Module 1. QUANTUM PHYSICS

Session 3 : Heisenberg Uncertainty Principle
Session 4 : Problems on HUP

Heisenberg Uncertainty Principle (HUP)

According to the classical mechanics, the position and momentum of the moving particle can be determined with great accuracy. However, when the particle is considered as a wave, it is not possible to know the exact location of the particle on the wave as the wave extends throughout the region in the space.

The de Broglie wavelength associated with a moving particle traveling with a uniform velocity 'v' is given by –

$$\lambda = \frac{h}{m\,v}$$

This is a monochromatic wave of infinite extent. The phase velocity 'v_p' of such monochromatic wave is given by –

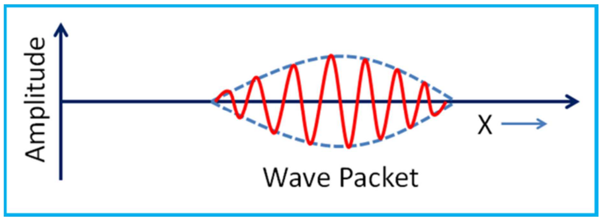
$$v_p = \nu\lambda = \frac{h\nu\lambda}{h} = \frac{h\nu}{\frac{h}{\lambda}} = \frac{E}{p} = \frac{mc^2}{mv} = \frac{c^2}{v} \qquad \text{--------------------(1)}$$

This phase velocity is always greater than the velocity of light in vacuum. Hence it is theoretically impossible for a monochromatic de Broglie wave-train to transport a particle or energy.

Further the stability of the material particle demands that it should be concentrated over a small region of space at any instant of time. Thus, mass of a particle is a localized entity whereas the de Broglie wave with which we represent the moving particle of infinite extent.

From Einstein's theory of velocity, it follows that the speed of light is maximum velocity that can be attained by a particle in nature. It means that the velocity of the particle 'v' is always less than the speed of light 'c'. From equation (1) it follows that the de Broglie wave velocity must be greater than 'c', which is not acceptable. Further, it follows from this result, that the wave associated with the particle would travel faster than the particle itself, thereby leaving the particle far behind. Hence it was concluded that a material particle would not be equivalent to a single wave-train.

Schrodinger solved this difficulty by postulating that a material particle in motion is equivalent to a wave packet rather than a single wave. A wave packet consists of a group of waves (each having slightly different velocity and wavelength). The phases and amplitudes of these waves are chosen in such a way that they undergo interference constructively over only a small region of space where the particle can be located. Outside this region, they undergo destructive interference so that that the amplitude reduces to zero rapidly. Such wave packet is shown in the following fugure-



This wave packet moves with its own velocity 'v_g', called as group velocity. The individual waves forming the packet have an average velocity 'v_p' called as a phase velocity. It can be proved that the velocity of the material particle is same as the group velocity of wave packet.

The phase velocity is the velocity with which a particular phase of the wave propagates in the medium.
Let the equation of the wave travelling in x-direction with vibrations in y-direction is –

$$y = A\sin(\omega t - kx)$$

where A = amplitude of vibration,

$k = \dfrac{2\pi}{\lambda}$ is propagation constant, $\omega = 2\pi\nu$ is angular frequency

$$\therefore \qquad \nu = \frac{\omega}{2\pi} \quad \text{and} \quad \lambda = \frac{2\pi}{k}$$

$$\text{phase velocity,} \quad v_p = \nu\lambda = \frac{\omega}{2\pi} \times \frac{2\pi}{k} = \frac{\omega}{k} \qquad \text{---- (1)}$$

wavelength of De Broglie wave associated with a particle of mass 'm' moving with velocity 'v' is given by -

$$\lambda = \frac{h}{m\,v} \quad \therefore\ k = \frac{2\pi}{\lambda} = \frac{2\pi\, m\, v}{h}$$

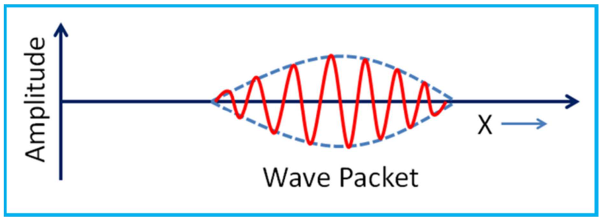
To find frequency (ν), let us equate energy e with relativistic total energy mc^2

$$\text{i.e.} \qquad h\nu = mc^2 \quad \therefore\ \nu = \frac{mc^2}{h} \quad \text{and} \quad \omega = 2\pi\nu = \frac{2\pi\, m\, c^2}{h}$$

$$\therefore \text{ from (1), phase velocity } (v_p) = \frac{\omega}{k} = \frac{\dfrac{2\pi\, m\, c^2}{h}}{\dfrac{2\pi\, m\, v}{h}} = \frac{c^2}{v}$$

Thus, the phase velocity of the wave under consideration is always greater than the velocity of light. Further the stability of the material particle demands that it should be concentrated over a small region of space at any instant of time. Thus, mass of a particle is a localized entity whereas the single monochromatic wave with which we represent the moving particle of infinite extent. So a single monochromatic wave cannot be associated with the moving particle.

Schrodinger postulated that a material particle in motion is equivalent to a wave packet rather than a single wave. A wave packet consists of a group of waves (each having slightly different velocity and wavelength). The phases and amplitudes of these waves are chosen in such a way that they undergo interference constructively over only a small region of space where the particle can be located. Outside this region, they undergo destructive interference so that that the amplitude reduces to zero rapidly. Such wave packet is shown in the following figure-



This wave packet moves with its own velocity 'v_g', called as group velocity. The individual waves forming the packet have an average velocity 'v_p' called as a phase velocity. It can be proved that the velocity of the material particle is same as the group velocity of wave packet.

If two waves have their angular velocities differing by $d\omega$ and propagation constants differing by dk (due to difference $d\lambda$ in their wavelengths), their equations can be written as -

$$y_1 = A \sin(\omega t - kx)$$

and $$y_2 = A \sin[(\omega + d\omega)t - (k + dk)x]$$

The resultant displacement 'y' at time 't' is - $y = y_1 + y_2$

$$y = 2A \sin\left(\frac{2\omega + d\omega}{2} t - \frac{2k + dk}{2} x\right) \cos\left(\frac{d\omega}{2} t - \frac{dk}{2} x\right)$$

As $d\omega$ and dk are very small compared to ω and k respectively,

$$y = 2A \sin(\omega t - kx) \cos\left(\frac{d\omega}{2} t - \frac{dk}{2} x\right)$$

The sine term in the above equation represents a wave of angular frequency ω and propagation constant k.

The cosine term modulates this wave with angular frequency $\frac{d\omega}{2}$ to produce wave groups traveling with velocity $v_g = \frac{d\omega}{dk}$ which is group velocity.

$$\therefore \ v_g = \frac{d\omega / dv}{dk / dv} \qquad \text{--------------(2)}$$

$$\text{Now,} \quad h\nu = mc^2 \qquad \therefore \nu = \frac{mc^2}{h} \quad \text{and} \quad \omega = 2\pi\nu = \frac{2\pi m c^2}{h}$$

$$\text{As} \quad m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad \therefore \ \omega = \frac{2\pi c^2}{h} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \frac{d\omega}{dv} = \frac{2\pi m_0}{h} v \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \qquad \text{-------(3)}$$

$$\text{Also, } k = \frac{2\pi}{\lambda} = \frac{2\pi\, m\, v}{h} = \frac{2\pi\, v}{h}\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

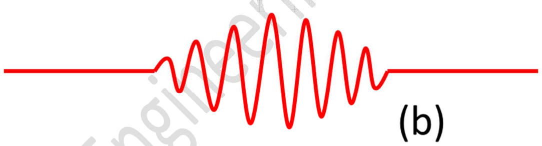
$$\therefore \frac{dk}{dv} = \frac{2\pi\ m_0}{h}\left(1-\frac{v^2}{c^2}\right)^{-3/2} \quad \text{-------(4)}$$

$$\text{From (2), (3) and (4), } v_g = \frac{d\omega/dv}{dk/dv} = v$$

Thus, de Broglie group velocity associated with a wave packet is equal to the velocity of the particle

The association of group of waves (wave packet) with a moving particle means that, the position of the particle at any instant of time cannot be specified with desired degree of accuracy. All that we can say is, the particle is somewhere within the wave packet. The probability of finding the particle at a point in a wave packet is directly proportional to the amplitude of the wave at that point.


(a)


(b)

If the width of the wave packet is small as shown in fig. (a) then the particle can be located somewhat accurately, but the determination of wavelength (And hence the momentum) becomes a problem. If width of the wave packet is more (fig. (b)), then wavelength measurement (and hence determination of momentum) is accurate. However, position of the particle cannot be determined accurately.

With this discussion, Heisenberg, put forward his uncertainty principle which states that –

It is impossible to determine simultaneously, the position and momentum of the moving particle accurately. In any simultaneous determination of position and momentum of the particle, the product of uncertainties is equal to or greater than $\frac{\hbar}{2}$ i.e. $\frac{h}{4\pi}$.

$$\boxed{\Delta x\ \Delta p_x \geq \frac{h}{4\pi}}$$

where Δx is the fundamental error or uncertainty in measurement of position and
Δp_x is fundamental error or uncertainty in measurement of momentum along X-axis.

Experimental Verification of HUP

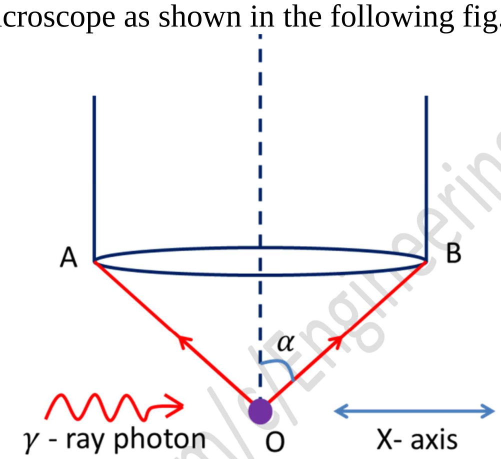
In order to confirm HUP, the following ideal experiments were performed.

1) Heisenberg's Gamma Ray Microscope Experiment and
2) Single Slit Diffraction Experiment.

As we do not have any ideal instruments, these experiments are performed in thought without violating any physics concepts. Hence, they are called as thought experiments or hypothetical experiments.

Heisenberg's Gamma Ray Microscope Experiment

In this experiment, we try to measure both position and momentum of electron. For this, let us set up a high power γ-ray microscope as shown in the following fig.



Let 'O' be the electron, 'λ' be the wavelength of γ-rays and 'α' be the semi-vertical angle of the cone of rays that enter the microscope objective. As the gamma ray photons collide with electrons, some of them bounce into the microscope and enable the observer to see the electron. According to the classical mechanics, the observer should be able to find out the exact position and momentum of the electron. However, there are two fundamental limitations in this experiment –

i) The accuracy in determining the position of electron by a microscope is limited by the laws of Optics. According to Optics, the resolving power of a microscope is given by –

$$\Delta x = \frac{\lambda}{2\sin\alpha}$$

where Δx is minimum distance between two points that can be distinguished as separate.
λ is wavelength of scattered gamma ray photon.

Due to this, if position of the electron changes by Δx, the microscope would not be able to detect it. To make Δx very small, radiation of very short wavelength such as X-rays or Gamma rays should be used. Thus Δx will be an error or uncertainty in determination of position of the electron.

$$\therefore \quad \Delta x = \frac{\lambda}{2\sin\alpha} \qquad \text{--------------------------------------(1)}$$

ii) While determining the momentum of the electron, the interaction of electron with gamma ray photon will result in change of momentum of electron because of its recoil.

In order that this change is to be as small as possible, consider a single gamma ray photon incident on an electron along the X-axis. A scattered photon of wavelength 'λ' will enter the objective anywhere between OA and OB. The momentum of scattered photon is p. If it enters the objective along OA, X-component of its momentum would be $+\ p\ \sin\alpha$

$$\text{Hence, momentum imparted to the electron along X-axis } = p' - p\ \sin\alpha \quad ----(2)$$

If scattered photon enters the microscope along OB, then X-component of its momentum would be $-\ p\ \sin\alpha$

The momentum given to the electron can therefore have any value between those given

$$\text{Hence, momentum imparted to the electron along X-axis } = p' - (-p\ \sin\alpha)$$
$$= p' + p\ \sin\alpha \quad ----(3)$$

by equation (2) and (3)

$$\therefore\ \Delta p_x = p' + p\ \sin\alpha - (p' - p\ \sin\alpha)$$

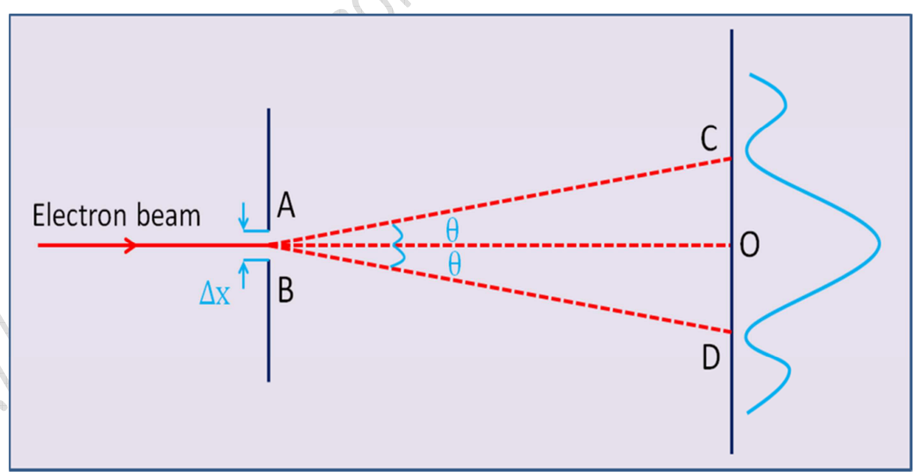
$$\therefore\ \Delta p_x = 2\ p\ \sin\alpha = 2\ \frac{h}{\lambda}\ \sin\alpha \quad ---------------------------(4)$$

From equation (1) and (4), the product of uncertainties is given by –

$$\Delta x\,.\,\Delta p_x = \frac{\lambda}{2\ \sin\alpha}\ .\ 2\ \frac{h}{\lambda}\ \sin\alpha$$
$$= h > \frac{h}{4\pi}$$

Which confirms the HUP.

Diffraction of abeam of electrons through a narrow slit



Consider a narrow slit AB of width Δx as shown in the figure. Let a beam of electrons fall on this slit. After passing through the slit, the electron beam produces a diffraction pattern containing a central maximum. The first minimum is obtained on either side of the central maximum at an angle θ given by the relation –

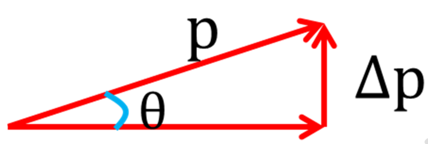
$$d \sin\theta = n\ \lambda \qquad \text{where, } d = \Delta x \quad \text{and} \quad n = 1$$
$$\therefore\ \Delta x \sin\theta = \lambda \qquad ------------------------------(1)$$

We cannot locate the exact position of the electrons in the beam when it passes through the slit. The uncertainty in the measurement of the position of the electron n the slit is equal to the width of the slit, which is given by –

$$\Delta x = \frac{\lambda}{\sin\theta} \qquad \text{------------------------}(2)$$

Let p be the momentum of the electron. The electron can be incident on the screen anywhere between central position and the first minimum. If the electron moves in the direction EC after diffraction, the change in momentum 'Δp' is given by

$$\Delta p = p\sin\theta \qquad \text{--------------------------------------}(3)$$



This Δp will be uncertainty in determination of the momentum. If we take product of uncertainties in the measurement of position and momentum, we get –

$$\Delta x \cdot \Delta p$$
$$= \frac{\lambda}{\sin\theta} \cdot p\sin\theta$$
$$= \lambda \cdot p = \frac{h}{p} \cdot p = h > \frac{h}{4\pi}$$

which confirms the HUP.

Why electron cannot exist in nucleus?

Approximate radius of nucleus r = 5 x 10^{-15} m

Therefore, uncertainty in position $\Delta x \approx 2r = 2 \times 5 \times 10^{-15}$ m

$$\Delta x\, \Delta p \geq \frac{h}{4\pi}$$

$$\therefore \Delta x\, m\Delta v \geq \frac{h}{4\pi}$$

$$\therefore \Delta v \geq \frac{h}{4\pi\, m\, \Delta x} = \frac{6.63 \times 10^{-34}}{4 \times 3.142 \times 9.1 \times 10^{-31} \times 10^{-14}} = 5.797 \times 10^{9}\ m/s$$

From this equation, the uncertainty of velocity is more than c ($3 \times 10^{8} m/s$). For this to happen, velocity of an electron must be greater that c. which is not possible. So the position of electron can't be in nucleus.

Time-Energy Uncertainty Relation :

Heisenberg's Uncertainty Principle (HUP) is applicable to all conjugate or complimentary pairs of physical variables whose product has the dimension of Planck's constant 'h'. Some common such pairs are Position-Linear momentum, Energy-Time, Angular momentum-Angular displacement etc.

The Time-Energy Uncertainty Principle states that –

In any simultaneous determination of the time and energy of the particle, the product of uncertainties is equal to or greater than Planck's constant 'h'.

$$\text{i.e.} \quad \Delta E \,.\, \Delta t \;\geq\; \frac{h}{4\pi}$$

where ΔE is the fundamental error or uncertainty in measurement of energy and
Δt is fundamental error or uncertainty in measurement of time.

It can be proved from HUP as follows –

Let us consider a particle of mass 'm' moving with a velocity 'v' so that its K.E. is –

$$E = \frac{1}{2} mv^2$$

$$\therefore \Delta E = \frac{1}{2} m\, 2\, v\, \Delta v$$

$$= v\, \Delta p \qquad (\because m\Delta v = \Delta p)$$

$$= \frac{\Delta x}{\Delta t} \Delta p \qquad (v = \frac{\Delta x}{\Delta t})$$

$$\therefore \Delta E \,.\, \Delta t = \Delta x \,.\, \Delta p \geq \frac{h}{4\pi}$$

Problems based on HUP:

1. A position and momentum of 1 keV electron are simultaneously measured. If position is located within 10 nm then what is the percentage uncertainty in its momentum?

$$Given: \quad E = 1\, keV = 1000 \times 10^{-19}\, J, \qquad \Delta x = 10 \times 10^{-9} m$$

$$\frac{\Delta p}{p} \times 100 = ?$$

$$p = \sqrt{2mE} = \sqrt{2 \times 9.1 \times 10^{-31} \times 1000 \times 10^{-19}} = 1.349 \times 10^{-23} \frac{kg.m}{s}$$

$$According\ to\ HUP, \quad \Delta x. \Delta p \geq \frac{h}{4\pi}$$

$$\therefore \Delta p \geq \frac{h}{4\pi} \times \frac{1}{\Delta x} = \frac{6.63 \times 10^{-34}}{4 \times 3.142 \times 10 \times 10^{-9}}$$

$$\Delta p = 5.275 \times 10^{-27} \frac{kg.m}{s}$$

$$\frac{\Delta p}{p} \times 100 = \frac{5.275 \times 10^{-27}}{1.349 \times 10^{-23}} \times 100 = 0.0391$$

2. An electron has a speed of 400 m/s with uncertainty of 0.01%. Find the accuracy in its position.

$$Given: \quad v = 400\frac{m}{s}, \qquad \frac{\Delta v}{v} = \frac{0.01}{100}, \qquad m = 9.1 \times 10^{-31}\, kg,$$

$$h = 6.63 \times 10^{-34}, \qquad \Delta x = ?$$

$$p = m\, v = 9.1 \times 10^{-31} \times 400 = 3.64 \times 10^{-28}\frac{kg.m}{s}$$

$$\Delta p = m\, \Delta v = m\, v\, \frac{\Delta v}{v} = p \times \frac{\Delta v}{v} = 3.64 \times 10^{-28} \times \frac{0.01}{100} = 3.64 \times 10^{-32}\frac{kg.m}{s}$$

$$According\ to\ HUP, \quad \Delta x.\Delta p \geq \frac{h}{4\pi}$$

$$\therefore \Delta x \geq \frac{h}{4\pi} \times \frac{1}{\Delta p} = \frac{6.63 \times 10^{-34}}{4 \times 3.142 \times 3.64 \times 10^{-32}}$$

$$\Delta x = 1.449 \times 10^{-3} m$$

3. An electron has a speed of 900 m/s with an accuracy of 0.001%. Calculate the uncertainty in the position of the electron.

$$Given: \quad v = 900\frac{m}{s}, \qquad \frac{\Delta v}{v} = \frac{0.001}{100}, \qquad m = 9.1 \times 10^{-31}\, kg,$$

$$h = 6.63 \times 10^{-34}, \qquad \Delta x = ?$$

$$p = m\, v = 9.1 \times 10^{-31} \times 900 = 8.19 \times 10^{-28}\frac{kg.m}{s}$$

$$\Delta p = m\, \Delta v = m\, v\, \frac{\Delta v}{v} = p \times \frac{\Delta v}{v} = 8.19 \times 10^{-28} \times \frac{0.001}{100} = 8.19 \times 10^{-33}\frac{kg.m}{s}$$

$$According\ to\ HUP, \quad \Delta x.\Delta p \geq \frac{h}{4\pi}$$

$$\therefore \Delta x \geq \frac{h}{4\pi} \times \frac{1}{\Delta p} = \frac{6.63 \times 10^{-34}}{4 \times 3.142 \times 8.19 \times 10^{-33}}$$

$$\Delta x = 6.44 \times 10^{-3} m$$

4. The speed of an electron is measured to within an uncertainty of 2 x 10^4 m/s. What is the minimum space required by the electron to be confined to an atom?

$$Given: \quad \Delta v = 2 \times 10^4 \frac{m}{s}, \qquad m = 9.1 \times 10^{-31}\, kg,$$

$$h = 6.63 \times 10^{-34}, \qquad \Delta x = ?$$

$$According\ to\ HUP, \quad \Delta x . \Delta p = \Delta x . m\, \Delta v \geq \frac{h}{4\pi}$$

$$\therefore \Delta x \geq \frac{h}{4\pi} \times \frac{1}{m\, \Delta v} = \frac{6.63 \times 10^{-34}}{4 \times 3.142 \times 9.1 \times 10^{-31} \times 2 \times 10^4}$$

$$\Delta x = 2.898 \times 10^{-9} m$$

5. An electron confined in a box of length 10^{-8} m. Calculate minimum uncertainty in its velocity.

$$Given: \quad \Delta x = 10^{-8} m, \qquad m = 9.1 \times 10^{-31}\, kg,$$

$$h = 6.63 \times 10^{-34}, \qquad \Delta v = ?$$

$$According\ to\ HUP, \quad \Delta x . \Delta p = \Delta x . m\, \Delta v \geq \frac{h}{4\pi}$$

$$\therefore \Delta v \geq \frac{h}{4\pi} \times \frac{1}{m\, \Delta x} = \frac{6.63 \times 10^{-34}}{4 \times 3.142 \times 9.1 \times 10^{-31} \times 10^{-8}}$$

$$\Delta v = 5797\, m/s$$