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P. R. N. | | | | | | | | |

Date | | | | | | | |

TEST - I / TEST - II

Programme : B. Tech / B. Plan / M. Tech / M. Plan / M. B. A.

Year : First / Second / Third / Final

Branch : Linear Algebra Semester : Odd / Even

Course Name : Quadratic forms.

Name & Signature of Invigilator : Rashmi kulkarni.

Mobile phones, programmable calculators, etc. are strictly banned in the Examination Hall.

Examiner

Q. No.	1	2	3	4	5	6	7	8	Total	Name and Sign. of Examiner
Marks										

Moderator

Q. No.	1	2	3	4	5	6	7	8	Total	Name and Sign. of Moderator
Marks										

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and further to find orthonormal set calculate $B' = \left\{ \frac{x_1'}{\|x_1''\|^2}, \frac{x_2'}{\|x_2''\|^2}, \ldots, \frac{x_n'}{\|x_n''\|^2} \right\}$

* Applications of eigen values and eigen vec.

* quadratic forms

eigen values and eigen vect. can be use to solve the rotation of axis problem. For that consider quadratic eq^n from conic section given by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad \text{— (A)}$$

if eq^n does not have xy term i.e. $b = 0$ it is a straigh forward eq^n if xy term is non-zero then classification acomplish mostly by performic rotation of axis that eliminate term xy. relative to new axis $x'y'$ then eq^n A will be of form.

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f = 0 \quad \text{— (B)}$$

the coeff. a' and c' are eigen values of mat. $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$

The expression $ax^2 + bxy + cy^2$ is called quadratic form associated with eq^n (A) and mat. A is called matrix of the quadratic form. moreover A is symmetric by def^n and A will be diagonal iff its corresponding quadratic form has no xy term

Find mat of quad. form associated with quad. eqn given by

1) $4x^2 + 9y^2 - 36 = 0$ 2) $13x^2 - 10xy + 13y^2 - 72 = 0$

→ ① because comparing given eqn with std. quadratic form we get mat.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

a symmetric mat. and can be written in std. form as

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

is an eqn of ellipse with centre ~~zero~~ (0,0)

to form quadratic form for eq ex ②

to eliminate xy term, we need to perform ~~with~~ be axis of rotation let $x = \begin{bmatrix} x \\ y \end{bmatrix}$ rotation.

then quad. eqn Ⓐ can be written as

$$x' AX + [d \;\; e] X + f \longrightarrow \text{Ⓒ}$$

$$\Rightarrow [x \;\; y] \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [d \;\; e] \begin{bmatrix} x \\ y \end{bmatrix} + f$$

$$\Rightarrow ax^2 + bxy + cy^2 + dx + ey + f \Rightarrow \text{Ⓐ}$$

For c if $b = 0$ then no rotation is necessary but if b is non zero, then because A is symm. we can conclude that ∃ orthogonal mat. P such that $P^{-1}AP = D$ which is diagonal

Let $P^T X = X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

it follows that $X = PX'$

$$X^T AX = (PX')^T A (PX') = X'^T D X'$$

$$= X'^T P^T A P X'$$

here choice of P that we made to use axis of rotation as orthogonal mat.

its determinant will be ± 1

If we choose $\det P = +1$ then by earlier knowledge of rotation mat our P will be

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ gives angle of rotation of conic measured from $+x$ axis in the dirn of $+$ve x axis

This leads to princ. axis thm.

for a conic section whose eqn given by A. the rotation given by $X = PX'$ eliminates xy term if P is ortho. with $|P| = 1$ and it diagonalize A in the form.

$$P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1, λ_2 are eigen val. of A.

and eqn of rotated conic is given by

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + [d \ e] P X' + f = 0$$

remark. Note that mat product $[d \ e] P X'$ has the form

$$[d \ e] \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\Rightarrow [d\cos\theta + e\sin\theta]\, x' + (-d\sin\theta + e\cos\theta)\, y'$$

$\therefore$ in order to solve pr. 2.

$$\text{mat } A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}$$

Finding eigenval. using cha. polynomial.

$|\lambda I - A| = \begin{bmatrix} \lambda-13 & 5 \\ 5 & \lambda-13 \end{bmatrix} = 0 \qquad (\lambda-13)^2 - 5^2 = 0$

$\cancel{(\lambda-13+5)}$

$(\lambda-13+5)(\lambda-13-5) = 0$

$\lambda_2 = 8, \; \lambda_1 = 18$

so the eqⁿ of rotated conic. is

$18(x')^2 + 8(y')^2 - 72 = 0$

$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$

$$P = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

Find. mat A of quad. form associated with eqⁿ. $13x^2 - 8xy + 7y^2 - 45 = 0$ then find orthogonal mat. P.

$\Rightarrow \quad A = \begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix}$

$|\lambda I - A| = \begin{bmatrix} \lambda-13 & 4 \\ 4 & \lambda-7 \end{bmatrix} = 0$

$\therefore (\lambda-13)(\lambda-7) - 16 = 0$

$\lambda^2 - 7\lambda - 13\lambda + 91 - 16 = 0$

$\lambda^2 - 20\lambda + 75 = 0$

$\lambda_1 = 5, \; \lambda_2 = 15$

$\therefore$ eqⁿ of rotated conic.

$5(x')^2 + 15(y')^2 - 45 = 0$

$\frac{(x')^2}{9} + \frac{(y')^2}{3} = 1 \Rightarrow \frac{(x')^2}{3^2} + \frac{(y')^2}{(\sqrt{3})^2} = 1$

Conic Sections & Rotation

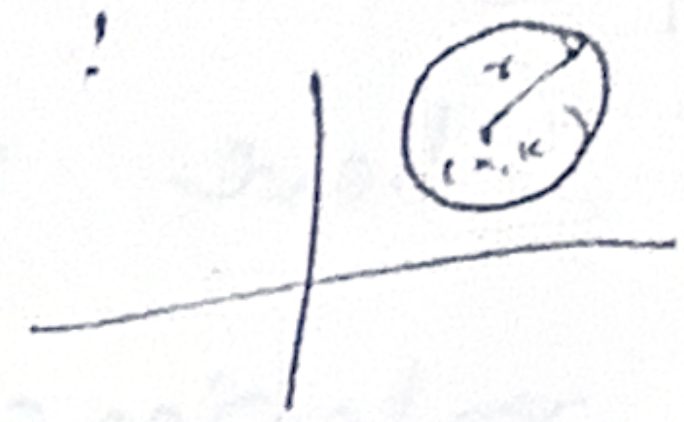
Every conic section in the xy-plane has an eq^n of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ — (*)

Identifying the graph of this equation is simple in case $b=0$ (i.e. xy term is not there). Because in that case conic axes are $\parallel^{el}$ to coordinate axes, and one can identify the conic by writing (*) in the standard form (i.e. completed square form) of conics:

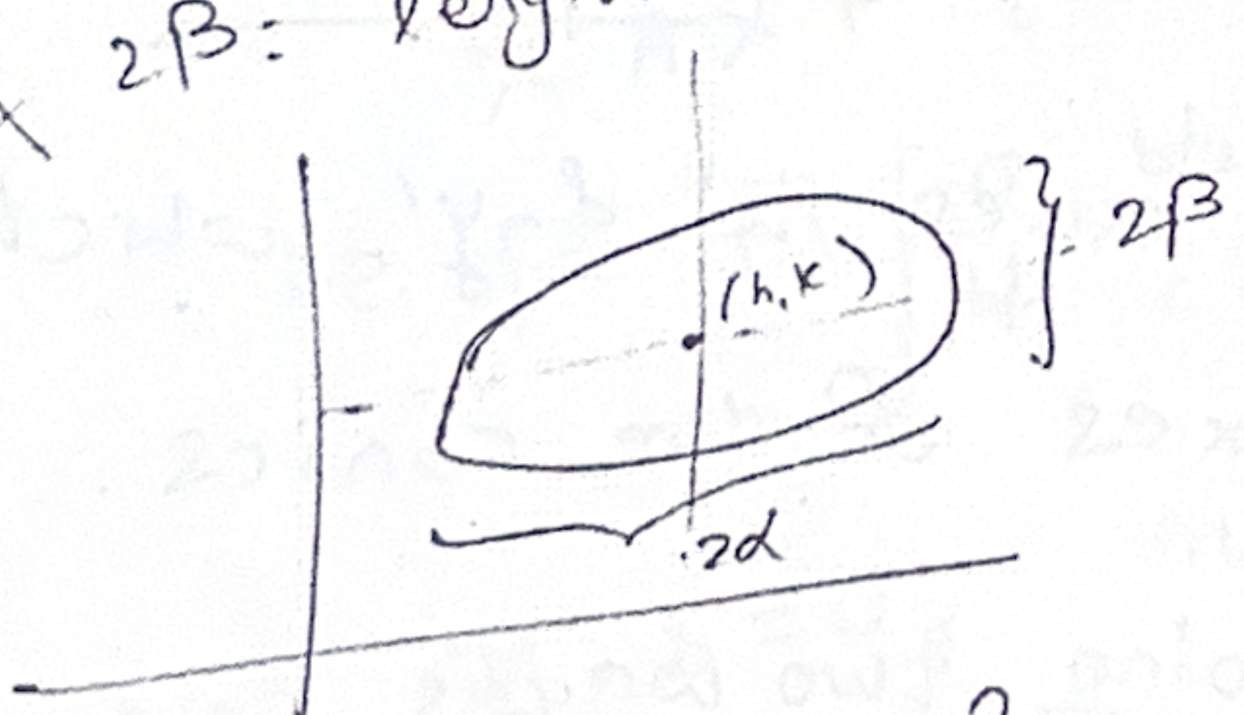
Recall: Standard forms of equations of conics:

① Circle with radius r & center (h,k):

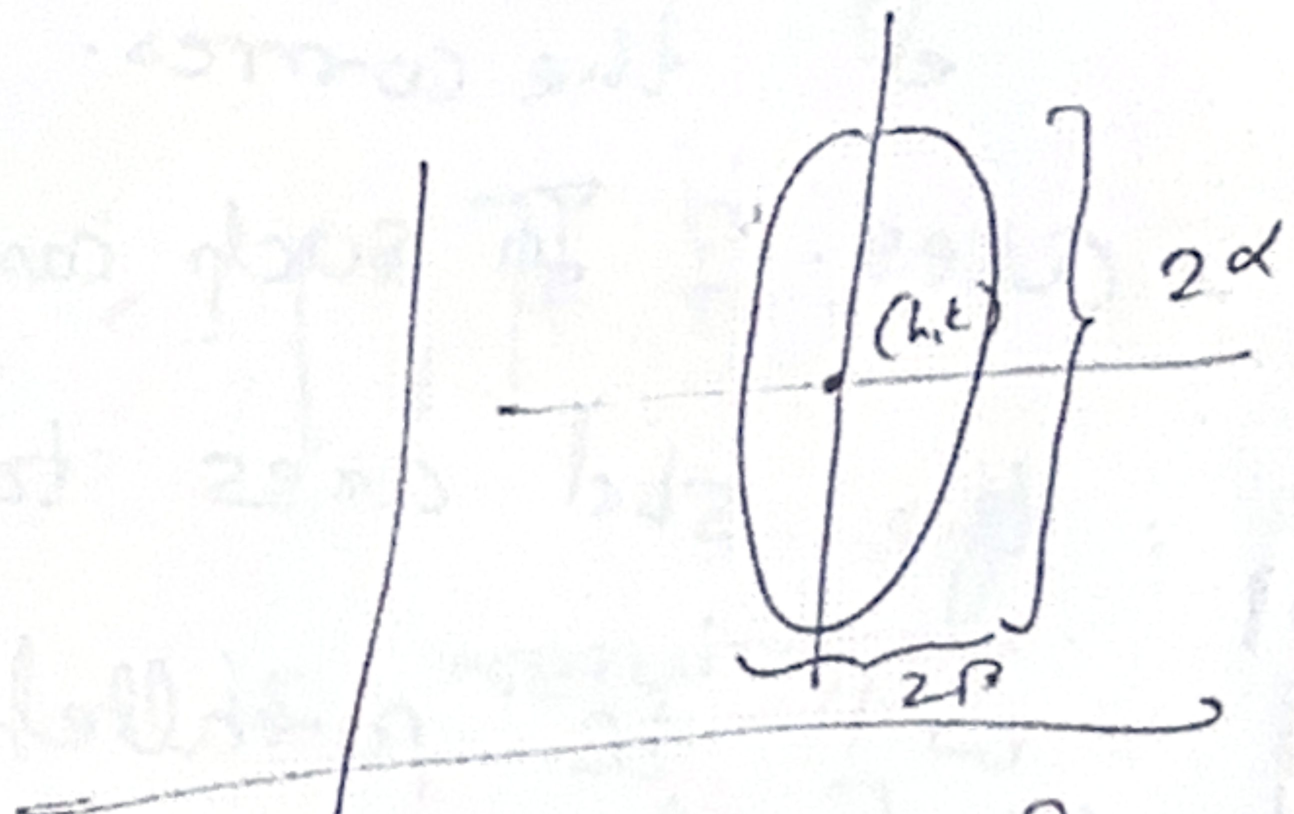
$$(x-h)^2 + (y-k)^2 = r^2$$



② Ellipse with center (h,k), 2α = length of major axis & 2β: length of minor axis.

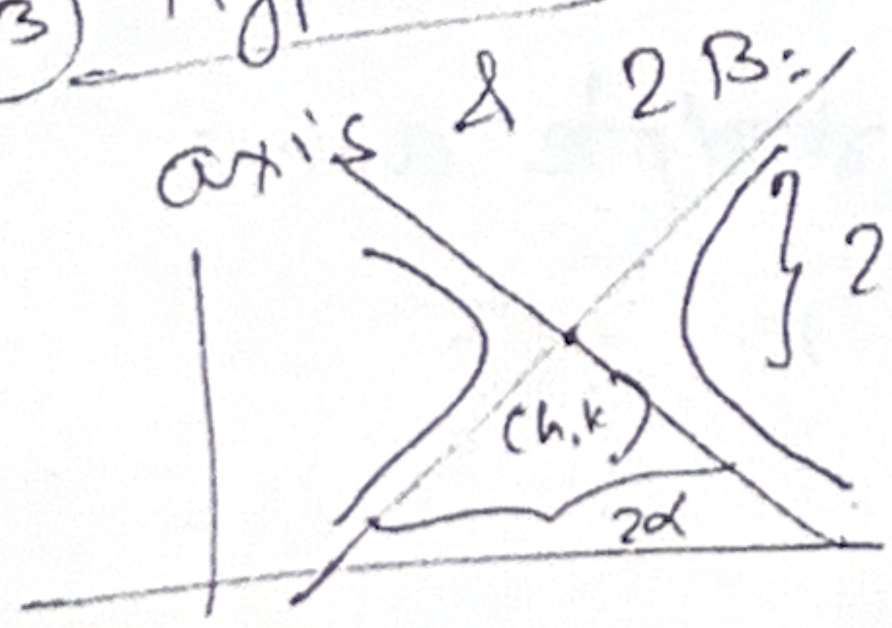


$$\frac{(x-h)^2}{\alpha^2} + \frac{(y-k)^2}{\beta^2} = 1$$

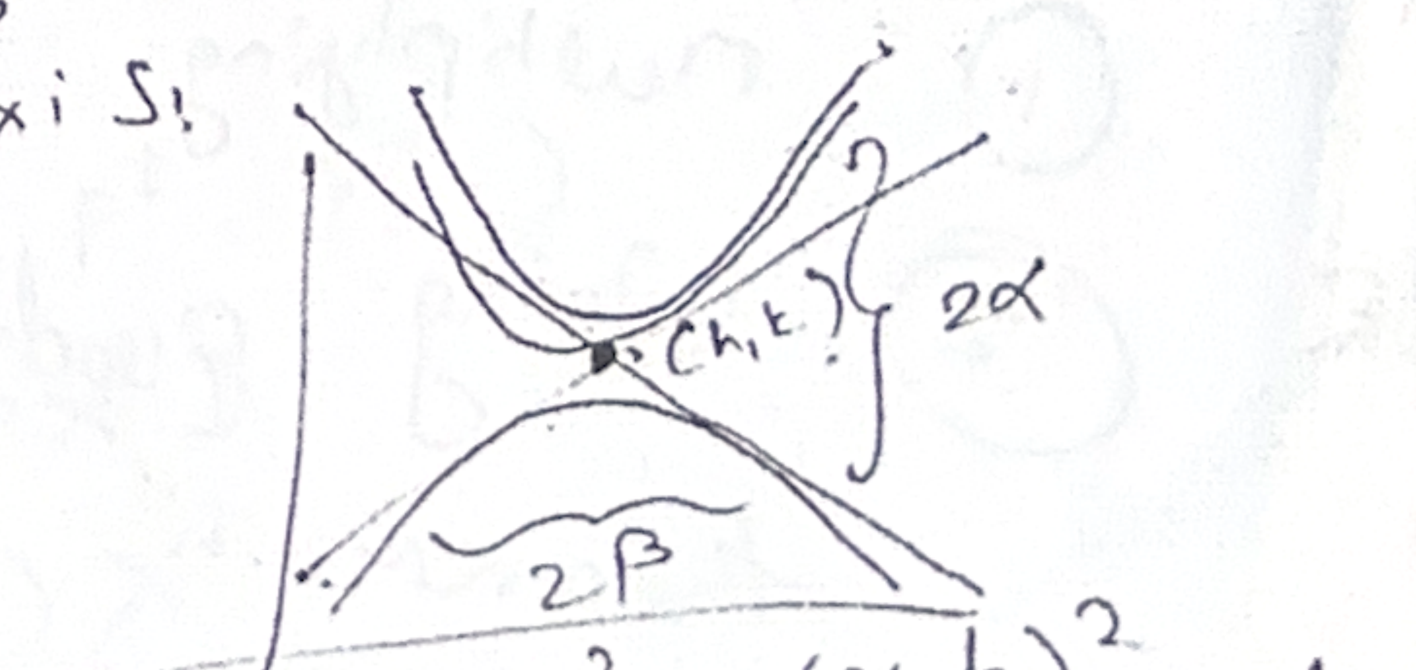


$$\frac{(x-h)^2}{\beta^2} + \frac{(y-k)^2}{\alpha^2} = 1$$

③ Hyperbola with center (h,k), 2α = length of transverse axis & 2β = length of minor axis:

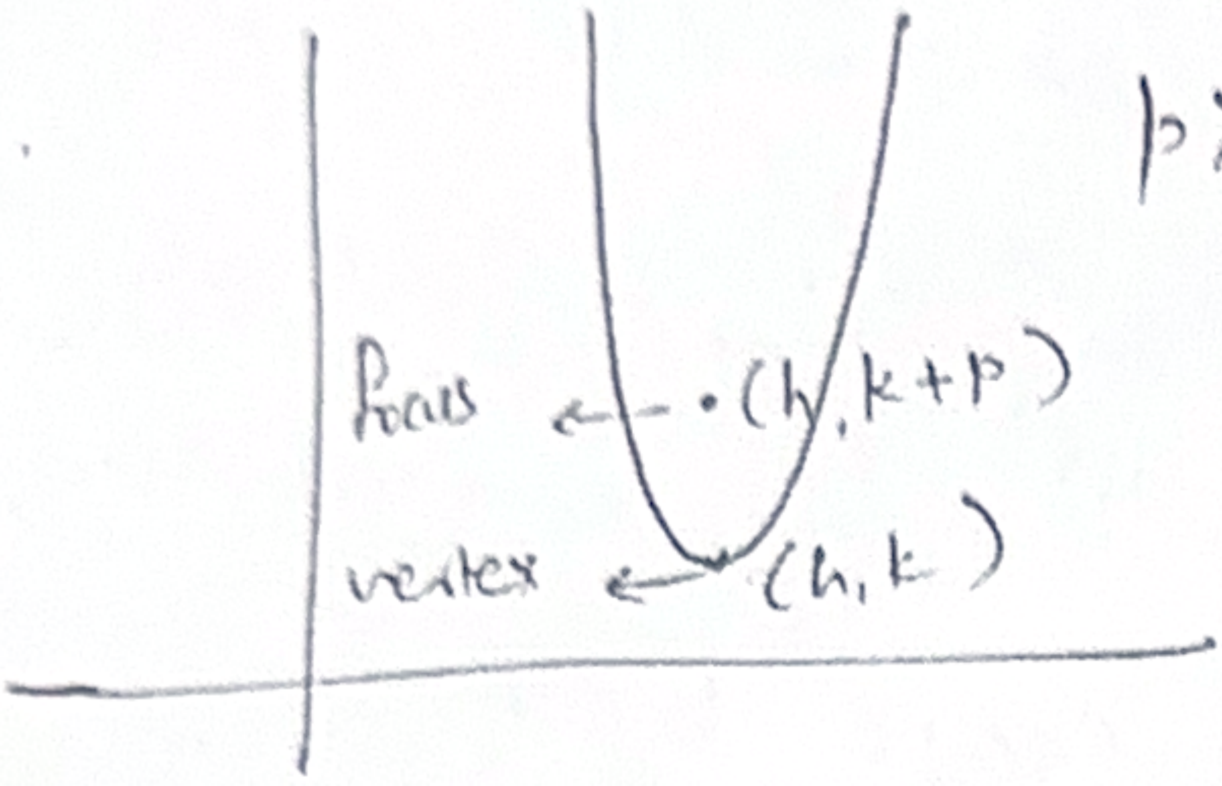


$$\frac{(x-h)^2}{\alpha^2} - \frac{(y-k)^2}{\beta^2} = 1$$

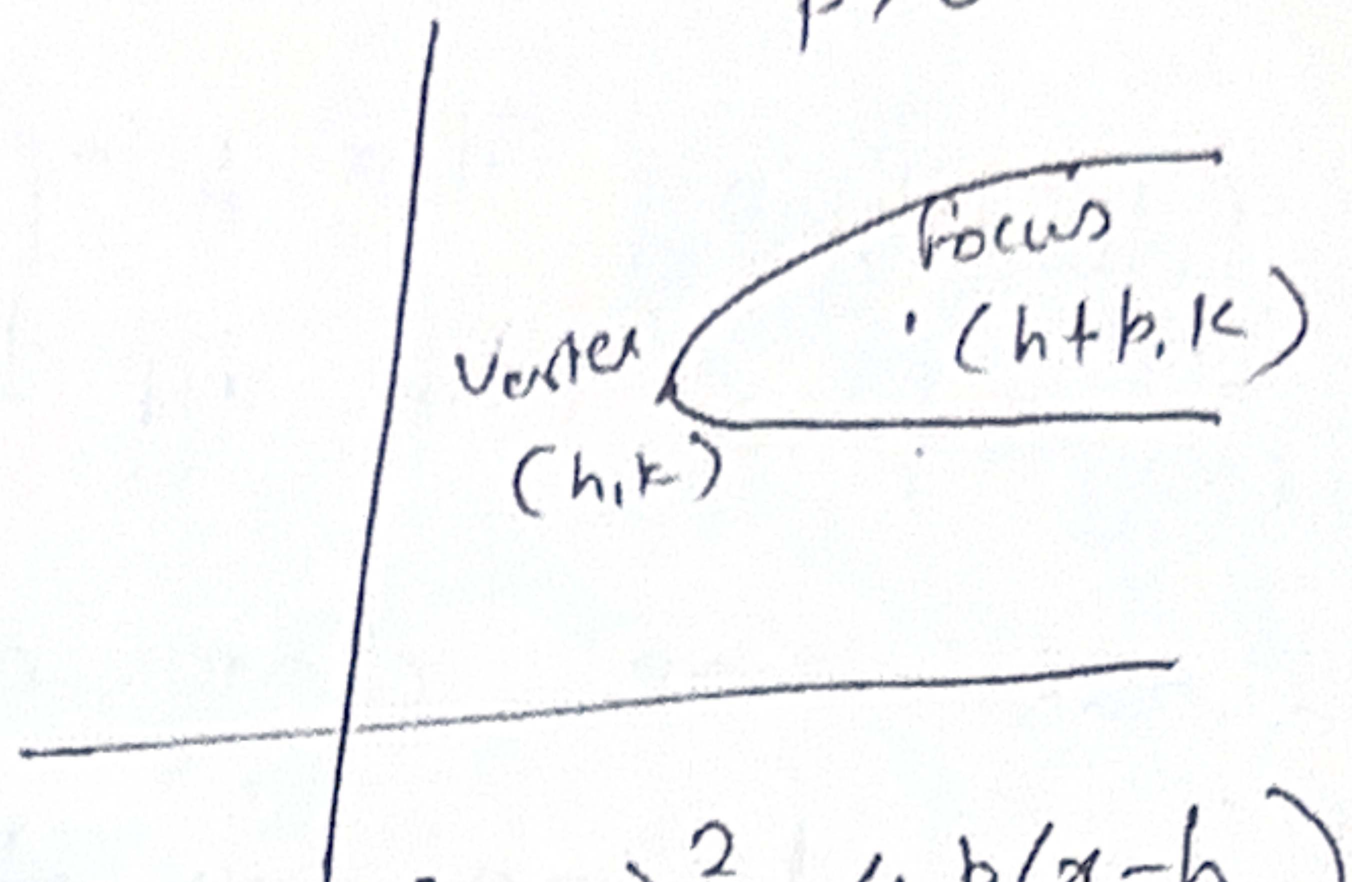


$$\frac{(y-k)^2}{\alpha^2} - \frac{(x-h)^2}{\beta^2} = 1$$

(4) <u>Parabola</u> with vertex at (h,k) & $p=$ directed distance from vertex to focus :



$(x-h)^2 = 4p(y-k)$



$(y-k)^2 = 4p(x-h)$

Note that none of the equations in standard form have xy term in them & their axes of rotations are parallel ~~match~~ with the coordinate axes

Whereas, 2nd deg. poly. having xy term, the axes of the corres. conics are not ||el to the coordinate axes. In such cases, one needs to <u>Rotate</u> the std axes to form new axes x' & y' which will be parallel to the axes of the conics.

This can be done ~~wi~~ using <u>two ways</u>

~~(1) Find angle θ through~~

(1) multiplying by rotation matrix

OR (2) Using Quadratic forms & principle axes theorem

The expression

$ax^2 + bxy + cy^2$ is called the quadratic form

associated with ⊛

Whereas, the matrix $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ is called the

matrix of the quadratic form associated with ⊛

Note that A is symmetric.

Moreover, A is diagonal iff its corres. quadratic

form has no xy-term.

Q. How to use matrix of quadratic form to perform rotation of axes?

→ let $X = \begin{bmatrix} x \\ y \end{bmatrix}$

Then $ax^2 + bxy + cy^2 + dx + ey + f$ can be written in

matrix form as

$X^t A X + [d \ \ e] X + f$

i.e. $[x \ \ y] \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [d \ \ e] \begin{bmatrix} x \\ y \end{bmatrix} + f$

Now, if $b = 0$, then no rotation is required.

But if $b \neq 0$, then for the symmetric matrix A, ∃ an

orthogonal matrix P such that $P^t A P = D$

∴ let $P^t X = X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Then, $X = PX'$ $(\because P^t = P^{-1})$

$\therefore X^t A X = (PX')^t A (PX')$

$= (X'^t)(P^t A P) X'$

$= (X')^t D X'$

Note : Choose P such that $|P| = 1$

$\therefore$ P is of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(If $|P| = -1$, multiply one of the columns of P by -1 to obtain the desired form)

Principal Axes Theorem:

For a conic whose equation is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

the rotation given by $X = PX'$ eliminates xy-term if P is an orthogonal matrix, with $|P| = 1$, that diagonalizes A.

i.e., $P^t A P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1, λ_2 are eigenvalues of A. The eq^n of the rotated conic is then given by

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + [d \;\; e] P X' + f = 0$$

Here, $[d \;\; e] P X'$ is of the form

$$(d\cos\theta + e\sin\theta)x' + (-d\sin\theta + e\cos\theta)y'.$$