Tutorial - 3.

Parity Operator:

$$P(\psi) = \epsilon \psi$$

where

$\epsilon = +1$: Even Parity

$\epsilon = -1$: Odd Parity

Ladder Operator:

It raises or lowers the quantum state of a system

$L_+ = L_x + iL_y$.

$L_- = L_x - iL_y$.

- $L_+ \psi_m = (m+1)\psi_m$
- $L_+ \, |L_z|\psi\rangle = (m+1)|\psi\rangle$

Unitary Operator:

$$PAP^+ = A$$

where $P^+ = (P^T)^*$

Properties:

1). $A = A^*$ $\Rightarrow$ REAL.

2). $A = A^+$ $\Rightarrow$ Hermitian.

3). $A = -A^+$ $\Rightarrow$ Skew - Hermitian

4). $A = A^T$ $\Rightarrow$ Symmetric.

5). $A = -A^*$ $\Rightarrow$ Imaginary.

6). $A^T = A^{-1}$ $\Rightarrow$ Orthogonal.

7). $A^+ = A^{-1}$ $\Rightarrow$ Unitary

* Questions:

1). Given : $|\psi\rangle = \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix}$ and $|\phi\rangle = \begin{bmatrix} 3 \\ 8i \\ -9i \end{bmatrix}$

Find i) $|\psi\rangle^*$

ii) $\langle\psi|$

iii) $\langle\psi|\psi\rangle$

iv) $\langle\phi|\psi\rangle$

v) $|\psi\rangle\langle\psi|$

→ i) $|\psi\rangle^* = \begin{bmatrix} -5i \\ 2 \\ i \end{bmatrix}$

ii) $\langle\psi| = [-5i \quad 2 \quad i]$

iii) $\langle\psi|\psi\rangle = [-5i \quad 2 \quad i] \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix} = 25 + 4 + 1$

$= 30.$

iv) $\langle\phi|\psi\rangle$

$\langle\phi| = [3 \quad -8i \quad 9i]$

$\langle\phi|\psi\rangle = [3 \quad -8i \quad 9i] \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix} = (15-16)i + 9$

$= 9 - i.$

v) $|\psi\rangle\langle\psi| = \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix} [-5i \quad 2 \quad i] = \begin{bmatrix} 25 & -10i & -5 \\ -10i & 4 & 2i \\ -5 & -2i & 1 \end{bmatrix}$

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2) Given: $A = \begin{bmatrix} 5 & 3+2i & 3i \\ -i & 3i & 8 \\ 1-i & 1 & 4 \end{bmatrix}$, $|\psi\rangle = \begin{bmatrix} -1+i \\ 3 \\ 2+3i \end{bmatrix}$, $|\phi\rangle = \begin{bmatrix} 6 \\ i \\ 5 \end{bmatrix}$

Find

i) $A|\psi\rangle = \begin{bmatrix} 5 & 3+2i & 3i \\ -i & 3i & 8 \\ 1-i & 1 & 4 \end{bmatrix} \begin{bmatrix} -1+i \\ 3 \\ 2+3i \end{bmatrix}$

$$= \begin{bmatrix} 5(i-1) + 3(3+2i) + 3i(2+3i) \\ -i(i-1) + 9i + 8(3i+2) \\ (1-i)(i-1) + 3 + 4(2+3i) \end{bmatrix}$$

$$= \begin{bmatrix} 17i - 5 \\ 34i + 17 \\ 14i + 11 \end{bmatrix}$$

ii) $\langle\phi|A = \begin{bmatrix} 6 & -i & 5 \end{bmatrix} \begin{bmatrix} 5 & 3+2i & 3i \\ -i & 3i & 8 \\ 1-i & 1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 30 - 1 + 5(1-i) \\ 6(3+2i) - i(3i) + 5 \\ 6(3i) \cancel{+40} + 20 - 8i \end{bmatrix}$$

$$= \begin{bmatrix} 34 - 5i \\ 26 + 12i \\ \overset{20}{\cancel{60}} + \overset{10}{\cancel{15}}i \end{bmatrix}$$

$$= \begin{bmatrix} 34 - 5i & 26 + 12i & \overset{20}{\cancel{60}} + \overset{10}{\cancel{18}}i \end{bmatrix}$$

iii) $\langle \phi | A | \psi \rangle = \begin{bmatrix} 34-5i & 26+12i & 20+10i \end{bmatrix} \begin{bmatrix} -1+i \\ 3 \\ 2+3i \end{bmatrix}$

$= (-34+3i+31i+5i+5+78+36i+40+60i+20i-30)$

$= 59+155i.$

iv) $|\psi\rangle\langle\phi| = \begin{bmatrix} i-1 \\ 3 \\ 2+3i \end{bmatrix} \begin{bmatrix} 6 & -i & 5 \end{bmatrix}$

$= \begin{bmatrix} 6(i-1) & -i(i-1) & 5(i-1) \\ 18 & -3i & 15 \\ 6(2+3i) & -i(2+3i) & 5(2+3i) \end{bmatrix}$

$= \begin{bmatrix} -6+6i & 1+i & -5+5i \\ 18 & -3i & 15 \\ 12+18i & 3-2i & 10+15i \end{bmatrix}$

v) $\langle \phi | \phi \rangle = \begin{bmatrix} 6 & -i & 5 \end{bmatrix} \begin{bmatrix} 6 \\ i \\ 5 \end{bmatrix} = 36 - i^2 + 25 = 62$

vi) Normalize $|\phi\rangle = |\phi_N\rangle = \frac{1}{\sqrt{\langle \phi | \phi \rangle}} |\phi\rangle = \frac{1}{\sqrt{36+1+25}} \begin{bmatrix} 6 \\ -i \\ 5 \end{bmatrix}$

$|\phi_N\rangle = 0.127 \begin{bmatrix} 6 \\ -i \\ 5 \end{bmatrix}$

vii) $|\phi\rangle\langle\phi| = \begin{bmatrix} 6 \\ i \\ 5 \end{bmatrix} \begin{bmatrix} 6 & -i & 5 \end{bmatrix} = \begin{bmatrix} 36 & -6i & 30 \\ 6i & 1 & 5i \\ 30 & -5i & 25 \end{bmatrix}$

3). $|\phi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle$ where $|\phi_n\rangle$ is orthonormal basis and $B|\phi_n\rangle = n^2|\phi_n\rangle$. Find $\langle B\rangle$.

→

$$\langle B\rangle = \frac{\langle\phi|B|\phi\rangle}{\langle\phi|\phi\rangle}$$

i) $\langle\phi|\phi\rangle = \left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)^T$

As $\langle\phi_x|\phi_x\rangle = 1$ and $\langle\phi_x|\phi_y\rangle = 0$
(Orthonormal)

$$= \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} * \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{10}} * \frac{1}{\sqrt{10}}$$

$$= 0.8.$$

ii) $\langle\phi|B|\phi\rangle \Rightarrow$

$$B|\phi\rangle = \sum_{n=1}^{3} n^2|\phi_n\rangle = (1)^2\cdot\left(\frac{1}{\sqrt{2}}|\phi_1\rangle\right) + (2)^2\left(\frac{1}{\sqrt{5}}|\phi_2\rangle\right) + (3)^2\left(\frac{1}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle$$

$$\langle\phi|B|\phi\rangle = \left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)^T\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} * \frac{4}{\sqrt{5}} + \frac{9}{\sqrt{10}} * \frac{1}{\sqrt{10}}$$

$$= 2.2.$$

$$\therefore \langle B\rangle = \frac{\langle\phi|B|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{2.2}{0.8} = 2.75$$

4). Find i). $[L_z, L_+]$
ii). $[L_+, L_-]$
iii). $[x, L_x]$.

→ i). $[L_z, L_+]$

$L_+ = L_x + iL_y$.

$[L_z, L_x + iL_y] = [L_z, L_x] + i[L_z, L_y]$.

We know,

$[L_x, L_y] = i\hbar L_z$

$[L_y, L_z] = i\hbar L_x$

$[L_z, L_x] = i\hbar L_y$.

$= i\hbar L_y + i(-i\hbar L_x)$

$= i\hbar L_y + \hbar L_x$

$= \hbar(L_x + iL_y)$

$= \hbar L_+$.

Hence, $[L_z, L_+] = \hbar L_+$.

ii). $[L_+, L_-]$

$L_+ = L_x + iL_y$, $L_- = L_x - iL_y$.

$[L_x + iL_y , L_x - iL_y] = [L_x, L_x] - i[L_x, L_y] + i[L_y, L_x] - i^2[L_y, L_y]$

$= 0 - i(i\hbar L_z) + i(-i\hbar L_z) - (-1)0$

$= 0 + \hbar L_z + \hbar L_z + 0$

$= 2\hbar L_z$.

Hence, $[L_+, L_1] = 2\hbar L_z$.

iii) $[x, L_x]$

We know , $[x L_x] = 0$

(Solved in Tutorial 1)