1.1. THE STERN-GERLACH EXPERIMENT

The example we concentrate on in this section is the Stern-Gerlach experiment, originally conceived by O. Stern in 1921 and carried out in Frankfurt by him in collaboration with W. Gerlach in 1922. This experiment illustrates in a dramatic manner the necessity for a radical departure from the concepts of classical mechanics. In the subsequent sections the basic formalism of quantum mechanics is presented in a somewhat axiomatic manner but always with the example of the Stern-Gerlach experiment in the back of our minds. In a certain sense, a two-state system of the Stern-Gerlach type is the least classical, most quantum-mechanical system. A solid understanding of problems involving two-state systems will turn out to be rewarding to any serious student of quantum mechanics. It is for this reason that we refer repeatedly to two-state problems throughout this book.

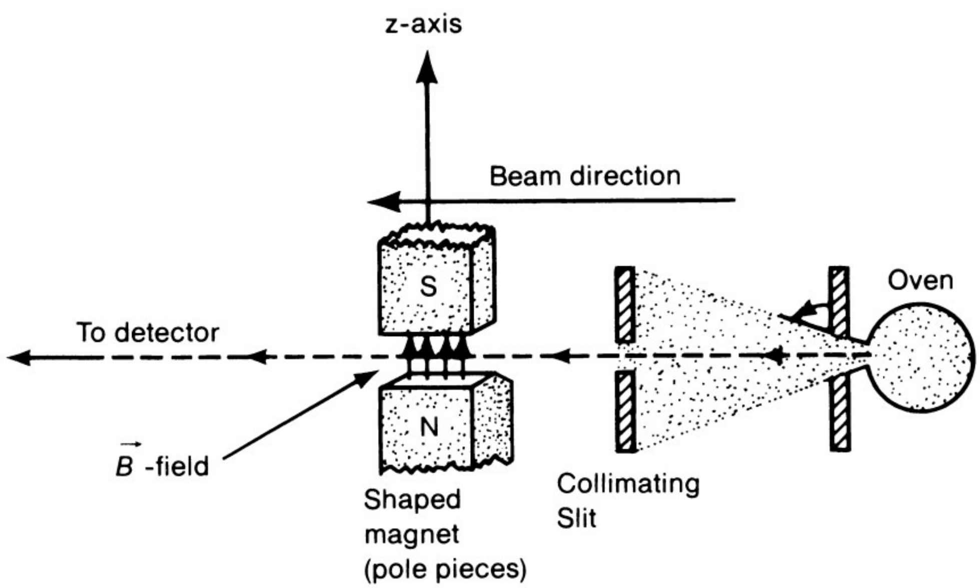


FIGURE 1.1. The Stern-Gerlach experiment.

proportional to the electron spin $\mathbf{S}$,

$$\boldsymbol{\mu} \propto \mathbf{S}, \tag{1.1.1}$$

where the precise proportionality factor turns out to be $e/m_e c$ ($e < 0$ in this book) to an accuracy of about 0.2%.

Because the interaction energy of the magnetic moment with the magnetic field is just $-\boldsymbol{\mu}\cdot\mathbf{B}$, the z-component of the force experienced by the atom is given by

$$F_z = \frac{\partial}{\partial z}(\boldsymbol{\mu}\cdot\mathbf{B}) \simeq \mu_z \frac{\partial B_z}{\partial z}, \tag{1.1.2}$$

where we have ignored the components of **B** in directions other than the z-direction. Because the atom as a whole is very heavy, we expect that the classical concept of trajectory can be legitimately applied, a point which can be justified using the Heisenberg uncertainty principle to be derived later. With the arrangement of Figure 1.1, the $\mu_z > 0$ ($S_z < 0$) atom experiences a downward force, while the $\mu_z < 0$ ($S_z > 0$) atom experiences an upward force. The beam is then expected to get split according to the values of μ_z. In other words, the SG (Stern-Gerlach) apparatus "measures" the z-component of $\boldsymbol{\mu}$ or, equivalently, the z-component of **S** up to a proportionality factor.

The atoms in the oven are randomly oriented; there is no preferred direction for the orientation of $\boldsymbol{\mu}$. If the electron were like a classical spinning object, we would expect all values of μ_z to be realized between $|\boldsymbol{\mu}|$ and $-|\boldsymbol{\mu}|$. This would lead us to expect a continuous bundle of beams coming out of the SG apparatus, as shown in Figure 1.2a. Instead, what we

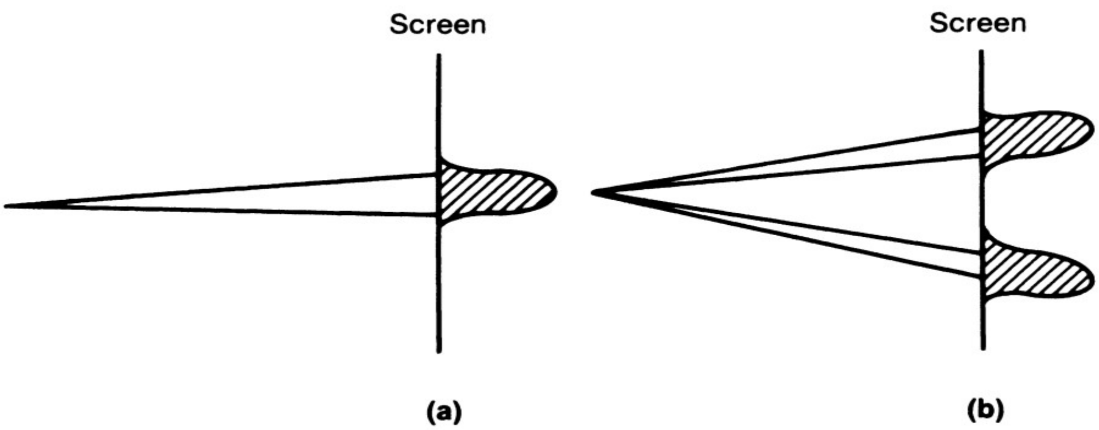


FIGURE 1.2. Beams from the SG apparatus; (a) is expected from classical physics, while (b) is actually observed.

experimentally observe is more like the situation in Figure 1.2b. In other words, the SG apparatus splits the original silver beam from the oven into *two distinct* components, a phenomenon referred to in the early days of quantum theory as "space quantization." To the extent that $\boldsymbol{\mu}$ can be identified within a proportionality factor with the electron spin **S**, only two possible values of the z-component of **S** are observed to be possible, S_z up and S_z down, which we call S_z+ and S_z-. The two possible values of S_z are multiples of some fundamental unit of angular momentum; numerically it turns out that $S_z = \hbar/2$ and $-\hbar/2$, where

$$\begin{aligned} \hbar &= 1.0546 \times 10^{-27} \text{ erg-s} \\ &= 6.5822 \times 10^{-16} \text{ eV-s} \end{aligned} \tag{1.1.3}$$

This "quantization" of the electron spin angular momentum is the first important feature we deduce from the Stern-Gerlach experiment.

Of course, there is nothing sacred about the up-down direction or the z-axis. We could just as well have applied an inhomogeneous field in a horizontal direction, say in the x-direction, with the beam proceeding in the y-direction. In this manner we could have separated the beam from the oven into an S_x+ component and an S_x- component.

Sequential Stern-Gerlach Experiments

Let us now consider a sequential Stern-Gerlach experiment. By this we mean that the atomic beam goes through two or more SG apparatuses in sequence. The first arrangement we consider is relatively straightforward. We subject the beam coming out of the oven to the arrangement shown in Figure 1.3a, where SG$\hat{\mathbf{z}}$ stands for an apparatus with the inhomogeneous magnetic field in the z-direction, as usual. We then block the S_z- component coming out of the first SG$\hat{\mathbf{z}}$ apparatus and let the remaining S_z+ component be subjected to another SG$\hat{\mathbf{z}}$ apparatus. This time there is only one beam component coming out of the second apparatus—just the S_z+ component. This is perhaps not so surprising; after all if the atom spins are up, they are expected to remain so, short of any external field that rotates the spins between the first and the second SG$\hat{\mathbf{z}}$ apparatuses.

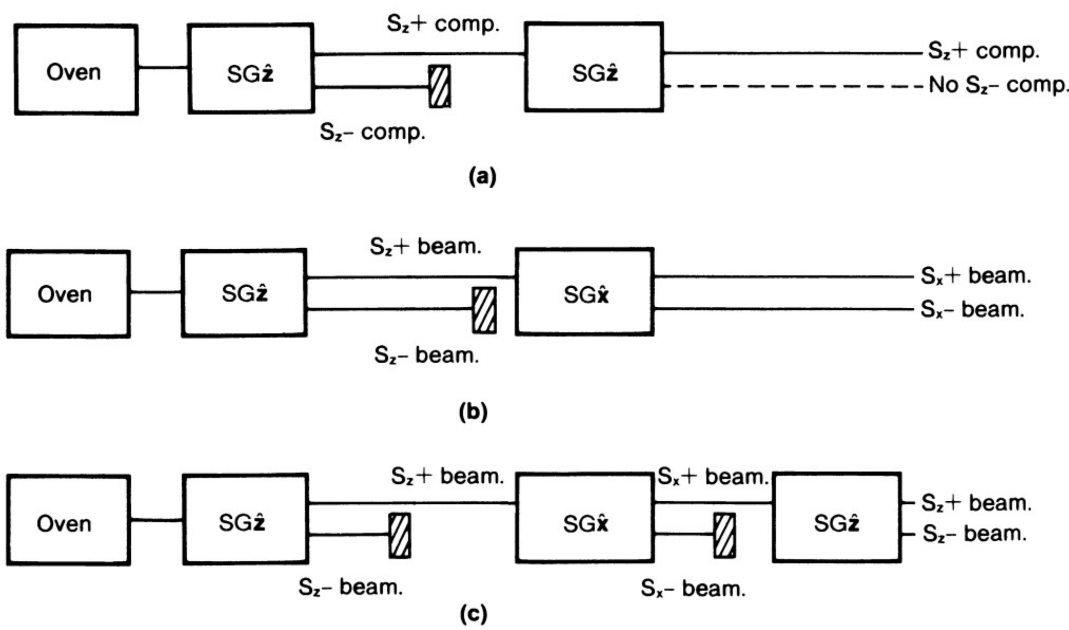


FIGURE 1.3. Sequential Stern-Gerlach experiments.

A little more interesting is the arrangement shown in Figure 1.3b. Here the first SG apparatus is the same as before but the second one (SG$\hat{\mathbf{x}}$) has an inhomogeneous magnetic field in the x-direction. The S_z+ beam that enters the second apparatus (SG$\hat{\mathbf{x}}$) is now split into two components, an S_x+ component and an S_x- component, with equal intensities. How can we explain this? Does it mean that 50% of the atoms in the S_z+ beam coming out of the first apparatus (SG$\hat{\mathbf{z}}$) are made up of atoms characterized by both S_z+ and S_x+, while the remaining 50% have both S_z+ and S_x-? It turns out that such a picture runs into difficulty, as will be shown below.

We now consider a third step, the arrangement shown in Figure 1.3(c), which most dramatically illustrates the peculiarities of quantum-mechanical systems. This time we add to the arrangement of Figure 1.3b yet a third apparatus, of the SG$\hat{\mathbf{z}}$ type. It is observed experimentally that *two* components emerge from the third apparatus, not one; the emerging beams are seen to have *both* an S_z+ component and an S_z- component. This is a complete surprise because after the atoms emerged from the first apparatus, we made sure that the S_z- component was completely blocked. How is it possible that the S_z- component which, we thought, we eliminated earlier reappears? The model in which the atoms entering the third apparatus are visualized to have both S_z+ and S_x+ is clearly unsatisfactory.

This example is often used to illustrate that in quantum mechanics we cannot determine both S_z and S_x simultaneously. More precisely, we can say that the selection of the S_x+ beam by the second apparatus (SG$\hat{\mathbf{x}}$) completely destroys any *previous* information about S_z.

Two Level- Spin ½ system

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$
$$[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x$$
$$[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

Eigen state : $|s, m_s\rangle = |s\rangle \otimes |m_s\rangle$

$s = 0, 1/2, 1, \ldots$

$m_s = -s, -s+1, \ldots, s-1, s$

$S^2|s, m_s\rangle = \hbar^2 s(s+1)|s, m_s\rangle$

$S_z|s, m_s\rangle = \hbar m_s|s, m_s\rangle$

$S_\pm|s, m_s\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s, m_s \pm 1\rangle$

$S_\pm = \hat{S}_x \pm i\hat{S}_y$

''

$s = \frac{1}{2}$

$m_s = -\frac{1}{2}, +\frac{1}{2}$

$|\frac{1}{2}\ +\frac{1}{2}\rangle \equiv |\uparrow\rangle \equiv |+\rangle \equiv |0\rangle$

$|\frac{1}{2}\ -\frac{1}{2}\rangle \equiv |\downarrow\rangle \equiv |-\rangle \equiv |1\rangle$

$\sum_a |a\rangle\langle a| = I$

spin $\frac{1}{2}$: $|+\rangle\langle +| + |-\rangle\langle -| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1)$

$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$|x\rangle = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} a \\ b \end{pmatrix}$

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle$$

$$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle.$$

Let $\hat{S}_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\hat{S}_z|+\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad a = \hbar/2$$
$$c = 0$$

$$\hat{S}_z|-\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\hbar/2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \quad c = 0$$
$$d = -\hbar/2$$

$$\frac{\hbar}{2}\hat{\sigma}_z = \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\frac{\hbar}{2}\hat{\sigma}_x = S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{\hbar}{2}\hat{\sigma}_y = S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$|\psi\rangle \rightarrow$ contains all information

$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle + \cdots$$

$$= \sum_i c_i|\phi_i\rangle$$

$$\sum_{i=1}^{n} |c_i|^2 = 1$$

$$|\psi\rangle = \{\underline{|\uparrow\rangle}, \underline{|\downarrow\rangle}\} \text{ or}$$

$$\{\underline{|+\rangle}, |-\rangle\} \text{ or } \{\underline{|0\rangle}, |1\rangle\}$$

$$|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$$

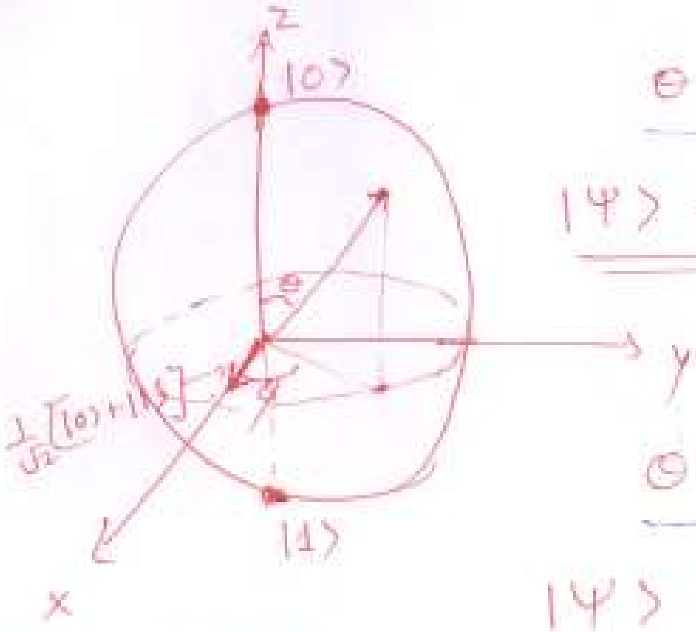
$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$|c_1|^2 + |c_2|^2 = 1$$

$$c_1 = \cos\theta/2 \quad \& \quad c_2 = \sin\theta/2 \, e^{i\phi}$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} \cdot e^{i\phi/2}|1\rangle$$



Bloch Sphere

$\theta = 0 \quad \phi = 0$

$|\psi\rangle = |0\rangle +$

$\theta = \pi \;;\; \phi = 0$

$|\psi\rangle = (+1)|1\rangle$

$\theta = \frac{\pi}{2}, \quad \phi = 0$

$$|\psi\rangle = \cos\frac{\pi}{4}|0\rangle + \sin\frac{\pi}{4}\, e^{i(0)}|1\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right]$$

→ Density Operator

$$\hat{\rho} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

Here

→ $|\psi_j\rangle$ is a state in the Hilbert space, $|\psi_j\rangle \in H$

→ $|\psi_j\rangle$ → appears with probability p_j

→ $\langle\psi_j|\psi_j\rangle = 1$

Verify $|\psi\rangle = \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix}$.

Is density operator.

$$\rho = |\psi\rangle\langle\psi|$$

$$= \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \end{pmatrix}$$

$$\rho = \begin{pmatrix} \cos^2\theta & e^{-i\phi}\cos\theta\sin\theta \\ e^{i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

- $Tr(\rho) = 1 = \sin^2\theta + \cos^2\theta.$
- $\rho = \rho^{\dagger}$
- $\rho^2 = \rho$

Verify the above three conditions. If density operator satisfies the above three conditions then it is a density operator.

(2) Is it mixed state or pure state?

(3) ~~$\langle \sigma_x \rangle = Tr(\rho \sigma_x)$~~

$$\rho = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{2}}{4} e^{-i\phi} \\ \frac{\sqrt{2}\, e^{i\phi}}{4} & 1/4 \end{bmatrix}$$

(1) Is it a density matrix?

Ans-1

(a) $Tr(\rho) = 1 = 3/4 + 1/4$

$\rho^2 = \rho$

$\rho^\dagger = \rho$

(b) $\rho = \rho^\dagger$

(3)
$$\rho^2 = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{2}}{4} e^{-i\phi} \\ \frac{\sqrt{2}}{4} e^{i\phi} & 1/4 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{2}}{4} e^{-i\phi} \\ \frac{\sqrt{2}}{4} e^{i\phi} & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{16} & \frac{\sqrt{2}}{4} e^{-i\phi} \\ \frac{\sqrt{2}}{4} e^{i\phi} & \frac{3}{16} \end{pmatrix} \neq \rho$$

(b) $Tr(\rho^2) = \frac{14}{16} < 1$ maximally mixed state

$$\begin{vmatrix} \frac{3}{4}-\lambda & \frac{\sqrt{2}}{4}e^{-i\phi} \\ \frac{\sqrt{2}}{4}e^{i\phi} & \frac{1}{4}-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = \frac{2+\sqrt{3}}{4}$$

$$\lambda_2 = \frac{2-\sqrt{3}}{4}$$

d) $\underline{\underline{\langle \sigma_x \rangle}} = Tr(\rho\sigma_x)$

$$Tr\begin{pmatrix} 3/4 & \frac{\sqrt{2}}{4}e^{-i\phi} \\ \frac{\sqrt{2}}{4}e^{i\phi} & 1/4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= Tr\begin{pmatrix} \frac{\sqrt{2}}{4}e^{i\phi} & 1/4 \\ 3/4 & \frac{\sqrt{2}}{4}e^{-i\phi} \end{pmatrix}$$

$$\langle \sigma_x \rangle = \mathrm{Tr}(\rho \sigma_x) = \frac{\cancel{2}\sqrt{2}}{4}\left(e^{i\phi} + e^{-i\phi}\right)$$

$$\frac{1}{\sqrt{2}}\cos\phi$$