yields $C_n = \sqrt{2/a}$ and hence $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$. The probability in the region $0 < x < a/2$ is given by

$$\frac{2}{a}\int_0^{a/2} \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{1}{a}\int_0^{a/2}\left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right] dx = \frac{1}{2}. \qquad (2.497)$$

This is expected since the total probability is 1: $\int_0^a \mid \psi_n(x) \mid^2 dx = 1$.

2.10 Exercises

Exercise 2.1
Consider the two states $\mid \psi\rangle = i \mid \phi_1\rangle + 3i \mid \phi_2\rangle - \mid \phi_3\rangle$ and $\mid \chi\rangle = \mid \phi_1\rangle - i \mid \phi_2\rangle + 5i \mid \phi_3\rangle$, where $\mid \phi_1\rangle$, $\mid \phi_2\rangle$ and $\mid \phi_3\rangle$ are orthonormal.

(a) Calculate $\langle \psi \mid \psi\rangle$, $\langle \chi \mid \chi\rangle$, $\langle \psi \mid \chi\rangle$, $\langle \chi \mid \psi\rangle$, and infer $\langle \psi + \chi \mid \psi + \chi\rangle$. Are the scalar products $\langle \psi \mid \chi\rangle$ and $\langle \chi \mid \psi\rangle$ equal?

(b) Calculate $\mid \psi\rangle\langle \chi \mid$ and $\mid \chi\rangle\langle \psi \mid$. Are they equal? Calculate their traces and compare them.

(c) Find the Hermitian conjugates of $\mid \psi\rangle$, $\mid \chi\rangle$, $\mid \psi\rangle\langle \chi \mid$, and $\mid \chi\rangle\langle \psi \mid$.

Exercise 2.2
Consider two states $|\psi_1\rangle = |\phi_1\rangle + 4i|\phi_2\rangle + 5|\phi_3\rangle$ and $|\psi_2\rangle = b|\phi_1\rangle + 4|\phi_2\rangle - 3i|\phi_3\rangle$, where $|\phi_1\rangle$, $|\phi_2\rangle$, and$|\phi_3\rangle$ are orthonormal kets, and where b is a constant. Find the value of b so that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

Exercise 2.3
If $\mid \phi_1\rangle$, $\mid \phi_2\rangle$, and $\mid \phi_3\rangle$ are orthonormal, show that the states $\mid \psi\rangle = i \mid \phi_1\rangle + 3i \mid \phi_2\rangle - \mid \phi_3\rangle$ and $\mid \chi\rangle = \mid \phi_1\rangle - i \mid \phi_2\rangle + 5i \mid \phi_3\rangle$ satisfy

(a) the triangle inequality and

(b) the Schwarz inequality.

Exercise 2.4
Find the constant α so that the states $\mid \psi\rangle = \alpha \mid \phi_1\rangle + 5 \mid \phi_2\rangle$ and $\mid \chi\rangle = 3\alpha \mid \phi_1\rangle - 4 \mid \phi_2\rangle$ are orthogonal; consider $\mid \phi_1\rangle$ and $\mid \phi_2\rangle$ to be orthonormal.

Exercise 2.5
If $\mid \psi\rangle = \mid \phi_1\rangle + \mid \phi_2\rangle$ and $\mid \chi\rangle = \mid \phi_1\rangle - \mid \phi_2\rangle$, prove the following relations (note that $\mid \phi_1\rangle$ and $\mid \phi_2\rangle$ are not orthonormal):

(a) $\langle \psi \mid \psi\rangle + \langle \chi \mid \chi\rangle = 2\langle \phi_1 \mid \phi_1\rangle + 2\langle \phi_2 \mid \phi_2\rangle$,

(b) $\langle \psi \mid \psi\rangle - \langle \chi \mid \chi\rangle = 2\langle \phi_1 \mid \phi_2\rangle + 2\langle \phi_2 \mid \phi_1\rangle$.

Exercise 2.6
Consider a state which is given in terms of three orthonormal vectors $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ as follows:

$$|\psi\rangle = \frac{1}{\sqrt{15}}|\phi_1\rangle + \frac{1}{\sqrt{3}}|\phi_2\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle,$$

where $|\phi_n\rangle$ are eigenstates to an operator $\hat{B}$ such that: $\hat{B}|\phi_n\rangle = (3n^2 - 1)|\phi_n\rangle$ with $n = 1, 2, 3$.

(a) Find the norm of the state $|\psi\rangle$.

(b) Find the expectation value of $\hat{B}$ for the state $|\psi\rangle$.

(c) Find the expectation value of $\hat{B}^2$ for the state $|\psi\rangle$.

Exercise 2.7
Are the following sets of functions linearly independent or dependent?
(a) $4e^x$, e^x, $5e^x$
(b) $\cos x$, e^{ix}, $3 \sin x$
(c) 7, x^2, $9x^4$, e^{-x}

Exercise 2.8
Are the following sets of functions linearly independent or dependent on the positive x-axis?
(a) $x, x+2, x+5$
(b) $\cos x$, $\cos 2x$, $\cos 3x$
(c) $\sin^2 x$, $\cos^2 x$, $\sin 2x$
(d) x, $(x-1)^2$, $(x+1)^2$
(e) $\sinh^2 x$, $\cosh^2 x$, 1

Exercise 2.9
Are the following sets of vectors linearly independent or dependent over the complex field?
(a) $(2, -3, 0)$, $(0, 0, 1)$, $(2i, i, -i)$
(b) $(0, 4, 0)$, $(i, -3i, i)$, $(2, 0, 1)$
(c) $(i, 1, 2)$, $(3, i, -1)$, $(-i, 3i, 5i)$

Exercise 2.10
Are the following sets of vectors (in the three-dimensional Euclidean space) linearly independent or dependent?
(a) $(4, 5, 6)$, $(1, 2, 3)$, $(7, 8, 9)$
(b) $(1, 0, 0)$, $(0, -5, 0)$, $(0, 0, \sqrt{7})$
(c) $(5, 4, 1)$, $(2, 0, -2)$, $(0, 6, -1)$

Exercise 2.11
Show that if $\hat{A}$ is a projection operator, the operator $1 - \hat{A}$ is also a projection operator.

Exercise 2.12
Show that $\mid \psi\rangle\langle\psi \mid / \langle\psi \mid \psi\rangle$ is a projection operator, regardless of whether $\mid \psi\rangle$ is normalized or not.

Exercise 2.13
In the following expressions, where $\hat{A}$ is an operator, specify the nature of each expression (i.e., specify whether it is an operator, a bra, or a ket); then find its Hermitian conjugate.
(a) $\langle\phi \mid \hat{A} \mid \psi\rangle\langle\psi \mid$
(b) $\hat{A} \mid \psi\rangle\langle\phi \mid$
(c) $\langle\phi \mid \hat{A} \mid \psi\rangle \mid \psi\rangle\langle\phi \mid \hat{A}$
(d) $\langle\psi \mid \hat{A} \mid \phi\rangle \mid \phi\rangle + i\hat{A} \mid \psi\rangle$
(e) $\left(\mid \phi\rangle\langle\phi \mid \hat{A}\right) - i\left(\hat{A} \mid \psi\rangle\langle\psi \mid\right)$

Exercise 2.14
Consider a two-dimensional space where a Hermitian operator $\hat{A}$ is defined by $\hat{A} \mid \phi_1\rangle = \mid \phi_1\rangle$ and $\hat{A} \mid \phi_2\rangle = - \mid \phi_2\rangle$; $\mid \phi_1\rangle$ and $\mid \phi_2\rangle$ are orthonormal.
(a) Do the states $\mid \phi_1\rangle$ and $\mid \phi_2\rangle$ form a basis?
(b) Consider the operator $\hat{B} = \mid \phi_1\rangle\langle\phi_2 \mid$. Is $\hat{B}$ Hermitian? Show that $\hat{B}^2 = 0$.

(c) Show that the products $\hat{B}\hat{B}^\dagger$ and $\hat{B}^\dagger\hat{B}$ are projection operators.
(d) Show that the operator $\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B}$ is unitary.
(e) Consider $\hat{C} = \hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}$. Show that $\hat{C} \mid \phi_1\rangle = \mid \phi_1\rangle$ and $\hat{C} \mid \phi_2\rangle = \mid \phi_2\rangle$.

Exercise 2.15
Prove the following two relations:
(a) $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{[\hat{A},\hat{B}]/2}$,
(b) $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \cdots$.
Hint: To prove the first relation, you may consider defining an operator function $\hat{F}(t) = e^{\hat{A}t}e^{\hat{B}t}$, where t is a parameter, $\hat{A}$ and $\hat{B}$ are t-independent operators, and then make use of $[\hat{A}, G(\hat{B})] = [\hat{A}, \hat{B}]dG(\hat{B})/d\hat{B}$, where $G(\hat{B})$ is a function depending on the operator $\hat{B}$.

Exercise 2.16
(a) Verify that the matrix
$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
is unitary.
(b) Find its eigenvalues and the corresponding normalized eigenvectors.

Exercise 2.17
Consider the following three matrices:
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
(a) Calculate the commutators $[A,\ B]$, $[B,\ C]$, and $[C,\ A]$.
(b) Show that $A^2 + B^2 + 2C^2 = 4I$, where I is the unity matrix.
(c) Verify that $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$.

Exercise 2.18
Consider the following two matrices:
$$A = \begin{pmatrix} 3 & i & 1 \\ -1 & -i & 2 \\ 4 & 3i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2i & 5 & -3 \\ -i & 3 & 0 \\ 7i & 1 & i \end{pmatrix}.$$
Verify the following relations:
(a) $\det(AB) = \det(A)\det(B)$,
(b) $\det(A^T) = \det(A)$,
(c) $\det(A^\dagger) = (\det(A))^*$, and
(d) $\det(A^*) = (\det(A))^*$.

Exercise 2.19
Consider the matrix
$$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$
(a) Find the eigenvalues and the normalized eigenvectors for the matrix A.

(b) Do these eigenvectors form a basis (i.e., is this basis complete and orthonormal)?

(c) Consider the matrix U which is formed from the normalized eigenvectors of A. Verify that U is unitary and that it satisfies

$$U^{\dagger} A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues of A.

(d) Show that $e^{xA} = \cosh x + A \sinh x$.

Exercise 2.20

Using the bra-ket algebra, show that $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}(\hat{C}\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{C}\hat{A})$ where $\hat{A}, \hat{B}, \hat{C}$ are operators.

Exercise 2.21

For any two kets $\mid \psi\rangle$ and $\mid \phi\rangle$ that have finite norm, show that $\mathrm{Tr}\,(\mid \psi\rangle\langle\phi \mid) = \langle\phi \mid \psi\rangle$.

Exercise 2.22

Consider the matrix $A = \begin{pmatrix} 0 & 0 & -1+i \\ 0 & 3 & 0 \\ -1-i & 0 & 0 \end{pmatrix}$.

(a) Find the eigenvalues and normalized eigenvectors of A. Denote the eigenvectors of A by $\mid a_1\rangle, \mid a_2\rangle, \mid a_3\rangle$. Any degenerate eigenvalues?

(b) Show that the eigenvectors $\mid a_1\rangle, \mid a_2\rangle, \mid a_3\rangle$ form an orthonormal and complete basis, i.e., show that $\sum_{j=1}^{3} \mid a_j\rangle\langle a_j \mid = I$, where I is the 3×3 unit matrix, and that $\langle a_j \mid a_k\rangle = \delta_{jk}$.

(c) Find the matrix corresponding to the operator obtained from the ket-bra product of the first eigenvector $P = \mid a_1\rangle\langle a_1 \mid$. Is P a projection operator?

Exercise 2.23

In a three-dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{\mid 1\rangle, \mid 2\rangle, \mid 3\rangle\}$, is

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Is A Hermitian? Calculate its eigenvalues and the corresponding normalized eigenvectors. Verify that the eigenvectors corresponding to the two nondegenerate eigenvalues are orthonormal.

(b) Calculate the matrices representing the projection operators for the two nondegenerate eigenvectors found in part (a).

Exercise 2.24

Consider two operators $\hat{A}$ and $\hat{B}$ whose matrices are

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

(a) Are $\hat{A}$ and $\hat{B}$ Hermitian?

(b) Do $\hat{A}$ and $\hat{B}$ commute?

(c) Find the eigenvalues and eigenvectors of $\hat{A}$ and $\hat{B}$.
(d) Are the eigenvectors of each operator orthonormal?
(e) Verify that $\hat{U}^\dagger \hat{B}\hat{U}$ is diagonal, $\hat{U}$ being the matrix of the normalized eigenvectors of $\hat{B}$.
(f) Verify that $\hat{U}^{-1} = \hat{U}^\dagger$.

Exercise 2.25
Consider an operator $\hat{A}$ so that $[\hat{A},\ \hat{A}^\dagger] = 1$.

(a) Evaluate the commutators $[\hat{A}^\dagger \hat{A},\ \hat{A}]$ and $[\hat{A}^\dagger \hat{A},\ \hat{A}^\dagger]$.

(b) If the actions of $\hat{A}$ and $\hat{A}^\dagger$ on the states $\{\mid a\rangle\}$ are given by $\hat{A} \mid a\rangle = \sqrt{a} \mid a-1\rangle$ and $\hat{A}^\dagger \mid a\rangle = \sqrt{a+1} \mid a+1\rangle$ and if $\langle a' \mid a\rangle = \delta_{a'a}$, calculate $\langle a \mid \hat{A} \mid a+1\rangle$, $\langle a+1 \mid \hat{A}^\dagger \mid a\rangle$ and $\langle a \mid \hat{A}^\dagger \hat{A} \mid a\rangle$ and $\langle a \mid \hat{A}\hat{A}^\dagger \mid a\rangle$.

(c) Calculate $\langle a \mid (\hat{A} + \hat{A}^\dagger)^2 \mid a\rangle$ and $\langle a \mid (\hat{A} - \hat{A}^\dagger)^2 \mid a\rangle$.

Exercise 2.26
Consider a 4×4 matrix

$$A = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find the matrices of $A^\dagger$, $N = A^\dagger A$, $H = N + \frac{1}{2}I$ (where I is the unit matrix), $B = A + A^\dagger$, and $C = i(A - A^\dagger)$.

(b) Find the matrices corresponding to the commutators $[A^\dagger, A]$, $[B,\ C]$, $[N,\ B]$, and $[N,\ C]$.

(c) Find the matrices corresponding to B^2, C^2, $[N,\ B^2 + C^2]$, $[H,\ A^\dagger]$, $[H,\ A]$, and $[H,\ N]$.

(d) Verify that $\det(ABC) = \det(A)\det(B)\det(C)$ and $\det(C^\dagger) = (\det(C))^*$.

Exercise 2.27
If $\hat{A}$ and $\hat{B}$ commute, and if $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$ are two eigenvectors of $\hat{A}$ with different eigenvalues ($\hat{A}$ is Hermitian), show that

(a) $\langle \psi_1 \mid \hat{B} \mid \psi_2\rangle$ is zero and

(b) $\hat{B} \mid \psi_1\rangle$ is also an eigenvector to $\hat{A}$ with the same eigenvalue as $\mid \psi_1\rangle$; i.e., if $\hat{A} \mid \psi_1\rangle = a_1 \mid \psi_1\rangle$, show that $\hat{A}(\hat{B} \mid \psi_1\rangle) = a_1\hat{B} \mid \psi_1\rangle$.

Exercise 2.28
Let A and B be two $n \times n$ matrices. Assuming that B^{-1} exists, show that $[A,\ B^{-1}] = -B^{-1}[A,\ B]B^{-1}$.

Exercise 2.29
Consider a physical system whose Hamiltonian H and an operator A are given, in a three-dimensional space, by the matrices

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Are H and A Hermitian?
(b) Show that H and A commute. Give a basis of eigenvectors common to H and A.

Exercise 2.30
(a) Using $[\hat{X},\ \hat{P}] = i\hbar$, show that $[\hat{X}^2,\ \hat{P}] = 2i\hbar\hat{X}$ and $[\hat{X},\ \hat{P}^2] = 2i\hbar\hat{P}$.
(b) Show that $[\hat{X}^2,\ \hat{P}^2] = 2i\hbar(i\hbar + 2\hat{P}\hat{X})$.
(c) Calculate the commutator $[\hat{X}^2,\ \hat{P}^3]$.

Exercise 2.31
Discuss the hermiticity of the commutators $[\hat{X},\ \hat{P}]$, $[\hat{X}^2,\ \hat{P}]$ and $[\hat{X},\ \hat{P}^2]$.

Exercise 2.32
(a) Evaluate the commutator $[\hat{X}^2,\ d/dx]$ by operating it on a wave function.
(b) Using $[\hat{X},\ \hat{P}] = i\hbar$, evaluate the commutator $[\hat{X}\hat{P}^2,\ \hat{P}\hat{X}^2]$ in terms of a linear combination of $\hat{X}^2\hat{P}^2$ and $\hat{X}\hat{P}$.

Exercise 2.33
Show that $[\hat{X},\ \hat{P}^n] = i\hbar\hat{X}\hat{P}^{n-1}$.

Exercise 2.34
Evaluate the commutators $[e^{i\hat{X}},\ \hat{P}]$, $[e^{i\hat{X}^2},\ \hat{P}]$, and $[e^{i\hat{X}},\ \hat{P}^2]$.

Exercise 2.35
Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

(a) Find the eigenvalues and the normalized eigenvectors of A.
(b) Do these eigenvectors form a basis (i.e., is this basis complete and orthonormal)?
(c) Consider the matrix U which is formed from the normalized eigenvectors of A. Verify that U is unitary and that it satisfies the relation

$$U^\dagger A U = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where λ_1, λ_2, and λ_3 are the eigenvalues of A.
(d) Show that $e^{xA} = \cosh x + A\sinh x$.
Hint: $\cosh x = \sum_{n=0}^{\infty} x^{2n}/(2n)!$ and $\sinh x = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$.

Exercise 2.36
(a) If $[\hat{A},\ \hat{B}] = c$, where c is a number, prove the following two relations: $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + c$ and $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-c/2}$.
(b) Now if $[\hat{A},\ \hat{B}] = c\hat{B}$, where c is again a number, show that $e^{\hat{A}}\hat{B}e^{-\hat{A}} = e^c\hat{B}$.

Exercise 2.37
Consider the matrix

$$A = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

(a) Find the eigenvalues of A and their corresponding eigenvectors.

(b) Consider the basis which is constructed from the three eigenvectors of A. Using matrix algebra, verify that this basis is both orthonormal and complete.

Exercise 2.38

(a) Specify the condition that must be satisfied by a matrix A so that it is both unitary and Hermitian.

(b) Consider the three matrices

$$M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Calculate the inverse of each matrix. Do they satisfy the condition derived in (a)?

Exercise 2.39

Consider the two matrices

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}.$$

(a) Are these matrices Hermitian?

(b) Calculate the inverses of these matrices.

(c) Are these matrices unitary?

(d) Verify that the determinants of A and B are of the form $e^{i\theta}$. Find the corresponding values of θ.

Exercise 2.40

Show that the transformation matrix representing a 90° counterclockwise rotation about the z-axis of the basis vectors $(\vec{i},\ \vec{j},\ \vec{k})$ is given by

$$U = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 2.41

Show that the transformation matrix representing a 90° clockwise rotation about the y-axis of the basis vectors $(\vec{i},\ \vec{j},\ \vec{k})$ is given by

$$U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Exercise 2.42

Show that the operator $(\hat{X}\hat{P} + \hat{P}\hat{X})^2$ is equal to $(\hat{X}^2\hat{P}^2 + \hat{P}^2\hat{X}^2)$ plus a term of the order of $\hbar^2$.

Exercise 2.43

Consider the two matrices $A = \begin{pmatrix} 4 & i & 7 \\ 1 & 0 & 1 \\ 0 & 1 & -i \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & i & 0 \\ -i & 0 & i \end{pmatrix}$. Calculate the products $B^{-1}A$ and $A\,B^{-1}$. Are they equal? What is the significance of this result?

Exercise 2.44
Use the relations listed in Appendix A to evaluate the following integrals involving Dirac's delta function:

(a) $\int_0^{\pi} \sin(3x)\cos^2(4x)\delta(x-\pi/2)\,dx$.
(b) $\int_{-2}^{2} e^{7x+2}\delta(5x)\,dx$.
(c) $\int_{-2\pi}^{2\pi} \sin(\theta/2)\delta''(\theta+\pi)\,d\theta$.
(d) $\int_0^{2\pi} \cos^2\theta\,\delta[(\theta-\pi)/4]\,d\theta$.

Exercise 2.45
Use the relations listed in Appendix A to evaluate the following expressions:

(a) $\int_0^5 (3x^2+2)\delta(x-1)\,dx$.
(b) $(2x^5-4x^3+1)\delta(x+2)$.
(c) $\int_0^{\infty}(5x^3-7x^2-3)\delta(x^2-4)\,dx$.

Exercise 2.46
Use the relations listed in Appendix A to evaluate the following expressions:

(a) $\int_3^7 e^{6x-2}\delta(-4x)\,dx$.
(b) $\cos(2\theta)\sin(\theta)\delta(\theta^2-\pi^2/4)$.
(c) $\int_{-1}^{1} e^{5x-1}\delta'''(x)\,dx$.

Exercise 2.47
If the position and momentum operators are denoted by $\hat{\vec{R}}$ and $\hat{\vec{P}}$, respectively, show that $\hat{\mathcal{P}}^{\dagger}\hat{\vec{R}}^{\,n}\hat{\mathcal{P}} = (-1)^n\hat{\vec{R}}^{\,n}$ and $\hat{\mathcal{P}}^{\dagger}\hat{\vec{P}}^{\,n}\hat{\mathcal{P}} = (-1)^n\hat{\vec{P}}^{\,n}$, where $\hat{\mathcal{P}}$ is the parity operator and n is an integer.

Exercise 2.48
Consider an operator

$$\begin{aligned}\hat{A} \;=\; & \mid\phi_1\rangle\langle\phi_1\mid + \mid\phi_2\rangle\langle\phi_2\mid + \mid\phi_3\rangle\langle\phi_3\mid - i\mid\phi_1\rangle\langle\phi_2\mid \\ & - \mid\phi_1\rangle\langle\phi_3\mid + i\mid\phi_2\rangle\langle\phi_1\mid - \mid\phi_3\rangle\langle\phi_1\mid,\end{aligned}$$

where $\mid\phi_1\rangle$, $\mid\phi_2\rangle$, and $\mid\phi_3\rangle$ form a complete and orthonormal basis.

(a) Is $\hat{A}$ Hermitian? Calculate $\hat{A}^2$; is it a projection operator?
(b) Find the 3×3 matrix representing $\hat{A}$ in the $\mid\phi_1\rangle$, $\mid\phi_2\rangle$, $\mid\phi_3\rangle$ basis.
(c) Find the eigenvalues and the eigenvectors of the matrix.

Exercise 2.49
The Hamiltonian of a two-state system is given by

$$\hat{H} = E\left(\mid\phi_1\rangle\langle\phi_1\mid - \mid\phi_2\rangle\langle\phi_2\mid - i\mid\phi_1\rangle\langle\phi_2\mid + i\mid\phi_2\rangle\langle\phi_1\mid\right),$$

where $\mid\phi_1\rangle$, $\mid\phi_2\rangle$ form a complete and orthonormal basis; E is a real constant having the dimensions of energy.

(a) Is $\hat{H}$ Hermitian? Calculate the trace of $\hat{H}$.
(b) Find the matrix representing $\hat{H}$ in the $\mid\phi_1\rangle$, $\mid\phi_2\rangle$ basis and calculate the eigenvalues and the eigenvectors of the matrix. Calculate the trace of the matrix and compare it with the result you obtained in (a).
(c) Calculate $[\hat{H},\ \mid\phi_1\rangle\langle\phi_1\mid]$, $[\hat{H},\ \mid\phi_2\rangle\langle\phi_2\mid]$, and $[\hat{H},\ \mid\phi_1\rangle\langle\phi_2\mid]$.

Exercise 2.50
Consider a particle which is confined to move along the positive x-axis and whose Hamiltonian is $\hat{H} = \mathcal{E} d^2/dx^2$, where $\mathcal{E}$ is a positive real constant having the dimensions of energy.

(a) Find the wave function that corresponds to an energy eigenvalue of $9\mathcal{E}$ (make sure that the function you find is finite everywhere along the positive x-axis and is square integrable). Normalize this wave function.

(b) Calculate the probability of finding the particle in the region $0 \leq x \leq 15$.

(c) Is the wave function derived in (a) an eigenfunction of the operator $\hat{A} = d/dx - 7$?

(d) Calculate the commutator $[\hat{H}, \ \hat{A}]$.

Exercise 2.51
Consider the wave functions:

$$\psi(x, y) = \sin 2x \ \cos 5x, \qquad \phi(x, y) = e^{-2(x^2+y^2)}, \qquad \chi(x, y) = e^{-i(x+y)}.$$

(a) Verify if any of the wave functions is an eigenfunction of $\hat{A} = \partial/\partial x + \partial/\partial y$.

(b) Find out if any of the wave functions is an eigenfunction of $\hat{B} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + 1$.

(c) Calculate the actions of $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ on each one of the wave functions and infer $[\hat{A}, \ \hat{B}]$.

Exercise 2.52

(a) Is the state $\psi(\theta, \phi) = e^{-3i\phi} \cos\theta$ an eigenfunction of $\hat{A}_\phi = \partial/\partial\phi$ or of $\hat{B}_\theta = \partial/\partial\theta$?

(b) Are $\hat{A}_\phi$ and $\hat{B}_\theta$ Hermitian?

(c) Evaluate the expressions $\langle \psi \mid \hat{A}_\phi \mid \psi \rangle$ and $\langle \psi \mid \hat{B}_\theta \mid \psi \rangle$.

(d) Find the commutator $[\hat{A}_\phi, \ \hat{B}_\theta]$.

Exercise 2.53
Consider an operator $\hat{A} = (\hat{X} d/dx + 2)$.

(a) Find the eigenfunction of $\hat{A}$ corresponding to a zero eigenvalue. Is this function normalizable?

(b) Is the operator $\hat{A}$ Hermitian?

(c) Calculate $[\hat{A}, \ \hat{X}]$, $[\hat{A}, \ d/dx]$, $[\hat{A}, \ d^2/dx^2]$, $[\hat{X}, \ [\hat{A}, \ \hat{X}]]$, and $[d/dx, \ [\hat{A}, \ d/dx]]$.

Exercise 2.54
If $\hat{A}$ and $\hat{B}$ are two Hermitian operators, find their respective eigenvalues such that $\hat{A}^2 = 2\hat{I}$ and $\hat{B}^4 = \hat{I}$, where $\hat{I}$ is the unit operator.

Exercise 2.55
Consider the Hilbert space of two-variable complex functions $\psi(x, y)$. A permutation operator is defined by its action on $\psi(x, y)$ as follows: $\hat{\pi}\,\psi(x, y) = \psi(y, x)$.

(a) Verify that the operator $\hat{\pi}$ is linear and Hermitian.

(b) Show that $\hat{\pi}^2 = \hat{I}$. Find the eigenvalues and show that the eigenfunctions of $\hat{\pi}$ are given by

$$\psi_+(x, y) = \frac{1}{2}\left[\psi(x, y) + \psi(y, x)\right] \quad \text{and} \quad \psi_-(x, y) = \frac{1}{2}\left[\psi(x, y) - \psi(y, x)\right].$$