Project 2: Chaotic Pendulum

Ordinary differential equations (ODE) are used in many fields, such as engineering and biology, and they describe how variable changes with respect to another variable, like position vs. velocity. When dealing with ODEs, they can be tricky to solve, especially when the solutions are not feasible, so they can be solved numerically using the Runge-Kutta 4th (RK4) order method.

The RK4 method is a numerical method used to solve ODEs. It calculates four intermediate estimates of the slope of each ODE at different points and they are obtained by evaluating the ODEs at various points using the current solution and the step size. The RK4 method then combines these intermediate estimates in a weighted manner to update the solution at each step. The final estimates of the solution are obtained by taking a weighted average of these intermediate estimates, and the process is repeated until the desired accuracy or number of steps is achieved. This method is widely used due to its simplicity and accuracy. For example, using RK4, the ODE for a pendulum can be solved using the following equations.

$$\frac{d^2\theta}{dt^2} + \sin(\theta) = 0 \quad (1)$$

$$\frac{d^2\theta}{dt^2} + \theta = 0 \quad (2)$$

$$\frac{d^2\theta}{dt^2} + \left(\theta - \frac{\theta^3}{6}\right) = 0 \quad (3)$$

The parameters a and b represent the intervals, while alpha1 and alpha2 are the initial conditions for position and velocity. RK4 returns a 2D array with the first column representing points for theta (θ) and the second column representing points for theta prime (θ'), which include the initial conditions. These columns can be used to create a phase portrait graph, also known as a position vs. velocity graph, that visually represents the behavior and dynamics of the system, and it will give information about the system's stability, periodicity, or chaos solutions. This is shown in Figure 1 below.

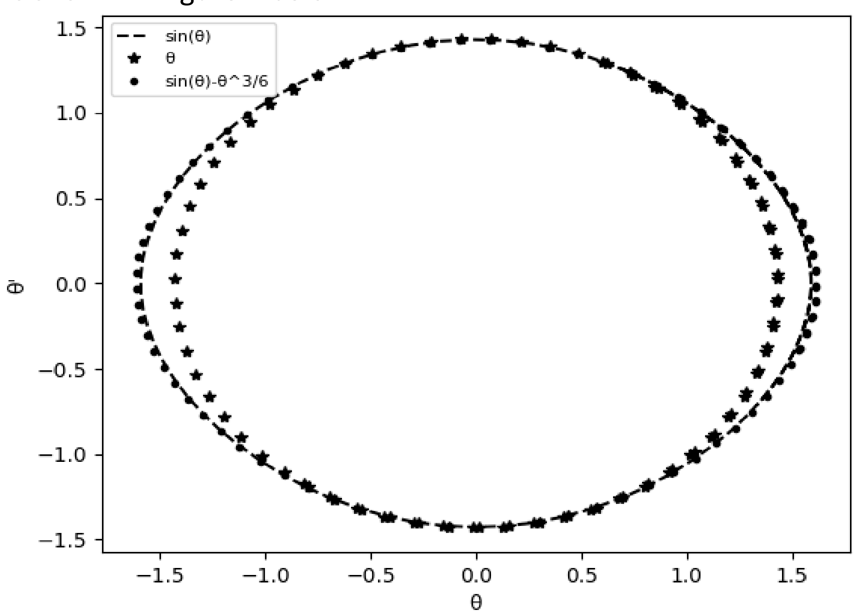

Figure 1: Parameters to make the graph are: a=0, b=10, alpha1=0.6, and alpha2=1.3

With the initial conditions in Figure 1, Equation 2 and especially Equation 3 are relatively close to Equation 1. However, when alpha2 is set to 0 and alpha1 is set to some number, all three equations drastically have a reduced error, as shown in Figure 2 below. The closer alpha1 is to 0, the more accurate the equations are.

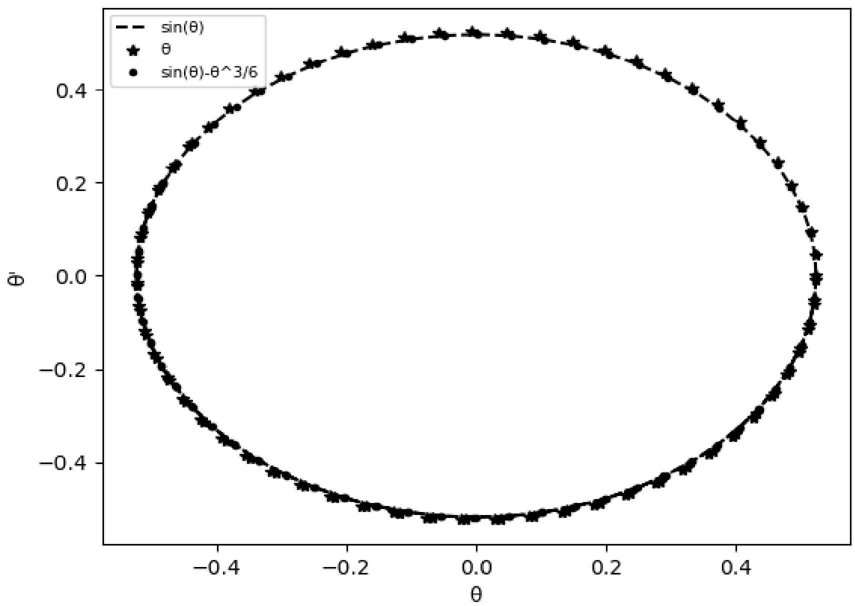

Figure 2: The initial conditions for this graph are: alpha1= $\frac{\pi}{6}$ and alpha2=0

Setting alpha1 and alpha2 to 0 will yield a single point in the graph. A periodic motion or a limit cycle is not expected when looking at a damped pendulum in comparison to Figure 1 and 2. Instead, erratic behavior is observed. This is mainly due to the sensitivity of the initial conditions.

$$\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \sin(\theta) = Fcos(t) \tag{4}$$

Using equation 4, the damped pendulum can be graphed using the RK4 method, as shown in Figure 3 below. The figure was created by plotting time against θ to show the erratic behavior of the time series. Because of varying amplitudes, the motion of the pendulum is unpredictable, because the angular velocity is different every time.

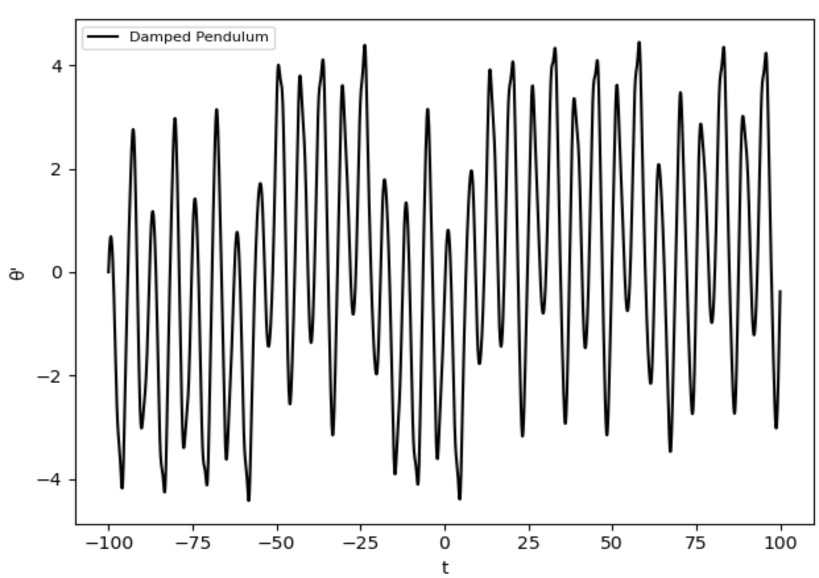

Figure 3: The unpredictable function is created using the following parameters: b=0.22, F=2.7, alpha1= $\frac{\pi}{6}$, and alpha2=0.

Project 2: Chaotic Pendulum

After analyzing the behavior effect of the damped pendulum, the Poincare section can help give a deeper understanding of its dynamics. The Poincare section samples data at a given point to understand a system's behavior. The Poincare section helps simplify the system's dynamics and provides insights into its stability and periodicity. Regarding the damped pendulum, the Poincare section helps visualize how the pendulum moves and what gives its chaotic behavior, shown in Figure 4 below.

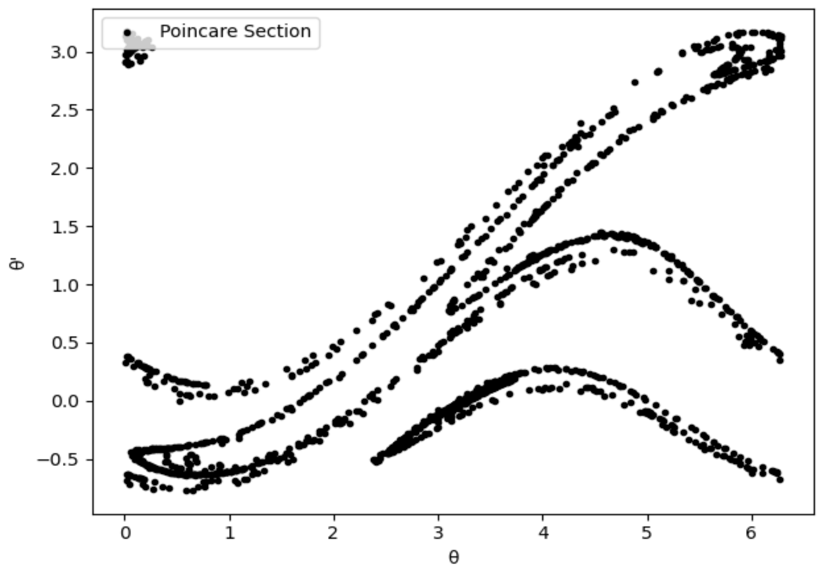

Figure 4: The Poincare section that is sampled from the RK4 function at every 2000 points from 0 to 1000, where θ%2π is plotted against θ'.

In conclusion, RK4 is used to help solve complex ODEs by calculating intermediate estimates at each step, which are obtained by calculating ODEs at numerous points. These estimates are used to update the final solution, and RK4 ensures reliability and efficiency for simulating ODEs. The Poincare section gives a more visual representation of how a system behaves and its long-term effects by taking data from RK4 and sampling them at specific points.