NASA Computational Case Study
The Flight of *Friendship 7*

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NASA Science Application: Astrodynamics, celestial mechanics, navigation, Project Mercury
Computational Algorithms: Modeling, time and coordinate conversions, root finding

Abstract

In this case study, we learn how to compute the position of an Earth-orbiting spacecraft as a function of time. As an exercise, we compute the position of John Glenn's Mercury spacecraft *Friendship 7* as it orbited the Earth during the third flight of NASA's Mercury program.

1 Introduction

On February 20, 1962, astronaut John Glenn became the third American astronaut in space and the first American to orbit the Earth in his spacecraft *Friendship 7*. Glenn's spacecraft was launched into orbit atop a Mercury-Atlas rocket at 9:47:39 am EST from Cape Canaveral, Florida (Figure 1). He completed three orbits around the Earth in just under five hours, after which his spacecraft splashed down in the Atlantic Ocean at 2:43:09 pm EST, about 260 miles north of San Juan, Puerto Rico. He was picked up by the destroyer USS *Noa* a few minutes after splashdown.

Friendship 7, like any Earth-orbiting spacecraft, traveled in an elliptical orbit about the Earth, with the Earth's center of mass at one focus of the ellipse. The point in the orbit closest to Earth is called *perigee*; the point farthest from Earth is called *apogee*. For *Friendship 7*, the perigee altitude was 86.92 nautical miles[1] above the Earth's surface, and the apogee altitude was 140.92 nautical miles. [1]

Calculating the position of *Friendship 7* at any time involves the application of some basic methods of celestial mechanics, which we'll examine here.

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[1] A unit of distance frequently used in spacecraft work: 1 nautical mile is exactly 1852 meters.

Figure 1: Launch of the Mercury-Atlas 6 mission on February 20, 1962. The *Friendship 7* spacecraft is the dark cone at the top of the Mercury-Atlas rocket. *(Credit: NASA)*

2 Time Measurement

In order to calculate the position of an Earth-orbiting spacecraft, we begin with finding a suitable method for measuring time. In everyday civil use, we measure time using a rather cumbersome system: years, months (of irregular length, between 28 and 31 days), and day of month. Within a single day, we divide time into hours, minutes, and seconds.

This system is not particularly convenient for use in calculations or plotting, where we would like to have time varying smoothly along some continuous time scale and measured with a single unit. Astronomers have developed such a uniform scale of time measurement, called the *Julian day*: it is defined to be the total number of days elapsed since *noon* on Monday, January 1, 4713 B.C. [2,3] For example, noon on January 1, 2000 is Julian day 2451545.0, since that's the number of days that had elapsed in the 6712 years since the beginning of 4713 B.C.

A fractional day may be added to the Julian day. Usually the fractional day is counted from midnight *Coordinated Universal Time* (UTC), which is five hours ahead of Eastern Standard Time (EST).[2]

The Julian day for any date on the Gregorian calendar may be found by consulting a table of Julian days [4], or computed using a simple algorithm. Let Y by the year M the month number (1 for January, 2 for February, etc., up to 12 for December), and D be the day of month (including a fractional day, counted from midnight). The algorithm for computing the Julian day (JD) is [3]:

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[2]Other time scales may be used as well. When it's important to be clear which time scale is used in computing the fractional day, the time scale is added in parentheses following JD: for example, "JD(UTC)".

- If $M > 2$, then leave Y and M unchanged.
  If $M = 1$ or $M = 2$, then replace Y with $Y - 1$, and M with $M + 12$.

- Calculate

$$A = \text{INT}\left(\frac{Y}{100}\right) \tag{1}$$

$$B = 2 - A + \text{INT}\left(\frac{A}{4}\right) \tag{2}$$

- Then the Julian day JD is found from

$$\text{JD} = \text{INT}[365.25(Y + 4716)] + \text{INT}[30.6001(M + 1)] + D + B - 1524.5 \tag{3}$$

Here $\text{INT}(x)$ indicates the greatest integer less than or equal to x.

EXAMPLE 1. Robert Goddard launched his first experimental rocket in Auburn, Massachusetts, on March 16, 1926, at 19:30 UTC. What was the corresponding Julian day?

Solution. The time 19:30 corresponds to a fractional day

$$\frac{19}{24} + \frac{30}{1440} = 0.8125 \text{ day},$$

where we have used 1 day = 24 hours = 1440 minutes. Now using the above algorithm to compute the Julian day, we find:

$$Y = 1926, \quad M = 3, \quad D = 16.8125, \quad A = 19, \quad B = -13,$$

and so

$$\text{JD} = 2425990 + 122 + 16.8125 - 13 - 1524.5 = \boxed{2424591.3125 \quad \text{JD}}$$

Activity 1. *Apollo 11* astronaut Neil Armstrong first set foot on the Moon on July 21, 1969, at 02:56 UTC. Find the Julian day corresponding to this time.

3 Reference Frames

In order to describe an orbit mathematically, it is necessary to introduce a *reference frame*, with respect to which the orientation of the orbit can be measured. Such a reference frame is determined by a *reference plane*, as well as a *reference direction* within that plane. Two reference planes are in common use:

- The *equator* is the plane of the Earth's equator. This is the reference plane usually chosen for bodies orbiting the Earth.

- The *ecliptic* is the plane of the Earth's orbit. This is the reference plane chosen in most other cases: finding positions of Sun-orbiting bodies such as planets, comets, asteroids, etc.[3]

In both cases, the reference *direction* is chosen to be the direction of the *vernal equinox*, which is in the direction from the Earth to the Sun on the first day of spring.[4] It is a fixed point in space in the constellation Pisces, and lies along the line of intersection between the planes of the equator and the ecliptic. The line pointing toward the vernal equinox is therefore common to both reference planes.

When an orbiting body crosses the reference plane from *south* to *north*, it is said to be at the *ascending node* of the orbit. Crossing the plane in the other direction, north to south, is the *descending node*.

Throughout this case study, we'll use the equator as the reference plane.

4 Orbital Elements

A spacecraft orbiting the Earth will travel in an elliptical orbit, with the center of mass of the Earth at one focus of the ellipse. To calculate the position of an orbiting body at any time, we need several parameters to describe the orbit:

- The *size* of the orbit is given by the *semi-major axis*[5] a of the ellipse.

- The *shape* of the orbit is given by the *eccentricity* e of the ellipse. For an ellipse with semi-major axis a and semi-minor axis b, the eccentricity is defined to be the ratio of the distance between the center and either focus $\sqrt{a^2 - b^2}$ to the semi-major axis a:

$$e = \frac{\sqrt{a^2 - b^2}}{a}. \tag{4}$$

  The eccentricity is a dimensionless number that varies between 0 (for a perfect circle) to almost 1 (for a long, elongated, cigar-shaped ellipse).

- The *orientation* of the orbit in space with respect to the equatorial reference frame is determined by three angles, as shown in Figure 2:

  - The *inclination* i of the orbit. This is the dihedral angle between the orbit plane and the plane of the Earth's equator.

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[3]Although it orbits the Earth, the position of the Moon is usually described using the ecliptic as the reference plane.

[4]That is, spring in the northern hemisphere—around March 21.

[5]The *semi-major axis* a is half the long axis of the ellipse; the *semi-minor axis* b is half the short axis.
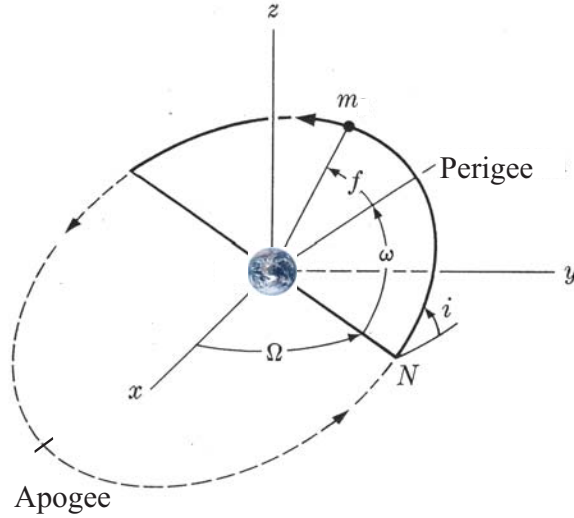
Figure 2: Orbital elements. In the case of an Earth-orbiting body, the xy plane is the plane of the equator, x is in the direction of the vernal equinox, and the z axis points northward along the Earth's rotation axis. Here the Earth is at the origin, and m is the orbiting body; i is the inclination, Ω is the longitude of the ascending node N, ω is the argument of perigee, and f is the true anomaly. (After [5].)

- The *longitude of the ascending node Ω.* This is the angle, measured in the plane of the Earth's equator, between the vernal equinox and the ascending node of the orbit.

- The *argument of perigee ω.* This is the angle, measured in the plane of the orbit, between the ascending node and the perigee point.

• The five elements given so far completely describe the orbit in space. We now only need to specify *where in the orbit* the spacecraft can be found at some given *epoch time T_0*. This is given by an angle called the *mean anomaly at the epoch time*, M_0. Angle M_0 is the angle, measured in the plane of the orbit, from the perigee point to the spacecraft position at epoch time T_0, assuming the spacecraft moves at a constant rate around the Earth.

5   Computation of the Position

We can now begin the orbit calculations, for which we'll use standard two-body orbit propagation methods. [5] Our goal will be to find the latitude φ and longitude L of an Earth-orbiting spacecraft for a given time t.

First, we will need to find the *mean daily motion* n of the orbit, which is just the number of revolutions (orbits) the spacecraft makes around the Earth per day. The mean daily motion may be given, or it may be computed from Kepler's third law if the semi-major axis a is known:

$$n = \frac{86400}{2\pi} \sqrt{\frac{GM_e}{a^3}}. \tag{5}$$

Here G is Newton's gravitational constant, M_e is the mass of the Earth, a is the semi-major axis of the orbit, and the factor $86400/2\pi$ converts from SI units of rad/s to units of rev/day. The product GM_e has a value[6,7] of 3.986004415×10^{14} m^3 s^{-2}.

Next, we begin by assuming the spacecraft is moving in a circular orbit at a constant rate; we'll correct for the eccentricity of the orbit later. Assuming a circular orbit, the mean anomaly M of the spacecraft at time t can be found from knowing the mean anomaly M_0 at the epoch time T_0:

$$M = M_0 + 2\pi n(t - T_0). \tag{6}$$

Here M is the mean anomaly at time t, M_0 is the mean anomaly at the epoch time T_0, and n is the mean daily motion. Both M and M_0 are in units of *radians*, both t and T_0 are specified as Julian days, and n has units of rev/day. (The factor of 2π converts the second term on the right from units of revolutions to radians.)

Next we introduce a correction for the eccentricity of the orbit. We do this by computing an angle called the *eccentric anomaly E*, which is found by solving *Kepler's equation*,

$$M = E - e\sin E. \tag{7}$$

Here both M and E must be in radians,[8] and e is the given eccentricity. The mean anomaly M is known from Eq. (6), and we wish to solve for the eccentric anomaly E. However, Kepler's equation cannot be solved explicitly for E; we must instead employ a numerical method to solve it. This is discussed in detail later.

The next step is to find another angle, the *true anomaly f*. This is the angle, measured in the plane of the orbit, from the perigee point to the spacecraft's position at time t, with the focus of the orbit at the vertex of the angle (Figure 2). The true anomaly f is found from

$$\tan\left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right). \tag{8}$$

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[6]As given in Ref. [4]. The *product* GM_e is known to better accuracy than either G or M_e individually, so it's best to use this product in computing Eq. (5).

[7]The reciprocal of n is the *orbital period*, or time required to orbit the Earth, in days.

[8]When an equation contains a "bare" angle that's not the argument of a trigonometric function—as M and the first E are here—the angle is assumed to be in radians.

We now need to compute the radial distance r from the orbit focus to the spacecraft. This is found from the eccentric anomaly E using

$$r = a(1 - e \cos E),\qquad(9)$$

where r will have the same units as the given semi-major axis a.

The quantities (r, f) are the polar coordinates of the spacecraft, in the plane of the orbit; the rest of the work is a series coordinate transformations, to go from plane polar coordinates in the plane of the orbit to spherical polar coordinates with the xy plane at the equator. We begin these coordinate transformations by defining the argument of latitude u by

$$u = \omega + f,\qquad(10)$$

where ω is the given argument of perigee and f is the true anomaly. Then the cartesian coordinates of the spacecraft in the equatorial frame at time t are given by

$$
\begin{aligned}
x &= r(\cos u \cos \Omega - \sin u \sin \Omega \cos i) & (11)\\
y &= r(\cos u \sin \Omega + \sin u \cos \Omega \cos i) & (12)\\
z &= r \sin u \sin i & (13)
\end{aligned}
$$

Converting from cartesian to spherical polar coordinates gives an azimuthal angle α called the *right ascension*, and a co-polar angle δ called the *declination* (Figure 3):

$$
\begin{aligned}
\tan \alpha &= \frac{y}{x} & (14)\\
\sin \delta &= \frac{z}{r} & (15)
\end{aligned}
$$

Finally, the longitude L and latitude φ are related to the right ascension and declination through a rotation about the z axis due to the Earth's rotation:

$$
\begin{aligned}
L &= \alpha - \text{GST} & (16)\\
\varphi &= \delta & (17)
\end{aligned}
$$

Here GST is the *Greenwich Sidereal Time*, and is the angle from the vernal equinox to the prime meridian (measured in the plane of the equator) at time t. The GST (in degrees) may be found from an empirical formula [3]:

$$
\begin{aligned}
\text{GST} &= 280.46061837 + 360.98564736629(t - 2451545.0)\\
&\quad + 0.000387933\,T^2 - T^3/38710000, & (18)
\end{aligned}
$$

where t is a Julian day, and T is the time in Julian centuries (of 36525 days) from noon, January 1, 2000:

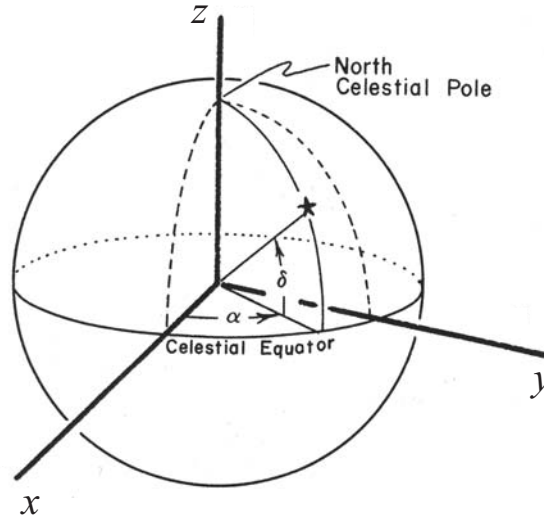$$T = \frac{t - 2451545.0}{36525}.\qquad(19)$$

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Figure 3: Right ascension α and declination δ of an object, shown here as a star. As in Fig. 2, the origin is at the center of the Earth, the x axis is toward the vernal equinox, and the z axis points northward along the Earth's rotation axis. (After [6].)

Equation (18) may return very large angles; you should add or subtract multiples of 360° as needed to bring the GST into the range $[\,0°, 360°)$.

Once you've computed the longitude L using Eq. (16), you should reduce it to the range $[-180°, +180°]$ by adding or subtracting multiples of 360° as needed. Then *positive L* corresponds to *east* longitude, and negative L to west longitude. The latitude φ should already be in the range $[-90°, +90°]$ after taking the inverse sine of both sides of Eq. (15); in this case *positive φ* corresponds to *north* latitude, and negative φ to south latitude.

Activity 2. The first satellite launched by the United States was *Explorer 1*, a 30-lb satellite in the shape of a cylinder about 5 feet long and 6 inches in diameter. It was launched into orbit by a Jupiter-C rocket on January 31, 1958, and was responsible for the discovery of the Van Allen radiation belts. It remained in orbit until 1970. [7]

The orbit of *Explorer 1* had a semi-major axis a of 7615.480 km, and an eccentricity e of 0.1155556. [8] (*a*) What was the mean daily motion n? (*b*) What was the orbital period? (*c*) What were the altitudes of apogee and perigee? (Assume for simplicity that the Earth is a sphere of radius 6378 km.)

Activity 3. (*a*) Referring to your solution to Activity 1, compute the Greenwich Sidereal Time (GST) for the moment Neil Armstrong first stepped onto the Moon. Reduce your answer to the range $[\,0°, 360°)$. (*b*) Given that the right ascension of the Moon at the time was $190.2°$, what was the geographic longitude of the Moon? (*c*) What part of the Earth would have been visible to Armstrong when he first looked back at the Earth from the Moon?

6   Solving Kepler's Equation

The calculations just described involved solving Kepler's equation, which relates the mean anomaly M to the eccentric anomaly E:

$$M = E - e \sin E, \tag{20}$$

where e is the eccentricity of the orbit, and both M and E must be in *radians*. We are given M and e, and wish to solve for E. This cannot be done in closed form, but must be done numerically.

Quite a few numerical methods for solving Kepler's equation have been developed over the years. [9] One simple method of solution is *Newton's method*.

Given a function $f(x)$, Newton's method is an iterative technique for finding the roots of $f(x)$, i.e. those values of x for which

$$f(x) = 0. \tag{21}$$

To find a root using Newton's method, we begin with an initial estimate x_0 of the root. Then successively better approximations to the root are found using the iterative formula [10]

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \tag{22}$$

where $f'(x)$ is the first derivative of $f(x)$. Starting with the initial estimate x_0, this formula gives a better estimate x_1. Substituting this x_1 back into the formula *again* gives an even better estimate x_2, and so on. Repeating this process gives better and better estimates of the root of $f(x)$; you simply continue the process until the answer converges to the desired accuracy (i.e. the difference between the current and previous iterations is below a certain threshold).

In the case of Kepler's equation, we wish to find the root E of the equation

$$f(E) = M - E + e \sin E = 0. \tag{23}$$

Therefore the iteration formula for the solution to Kepler's equation using Newton's method becomes

$$E_{n+1} = E_n - \frac{M - E_n + e \sin E_n}{e \cos E_n - 1}. \tag{24}$$

Newton's method requires an initial estimate for E, for which you can just use the mean anomaly M, so that $E_0 = M$.

Activity 4. Derive Eq. (24) from Eqs. (22) and (23).

Activity 5. Halley's comet is in a highly elongated elliptical orbit, with eccentricity $e = 0.967$. If the mean anomaly of Halley's comet is $M = 215°$, then use Newton's method to solve Kepler's equation for the eccentric anomaly E.
Hint 1: You will need to work this problem in radians. Begin by converting M from degrees to radians (by multiplying by $\pi/180$).
Hint 2: Use M as the initial estimate of E, then apply Eq. (24) repeatedly. It should only take 3 iterations for the answer to converge to 11 significant digits.
Hint 3: At the end, convert your final answer E from radians back to degrees (by multiplying by $180/\pi$).

Challenge. Now let's compute the position of the *Friendship 7* spacecraft at a specific time. During his first orbit, John Glenn reported seeing what he called "fireflies"—small lighted objects swirling outside the spacecraft:

> "I am in a big mass of some very small particles, that are brilliantly lit up like they're luminescent. I never saw anything like it. They round a little; they're coming by the capsule, and they look like little stars. A whole shower of them coming by. They swirl around the capsule and go in front of the window and they're all brilliantly lighted. They probably average maybe 7 or 8 feet apart, but I can see them all down below me, also."

It was later determined that the most likely cause of these lights was bits of paint or other material flaking off the spacecraft.

Glenn reported seeing these "fireflies" at around time 16:03:03 UTC. Find the latitude and longitude of the *Friendship 7* spacecraft at this time, given the orbital elements of the spacecraft in Table 1 below.

Table 1. Orbital elements of *Friendship 7*.

Orbital element	Symbol	Value
Semi-major axis	a	6589.116 km
Eccentricity	e	0.007589
Inclination	i	32.54°
Longitude of ascending node	Ω	235.2°
Argument of perigee	ω	181.2°
Mean anomaly at epoch	M_0	228.5°
Epoch time	T_0	2437716.11642 JD

If you write a computer program to compute the positions of the spacecraft at, for example, 10-second intervals during the entire mission—from launch to splashdown—and plot them on a map of the Earth, you'll see the sinusoidal *ground track* of *Friendship 7*, which shifts westward each orbit due to the rotation of the Earth.

About the Author

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References

[1] National Aeronautics and Space Administration. *Results of the First United States Manned Orbital Space Flight: February 20, 1962*. U.S. Government Printing Office, Washington, 1962.

[2] P.K. Seidelmann (ed.). *Explanatory Supplement to the Astronomical Almanac*. University Science Books, Mill Valley, Calif., 1992.

[3] J. Meeus. *Astronomical Algorithms*. Willmann-Bell, second edition, Richmond, 1998.

[4] *The Astronomical Almanac* (annual). U.S. Government Printing Office, Washington.

[5] S.W. McCuskey. *Introduction to Celestial Mechanics*. Addison-Wesley, Reading, Mass., 1963.

[6] R.B. Bate, D.D. Mueller, and J.E. White. *Fundamentals of Astrodynamics*. Dover, New York, 1971.

[7] *Explorer 1*. Retrieved May 23, 2012, from http://www.jpl.nasa.gov/explorer.

[8] Orbital elements for *Explorer 1* retrieved via special data request on May 11, 2012, from http://www.celestrak.com. Elements are for the beginning of the mission.

[9] P. Colwell. *Solving Kepler's Equation over Three Centuries*. Willmann-Bell, Richmond, 1993.

[10] R.W. Hamming. *Numerical Methods for Scientists and Engineers*. McGraw-Hill, New York, 1962.