# Examples for Week 01 (1)

## Functions of One Random Variable

Example 1:

| $x$ | $p_X(x)$ |
|---|---|
| 1 | 0.2 |
| 2 | 0.4 |
| 3 | 0.3 |
| 4 | 0.1 |

$\mathrm{Y} = \mathrm{X}^2$

| $y = x^2$ | $p_Y(y) = p_X(\sqrt{y})$ |
|---|---|
| 1 | 0.2 |
| 4 | 0.4 |
| 9 | 0.3 |
| 16 | 0.1 |

Example 2:

| $x$ | $p_X(x)$ |
|---|---|
| $-2$ | 0.2 |
| 0 | 0.4 |
| 2 | 0.3 |
| 3 | 0.1 |

$\mathrm{Y} = \mathrm{X}^2$

| $y$ | $p_Y(y)$ |
|---|---|
| 0 | $p_X(0) = 0.4$ |
| 4 | $p_X(-2) + p_X(2) = 0.5$ |
| 9 | $p_X(3) = 0.1$ |

Example 3:

X ~ Poisson( $\lambda$ ): $$p_X(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \quad x = 0, 1, 2, 3, 4, 5, 6, \ldots .$$

$\mathrm{Y} = \mathrm{X}^2 \quad \Rightarrow$ $$p_Y(y) = \frac{\lambda^{\sqrt{y}} \cdot e^{-\lambda}}{(\sqrt{y})!}, \quad y = 0, 1, 4, 9, 16, 25, 36, \ldots .$$

Let X be a continuous random variable.

Let $Y = g(X)$. What is the probability distribution of Y?

Cumulative Distribution Function approach:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = \int_{\{x:\, g(x) \le y\}} f_X(x)\,dx = \ldots$$

Moment-Generating Function approach:

$$M_Y(t) = E(e^{Y \cdot t}) = E(e^{g(X) \cdot t}) = \int_{-\infty}^{\infty} e^{g(x) \cdot t} f_X(x)\,dx = \ldots$$

**1.** Let U be a Uniform(0, 1) random variable:

$$f_U(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_U(u) = \begin{cases} 0 & u < 0 \\ u & 0 \le u < 1 \\ 1 & u \ge 1 \end{cases}$$

Consider $Y = U^2$. What is the probability distribution of Y?

$0 < u < 1 \qquad Y = U^2 \qquad \Rightarrow \qquad 0 < y < 1.$

$$F_Y(y) = P(Y \le y) = P(U^2 \le y) = \ldots$$

$y < 0 \qquad P(U^2 \le y) = 0 \qquad F_Y(y) = 0.$

$y \ge 1 \qquad P(U^2 \le y) = 1 \qquad F_Y(y) = 1.$

$0 \le y < 1 \qquad P(U^2 \le y) = P(U \le \sqrt{y}\,) = F_U(\sqrt{y}\,) = \sqrt{y}.$

OR $\qquad P(U^2 \le y) = P(U \le \sqrt{y}\,) = \int_0^{\sqrt{y}} 1\,du = \sqrt{y}.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$

$$f_Y(y) = F_Y'(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

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Change-of-Variable Technique:

**Theorem 1.7.1** X – continuous r.v. with p.d.f. $f_X(x)$.

$Y = g(X)$ $\qquad$ $g(x)$ – one-to-one, differentiable

$$\frac{dx}{dy} = \frac{d}{dy}\, g^{-1}(y)$$

$$\Rightarrow \qquad f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

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**1.** Let U be a Uniform(0, 1) random variable.
Consider $Y = U^2$. What is the probability distribution of Y?

$f_U(u) = 1, \qquad 0 < u < 1.$

$g(u) = u^2 \qquad g^{-1}(y) = \sqrt{y} = y^{1/2} \qquad \dfrac{du}{dy} = \dfrac{1}{2}\, y^{-1/2}$

$$f_Y(y) = f_U(g^{-1}(y)) \left| \frac{du}{dy} \right| = (1) \left| \frac{1}{2}\, y^{-1/2} \right| = \frac{1}{2}\, y^{-1/2}, \qquad 0 < y < 1.$$

**1.5.** Consider a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} 6x^5 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the probability distribution of $Y = {}^1\!/_{X^2}$.

Support of X $= \{\, 0 < x < 1 \,\}$

$Y = {}^1\!/_{X^2} \quad \Rightarrow \quad$ Support of Y $= \{\, y > 1 \,\}$

$$g(x) = {}^1\!/_{x^2} \qquad\qquad g^{-1}(y) = \frac{1}{\sqrt{y}} = y^{-1/2} \qquad\qquad \frac{dx}{dy} = -\frac{1}{2}\,y^{-3/2}$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = (6y^{-5/2})(\tfrac{1}{2}\,y^{-3/2}) = 3y^{-4} \qquad y > 1.$$

OR

$$f_X(x) = \begin{cases} 6x^5 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \qquad\qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x^6 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$F_Y(y) = P(Y \le y) = P({}^1\!/_{X^2} \le y) = P(X \ge {}^1\!/_{\sqrt{y}}) = 1 - F_X({}^1\!/_{\sqrt{y}})$$

$$= 1 - y^{-3}, \qquad\qquad y > 1.$$

$$f_Y(y) = F'_Y(y) = 3y^{-4}, \qquad\qquad y > 1.$$

**2.** Consider a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

a) Find the probability distribution of $Y = \sqrt{X}$.

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$0 < x < 1 \qquad \Rightarrow \qquad 0 < y < 1$$

$y < 0$ $\quad F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = 0.$

$y \ge 0$ $\quad F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = F_X(y^2).$

$0 \le y < 1$ $\quad F_Y(y) = F_X(y^2) = y^4.$

$y \ge 1$ $\quad F_Y(y) = F_X(y^2) = 1.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^4 & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases} \qquad f_Y(y) = F_Y'(y) = \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

OR

$$g(x) = \sqrt{x} \qquad g^{-1}(y) = y^2 \qquad \frac{dx}{dy} = 2y$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (2y^2)(2y) = 4y^3, \qquad 0 < y < 1.$$

b) Find the probability distribution of $W = \frac{1}{X+1}$.

$$0 < x < 1 \qquad \Rightarrow \qquad \frac{1}{2} < w < 1$$

$$F_W(w) = P(W \le w) = P\left(\frac{1}{X+1} \le w\right) = P\left(X \ge \frac{1}{w} - 1\right) = 1 - F_X\left(\frac{1}{w} - 1\right)$$

$$= 1 - \left(\frac{1}{w} - 1\right)^2 = \frac{2}{w} - \frac{1}{w^2}, \qquad \frac{1}{2} < w < 1.$$

$$f_W(w) = F'_W(w) = -\frac{2}{w^2} + \frac{2}{w^3} = \frac{2 - 2w}{w^3}, \qquad \frac{1}{2} < w < 1.$$

OR

$$g(x) = \frac{1}{x+1} \qquad g^{-1}(w) = \frac{1}{w} - 1 \qquad \frac{dx}{dw} = -\frac{1}{w^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \left[2\left(\frac{1}{w} - 1\right)\right]\left(\frac{1}{w^2}\right) = \frac{2 - 2w}{w^3}, \qquad \frac{1}{2} < w < 1.$$

**3.** Consider a continuous random variable X with the p.d.f. $f_X(x) = \frac{24}{x^4}$, $x > 2$.

a) Let $Y = 1/X$. Find the p.d.f. of Y, $f_Y(y)$.

Support of X = $\{ x > 2 \}$

$Y = 1/X \quad \Rightarrow \quad$ Support of Y = $\{ 0 < y < 1/2 \}$

$$g(x) = 1/x \qquad g^{-1}(y) = 1/y \qquad \frac{dx}{dy} = -1/y^2$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (24y^4)(y^{-2}) = 24y^2, \quad 0 < y < 1/2.$$

OR

$$F_X(x) = 1 - \frac{8}{x^3}, \qquad x > 2.$$

$$F_Y(y) = P(Y \le y) = P(1/X \le y) = P(X \ge 1/y) = 1 - F_X(1/y) = 8y^3,$$

$$0 < y < 1/2.$$

$$f_Y(y) = 24y^2, \; 0 < y < 1/2.$$

b) Find the probability distribution of $Y = {}^{1}\!/_{X^2}$.

Support of $X = \{ x > 2 \}$

$Y = {}^{1}\!/_{X^2} \quad \Rightarrow \quad$ Support of $Y = \{ 0 < y < {}^{1}\!/_{4} \}$

$$g(x) = {}^{1}\!/_{x^2} \qquad g^{-1}(y) = {}^{1}\!/_{\sqrt{y}} = y^{-1/2} \qquad \frac{dx}{dy} = -\frac{1}{2}\, y^{-3/2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (24y^2)\left(\frac{1}{2}\, y^{-3/2}\right) = 12\, y^{1/2} = 12\sqrt{y}, \qquad 0 < y < {}^{1}\!/_{4}.$$

OR

$$F_X(x) = 1 - \frac{8}{x^3}, \qquad x > 2.$$

$$F_Y(y) = P(Y \le y) = P\left({}^{1}\!/_{X^2} \le y\right) = P\left(X \ge {}^{1}\!/_{\sqrt{y}}\right)$$

$$= 1 - F_X\left({}^{1}\!/_{\sqrt{y}}\right) = 8\, y^{3/2}, \qquad 0 < y < {}^{1}\!/_{4}.$$

$$f_Y(y) = 12\, y^{1/2} = 12\sqrt{y}, \quad 0 < y < {}^{1}\!/_{4}.$$