2 - DOF planar RR with joint angles $\theta = [\theta_1, \theta_2]^T$

L_1, L_2 = link lengths ; ℓ_1, ℓ_2 ($\ell_i = L_i/2$) = Link center-of-mass distances

m_1, m_2 = link masses, rotational inertias about COM I_1, I_2

Gravity = g = 9.81 ($-y$ direction)

Lagrange - Euler : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$; $(q = \theta)$; $L(\theta, \dot{\theta}) = T(\theta, \dot{\theta}) - V(\theta)$

Manipulator Dynamics : $\tau = M(q)\ddot{q} + h(q, \dot{q})$ where $M(q)$ = mass matrix, $h(q)$ = Coriolis + centripetal + gravity

Kinematics :

Link 1 COM - $r_{c1} = \begin{bmatrix} \ell_1\cos\theta_1 \\ \ell_1\sin\theta_1 \end{bmatrix}$

Link 2 COM - $r_{c2} = \begin{bmatrix} L_1\cos\theta_1 + \ell_2\cos(\theta_1+\theta_2) \\ L_1\sin\theta_1 + \ell_2\sin(\theta_1+\theta_2) \end{bmatrix}$

Link 1 COM Velocity - $\dot{r}_{c1} = \begin{bmatrix} -\ell_1\dot{\theta}_1\sin\theta_1 \\ \ell_1\dot{\theta}_1\cos\theta_1 \end{bmatrix}$

Link 2 COM Velocity - $\dot{r}_{c2} = \begin{bmatrix} -L_1\dot{\theta}_1\sin\theta_1 - \ell_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \\ L_1\dot{\theta}_1\cos\theta_1 + \ell_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) \end{bmatrix}$

Angular velocities : $\omega_1 = \dot{\theta}_1$, $\omega_2 = \dot{\theta}_1 + \dot{\theta}_2$

Energies :

Kinetic (translational + rotational) :

$$T = \tfrac{1}{2}m_1\dot{r}_{c1}^T\dot{r}_{c1} + \tfrac{1}{2}I_1\dot{\theta}_1^2 + \tfrac{1}{2}m_2\dot{r}_{c2}^T\dot{r}_{c2} + \tfrac{1}{2}I_2(\dot{\theta}_1+\dot{\theta}_2)^2$$

Potential (Gravity downwards) : $V = m_1 g\ell_1\sin\theta_1 + m_2 g(L_1\sin\theta_1 + \ell_2\sin(\theta_1+\theta_2))$

Euler-Lagrange:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

Final Equations of Motion:

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}$$

$$M_{11} = I_1 + I_2 + m_1 \ell_1^2 + m_2\left(L_1^2 + \ell_2^2 + 2L_1\ell_2\cos\theta_2\right)$$

$$M_{12} = I_2 + m_2\left(\ell_2^2 + L_1\ell_2\cos\theta_2\right)$$

$$M_{22} = I_2 + m_2\ell_2^2$$

Coriolis/Centripetal Matrix $C(q,\dot{q})$

$$h = -m_2 L_1 \ell_2 \sin\theta_2$$

$$C(\theta,\dot{\theta}) = \begin{bmatrix} h\dot{\theta}_2 & h(\dot{\theta}_1 + \dot{\theta}_2) \\ -h\dot{\theta}_1 & 0 \end{bmatrix}$$

Gravity Vector $G(q)$:

$$G(q) = \begin{bmatrix} (m_1\ell_1 + m_2 L_1) g\cos\theta_1 + m_2\ell_2 g\cos(\theta_1 + \theta_2) \\ m_2\ell_2 g\cos(\theta_1 + \theta_2) \end{bmatrix}$$