Permutations & Combinations

First Year Computer Engineering

Discrete Structures

Permutations & Combinations

Permutations : set of distinct objects / ordered arrangement of objects.

S= {1,2,3}   then (3,2,1) is a 3-permutation of S

(2,3) is a 2-permutation of S.

Combinations: r-combinations of elements is a set of unordered selection of r elements form the set.

R-combination is a subset of r elements in the Set S.

C(4,2) = 2 combination of 4 elements {a,b,c,d}

Questions

Q1. How many 3 letter with or without meaning can be formed from **'LOGARITHMS'** if repetition is not allowed.

Ans 1] Types of 3 letter words : LOG, OGA, ART, ARH

so the number of permutation is

$^{n}P_{r} = {}^{10}P_{3}$

Q2. In a cricket championship there are **21 matches** if each team plays one match with every other team. What is the total number of teams.

Ans 2]
21 matches played
Each team played with other team
Hence number of teams in ***n.***

- ***$^{n}C_2 = 21$***

Q3. A question paper consists of 10 questions divided into part A and part B. Each part has 5 questions . A candidate has to answer 6 questions in all atleast 2 should be from part A and 2 should be from part B. How many ways can a student select the questions if he can answer all equally well.

- Possibilities:

A	B	Total
2	4	6
3	3	6
4	2	6

- Number of ways to choose 2 from part A and 4 from part B= $\binom{5}{2}\times\binom{5}{4}=50$.
- Number of ways to choose 3 from part A and 3 from part B= $\binom{5}{3}\times\binom{5}{3}=100$.
- Number of ways to choose 4 from part A and 2 from part B= $\binom{5}{2}\times\binom{5}{4}=50$.
- So the total number of ways = 50 + 100 + 50 = 200.

Q4.How many permutations of the letters **ABCDEFGH** contain the string **ABC**.

A4] ABC must occur as a block , so find the number of permutations of 6 objects. 6!=720

Q5. Find the number of permutations of the word **'CLIMATE'** such that the vowels should always occur in odd places.

Ans 5]

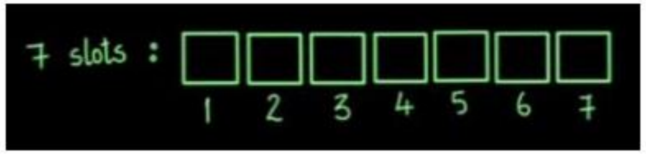
The word here is CLIMATE and it has no repetitions. But the condition given is the vowels should always occur in odd places.

the word climate has:

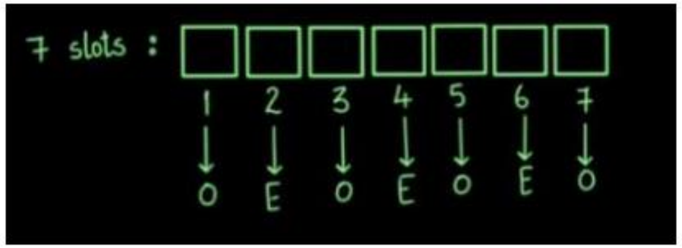
4 consonants: C, L, M, T

3 vowels: I, A, E

It has the total 7 letters. So have seven slots.



There are 4 odd slots and 3 even slots.



3 vowels should occur in the odd places.

odd places → 4

vowels → 3.

$$^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1} = 4 \times 3 \times 2 = 24.$$

So in 24 ways the 3 vowels can occur in 4 odd slots.

There are only 3 vowels here and 4 consonants. So 4 places remaining after the 3 vowels take 3 places. So the 4 consonants can take rest of the 4 places in $^4P_4$ ways.

$$^4P_4 = \frac{4!}{0!} = \frac{4!}{1} = 4! = 24.$$

So in 24 ways the 4 consonants can take 4 slots.

So by the rule of product,

the total number of permutations = $24 \times 24 = 576$

Q6. Suppose a saleswomen has to visit **8 different cities.** She must begin her trip in a specified city, but she can visit other cities in any order that she wishes. How many possible orders can he saleswomen use when visiting the cities.

A6] Number of possible paths between the cities is the permutation of 7 elements, because the first city is determined the remaining 7 cities can be ordered arbitrarily. Hence 7! = 5040 paths

Difference between permutation and combinations

Permutations

- Arrangement
- Order is important

Combinations

- Selection
- Order is not important

Binomial Coefficients

- The number of ***r-combinations*** for a set of ***n*** elements is also called

$$\binom{n}{r}$$

These numbers occur as coefficients in the expansion of powers of binomal expression.

$$(a+b)^n$$

Binomial Theorem

BINOMIAL THEOREM:

Let a and b be variables and n be a non negative integer. Then

$$\begin{aligned}(a+b)^n &= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n\end{aligned}$$

Binomial Theorem

$$(a+b)^3 = (a+b)\,(a+b)\,(a+b)$$

How many possible ways to pick an element per (a+b) to obtain a term on the right-hand side.

$\mathbf{a^3}$ $\mathbf{ab^2}$ $\mathbf{a^2b}$ $\mathbf{b^3}$

Binomial Theorem

$$(a+b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$$

a^2b = choose 1 b from 3 sets of $(a+b)$

ab^2 =choose 2 b from 3 sets of $(a+b)$

Binomial Theorem

Property 1: if x = 1 and y = x in the expression $(x+y)^n$ **(Next Slide)**

Property 2: if x = 1 and y = 1 in the expression $(x+y)^n$ i.e. $(1+1)^n = 2^n$

Property 3: r^{th} term in the Binomial Coefficient **(Next Slide)**

Property 4: n is odd there are 2 middle terms $\left(\frac{n+1}{2}\right)^{th}$ & $\left(\frac{n+1}{2}^{th} + 1\right)$

If n is even then 1 middle term $\left(\frac{n+2}{2}\right)^{nd}$ **(Next Slide)**

Binomial Theorem Property 1:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \,.$$

replace x= 1 and y = x. So

$$(1+x)^n$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$$

So whenever we have $(1+x)^2$ or $(1+x)^3$ can directly write the terms as

$$\binom{n}{k} x^k$$

Binomial Theorem Property 3: r^th term in the Binomial Coefficient

$$\text{expand}\,(x+y)^n$$

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}x^0 y^n$$

$$= x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + y^n$$

The r^{th} term

$$r^{th}\ \ term = \binom{n}{r-1}x^{n-r+1}y^{r-1}$$

Binomial Theorem Property 4:

n is even. Then there is only one middle term which is the $\frac{n+2}{2}^{th}$ term.

n is odd. Then you will have two middle terms.

example $(x+y)^3$.

$$(x+y)^3 = x + 3x^2y + 3xy^2 + y^3$$

have two middle terms $3x^2y$ and $3xy^2$

So consider both of them therefore if n is odd

$\frac{n+1}{2}^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$

2 middle terms as middle terms

Problems

Q1. What is the expansion of $(x+y)^4$

Q2. What is the coefficient of $x^{12}y^{13}$ in the expression $(2x-3y)^{25}$

Problems

Q3] Expand
$(a+b)^6$

Q4] Expand
$(1.04)^4$

Q5] Find the middle term in $(3x-4)^6$

Pascal's Identity and Triangle

- This is the basis for a geometric arrangement of binomial coefficients in a triangle
- The nth row in the triangle consists of the binomial coefficients of

$$\binom{n}{r} \quad r = 0, 1, \ldots. n.$$

Pascal's Triangle

Pascal's identity together with initial conditions (n 0) = (n n) = 1 for all integers n can be used to recursively define binomial coefficients.

Pascals Identity is the basis of geometric arrangement of the binomial coefficients in a triangle .

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}
\end{array}
$$

Fun Facts of the Pascal's triangle

The second diagonal highlighted in the figure in red, goes on as 1, 2, 3, 4, 5, 6 and so on and this diagonal represents the natural numbers.

1
1 **1**
1 **2** 1
1 **3** 3 1
1 **4** 6 4 1
1 **5** 10 10 5 1
1 **6** 15 20 15 6 1
1 **7** 21 35 35 21 7 1

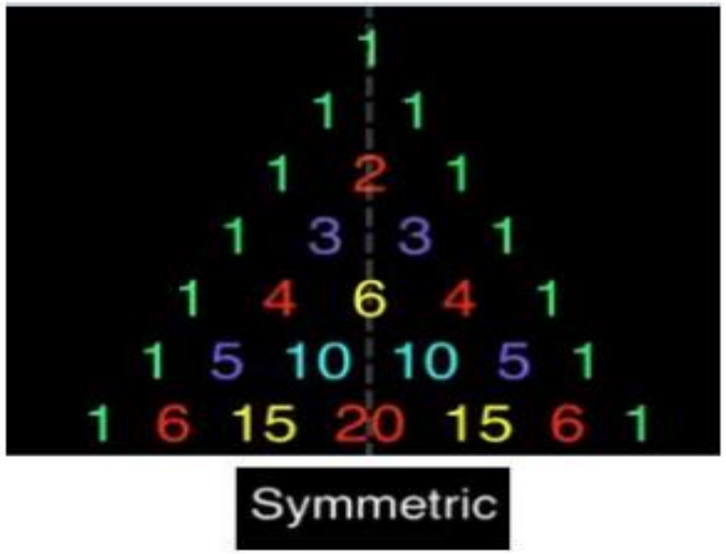
The second diagonal representing natural numbers

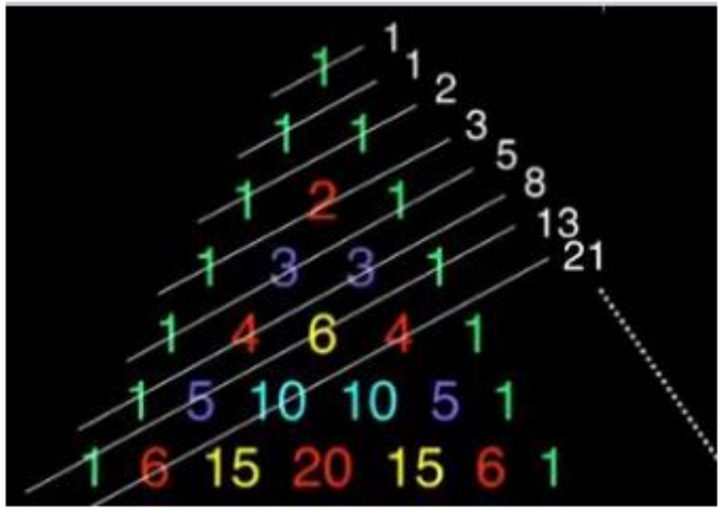
Pascal's Triangle	Sum	2^n
1	1	2^0
1 1	1+1=2	2^1
1 2 1	1+2+1=4	2^2
1 3 3 1	1+3+3+1=8	2^3
1 4 6 4 1	1+4+6+4+1=16	2^4
1 5 10 10 5 1	1+5+10+10+ 5+1=32	2^5

Pascal's Triangle

If we take each row as a number, it represents a power of 11.

Pascal's Triangle	Number	11^n
1	1	11^0
1 1	11	11^1
1 2 1	121	11^2
1 3 3 1	1331	11^3
1 4 6 4 1	14641	11^4
1 5 10 10 5 1	151010 5132	11^5





The Fibonacci sequence is seen like this: 1, 1, 2, 3, 5, 8, 13, 21, ….

Generalized Permutations and Combinations

- In many counting problems elements will be repeated.
- Eg. Numbers in a license plate
- So we consider permutations and combination where items are used more than once.
- i.e counting involves indistinguishable objects (contrast to all problems that was distinguishable)
- Eg.

  Letters on the work SUCCESS

Permutations with repetitions

- **Example 1:** How many strings of length r can be formed from the English alphabet?

Solution:

By product rule, as there are 26 letters & each letter can be repeated – there are 26^r strings of length r.

Combinations with repetitions

Example 1: How may ways can 100 be written as a sum of 4-non negative integers.

Solution: A+B+C+D = 100.

We can have: 30+30+20+20 = 100

So, $100 + 0 + 0 + 0 = 100$

$0 + 100 + 0 + 0 = 100$

$0 + 0 + 100 + 0 = 100$

$0 + 0 + 0 + 100 = 100$

So these are 4 different possibilities.

n=100 r=4

$${}^{n+r-1}C_r = {}^{100+4-1}C_4$$