5. Queuing theory

Queues are systematic waiting lines to get the serv...

Operating Characteristics:

(i) Arrival Pattern.

(ii) Service Pattern

(iii) Queue discipline.

(iv) Customer behaviour [Balking, Reneging, Jockey]

Pure birth process: birth means arrival.

Probability of one arrival during Δt is $\lambda \Delta t$.

Probability of more than one arrival during Δt is ze...

The no. of arrivals b/w 2 non-overlapping intervals are statistically independent, then

$$\boxed{P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

Notations: λ : arrival rate.

μ : service rate

$P_n(t)$: probability that there are 'n' customers at the time 't'.

P_n : probability that there are n customers

n : no. of customers in the system.

m : no. of customers in the queue $= n-1$

L_s : length of the system = Expected value of n = $E(n)$

L_q : length of the queue = $E(m)$.

W_s : waiting time in the system.

W_q : waiting time in the queue

ρ : traffic intensity = λ/μ.

Pure death process: death means departure.

(i) The probability of departure during δt is $\mu\delta t$.

(ii) Probability of ~~departure~~ more than 1 departure during δt is zero.

(iii) No. of departures b/n. & non overlapping intervals are statistically independent. Then

$$P_n(t) = e^{-\mu t}(\mu t)^n / n!$$

Kendal's Notation: Useful to represent a queuing system

$$(a/b/c) = (d/e).$$

a → shows the distribution of arrival time.

b → shows the " " service time.

c → no. of services. d → system capacity.

e → queue discipline.

Eg:- $(M/M/1) = (\infty/FIFO)$
$(M/M/1) = (N/FIFO)$ } birth and death models.

Note: Studying the constants wrt time is known as "Transcend state" Expression" & without time is known as "steady state expression".

$(M/M/1):(\infty/FIFO)$:- To derive the expression for:

$\boxed{P_n = \rho^n P_0}$ where $\boxed{\rho = \lambda/\mu}$ & $\boxed{P_0 = 1-\rho}$

Constants: $P(n \text{ customers}) = P_n = \rho^n P_0$

$P(\text{system is empty}) = P_0 = 1-\rho$

$$P(n \text{ or more customers}) = P_n + P_{n+1} + P_{n+2} + \dots$$

$$= \rho^n P_0 + \rho^{n+1} P_0 + \rho^{n+2} P_0 + \dots = (\rho^n + \rho^{n+1} + \rho^{n+2} + \dots) P_0$$

$$= (1 + \rho + \rho^2 + \dots)\rho^n P_0 = P_0\left[\frac{1}{1-\rho}\right]\rho^n$$

$$= \cancel{(1-\rho)}\left[\frac{1}{\cancel{1-\rho}}\right]\rho^n = \rho^n$$

$$P(\text{more than } n \text{ customers}) = P_{n+1} + P_{n+2} + P_{n+3} + \dots$$

$$= \rho^{n+1} P_0 + \rho^{n+2} P_0 + \dots = \rho^{n+1} P_0 (1 + \rho + \rho^2 + \dots)$$

$$= \rho^{n+1} P_0 \left(\frac{1}{1-\rho}\right) = \rho^{n+1}\left(\frac{\cancel{1-\rho}}{\cancel{1-\rho}}\right) = \underline{\underline{\rho^{n+1}}}$$

$$L_s = E(n) = \rho/1-\rho$$

$$E_n = \sum_{n=0}^{\infty} n P_n$$

$$= P_1 + 2P_2 + 3P_3 + \dots$$

$$= \rho P_0 + \rho^2 P_0 + \rho^3 P_3 + \dots$$

$$= P_0 \rho (1 + 2\rho + 3\rho^2 + \dots)$$

$$= P_0 \rho \left[\frac{1}{(1-\rho)^{\cancel{2}}}\right] \qquad [\because P_0 = (1-\rho)]$$

$$= \underline{\underline{\rho/1-\rho}}$$

6) $L_q = E(m) = \dfrac{\rho^2}{1-\rho}$

$= \sum (n-1) P_n$.

$W_s = 1/(\mu - \lambda)$

$W_q = \rho/(\mu - \lambda)$.

<u>Note:</u> $\boxed{\begin{array}{l} L_q = \rho L_s \\ W_q = \rho W_s \end{array}} \longrightarrow$ Little's formula.

Expected length of non empty queue.

$$E(m/m>0) = \frac{E(m)}{P(m>0)} = \frac{1}{1-\rho}\left[\because \frac{\frac{\cancel{\rho^2}}{1-\rho}}{\cancel{\rho^2}}\right]$$

$$P(m>0) = P(n-1>0) = P(n>1) = \rho^2.$$

$$V(n) = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

$$V(n) = E(n^2) - (E(n))^2.$$

P(To wait more than a time w) $= \rho e^{-(\mu-\lambda)w}$.