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1 Introduction

In this report, we are asked to price a bond contract in which the holder has the option to choose between receiving the principle F or alternatively R underlying stocks with price S at time $t = T$.

W calculate the value of this option using the finite-difference method with a Crank-Nicolson Scheme. We then explore the effects of diting β and σ. Finally we explore the effect of i_{max},j_{max} and S_{max} on the solution.

we then move on to valuing American options, and look at the effect of changing r on the American option value.

2 European Option

The market value of the European option with a continuous coupon is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 V}{\partial S^2} + \kappa(\theta(t) - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0 \tag{1}$$

where $\theta(t)$ is

$$\theta(t) = (1+\mu)Xe^{\mu t}. \tag{2}$$

2.1 S=0 Numerical Scheme

The boundary condition at $S = 0$ is:

$$\frac{\partial V}{\partial t} + \kappa\theta(t)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0. \tag{3}$$

We can then use the finite difference method to estimate the partial derivatives, and allow us to use a matrix method to calculate them.

The partial derivatives for S = 0 are calculated as follows:

$$\frac{\partial V}{\partial t} = \frac{V_j^{i+1} - V_j^i}{\Delta t} \tag{4}$$

$$\frac{\partial V}{\partial S} = \frac{V_{j+1}^{i} - V_j^i}{\Delta S} \tag{5}$$

$$V = \frac{1}{2}(V_j^{i+1} + V_j^i). \tag{6}$$

When these are substituted into the PDE at S=0, we get

$$(-\frac{1}{\Delta t} - \frac{\kappa\theta(t)}{\Delta S} - \frac{r}{2})V_j^i + \frac{\kappa\theta(t)}{\Delta S}V_{j+1}^i = -(\frac{1}{\Delta t} - \frac{r}{2})V_j^{i+1} - Ce^{-\alpha t} \tag{7}$$

Therefore, the numerical scheme is:

$$a_0 = 0 \tag{8}$$

$$b_0 = -\frac{1}{\Delta t} - \frac{\kappa\theta(t)}{\Delta S} - \frac{r}{2} \tag{9}$$

$$c_0 = \frac{\kappa\theta(t)}{\Delta S} \tag{10}$$

$$d_0 = -(\frac{1}{\Delta t} - \frac{r}{2})V_j^{i+1} - Ce^{-\alpha t}. \tag{11}$$

2.2 Intermediate points numerical scheme

For the intermediate points, we can use a different set of approximations for the partial derivatives.

$$\frac{\partial V}{\partial t} = \frac{V_j^{i+1} - V_j^i}{\Delta t} \tag{12}$$

$$\frac{\partial V}{\partial S} = \frac{1}{4\Delta S}(V_{j+1}^i - V_{j-1}^i + V_{j+1}^{i+1} - V_{j-1}^{i+1}) \tag{13}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{2\Delta S^2}(V_{j+1}^i - 2V_j^i + V_{j-1}^i + V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}) \tag{14}$$

$$V = \frac{1}{2}(V_j^{i+1} + V_j^i). \tag{15}$$

Using these estimates and putting them into the PDE, and then rearranging, we get

$$
\begin{aligned}
&(\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V^i_{j-1} + \\
&\qquad (\frac{-1}{\Delta t} - \frac{1}{\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2})V^i_j + \\
&\qquad (\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V^i_{j+1} = \\
&\qquad -(\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V^{i+1}_{j-1} - \\
&\qquad (\frac{-1}{2\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2} + \frac{1}{\Delta t})V^{i+1}_j - \\
&\qquad (\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V^{i+1}_{j+1} \\
&\qquad - Ce^{-\alpha t}. \qquad (16)
\end{aligned}
$$

We can again extract the numerical scheme from this

$$a_j = \frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S) \qquad (17)$$

$$b_j = \frac{-1}{\Delta t} - \frac{1}{2\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2} \qquad (18)$$

$$c_j = \frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S) \qquad (19)$$

$$
\begin{aligned}
d_j = &-(\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V^{i+1}_{j-1} - \\
&\qquad (\frac{-1}{2\Delta S^2}\sigma^2(j\Delta S)^{2\beta} - \frac{r}{2} + \frac{1}{\Delta t})V^{i+1}_j - \\
&\qquad (\frac{1}{4\Delta S^2}\sigma^2(j\Delta S)^{2\beta} + \frac{1}{4\Delta S}\kappa(\theta(t) - j\Delta S))V^{i+1}_{j+1} \\
&\qquad - Ce^{-\alpha t}. \qquad (20)
\end{aligned}
$$

2.3 Large S limit

At very large S, the PDE simplifies slightly to

$$\frac{\partial V}{\partial t} + \kappa(X - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0. \qquad (21)$$

In the case of large S, we can assume the solution to the equation is of the form

$$V(S,t) = SA(t) + B(t). \qquad (22)$$

By substituting this in to equation 4, we can extract the form of A and B. Through this, we calculated

$$A(t) = Re^{(\kappa+r)(t-T)} \tag{23}$$

$$B(t) = -XRe^{(k+r)(t-T)} + \frac{C}{\alpha+r}e^{-\alpha t} + \frac{C}{\alpha+r}e^{-(\alpha+r)T+rt} + XRe^{r(t-T)} \tag{24}$$

For the upper bound $S = S_{max}$ we have found the analytical solution and from this we can calculate the upper bound.

$$a_{j_{max}} = 0 \tag{25}$$
$$b_{j_{max}} = 1 \tag{26}$$
$$c_{j_{max}} = 0 \tag{27}$$
$$d_j = A(t)S + B(t). \tag{28}$$

2.4 Option value as a function of underlying asset price for two cases

We explore the effect of varying the underlying asset price for different β and σ.

We can resolve the convertible bond into three parts. The coupon payment, modelled in continuous time, the bond part and the stock part. Ignoring the coupon payment temporarily, the final payment can be written as

$$V(S,T) = max(F,RS). \tag{29}$$

This can be rewritten by splitting it into a bond and call option.

$$V(S,T) = N + Rmax(0, S - C_p). \tag{30}$$

where the strike of each call option is F/R.

A graph for convertible bond price against share price would show a linear behaviour for high share price levels. This is because it becomes much more likely the holder will want the shares, and therefore behaves more like a call option.

At low share prices, the holder of the bond will likely not convert it to stock, and therefore it acts like a simple bond. At low share prices, the value of the convertible bond approaches the bond floor, which is the sum of the discounted cash flows distributed by the bond.

β is known as the elasticity of variance in the market. A $\beta < 1$ means the asset price has a variance which increases as S decreases. This allows us to model shares where the volatility of the price increases as the price moves down.

A $\beta = 1$ reverts the theory to standard geometric motion, whilst a $\beta = 0.5$ is included in the Cox-Ingersoll-Ross model. This allows us to include a mean -reversion property into the stock price.

3 Appendix