COEP Technological University, Pune
Department of Mathematics
MA : Linear Algebra (LA)
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Tutorial 4 : Vector spaces and subspaces
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1. Define a vector space V over a field F.

   Consider vector spaces over $\mathbb{R}$. If nothing is mentioned about the operations on the vector spaces, assume them to be standard addition and scalar multiplication.
   $M_{m,n}$ denotes the vector space of all $m \times n$ matrices with entries from $\mathbb{R}$.
   P_n denotes the vector space of polynomials of degree at most n with real coefficients.
   $\mathbb{C}[a, b]$ denotes set of all continuous functions on $[a, b]$.

2. Determine whether the set, together with the corresponding standard operations, is a vector space. If it is not, identify at least one of the ten vector space axioms that fails:

   (a) $M_{2\times 6}$.

   (b) The set of third-degree polynomials.

   (c) The set of first-degree polynomial functions $ax + b,\ a \neq 0$, whose graphs pass through the origin.

   (d) The set $\{(x, y) : x \geq 0,\ y \text{ is a real number}\}$.

   (e) The set of 2×2 matrices of the form $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$.

3. Determine whether $\mathbb{R}^2$ with operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),\ c(x, y) = (cx, y)$ is a vector space. If it is not, identify at least one of the ten vector space axioms that fails.

4. Determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text:

   (a) The set of integers is a vector space over $\mathbb{R}$.

(b) To show that a set is not a vector space, it is sufficient to show that just one axiom is not satisfied.

(c) In a vector space, the zero vector must exist and it should be unique.

(d) In a vector space, additive inverse of an element need not be unique.

(e) Set of integer valued functions form a vector space.

5. Define linear combination of vectors in a vector space. What is meant by span of a subset?

6. Define subspace of a vector space and state the necessary and sufficient conditions for a subset of a vector space to be a subspace.

7. Consider the vector space $M_{2,2}$. Which of the following subsets are subspaces of $M_{2,2}$?

   (a) Set of symmetric matrices.

   (b) Set of singular matrices.

   (c) Set of scalar matrices.

   (d) Set of triangular matrices.

8. Determine if the given set is subspace of the mentioned vector space. Justify.

   (a) Set $\{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ in the vector space $\mathbb{R}^2$.

   (b) Set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in the vector space $\mathbb{R}^3$.

   (c) Set $\mathbb{C}[0, 1]$ in the vector space of all integrable functions on $[0, 1]$.

9. Let V be the set of all solutions of a system of m homogeneous linear equations in n variables with real coefficients. Is it a subspace of $\mathbb{R}^n$? Justify.

10. Let W_1 and W_2 be subspaces of a vector space V. Which of the following are subspaces of V?
(a) $W_1 \cup W_2$, (b) $W_1 \cap W_2$, (c) $W_1 + W_2$.

11. Determine whether the set S spans $\mathbb{R}^2$. If the set does not span $\mathbb{R}^2$, give a geometric description of the subspace spanned by S.
(i) $S = \{(2, 1), (-1, 2)\}$, (ii) $S = \{(-3, 5)\}$, (iii) $S = \{(-1, 2), (4, -8)\}$.

12. Determine whether $S = \{1, x^2, x^2 + 2\}$ spans P_2.

13. Check if $\begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$ is linear combination of $\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$

14. Check if $(1, -1, 0, 2)$ is a linear combination of $(0, 1, -1, 3)$ and $(1, 4, 2, 3)$.

15. write $0.25 + x^2 - 3x$ as a linear combination of $\{3, 2x, -x^2\}$

16. Write the solution set of any system of equations in 3 unknowns from tutorial 3 as a linear combination of vectors in $\mathbb{R}^3$.

17. Prove that the solution set of a homogeneous system of equations is a vector space.

Note: If you find any mistake please upload corrected question and your solution on moodle for others to follow/check.