Potential Barrier

$V(x) = 0$	$x \leq 0$
$V(x) = V_o$	$0 < x < a$
$V(x) = 0$	$x \geq a$

$E > V_o$????

$E < V_o$????

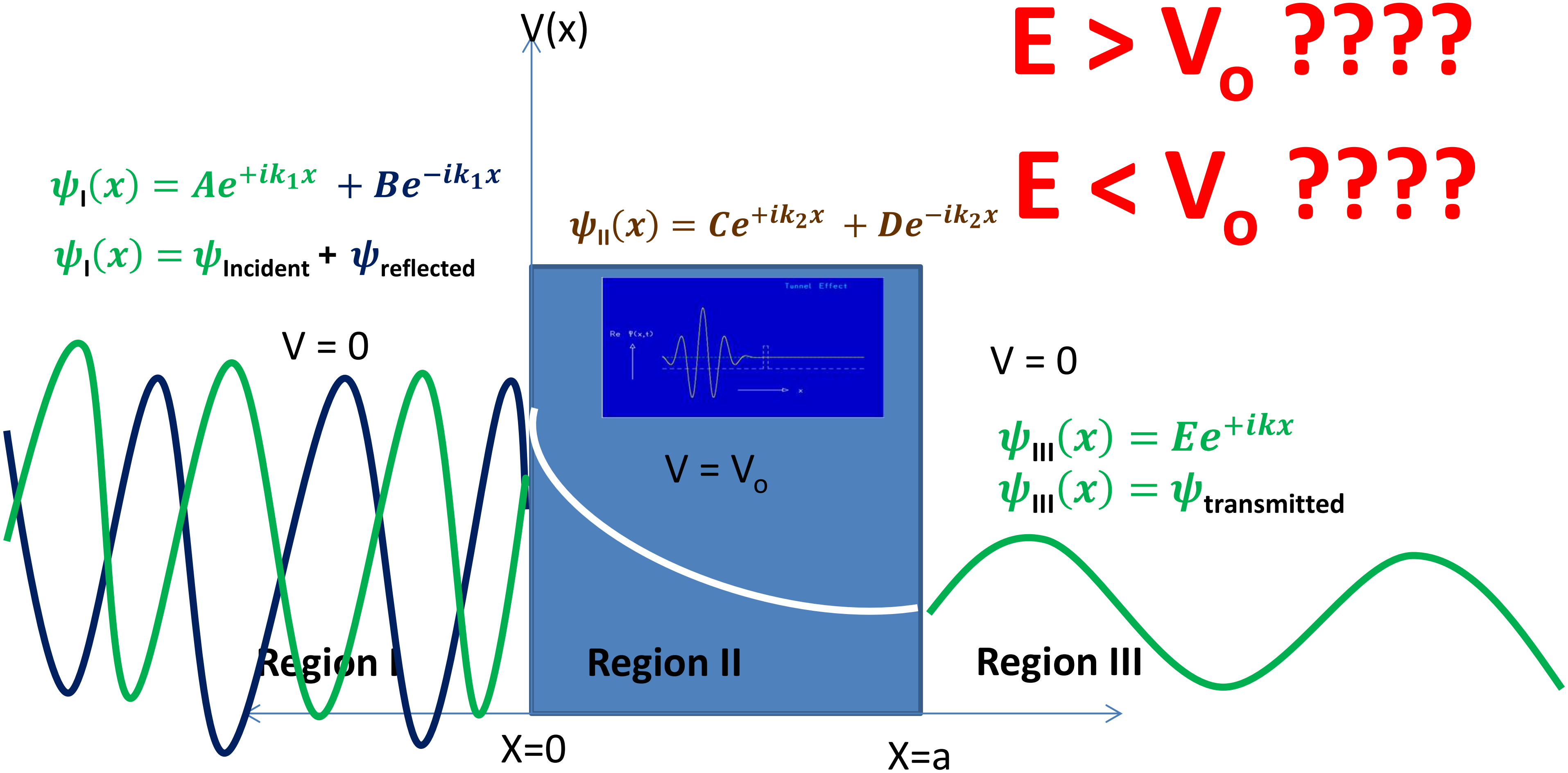
$$\psi_{I}(x) = Ae^{+ik_1x} + Be^{-ik_1x}$$

$$\psi_{I}(x) = \psi_{\text{Incident}} + \psi_{\text{reflected}}$$

$$\psi_{II}(x) = Ce^{+ik_2x} + De^{-ik_2x}$$

$$\psi_{III}(x) = Ee^{+ikx}$$

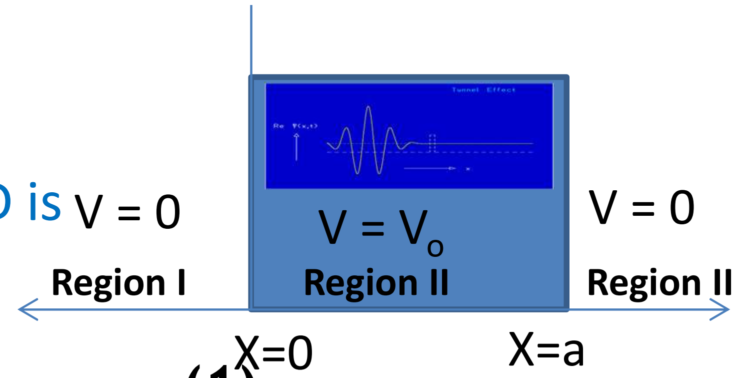
$$\psi_{III}(x) = \psi_{\text{transmitted}}$$



- Scanning Tunneling Microscopy
- Tunnel diode

- **For Region I**

- Schrodinger time independent Wave equation in 1D is



$$\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}(E - V)\,\psi(x) = 0 \text{ ----------------------(1)}$$

- In region **I**, V = 0.

$$\frac{d^2}{dx^2}\psi_I(x) + \frac{2m}{\hbar^2}E\,\psi_I(x) = 0 \text{ --- -----------------(2)}$$

- Let, $k_1^2 = \frac{2m}{\hbar^2}E$ ---(3)

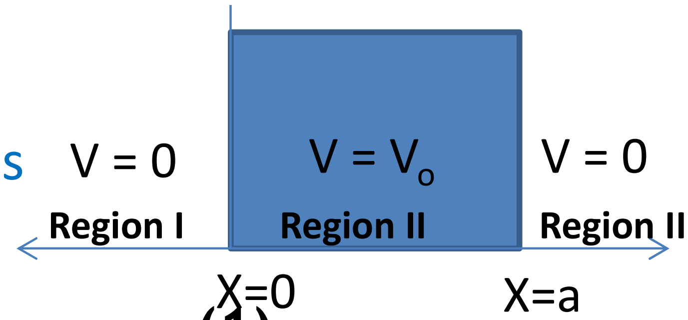
$$(\nabla^2 + k_1^2)\,\psi_I(x) = 0$$

$$(\nabla^2 - i^2 k_1^2)\,\psi_I(x) = 0 \text{ ------ ----------------------------(4)}$$

$$\psi_I(x) = Ae^{+ik_1x} + Be^{-ik_1x} \text{---------------------------- --(5)}$$

$$\psi_I(x) = \psi_{\text{Incident}} + \psi_{\text{reflected}} \text{ -----------------------------(6)}$$

- **For Region II**
- Schrodinger time independent Wave equation in 1D is



$$\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}(E - V)\,\psi(x) = 0 \quad \text{--------------------(1)}$$

- **In region II, V = V$_o$.**

$$\frac{d^2}{dx^2}\psi_{II}(x) + \frac{2m}{\hbar^2}(E - V_o)\,\psi_{II}(x) = 0 \quad \text{--------------------(2)}$$

- Let,

$$k_2^2 = \frac{2m}{\hbar^2}(E - V_o) \text{--(3)}$$

- Eqn 2 can be

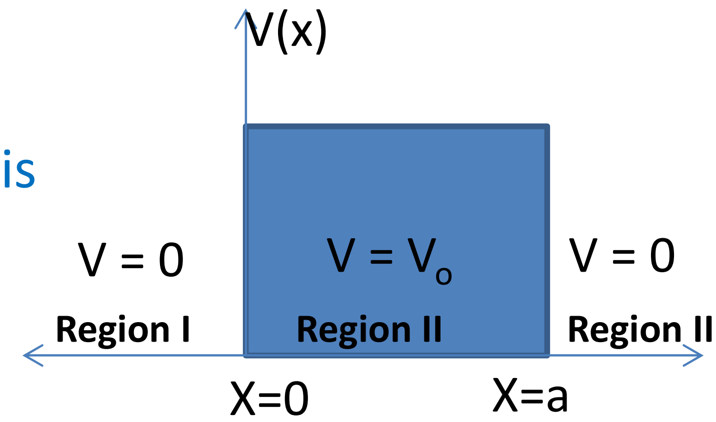
$$\left(\frac{d^2}{dx^2} + k_2^2\right)\psi_{II}(x) = 0 \quad \text{--- ------------------------------(4)}$$

$$(\nabla^2 - \boldsymbol{i}^2 k_2^2)\,\psi_{II}(x) = 0$$

$$\psi_{II}(x) = Ce^{+ik_2x} + De^{-ik_2x} \text{--- ------------------------(5)}$$

- **For Region III**

- Schrodinger time independent Wave equation in 1D is



$$\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}(E - V)\,\psi(x) = 0 \text{ ----------(1)}$$

- In region **III**, V = 0.

$$\frac{d^2}{dx^2}\psi_{III}(x) + \frac{2m}{\hbar^2}E\,\psi_{III}(x) = 0 \text{ ------(2)}$$

- Let, $k_1^2 = \frac{2m}{\hbar^2}E$ --------------------------(3)

$$(\nabla^2 + k_1^2)\,\psi_{III}(x) = 0$$

$$(\nabla^2 - i^2 k_1^2)\,\psi_{III}(x) = 0 \text{ ------ -----------------------------(4)}$$

$$\psi_{III}(x) = Ee^{+ikx} \text{ --- ---------------------- --(5)}$$

$$\psi_{III}(x) = Ee^{+ikx} = \psi_{transmitted} \text{ ---------------------------(6)}$$

At $x = 0$, $\psi_{I}(x)|_{x=0} = \psi_{II}(x)|_{x=0}$

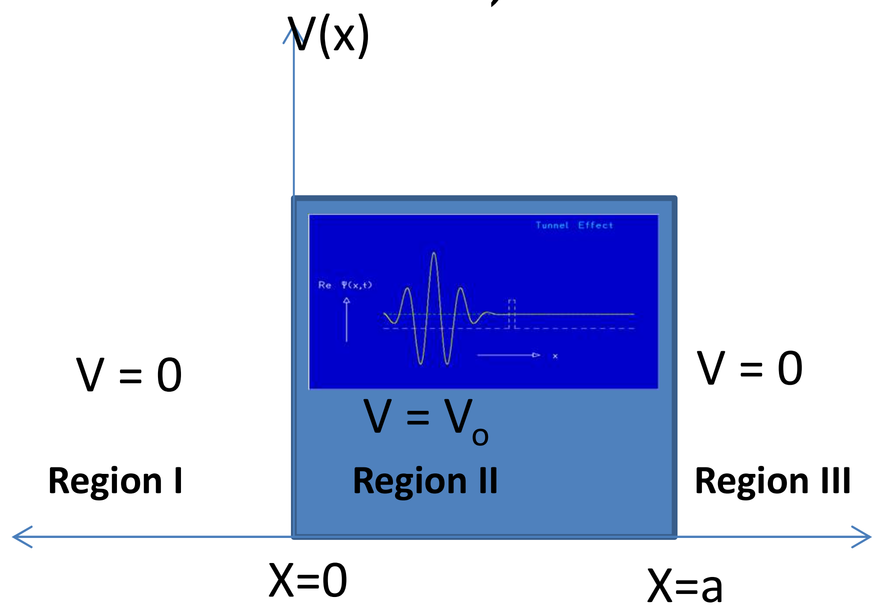
$$Ae^{+ik_1(0)} + Be^{-ik_1(0)} = Ce^{+ik_2(x=0)} + De^{-ik_2(x=0)}$$

$$A + B = C + D$$

$$\frac{d}{dx}\psi_{I}\,|_{x=0} = \frac{d}{dx}\psi_{II}\,|_{x=0}$$

$$\frac{d}{dx}\left(Ae^{+ik_1(x=0)} + Be^{-ik_1(x=0)}\right) = \frac{d}{dx}\left(Ce^{+ik_2(x=0)} + De^{-ik_2(x=0)}\right)$$

$$k_1 (A - B) = k_2 (C - D)$$



At $x = a$, $\psi_{II}(x)|_{x=a} = \psi_{III}(x)|_{x=a}$

$$Ce^{+ik_2(x=a)} + De^{-ik_2(x=a)} = Ee^{+ik_1(x=a)}$$

$$Ce^{+ik_2 a} + De^{-ik_2 a} = Ee^{+ik_1 a}$$

$$\frac{d}{dx}\psi_{II}\,|_{x=a} = \frac{d}{dx}\psi_{III}\,|_{x=a}$$

$$\frac{d}{dx}\left(Ce^{+ik_2(x=a)} + De^{-ik_2(x=a)}\right) = \frac{d}{dx}\left(E\,e^{+ik_1(x=a)}\right)$$

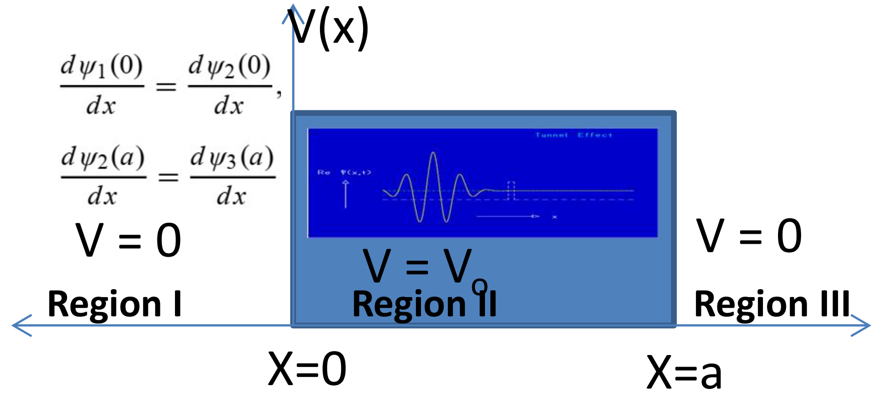
$$k_2 \left(Ce^{+ik_2 a} + De^{-ik_2 a}\right) = k_1\, E\, e^{+ik_1 a}$$

$$A + B = C + D \quad \text{\_\_\_\_\_\_\_\_\_\_}(1)$$

$$k_1\,(A - B) = k_2\,(C - D) \quad \text{\_\_\_\_\_\_\_\_\_\_}(2)$$

$$\psi_1(0) = \psi_2(0), \qquad \frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx},$$

$$\psi_2(a) = \psi_3(a), \qquad \frac{d\psi_2(a)}{dx} = \frac{d\psi_3(a)}{dx}$$



Multiply eqn (1) by k_1 and add & subtract from eqn (2)

$$k_1 A + k_1 B = k_1 C + k_1 D$$
$$k_1 A - k_1 B = k_2 C - k_2 D)$$
$$+ \qquad + \qquad = + \qquad +$$

$$2k_1 A = (k_1 + k_2)C + (k_1 - k_2)D$$

$$\Rightarrow \quad A = \left(\frac{k_1 + k_2}{2k_1}\right) C + \left(\frac{k_1 - k_2}{2k_1}\right) D$$

$$k_1 A + k_1 B = k_1 C + k_1 D$$
$$k_1 A - k_1 B = k_2 C - k_2 D)$$
$$- \qquad - \qquad = - \qquad -$$

$$2k_2 B = (k_1 - k_2)C + (k_1 + k_2)D$$

$$\Rightarrow \quad B = \left(\frac{k_1 - k_2}{2k_1}\right) C + \left(\frac{k_1 + k_2}{2k_1}\right) D$$

$$Ce^{+ik_2a} + De^{-ik_2a} = Ee^{+ik_1a} \quad \text{\_\_\_\_\_\_\_\_\_\_}(3)$$

$$k_2\,(Ce^{+ik_2a} - De^{-ik_2a}) = k_1\,E\,e^{+ik_1a} \quad \text{\_\_\_\_\_\_\_\_\_\_}(4)$$

Multiply eqn (3) by k_2 and add & subtract from eqn (4)

$$k_2 Ce^{+ik_2a} + k_2 De^{-ik_2a} = k_2 Ee^{+ik_1a}$$
$$k_2 Ce^{+ik_2a} - k_2 De^{-ik_2a} = k_1\,E\,e^{+ik_1a}$$
$$\pm \qquad \pm \qquad = \pm$$

$$2\,k_2 Ce^{+ik_2a} = (k_1 + k_2)\,E\,e^{+ik_1a}$$

$$\Rightarrow \quad C = \left(\frac{k_1 + k_2}{2k_2}\right) E\,e^{+ik_1a}\;e^{-ik_2a}$$

$$\Rightarrow \quad D = \left(\frac{k_2 - k_1}{2k_2}\right) E\,e^{+ik_1a}\;e^{+ik_2a}$$

$$A + B = C + D \quad \text{\_\_\_\_\_\_\_\_\_\_(1)}$$

$$k_1 (A - B) = k_2 (C - D) \quad \text{\_\_\_\_\_\_\_\_ (2)}$$

$$Ce^{+ik_2a} + De^{-ik_2a} = Ee^{+ik_1a} \quad \text{\_\_\_\_\_\_(3)}$$

$$k_2 (Ce^{+ik_2a} - De^{-ik_2a}) = k_1 E\, e^{+ik_1a} \quad \text{\_\_\_\_\_\_\_(4)}$$

$$A = \left(\frac{k_1 + k_2}{2k_1}\right) C + \left(\frac{k_1 - k_2}{2k_1}\right) D \quad \text{\_\_\_\_\_(1')}$$

$$B = \left(\frac{k_1 - k_2}{2k_1}\right) C + \left(\frac{k_1 + k_2}{2k_1}\right) D \quad \text{\_\_\_\_\_(2')}$$

$$C = \left(\frac{k_1 + k_2}{2k_2}\right) E\, e^{+ik_1a}\, e^{-ik_2a} \quad \text{\_\_\_\_\_(3')}$$

$$D = \left(\frac{k_2 - k_1}{2k_2}\right) E\, e^{+ik_1a}\, e^{+ik_2a} \quad \text{\_\_\_\_\_(4')}$$

$$A = \left(\frac{k_1 + k_2}{2k_1}\right) C + \left(\frac{k_1 - k_2}{2k_1}\right) D$$

$$A = \left(\frac{k_1 + k_2}{2k_1}\right)\left(\frac{k_1 + k_2}{2k_2}\right) E\, e^{+ik_1a}\, e^{-ik_2a} + \left(\frac{k_1 - k_2}{2k_1}\right)\left(\frac{k_2 - k_1}{2k_2}\right) E\, e^{+ik_1a}\, e^{+ik_2a}$$

$$A = \left(\frac{(k_1 + k_2)^2}{4\, k_1 k_2}\right) E\, e^{+ik_1a}\, e^{-ik_2a} - \left(\frac{(k_1 - k_2)^2}{4 k_1 k_2}\right) E\, e^{+ik_1a}\, e^{+ik_2a}$$

$$A = \frac{E\, e^{+ik_1a}}{4\, k_1 k_2}\left((k_1 + k_2)^2\, e^{-ik_2a} - (k_1 - k_2)^2\, e^{+ik_2a}\right)$$

$$A * = \frac{E\, e^{-ik_1a}}{4\, k_1 k_2}\left((k_1 + k_2)^2\, e^{+ik_2a} - (k_1 - k_2)^2\, e^{-ik_2a}\right)$$

$$A + B = C + D \quad \text{\_\_\_\_\_\_\_\_\_\_\_(1)}$$

$$k_1 (A - B) = k_2 (C - D) \quad \text{\_\_\_\_\_\_\_ (2)}$$

$$Ce^{+ik_2a} + De^{-ik_2a} = Ee^{+ik_1a} \quad \text{\_\_\_\_\_(3)}$$

$$k_2 (Ce^{+ik_2a} - De^{-ik_2a}) = k_1 E e^{+ik_1a} \quad \text{(4)}$$

$$A = \left(\frac{k_1 + k_2}{2k_1}\right) C + \left(\frac{k_1 - k_2}{2k_1}\right) D \quad \text{\_\_\_\_(1')}$$

$$B = \left(\frac{k_1 - k_2}{2k_1}\right) C + \left(\frac{k_1 + k_2}{2k_1}\right) D \quad \text{\_\_\_\_(2')}$$

$$C = \left(\frac{k_1 + k_2}{2k_2}\right) E e^{+ik_1a} e^{-ik_2a} \quad \text{\_\_\_\_(3')}$$

$$D = \left(\frac{k_2 - k_1}{2k_2}\right) E e^{+ik_1a} e^{+ik_2a} \quad \text{\_\_\_\_(4')}$$

$$B = \left(\frac{k_1 - k_2}{2k_1}\right) C + \left(\frac{k_1 + k_2}{2k_1}\right) D$$

$$B = \left(\frac{k_1 - k_2}{2k_1}\right)\left(\frac{k_1 + k_2}{2k_2}\right) E e^{+ik_1a} e^{-ik_2a} + \left(\frac{k_1 + k_2}{2k_1}\right)\left(\frac{k_2 - k_1}{2k_2}\right) E e^{+ik_1a} e^{+ik_2a}$$

$$B = \left(\frac{(k_1^2 - k_2^2)}{4 k_1 k_2}\right) E e^{+ik_1a} e^{-ik_2a} - \left(\frac{(k_1^2 - k_2^2)}{4 k_1 k_2}\right) E e^{+ik_1a} e^{+ik_2a}$$

$$B = \frac{E e^{+ik_1a}}{4 k_1 k_2} (k_2^2 - k_1^2)\left(e^{+ik_2a} - e^{-ik_2a}\right)$$

$$B = \frac{E e^{+ik_1a}}{4 k_1 k_2} (k_2^2 - k_1^2)\, 2i \sin k_2 a$$

$$B^* = \frac{E e^{-ik_1a}}{4 k_1 k_2} (k_2^2 - k_1^2)(-2i) \sin k_2 a$$

$$A = \frac{E\,e^{+ik_1a}}{4\,k_1k_2}\left((k_1 + k_2)^2\,e^{-ik_2a} - (k_1 - k_2)^2\,e^{+ik_2a}\right)$$

$$A^* = \frac{E^*\,e^{-ik_1a}}{4\,k_1k_2}\left((k_1 + k_2)^2\,e^{+ik_2a} - (k_1 - k_2)^2\,e^{-ik_2a}\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left((k_1 + k_2)^4 + (k_1 - k_2)^4 - (k_1 + k_2)^2(k_1 - k_2)^2e^{-2ik_2a} - (k_1 + k_2)^2(k_1 - k_2)^2e^{+2ik_2a}\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left((k_1 + k_2)^4 + (k_1 - k_2)^4 - (k_1 + k_2)^2(k_1 - k_2)^2(e^{-2ik_2a} + e^{+2ik_2a})\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left((k_1 + k_2)^4 + (k_1 - k_2)^4 - 2(k_1 + k_2)^2(k_1 - k_2)^2\cos 2k_2a\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left((k_1 + k_2)^4 + (k_1 - k_2)^4 - 2(k_1 + k_2)^2(k_1 - k_2)^2(1 - 2\sin^2 k_2a)\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left((k_1 + k_2)^4 + (k_1 - k_2)^4 - 2(k_1 + k_2)^2(k_1 - k_2)^2 + 4(k_1 + k_2)^2(k_1 - k_2)^2\sin^2 k_2a\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left(\{(k_1 + k_2)^2 - (k_1 - k_2)^2\}^2 + 4(k_1 + k_2)^2(k_1 - k_2)^2\sin^2 k_2a\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left(\{(k_1)^2 + (k_2)^2 + 2\,k_1k_2) - (k_1)^2 - (k_2)^2 + 2\,k_1k_2)\}^2 + 4(k_1 + k_2)^2(k_1 - k_2)^2\sin^2 k_2a\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left(16\,(k_1)^2(k_2)^2 + 4(k_1 + k_2)^2(k_1 - k_2)^2\sin^2 k_2a\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1k_2)^2}\left(16\,(k_1)^2(k_2)^2 + 4(k_1^2 - k_2^2)^2\sin^2 k_2a\right)$$

$$AA^* = \frac{EE^*}{(4\,k_1 k_2)^2}\left(16\,(k_1)^2(k_2)^2 + 4(k_1^2 - k_2^2)^2 \sin^2 k_2 a)\right)$$

$$EE^* = \frac{16\,(k_1)^2(k_2)^2 AA^*}{\left(16\,(k_1)^2(k_2)^2 + 4(k_1^2 - k_2^2)^2 \sin^2 k_2 a)\right)}$$

$$EE^* = \frac{AA^*}{\left(1 + \frac{(k_1^2 - k_2^2)^2}{16 k_1^2 k_2^2} 4 \sin^2 k_2 a)\right)}$$

$$B = \frac{E\,e^{+ik_1 a}}{4\,k_1 k_2}(k_2^2 - k_1^2)\ 2i \sin k_2 a$$

$$B^* = \frac{E\,e^{-ik_1 a}}{4\,k_1 k_2}(k_2^2 - k_1^2)(-2i)\sin k_2 a$$

$$BB^* = \frac{EE^*}{16 k_1^2 k_2^2}(k_2^2 - k_1^2)^2\, 4 \sin^2 k_2 a$$

$$AA^* = \frac{EE^*}{(4\,k_1 k_2)^2}\left(16\,(k_1)^2(k_2)^2 + 4(k_1^2 - k_2^2)^2 \sin^2 k_2 a\right)$$

$$BB^* = \frac{EE^*}{16 k_1^2 k_2^2}(k_2^2 - k_1^2)^2\, 4 \sin^2 k_2 a$$

$$k_1^2 = \frac{2m}{\hbar^2} E$$

$$EE^* = \frac{AA^*}{\left(1 + \frac{(k_1^2 - k_2^2)^2}{16 k_1^2 k_2^2} 4 \sin^2 k_2 a\right)}$$

$$k_2^2 = \frac{2m}{\hbar^2}(E - V_o)$$

$$\psi_{\text{Incident}} = Ae^{+ik_1 x} \Rightarrow J_{incident} = \frac{i\hbar}{2m}\left(\psi_i(x)\frac{d\psi_i^*(x)}{dx} - \psi_i^*(x)\frac{d\psi_i(x)}{dx}\right) = \frac{\hbar k_1}{m}|A|^2$$

$$\psi_{\text{reflected}} = Be^{-ik_1 x} \Rightarrow J_{reflected} = \frac{\hbar k_1}{m}|B|^2 \Rightarrow R = \frac{J_{reflected}}{J_{incident}} \Rightarrow R = \frac{|B|^2}{|A|^2}$$

$$\psi_{\text{transmitted}} = Ee^{+ikx} \Rightarrow J_{transmitted} = \frac{\hbar k_1}{m}|E|^2 \Rightarrow T = \frac{J_{transmitted}}{J_{incident}} \Rightarrow T = \frac{|E|^2}{|A|^2}$$

$$\frac{4(k_1^2 - k_2^2)^2}{16 k_1^2 k_2^2} = \frac{\left(\frac{2m}{\hbar^2}E - \frac{2m}{\hbar^2}(E - V_o)\right)^2}{16\left(\frac{2m}{\hbar^2}E\right)\left(\frac{2m}{\hbar^2}(E - V_o)\right)} = \frac{V_o^{\,2}}{4E(E - V_o)}$$

$$AA^* = \frac{EE^*}{(4\,k_1 k_2)^2}\left(16\,(k_1)^2(k_2)^2 + 4(k_1^2 - k_2^2)^2 \sin^2 k_2 a\right)$$

$$BB^* = \frac{EE^*}{16 k_1^2 k_2^2}(k_2^2 - k_1^2)^2\, 4 \sin^2 k_2 a$$

$$EE^* = \frac{AA^*}{\left(1 + \frac{(k_1^2 - k_2^2)^2}{16 k_1^2 k_2^2} 4 \sin^2 k_2 a\right)}$$

$$\frac{4(k_1^2 - k_2^2)^2}{16 k_1^2 k_2^2} = \frac{V_o^{\,2}}{4E(E - V_o)}$$

$$T = \frac{EE^*}{AA^*} = \frac{1}{\left(1 + \frac{(k_1^2 - k_2^2)^2}{16 k_1^2 k_2^2} 4 \sin^2 k_2 a\right)} = \frac{1}{\left(1 + \frac{V_o^{\,2}}{4E(E - V_o)} \sin^2 k_2 a\right)}$$

$$R = \frac{BB^*}{AA^*} = \frac{\frac{EE^*}{16 k_1^2 k_2^2}(k_2^2 - k_1^2)^2\, 4 \sin^2 k_2 a}{\frac{EE^*}{(4\,k_1 k_2)^2}\left(16\,(k_1)^2(k_2)^2 + 4(k_1^2 - k_2^2)^2 \sin^2 k_2 a\right)} = \frac{1}{1 + \frac{16\,(k_1)^2(k_2)^2 +}{(4(k_1^2 - k_2^2)^2 \sin^2 k_2 a))}}$$

$$T = \frac{EE^*}{AA^*} = \frac{1}{\left(1 + \frac{V_o^2}{4\,E\,(E - V_o)}\sin^2 k_2 a\right)}$$

$$R = \frac{BB^*}{AA^*} = \frac{\frac{EE^*}{16 k_1^2 k_2^2}(k_2^2 - k_1^2)^2\, 4\, \sin^2 k_2 a}{\frac{EE^*}{(4\, k_1 k_2)^2}\left(16\,(k_1)^2 (k_2)^2 + 4(k_1^2 - k_2^2)^2 \sin^2 k_2 a\right)} = \frac{1}{1 + \frac{16\,(k_1)^2 (k_2)^2 +}{(4(k_1^2 - k_2^2)^2 \sin^2 k_2 a)}}$$

$$R = \frac{BB^*}{AA^*} = \frac{1}{1 + \frac{16\,(k_1)^2 (k_2)^2 +}{\left(4(k_1^2 - k_2^2)^2 \sin^2 k_2 a\right)}} = \frac{1}{\left(1 + \frac{4E(E - V_o)}{V_o^2 \sin^2 k_2 a}\right)}$$

$$R = \frac{BB^*}{AA^*} = \frac{1}{\left(1 + \frac{4\,E\,(E - V_o)}{V_o^2 \sin^2 k_2 a}\right)}$$

$$\boxed{R + T = \frac{1}{\left(1 + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a}\right)} + \frac{1}{\left(1 + \frac{V_o^2}{4\,E\,(E - V_o)}\,\sin^2 k_2 a\right)}}$$

$$R + T = \frac{\left(1 + \frac{V_o^2 \sin^2 k_2 a}{4\,E\,(E - V_o)}\right) + \left(1 + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a}\right)}{\left(1 + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a}\right)\left(1 + \frac{V_o^2}{4\,E\,(E - V_o)}\,\sin^2 k_2 a\right)}$$

$$R + T = \frac{1 + \frac{V_o^2 \sin^2 k_2 a}{4\,E\,(E - V_o)} + 1 + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a}}{1 + \frac{V_o^2 \sin^2 k_2 a}{4\,E\,(E - V_o)} + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a} + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a}\,\frac{V_o^2 \sin^2 k_2 a}{4\,E\,(E - V_o)}}$$

$$R + T = 1 = \frac{2 + \frac{V_o^2 \sin^2 k_2 a}{4\,E\,(E - V_o)} + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a}}{2 + \frac{V_o^2 \sin^2 k_2 a}{4\,E\,(E - V_o)} + \frac{4\,E\,(E - V_o)}{V_o^2\,\sin^2 k_2 a}}$$

$$\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}(E - V)\,\psi(x) = 0$$

$$\psi(x) = \begin{cases} \psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}, & x \le 0, \\ \psi_2(x) = Ce^{ik_2x} + De^{-ik_2x}, & 0 < x < a, \\ \psi_3(x) = Ee^{ik_1x}, & x \ge a, \end{cases}$$

$$k_1 = \sqrt{2mE/\hbar^2} \text{ and } k_2 = \sqrt{2m(E - V_0)/\hbar^2}$$

$$\psi_1(0) = \psi_2(0), \qquad \frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx}$$

$$\psi_2(a) = \psi_3(a), \qquad \frac{d\psi_2(a)}{dx} = \frac{d\psi_3(a)}{dx}$$

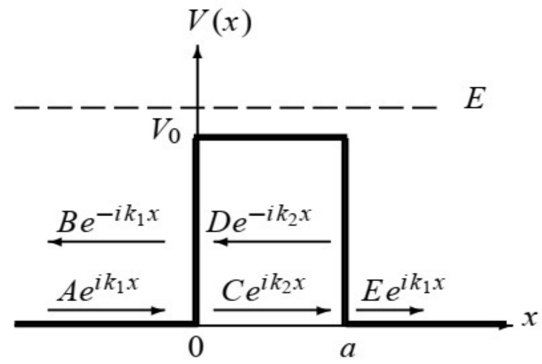
$$A + B = C + D, \qquad ik_1(A - B) = ik_2(C - D),$$

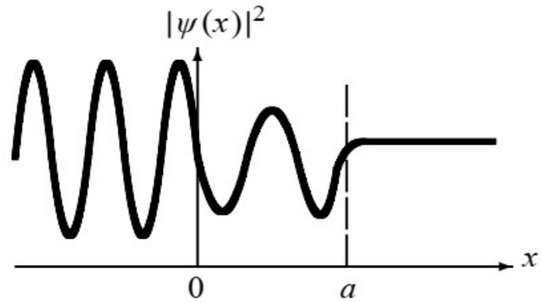
$$Ce^{ik_2a} + De^{-ik_2a} = Ee^{ik_1a}, \qquad ik_2\left(Ce^{ik_2a} - De^{-ik_2a}\right) = ik_1Ee^{ik_1a}$$

$$\boxed{T = \frac{k_1EE^*}{k_1AA^*} = \frac{1}{\left(1 + \dfrac{V_o^2 \sin^2 k_2a}{4E(E - V_o)}\right)}}$$

Using the notation $\lambda = a\sqrt{\frac{2mV_0}{\hbar^2}}$ *and* $\varepsilon = \frac{E}{V_0}$, rewrite T as

$$T = \frac{1}{\left(1 + \dfrac{\sin^2 \lambda\sqrt{\varepsilon - 1}}{4\varepsilon(\varepsilon - 1)}\right)}$$





$E > V_0$

Potential barrier and propagation directions of the incident, reflected, and transmitted waves, plus their probability densities $E > V_0$

$$T = \left(1 + \frac{\sin^2 \lambda\sqrt{\varepsilon - 1}}{4\varepsilon(\varepsilon - 1)}\right)^{-1}$$

$$\boxed{R = \frac{BB^*}{AA^*} = \frac{1}{\left(1 + \dfrac{4E(E - V_o)}{V_o^2 \sin^2 k_2a}\right)}}$$

$$\boxed{R = \left(1 + \frac{4\varepsilon(\varepsilon - 1)}{\sin^2 \lambda\sqrt{\varepsilon - 1}}\right)^{-1}}$$

Special cases

1. If $E >> V_0$ or $\varepsilon >> 1$, the transmission coefficient T becomes asymptotically equal to unity, $T \sim 1$, and $R \sim 0$ as $E \to \infty$. So, at very high energies and weak potential barrier, the particles would not feel the effect of the barrier; we have total transmission. However kinetic energy of the particle reduced as long as particles are in region II.

$$\boldsymbol{T} = \left(\mathbf{1} + \frac{\boldsymbol{sin^2} \lambda \sqrt{\varepsilon - 1}}{4\, \varepsilon\, (\varepsilon - 1)} \right)^{-\mathbf{1}} \longrightarrow 1$$

Special cases POTENTIAL BARRIER $E > V_o$

2. We also have total transmission when $\sin k_2 a = 0$ or $k_2 a = 0$ or $\lambda\sqrt{\varepsilon - 1} = 0$. As shown in Figure, the total transmission, $T(\mathcal{E}_n) = 1$, occurs whenever the incident energy of the particle is

$$\mathcal{E}_n = (E_n/V_0) = n^2\pi^2\hbar^2/(2ma^2V_0) + 1 \quad \text{or}$$

$$E_n = V_0 + n^2\pi^2\hbar^2/(2ma^2)$$

with n 1, 2, 3,…

- The variation of T w.r.t $k_2 a/\pi$ is shown in the Figure. The coefficient of transmission goes through alternate maxima and minima. The maxima always corresponds to $T \to 1, \frac{k_2 a}{\pi} \to \infty$. When T is maximum, R is minimum and vice versa. However, R_{max} will not be 1. It will always less than 1.
- The maxima of the transmission coefficient coincide with the energy eigenvalues of the infinite square well potential; these are known as resonances.
- This resonance phenomenon, which does not occur in classical physics, results from a constructive interference between the incident and the reflected waves.
- This phenomenon is observed experimentally in a number of cases such as when scattering low-energy (E ~0.1 eV) electrons off noble atoms (known as the *Ramsauer–Townsend effect*, a consequence of symmetry of noble atoms) and neutrons off nuclei.

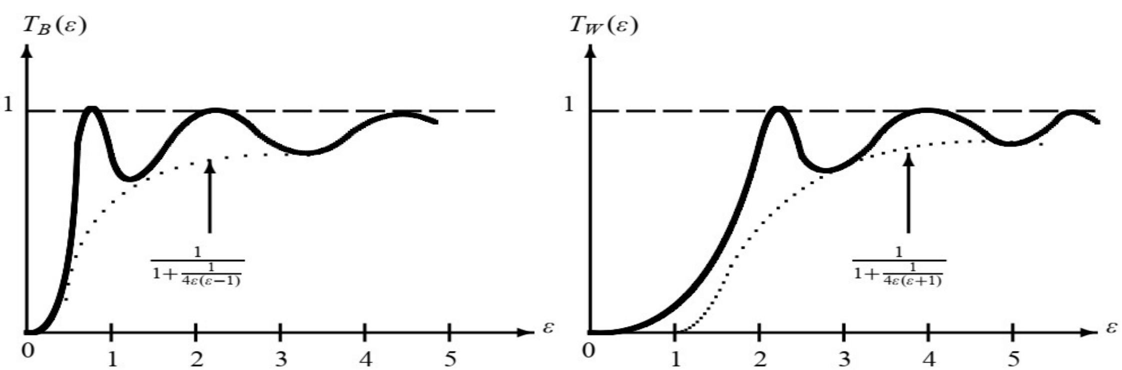


Figure 4.4 Transmission coefficients for a potential barrier, $T_B(\varepsilon) = \frac{4\varepsilon(\varepsilon-1)}{4\varepsilon(\varepsilon-1)+\sin^2(\lambda\sqrt{\varepsilon-1})}$, and for a potential well, $T_W(\varepsilon) = \frac{4\varepsilon(\varepsilon+1)}{4\varepsilon(\varepsilon+1)+\sin^2(\lambda\sqrt{\varepsilon+1})}$.

$$T = \frac{1}{\left(1 + \frac{V_o^2 \sin^2 k_2 a}{4E(E - V_o)}\right)}$$

$$T = \left(1 + \frac{\sin^2\lambda\sqrt{\varepsilon - 1}}{4\varepsilon(\varepsilon - 1)}\right)^{-1}$$

$$\boxed{R = \left(1 + \frac{4\varepsilon(\varepsilon - 1)}{\sin^2\lambda\sqrt{\varepsilon - 1}}\right)^{-1}}$$

Special cases

3. In the limit $\varepsilon \to 1$ we have $\sin(\lambda\sqrt{\varepsilon - 1}) \sim \lambda\sqrt{\varepsilon - 1}$, hence (4.44) and (4.45) become

$$T = \left(1 + \frac{ma^2 V_0}{2\hbar^2}\right)^{-1}, \qquad R = \left(1 + \frac{2\hbar^2}{ma^2 V_0}\right)^{-1}. \tag{4.46}$$

POTENTIAL BARRIER $E < V_o$

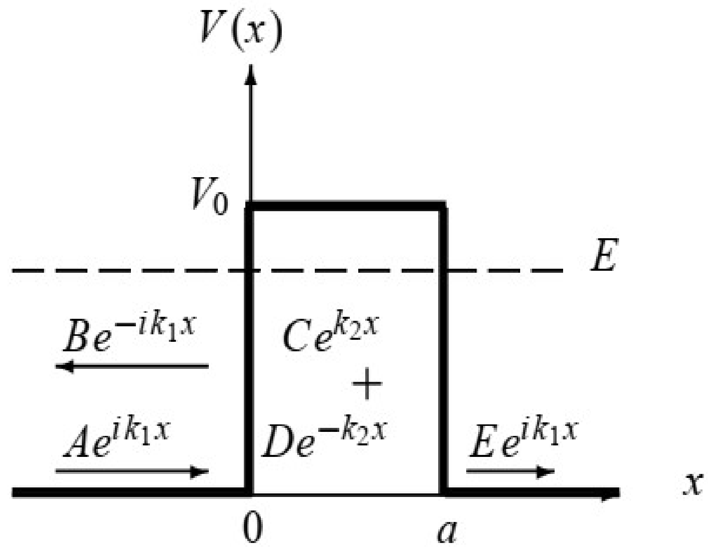
$$\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}(E - V)\,\psi(x) = 0$$

$$\psi_{I}(x) = Ae^{+ik_1x} + Be^{-ik_1x} \quad for\ x \le 0$$

$$\psi_{II}(x) = Ce^{+k_2'x} + De^{-k'_2x} \quad for\ 0 < x < a$$

$$\psi_{III}(x) = Ee^{+ik_1x} \quad for\ x \ge a$$

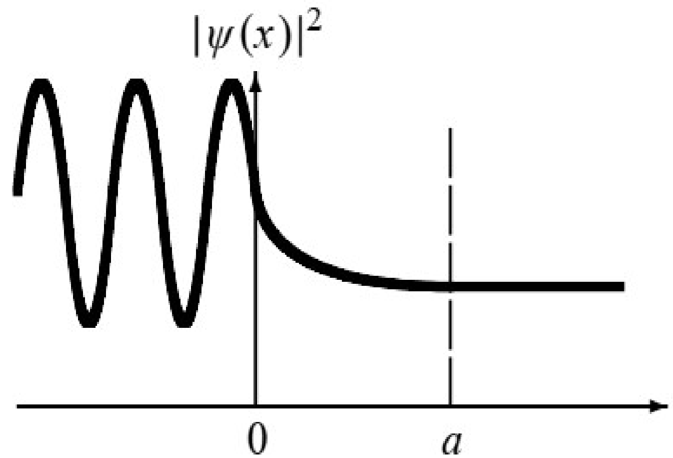
$$\boxed{k_1^2 = \frac{2m}{\hbar^2}E} \qquad \boxed{k_2^2 = \frac{2m}{\hbar^2}(E - V_o)}$$



For, $E<V_o$

$$\boxed{Let,\ k'^2_2 = \frac{2m}{\hbar^2}(V_o - E)}$$

$$\boxed{k'^2_2 = -k_2^2 = i^2k_2^2}$$



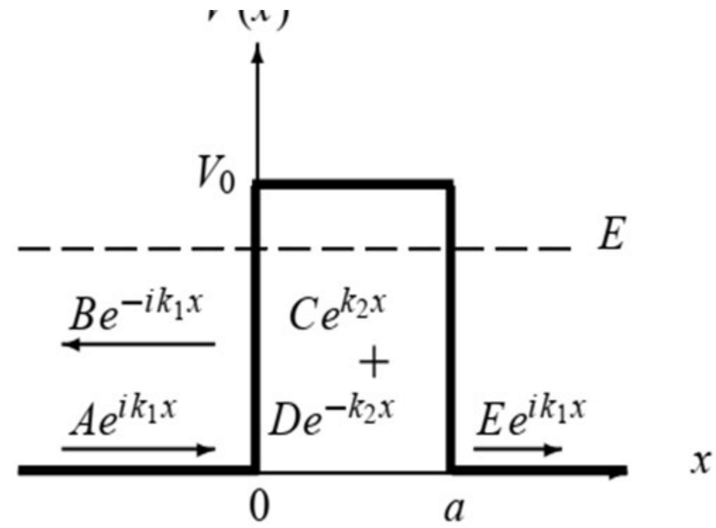
$E < V_0$

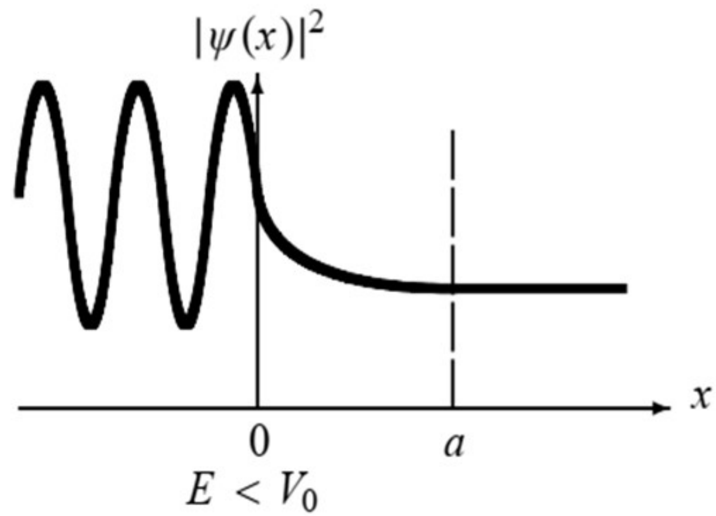
$$T = \frac{1}{\left(1 + \frac{V_o^2 \sin^2 ik'_2a}{4E(V_o - E)}\right)} = \frac{1}{\left(1 + \frac{V_o^2 \sinh^2 k'_2a}{4E(V_o - E)}\right)}$$

$$R = \frac{1}{\left(1 + \frac{4E(V_o - E)}{V_o^2 \sin^2 ik'_2a}\right)} = \frac{1}{\left(1 + \frac{4E(V_o - E)}{V_o^2 \sinh^2 k'_2a}\right)}$$

Potential barrier and propagation directions of the incident, reflected, and transmitted waves, plus their probability densities $E < V_0$

- Note that T is *finite* for $E<V_0$. This means that the probability for the transmission of the particles into the region $x \geq a$ is *not zero* (in classical physics, however, the particle can in no way make it into the $x \geq 0$ region).
- This is a purely quantum mechanical effect which is due to the *wave aspect* of microscopic objects; it is known as the *tunneling effect*: *quantum mechanical objects can tunnel through classically impenetrable barriers*.
- This *barrier penetration* effect has important applications in various branches of modern physics ranging from particle and nuclear physics to semiconductor devices. For instance, radioactive decays and charge transport in electronic devices are typical examples of the tunneling effect.





$$T = \left(1 + \frac{V_o^2 \sinh^2 k'_2 a}{4E(V_o - E)}\right)^{-1}$$

$$R = \left(1 + \frac{4E(V_o - E)}{V_o^2 \sinh^2 k'_2 a}\right)^{-1}$$

Exercise 1.

Two copper wires of uniform cross sectional area are separated by an oxide layer (CuO). Treat oxide layer as square barrier of height 10 eV and estimate the transmission co-efficient for penetration by 7eV electrons if the layer thickness is

(1) 5 nm and

(2) 1 nm.

(3) What will be the parameter λ? Determine for both the cases.

$$\boxed{Let, \boldsymbol{k'}_2^2 = \frac{2m}{\hbar^2}(V_o - E)}$$

$$\boldsymbol{T} = \left(1 + \frac{V_o^2 \, \boldsymbol{\sin h\ 2\ k'_2 a}}{4\ E\ (V_o\ - E)}\right)^{-1}$$

Exercise 2.

Determine the reflection coefficient for monoenergetic beam of electrons with energy 0.04eV incident on a potential barrier of height 0.08 eV of length 10 angstrom.

$$\boxed{\boldsymbol{k'}_2^2 = \frac{2m}{\hbar^2}(V_o - E)}$$

$$\boldsymbol{R} = \frac{\mathbf{1}}{\left(\mathbf{1} + \frac{4\,E\,(V_o\,-E)}{{V_o}^2\,\boldsymbol{sin\,h\,2}_{\boldsymbol{k'_2 a}}}\right)}$$

Exercise 3.

Determine the transmission coefficient for 1 MeV proton through a potential barrier of 4 MeV and length 0.01 angstrom.