Dynamical Systems:
Linearization, Equilibrium Points & Stability

Equilibrium Points

Consider the following nonlinear system:

$$\dot{q}_1 = aq_1 - bq_1q_2$$

$$\dot{q}_2 = bq_1q_2 - cq_2$$

where $q_1, q_2 \geq 0$ and $a, b, c \geq 0$ are positive constants. This can be represented as:

$$\frac{dx}{dt} = f(x);$$

$$x = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} aq_1 - bq_1q_2 \\ bq_1q_2 - cq_2 \end{bmatrix}$$

Equilibrium points occur when our system does not change with time. This occurs when:

$$\dot{q}_1 = 0$$

$$\dot{q}_2 = 0$$

Therefore, equilibrium points occur when:

$$q_1(a - bq_2) = 0$$

$$q_2(bq_1 - c) = 0$$

We can solve and show that there are two equilbrium points:

- Equilibrium Point 1:
$$q_1 = 0 \ , \ q_2 = 0$$

- Equilibrium Point 2:
$$q_1 = \frac{c}{b} \ , \ q_2 = \frac{a}{b}$$

Jacobian Linearization Around an Equilibrium Point

If we want to study the local behavior of our system close to an equilibrium point, the nonlinear perturbations (higher-order) can be ignored as they are small in comparison to the lower-order linear terms that are derived via Taylor series expansion. Therefore, the Jacobian linearization of our nonlinear system is:

$$\frac{dx}{dt} = Ax$$

where A is the Jacobian of our matrix $f(x)$. Let $f_1 = \dot{q}_1$, $f_2 = \dot{q}_2$, and $x = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$.

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix}$$

- Evaluate at Equilibrium Point #1:

  $$A = \begin{bmatrix} a - bq_2 & -bq_1 \\ bq_2 & bq_1 - c \end{bmatrix}$$

  At Equilibrium Point 1, $q_1 = 0$ and $q_2 = 0$.

  $$A = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

  $$det(A - \lambda I) = 0$$

  Therefore, $\lambda_1 = a$ and $\lambda_2 = -c$. Since λ_1 is positive, the system is unstable around equilibrium point #1.

- Evaluate at Equilibrium Point #2:

  $$A = \begin{bmatrix} a - bq_2 & -bq_1 \\ bq_2 & bq_1 - c \end{bmatrix}$$

  At Equilibrium Point 2, $q_1 = \frac{c}{b}$ and $q_2 = \frac{a}{b}$.

  $$A = \begin{bmatrix} 0 & -c \\ a & 0 \end{bmatrix}$$

  $$det(A - \lambda I) = 0$$

  Therefore, $\lambda_1 = +i\sqrt{ac}$ and $\lambda_2 = -i\sqrt{ac}$. Since λ_1 and λ_2 are complex conjugates, the system is marginally stable around this equilibrium point.

Simulation of System

Choosing arbitrary constants a, b, c, we can simulate the response of this system for varying initial conditions. We choose a small time step, Δt, and assume that f is constant during the interval. Under these assumptions, q can be iteratively updated by

$$q_{t+\Delta t} = q_t + \Delta t \dot{q}_t = q_t + \Delta t f(q)$$

We choose a step size of $\Delta t = 0.1$, set constants $a = b = c = 1$, and plot the trajectory of q_1 vs q_2 over 20s for the following initial conditions:

- $q_1 = 1$, $q_2 = 3$
- $q_1 = 1.1$, $q_2 = 1$
- $q_1 = 1$, $q_2 = 1.5$