Coherent Structures and Flow Topology:

1. **Answer**: eigenvalues are the roots of the characteristic equation of A_{ij} which can be written as

$$\lambda^3+P\lambda^2+Q\lambda+R=0$$

where *P*, *Q* and *R* are the invariants of the velocity gradient tensor. These invariants are

$$\begin{aligned}
P &= -A_{ii}, \\
Q &= \tfrac{1}{2}P^2 - \tfrac{1}{2}A_{ik}A_{ki}, \\
R &= -\tfrac{1}{3}P^3 + PQ - \tfrac{1}{3}A_{ik}A_{kn}A_{ni}.
\end{aligned}$$

- equations from research paper : 2000-Chacin.Cantwell-JFM-Dynamics of a low Reynolds number turbulent boundary layer

Now three invariants P, Q, R in terms of the eigenvalues, and in terms of S_{ij} and R_{ij} :

A.1 Characteristic Equation of A:

$$|A - \lambda I| = \lambda^3 + P\lambda^2 + Q\lambda + R = 0$$

$\therefore P = -\sum \lambda_i$, $Q = \sum_{1 \le i < j \le 3} \lambda_i \lambda_j$, $R = -\lambda_1 \lambda_2 \lambda_3$

$\therefore P = -(\lambda_1 + \lambda_2 + \lambda_3) = Tr(A)$

$\therefore Q = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \frac{1}{2}\left[(\lambda_1+\lambda_2+\lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\right]$

$\therefore R = -\lambda_1\lambda_2\lambda_3 = \det(A)$

$\rightarrow A_{ij} = S_{ij} + R_{ij}$

$S_{ij} = \frac{1}{2}\left(\partial_j v_i + \partial_i v_j\right)$, $R_{ij} = \frac{1}{2}\left(\partial_j v_i - \partial_i v_j\right)$.

$\Rightarrow P = -Tr(A) = -A_{ii} = -\cancel{S_{ij}}\ -S_{ii}$.

$\Rightarrow Q = \frac{1}{2}\left[(\lambda_1+\lambda_2+\lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\right]$

$\therefore A = S + R$

$A^2 = S^2 + R^2$

$\Rightarrow tr(A^2) = tr(S^2) + tr(R^2)$

$= tr\left[S_{ij}S_{ij} + R_{ij}R_{ij}\right]$

So, $Q = \frac{1}{2}\left[(tr(A))^2 - tr\left(S_{ij}S_{ij} + R_{ij}R_{ij}\right)\right]$

on $Q = \frac{1}{2}P^2 - \frac{1}{2}A_{ik}A_{ki}$

$= \frac{1}{2}P^2 - \frac{1}{2}(S_{ik} + R_{ik})(S_{ki} + R_{ki})$

$= \frac{1}{2}P^2 - \frac{1}{2}(S_{ik}S_{ki} + S_{ik}R_{ki} + R_{ik}S_{ki} + R_{ik}R_{ki})$

$= \frac{1}{2}S_{ii}^2 - \frac{1}{2}(S_{ik}S_{ki} - R_{ik}R_{ik})$

$\rightarrow R = -\frac{1}{3}P^3 + PQ - \frac{1}{3}A_{ik}A_{kn}A_{ni}$

$= -\frac{1}{3}P^3 + P\left(\frac{1}{2}P^2 - \frac{1}{2}\cancel{P}(S_{ik}S_{ki} - R_{ik}R_{ik})\right)$

$- \frac{1}{3}(S_{ik} + R_{ik})(S_{kn} + R_{kn})(S_{ni} + R_{ni})$

$= -\frac{1}{3}P^3 + \frac{P^3}{2} - \frac{P}{2}(S_{ik}S_{ki} - R_{ik}R_{ik})$

$- \frac{1}{3}(S_{ik} + R_{ik})(S_{kn}S_{ni} + S_{kn}R_{ni} + R_{kn}S_{ni} + R_{kn}R_{ni})$

$= \frac{P^3}{6} - \frac{P}{2}(S_{ik}S_{ik} - R_{ik}R_{ik})$

$-\frac{1}{3}(S_{ik}S_{kn}S_{ni} + S_{ik}S_{kn}R_{ni} + S_{ik}R_{kn}S_{ni} + S_{ik}R_{kn}R_{ni}$
$+ R_{ik}S_{kn}S_{ni} + R_{ik}S_{kn}R_{ni} + R_{ik}R_{kn}S_{ni} + R_{ik}R_{kn}R_{ni})$

$= \frac{P^3}{6} - \frac{P}{2}(S_{ik}S_{ik} - R_{ik}R_{ik})$

$- \frac{1}{3}(S_{ik}S_{kn}S_{ni} - 3R_{ij}R_{jk}S_{ki})$

$= \frac{S_{ii}^3}{6} - \frac{S_{jj}}{2}(S_{ik}S_{ik} - R_{ik}R_{ik}) - \frac{1}{3}(S_{ik}S_{kn}S_{ni} - 3R_{ij}R_{jk}S_{ki})$

2. **Answer**: To calculate the velocity gradient tensor , find eigan values and plot PDF of the eigan values:

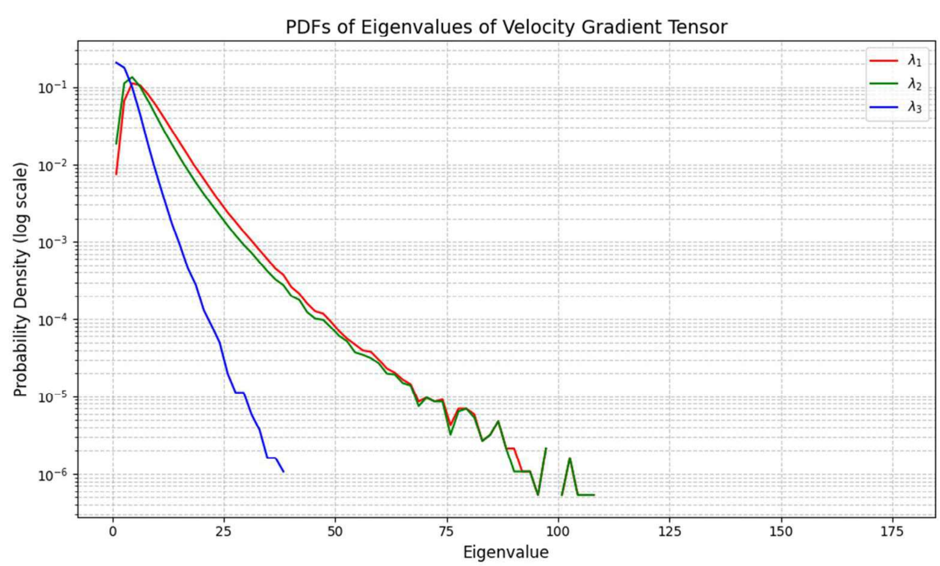
   The velocities in x,y,z direction are extracted from the file isotropic1024_sstaks3.npz choosing the middle plane.

   The velocity gradients in x,y directions are calculated using np.roll in axis 0,1. IN the z direction gradients are calculated using central difference method.

   After finding the gradients the values are stored in a fouth dimensional matrix i.e. A = np.empty((Nx, Nx, 3, 3)), where

   - The first two dimensions (Nx, Nx) represent the spatial grid (say, in x and y directions).
   - The last two dimensions (3, 3) represent a **3×3 tensor** (or matrix) at each point in the grid.
   - If your spatial domain is 3D (Nx, Ny, Nz), you need to store one 3×3 matrix per point — hence a shape like (Nx, Ny, Nz, 3, 3).

   Complex eigenvalues are converted to it's absolute value



<u>Probability density vs eigenvalue</u>

3. **Answer**: Verify whether mean of absolute values of P, $\langle|P|\rangle \approx 0$ over the 2D domain:
The value of P is calculated as −1 times trace of the matrix A, i.e. P= -(ux_x + uy_y + uz_z) and $\langle|P|\rangle$ is found by np.mean((np.abs(P)))

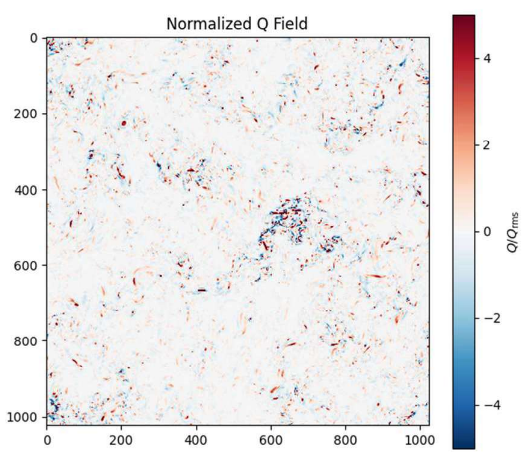
Then the mean of absolute values of P is obtained as 0.33707765285807023

4. **Answer**: Plot the fields of Q/Qrms and R/Rrms:

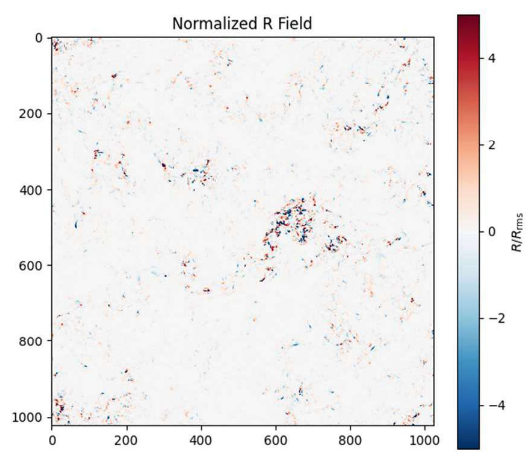
As we found in the first answer $Q = (1/2)(P^2 - tr(A^2))$
Considering the P value 0, then $Q = (1/2)(-tr(A^2))$

P=0 that implies $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and if a+b+c =0 then $a^3+b^3+c^3=3abc$.
So $\lambda_1 * \lambda_2 * \lambda_3 = (1/3) * [\lambda_1**3 + \lambda_2**3 + \lambda_3**3]$
Therefore R = -(1/3) * trace(A^3)

The Q an R are normalized by their rms values.



Q/Q_{rms} field



R/R_{rms} field

Observations about the spatial organization of the normalized Q and R fields:

1. **Structural Similarity in Spatial Patterns**:
   - Q/Qrms and R/Rrms both exhibit filamentary and patchy structures, typical of turbulence.
   - High absolute values of both Q and R tend to cluster in coherent regions (vortex tubes, sheets, or edges of turbulent eddies).
   - These structures are not randomly distributed; they form organized, extended patterns.

2. **Correlation** :
   - Regions of high Q>0 typically indicate vortical dominance (rotation stronger than strain).
   - These often correspond to positive or negative R, depending on the type of vortex stretching or compression.
   - However, Q and R are not always aligned in sign or magnitude
   - For instance, two regions may both have Q>0, but one may have R>0 (vortex stretching), and the other R<0 (vortex compression).

3. **Role of R in Distinguishing Flow Topologies**:
   - R adds a layer of insight, it helps distinguish the nature of local strain-vorticity interaction.
   - R>0: vortex stretching
   - R<0: vortex compression
   - So even if Q identifies regions with similar magnitudes, R helps classify what kind of dynamics occur there.

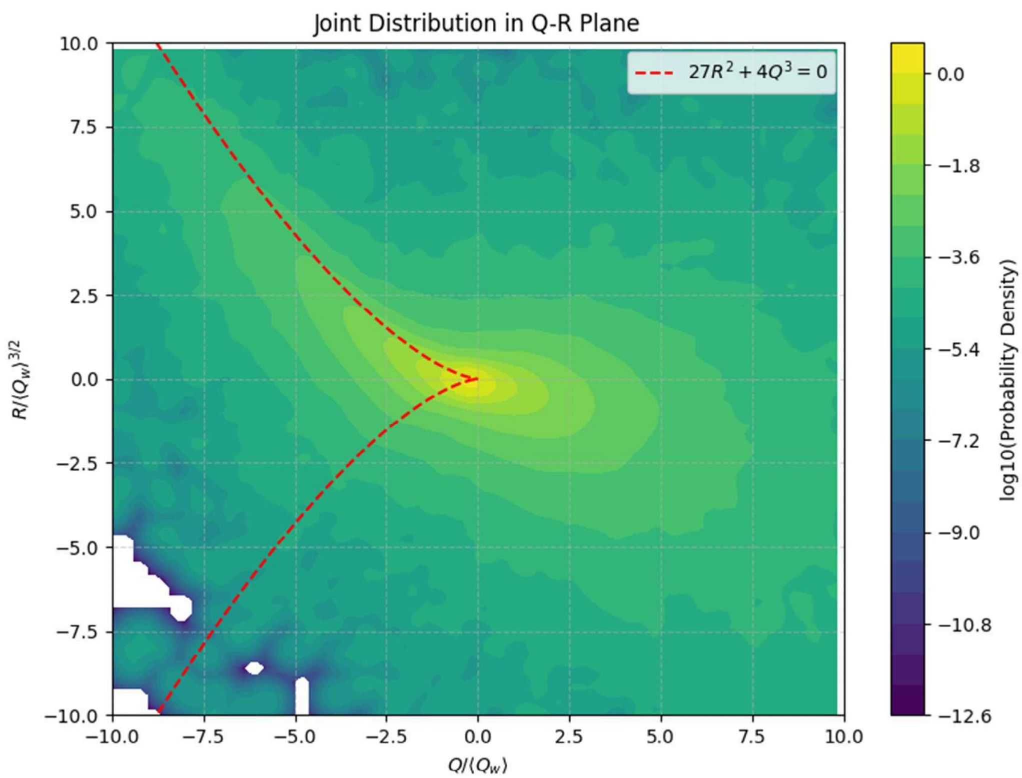
4. **Incompressible Flow Consistency**:

- Since the velocity field is nearly divergence-free ($\langle|\nabla\cdot u|\rangle \approx 0$, we expect a strong relation between the balance of strain and rotation.
- This is visible in how Q and R form complementary but intertwined patterns, consistent with known turbulence dynamics (especially in isotropic turbulence).

Conclusion:

- Q reveals where rotation dominates over strain (or vice versa)
- R reveals how strain and vorticity interact — particularly whether the flow is undergoing vortex stretching or compression.
- Together, Q and R provide a richer picture of local flow topology, and their spatial organization reflects the multiscale, coherent structure characteristic of turbulent flows.

5. **Answer**: Make a scatter-plot/joint-distribution of $Q/\langle Q_w \rangle$ v/s $R/\langle Q_w \rangle^{3/2}$, where $\langle Q_w \rangle = \langle \omega^2 \rangle/4$ in Q-R plane and draw the discriminant line $27R^2 + 4Q = 0$.



<u>**$Q/\langle Q_w \rangle$ v/s $R/\langle Q_w \rangle^{3/2}$**</u>

Prominent Flow Topologies in the Tear-Drop Plot

In the Q-R plane (tear-drop plot), two flow topologies dominate:

1. **Vortex Stretching** ($Q>0$, $R<0$)
   - **Physics**: Swirling motion with stretching along the vortex axis (e.g., turbulent eddies).
   - **Eigenvalues**: One positive ($\lambda_1>0$) and two complex-conjugate eigenvalues.
   - **Visualization**: Located in the **upper-left quadrant** of the tear-drop (negative R, positive Q).

2. **Biaxial Strain** ($Q<0$, $R>0$)
    - **Physics**: Fluid is compressed along two axes and stretched along one (dissipation-dominated).
    - **Eigenvalues**: One negative ($\lambda_3<0$) and two complex-conjugate eigenvalues.
    - **Visualization**: Located in the **lower-right quadrant** of the tear-drop (positive R, negative Q).

Link to the Enstrophy Equation

The enstrophy, ($\Omega=0.5\langle\omega_i\omega_i\rangle$) evolution equation is:

$D\Omega/Dt = \omega_i\omega_j\, S_{ij} - \nu\langle(\nabla\times\omega)^2\rangle$ + viscous diffusion,

where S_{ij} is the strain-rate tensor.

Connection to Q and R:

1. **Vortex Stretching ($Q>0$)**
    - Dominated by $\omega_i\omega_j\, S_{ij} > 0$: Vorticity is stretched, increasing enstrophy.
    - Corresponds to **enstrophy production** (positive term in the equation).
2. **Biaxial Strain ($Q<0$)**
    - Dominated by $\omega_i\omega_j\, S_{ij}$: Vorticity is compressed, reducing enstrophy.
    - Associated with **dissipation** (negative term balanced by viscous effects).

Mathematical Link:

- $Q=(1/2)(\|R\|^2-\|S\|^2)$ where $\|R\|^2$ is enstrophy and $\|S\|$ is strain.
  - $Q>0$: Rotation dominates ($\|R\|^2>\|S\|^2$).
  - $Q<0$: Strain dominates ($\|S\|^2>\| R \|^2$).
- R determines the **sign of vortex stretching**:
  - $R<0$: Vortex stretching (positive enstrophy production).
  - $R>0$: Vortex compression (enhanced dissipation).