PSE 2024-25 ESE Solution

Q.1 (a) $\mu = p$, $\sigma^2 = pq$

(b) (i) Poisson

(ii) 1.73

(iii) 0

Q.2 (a) $P(X \geq 30) = P\left(\frac{X-\mu}{\sigma} \geq \frac{30-24}{3.8}\right)$

$= P(Z \geq 1.58)$

$= 1 - P(Z < 1.58)$

$= 1 - 0.9429$

$= 0.0571$

R command : 1 − pnorm(30, 24, 3.8) or
1 − pnorm(1.58)

(b) $P(X > 15) = P\left(Z > \frac{15-24}{3.8}\right)$

$= P(Z > -2.37)$

$= 1 - P(Z \leq -2.37)$

$= 1 - 0.0089$

$= 0.9911 \approx 99.11$ % of time he will be late for the work.

R Command : 1 − Pnorm(15, 24, 3.8)
or 1 − Pnorm(−2.37)

(c) $P(X > 25) = P\left(Z > \frac{25-24}{3.8}\right)$

$= P(Z > 0.26)$

$= 1 - P(Z \leq 0.26)$

$= 1 - 0.6026$

$= 0.3974$

R command: 1 − Pnorm(0.26) or
1 − Pnorm(25, 24, 3.8)

(d) find 'x' s.t. $P(X > x) = 0.15$

$$\Rightarrow 1 - P(X \leq x) = 0.15$$

$$\Rightarrow P(X \leq x) = 1 - 0.15$$

$$P(X \leq x) = 0.85$$

R ⇒ qnorm(0.85, 24, 3.8)

$$\Rightarrow z = 1.04 \Rightarrow x = \mu + \sigma z$$

$$= 24 + 3.8 \times 1.04$$

$$\Rightarrow \boxed{x = 27.952} \text{ minutes.}$$

(e) $P(Y=2) = {}^{3}C_{2} * (0.0571)^2 * (1 - 0.0571)^1$

$$= 9.222 \times 10^{-3}$$

R ⇒ dbinom(2, 3, 0.0571)

Q.3 $\lambda = \frac{1}{10000}$

$F(x) = P(X \leq x) = 1 - e^{-\lambda x}$

$\therefore \boxed{P(X > x) = e^{-\lambda x}}$

Prob(remaining lifetime > 5000)

$$= 1 - F(5000) = 1 - (1 - e^{-\lambda x})$$

$$= e^{-\lambda x}$$

$$= e^{-\frac{5000}{10000}}$$

$$= e^{\frac{-5000}{10000}}$$

$$= e^{-1/2} = 0.604$$

Q.4) $p = \frac{1}{1000} = 10^{-3}$ (small)

$n = 1000$ (large)

$\lambda = np = 10^{-3} \times 10^{3} = 1$

$$P(X < 7) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \frac{e^{-1} 1^6}{6!} \sim$$

R ⇒ ppois(6, 1)

II

$\lambda = ?$ when $n = 10,000$

$\lambda = 1$ for 1000

$\therefore \lambda = ?$ for 10000

$\Rightarrow \boxed{\lambda = 10 \text{ for } n = 10,000}$

Q5 (a) if $E(\eta) = \theta$ then η is called as an unbiased estimator of the population parameter θ where η: sample parameter

(b) if $E(\eta) \neq \theta$ then η is said to be a 'biased' estimator' of θ.

(c) Unbiased estimator of σ^2 is s^2.

justification:

Note : $s^2 = \dfrac{\sum_i (X_i - \bar{X})^2}{n-1}$

where $\{x_1, x_2, \ldots x_n\}$ are sample points/elements

n: sample size

s^2: sample variance

Note $\sum (X_i - \mu)^2 = \sum (x_i - \bar{x} + \bar{X} - \mu)^2$

$= \sum (X_i - \bar{X})^2 + 2 \sum (x_i - \bar{x})(\bar{x} - \mu) + \sum (\bar{x} - \mu)^2$

$\Rightarrow \sum (X_i - \mu)^2 = \sum (x_i - \bar{x})^2 + \sum (\bar{x} - \mu)^2$

$\Rightarrow \sum (X_i - \mu)^2 = \sum (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$

$\Rightarrow \dfrac{\sum (X_i - \mu)^2}{(n-1)} = \dfrac{\sum (X_i - \bar{X})^2}{n-1} + \dfrac{n(\bar{x} - \mu)^2}{n-1}$

$$\therefore\ s^2 = \frac{\sum (x_i - \bar{x})^2}{(n-1)} - \frac{n(\bar{x}-\mu)^2}{(n-1)}$$

$$\therefore\ E(s^2) = E\left(\frac{\sum (x_i - \bar{x})^2}{(n-1)}\right) - \frac{n}{n-1} E(\bar{x}-\mu)^2$$

$$= \frac{\sum E(x_i - \bar{x})^2}{(n-1)} - \frac{n}{n-1} E(\bar{x}-\mu)^2$$

$$= \frac{\sum_i \mathrm{Var}(x_i)}{(n-1)} - \frac{n}{n-1} \mathrm{var}(\bar{x})$$

$$= \frac{\sum \sigma^2}{n-1} - \frac{n}{(n-1)} \frac{\sigma^2}{n}$$

$$= \frac{n\sigma^2}{n-1} - \frac{n\sigma^2}{n(n-1)}$$

$$= \frac{n\sigma^2}{(n-1)} \left[\frac{1}{\cancel{~}1} - \frac{1}{n}\right]$$

$$\boxed{E(s^2) = \frac{\cancel{n}\sigma^2}{(\cancel{n-1})} \times \frac{\cancel{n-1}}{\cancel{n}} = \sigma^2}$$

Central limit Thm :

Q.6 (a) Let $\{X_1, X_2, \ldots X_n\}$ be a sample of 'n' elements s.t. $E(x_i) = \mu$ & $\mathrm{Var}(x_i) = \sigma^2\ \forall\ i$ then $E(\bar{X}) = \mu$ & $\mathrm{Var}(\bar{X}) = \sigma^2/n$ & hence

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$ where Z: standard normal r.v.

(b) X: No. of cherries in cherry Puff.

$$E(X) = E(\bar{X}) = \mu = \sum_{i=1}^{4} x_i f(x_i)$$

$$= (4 \times 0.2) + (5 \times 0.4) + (6 \times 0.3) + (7 \times 0.1)$$

$$= 5.3$$

(III)

$$Var(\bar{X}) = \frac{Var(X)}{n} = \frac{E(X^2) - (E(X))^2}{n}$$

$$= (16 \times 0.2) + (25 \times 0.4) + (36 \times 0.3) + (49 \times 0.1) - 5.3^2 / n$$

$$= (3.2 + 10 + 10.8 + 4.9) - 5.3^2 / 36$$

$$= 28.9 - 5.3^2 / 36 = \frac{0.81}{36} = 0.0225$$

$$\therefore \boxed{Var(\bar{X}) = 0.0225}$$

(ii) $P(\bar{X} < 5.5) = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{5.5 - 5.3}{\sqrt{0.0225}}\right)$

$$= P\left(Z < \frac{0.2}{0.15}\right)$$

$$= P(Z < 1.33)$$

$$\boxed{Ans = 0.9082}$$

(Q.7) $n = 64$

$H_0 : p = 0.5$ (coin is fair)

$H_1 : p \neq 0.5$ (coin is unfair)

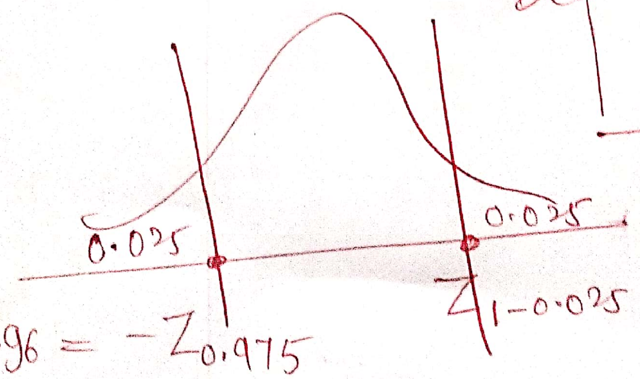
$\alpha = 0.05$

Decision rule = ?

$\boxed{\text{Let } X : \text{No. of heads in 64 tosses}}$

Decision

$\boxed{\text{Accept } H_0 \text{ if } -1.96 < Z < 1.96}$



$-1.96 = -Z_{0.975}$ $\quad Z_{1-0.025} = Z_{0.975} = 1.96$

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ie. if

$$-1.96 < \frac{X-\mu}{\sigma} < 1.96$$

$$\Rightarrow -1.96 < \left(\frac{X-np}{\sqrt{npq}}\right) < 1.96$$

$$\Rightarrow -1.96 < \left(\frac{X-64\times0.5}{\sqrt{64\times0.5\times0.5}}\right) < 1.96$$

$$\Rightarrow -1.96 < \frac{X-32}{4} < 1.96$$

$$\Rightarrow -7.84 < X-32 < 7.84$$

$$\boxed{24.16 < X < 39.84}$$ → Accept H_0 that coin is fair otherwise reject.

$\uparrow$ No. of 'Heads'

OR

$n = 20$ packages

$\sigma_0 = 0.25$ kg

$s = 0.32$

$\alpha = 0.01$

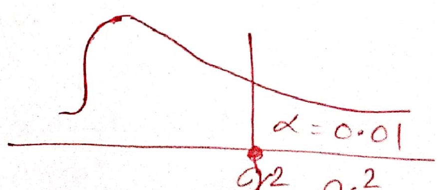
(I) $H_0: \sigma \le \sigma_0 \Rightarrow \sigma \le 0.25$

$H_1: \sigma > \sigma_0 \Rightarrow \sigma > 0.25$

(II) $\alpha = 0.01$

(III) $$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{19\times0.32^2}{0.25^2} = 31.1296$$

(IV) Critical region:



$\chi^2_\alpha = \chi^2_{0.01} = 36.191$

$\chi^2 > 36.191$

(V) $31.1296 \notin$ Critical region

$\therefore$ fail to reject H_0.

8) $n = 100$
$\bar{X} = 39350$
$\sigma = 3260$
$\alpha = 0.01$
$\mu_0 = 40000$

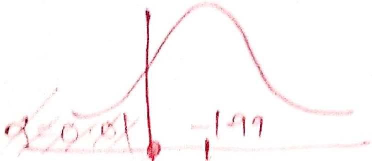
(I) $H_0 : \mu \geq 40{,}000$
$H_1 : \mu < 40{,}000$

(II) $\alpha = 0.01$

(III) $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{39350 - 40{,}000}{\frac{3260}{\sqrt{100}}}$

$\boxed{z = -1.99386}$

(IV) Critical region



$Z_\alpha = Z_{0.01}$
$= -2.33$

$\boxed{\text{Critical region is } Z < -2.33}$

$-1.99 \notin$ critical region

(V) fail to reject H_0.

$\boxed{\text{OR}}$

OR

(I) $H_0 : \tilde{\mu} = 12$

$H_1 : \tilde{\mu} \neq 12$ $\Big\}$ $\frac{1}{2}$

(II) $\alpha = 0.02$

(III) test statistic: Binomial r.v. X with $p = \frac{1}{2}$

(IV) $-, 0, +, +, 0, +, 0, -, +, +, +, +, +, -,$
$+, -, -, -$ $\Big\}$ $\frac{1}{2}$

$n = 15$ (ommitting '0's)

$x = 9$ & $n/2 = 7.5$

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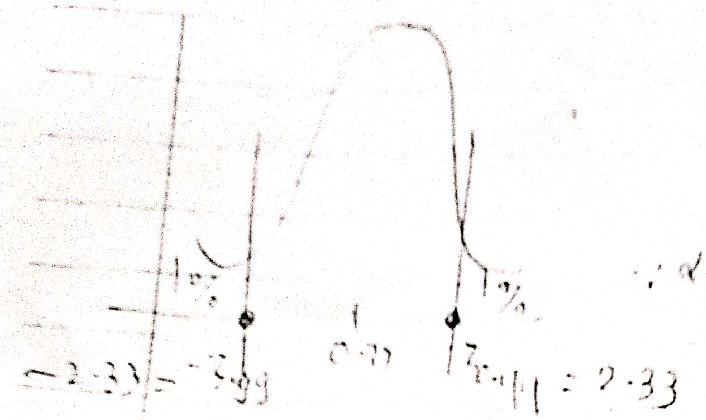

$\therefore \alpha = 2\%$

$P = 2P(X \leq 9 \,;\, P = \tfrac{1}{2})$

$$z = \frac{X - np}{\sqrt{npq}} = \frac{9 - 15 * \frac{1}{2}}{\sqrt{15 * \frac{1}{2} * \frac{1}{2}}}$$

$$= \frac{9 - 7.5}{\sqrt{3.75}}$$

$$= 0.7745 \in \text{Acceptance Region}$$

$\therefore$ Accept H_0.

Q.15 $n = 7$

Usin(g) R

$y \approx b_0 + b_1 x_1 + b_2 x_2$

A:

$$\boxed{\begin{aligned} b_0 &= 51.7152 \\ b_1 &= 0.5810 \\ b_2 &= -0.6899 \end{aligned}}$$

$y = 51.7152 + (0.5810 * 60) - 0.6899 * 4$

$\Rightarrow \boxed{y = 83.8156}$

estimate for $x_1 = 60$ &

$x_2 = 4$

4) (a) $\mu = E(X) = (-1 * \frac{1}{2}) + (1 * \frac{1}{2})$

$= 0$ (independent of t) yes! It is

(b) $E(X^2) = ((-1)^2 * \frac{1}{2}) + (1^2 * \frac{1}{2})$

$= \frac{1}{2} + \frac{1}{2} = 1$

$\frac{1}{2}$

(c) Yes! stationary in wide Sense!

2

Variables $X_0, X_1, \ldots$ ... of possible values of these random variables is $\{0, 1, 2, \ldots, M\}$. It is helpful to interpret X_n as being state of some system at time n.

therefore we say that system is in state i at time n if $X_n = i$. A system of Markov Chain is a sequence of random variables if each time system is in state i there is some fixed probability say P_{ij} for which the system will be in next state j

ie $P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$

$= P_{ij} \longrightarrow (M)$

$0 \le i \le M$ & $0 \le j \le N$.

P_{ij} : transition Probability

$\therefore P_{00} = P\{X_1 = 0 \mid X_0 = 0\}$

$=$ Prob{raining today when tomorrow when it was raining today}

$\boxed{P_{00} = \alpha}$

$P_{01} = P\{X_1 = 1 \mid X_0 = 0\}$

= Prob{not raining tomorrow ...

$P_{10} = P\{X_1 = 0 \mid X_0 = 1\}$

Prob (of raining tomorrow when it is not raining today)

$= \beta$

$P_{11} = P(X_1 = 1 \mid X_0 = 1)$

$= P$ (not raining tomorrow when it was not raining today)

$P_{11} = 1-\beta$

∴ Transition matrix is $\begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$