Sets and Functions

Functions

- <u>Assign</u> <u>to each element of a set</u> a particular element of another set.
- Concept of function – important in computer science.
  - In DS it is used to definition of sequences and strings
  - tells us how long it takes for a computer to solve problems
- Definition: A and B are non –empty sets

- $A \rightarrow B$ assigns element of B to an element of A
- $F: A \rightarrow B$
- Also called <u>mappings and transformations</u>

Functions

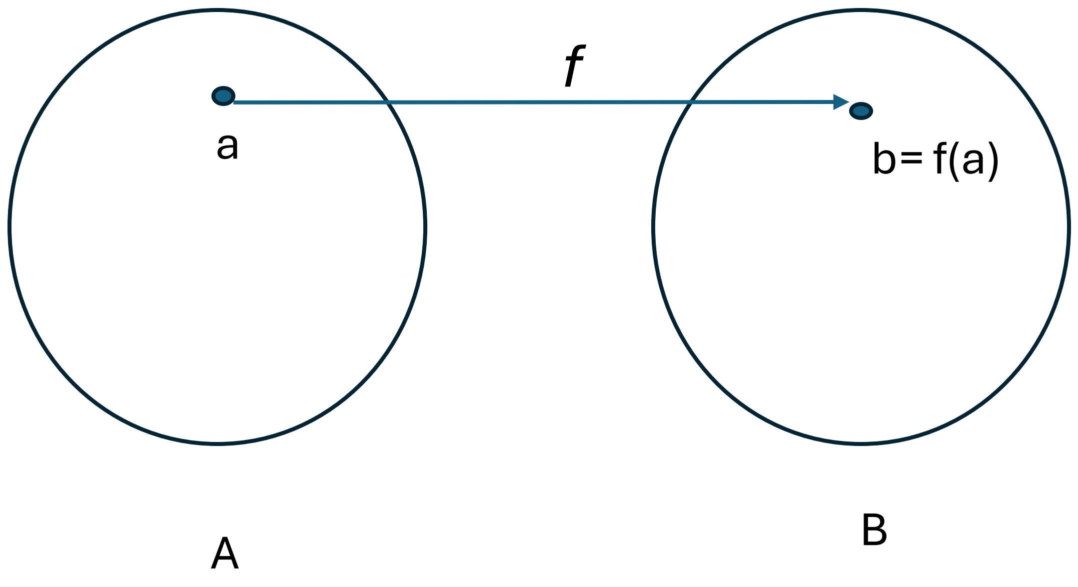
- A : is called as the <u>Domain</u>
- B is called as the <u>co-domain</u>

  f(a) = b

  b is called the <u>image of a</u>

  a is called the <u>preimage of b</u>

AXB = ordered pair (a, b) where $a \in A$.



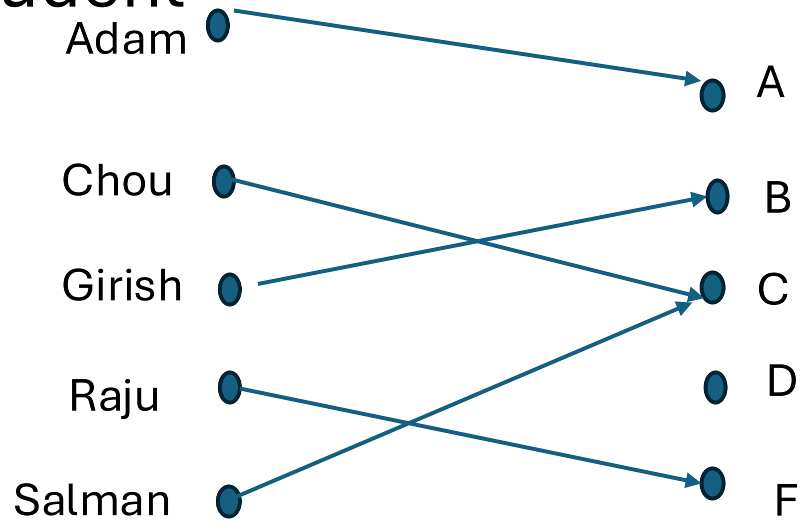
Functions

Examples

1. Students S = { Adam, Chou, Girish, Raju, Salman}, Grades = { A ,B, C ,D, F}

Range of Grades = { A, B, C, F}

Each grade except D assigned to a student



Functions

Two real valued function can be added and multiplied.

Eg. f1 and f2 be the functions from A to R.

Then f1 + f2 and f1*f2 are also functions from A to R. defined by

$(f1+f2)(x) = f1(x) + f2(x)$

$(f1f2)(x) = f1(x) + f2(x)$

Functions

- **Example:**

$f1(x) = x^2$ and $f2(x) = x – x^2$

$(f1+f2)\ (x) = f1(x) + f2(x) = x^2 + (x – x^2) = x$

$(f1f2)\ (x) = x^2\ (x – x^2) = x^3 – x^4$

- **Example:** A = {a, b, c, d, e} B = {1, 2, 3, 4}   f(a) = 2, f(b)=1, f(c)=4, f(d) = 1, f(e). What is the image of subset S ={b, c, d} is f(S)?

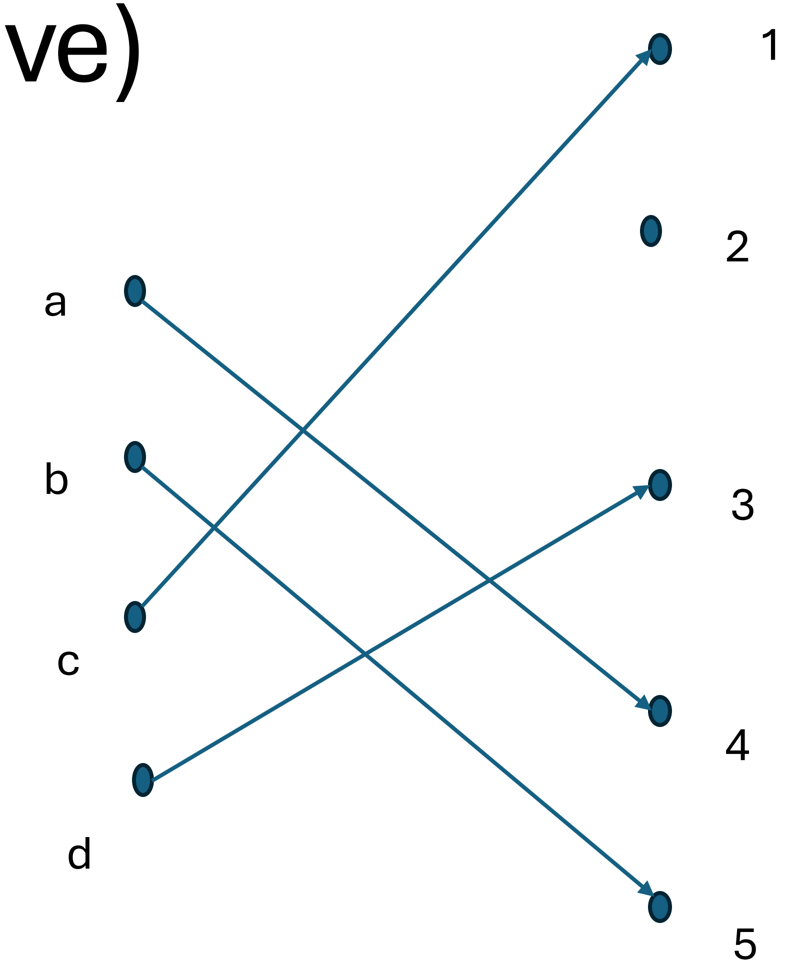
One- to-One Functions (injective)

- Same values never assigned to different domain elements.

Definition: Function *f* is <u>one-to-one or injective</u> if and only if f(a) = f(b) then a = b.

$\forall a\, \forall b\ (f(a) = f(b) \rightarrow a = b)$

Example:



One-to-one functions

- **Example 1:** $f(x) = x^2$ from a set of integers to a set of integers is it one –to-one?

  In the example $f(1) = 1$ and $f(-1) = 1$

  Hence not one-to-one.

- **Example 2:** Determine if $f(x) = x+1$ is a one-to-one function?

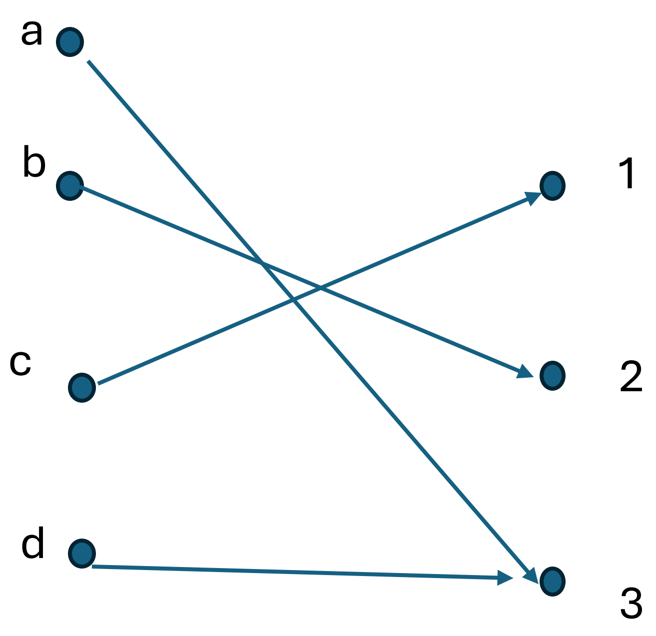
Onto Function (surjective)

- Every member of the co-domain is an image of some element in the domain.

Definition : *f* from A to B is called onto or surjective if and only if for every element b ∈ B there is an a ∈ A with f(a) = b.

∈

On-to Function (surjective)



Example 1: f be a function from {a, b, c, d} to {1, 2, 3} and f(a) = 3, f(b) = 2, f(c)= 1 and f(d) = 3. Is this an onto function.

Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Example 3: Is the $f(x) = x + 1$ from the set of integers to the set of integers onto?

One-to-one correspondence (bijection)

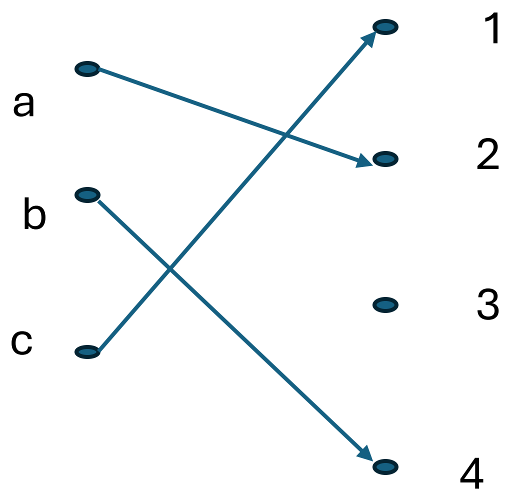
- It is both one-to-one and onto then it is called one-to-one correspondence or bijection.
- **Example1**

Let f be a function from {a, b, c, d} to {1, 2, 3, 4} with f(a) = 4, f(b) = 2, f(c) = 1 and f(d) = 3. Is it a bijection?

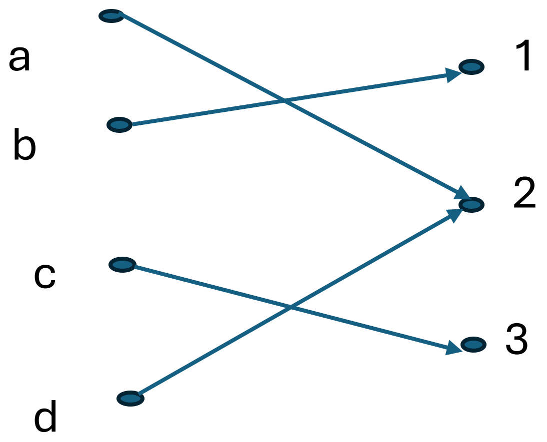
Example 2

Let A be a set. The identity function on A is the function i^A : A -> A for all $x \in A$. Is it a bijection?

Examples



One-to-one but not on-to



Onto but not one-to-one

Inverse Function (f^{-1})

- We already know that
  - If f is an onto function , then every element of B is the image of every element in A.
  - If f is a one-to-one function, then every element of B is the image of a unique element in A.
- Define a new function from B to A that reverses the correspondence given by *f*.
- **Definition:** let *f* be one-to-one correspondence from set A to set B.

Inverse function one that assigns and element of b belonging to B to unique element of A, so that f(a) = b.

Hence $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Function (f^{-1})

- If function is <u>not one-to-one correspondence</u>, then inverse function <u>cannot be defined</u>.
- If it is not one-to-one correspondence, then it is not onto and one-to-one.
- If f is not one-to-one then , some element b in the co-domain is the image of more than one element in the domain.
- If f is not onto then, for some element b in the co-domain no element a in the domain exists.

Inverse Function (f^{-1})

Example 1: Let f be a function from {a,b, c} to {1, 2, 3} with f(a) = 2, f(b) = 3 and f(c) = 1. Is the function invertible, and if so what is the inverse.

Example 2: Let f: Z->Z be such that f(x) = x + 1 . Is f invertible and what is the inverse.

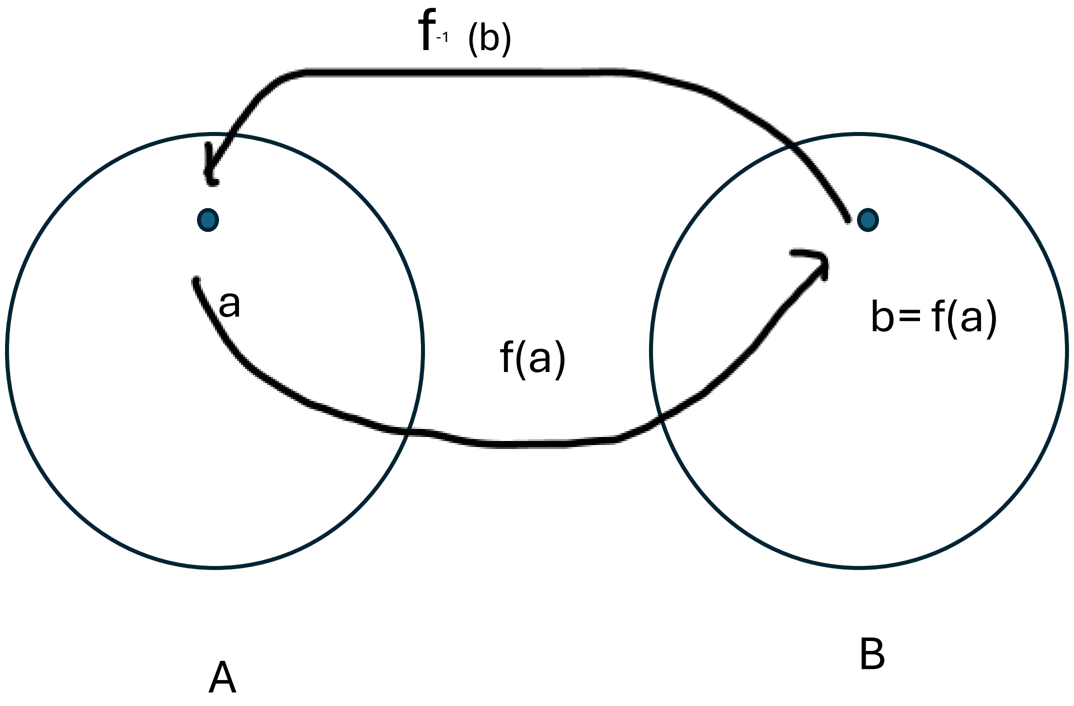
Solution 2: f has an inverse as f is one-to-one correspondence,

Let y be the image of x so that y = x+1

So x = y-1

Hence $f^{-1}(y) = y - 1$

Example 3: Let f be a function from R to R with $f(x) = x^2$. Is f invertible?



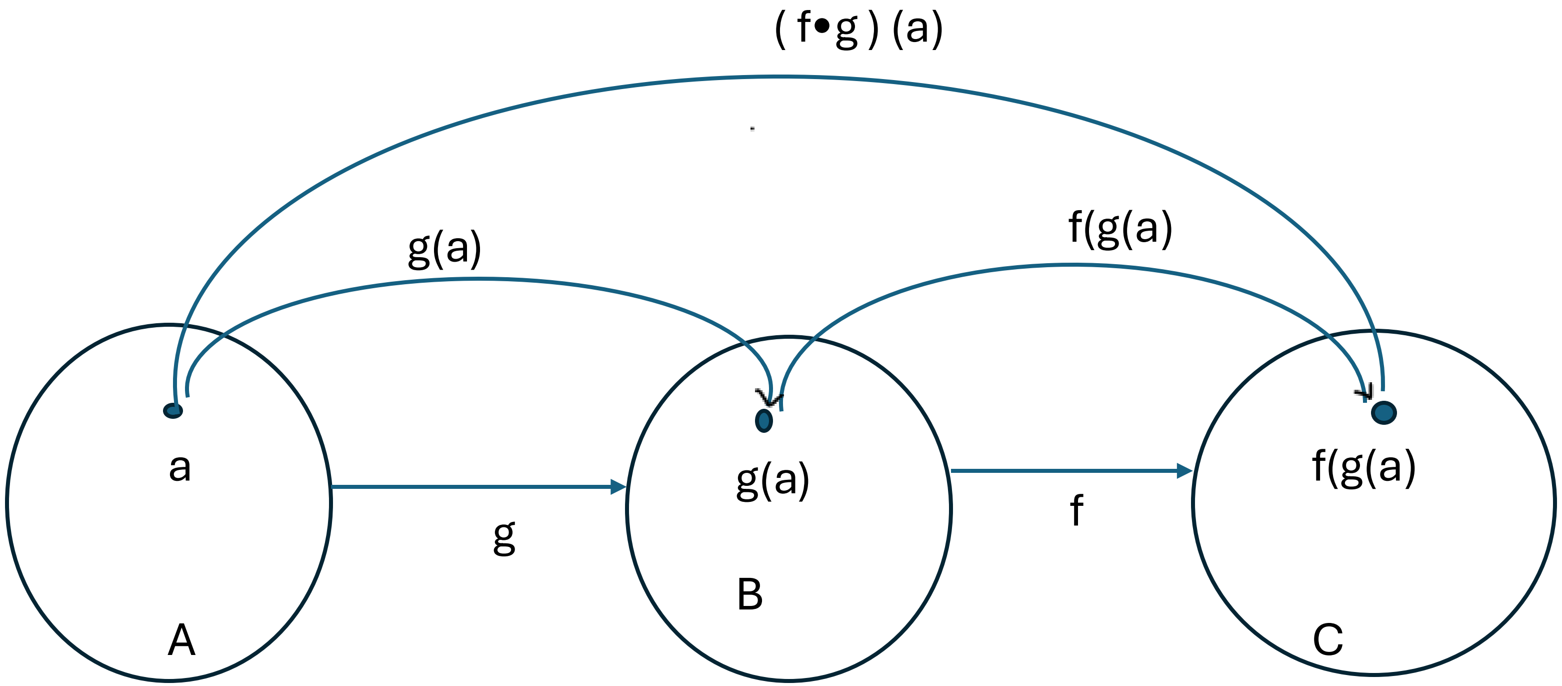
Composition function

- **Definition:** Let *g* be function from the set A to set B and let *f* be a function from set B to set C. The composition of a function is denoted by f•g i.e.

(f•g) (a) = f(g(a))

- First apply function g to a to obtain g(a) and then
- Apply the function f to the result fo g(a) to obtain (f•g) (a) = f(g(a)).

Composition function



Composition function

- **<u>Example 1:</u>** Let *g* be function from the set { a, b, c} <u>to itself</u> such that g(a) = b, g(b) =c and g(c) = a. Let *f* be the function from the set {a, b, c} to the set { 1, 2, 3} such that f(a) = 3, f(b) = 2 and f(c) = 1. what is the composition of f and g and what is the composition of g and f.
- **Solution 1:**

$(f \circ g)(a) = f(g(a)) = f(b)=2,$

$=f(g(b)) = f(c)=1$

$=f(g(c))=f(a)=3$

Examples

- **Example 2:** Let *g* and *f* be from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of *f* and *g*? what is the composition of *g* and *f*?
- Solution :

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x +2) +2 = 6x + 7$$

$$(g \circ f)(x) = g(f(x)) = g(2x+ 3) = 2(3x+2)+3 =$$

Relations

- Types of relations : employee – salary, business – phone number,
- In mathematics : positive integer and divisor
- Relation between elements of a set represented using a structure called a <u>relation - also subset of a cartesian product of the set.</u>
- Relations also used to solve problems

Relations and its properties

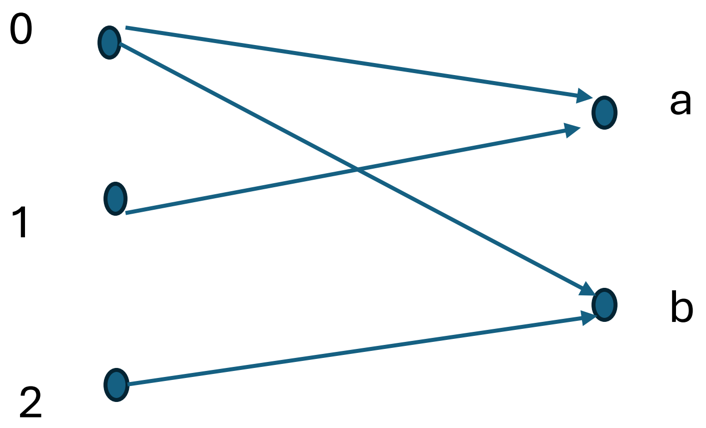
- Represented by an ordered pair made up of related elements.
- Set of ordered pairs called as binary relations.
- **Definition:** Let A and B be set. A binary relation from A to B is a subset of (A X B).

- aRb notation given as $(a, b) \in R$ and $a \not R b$ $(a, b) \notin R$
- Binary relationships are relationships between two sets.
- Example :Let A be a set of Cities and B be a set of States. Then define Relation R such that (a, b) where *a* is a city in a particular state.
- Eg. (Surat, Gujarat)

Functions as Relations

- Function assigns one element of B to each element of A.
- Graph of f is an ordered pairs of (a, b) such that b = f(a).
- Relations can be used to express a one-to-many relationship between the elements of the sets A and B.

R	a	b
0	x	x
1	x	
2		x



Relations on a Set

- Relations from a set to itself are of special interest.
- **Definition** : Relation on the set A is a subset of A X A.

- Example: let A be the set {1, 2, 3, 4}. What ordered pairs are there in the relation R = { (a,b) | a divides b}

Solution:{ (1,1), (1,2),(1,3), (1,4), (2,2), (2,4), (3,3), (4, 4)}. It can be represented in both tabular form and graphical form.

Example 6: how many relations are there on a set with n elements.

2^(n^2)

Properties of Relations

- Some relations in element is always related to itself.

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

$\forall a ((a,a) \in R)$ where universe of discourse is the set of all elements of A.

Examples of Reflexive relation

- Example which of the following relations are reflexive.

$R1 = \{ (a,b) \mid a <= b\}$

$R2 = \{ (a,b) \mid a > b\}$

$R3 = \{ (a,b) \mid a = b \text{ or } a = -b\}$

$R4 = \{ (a,b) \mid a = b\}$

$R5 = \{ (a,b) \mid a = b + 1\}$

$R6 = \{ (a,b) \mid a + b <= 3\}$

- Example : Is the 'divides' relation on the set of positive integers reflexive.

Properties on Relations

- In some relations an element is related to the second element if the second element is also related to the first element.

Definition:

- A relation R on a set A is called symmetric if $(b,a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- A relation R on a set A such that for all $a,b \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

Using quantifiers:

$\forall a \forall b\, ((a,b) \in R) \rightarrow ((b,a) \in R)$ (symmetric)

$\forall a \forall b\, ((a,b) \in R) \land ((b,a) \in R) \rightarrow (a = b)$ (anti-symmetric)

Properties of Relations

- Transitive property

Definition: A relation R on a set A is called <u>transitive</u> if whenever (a, b) ∈ R and (b, c) ∈ R then (a, c) ∈ R for all (a, b, c) ∈ R.

For $\forall a \; \forall b \; \forall c \; (((a,b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$

Example: How many reflexive relations are there on a set with n elements.

Examples Transitive relations

- Example 1: Is the divides relation on the set of positive integers transitive.

- Example 2: How many reflexive relations are there on a set with n elements.
- Solution: AXA – n^2 ordered pairs,
- By product rule of counting there are $2^{n(n-1)}$ reflective relations

Combining Relations

- Example: What is the composite of two relations R and S, where R is the relation from {1, 2, 3} to { 1, 2, 3, 4} with R = { (1,1), (1,4), (2, 3), (3, 1), (3,4)} and S is the relation from { 1, 2, 3, 4} and { 0,1,2} with S = {(1,0), (2,0), (3, 1), (3, 2), (4,1)}.