Home work assignment-1

Problem - 1

Given: $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

$$X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1K} \\ 1 & x_{21} & \cdots & x_{2K} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n1} & \cdots & x_{nK} \end{pmatrix}$$

$$\beta = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_K \end{pmatrix}$$

sum of squared errors $= \epsilon . \epsilon^T = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} . \begin{pmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_n \end{pmatrix}$

$=$ Scaler number

Multiple linear regression model in matrix form:

$$Y = X\beta + \epsilon$$

(i) Confirming dimension of matrices both sides

$$Y_{dimension} = (n \times 1)$$

dimension of $X\beta + \epsilon$

$$\Rightarrow (n \times (K+1))((K+1)\times 1) + (n \times 1)$$

$$\Rightarrow (n \times 1) + (n \times 1)$$

$$\Rightarrow (n \times 1)$$

Therefore dimension of both sides are same i.e $= (n \times 1)$.

(ii) Finding sum of squared residuals (or MSE) in matrix form.

We have, $Y = X\beta + \epsilon$

$$\epsilon = Y - X\beta \qquad \text{—— (1)}$$

$$\epsilon = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n1} & \cdots & x_{nk} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_k \end{bmatrix}$$

i^{th} element of ϵ can be shown as

$$\epsilon_i = y_i - \sum_{j=0}^{k} x_{ij} \beta_j$$

Sum of squared residuals (SSR)

$$SSR = \sum_{i=0}^{k} \epsilon_i^2$$

$$SSR = \left[e_1 \times e_1 + e_2 \times e_2 + \cdots + e_k \times e_k \right]$$

$$SSR = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_k \end{bmatrix} \cdot \begin{bmatrix} e_1 & e_2 & \cdots & e_k \end{bmatrix}$$

$$SSR = \epsilon . \epsilon^T \qquad \text{———} \qquad (2)$$

substituting eqn (1) to eqn (2)

$$\boxed{SSR = (Y - X\beta)(Y - X\beta)^T}$$

(iii) Deriving $X^T X \beta^* = X^T Y$

$$SSR = \epsilon^T \epsilon = (Y - X\beta)^T (Y - X\beta)$$

$$\frac{\partial(SSR)}{\partial\beta} = \frac{\partial}{\partial\beta}(Y - X\beta)^T (Y - X\beta)$$

$$\frac{\partial(SSR)}{\partial\beta} = -2X^T(Y - X\beta^*)$$

equating to zero to get β^*

$$\Rightarrow \quad -2X^T(Y - X\beta^*) = 0$$

$$\Rightarrow \quad \boxed{X^T X \beta^* = X^T Y} \quad \text{—— (3)}$$

Hence verified

(iv) Deriving $\beta^* = (X^T X)^{-1}(X^T Y)$

from eq^n (3) we have

$$X^T X \beta^* = X^T Y$$

multiply both sides By $(X^T X)^{-1}$

$$\Rightarrow \quad (X^T X)^{-1} X^T X \beta^* = \cancel{X^T} (X^T X)^{-1} X^T Y$$

Since $A^{-1} A = I$ (identity matrix)

$$\Rightarrow \quad I \beta^* = (X^T X)^{-1} X^T Y$$

$$\Rightarrow \quad \boxed{\beta^* = (X^T X)^{-1} (X^T Y)}$$

Hence derived