 0.6 \vec{k}$$

$$\therefore \vec{v_3} = v_3 . \hat{n}_3$$

$$= 5(-0.8 \vec{j} + 0.6 \vec{k})$$

$$= -4 \vec{j} + 3 \vec{k}$$

Unit vector along $v_4 = \vec{j}$

$$\vec{v_4} = v_4 . \vec{j} = 1.5 \vec{j}$$

Total linear momentum

$$\vec{L} = \Sigma m \vec{v}$$

$$= m_1 \vec{v_1} + m_2 \vec{v_2} + m_3 \vec{v_3} + m_4 \vec{v_4}$$

$$= 0.5(3.703 \vec{i} + 5.929 \vec{j}) + (1.5)\, 6 \vec{i} + 1.(-4 \vec{j} + 3 \vec{k})$$

$$+ (0.5).1.5 \vec{j}$$

$$= 10.85\,\vec{i} - 0.29\,\vec{j} + 3\,\vec{k} \text{ kg m/s}$$

$$|\vec{L}| = \sqrt{(10.85)^2 + (0.29)^2 + 3^2} = 11.2608 \text{ kg m/s}$$

Angular momentum about O

$$\vec{H}_O = \Sigma(\vec{r} \times m\vec{v})$$

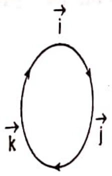
$$r_1 = r_{m1/O} = m_1 - O = (0, 0, 3) - (0, 0, 0) = 3\,\vec{k}$$

$$r_2 = r_{m2/O} = m_2 - O = (0, 4, 0) - (0, 0, 0) = 4\,\vec{j}$$

$$r_3 = r_{m3/O} = m_3 - O = (2.5, 4, 0) - (0, 0, 0) = 2.5\,\vec{i} + 4\,\vec{j}$$

$$r_4 = r_{m4/O} = m_4 - O = (0, 0, 0) - (0, 0, 0) = 0$$

$$\vec{H}_O = \Sigma\,\vec{r} \times m\,\vec{v}$$



$$= 3\,\vec{k} \times 0.5(3.703\,\vec{i} + 5.929\,\vec{j}\,)$$

$$+ 4\,\vec{j} \times (1.5)\,6\,\vec{i} + (2.5\,\vec{i} + 4\,\vec{j}\,)$$

$$\times 1.(-4\,\vec{j} + 3\,\vec{k}\,) + 0$$

$$= 5.55\,\vec{j} - 8.89\,\vec{i} - 36\,\vec{k} - 10\,\vec{k} - 7.5\,\vec{j} + 12\,\vec{i}$$

$$= 3.106\,\vec{i} - 1.95\,\vec{j} - 46\,\vec{k}$$

$$|\vec{H}_O| = \sqrt{(3.106)^2 + (1.95)^2 + (46)^2} = 46.146 \text{ kg m}^2/\text{s}^2$$

Angular momentum about a

$$\vec{H}_a = \Sigma\,\vec{r} \times m\,\vec{v}$$

$$r_1 = r_{m1/a} = m_1 - a = (0, 0, 3) - (2.5, 4, 3) = -2.5\,\vec{i} - 4\,\vec{j}$$

$$r_2 = r_{m2/a} = m_2 - a = (0, 4, 0) - (2.5, 4, 3) = -2.5\,\vec{i} - 3\,\vec{k}$$

$$r_3 = r_{m3/a} = m_3 - a = (2.5, 4, 0) - (2.5, 4, 3) = -3\,\vec{k}$$

$$r_4 = r_{m4/a} = m_4 - a = (0, 0, 0) - (2.5, 4, 3) = -2.5\vec{i} - 4\vec{j} - 3\vec{k}$$

$$\begin{aligned}
H_a &= \Sigma \vec{r} \times m\vec{v} \\
&= (-2.5\vec{i} - 4\vec{j}) \times (0.5)(3.703\vec{i} + 5.929\vec{j}) \\
&\quad + (-2.5\vec{j} - 3\vec{k}) \times (1.5) \,.\, \vec{i} + -3\vec{k} \times 1.(-4\vec{j} + 3\vec{k}) \\
&\quad + (-2.5\vec{i} - 4\vec{j} - 3\vec{k}) \times (0.5)(1.5\vec{j})
\end{aligned}$$

$$\vec{H}_a = -9.75\vec{i} - 27\vec{j} - 1.875\vec{k}$$

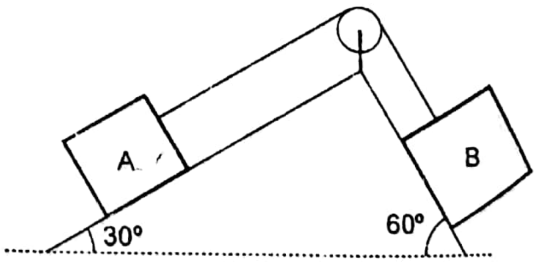
$$|\vec{H}_a| = \sqrt{(9.75)^2 + 27^2 + (1.875)^2} = 28.767 \text{ kg m}^2/\text{s}^2$$

Practice Question

1. A force given by $\vec{F} = 3t^2\vec{i} + 5t\vec{j} - (8t^3 + 400)\vec{k}$ acts from t = 0 to t = 10 sec. Determine the impulse of force.

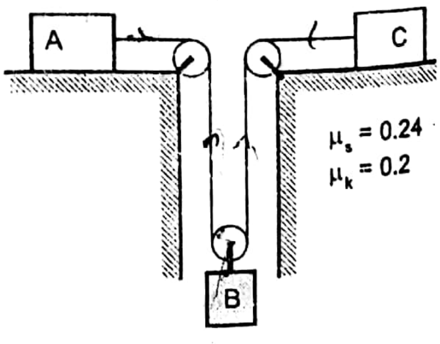
(Ans. $1000\vec{i} + 250\vec{j} - 24000\vec{k}$ NS)

2. Two rough plane inclined at 30° and 60° to the horizontal and same height are placed back to back. Mass of A is 12 kg and B is 24 kg and are connected by string. If μ = 0.6, find resulting acceleration.



(Ans. 0.3677 m/s²)

3. Three blocks A (m_A = 5 kg), B (m_B = 10 kg), C (m_C = 10 kg) are connected by rope and pulley arrangement as shown in figure. Neglecting mass of pulley, determine
   a) acceleration of each block
   b) tension in each cable.

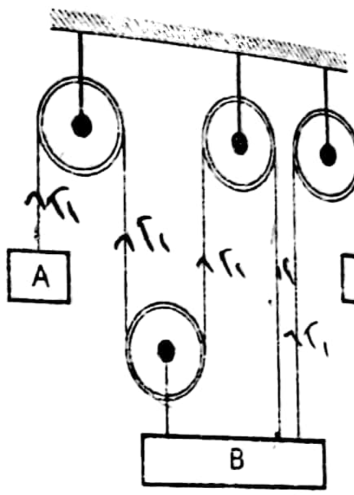


(Ans. $a_A = 4.76$, $a_B = 3.08$, $a_C = 1.401$ m/s²; $T_1 = 33.6$ N, $T_2 = 67.2$ N)

4. The system shown is initially at rest. Neglecting the masses of the pulleys and effect of friction in the pulley. $m_A = 9$ kg, $m_B = 27$ kg, $m_C = 9$ kg. Determine

a) acceleration of each block

b) tension in each cable.



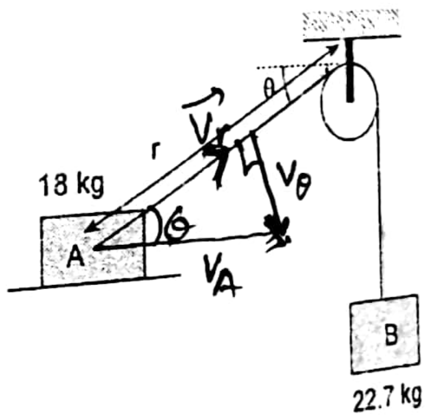
(Ans. $a_A = 2.25$ m/s^2, $a_B = 0.75$ m/s^2, $a_C = 0.75$ m/s^2, $T_1 = 67$ N, $T_2 = 81$ N)

5. The velocity of the block A is 1.8 m/s to the right at the instant when r = 0.73 and $\theta = 30°$. Neglecting the mass of pulley and effect of friction, determine at this instant

a) tension in the cable

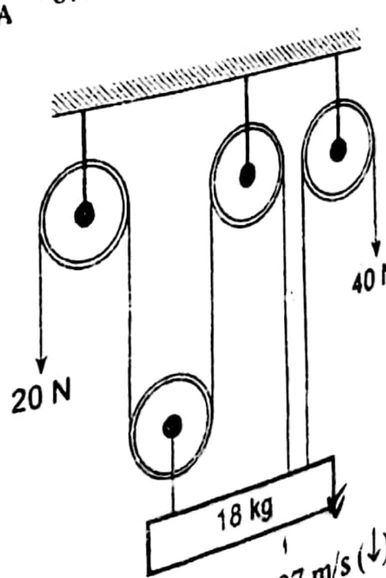
b) acceleration of block A

c) acceleration of block B



(Ans. T = 127.37 N, $a_A = 6.128$ m/s^2, $a_B = 4.199$ m/s^2)

6. The 18 kg block starts from rest and moves upward when constant forces of 20 N and 40 N are applied to supporting ropes. Neglecting the masses of the pulley and the affect of friction, determine the speed of the block after it has moved 0.457 m.

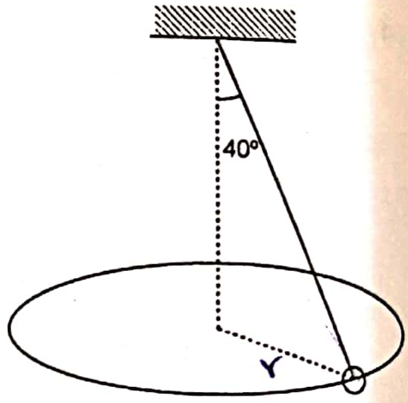


(Ans. 1.97 m/s (↓))

7. A small bob of mass 5 kg is attached to a string of length 2 m as shown. It is allowed to revolve in a horizontal circle at constant speed of v_0. Determine

a) tension in string

b) speed v_0 of the bob



(Ans. a) 64.09 N, b) 3.255 m/s)