ME647 - Introduction to Turbulence

Assignment 1

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by:

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Ans. $\frac{\partial u_i}{\partial t} + u_j \partial_j u_i = -\partial_i \left(P/\rho\right) + \nu \partial_j \partial_j u_i$

1-1

& $\partial_i u_i = 0$

$k = \frac{u_i u_i}{2}$

multiply by u_i

$= u_i \frac{\partial u_i}{\partial t} + u_i u_j \partial_j u_i = -u_i \partial_i (P/\rho) + \nu u_i \partial_j \partial_j u_i$

$= k = \frac{1}{2} u_i u_i \quad ; \quad \frac{\partial k}{\partial t} = \frac{u_i \partial u_i}{\partial t}$

1st term :- $\frac{u_i \partial u_i}{\partial t} = \frac{\partial k}{\partial t}$

2nd term :- convective : $u_i u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k u_j\right)$

3rd Pressure term : $-u_i \frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) = -\frac{\partial}{\partial x_i}\left(u_i \frac{P}{\rho}\right) + \underbrace{\frac{P}{\rho}\frac{\partial u_i}{\partial x_i}}_{= 0}$

4th viscous term :

$$\nu u_i \frac{\partial^2 u_i}{\partial x_j^2} = \nu \left[\frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_i}{\partial x_j}\right) - \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right]$$

Now, integrating over a Volume V bounded by a surface $d\Omega$

$$\rightarrow \int_V \frac{\partial k}{\partial t} dv + \int_V \frac{\partial}{\partial x_j}(k u_j) dv = -\int_V \frac{\partial}{\partial x_i}\left(u_i \frac{P}{\rho}\right) dv + \nu \int \left(\frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_i}{\partial x_j}\right) - \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right) dv$$

- using gauss divergence theorem, the divergence terms can be converted into surface integrals over $d\Omega$.

$$= \int_V \frac{\partial k}{\partial t} dv + \int_{d\Omega} k U_j n_j dS = -\int_{d\Omega} u_i \frac{P}{\rho} n_i ds + \nu \int_{d\Omega} u_i \frac{\partial U_i}{\partial x_j} n_j ds - \nu \int_V \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} dv$$

unsteady term | convective flux | Pressure work | viscous flux | Dissipation

Neglected Terms in a Turbulent flow far from walls:

1. Surface terms ($\int_{d\Omega}$) can often be neglected in a unbounded turbulent region far from walls, as the influence of boundary diminishes.

2. The viscous dissipation term $-\nu \int_V \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} dv$ remains significant as it represents energy dissipation.

3. The pressure term $\int_{d\Omega} u_i \frac{P}{\rho} d\Omega$ can be neglected if pressure fluctuations are small.

Thus, the final Energy Equation simplifies to:

$$\frac{d}{dt} \int_V k dv = -\nu \int_V \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} dv$$

- states that kinetic Energy is dissipated due to viscosity.

Neglected terms in the Middle of a Turbulent flow :

In the middle of a turbulent flow, far from physical walls, several terms can be neglected due to nature of turbulence and the assumptions about large scale homogeneity

→ Convective Transport : $\frac{\partial}{\partial x_j}(k u_j)$

- In an unbounded region, far from solid boundaries, the spatial variations in kinetic energy are relatively small when averaged over a sufficiently large region.
- if turbulence is statistically homogeneous, the divergence of the convective term averages to zero, meaning that large-scale advection effects are negligible.

→ Pressure Transport term $\frac{\partial}{\partial x_i}\left(u_i \frac{p}{\rho}\right)$:

- In homogeneous turbulence, where statistical averages do not change in space, the divergence of this term vanishes.
- In the middle of a turbulent flow (away from walls or obstacles), pressure fluctuations tend to balance out over large volumes.

→ Viscous Transport term : $\nu \frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_i}{\partial x_j}\right)$

- In the bulk of a turbulent flow, far from boundaries, viscosity effets are generally small compared to inertial forces.
- At very small scales (kolmogorov scales), viscocity is significant. But in the middle of a large turbulent flow, this term is negligible at larger scales.

Ans 1-a:

$$NSE = \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j}$$

Vorticity definition : $\omega_p = \epsilon_{pqi} \frac{\partial U_i}{\partial x_q}$

Taking curl ($\epsilon_{pqi}\partial_q$) of NSE :-

$$\Rightarrow \epsilon_{pqi}\partial_q\left(\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j}\right) = \epsilon_{pqi}\partial_q\left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}\right).$$

The above NSE can be written as

$$\Rightarrow \epsilon_{pqi}\partial_q\left(\underset{1}{\frac{\partial U_i}{\partial t}} - \underset{2}{\epsilon_{ijk}U_j\omega_k}\right) = \epsilon_{pqi}\partial_q\left(\underset{3}{-\partial_i\left(\frac{p}{\rho}+\frac{U^2}{2}\right)} + \underset{4}{\nu\partial_j\partial_j U_i}\right)$$

1) $\epsilon_{pqi}\partial_q\left(\frac{\partial U_i}{\partial t}\right) = \frac{\partial \omega_p}{\partial t}$

4) $\nu\epsilon_{pqi}\partial_q\left[\partial_j\partial_j U_i\right] = \nu\partial_j\partial_j\left[\epsilon_{pqi}\partial_q U_i\right] = \nu\partial_j\partial_j\omega_p$

3) 0, curl of scalar $= 0$

2) $-\epsilon_{pqi}\partial_q\left[\epsilon_{ijk}U_j\omega_k\right]$

$= \epsilon_{iqp}\epsilon_{ijk}\partial_q U_j\omega_k$

$= (\delta_{qj}\delta_{pk} - \delta_{qk}\delta_{pj})\partial_q U_j\omega_k$

$= \partial_j(U_j\omega_p) - \partial_k(U_p\omega_k)$

$= U_j\partial_j\omega_p + \omega_p\partial_j U_j - \omega_k\partial_k U_p - U_p\partial_k\omega_k$

$\omega_p\partial_j U_j$: $= 0$: continuity

$U_p\partial_k\omega_k$: $= 0$: div. of curl

- Substituting all the terms

$$= \frac{\partial \omega_p}{\partial t} + u_j \frac{\partial \omega_p}{\partial x_j} = \omega_j \frac{\partial u_p}{\partial x_j} + \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

$$= \frac{D\omega_p}{Dt} = \omega_j \frac{\partial u_p}{\partial x_j} + \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

Enstrophy Equation :

Multiply the vorticity equation with ω_p

$$= \omega_p \frac{D\omega_p}{Dt} = \omega_p \omega_j \frac{\partial u_p}{\partial x_j} + \nu \omega_p \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

$$\Rightarrow \frac{D}{Dt}\left(\frac{1}{2}\omega_p \omega_p\right) = \omega_p \omega_j \frac{\partial u_p}{\partial x_j} + \nu \omega_p \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

Expand the viscous term :

$$\nu \omega_p \frac{\partial^2 \omega_p}{\partial x_j \partial x_j} = \nu \frac{\partial}{\partial x_j}\left(\omega_p \frac{\partial \omega_p}{\partial x_j}\right) - \nu \left(\frac{\partial \omega_p}{\partial x_j}\right)^2$$

$$= \nu \frac{\partial}{\partial x_j \partial x_j}\left(\frac{1}{2}\omega_p \omega_p\right) - \nu \left(\left(\frac{\partial \omega_p}{\partial x_j}\right)\left(\frac{\partial \omega_p}{\partial x_j}\right)\right)$$

Final Enstrophy Equation :

$$\frac{D}{Dt}\left(\frac{1}{2}\omega_p \omega_p\right) = \underbrace{\omega_p \omega_j \frac{\partial u_p}{\partial x_j}}_{\text{Vortex stretching}} + \underbrace{\nu \frac{\partial}{\partial x_j \partial x_j}\left(\frac{1}{2}\omega_p \omega_p\right)}_{\text{Vortex diffusion}} - \underbrace{\nu \left(\frac{\partial \omega_p}{\partial x_j}\right)\left(\frac{\partial \omega_p}{\partial x_j}\right)}_{\text{Enstrophy dissipation}}$$

Kinetic Energy Equation from NSE :-

$$\frac{D}{Dt}\left(\frac{1}{2}u_i u_i\right) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}(p u_i) + \nu\frac{\partial}{\partial x_j \partial x_j}\left(\frac{1}{2}u_i u_i\right) - \nu\left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right)$$

Pressure work   Viscous diffusion   Kinetic Energy dissipation

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Comparison of Terms :-

Term	Enstrophy Equation	Kinetic Energy Equation
Material Derivative	$\frac{D}{Dt}\left(\frac{1}{2}\omega^2\right)$	$\frac{D}{Dt}\left(\frac{1}{2}u^2\right)$
Production	Vortex Stretching : $\omega_p \omega_j \frac{\partial u_p}{\partial x_j}$	Pressure work : $-\frac{1}{\rho}\nabla(pu)$
Diffusion	$\nu\nabla^2\left(\frac{1}{2}\omega^2\right)$	$\nu\nabla^2\left(\frac{1}{2}u^2\right)$
Dissipation	$-\nu\left(\frac{\partial \omega_p}{\partial x_j}\right)^2$	$-\nu\left(\frac{\partial u_i}{\partial x_j}\right)^2$

Key Differences :-

1. Production Mechanisms :-
   - → Enstrophy grows via vortex stretching
   - → Kinetic energy redistributes via pressure work
2. Dissipation :
   - Enstrophy dissipates through Vorticity gradients
   - Kinetic energy dissipates through velocity gradients
3. Pressure :-
   - Pressure directly affects kinetic energy but not enstrophy.

Answer 2:

Flow Characteristics

- Reynolds Number (Re): 4477.86
- Kolmogorov Scales:
  - Length Scale (η): 0.002492
  - Velocity Scale (u_η): 0.005512
  - Time Scale (τ_η): 0.033562
- Grid Resolution:
  - Grid Cell Size / Kolmogorov Length: 2.46

Velocity Component Statistics

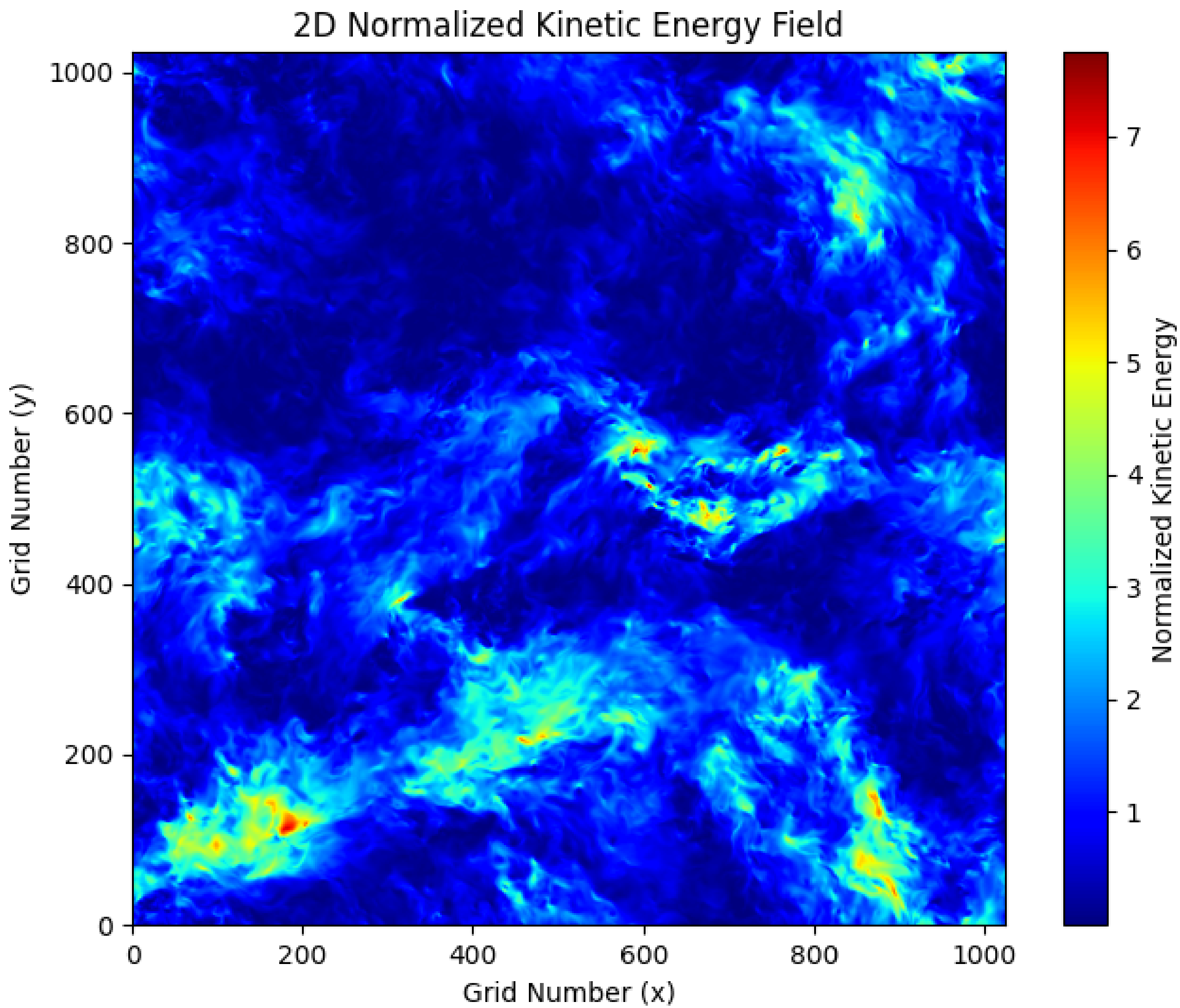
- Velocity Component (u):
  - Skewness: 0.171909
  - Kurtosis: 0.045412
- Velocity Component (v):
  - Skewness: 0.091369
  - Kurtosis: -0.408890
- Velocity Component (w):
  - Skewness: -0.220180
  - Kurtosis: -0.417023

Vorticity and Enstrophy

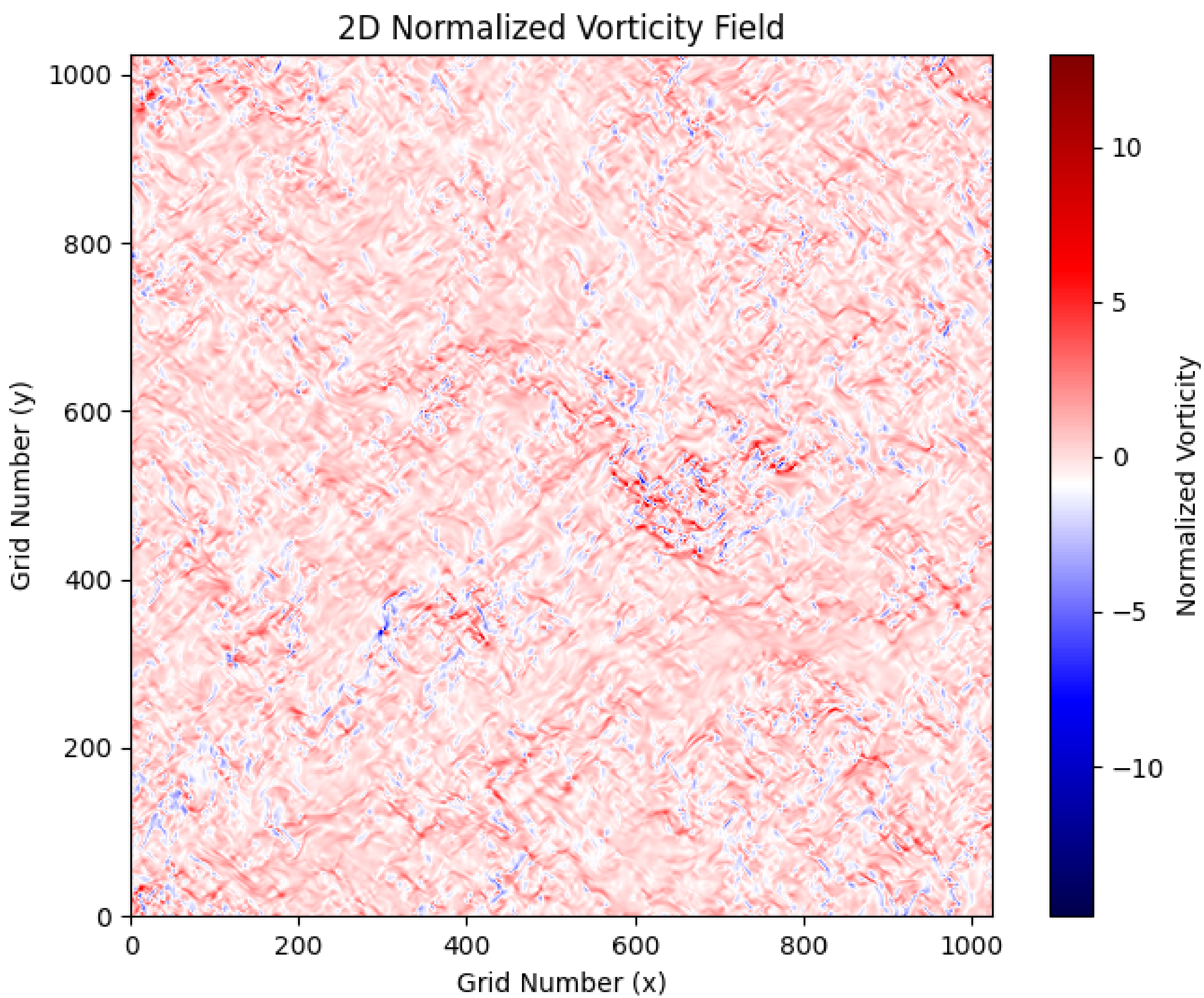
- Vorticity Component (ω_z):
  - Skewness: 0.056656
  - Kurtosis: 4.599406
- Enstrophy Component (ω_z * w_z):
  - Skewness: 12.964806
  - Kurtosis: 432.338006

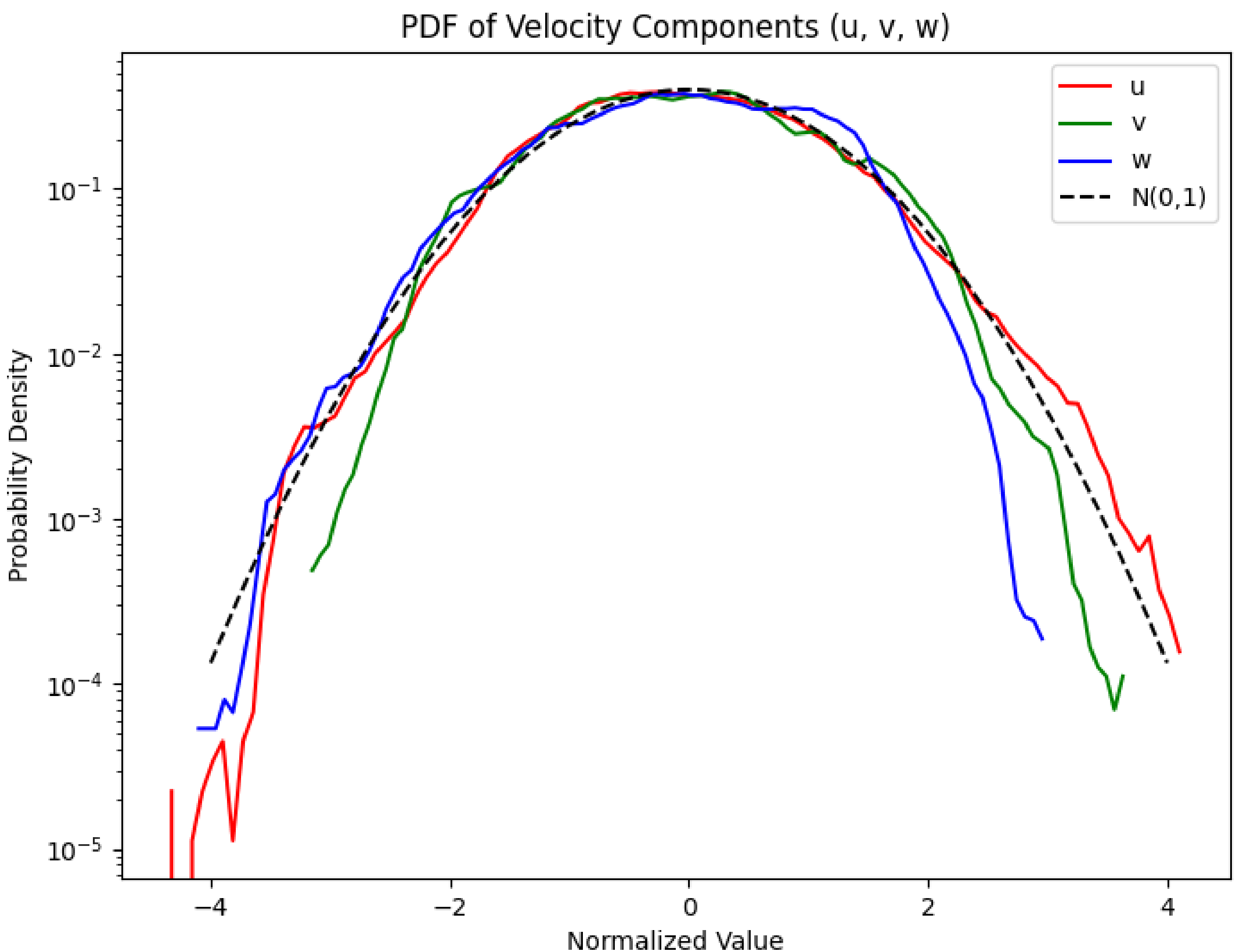
Velocity Gradient Statistics

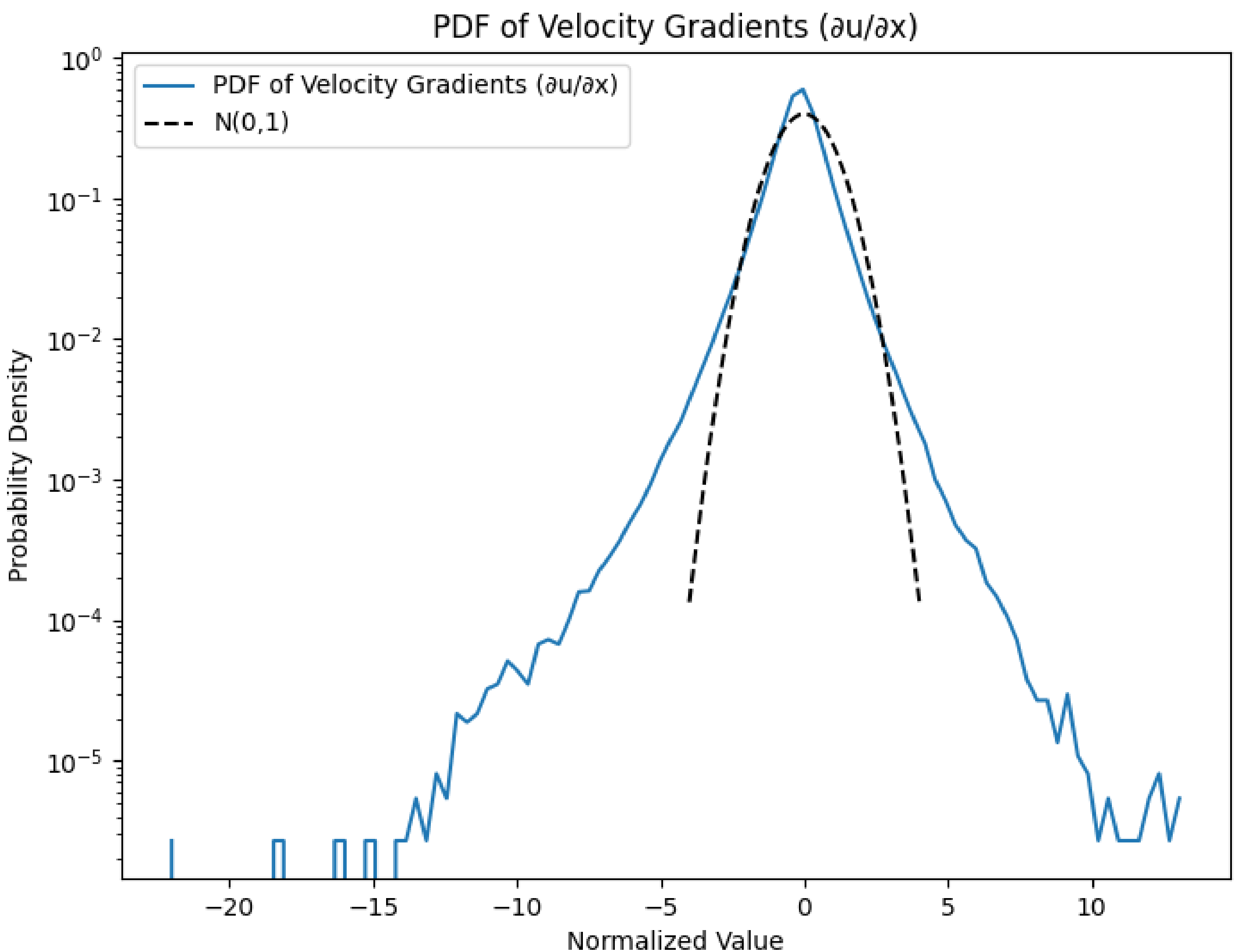
- Velocity Gradient (∂u/∂x):
  - Skewness: -0.332042
  - Kurtosis: 7.788521
- Velocity Gradient (∂v/∂y):
  - Skewness: 0.117336
  - Kurtosis: 7.579237



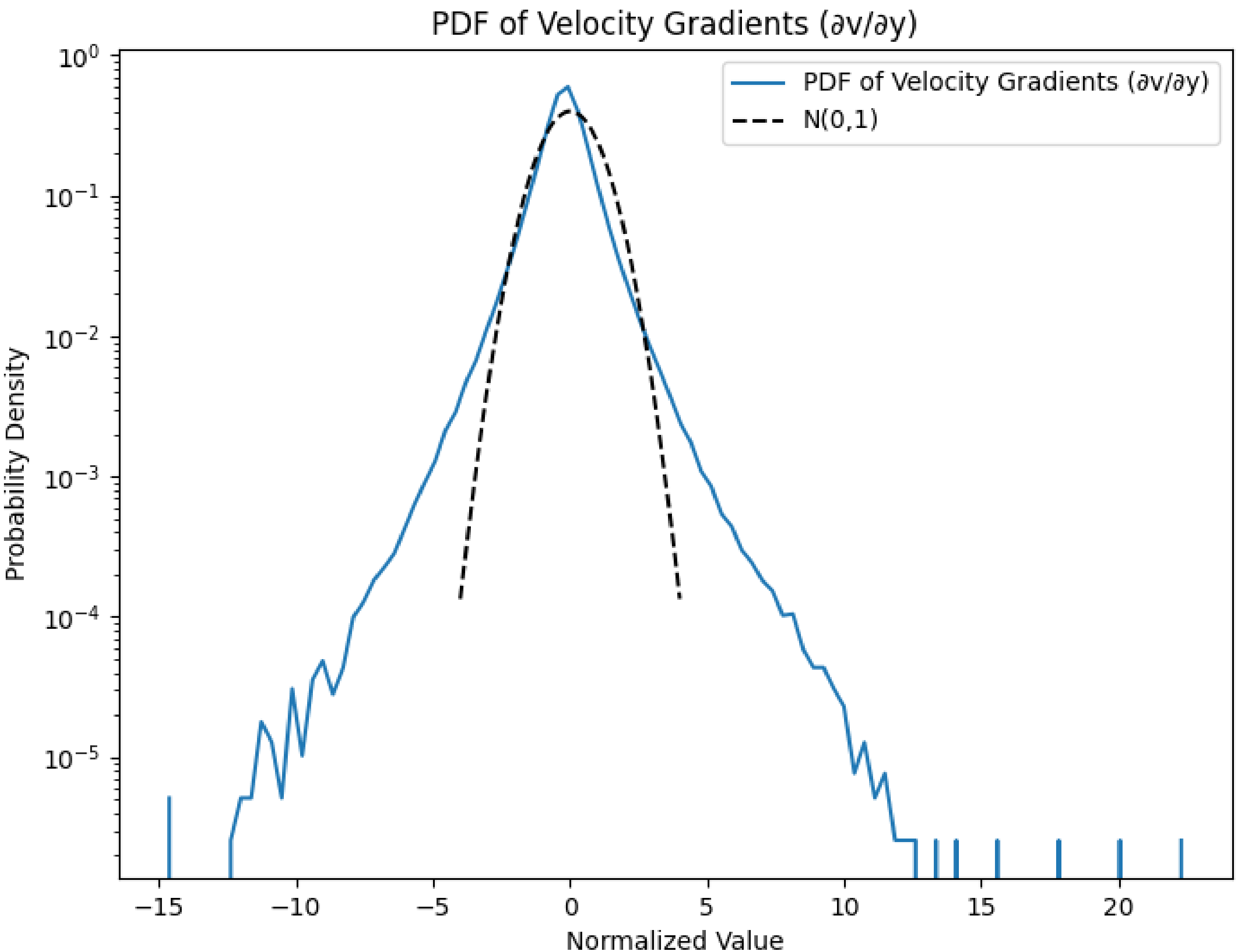
2D Normalized Kinetic Energy Field
Grid Number (y)
Grid Number (x)
0
200
400
600
800
1000
Normalized Kinetic Energy
1
2
3
4
5
6
7


2D Normalized Vorticity Field
Grid Number (y)
Grid Number (x)
Normalized Vorticity
0
200
400
600
800
1000
10
5
−5
−10

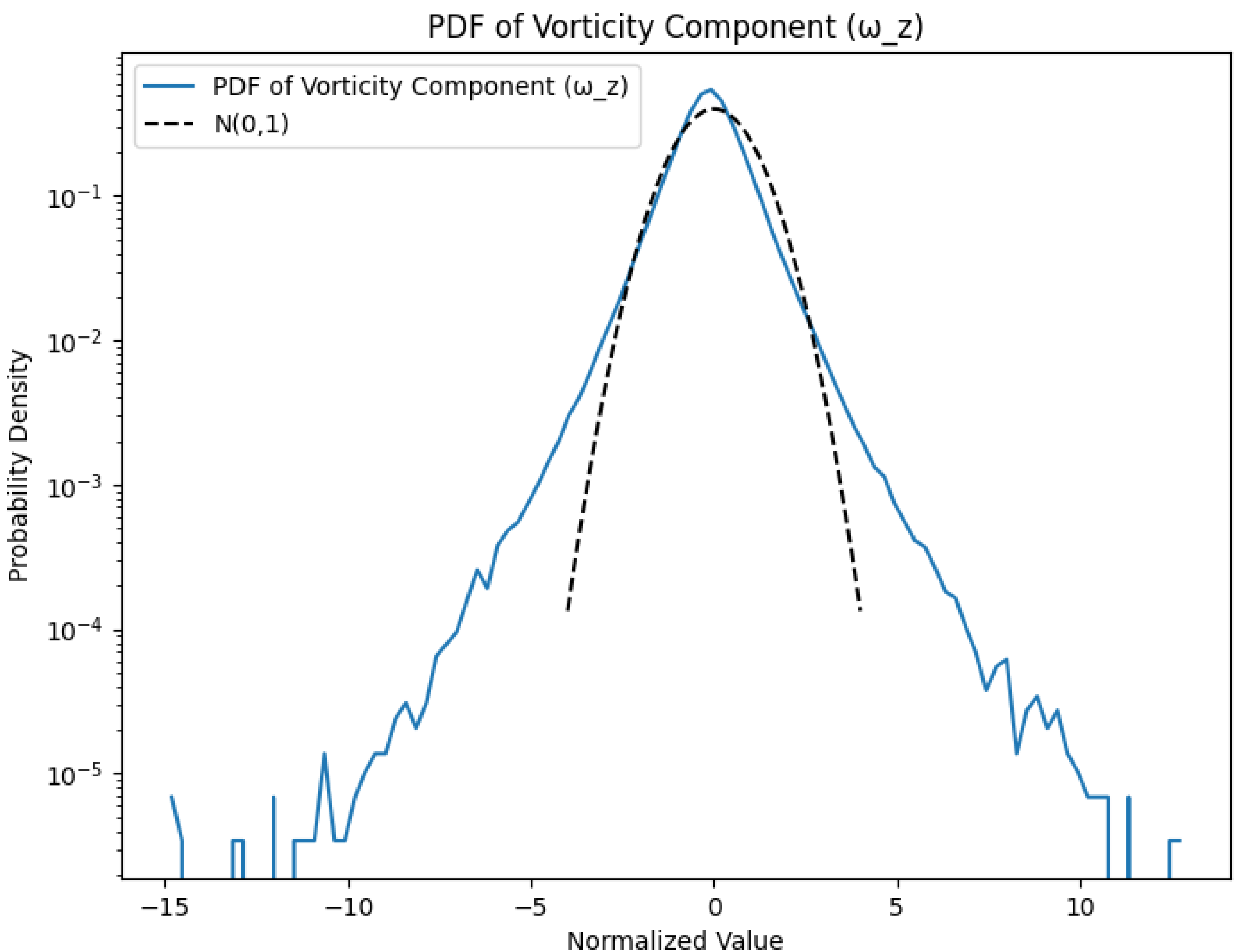

PDF of Velocity Components (u, v, w)
u
v
w
N(0,1)
Probability Density
Normalized Value
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
−4
−2
0
2
4

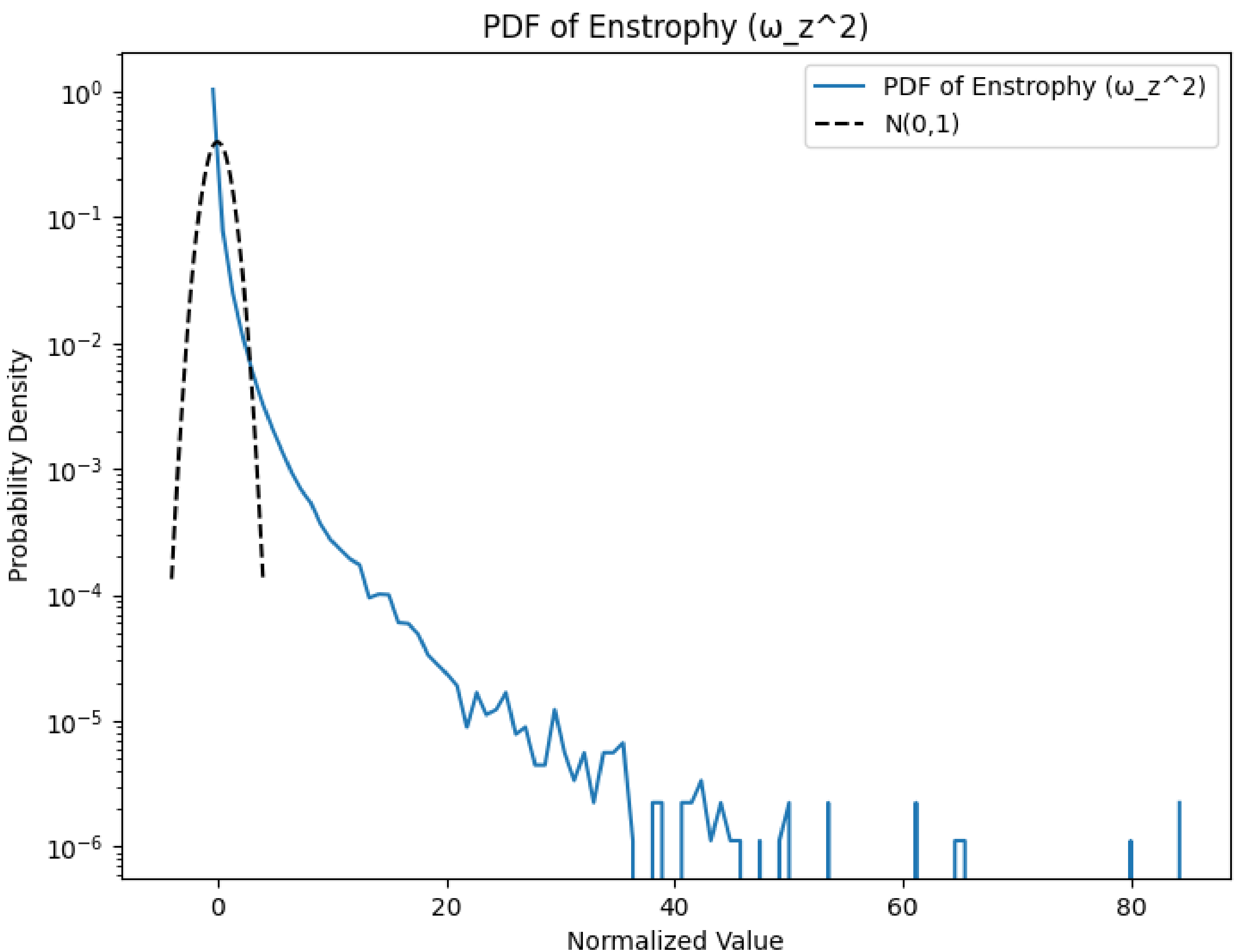


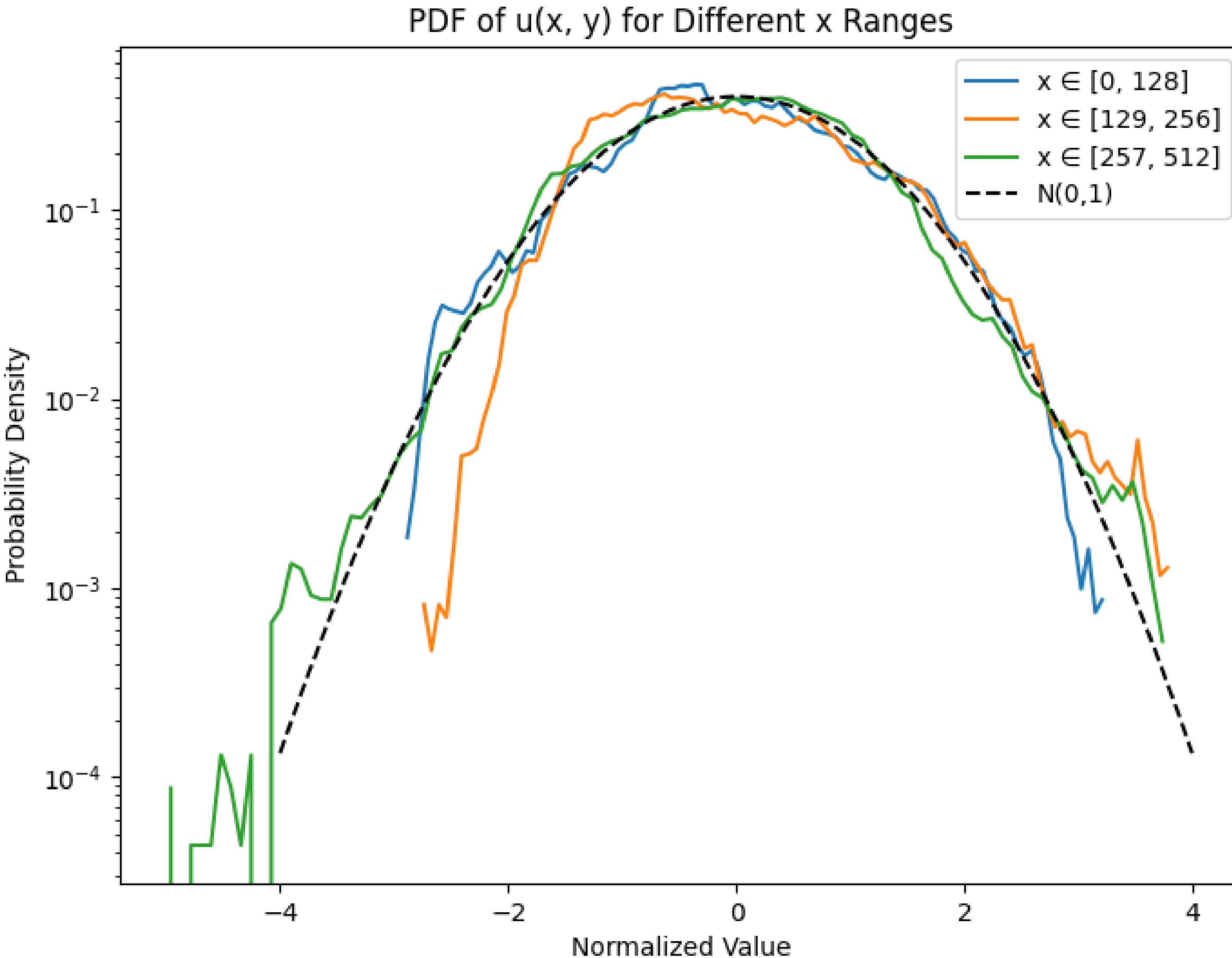
PDF of Velocity Gradients (∂u/∂x)
PDF of Velocity Gradients (∂u/∂x)
N(0,1)
Probability Density
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
−20
−15
−10
−5
0
5
10
Normalized Value



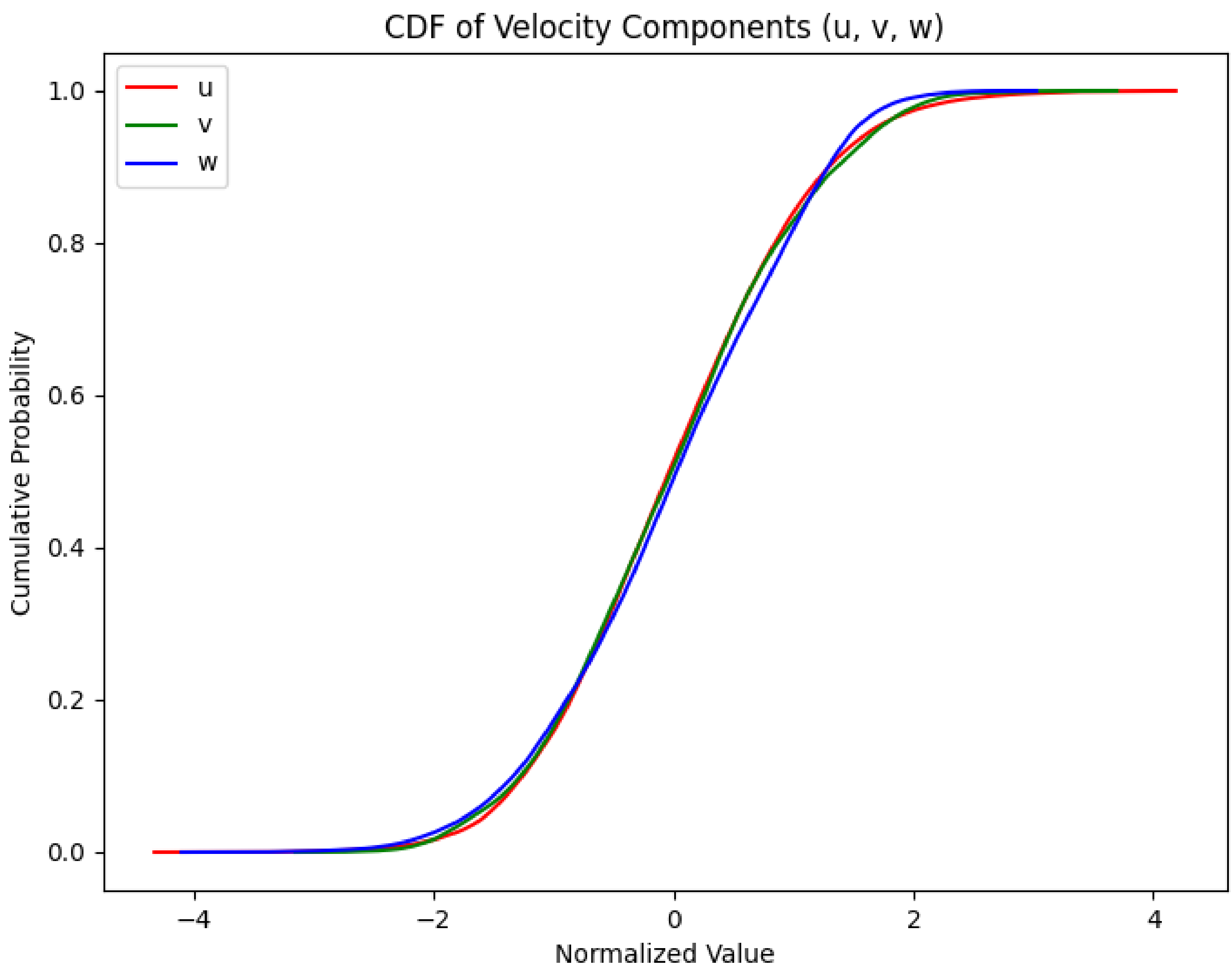
PDF of Velocity Gradients (∂v/∂y)
PDF of Velocity Gradients (∂v/∂y)
N(0,1)
Probability Density
Normalized Value

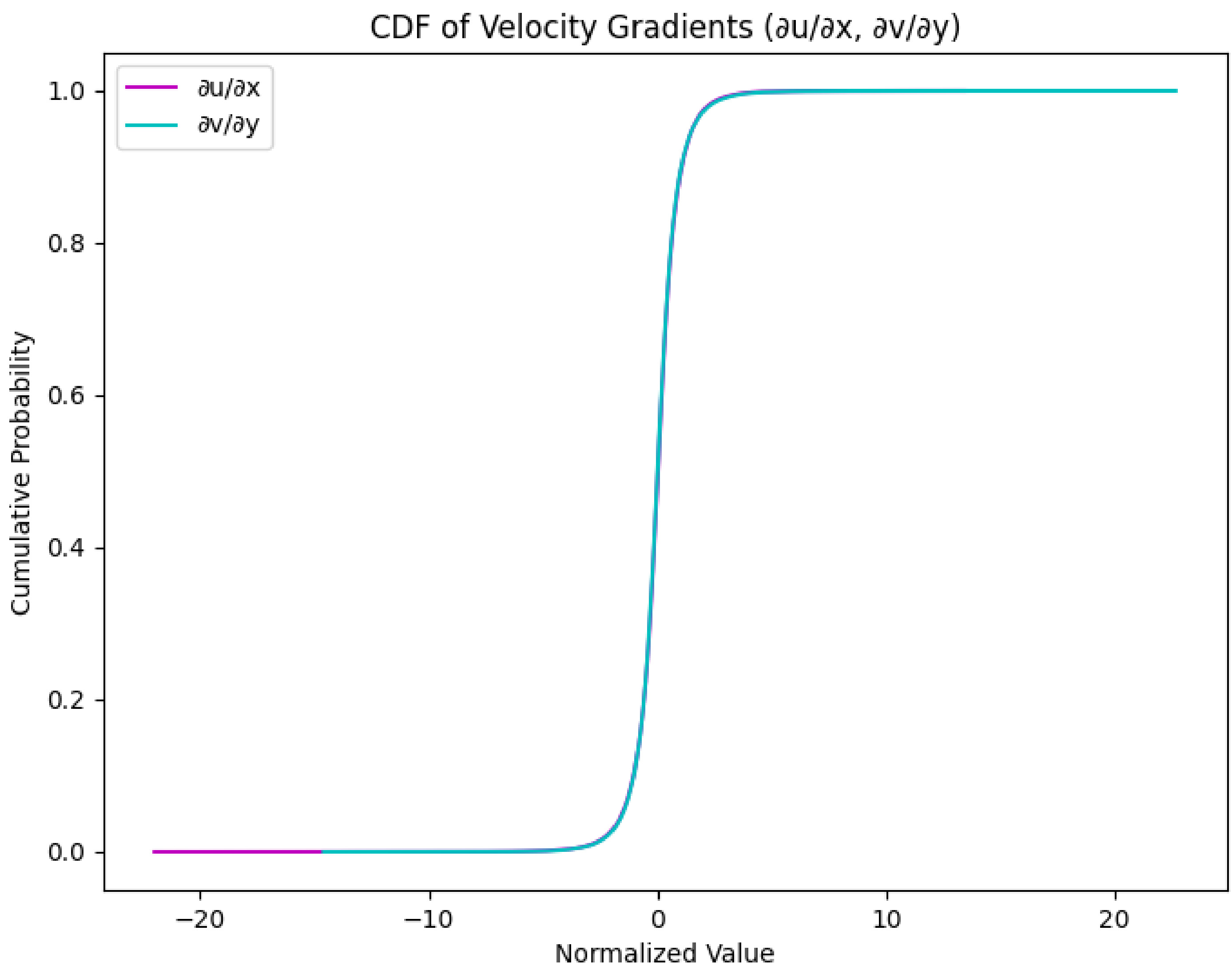

PDF of Vorticity Component (ω_z)
PDF of Vorticity Component (ω_z)
N(0,1)
Probability Density
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
−15
−10
−5
0
5
10
Normalized Value


PDF of Enstrophy (ω_z^2)
PDF of Enstrophy (ω_z^2)
N(0,1)
Probability Density
Normalized Value
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
20
40
60
80

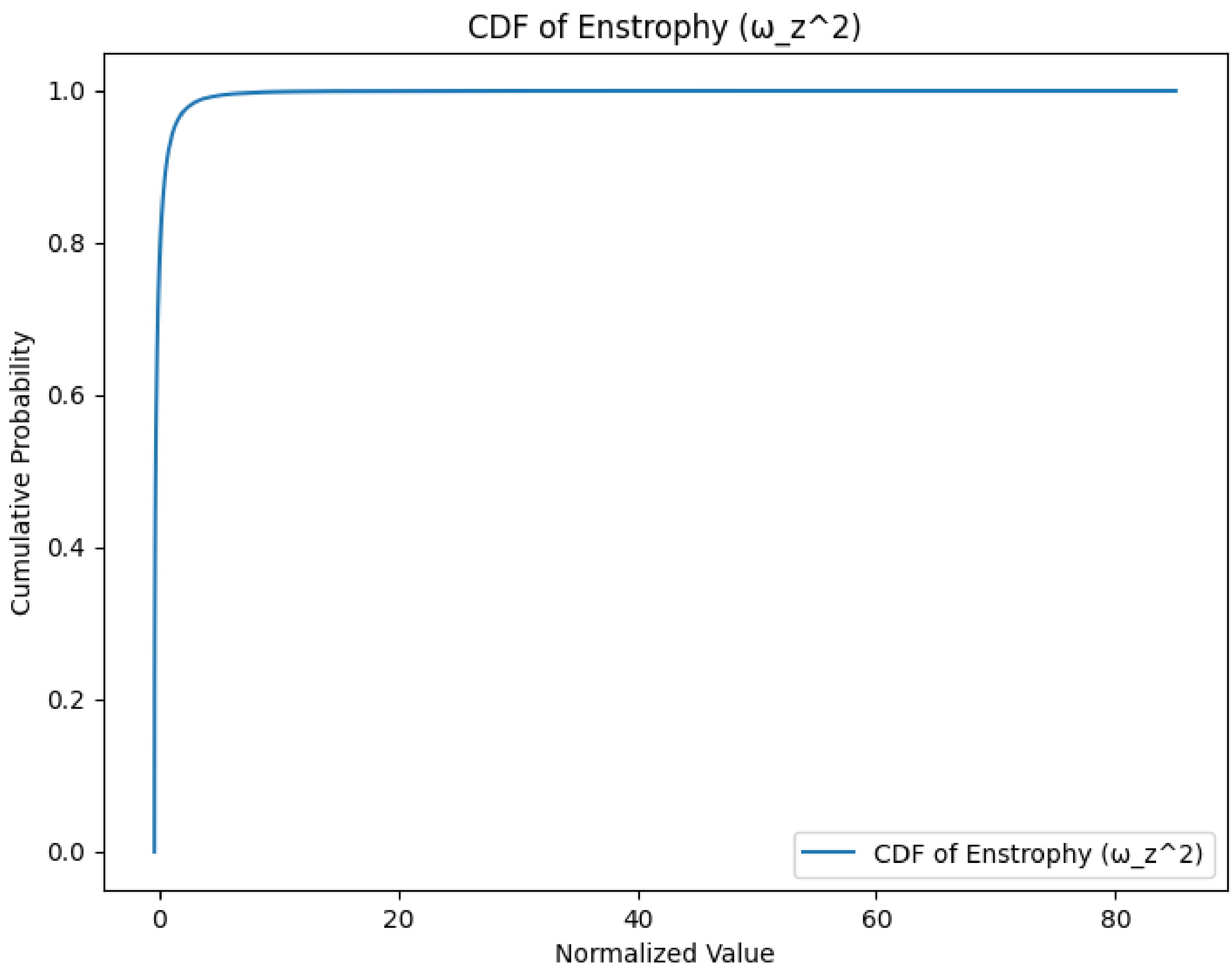


PDF of u(x, y) for Different x Ranges
x ∈ [0, 128]
x ∈ [129, 256]
x ∈ [257, 512]
N(0,1)
Probability Density
Normalized Value
10^{-1}
10^{-2}
10^{-3}
10^{-4}
−4
−2
0
2
4

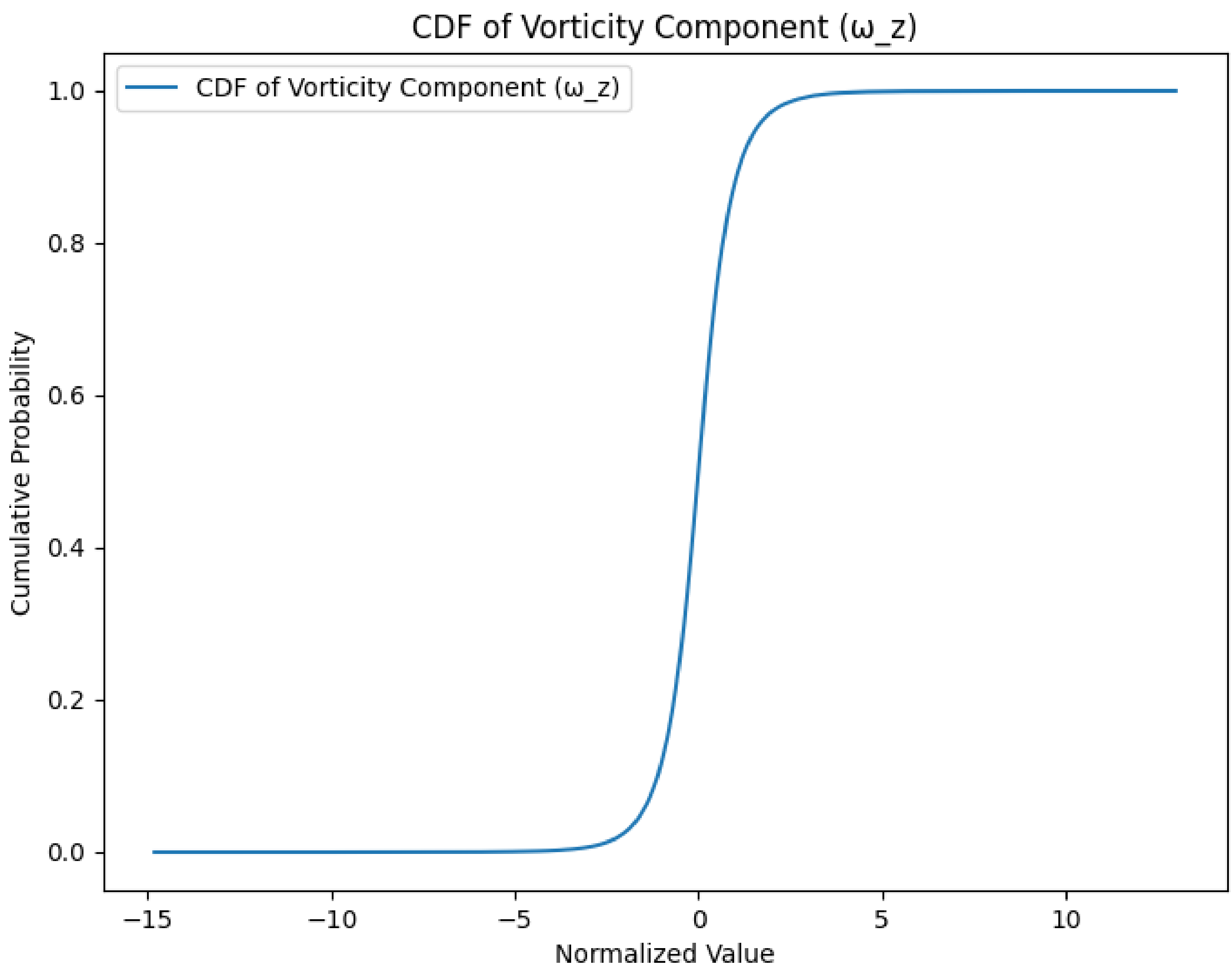

CDF of Velocity Components (u, v, w)
u
v
w
1.0
0.8
0.6
0.4
0.2
0.0
Cumulative Probability
−4
−2
0
2
4
Normalized Value

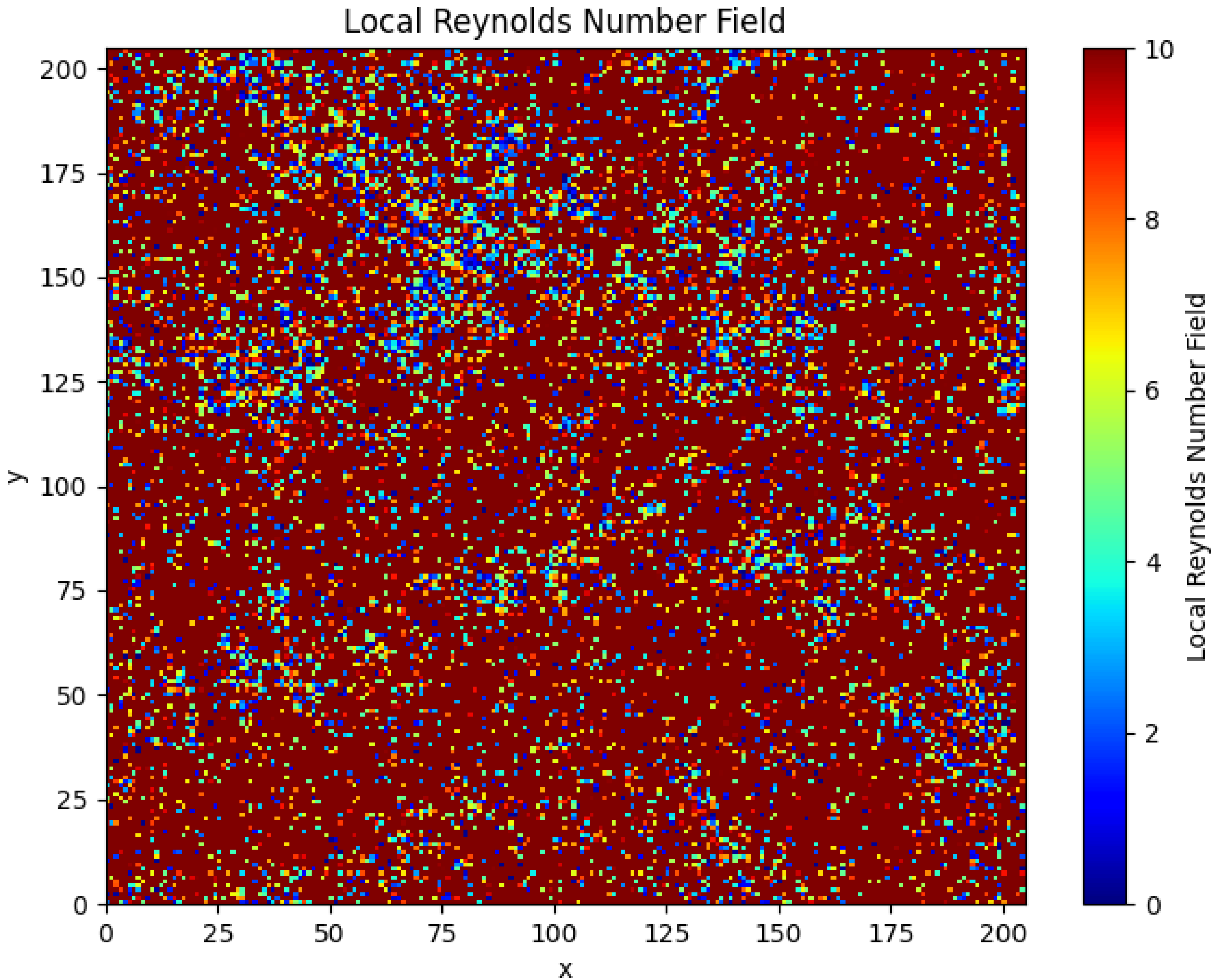


CDF of Velocity Gradients (∂u/∂x, ∂v/∂y)
∂u/∂x
∂v/∂y
1.0
0.8
0.6
0.4
0.2
0.0
Cumulative Probability
−20
−10
0
10
20
Normalized Value



CDF of Enstrophy (ω_z^2)
1.0
0.8
0.6
0.4
0.2
0.0
Cumulative Probability
0
20
40
60
80
Normalized Value
CDF of Enstrophy (ω_z^2)


CDF of Vorticity Component (ω_z)
CDF of Vorticity Component (ω_z)
1.0
0.8
0.6
0.4
0.2
0.0
Cumulative Probability
−15
−10
−5
0
5
10
Normalized Value



Local Reynolds Number Field
y
x
0
25
50
75
100
125
150
175
200
0
2
4
6
8
10
Local Reynolds Number Field

Ans
2-5. Comparison of Local Reynolds Number, kinetic Energy and vorticity fields:

- From the plots obtained from previous questions.

1. Kinetic Energy vs local Reynolds Number:

- The kinetic energy field shows a structured turbulence pattern with regions of high kinetic energy forming filament-like structures.
- The local Reynolds number field, on the other hand, appears more scattered, with high values spread in a more granular, discrete manner.
- While both fields show some correlation (regions of high kinetic energy tend to have higher local Re), the kinetic energy field has smoother transitions, whereas the Reynolds number field has a more stochastic distribution.

2. Vorticity vs Local Reynolds Number:

- The vorticity field shows fine-scale structures with alternating positive and negative vorticity.
- Comparing it with the local Re field, we see that strong vorticity regions often coincide with elevated Reynolds number values, especially in the turbulent zones.
- However, unlike the kinetic energy field, the vorticity structures show more intricate, small-scale variations, whereas the Reynolds number field is still more patchy.

3. Overall Trend:

- The local Reynolds number appears to be influenced by both kinetic energy and vorticity but is more randomly distributed.
- High kinetic energy regions generally correspond to higher Reynolds numbers, but not all high Reynolds number regions have high kinetic energy.
- Vorticity has a strong influence on localized Reynolds number variations, especially in areas where turbulent eddies and vortices exist.

Conclusion:

- The kinetic energy field provides a large-scale view of turbulence intensity, while vorticity highlights small-scale rotational structures.
- The local Reynolds number reflects aspects of both fields but in a more sporadic manner.
- Overall, kinetic energy and vorticity contribute to regions of high Reynolds numbers, but their relationship is not strictly one-to-one.