* D.M. Unit 3 Tutorial *

Q1. Consider the following relations on the set $A = \{1,2,3,$
$R_1 = \{(1,1),(1,2),(2,3),(1,3),(4,4)\}$
$R_2 = \{(1,1)(1,2)(2,1)(2,2)(3,3)(4,4)\}$
$R_3 = \{(1,3),(2,1)\}$
$R_4 = \phi$, the empty relation.
Determine which of the relation are reflexive.

Sol: i] $R_1 = \{(1,1),(1,2),(2,3),(1,3),(4,4)\}$
$\therefore$ R is not reflexive.

ii] $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$
$\therefore$ R_2 is reflexive

iii] $R_3 = \{(1,3)(2,1)\}$
$\therefore$ R_3 is not reflexive

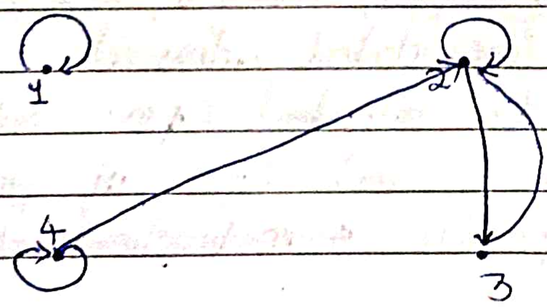
iv] $R_4 = \phi$, the empty relation $\therefore$ not reflexive

v] $R_5 = A \times A$, the universal relation
$\therefore$ R_5 is reflexive.

Q2. Consider the relation $R = \{(1,1),(2,2),(2,3)(3,2)(4,2)(4,4)\}$ on $A = \{1,2,3,4\}$.
a.] Draw its direct graph b.] Find $R^2 = R \circ R$.

Sol:

a]



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b] $R^{\wedge 2} - R \circ R$

$= \{(1,1),(2,2),(2,3),(3,2),(4,2)(4,4)\}$
$\{(1,1),(2,2),(2,3)(3,2)(4,2)(4,4)\}$

$= \{(1,1),(2,2),(3,2),(4,2),(3,3),(4,3),(4,2),(4,4)\}$

3 Let R and S be the following relations on $A = \{1,2,3\}$:

$R = \{(1,1),(1,2)(2,3)(3,1)(3,3)\}$

$S = \{(1,2),(1,3)(2,1),(3,3)\}$

Find (a) $R \cup S$, $R \cap S$, $R^{\wedge}C$; (b) $R \circ S$; (c) $S^2 = S \circ S$.

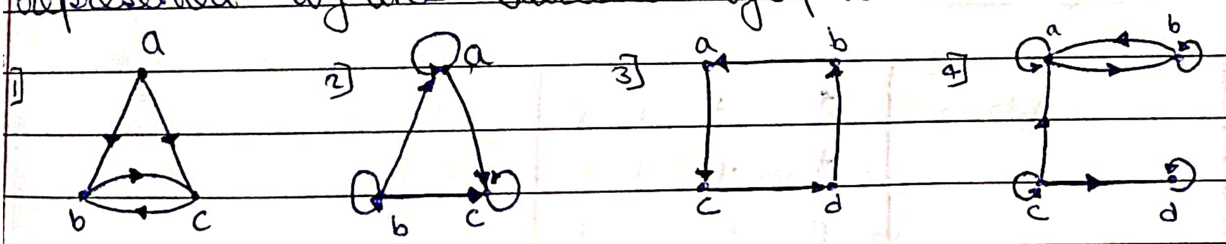
a] $R \cup S = \{(1,1),(1,2),(2,3),(3,1),(3,3)\} \cup \{(1,2),(1,3),(2,1),(3,3)\}$
$= \{(1,1),(1,2)(1,3),(2,1),(2,3),(3,1)(3,3)\}$

$R \cap S = \{(1,1),(1,2),(2,3)(3,1),(3,3)\} \cap \{(1,2),(1,3),(2,1),(3,3)\}$
$= \{(1,2),(3,3)\}$

b]. $R \circ S = \{(1,1)(1,2)(2,3)(3,1),(3,3)\} \circ \{(1,2)(1,3),(2,1),(3,3)\}$
$= \{(1,2),(1,3),(3,2),(3,3),(1,1)(2,3),(3,3)\}$

c] $S^2 = S.S = \{(1,1),(1,2)(2,3),(3,1),(3,3)\} . \{(1,1),(1,2),(2,3),(3,1),(3,3)\}$
$= \{(1,1),(3,1),(1,2),(3,2),(1,3),(2,1),(3,1),(2,3),(3,3)\}$

In the following list the ordered pairs in the relations represented by the directed graphs.



1] $\{(a,b),(a,c),(b,c),(c,b)\}$
2] $\{(a,a),(b,b),(c,c),(a,c),(b,a),(b,c)\}$
3] $\{(a,c),(c,d),(d,b),(b,a)\}$
4] $\{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(c,a),(c,d)\}$

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Q5. Use Warshall's algorithm to find the transitive closure of the relation on {1,2,3,4}
a] {(1,2), (2,1), (2,3), (3,4), (4,1)}
b] {(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)}

Sol: a]

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = W_0$$

Row 1 = 2 Column 1 = 2,4

$A_1 = \{(2,2), (4,2)\}$

Warshall 1 :-

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Row 2 = 1,2,3 Column 2 = 1,2,4

$A_2 = \{(1,1), (1,2) (1,3) (2,1), (2,2), (2,3), (4,1), (4,2), (4,3)\}$

Warshall 2 :-

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Row 3 = 4 Column = 1,2,4

$A_3 = \{(1,4), (2,4), (4,4)\} (4,1), (4,3)\}$

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Warshal's 3 :-

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad W_4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Row 4 = 1, 2, 3, 4 Column = 1, 2, 3, 4

$A_4 = \{(1,1)(1,2), (1,3), (1,4), (2,1)(2,2)(2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3)(4,4)\}$

W_4 is the transitive closure.

b]. $W_0 = M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Row 1 = - Column 1 = 2, 3, 4

Warshal 1 :

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Row 2 = 1, 3 Column 2 = -

$B_2 = \{\}$

Warshal 2:

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Row 3 = 1, 4

Column 3 = 2, 4

$B_3 = \{(2,1), (2,4), (4,1), (4,4)\}$

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Warshal 3:

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Row 4 = 1,3,4   Column = 2,3,4

$B_4 = \{(2,1),(2,3),(2,4),(3,1),\cancel{(3,3)},(3,4),(4,1),(4,3),(4,4)\}$

Warshal 4:

$$W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

~~Row 5~~ So, W_4 is transitive

Q6. Use Warshall's algorithm to find the transitive closures of these relations on {(a,b,c,d,e)}

~~Sol~~: a] {(a,c),(b,d),(c,a),(d,b),(e,d)}

b] {(b,c),(b,e),(c,e),(d,a),(e,b),(e,c)}

Sol: $W_0 = M_R$

$$W_0 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Row 1 = c

Column 1 = c

$A_1 = \{(c,c)\}$

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Warshal 1

$$W_1 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 1 & 0 \end{array}$$

Row 2 = d
Column 2 = d
$A_2 = \{(d,d)\}$

Warshal 2

$$W_2 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 1 & 0 \end{array}$$

Row 3 = a, c
Column = a, c
$A_3 = \{(a,a), (a,c), (c,a), (c,c)\}$

Warshal 3

$$W_3 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 1 & 0 \end{array}$$

Row 4 = b,d
Column 4 = b,d,e
$A_4 = \{(b,b), (b,d) (d,b) (d,d) (e,b) (e,d)\}$

Warshal 4

$$W_4 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 1 & 0 \\ e & 0 & 1 & 0 & 1 & 0 \end{array}$$

Row 5 = b,d
Column 5 = –
$A_5 = \{\}$

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Warshal 5

$$W_5 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 1 & 0 \\ e & 0 & 1 & 0 & 1 & 0 \end{array}$$

So W_5 is the transitive closures.

5]

$$W_0 = M_R = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 1 & 0 & 0 \end{array}$$

Row 1 = --
Column 1 = d
B1 = {}

Warshal's 1 =

$$W_1 = \begin{array}{c|ccccc} a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 1 & 0 & 0 \end{array}$$

Row 2 = c, e
Column 2 = e
B2 = {(e,c), (e,e)}

Warshall 2 = W_2 =

$$\begin{array}{c|ccccc} a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 1 & 0 & 1 \end{array}$$

Row 3 = e
Column 3 = b, e
B3 = {(b,e), (e,e)}

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Warshals 3 :

$$W_3 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 1 & 0 & 1 \end{array}$$

Row 4 = d
Column 4 = -
$B_4 = \{\}$

Warshals 4 :-

$$W_4 = \begin{array}{c|ccccc} a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 1 & 0 & 1 \end{array}$$

Row 5 = b, c, e
Colum 5 = b, c, e
$B_5 = \{(b,b), (b,c), (b,e), (c,b), (c,c), (c,e), (e,b), (e,c), (e,e)\}$

Warshals 5 :-

$$W_5 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 1 & 0 & 1 \end{array}$$

So, W_5 is transitive closures.

Q7 Draw the Hasse diagram representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on $\{1,2,3,4,6,8,12\}$

Sol: Partial Ordering

$R = \{(a,b) \mid a \text{ divides } b\}$ on $\{1,2,3,4,6,8,12\}$

$R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),(2,12),$
$(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),$
$(12,12)\}$

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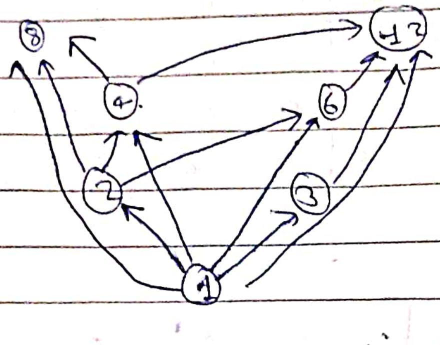
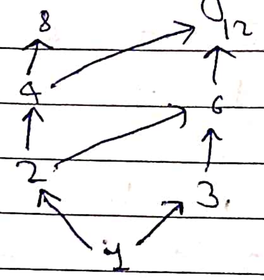
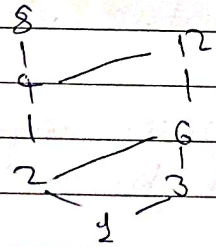
Step1: We construct a directed graph corresponding a relation R.



Step2: We remove all loops from the diagram & all transitive edge.



Step3: We make sure that the initial vertex is below the terminal vertex and remove all arrows.



Q.8- Determine whether the posets (1, 2, 3, 4, 5) and {1, 2, 4, 8, 16}, | are lattices.

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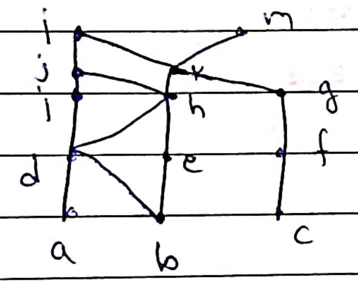
sol: Posets ({1,2,3,4,5}, |) and ({1,2,4,8,16}, |)

Because 2 and 3 have no upper bounds in ({1,2,3,4,5}, |) they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound & a greatest lower bound the least upper bound of two elements in this poset is the larger of the element & the greatest lower bound of the element is the smaller of the elements as the reader should verify. Hence second poset is a lattice.

Q9.15 Answer these questions for the partial order represented by this Hasse diagram



a. Find the maximal elements → l, m

b. Find the minimal elements → a, b, c

c. Is there a greatest element → No

d. Is there a least element → No

e. Find all upper bounds of {a,b,c} → l, k, m

f. Find the least upper bound of {a,b,c}, if it exists. → k

g. Find all lower bounds of {f,g,h} → Not exists

h. Find the greatest lower bound of {f,g,h}, if it exists.
→ Not exists.

Sol(10) Let a and b be positive integers, and suppose Q is define recursively as follows:

$Q(a,b) = 0$ if $a < b$

$Q(a-b,b)+1$ if $b \le a$

a] Find (i) $Q(2,5)$; (ii) $Q(12,5)$

b] What does this function Q do? Find $Q(5861,7)$:

Sol:

i] $Q(2,5) = Q(a,b) = 0$

ii] $Q(12,5) = 0$

if $b \le a$

$Q(a-b,b)+1$

$Q(12,5) = Q(12-5,5)+1$

$Q(7,5) = Q(7-5,5)+1$

$Q(2,5) = 0$

$Q(7,5) = 0+1 = 1$

$Q(12,5) = 1+1 = 2$

iii] $Q(5861,7)$

if $b \le a$

the $Q(a-b,b)+1$

$Q(5861,7) = Q(5861-7,7)+1$

$= Q(5854,7)+2$

$= Q(5847,7)+3$

$= Q(5840,7)+4$

$= Q(5833,7)+5 = Q(5826,7)+6$

$= Q(5819,7)+7 = Q(5812,7)+8$

$= Q(5805,7)+9$

$\vdots$

$= Q$

$Q(5819,7) = 837$