Lagrangian Aspects of Turbulence:

1. **Answer**: Plot Lagrangian trajectories for Np = 20
   Steps followed in solving the Lagrangian trajectories:

   - **Calculation of τη:**
   The code computes spatial gradients (using periodic central differences) for u and v, constructs the symmetric strain rate tensor components S_{ij}, and estimates the energy dissipation rate ε. Then, τη is computed as

   $$\tau_\eta = \sqrt{\frac{\nu}{\epsilon}}$$

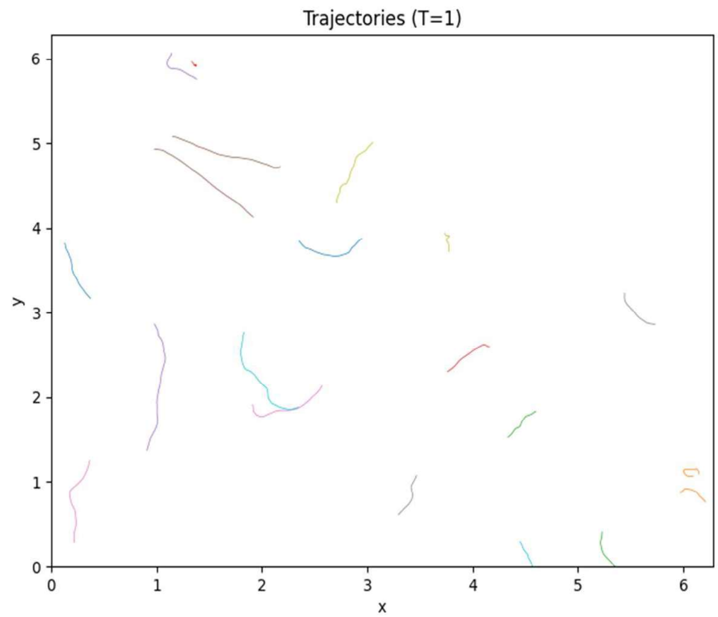
   and the time step is set as Δt = τη/20.

   - **Bilinear Interpolation:**
   The velocity function computes the velocity at any arbitrary (x, y) by converting the physical coordinates into grid-index space and then performing bilinear interpolation. For the velocity lookup, the (x,y) positions are wrapped with periodic conditions.

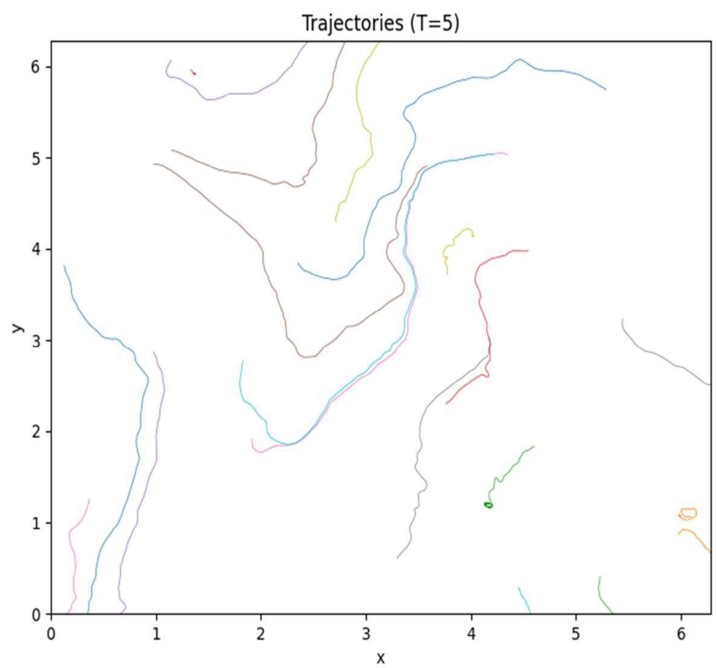
   - **Lagrangian Integration:**
   The function integrate_particles method initializes N_p=20 particles uniformly randomly within the physical domain. It then uses an Euler scheme to update positions at every time step using the local velocity (obtained by periodic lookup).

   - **Plotting:**
   Trajectories for three total integration times, T = 1, 5, and 10, are plotted.

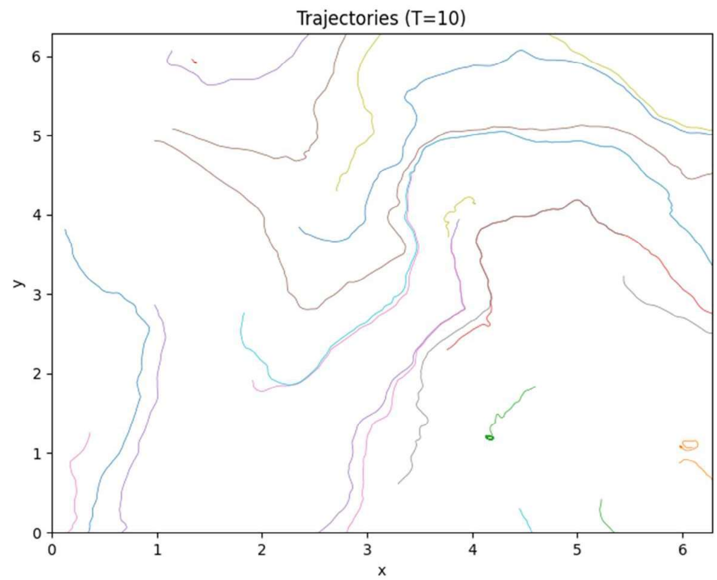
   Computed τη = 0.05890987513673654
   and Δt = 0.002945493756836827



Lagrangian trajectories for Np = 20 at T=1

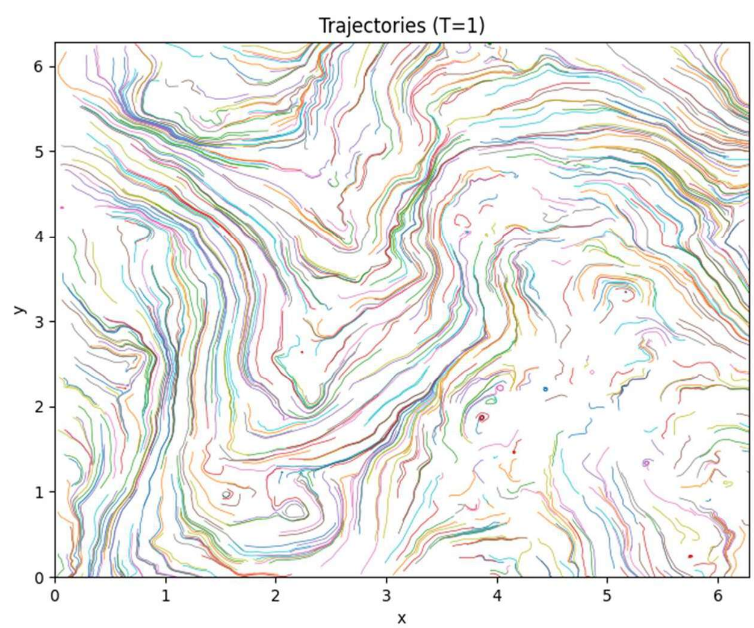


Lagrangian trajectories for Np = 20 at T=5

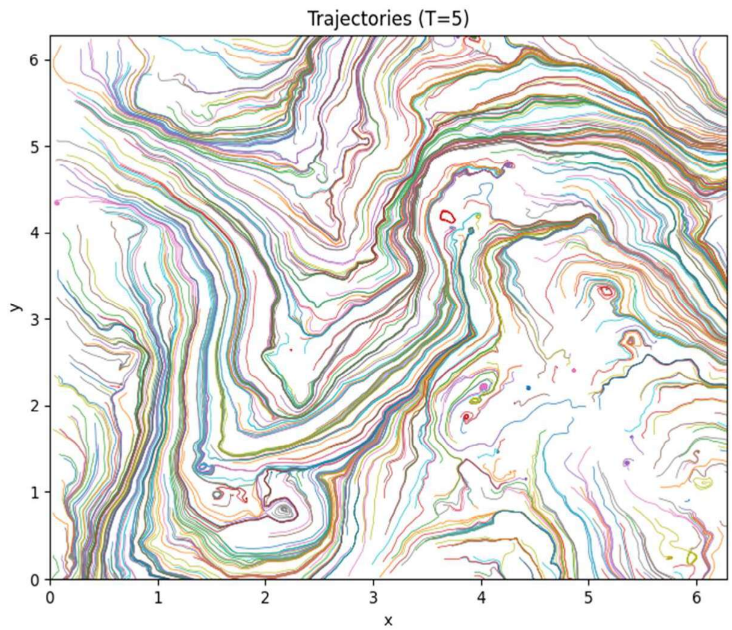


Lagrangian trajectories for Np = 20 at T=10

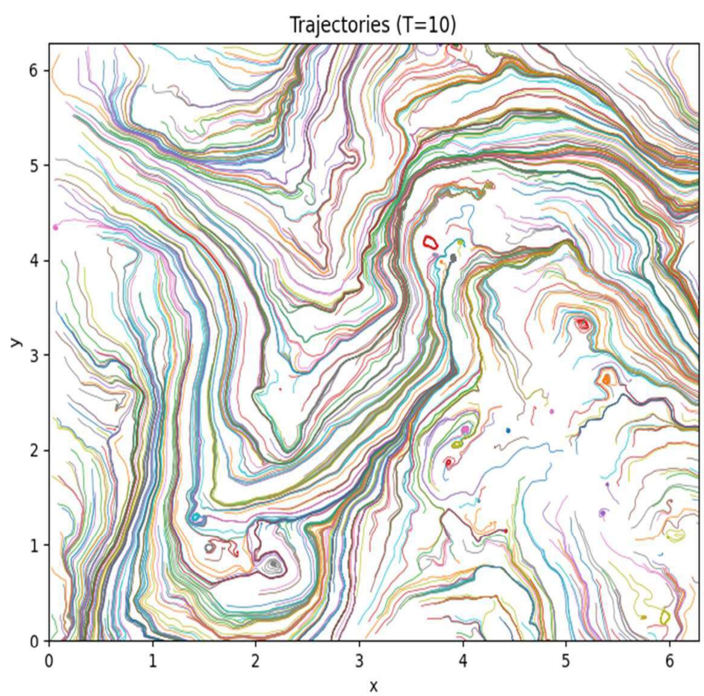
2. **Answer:** Np is increased to 1000



Lagrangian trajectories for Np = 1000 at T=1



Lagrangian trajectories for Np = 1000 at T=5



Lagrangian trajectories for Np = 1000 at T=10

When we increase Np to 1000 (see a plot of it), some particles will appear "trapped" because, we start sampling a lot more of the fine-scale structure in the (static) velocity field—and we'll inevitably drop particles right into closed recirculation regions (vortical "cells") where the local streamlines loop back on themselves. Owing to the file contains only a single snapshot of u,v,w, our integration is actually following *streamlines* of a frozen flow—those streamlines often form closed loops, so any particle initialized inside one of them simply circulates forever and looks "trapped."

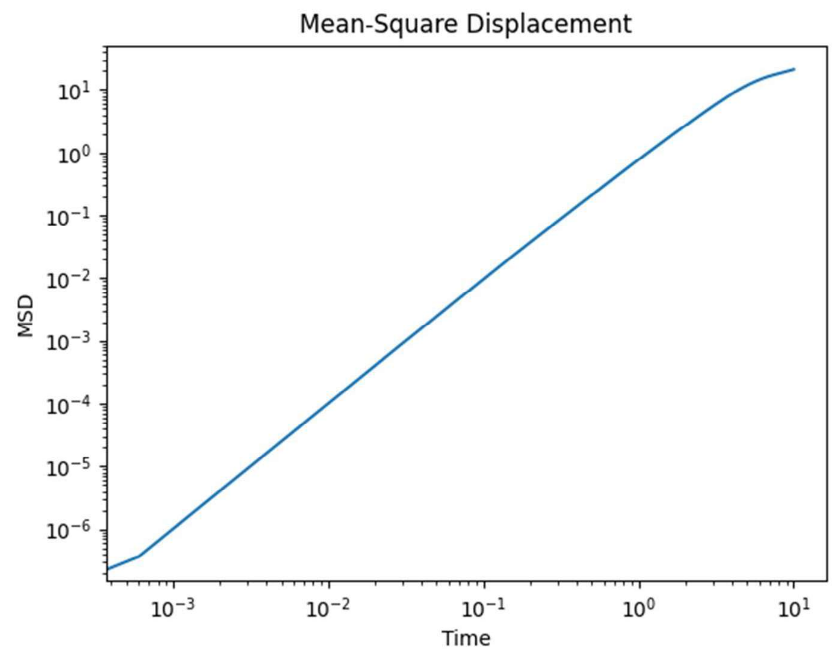
Particles may appear trapped due to being caught in coherent structures like vortices where the velocity field causes recirculating motion

On top of that, the combination of bilinear interpolation + an explicit Euler step at finite Δt can create tiny numerical stagnation zones near grid-node locations (where interpolated velocities nearly vanish). Particles that wander into those zones move so little that, over the course of our plot, they appear stuck.

How to mitigate:

- Using a *time-resolved* velocity dataset (so particles can escape recirculation as the flow evolves).
- Switching to a higher-order integrator (e.g. RK4) and/or a higher-order interpolation scheme to reduce artificial trapping near grid points

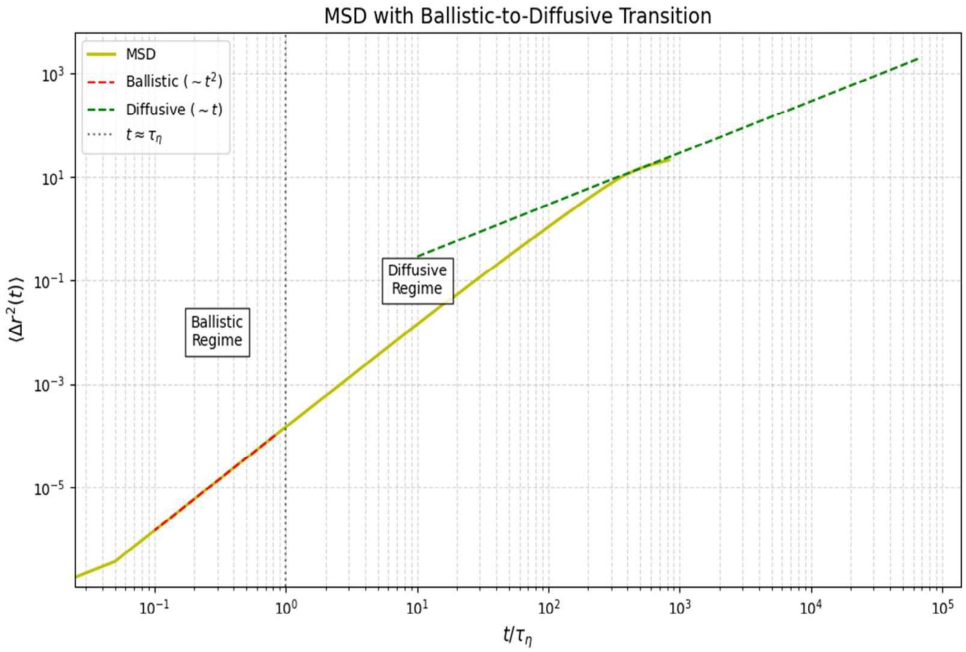
3. **Answer**: Plot the mean-square-displacement as a function of (pseudo)time, $\langle \Delta x^2(t) \rangle = \langle |x(t) - x(0)|^2 \rangle$



Mean-Square Displacement vs time for N_p=10000

Observations:

The scaling appeared to be t^2 in the ballistic and t in the diffusive regimes, this is captured by showing the lines in the plot below:



MSD vs t/t_eta with t^2 and t curves

4. **Answer**:To Calculate the turbulent diffusivity coefficient DT , by equating $\langle \Delta x^2(t) \rangle = D_T\, t$.

   Turbulent Diffusivity found to be 2.6099633996291574

   The ratio D_T/D_{mol} shows how much turbulence enhances mixing compared to molecular diffusion

   The diffusivity coefficient of a dye in water is 1.7×10^{-10} to 4.4×10^{-8}

   The turbulent diffusivity is 10^8 times more than the diffusivity coefficient.

5. Answer: Richardson pair dispersion:

This measures the average separation of pairs of particles,

initially separated over a distance ϵ, i.e.

$$\Delta r^2(t) = \langle |x_1(t) - x_2(t)|^2 \rangle_{Np} ,$$

where Np now is the number of particle pairs.

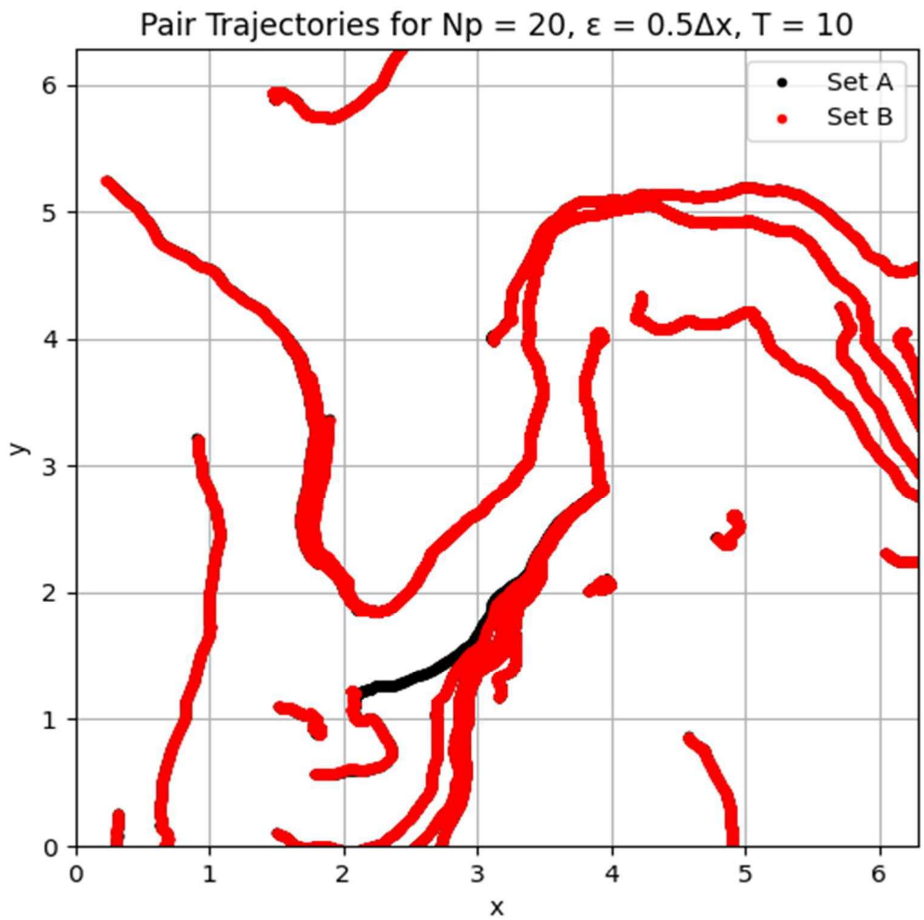
So, $\Delta r^2(t)=\langle(x1(t)-x2(t))^2\rangle+\langle(y1(t)-y2(t))^2\rangle+\langle(z1(t)-z2(t))^2\rangle$ is done for all particles

For this, initialize two-sets of Np particles each, say Set A and Set B. All particles of Set B are first identically initialized as Set A, and then their locations are perturbed, such that

$$x_{B,i} = x_{A,i} + \epsilon_i$$

and $|\epsilon_i| = \epsilon$ is a two-dimensional perturbation of fixed amplitude but random direction

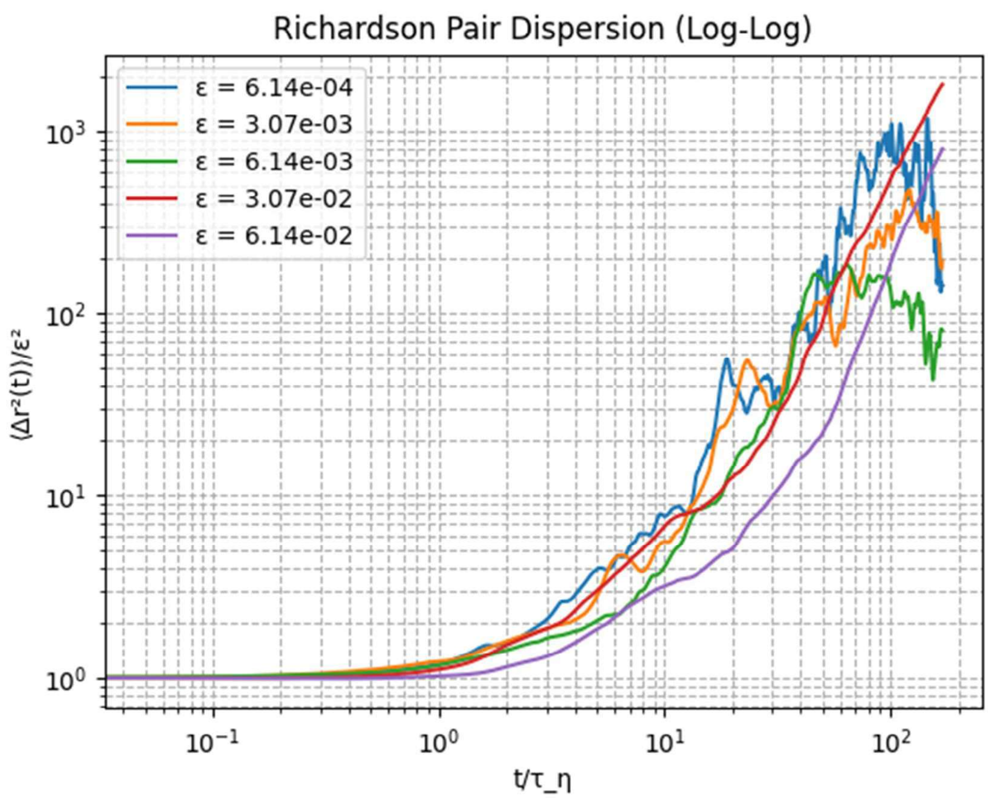
5a. Answer: Plot pair trajectories for Np = 20, ϵ = 0.5Δx (with Δx = 2π/Nx) and T = 10



<u>Plot pair trajectories for Np = 20 at T=10</u>

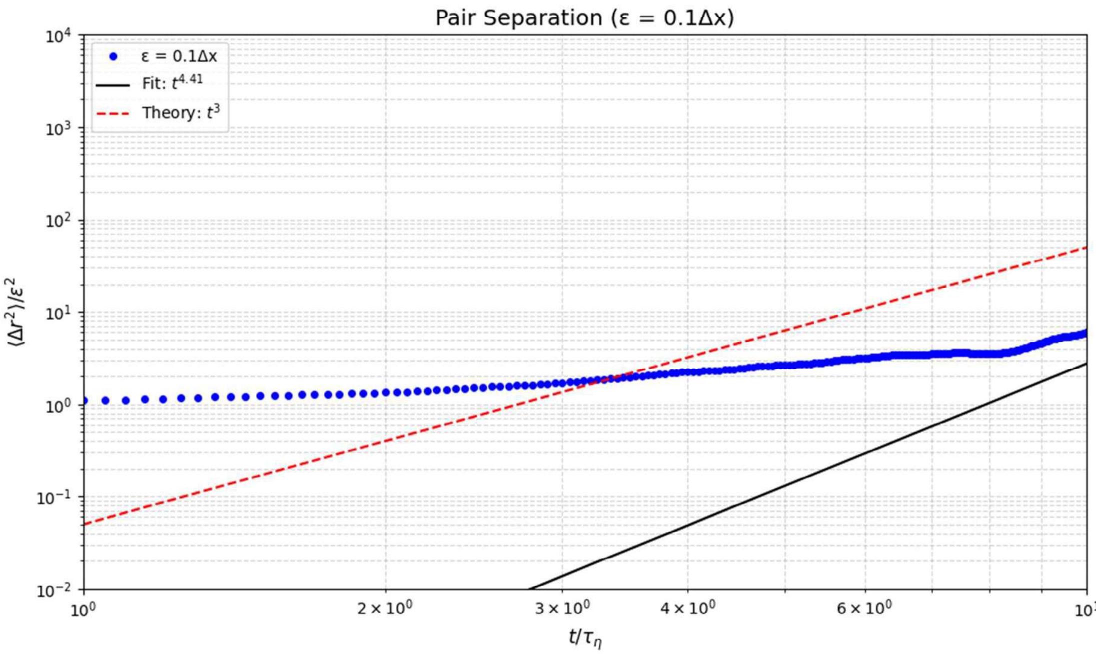
5b. Answer: Plot the pair separation of trajectories ($\Delta r^2(t)/\epsilon^2$) v/s $t/\tau\eta$, on a loglog scale, for $\epsilon/\Delta x$ = {0.1, 0.5, 1.0, 5.0, 10} for T = 10

Took Np = 100 because of constraint in computational power

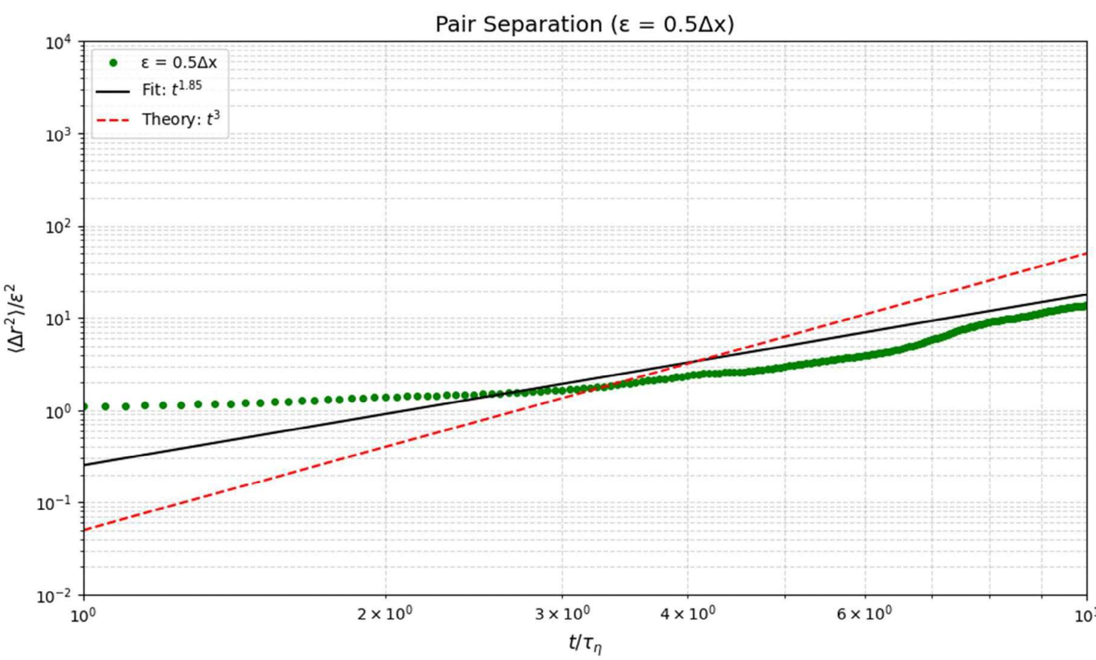


<u>Δr2(t)/ϵ2) v/s t/τη (loglog scale)</u>

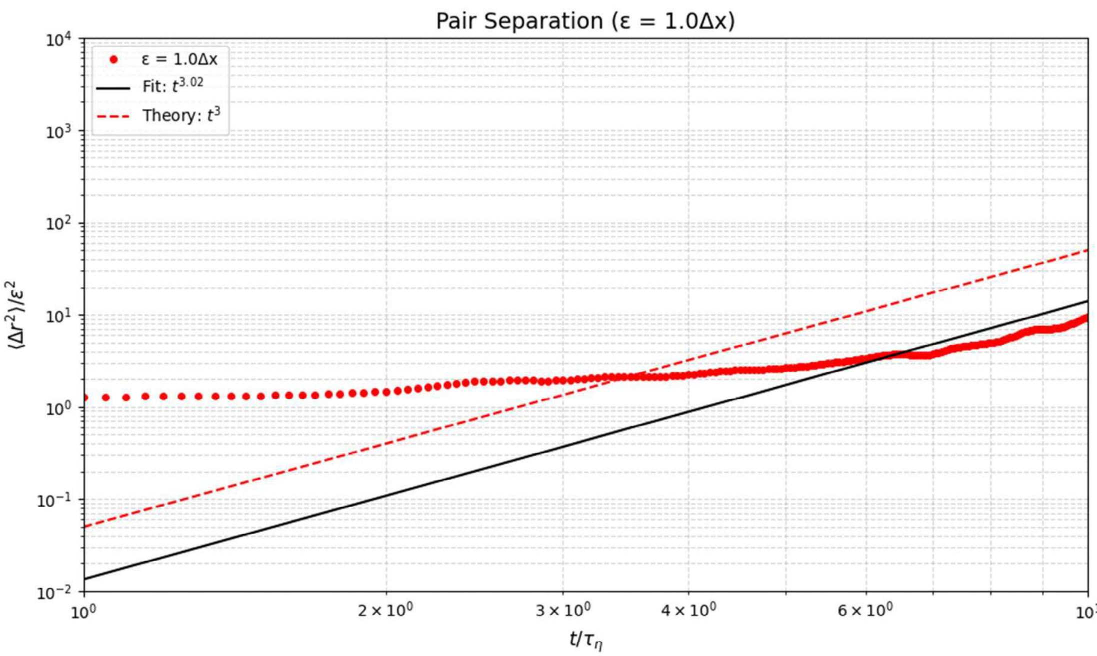
Plots to check to observe a $\Delta r^2 \sim t^3$ scaling in the inertial range with a line fit to the loglog figure



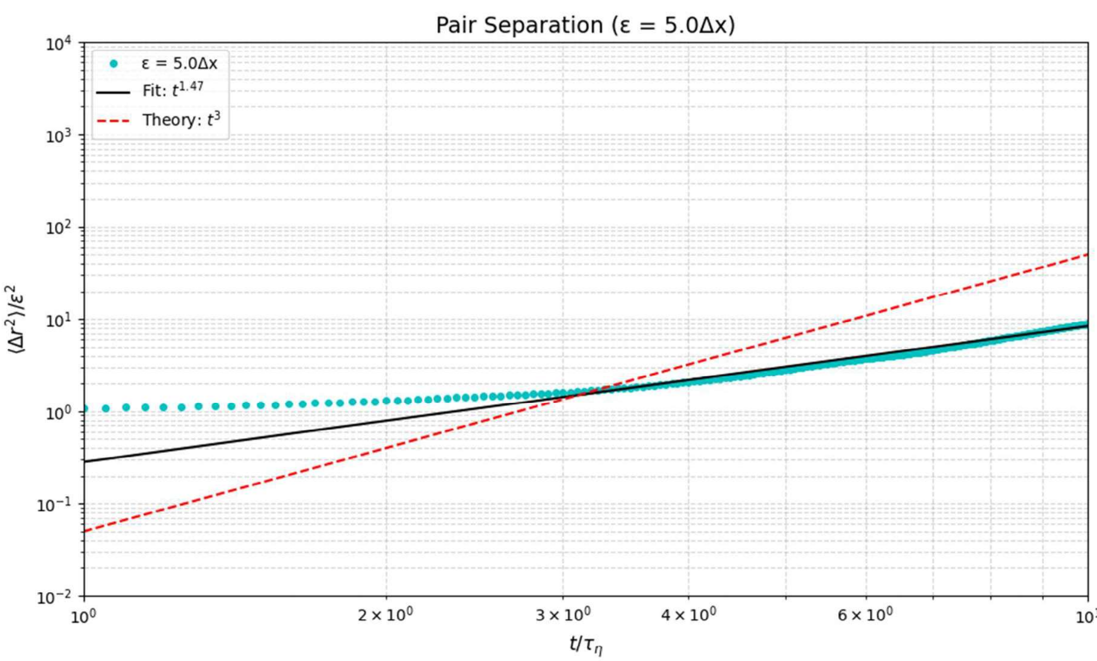
$\Delta r2(t)/\epsilon 2)$ v/s $t/\tau\eta$ for $\epsilon = 0.1\Delta x$



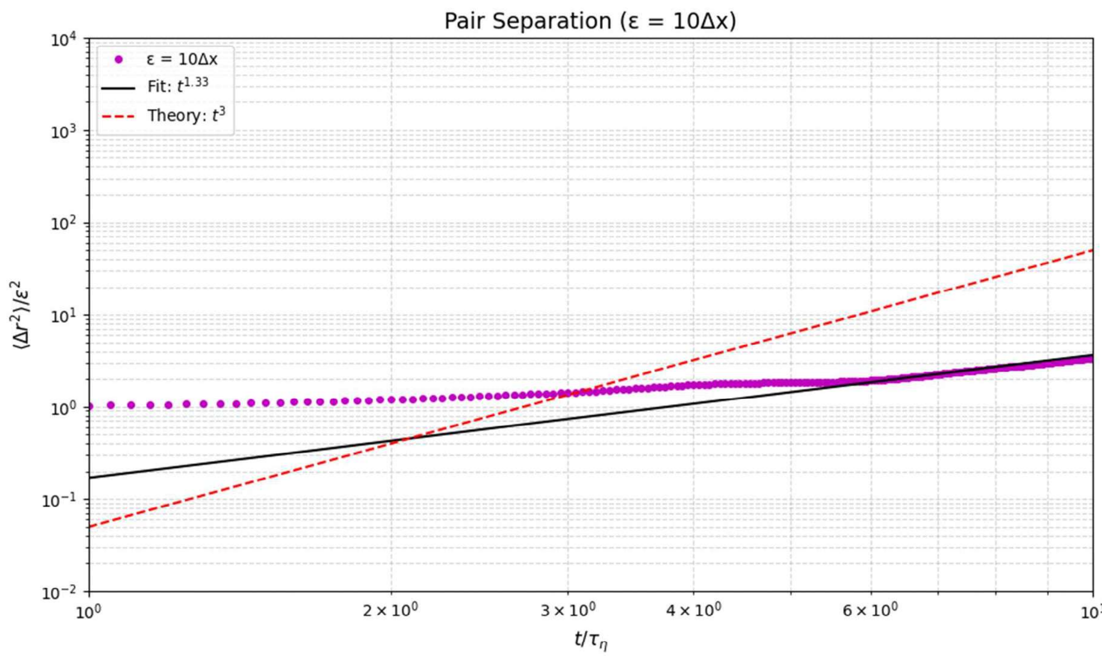
$\Delta r2(t)/\epsilon 2)$ v/s $t/\tau\eta$ for $\epsilon = 0.5\Delta x$



Δr2(t)/ϵ2) v/s t/τη for ϵ= 1Δx



Δr2(t)/ϵ2) v/s t/τη for ϵ= 5Δx



<u>Δr2(t)/ϵ2) v/s t/τη for ϵ= 10Δx</u>

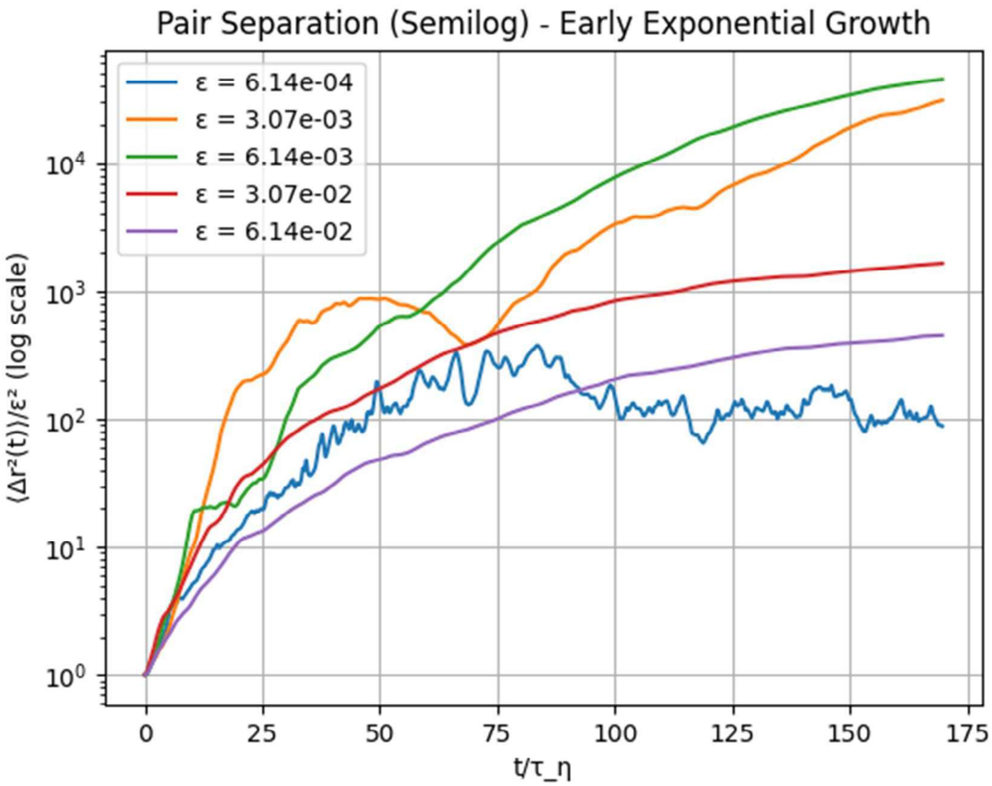
The log-log plot with t^3 fits quantitatively tests Richardson's theory. The data seems to align with the t^3 line in the inertial range, the flow follows classical turbulence scaling. Deviations reveal viscous contamination, insufficient scale separation, or statistical noise.

So, if slope = 3, Richardson scaling holds

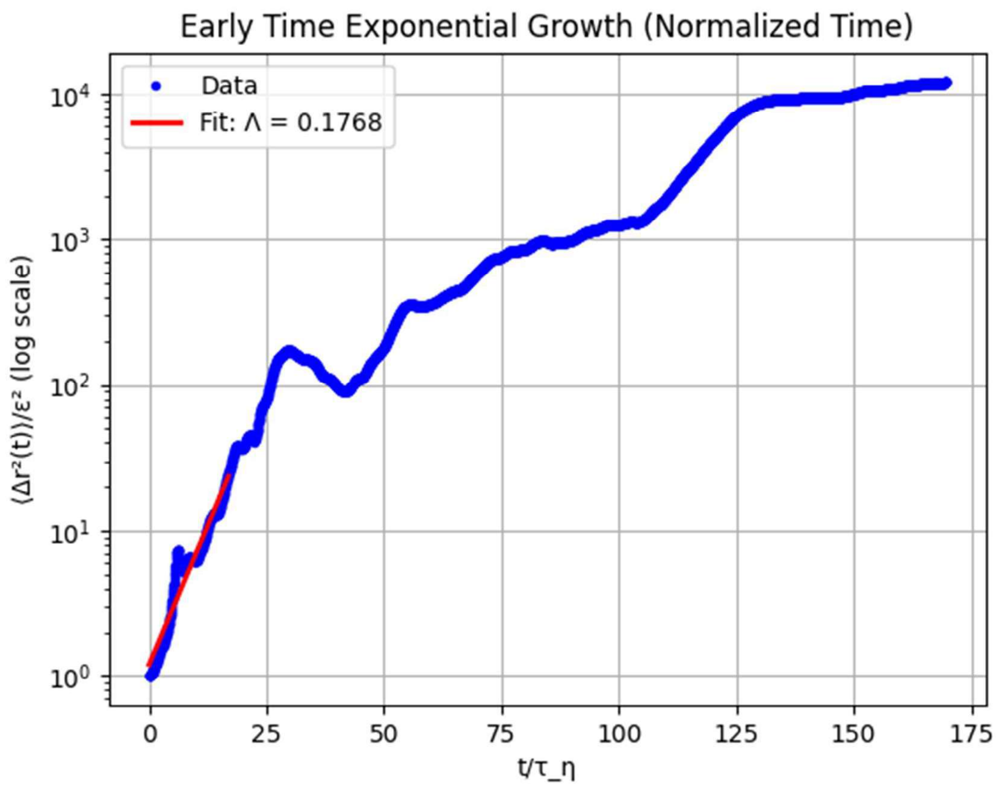
slope < 3, viscous effects are dominant

slope > 3, Large scale staining dominates

5c Answer: Plot the pair separation of trajectories ($\Delta r^2(t)/\epsilon^2$) v/s t/τη on a semilog scale



Δr2(t)/ϵ2) v/s t/τη (semilog scale)



<u>$\Delta r2 \sim \exp(\Lambda t)$, with line-fit</u>

The value of the Lyapunov exponent Λ, which marks the region of chaotic growth of particle separation is found to be 0.1768