EE2703 : Applied Programming Lab
Assignment 5
Laplace equation

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Introduction

Our goal will be to find the flow of currents in a resistor and visualize how the resistor heats up. We will use the potential distributions to determine the current flow, and further use that to plot temperature contours. The resistor is assumed to be a square with a dimension of 1 cm by 1 cm, and is floating in midair, with one side of it being grounded and one face of the plate being connected to the wire of the given radius.

Theory

The center of the copper plate is held at 1V initially. The plate is 1cm by 1cm in size, and is floating midair with the bottom side of it being grounded.

Conductivity formula:

$$\vec{j} = \sigma \vec{E}$$

And electric field is the gradient of the potential, so

$$\vec{E} = -\nabla \phi.$$

Continuity of charge gives us

$$\nabla . \vec{j} = -\frac{\partial \rho}{\partial t}$$

Thus, we get:

$$\nabla . (-\sigma \nabla \phi) = -\frac{\partial \rho}{\partial t}$$

Assuming our conductivity is a constant and the current used is a DC current,

$$\nabla^2 \phi = 0$$

This is the derivation for Laplace's Equation and upon reducing it to a 2D differential equation in Cartesian coordinates we get:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Solving this numerically we get:

$$\phi_{i,j} = \frac{\phi(x_{i+1},y_j) - 2\phi(x_i,y_j) + \phi(x_{i-1},y_j)}{4}$$

Interestingly, this appears to be a case where the potential at any point is determined by the average of its neighbors. At boundaries which are floating, the current must be tangential. Thus, potential should not vary normally, and has been implemented as such.

The equations for the current densities are:

$$\vec{j_x} = -\partial\phi/\partial x$$

$$\vec{j_y} = -\partial\phi/\partial y$$

Numerically,

$$J_{x,ij} = \frac{1}{2}(\phi_{i,j-1} - \phi_{i,j+1})$$

$$J_{y,ij} = \frac{1}{2}(\phi_{i-1,j} - \phi_{i+1,j})$$

The current flows from higher potential to lower potential so most of the current will flow from the wire portion to the lower boundary. Ideally, no current will flow out of the top part of the plate. The lower region will also get strongly heated. To find the temperature we use the following equation:

$$\nabla.(\kappa\nabla T) = q = \frac{1}{\sigma}|J|^2$$

where the right side represents heat generated from ohmic loss. The boundary condition her is that T=300 at the wire and the ground while $\partial T/\partial n = 0$ at the other edges.

Execution and Analysis of plots

The code should be run with 4 parameters, in the following order:

- N_x (Number of divisions along the x-axis)
- N_y (Number of divisions along the y-axis)
- Radius
- Number of iterations

The example here uses 25 25 8 1500 as its parameters.

Base case

We set the potential to be 1.0V within the given radius.

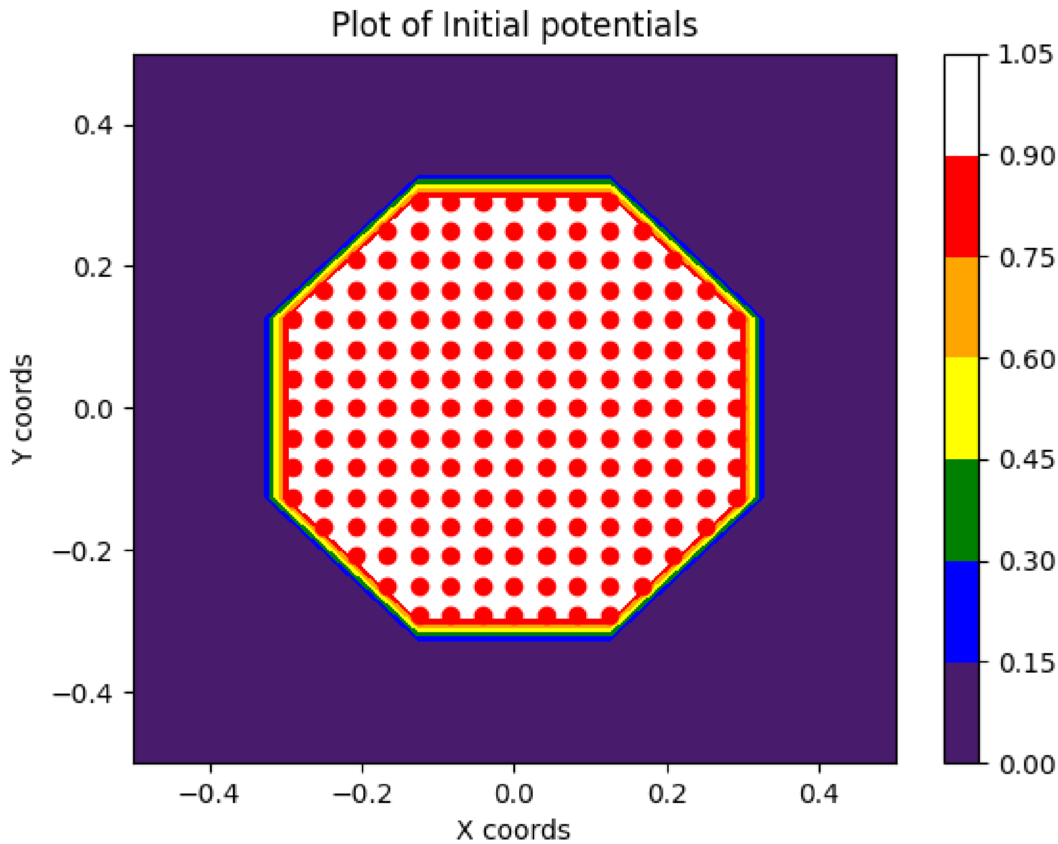

Figure 1: Base case

Next, we need to generate the new value of phi from older values, and we must perform this operation N_{iter} times. We do this process by finding the new value for each element as the average of its surrounding 4 elements. Boundary conditions must also be dealt with. The bottom side is obviously grounded, while the other 3 are set to value the element opposite to the edge posses.

We use vectorized code as opposed to a double for loop to increase the speed of the program. The purpose is the same, however the vectorized code is executed in C, thus is a lot faster.

Error analysis

Let's plot the errors in semilog scale and loglog scale.

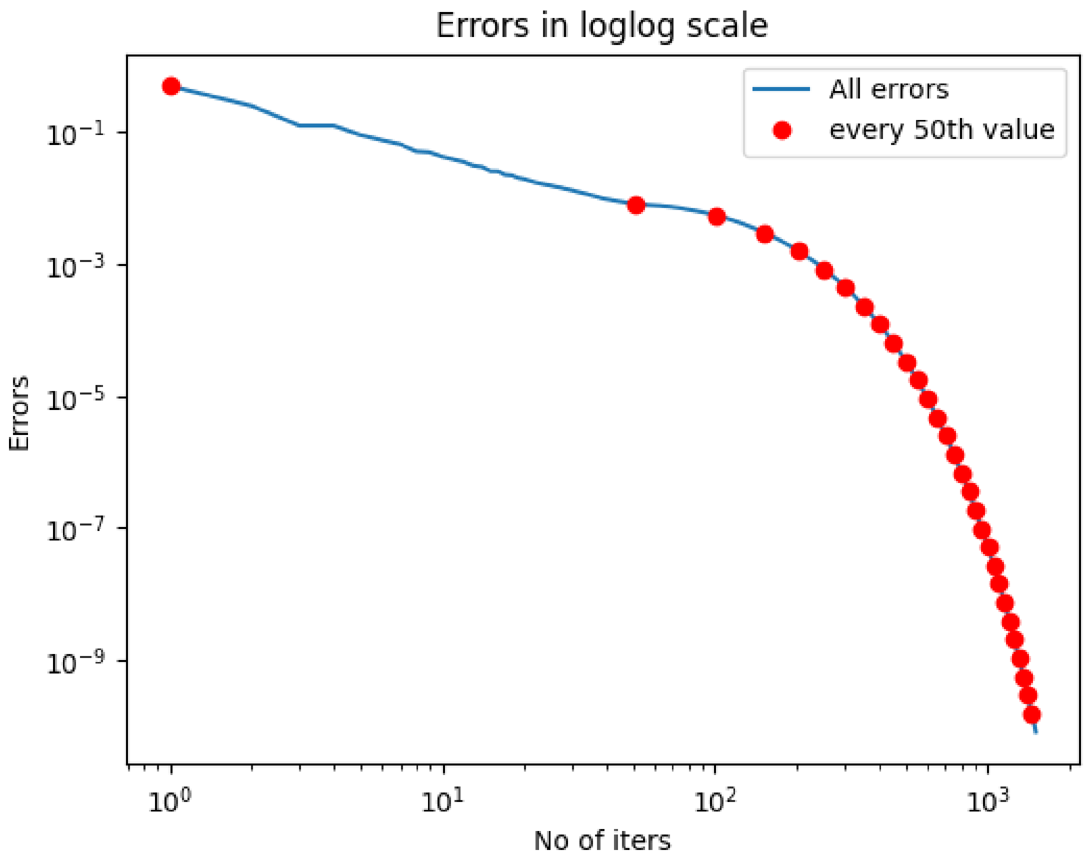

Figure 2: Errors in loglog scale

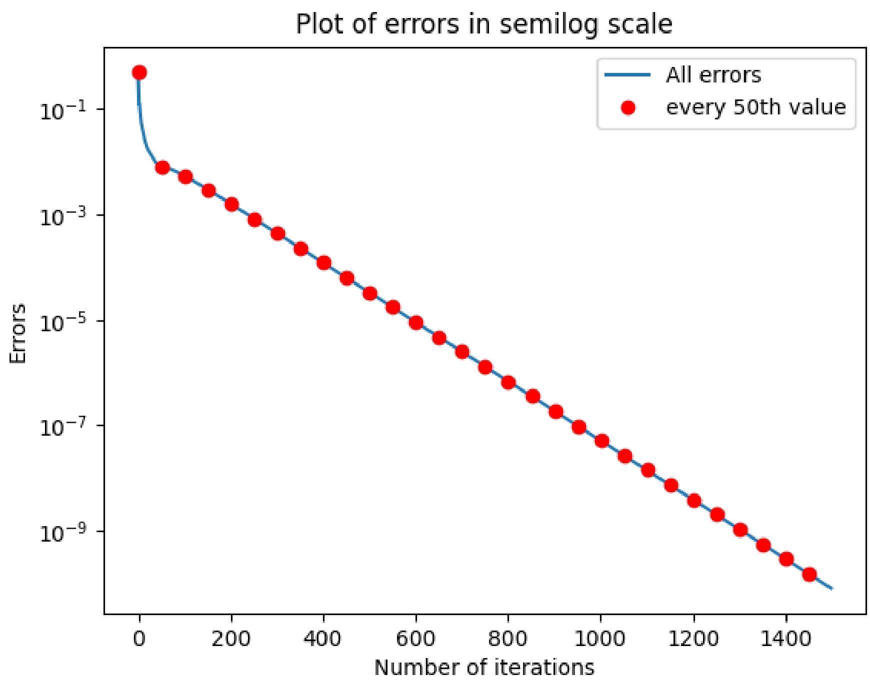

Figure 3: Errors in semilog scale

Error fitting using lstsq method

By looking at the semilog scale, it seems that the function should be exponential in nature. So let's try to fit the errors in the form e^{A+Bx}.

$$y = e^{A+B*x}$$

$$logy = A + B * x$$

For fitting we need to fit the best values of A and B such that the error in determining y is minimum. For fitting we can use Least Squares Method to estimate A and B.

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} logy_1 \\ logy_2 \\ \vdots \\ logy_n \end{pmatrix}$$

The error in estimating y can be expected to reduce with increasing x. So we shall fit for the whole vector as well as after 500 iterations. (fit2 is for after 500 iterations.)

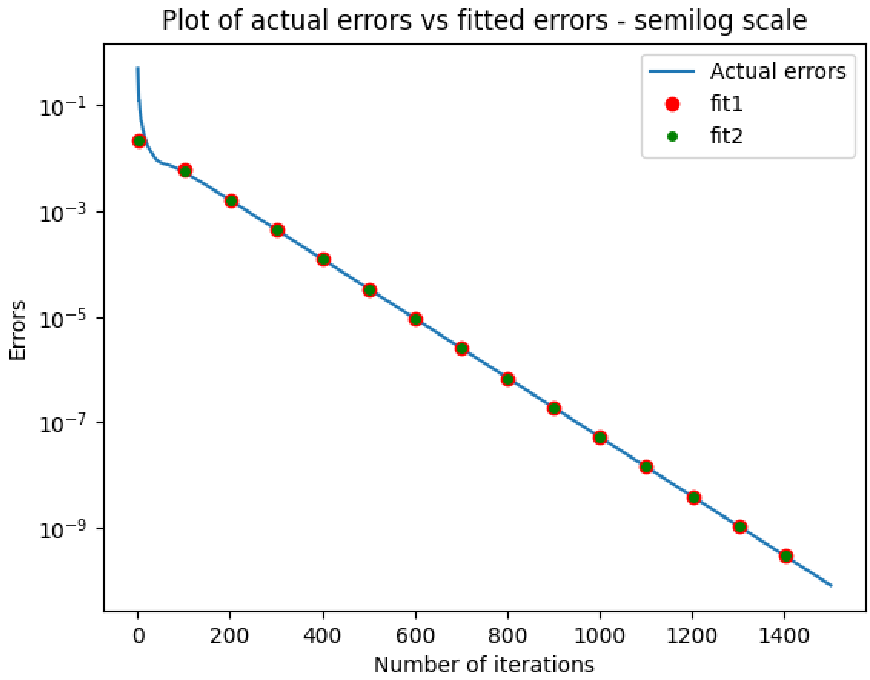

Figure 4: Best fit for error in semilog scale

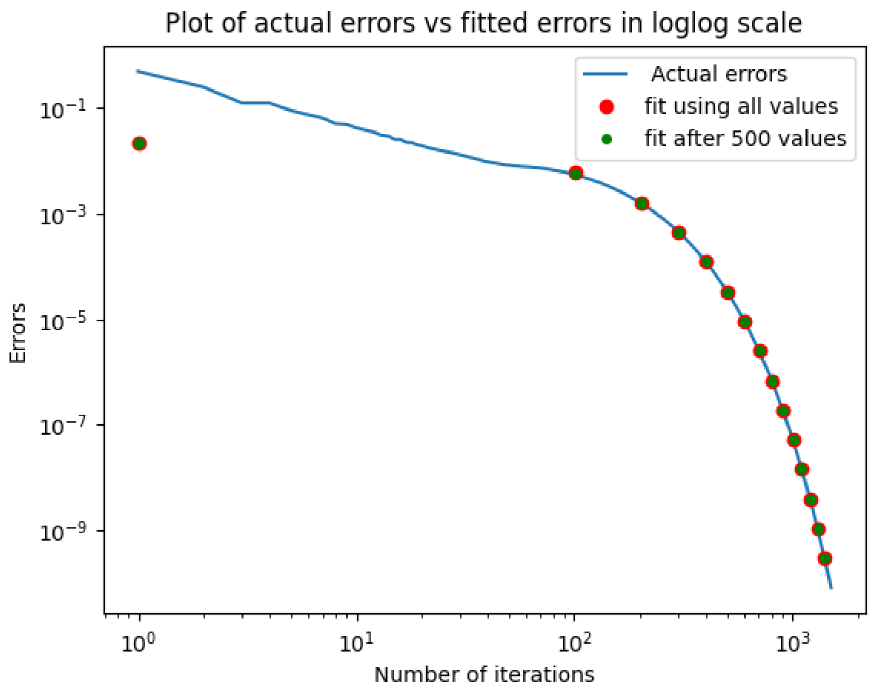

Figure 5: Best fit for error in loglog scale

Also, a plot for cumulative errors has been included as well to estimate how much error we may get after N_{iter} calculations.

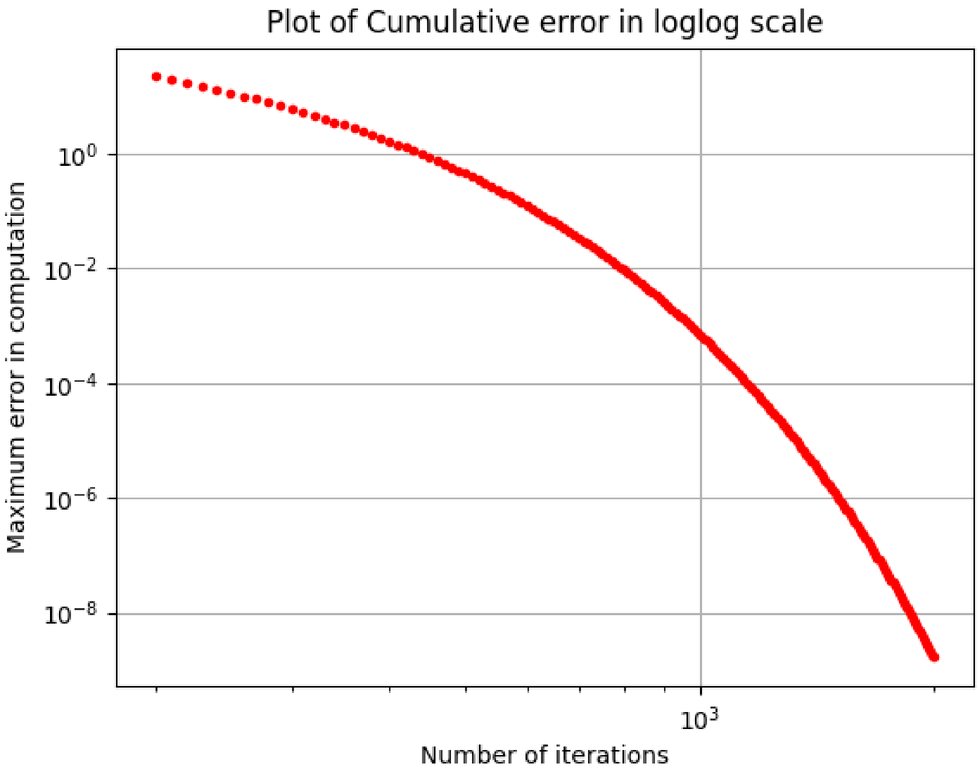

Figure 6: Plot of Cumulative error

Surface and contour plots for potential

We will now plot the 3-D surface and contour plots for potential.

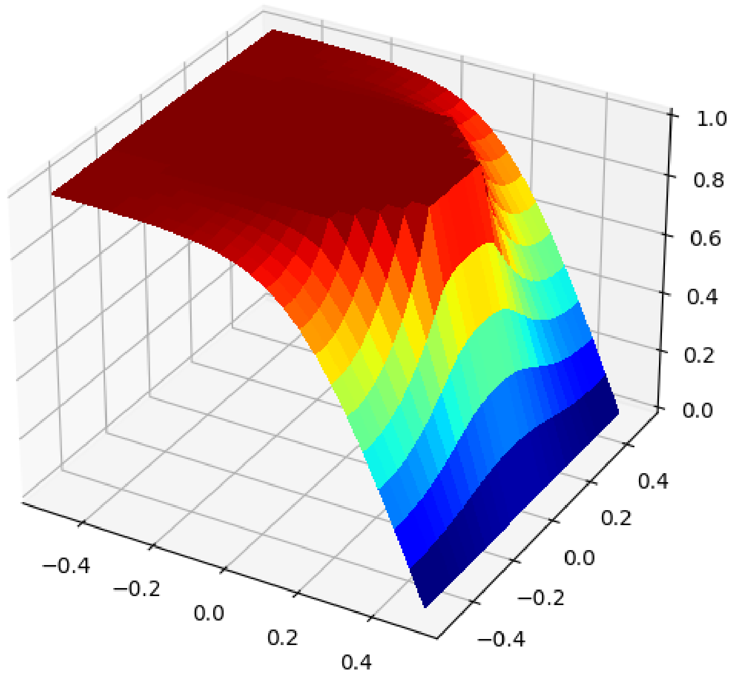

Figure 7: 3D surface plot

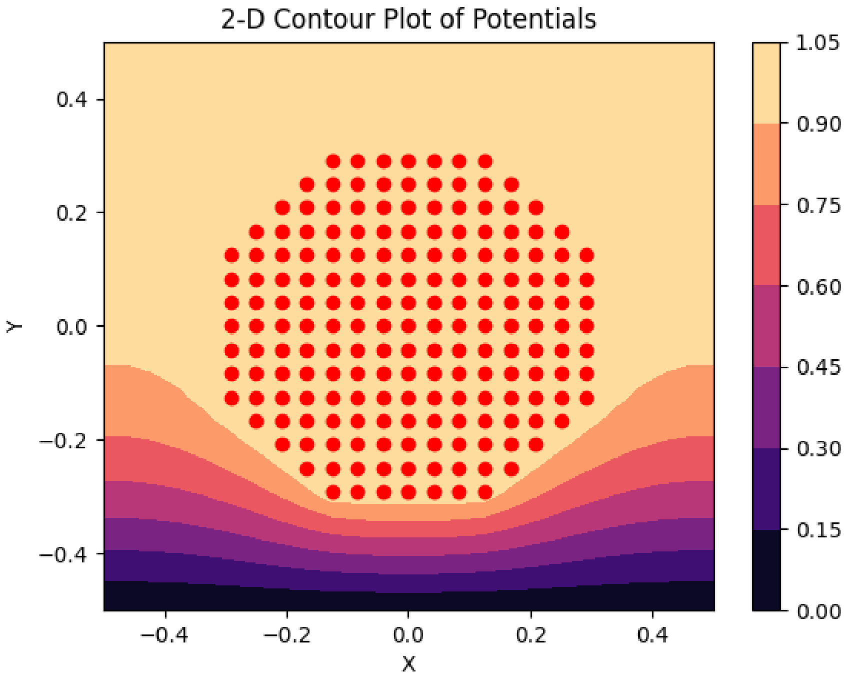

Figure 8: Contour plot

Vector Plot of Currents:

Now let's obtain the currents. Here we shall set σ as unity as it's value won't affect the shape of the current profile. We shall create the arrays J_x and J_y and then use the quiver command as instructed. This creates a vector current plot.

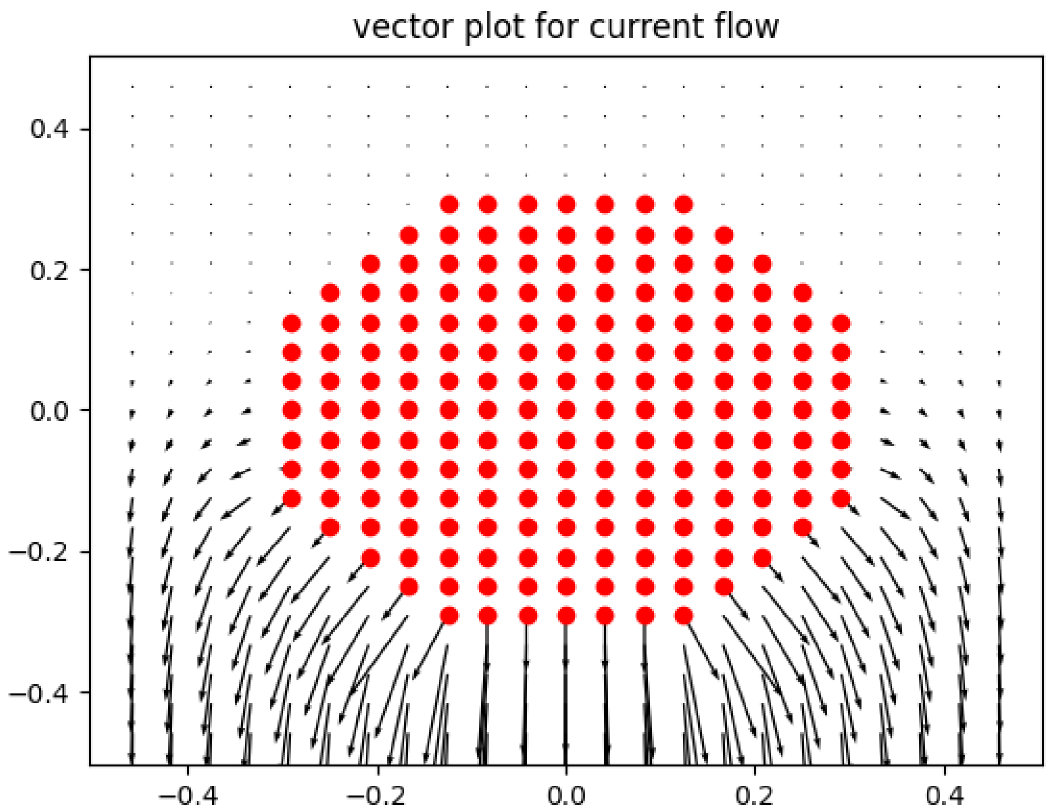

Figure 9: Vector current plot

Heat map

Lastly, we need to find the heat map of our plate. It is quite similar to the potential calculations. The bottom side has been set to 300K for our calculations.

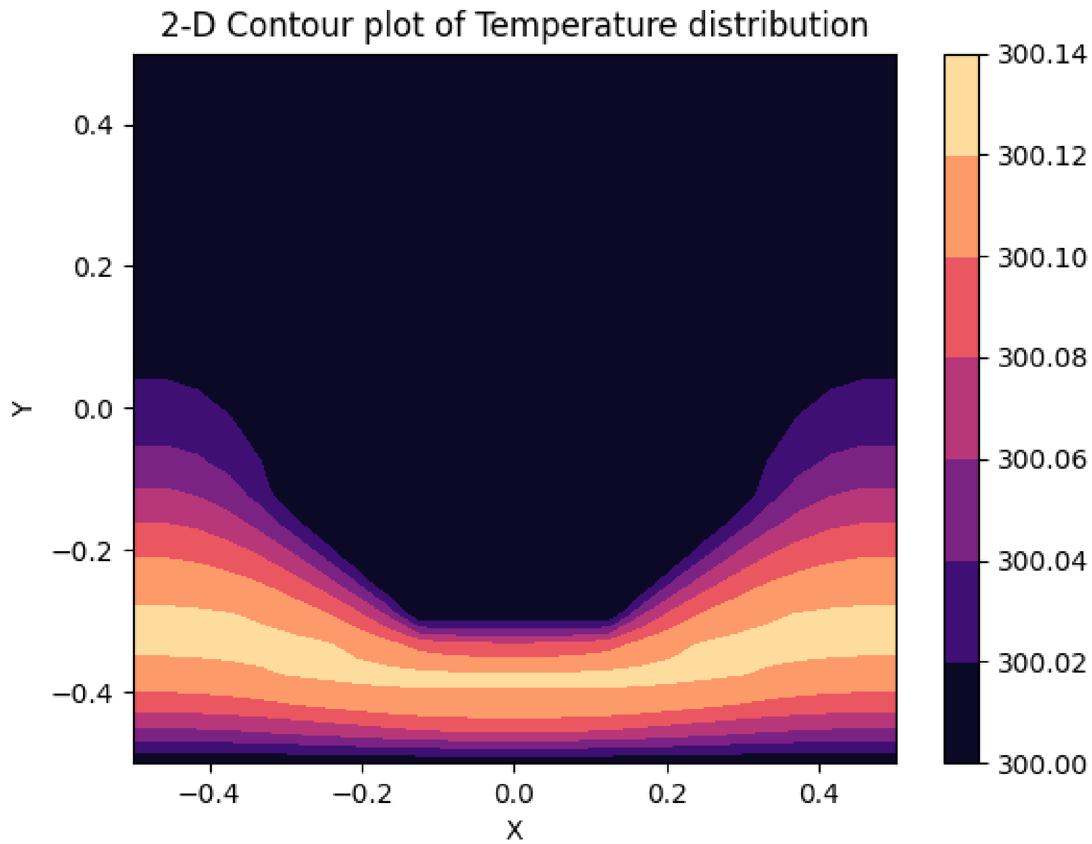

Figure 10: Heat Map

Conclusion:

Solution for the Laplace's equation for the given system is found numerically. Errors where found to have an exponential decay, and as the decay was quite slow, the method itself is very inefficient. The vector plot of the currents are plotted, and we find that the current flow is towards the bottom part of the plane - thus heating up that region. The 3D plot and contour plots for potential have also been plotted.