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1 Introduction

In this report, we are asked to price a bond contract in which the holder has the option to choose between receiving the principle F or alternatively R underlying stocks with price S at time $t = T$.

W calculate the value of this option using the finite-difference method with a Crank-Nicolson Scheme. We then explore the effects of diting β and σ. Finally we explore the effect of i_{max},j_{max} and S_{max} on the solution.

we then move on to valuing American options, and look at the effect of changing r on the American option value.

2 European Option

The market value of the European option with a continuous coupon is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 V}{\partial S^2} + \kappa(\theta(t) - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0 \tag{1}$$

where $\theta(t)$ is

$$\theta(t) = (1+\mu)Xe^{\mu t}. \tag{2}$$

The boundary condition at $S = 0$ is:

$$\frac{\partial V}{\partial t} + \kappa\theta(t)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0. \tag{3}$$

2.1 Large S limit

At very large S, the PDE simplifies slightly to

$$\frac{\partial V}{\partial t} + \kappa(X - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0. \tag{4}$$

In the case of large S, we can assume the solution to the equation is of the form

$$V(S,t) = SA(t) + B(t). \tag{5}$$

By substituting this in to equation 4, we can extract the form of A and B. Through this, we calculated

$$A(t) = Re^{(\kappa+r)(t-T)} \tag{6}$$

$$B(t) = XRe^{(k+r)(t-T)} - \frac{C}{\alpha+r}e^{-\alpha t} + \frac{C}{\alpha+r}e^{-(\alpha+r)T} - XRe^{-rT} \tag{7}$$

2.2 S=0 Numerical Scheme

We can then use the finite difference method to estimate the partial derivatives, and allow us to use a matrix method to calculate them.

The partial derivatives for S = 0 are calculated as follows:

$$\frac{\partial V}{\partial t} = \frac{V_j^{i+1} - V_j^i}{\Delta t} \tag{8}$$

$$\frac{\partial V}{\partial S} = \frac{V_{j+1}^i - V_j^i}{\Delta S} \tag{9}$$

$$V = \frac{1}{2}(V_j^{i+1} + V_j^i). \tag{10}$$

When these are substituted into the PDE at S=0, we get

$$(-\frac{1}{\Delta t} - \frac{\kappa\theta(t)}{\Delta S} - \frac{r}{2})V_j^i + \frac{\kappa\theta(t)}{\Delta S}V_{j+1}^i = -(\frac{1}{\Delta t} - \frac{r}{2})V_j^{i+1} - Ce^{-\alpha t} \tag{11}$$

Therefore, the numerical scheme is:

$$a_0 = 0 \tag{12}$$

$$b_0 = -\frac{1}{\Delta t} - \frac{\kappa\theta(t)}{\Delta S} - \frac{r}{2} \tag{13}$$

$$c_0 = \frac{\kappa\theta(t)}{\Delta S} \tag{14}$$

$$d_0 = -(\frac{1}{\Delta t} - \frac{r}{2})V_j^{i+1} - Ce^{-\alpha t}. \tag{15}$$

2.3 Intermediate points numerical scheme

For the intermediate points, we can use a different set of approximations for the partial derivatives.

$$\frac{\partial V}{\partial t} = \frac{V_j^{i+1} - V_j^i}{\Delta t} \tag{16}$$

$$\frac{\partial V}{\partial S} = \frac{1}{4\Delta S}(V_{j+1}^i - V_{j-1}^i + V_{j+1}^{i+1} - V_{j-1}^{i+1}) \tag{17}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{2\Delta S^2}(V_{j+1}^i - 2V_j^i + V_{j-1}^i + V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}) \tag{18}$$

$$V = \frac{1}{2}(V_j^{i+1} + V_j^i). \tag{19}$$

Using these estimates and putting them into the PDE, and then rearranging, we get

$$\begin{aligned}
&(\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V_{j-1}^i + \\
&\qquad (\frac{-1}{\Delta t} - \frac{1}{\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2})V_j^i + \\
&\qquad (\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V_{j+1}^i = \\
&\qquad -(\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V_{j-1}^{i+1} - \\
&\qquad (\frac{-1}{2\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2} + \frac{1}{\Delta t})V_j^{i+1} - \\
&\qquad (\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V_{j+1}^{i+1} \\
&\qquad - Ce^{-\alpha t}.
\end{aligned} \tag{20}$$

We can again extract the numerical scheme from this

$$a_j = \frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S) \tag{21}$$

$$b_j = \frac{-1}{\Delta t} - \frac{1}{\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2} \tag{22}$$

$$c_j = \frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S) \tag{23}$$

$$
\begin{aligned}
d_j = -(\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V_{j-1}^{i+1} - \\
(\frac{-1}{2\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2} + \frac{1}{\Delta t})V_j^{i+1} - \\
(\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V_{j+1}^{i+1} \\
-Ce^{-\alpha t}.
\end{aligned}
\tag{24}
$$

2.4 Upper bound numerical scheme

For the upper bound $S = S_{max}$ we have found the analytical solution and from this we can calculate the upper bound.

$$a_{j_{max}} = 0 \tag{25}$$

$$b_{j_{max}} = 1 \tag{26}$$

$$c_{j_{max}} = 0 \tag{27}$$

$$d_j = A(t)S + B(t). \tag{28}$$

3 Appendix