Quantum walks and application to search problems

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Overview

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Random walks on a Line

Introduction

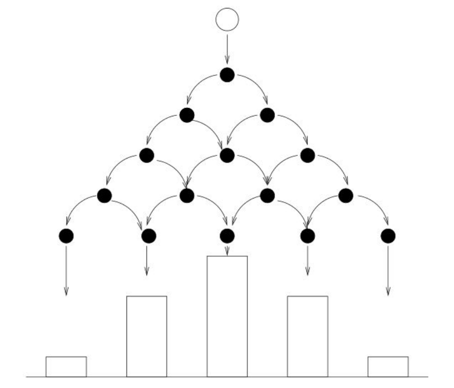
- Consider a line, with grid-length 1 and a particle starting off at $x = 0$
- The probability of moving left or right is $\frac{1}{2}$
- 

Figure 1: Classical distribution

Quantum walk on a line

- The basis states for $\mathcal{H}_P$ are $\{|i\rangle : i \in \mathbb{Z}\}$
- A coin space $\mathcal{H}_C$ augments $\mathcal{H}_P$, basis states for which are $\{|left\rangle, |right\rangle\}$. State for the total system $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$
- The conditional shift operator S, transforms $|left\rangle \otimes |i\rangle$ to $|left\rangle \otimes |i-1\rangle$ and $|right\rangle \otimes |i\rangle$ to $|right\rangle \otimes |i+1\rangle$
- Equivalently $S = \sum_i |right\rangle \otimes |i+1\rangle \langle right| \otimes \langle i| + \sum_i |left\rangle \otimes |i-1\rangle \langle left| \otimes \langle i|$

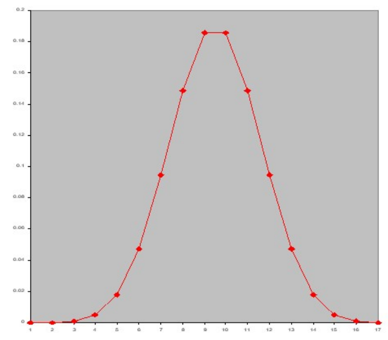
The Unitary

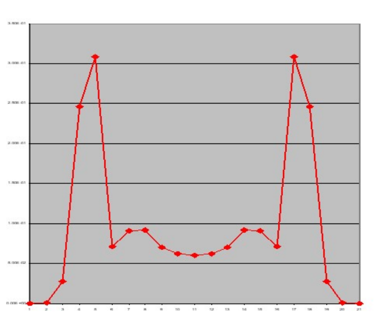
- Quantum walk for t steps is given by U^t, where $U = S \cdot (C \otimes I)$
- Coin flip acts just on $\mathcal{H}_C$, The choice for the coin is:

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

- For the starting state $|left\rangle |0\rangle \xrightarrow{S \cdot (C \otimes I)} \frac{1}{\sqrt{2}} |right\rangle |1\rangle + \frac{i}{\sqrt{2}} |left\rangle |-1\rangle$

Distribution Comparison





- Classical walk after T steps has a $\sigma = \Theta(\sqrt{T})$ whereas quantum walk has $\sigma = \Theta(T)$
- Quadratically faster propogation.

Quantum walk for database search

Quantum walk search algorithm

- Search space : n-bit binary strings, $x \in \{0,1\}^n$
- $f(\vec{x}) \in \{0,1\}$ is such that $f(x) = 1$ for exactly one input $\vec{x}_{target}$
- Goal: Find $\vec{x}_{target}$
- Using the mapping of n-bit binary string to nodes on the hypercube, the search problem is then equivalent to searching for a single marked node amongst the $N = 2^n$ nodes.

Shift and Coin operators

- $|d, \vec{x}\rangle \xrightarrow{S} |d, \vec{x} \oplus \vec{e_d}\rangle$

$$S = \sum_{d=0}^{n-1} \sum_{\vec{x}} |d, \vec{x} \oplus \vec{e_d}\rangle \langle d, \vec{x}| \tag{1}$$

- The coin operator to be applied is perturbed,

$$C_{pert} = C_0 \otimes I - (I + C_0) \otimes |\vec{0}\rangle \langle \vec{0}| \,. \tag{2}$$

Algorithm 1 Search with coin oracle

1: **procedure** FIND MARKED
2: $\psi_0 \leftarrow |s^C\rangle \otimes |s^S\rangle$ i.e equal superposition over all states
3: Compute $U_{pert}^t |\psi_0\rangle$ where $t = \frac{\pi}{2}\sqrt{2^n}$
4: Measure in $|d, \vec{x}\rangle$ basis

Outcome on measurement

- Wp $\frac{1}{2} - O(1/n)$, the measured outcome is the marked state.
- The proof for the earlier algorithm: is that the walk can be *approximately* mapped onto a 2D subspace.
- The approximations for the initial state and final state are $\pi/2$ apart and each application of U corresponds to a rotation angle of $1/\sqrt{2^{n-1}}$. So the search is completed after $\frac{\pi}{2}\sqrt{2^{n-1}}$ steps.
- Running time is $\mathcal{O}\left(\sqrt{N}\right)$

Quantum walk for Spatial search

Spatial search, Quadratic speedup for finding marked vertices

- Finding a marked vertex in a general graph, when multiple marked vertices exist was an open problem.
- Ambainis et al find an $\widetilde{\mathcal{O}}\left(\sqrt{HT}\right)$ algorithm.
- Key components for this result are **Interpolated walk** and **Quantum fast forwarding** this walk.

Ergodic chain, Discriminant Matrix

- $\mathcal{P}$ is *ergodic* if for a large enough $t \in \mathbb{N}$ all elements of $\mathcal{P}^t$ are non-zero.
- The corresponding *time-reversed* Markov chain is defined as $\mathcal{P}^* := \operatorname{diag}(\pi)^{-1} \cdot \mathcal{P}^T \cdot \operatorname{diag}(\pi)$. $\mathcal{P}$ is *reversible* if it is ergodic and $\mathcal{P}^* = \mathcal{P}$.
- Discriminant matrix D's xy-entry is $\sqrt{\mathcal{P}_{xy}\mathcal{P}^*_{yx}}$.

$$D = \operatorname{diag}(\pi)^{\frac{1}{2}} \cdot \mathcal{P} \cdot \operatorname{diag}(\pi)^{-\frac{1}{2}} . \tag{3}$$

Interpolated walk

- For $\mathcal{P}$ the marked set $M \subset X$. $\quad\mathcal{P}'$ is the absorbing markov chain.

$$\mathcal{P} := \begin{pmatrix} \mathcal{P}_{UU} & \mathcal{P}_{UM} \\ \mathcal{P}_{MU} & \mathcal{P}_{MM} \end{pmatrix}, \qquad \mathcal{P}' := \begin{pmatrix} \mathcal{P}_{UU} & \mathcal{P}_{UM} \\ 0 & I \end{pmatrix}.$$

- The *interpolated walk* operator, for $s \in [0, 1)$:
  $\mathcal{P}(s) := (1 - s)\mathcal{P} + s\mathcal{P}'$

Quantum Fast forwarding

The following result is by Apers and Sarlette:

Theorem

Let $\varepsilon \in (0,1)$*,* $s \in [0,1]$ *and* $t \in \mathbb{N}$*. Let* $\mathcal{P}$ *be any reversible Markov chain on state space* X*, and let* Q *be the cost of implementing the (controlled) quantum walk operator* $W(s)$*. There is a quantum algorithm with complexity* $\mathcal{O}\left(\sqrt{t\log(1/\varepsilon)}\mathsf{Q}\right)$ *that takes input* $|\bar{0}\rangle|\psi\rangle \in \mathrm{span}\{|\bar{0}\rangle|x\rangle : x \in X\}$*, and outputs a state that is* ε*-close to a state of the form*

$$|0\rangle^{\otimes a}|\bar{0}\rangle D^t|\psi\rangle + |\Gamma\rangle$$

where $a = \mathcal{O}(\log(t\log(1/\varepsilon)))$ *and* $|\Gamma\rangle$ *is some garbage state that has no support on states containing* $|0\rangle^{\otimes a}|\bar{0}\rangle$ *in the first two registers.*

Main result by Ambainis et al

Theorem

Let $\mathcal{P}$ be a reversible ergodic Markov chain, and let π be its stationary distribution. If $p_M \leq 1/9$ and $T \geq 3HT$, then choosing $s \in S = \{1 - \frac{1}{r} : r \in R\}$ and $t \in [24T]$ uniformly at random we get, that

$$\mathbb{E}\left[\left\|\Pi_M D^t(s)\,|\sqrt{\pi_U}\rangle\right\|^2\right] = \Omega\left(\frac{1}{\log(T)}\right).$$

FF-based search

Algorithm 2 Fast-forwarding-based search algorithm

Search($\mathcal{P}$, M, T)

Use $\mathcal{O}\left(\sqrt{\log(T)}\right)$ rounds amplitude amplification to amplify the success probability of steps 1-3:

1. Use `Setup`($\mathcal{P}$) to prepare the state
$$\sum_{t=1}^{T} \frac{1}{\sqrt{T}} |t\rangle \sum_{s \in S} \frac{1}{\sqrt{|S|}} |s\rangle |\sqrt{\pi}\rangle .$$
2. Measure $\{\Pi_M, I - \Pi_M\}$ on the last register. If the outcome is "marked", measure in the computational basis, and output the entry in the last register. Otherwise continue with the (subnormalized) post-measurement state
$$\sum_{t=1}^{T} \frac{1}{\sqrt{T}} |t\rangle \sum_{s \in S} \frac{1}{\sqrt{|S|}} |s\rangle |\sqrt{\pi_U}\rangle .$$
3. Use quantum fast-forwarding, controlled on the first two registers, to map $|t\rangle |s\rangle |\sqrt{\pi_U}\rangle$ to $|1\rangle |t\rangle |s\rangle D^t(s) |\pi_U\rangle + |0\rangle |\Gamma\rangle$ for some arbitrary $|\Gamma\rangle$, with precision $\mathcal{O}\left(\frac{1}{\log(T)}\right)$. Finally, measure the last register and output its content if marked, otherwise output `No marked vertex`.

Runtime

The complexity of steps 1-3 is $\mathcal{O}\left(\mathsf{S} + \sqrt{T \log\log(T)}(\mathsf{U}+\mathsf{C})\right)$
Amplitude amplification gives a $\sqrt{\log(T)}$ multiplicative overhead, giving the final complexity

$$\mathcal{O}\left(\mathsf{S}\sqrt{\log(T)} + \sqrt{T}(\mathsf{U}+\mathsf{C})\sqrt{\log(T)\log\log(T)}\right)$$

Which is $\widetilde{\mathcal{O}}\left(S + \sqrt{HT}(U+C)\right)$

References

[1] Andris Ambainis, András Gilyén, Stacey Jeffery, and Martins Kokainis. Quadratic speedup for finding marked vertices by quantum walks. *ArXiv*, abs/1903.07493, 2019.

[2] Neil Shenvi, J. Kempe, and K. Birgitta Whaley. Quantum random-walk search algorithm. 2003.