

# Dynamical Systems: Linearization, Equilibrium Points & Stability

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## Equilibrium Points

Consider the following nonlinear system:

$$\dot{q}_1 = aq_1 - bq_1q_2$$

$$\dot{q}_2 = bq_1q_2 - cq_2$$

where  $q_1, q_2 \geq 0$  and  $a, b, c \geq 0$  are positive constants. This can be represented as:

$$\frac{dx}{dt} = f(x);$$

$$x = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} aq_1 - bq_1q_2 \\ bq_1q_2 - cq_2 \end{bmatrix}$$

Equilibrium points occur when our system does not change with time. This occurs when:

$$\dot{q}_1 = 0$$

$$\dot{q}_2 = 0$$

Therefore, equilibrium points occur when:

$$q_1(a - bq_2) = 0$$

$$q_2(bq_1 - c) = 0$$

We can solve and show that there are two equilibrium points:

- Equilibrium Point 1:

$$q_1 = 0, q_2 = 0$$

- Equilibrium Point 2:

$$q_1 = \frac{c}{b}, q_2 = \frac{a}{b}$$

# Jacobian Linearization Around an Equilibrium Point

The Jacobian linearization of our nonlinear system is:

$$\frac{dx}{dt} = Ax$$

where  $A$  is the Jacobian of our matrix  $f(x)$ . Let  $f_1 = \dot{q}_1$ ,  $f_2 = \dot{q}_2$ , and  $x = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ .

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix}$$

- Evaluate at Equilibrium Point #1:

$$A = \begin{bmatrix} a - bq_2 & -bq_1 \\ bq_2 & bq_1 - c \end{bmatrix}$$

At Equilibrium Point 1,  $q_1 = 0$  and  $q_2 = 0$ .

$$A = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

Therefore,  $\lambda_1 = a$  and  $\lambda_2 = -c$ . Since  $\lambda_1$  is positive, the system is unstable around equilibrium point #1.

- Evaluate at Equilibrium Point #2:

$$A = \begin{bmatrix} a - bq_2 & -bq_1 \\ bq_2 & bq_1 - c \end{bmatrix}$$

At Equilibrium Point 2,  $q_1 = \frac{c}{b}$  and  $q_2 = \frac{a}{b}$ .

$$A = \begin{bmatrix} 0 & -c \\ a & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

Therefore,  $\lambda_1 = +i\sqrt{ac}$  and  $\lambda_2 = -i\sqrt{ac}$ . Since  $\lambda_1$  and  $\lambda_2$  are complex conjugates, the system is marginally stable around this equilibrium point.

## Simulation of System

Choosing arbitrary constants  $a, b, c$ , we can simulate the response of this system for varying initial conditions. We choose a small time step,  $\Delta t$ , and assume that  $f$  is constant during the interval. Under these assumptions,  $q$  can be iteratively updated by

$$q_{t+\Delta t} = q_t + \Delta t \dot{q}_t = q_t + \Delta t f(q)$$

We choose a step size of  $\Delta T = 0.1$ , set constants  $a = b = c = 1$ , and plot the trajectory of  $q_1$  vs  $q_2$  over 20s for the following initial conditions:

- $q_1 = 1, q_2 = 3$
- $q_1 = 1.1, q_2 = 1$
- $q_1 = 1, q_2 = 1.5$