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Predictive Path Following of Mobile Robots without Terminal Stabilizing Constraints ⋆

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Abstract: This paper considers model predictive path-following control for differentially driven mobile robots and state-space paths. In contrast to previous works, we analyze stability of model predictive path-following control without stabilizing terminal constraints or terminal costs. To this end, we verify cost controllability assumptions and compute bounds on the stabilizing horizon length. Finally, we draw upon simulations to verify our stability results.

Keywords: nonlinear model predictive control, mobile robots, path following, stability.

1. INTRODUCTION

Recently, nonholonomic mobile robots have attracted considerable interest as they are increasingly used in industry, for discovery and observation purposes, and in autonomous services. Often, the differential drive model, i.e. the unicycle, is used to describe the kinematics of nonholonomic robots. In applications, different control tasks, such as set-point stabilization/regulation, trajectory tracking, and path following, arise. Set-point stabilization refers to the control task of stabilizing a given setpoint. In case the reference is time-varying, the control task is referred to as trajectory tracking. Control tasks in which a geometric reference is to be followed, while the speed to move along the reference is not given apriori, are commonly referred to as path-following problems, cf. Aguiar et al. (2008); Faulwasser and Findeisen (2016). We refer to Faulwasser (2013) for a literature review on this subject.

Nonlinear Model Predictive Control (NMPC) is of a particular interest in robot control as it can handle constrained nonlinear systems. In NMPC, a finite-horizon Optimal Control Problem (OCP) is solved in a receding horizon fashion, and the first part of the optimal control is applied to the plant. Several successful NMPC approaches to path-following problems have been presented in the literature. An early numerical investigation for nonholonomic robots is presented in Faulwasser and Findeisen (2009). Generalizations and extensions are discussed in follow-up papers such as Alessandretti et al. (2013); Faulwasser (2013); Faulwasser and Findeisen (2016); Lam et al. (2013); Yu et al. (2015). Therein, stability and path convergence are enforced using additional stabilizing terminal constraints and/or terminal costs. Implementation results are dicussed in Faulwasser et al. (2016).

The present paper discusses the stability of Model Predictive Path-Following Control (MPFC) as proposed in Faulwasser and Findeisen (2016). Similar to Faulwasser and Findeisen (2009), we investigate paths defined in the state space. However, we try to explicitly avoid the use of stabilizing terminal constraints and terminal penalties. Instead, we work in the framework of controllability assumptions (Grüne et al., 2010), which allows to guarantee stability by appropriately choosing the prediction horizon length. Recently, these techniques have been extended to a continuous time setting by Reble and Allgöwer (2012); Worthmann et al. (2014). The pursued design also allows the estimation of bounds on the infinite horizon performance of the closed loop, see Grüne (2009).

In the present paper, we first reformulate the path-following problem as the set-point stabilization of an augmented system, which combines the robot model and a timing law. The augmented state is subject to a specific constraint such that close to the end of the path the robot has to be exactly on the geometric reference. We show that this particular structure of the state constraint simplifies the verification of the controllability assumption. Specifically, we verify it by designing an open-loop control maneuver such that a robot is steered from any arbitrary initial position to the end point of the reference path. This control maneuver leads to bounds on the value function, which allow to derive a stabilizing horizon length. In essence, we extend the set-point stabilization analysis of Worthmann et al. (2016, 2015) to path-following problems.

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The paper is organized as follows: in Section 2, a brief description of the considered model predictive path-following control scheme is given. In the subsequent Section 3, stability results from Reble and Allgöwer (2012) are recalled. Then, in Section 4, a growth bound on the value function is derived based on feasible open-loop trajectories, which is used in Section 5 to determine a stabilizing prediction horizon length. Finally, we draw upon numerical simulations to assess the closed-loop performance.

Notation: In this paper, piecewise continuous functions are denoted by $\mathcal{PC}([0,T),\mathbb{R}^m)$ for $T \in \mathbb{R}_{>0} \cup \{\infty\}$ and $m \in \mathbb{N}$. A continuous function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is zero at zero and strictly monotonically increasing. If it is, in addition, unbounded it is called a class $\mathcal{K}_\infty$-function.

2. PROBLEM FORMULATION

We recall the model of a differentially driven robot as well as the path-following problem and the MPFC formulation from Faulwasser and Findeisen (2009).

The kinematic model of the considered robot is

$$\dot{x}(t) = f(x(t), u(t)) = \begin{pmatrix} u_1(t)\cos(x_3(t)) \\ u_1(t)\sin(x_3(t)) \\ u_2(t) \end{pmatrix} \tag{1}$$

with vector field $f : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ and initial condition $x(0) = x_0$. The state vector $x = (x_1, x_2, x_3)^\top$ contains the robot's posture variables, i.e. the spacial components $x_1\,[\mathrm{m}]$, $x_2\,[\mathrm{m}]$, and the orientation $x_3\,[\mathrm{rad}]$. The control input $u = (u_1, u_2)^\top$ consists of the linear and the angular speeds of the robot $u_1\,[\mathrm{m/s}]$ and $u_2\,[\mathrm{rad/s}]$, respectively. The state and input constraints are given by $X_\varepsilon = [-\bar{x}_1, -\varepsilon] \times [-\bar{x}_2, \bar{x}_2] \times \mathbb{R} \subset \mathbb{R}^3$, $U = [-\bar{u}_1, \bar{u}_1] \times [-\bar{u}_2, \bar{u}_2] \subset \mathbb{R}^2$ with $\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2 > 0$. Lateron, the design parameter $\varepsilon \in [0, \bar{x}_1)$ is employed to construct an extended set Z_ε (introduced later) taking the particular structure of the path-following problem into account.

2.1 Path-following problem

The state-space path-following problem is to steer system (1) along a geometric reference curve $\mathcal{P} \subset X_0$. To this end, a parametrization $p : [\bar{\theta}, 0] \to \mathbb{R}^3$ with $\bar{\theta} = -\bar{x}_1$ and satisfying $p(0) = 0$, specifies $\mathcal{P}$. Here, the scalar variable $\theta \in \mathbb{R}$ is called the path parameter. In other words, the reference is given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^3 \,|\, \exists\, \theta \in [\bar{\theta}, 0] : p(\theta) = x \right\}.$$

In path-following problems, *when to be where* on $\mathcal{P}$ is not a strict requirement. Nonetheless, the path parameter θ is time dependent with unspecified time evolution $t \mapsto \theta(t)$. Therefore, the control function $u \in \mathcal{PC}([0,\infty),\mathbb{R}^2)$ and the timing of $\theta : [0,\infty) \to [\bar{\theta}, 0]$ are chosen such that the path $\mathcal{P}$ is followed as closely as possible while maintaining feasibility with respect to the state and control constraints. The state-space path-following problem is summarized as follows, cf. (Faulwasser, 2013, Chap. 5.1):

Problem 1. (State-space path following)

(1) Convergence to path: The robot state x converges to the path $\mathcal{P}$ such that
$$\lim_{t\to\infty} \|x(t) - p(\theta(t))\| = 0.$$
(2) Convergence on path: The robot moves along $\mathcal{P}$ in the direction of increasing θ such that $\dot{\theta}(t) \geq 0$ holds and $\lim_{t\to\infty} \theta(t) = 0$.
(3) Constraint satisfaction: The state and control constraints X_0 and U are satisfied for all $t \geq 0$. □

Here, we consider paths parameterized by

$$p(\theta) = \left(\theta, \rho(\theta), \arctan\left(\tfrac{\partial \rho}{\partial \theta}\right)\right)^\top, \tag{2}$$

whereby ρ is at least twice continuously differentiable. A basic requirement for the path-following problem to be feasible, is the consistency of the path with the state constraints, i.e. we assume that $\mathcal{P} \subset X_0$.

Similar to Faulwasser and Findeisen (2009), we treat the path parameter θ as a virtual state, whose time evolution $t \mapsto \theta(t)$ is governed by an additional (virtual) control input $v \in \mathbb{R}$. Therefore, the dynamics of θ, which is an extra degree of freedom in the controller design, is described by a differential equation referred to as *timing law*. Here, we define the timing law as a single integrator

$$\dot{\theta}(t) = v(t), \qquad \theta(0) = \theta_0, \tag{3}$$

$\theta_0 \in [\bar{\theta}, 0]$. The virtual input of the timing law is assumed to be piecewise continuous and bounded, i.e. for all $t \geq 0$, $v(t) \in V := [0, \bar{v}] \subset \mathbb{R}$. Using systems (1) and (3), the path-following problem is analyzed via the following augmented system

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} f(x,u) \\ v \end{pmatrix} = f_g(z, w). \tag{4}$$

The augmented state vector $z := (x^\top, \theta)^\top \in \mathbb{R}^4$ embodies the state of the robot as well as the virtual state θ. For the parameter $\varepsilon \in [0, \bar{x}_1)$, the constraint set of the augmented state variable z is defined as

$$Z_\varepsilon = X_\varepsilon \times [\bar{\theta}, -\varepsilon] \cup \left\{ (p(\theta)^\top, \theta)^\top \,|\, \theta \in (-\varepsilon, 0] \right\} \subset \mathbb{R}^4. \tag{5}$$

In other words, the robot is forced to be on the reference path $\mathcal{P}$ if the path parameter θ satisfies $\theta > -\varepsilon$. This particular structure of Z_ε serves as a kind of *stabilizing constraint* mainly used to simplify the later derivations. We remark that while Z_ε imposes a constraint on the robot motion close to the end of the path $\mathcal{P}$, it will be *not* enforced at the end of each prediction horizon, i.e. it is not a stabilizing terminal constraint in the classical sense.

Additionally, we define the vector of augmented control actions $w := (u^\top, v)^\top \in \mathbb{R}^3$, which contains the robot control input as well as the virtual control. The input constraint of w is $W = U \times V$. The path-following problem is reduced now to the point stabilization of the augmented system (4), cf. Faulwasser and Findeisen (2009). This allows us to directly use the stability results of Reble and Allgöwer (2012) and the techniques presented in Worthmann et al. (2015) as will be shown in the following section.

For system (4), we use $z(\cdot; z_0, w)$ to denote a trajectory originating at z_0 and driven by the input $w \in \mathcal{PC}([0,T),\mathbb{R}^3)$, $T \in \mathbb{R}_{>0} \cup \{\infty\}$. Additionally, the control function w is called admissible on the interval $[0,T)$ if

$$w(t) \in W, t \in [0,T), \quad \text{and} \quad z(t; z_0, w) \in Z_\varepsilon, t \in [0,T],$$

hold. We denote the set of all admissible control functions w for initial value z_0 on $[0,T)$ by $\mathcal{W}_T(z_0)$.

System (1) is *differentially flat*, and $(x_1,x_2)^\top$ is one of its flat outputs. [1] Hence, an input u_{ref} ensuring that the system follows the path (2) exactly for a given timing $\theta(t)$ can be obtained using ideas from Faulwasser et al. (2011).

First, by investigating the first two equations of (1), $u_{1,ref}$ is obtained as

$$u_{1,ref}(\theta,\dot\theta) = \dot\theta \cdot \sqrt{1+\left(\tfrac{\partial\rho(\theta)}{\partial\theta}\right)^2}. \tag{6a}$$

Similarly, using the last equation in model (1), $u_{2,ref}$ is computed as

$$\begin{aligned} u_{2,ref}(\theta,\dot\theta) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\arctan\left(\tfrac{\partial\rho(\theta)}{\partial\theta}\right)\right) \\ &= \dot\theta\cdot\left(1+\left(\tfrac{\partial\rho(\theta)}{\partial\theta}\right)^2\right)^{-1}\left(\tfrac{\partial^2\rho(\theta)}{\partial\theta^2}\right). \end{aligned} \tag{6b}$$

Note that due to (3), u_{ref} can also be regarded as a function of θ and the virtual control v. The largest value of v for which u_{ref} is admissible is given by

$$\hat v := \max\{v\in[0,\bar v]\,|\,u_{ref}(\theta,v)\in U \quad \forall\,\theta\in[\bar\theta,0]\}. \tag{7}$$

It is readily seen that, for any twice continuously differentiable $\rho(\theta)$ in (2), the timing law $\dot\theta = v$, the input constraint U and the structure of (6) imply $\hat v > 0$.

2.2 Model Predictive Path-Following Control (MPFC)

Here, we recall the MPFC scheme for state-space paths as proposed in Faulwasser and Findeisen (2009). We refer to Faulwasser (2013); Faulwasser and Findeisen (2016) for extension to paths defined in output spaces. MPFC solves Problem 1 via continuous-time sampled-data NMPC.

To this end, we consider a continuous running (stage) cost $\ell:\mathbb{R}^4\times\mathbb{R}^3\to\mathbb{R}_{\geq 0}$ satisfying

$$\ell(z^\star,0)=0 \quad \text{and} \quad \inf_{w\in W}\ell(z,w)>0 \quad \forall\, z\in\mathbb{R}^4\setminus z^\star,$$

where $z^\star$ is the path end point we would like to stabilize the dynamics (4) at. That is, $z^\star$ is given by

$$z^\star := \left(0,\rho(0),\arctan\left(\tfrac{\partial\rho(\theta)}{\partial\theta}\right)|_{\theta=0},0\right)^\top. \tag{8}$$

Similar to Faulwasser and Findeisen (2009), ℓ is chosen as

$$\ell(z,w) = \left\|\begin{matrix} x-p(\theta)\\ \theta\end{matrix}\right\|_Q^2 + \left\|\begin{matrix} u-u_{ref}(\theta,v)\\ v\end{matrix}\right\|_R^2. \tag{9}$$

$Q=\mathrm{diag}(q_1,q_2,q_3,\hat q), R=\mathrm{diag}(r_1,r_2,\hat r)$ are positive definite weighting matrices and $u_{ref}(\theta,v)$ is from (6). The objective functional to be minimized in MPFC reads

$$J_T(z_k,w) := \int_{t_k}^{t_k+T}\ell(z(\tau;z_k,w),w(\tau))\,\mathrm{d}\tau$$

with $T\in\mathbb{R}_{>0}$. Hence, the MPFC scheme is based on repeatedly solving the following OCP [2]

$$\begin{aligned} V_T(z_k) = \min_{w\in\mathcal{PC}([t_k,t_k+T),\mathbb{R}^3)} \quad & J_T(z_k,w) \\ \text{subject to } & z(t_k)=z_k, \\ & \dot z(\tau) = f_g(z(\tau),w(\tau)) \quad \forall\tau\in[t_k,t_k+T) \\ & z(\tau)\in Z_\varepsilon \quad \forall\tau\in[t_k,t_k+T] \\ & w(\tau)\in W \quad \forall\tau\in[t_k,t_k+T) \end{aligned} \tag{10}$$

For a given time t_k and initial value $z(t_k)=z_k\in Z_\varepsilon$, the minimum of this OCP, i.e. $V_T(z_k)$, does actually not depend on t_k. Hence, $V_T:\mathbb{R}^4\to\mathbb{R}_{\geq 0}\cup\{\infty\}$ is the optimal value function.

The solution of (10) results in the optimal control function $w^\star\in\mathcal{W}_T(z_0)$. Then, for a sampling period $\delta\in(0,T)$, the MPFC feedback applied to the robot is given by

$$u(t) = \left(w_1^\star(t,z(t_k)),\, w_2^\star(t,z(t_k))\right)^\top, \quad t\in[t_k,t_k+\delta). \tag{11}$$

Note that the initial condition z_k in the optimization is composed of the robot state $x(t_k)$ and the path parameter $\theta(t_k)$. Similar to Faulwasser and Findeisen (2009), the initial condition is $\theta(t_k)=\theta(t_k;\theta_{k-1},v^\star)$, i.e. the corresponding value of the last predicted trajectory. We remark that in contrast to standard NMPC schemes, this implies that the MPFC scheme is a *dynamic* feedback, cf. Faulwasser and Findeisen (2016). Furthermore, Z_ε from (5) requires the robot to follow the final part of the path exactly. However, OCP (10) does not involve any terminal constraint.

3. STABILITY AND PERFORMANCE BOUNDS

We first recall an NMPC stability result presented in Reble and Allgöwer (2012) in a slighly reformulated version, see Worthmann et al. (2015). Then, we show that, if this stability result is verified for the MPFC scheme, for the considered initial conditions, Problem 1 is solved.

We consider the nominal case (no plant-model mismatch). For a specific choice of δ and T, let the subscript $(\cdot)_{T,\delta}$ denote the NMPC closed-loop variables (states and inputs) of the augmented system (4), where $w^\star$ is applied in the sense of (11).

Theorem 1. Assume existence of a monotonically increasing and bounded function $B:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ satisfying, for all $z_0\in Z_\varepsilon$,

$$V_t(z_0)\leq B(t)\cdot\ell^\star(z_0) \qquad \forall\, t\geq 0 \tag{12}$$

where $\ell^\star(z_0):=\inf_{w\in W}\ell(z_0,w)$. Let the sampling period $\delta>0$ and the prediction horizon $T>\delta$ be chosen such that the condition $\alpha_{T,\delta}>0$ holds for

$$\alpha_{T,\delta} = 1 - \frac{e^{-\int_\delta^T B(t)^{-1}\,\mathrm{d}t}\cdot e^{-\int_{T-\delta}^T B(t)^{-1}\,\mathrm{d}t}}{\left[1-e^{-\int_\delta^T B(t)^{-1}\,\mathrm{d}t}\right]\left[1-e^{-\int_{T-\delta}^T B(t)^{-1}\,\mathrm{d}t}\right]}.$$

Then, for all $z\in Z_\varepsilon$, the relaxed Lyapunov inequality

$$V_T(z_{T,\delta}(\delta;z))\leq V_T(z)-\alpha_{T,\delta}\int_0^\delta \ell(z_{T,\delta}(t;z),w_{T,\delta}(t;z))\,\mathrm{d}t$$

as well as the performance estimate

$$V_\infty^{w_{T,\delta}}(z)\leq\alpha_{T,\delta}^{-1}\cdot V_\infty(z) \tag{13}$$

are satisfied, whereby $V_\infty^{w_{T,\delta}}(z)$ is the NMPC value function

$$V_\infty^{w_{T,\delta}}(z) := \int_0^\infty \ell(z_{T,\delta}(t;z),w_{T,\delta}(t;z))\,\mathrm{d}t.$$

[1] We refer to Martin et al. (2003) for more details on flatness.

[2] To avoid cumbersome technicalities, we assume that the minimum exists and is attained.

If, in addition, there exists $\mathcal{K}_\infty$-functions $\underline{\eta}, \bar{\eta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\underline{\eta}(\|z - z^\star\|) \leq \ell^\star(z) \leq \bar{\eta}(\|z - z^\star\|) \qquad \forall\, z \in Z_\varepsilon, \tag{14}$$

the NMPC closed loop is asymptotically stable. □

While condition (14) holds trivially for the chosen running cost (9), it is crucial to verify the growth condition (12). The next result provides the connection between Theorem 1 and Problem 1.

Proposition 2. Let δ, T in MPFC scheme based on OCP (10) be chosen such that the conditions of Theorem 1 hold. Then, the MPFC feedback (11) solves Problem 1. □

Proof. Recursive feasibility of the optimization follows from the observation that the state constraint set Z_ε can be rendered forward invariant by means of inputs satisfying the input constraint W. Furthermore, the chosen running cost (9) implies that, as z goes to $z^\star$, the robot state converges to the path and θ converges to 0. Hence, stabilizing z at $z^\star$ in an admissible way implies solving Problem 1. ■

In the next section, we show a method to construct a function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that all assumptions of Theorem 1 hold. In our construction, we exploit the structure of the state constraint Z_ε from (5), i.e. we rely on the restriction of the robot motion close to the final equilibrium point. We remark that the simplification induced by the structure of Z_ε is inline with observations in Boccia et al. (2014), wherein it is shown that the local analysis of growth bounds can be difficult.

4. COST CONTROLLABILITY ANALYSIS OF MPFC

Here, we present a method to compute a stabilizing horizon length for model predictive path following control (MPFC). This is achieved by deriving the growth bound (12) of Theorem 1 for the MPFC scheme of Section 2.2. The following theorem summarizes the main result of this section.

Theorem 3. Let $\varepsilon > 0$ be given and let the relation $r_2 \leq \frac{q_3}{2}$ hold. Then, for all $z_0 \in Z_\varepsilon$, the function $B \in \mathcal{PC}([0,\infty), \mathbb{R}_{\geq 0})$, $t \mapsto \min\{t, \int_0^t c(s)\,\mathrm{d}s\}$ with the function $c(t)$ defined by

$$c(t) = \begin{cases} 1 + \dfrac{q_3\pi^2(t^2 + 6t_r t + 0.5)}{4t_r^2 \cdot \hat{q}\varepsilon^2} & t \in \mathcal{I}_I \\ 1 + \dfrac{q_3(3\pi/2)^2 + r_1\bar{u}_1^2}{\hat{q}\varepsilon^2} & t \in \mathcal{I}_{II} \\ 1 + \dfrac{q_3\pi^2((4t_r + t_l - t)^2 + 0.5)}{4t_r^2 \cdot \hat{q}\varepsilon^2} & t \in \mathcal{I}_{III} \\ t_v^{-2}((4t_r + t_l + t_v - t)^2 + \hat{r}\hat{q}^{-1}) & t \in \mathcal{I}_{IV} \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

whereby

$$\mathcal{I}_I = [0, t_r), \qquad \mathcal{I}_{II} = [t_r, t_r + t_l),$$
$$\mathcal{I}_{III} = [t_r + t_l, 4t_r + t_l), \quad \mathcal{I}_{IV} = [4t_r + t_l, 4t_r + t_l + t_v),$$

and

$$t_r = \pi/(2\bar{u}_2), \quad t_l = \sqrt{4\bar{x}_2^2 + (\bar{\theta} + \varepsilon)^2}/\bar{u}_1, \quad t_v = |\bar{\theta}|/\hat{v}$$

with $\hat{v}$ given by (7), satisfies the growth bound (12) of Theorem 1 for ℓ from (9). □

Proof. Firstly, we construct a bounded function $\tilde{B}$, which already satisfies Inequality (12). Moreover, since the constant input function $w \equiv 0_{\mathbb{R}^3}$ is admissible on the infinite time horizon, Inequality (12) is also satisfied for the identity $\hat{B}(t) = t$. In summary, the function B defined in Theorem 3 will yield Inequality (12) for all $z_0 \in Z_\varepsilon$.

We first consider initial conditions

$$z_0 = (x_{1,0}, x_{2,0}, x_{3,0}, \theta_0)^\top \in X_\varepsilon \cup [\bar{\theta}, -\varepsilon] \subsetneq Z_\varepsilon. \tag{16}$$

We derive an admissible (open-loop) control function $w_{z_0} \in \mathcal{W}_\infty(z_0)$ steering the robot, in finite time, to the end point of the considered path, i.e. to $p(0) = 0$. This function w_{z_0} yields (suboptimal) running cost $\ell(z(t; z_0, w_{z_0}), w_{z_0}(t))$. Then, we uniformly estimate the quotient

$$\frac{\ell(z(t; z_0, w_{z_0}), w_{z_0}(t))}{\ell^\star(z_0)} \leq c(t) \qquad \forall\, z \in X_\varepsilon \cup [\bar{\theta}, -\varepsilon] \tag{17}$$

for all $t \geq 0$ with $c \in \mathcal{PC}(\mathbb{R}_{>0}, \mathbb{R}_{\geq 0})$. Then, since the maneuver is carried out in finite time and, thus, there exists a $\bar{t} > 0$ such that $c(t) = 0$ for all $t \geq \bar{t}$, and a bounded growth function $\tilde{B}$ given by $\tilde{B}(t) := \int_0^t c(\tau)\,\mathrm{d}\tau$.

As the robot can move forward and backward, it suffices to consider the case $x_{2,0} < \rho(\theta_0)$. Moreover, note that $\ell^\star(z_0)$ is given by

$$q_1(x_{1,0} - \theta_0)^2 + q_2(x_{2,0} - \rho(\theta_0))^2 + q_3(x_{3,0} - \hat{x}_3)^2 + \hat{q}\theta_0^2,$$

with angle $\hat{x}_3 = \arctan\left(\frac{\partial\rho}{\partial\theta}(\theta_0)\right) \in (-\pi/2, \pi/2)$. We emphasize that using the virtual input $v = 0$ implies that $u_{i,ref}(\theta, v) = 0$, $i \in \{1, 2\}$. It's important to keep this in mind for the subsequent construction of the function (15). All initial conditions (16) satisfy $\theta_0 \leq -\varepsilon$. Hence, $\ell^\star(z_0)$ is uniformly bounded from below by $\hat{q}\varepsilon^2$.

The blueprint for the construction of the control function w_{z_0} can be summarized as follows, see also Fig. 1:

I. Define the angle
$$\phi_{z_0} = \operatorname{atan2}\left(\tfrac{\rho(\theta_0) - x_{2,0}}{\theta_0 - x_{1,0}}\right) \in [-\pi, \pi)$$
and turn the robot such that the condition $x_3(t) = \phi_{z_0}$ or $x_3(t) = \phi_{z_0} - \pi$ is satisfied. At the end of this step, the robot points towards (or in the opposite direction to) the path point $(\theta_0, \rho(\theta_0))^\top$.

II. Drive directly (forward or backwards) until the robot reaches the path at $(\theta_0, \rho(\theta_0))^\top$.

III. Turn the robot until its orientation becomes tangent to the path at the selected point, i.e. until angle $\hat{x}_3$ is reached.

IV. Drive the robot along the path until the end of the path, i.e. until $\theta = 0$.

The time needed to complete this maneuver depends on the constraints X_0 and W: the minimal time required to turn the vehicle by 90 degrees is t_r, the minimal time required to drive the vehicle between the two corners of the box $[\bar{\theta}, -\varepsilon] \times [-\bar{x}_2, \bar{x}_2]$ is t_l, and the minimum time to drive the path variable θ, with a constant control v, from its limit $\bar{\theta}$ to 0 is t_v. The proposed maneuver actions I–IV are as follows.

Part I. First, the vehicle turns until time t_r such that $x_3(t_r) = \phi_{z_0}$ or $x_3(t_r) = \phi_{z_0} - \pi$. This is achieved by applying the constant input $w_{z_0}(t) \in W$, $t \in [0, t_r)$

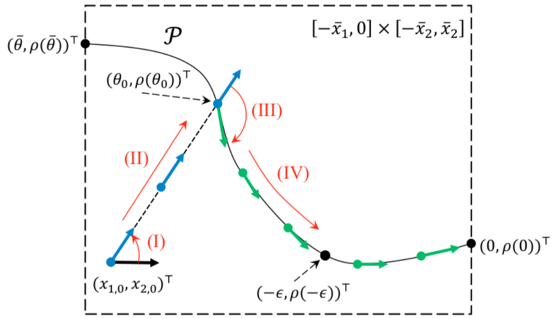

Fig. 1. Proposed maneuver for initial conditions characterized by (16).

$$w_{z_0} \equiv \begin{cases} \pm(0, (\phi_{z_0} - x_{3,0})t_1^{-1}, 0)^\top & \text{if } |\phi_{z_0} - x_{3,0}| \le \dfrac{\pi}{2}, \\ \pm(0, (\phi_{z_0} - \pi - x_{3,0})t_1^{-1}, 0)^\top & \text{otherwise,} \end{cases}$$

where the control sign is adjusted such that the robot rotation is achieved in the proper direction, i.e. clock wise or counter clock wise. This yields, for $\ell(z(t; z_0, w_{z_0}), w_{z_0}(t))$, the expression [3]

$$\ell^\star(z_0) + q_3 \left[\left(\frac{(t_r - t)x_{3,0} + t\phi_{z_0}}{t_r} - \hat{x}_3 \right)^2 - (x_{3,0} - \hat{x}_3)^2 \right] + r_2 \left(\frac{\phi_{z_0} - x_{3,0}}{t_r} \right)^2 .$$

When expanding the square bracket, the terms $(t/t_r)^2(\phi_{z_0} - x_{3,0})^2$ and $(2t/t_r)(\phi_{z_0} - x_{3,0})(x_{3,0} - \hat{x}_3)$ appear. Since $|\phi_{z_0} - x_{3,0}| \le \pi/2$, the term $(\phi - x_{3,0})^2$ is upper bounded by $(\pi/2)^2$. Moreover, it is straightforward to show that the product $(\phi_{z_0} - x_{3,0})(x_{3,0} - \hat{x}_3)$ is bound from above by $(\pi/2 \cdot 3\pi/2)$. Using these upper bounds along with our estimate for $\ell^\star(z_0)$ and the assumption on r_2, Inequality (17) holds for $t \in [0, t_r)$ with c defined by (15).

<u>Part II.</u> Next, the vehicle drives to the point $(\theta_0, \rho(\theta_0))^\top$ until time $t_r + t_l$ with a constant control $w(t)$, $t \in [t_r, t_r + t_l)$, defined as

$$w_{z_0} \equiv \begin{cases} \left(Dt_l^{-1}, 0, 0\right)^\top & \text{if } |\phi_{z_0} - x_{3,0}| \le \dfrac{\pi}{2} \\ -\left(Dt_l^{-1}, 0, 0\right)^\top & \text{otherwise} \end{cases}$$

with $D = \sqrt{(x_{1,0} - \theta_0)^2 + (x_{2,0} - \rho(\theta_0))^2}$. This yields equality of $\ell(z(t; z_0, w_{z_0}), w_{z_0}(t))$ and

$$((t_r + t_l - t)t_l^{-1})^2 \left[q_1(x_{1,0} - \theta_0)^2 + q_2(x_{2,0} - \rho(\theta_0))^2 \right] + q_3(\phi_{z_0} - \hat{x}_3)^2 + \hat{q}\theta_0^2 + \frac{r_1}{t_l^2} D^2$$

for $t \in [t_r, t_r + t_l)$. Since $|\phi_{z_0} - \hat{x}_3| \le 3\pi/2$, and the magnitude of the control effort is smaller than $\bar{u}_1$, Inequality (17) holds with $c(t)$ defined by (15).

<u>Part III.</u> The vehicle turns until its orientation becomes $\hat{x}_3$ at time $4t_r + t_l$, i.e. a turn of a maximum 270 degrees. This is achieved using the (constant) control $w_{z_0} \equiv (0, -(\phi_{z_0} - \hat{x}_3)(3t_r)^{-1}, 0)^\top$, $t \in [t_r + t_l, 4t_r + t_l)$. Thus, we have $\ell(z(t; z_0, w_{z_0}), w_{z_0}(t))$ given by

$$q_3 \left[(\phi_{z_0} - \hat{x}_3) \left(\frac{4t_r + t_l - t}{3t_r} \right) \right]^2 + \hat{q}\theta_0^2 + r_2 \left[\frac{(\phi_{z_0} - \hat{x}_3)}{-3t_r} \right]^2 .$$

[3] Without loss of generality, the term $(\phi_{z_0} - x_{3,0})$ can be replaced by $(\phi_{z_0} - \pi - x_{3,0})$ in the considered $\ell(\cdot, \cdot)$ because the norm of either term is upper bounded by $\pi/2$.

Similar to the considerations above, we use $|\phi_{z_0} - \hat{x}_3| \le 3\pi/2$. Thus, using the assumption on r_2 and $\ell^\star(z_0) \ge \hat{q}\varepsilon^2$, ensures (17) with $c(t)$ defined by (15).

<u>Part IV.</u> Finally, the robot drives along the path until it reaches its end point, i.e. $z^\star$, at time $4t_r + t_l + t_v$. This is achieved with $w_{z_0}(t) = (u_{1,ref}, u_{2,ref}, -\theta_0 t_v^{-1})^\top$, $t \in [4t_r + t_l, 4t_r + t_l + t_v)$, where $u_{i,ref}$, $i \in \{1, 2\}$, has the argument $(z_4(t; z_0, w_{z_0}), -\theta_0 t_v^{-1})$ at time t. This results in running cost $\ell(z(t; z_0, w_{z_0}), w_{z_0}(t))$ given by

$$\hat{q} \left(\frac{\theta_0(2t_r + t_l + t_v - t)}{t_v} \right)^2 + \hat{r} \left(\frac{\theta_0}{t_v} \right)^2 .$$

Then, Inequality (17) is ensured with $c(t)$ given by (15).

Moreover, for $t \ge 2t_r + t_l + t_v$, the function $c(t)$ is defined as $c(t) = 0$. Note that the function c is independent of the particular initial condition z_0. Additionally, the proposed maneuver ensures the satisfaction of the state constraints, i.e. $z(t; z_0, w_{z_0}) \in Z_\varepsilon$ and the selection of the times t_l, t_r, and t_v ensures satisfaction of the control constraints. Clearly, c is a piecewise continuous function with at most four points of discontinuity. Additionally, since $c(t) = 0$ for all $t \ge 2t_r + t_l + t_v$, it is bounded and integrable on $t \in [0, \infty)$. Therefore, the growth bound B is well defined.

Furthermore, for initial conditions $z_0 \in Z_0$ with $\theta_0 > -\varepsilon$, the proposed maneuver implies waiting ($w_{z_0} \equiv 0$ on $[0, 4t_r + t_l)$ before the mobile robots travels along the final segment of the path. Again, the constructed function c satisfies (17). In conclusion, the deduced function B satisfies Condition (12) for all $z_0 \in Z_0$. ■

The idea of using an integrable function, which is zero after a finite time interval goes back to Grüne (2009), see Reble and Allgöwer (2012) for the continuous-time case. We remark that, for given $\delta > 0$, the existence of a prediction horizon $T > 0$ such that the stability condition $\alpha_{T,\delta} > 0$ holds, is shown in Grüne et al. (2010).

5. NUMERICAL RESULTS

Next, using the bound B derived in Section 4, we first determine a horizon length $\hat{T}$ such that the MPFC closed loop is asymptotically stable. The horizon $\hat{T}$ is then employed for closed-loop simulations.

The derived growth function B of the previous section is employed to determine $\hat{T}$ which is defined as

$$\hat{T} = \min \{T > 0 \,|\, \exists\, N \in \mathbb{N} : T = N\delta \text{ and } \alpha_{T,\delta} > 0\} .$$

The horizon $\hat{T}$ depends on the constraints Z_0 and W, the sampling period δ, the weights of the running cost (9), the parameter ε, and the shape of the path to be followed. The path to be followed is given by $\rho(\theta) = 0.6\sin(0.25 \cdot \theta)$. The sampling rate δ is set as $\delta = 0.1$ (seconds). The constraints are $X_0 = [-20, 0] \times [-1, 1] \times \mathbb{R}$, $[\bar{\theta}, 0] = [-20, 0]$, $W = [-4, 4] \times [-\pi/2, \pi/2] \times [0, 4]$. Additionally, we choose the parameters of ℓ as $q_1 = q_2 = 10^4$, $q_3 = 0.01$, $r_1 = 0.1$, $r_2 = q_3/2$, and $\hat{r} = 0.1$ while $\hat{q}$ is investigated over the set $\{0.1, 0.2, 20\}$. For $\varepsilon = 2$, Fig. 2 (left) shows the effect of changing $\hat{q}$ on $\alpha_{T,\delta}$. As one can see, increasing $\hat{q}$ reduces the stabilizing prediction horizon length. However, setting $\hat{q}$ above 20 does not make a noticeable improvement to the stabilizing prediction horizon length. For $\hat{q} = 20$, Fig. 2

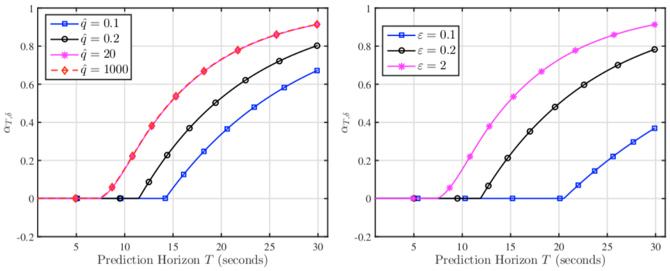

Fig. 2. Left: Effect of changing $\hat{q}$ on $\alpha_{T,\delta}$ for $\varepsilon = 2$. Right: Effect of changing ε on $\alpha_{T,\delta}$ for $\hat{q} = 20$.

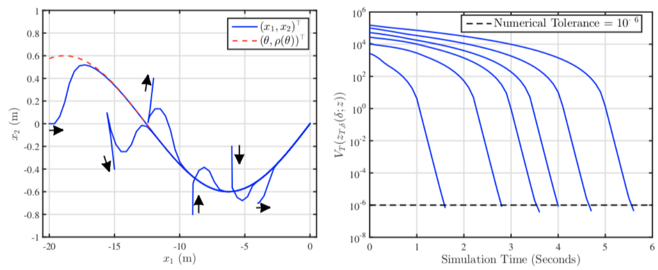

Fig. 3. Left: Closed-loop trajectories for six initial conditions. Right: Value function $V_T(t)$ for closed-loop trajectories. In all simulations: $\hat{q} = 20$, $\varepsilon = 2$, $T = 7.5$.

(right) shows the effect of changing ε. As shown, increasing ε reduces the stabilizing prediction horizon length.

Next, we use $\hat{T} = 7.5$ (seconds) for MPFC closed-loop simulations. Here, we investigate the MPFC scheme performance by considering six initial conditions of the robot. Following Faulwasser and Findeisen (2009), θ_0 is set as

$$\theta_0 = \underset{\theta \in [\bar{\theta}, -\varepsilon]}{\arg\min} \|x_0 - p(\theta)\|,$$

i.e. θ_0 is chosen such that the distance $\|x_0 - p(\theta)\|$ is minimized. All simulations have been run utilizing the interior-point optimization method provided by the IPOPT package, see Wächter and Biegler (2006), coupled with MATLAB via the CasADi toolbox, cf. Andersson (2013). Closed-loop trajectories are considered until the condition $V_T(z_{T,\delta}(\delta; z)) \leq 10^{-6}$ is met. Fig. 3 (left) shows the closed-loop trajectories exhibited by the robot for the considered initial conditions. In all cases, MPFC successfully steered the robot to the end point on the considered path. Moreover, as can be noticed from Fig. 3 (right), the evolution of the value function V_T exhibited a strictly monotonically decreasing behavior for all considered initial conditions. This demonstrates that the relaxed Lyapunov inequality presented in Theorem 1 is satisfied and, thus, the closed-loop stability is verified under the chosen prediction horizon length.

6. CONCLUSION

In this paper, asymptotic stability of model predictive path-following control (MPFC) for non-holonomic mobile robots and state-space paths has been rigorously proven without stabilizing terminal constraints or costs. We have demonstrated how the main assumptions, i.e. the growth bound on the value function, of the stability result can be derived. We also studied the effect of the tuning parameters of MPFC on the horizon length.

Future work aims at improving the estimates of the stabilizing horizon length obtained here. This will be achieved by adopting less conservative maneuvers in finding upper bounds on the optimal value function.

REFERENCES

Aguiar, A.P., Hespanha, J.P., and Kokotović, P.V. (2008). Performance limitations in reference tracking and path following for nonlinear systems. *Automatica*, 44(3), 598 – 610.

Alessandretti, A., Aguiar, A.P., and Jones, C. (2013). Trajectory-tracking and Path-following Controllers for Constrained Underactuated Vehicles using Model Predictive Control. In *Proc. European Control Conf.*, 1371–1376. Zürich, Switzerland.

Andersson, J. (2013). *A General-Purpose Software Framework for Dynamic Optimization*. PhD thesis, Arenberg Doctoral School, KU Leuven, Dept. Elect. Eng., Belgium.

Boccia, A., Grüne, L., and Worthmann, K. (2014). Stability and feasibility of state constrained MPC without stabilizing terminal constraints. *Syst. Control Lett.*, 72(8), 14–21.

Faulwasser, T. (2013). *Optimization-based Solutions to Constrained Trajectory-tracking and Path-following Problems*. Shaker, Aachen, Germany.

Faulwasser, T. and Findeisen, R. (2016). Nonlinear model predictive control for constrained output path following. *IEEE Trans. Automat. Control*, 61(4), 1026–1039.

Faulwasser, T., Hagenmeyer, V., and Findeisen, R. (2011). Optimal exact path-following for constrained differentially flat systems. In *Proc. 18th IFAC World Congr., Milano, Italy*, 9875–9880.

Faulwasser, T., Weber, T., Zometa, P., and Findeisen, R. (2016). Implementation of nonlinear model predictive path-following control for an industrial robot. *IEEE Trans. Contr. Syst. Techn. In press.* doi:10.1109/TCST.2016.2601624.

Faulwasser, T. and Findeisen, R. (2009). Nonlinear model predictive path-following control. In L. Magni, D. Raimondo, and F. Allgwer (eds.), *Nonlinear Model Predictive Control*, volume 384 of *Lecture Notes in Control and Inform. Sciences*, 335–343. Springer Berlin Heidelberg.

Grüne, L. (2009). Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems. *SIAM J. Control Optimization*, 48, 1206–1228.

Grüne, L., Pannek, J., Seehafer, M., and Worthmann, K. (2010). Analysis of unconstrained nonlinear MPC schemes with varying control horizon. *SIAM J. Control and Optimization*, 48(8), 4938–4962.

Lam, D., Manzie, C., and Good, M.C. (2013). Multi-axis model predictive contouring control. *Int. J. Control*, 86(8), 1410–1424.

Martin, P., Murray, R.M., and Rouchon, P. (2003). Flat systems, equivalence and trajectory generation.

Reble, M. and Allgöwer, F. (2012). Unconstrained model predictive control and suboptimality estimates for nonlinear continuous-time systems. *Automatica*, 48(8), 1812–1817.

Wächter, A. and Biegler, T.L. (2006). On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math. Programming*, 106(1), 25–57.

Worthmann, K., Mehrez, M.W., Zanon, M., Mann, G.K.I., Gosine, R.G., and Diehl, M. (2015). Regulation of Differential Drive Robots using Continuous Time MPC without Stabilizing Constraints or Costs. *IFAC-PapersOnLine: 5th IFAC Conf. Nonlinear Model Predictive Control, Seville, Spain*, 48(23), 129–135.

Worthmann, K., Mehrez, M.W., Zanon, M., Mann, G.K.I., Gosine, R.G., and Diehl, M. (2016). Model predictive control of nonholonomic mobile robots without stabilizing constraints and costs. *IEEE Trans. Control Syst. Technol.*, 24(4), 1394–1406.

Worthmann, K., Reble, M., Grüne, L., and Allgöwer, F. (2014). The role of sampling for stability and performance in unconstrained nonlinear model predictive control. *SIAM J. Control and Optimization*, 52(1), 581–605.

Yu, S., Li, X., Chen, H., and Allgöwer, F. (2015). Nonlinear model predictive control for path following problems. *Int. J. Robust and Nonlinear Control*, 25(8), 1168–1182.