Unit-1:Determinants & Matrix

Introduction and Examples

DEFINITION: A matrix is defined as an ordered rectangular array of numbers. They can be used to represent systems of linear equations, as will be explained below.Here are a example $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -5 \\ 3 & -5 & 6 \end{bmatrix}$

Square

A **square** matrix has the same number of rows as columns.

$$\begin{bmatrix} 2 & 0 \\ 1 & 8 \end{bmatrix}$$

A square matrix (2 rows, 2 columns)

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \\ 3 & 0 & 7 \end{bmatrix}$$

Also a square matrix (3 rows, 3 columns)

Identity Matrix

An **Identity Matrix** has **1**s on the main diagonal and **0**s everywhere else:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3×3 Identity Matrix

- It is square (same number of rows as columns)
- It can be large or small (2×2, 100×100, ... whatever)
- Its symbol is the capital letter **I**

It is the matrix equivalent of the number "1", when we multiply with it the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Diagonal Matrix

A diagonal matrix has zero anywhere not on the main diagonal:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A diagonal matrix

Scalar Matrix

A scalar matrix has all main diagonal entries the same, with zero everywhere else:

Triangular Matrix

Lower triangular is when all entries above the main diagonal are zero:

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & 6 & -3 \end{bmatrix}$$

A lower triangular matrix

Upper triangular is when all entries below the main diagonal are zero:

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 4 & 11 \\ 0 & 0 & 5 \end{bmatrix}$$

An upper triangular matrix

Zero Matrix (Null Matrix)

Zeros just everywhere:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Zero matrix

Matrix Addition and Subtraction

DEFINITION: Two matrices A and B can be added or subtracted if and only if their dimensions are the same (i.e. both matrices have the same number of rows and columns. Take:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

Addition

If A and B above are matrices of the same type then the sum is found by adding the corresponding elements $a_{ij} + b_{ij}$.

Here is an example of adding A and B together.

$$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 2 & 0 & 5 \end{pmatrix}$$

Subtraction

If A and B are matrices of the same type then the subtraction is found by subtracting the corresponding elements $a_{ij} - b_{ij}$.

Here is an example of subtracting matrices.

$$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Matrix Multiplication

DEFINITION: When the number of columns of the first matrix is the same as the number of rows in the second matrix then matrix multiplication can be performed.

Here is an example of matrix multiplication for two 2×2 matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{pmatrix}$$

Here is an example of matrix multiplication for two 3×3 matrices.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{pmatrix}$$

Note: That A×B is not the same as B×A

Transpose of Matrices

DEFINITION: The transpose of a matrix is found by exchanging rows for columns i.e. Matrix A = (a_{ij}) and the transpose of A is:

A^T = (a_{ji}) where j is the column number and i is the row number of matrix A.

For example, the transpose of a matrix would be:

$$A = \begin{pmatrix} 5 & 2 & 3 \\ 4 & 7 & 1 \\ 8 & 5 & 9 \end{pmatrix} \quad A^T = \begin{pmatrix} 5 & 4 & 8 \\ 2 & 7 & 5 \\ 3 & 1 & 9 \end{pmatrix}$$

In the case of a square matrix (m = n), the transpose can be used to check if a matrix is symmetric. For a symmetric matrix A = A^T.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = A^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = A$$

The Determinant of a Matrix

DEFINITION: Determinants play an important role in finding the inverse of a matrix and also in solving systems of linear equations. In the following we assume we have a square matrix (m = n). The determinant of a matrix A will be denoted by det(A) or |A|. Firstly the determinant of a 2×2 and 3×3 matrix will be introduced, then the n×n case will be shown.

Determinant of a 2×2 matrix

Assuming A is an arbitrary 2×2 matrix A, where the elements are given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of a 3×3 matrix

The determinant of a 3×3 matrix is a little more tricky and is found as follows (for this case assume A is an arbitrary 3×3 matrix A, where the elements are given below).

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} -2 & 3 & -1 \\ 5 & -1 & 4 \\ 4 & -8 & 2 \end{vmatrix} = -2\begin{vmatrix} -1 & 4 \\ -8 & 2 \end{vmatrix} -5\begin{vmatrix} 3 & -1 \\ -8 & 2 \end{vmatrix}$$

$$+4\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix}$$

$$= -2[(-1)(2) - (-8)(4)] - 5[(3)(2)$$
$$-(-8)(-1)] +4[(3)(4) \ -(-1)(-1)]$$

$$= -2(30) - 5(-2) + 4(11)$$

$$= -60 + 10 + 44$$

$$= -6$$

Here, we are **expanding by the first column.** We can do the expansion by using the first row and we will get the same result.

The Inverse of a Matrix

DEFINITION: Assuming we have a square matrix A, which is non-singular (i.e. det(A) does not equal zero), then there exists an n×n matrix A^{-1} which is called the inverse of A, such that this property holds:

$AA^{-1} = A^{-1}A = I$, where I is the identity matrix.

The inverse of a 2×2 matrix

Take for example a arbitury 2×2 Matrix A whose determinant (ad − bc) is not equal to zero.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a,b,c,d are numbers, The inverse is:

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse Matrix Method

DEFINITION: The inverse matrix method uses the inverse of a matrix to help solve a system of equations, such like the above Ax = b. By pre-multiplying both sides of this equation by A^{-1} gives:

$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b$$

or alternatively

$$x = A^{-1}b$$

So by calculating the inverse of the matrix and multiplying this by the vector b we can find the solution to the system of equations directly. And from earlier we found that the inverse is given by

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

From the above it is clear that the existence of a solution depends on the value of the determinant of A. There are three cases:

1. If the det(A) does not equal zero then solutions exist using

2. If the det(A) is zero and b=0 then the solution will be not be unique or does not exist.
3. If the det(A) is zero and b=0 then the solution can be x = 0 but as with 2. is not unique or does not exist.

Looking at two equations we might have that

$$ax + by = c$$
$$dx + ey = f$$

Written in matrix form would look like

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

and by rearranging we would get that the solution would look like

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}\begin{pmatrix} c \\ f \end{pmatrix}$$

Cofactors

The 2 × 2 determinant

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

is called the **cofactor** of a_1 for the 3 × 3 determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The cofactor is formed from the elements that are not in the same row as a_1 and not in the same column as a_1.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Similarly, the determinant

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

is called the **cofactor** of a_2. It is formed from the elements not in the same row as a_2 and not in the same column as a_2.

We continue the pattern for the cofactor of a_3.

Cramer's Rule to Solve 3 × 3 Systems of Linear Equations

We can solve the general system of equations,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

by using the determinants:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\Delta}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\Delta}$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example 4

Solve, using Cramer's Rule:

$$2x + 3y + z = 2$$

$$-x + 2y + 3z = -1$$

$$-3x - 3y + z = 0$$

$$x = \frac{\begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 0 & -3 & 1 \end{vmatrix}}{\Delta}$$

$$y = \frac{\begin{vmatrix} 2 & 2 & 1 \\ -1 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix}}{\Delta}$$

$$z = \frac{\begin{vmatrix} 2 & 3 & 2 \\ -1 & 2 & -1 \\ -3 & -3 & 0 \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ -3 & -3 & 1 \end{vmatrix} = 2(11) + 1(6) - 3(7)$$
$$= 7$$

So

$$x = \frac{2(11) + 1(6) + 0}{7} = \frac{28}{7} = 4$$

$$y = \frac{2(-1) + 1(2) - 3(7)}{7} = -\frac{21}{7} = -3$$

$$z = \frac{2(-3) + 1(6) - 3(-7)}{7} = \frac{21}{7} = 3$$

Checking solutions:

[1] $2(4) + 3(\text{-}3) + 3 = 2$ OK

[2] $-(4) + 2(\text{-}3) + 3(3) = \text{-}1$ OK

[3] $-3(4) - 3(-3) + 3 = 0$ OK

So the solution is $(4, -3, 3)$.

Rank of a Matrix

The rank of a matrix with m rows and n columns is a number **r** with the following properties:

- r is less than or equal to the smallest number out of m and n.
- r is equal to the order of the greatest minor of the matrix which is not 0.

Determining the Rank of a Matrix

- We pick an element of the matrix which is not 0.
- We calculate the order 2 minors which contain that element until we find a minor which is not 0.
- If every order 2 minor is 0, then the rank of the matrix is 1.
- If there is any order 2 minor which is not 0, we calculate the order 3 minors which contain the previous minor until we find one which is not 0.

- If every order 3 minor is 0, then the rank of the matrix is 2.
- If there is any order 3 minor which is not 0, we calculate the order 4 minors until we find one which is not 0.
- We keep doing this until we get minors of an order equal to the smallest number out of the number of rows and the number of columns.

Example

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & 5 \end{pmatrix}$$

The matrix has 2 rows and 3 columns, so the greatest possible value of its rank is 2. We pick any element which is not 0.

$$\begin{pmatrix} \color{red}{1} & 2 & 4 \\ 3 & 6 & 5 \end{pmatrix}$$

We form an order 2 minor containing 1.

$$\begin{pmatrix} \color{red}{1} & \color{red}{2} & 4 \\ \color{red}{3} & \color{red}{6} & 5 \end{pmatrix}$$

We calculate this minor.

$$\begin{vmatrix} \color{red}{1} & \color{red}{2} \\ \color{red}{3} & \color{red}{6} \end{vmatrix} = 6 - 6 = 0$$

We form another order 3 minor containing 1. $A =$

$$\begin{pmatrix} \color{blue}{1} & 2 & \color{blue}{4} \\ \color{blue}{3} & 6 & \color{blue}{5} \end{pmatrix}$$

We calculate this minor.

$$\begin{vmatrix} \color{blue}{1} & \color{blue}{4} \\ \color{blue}{3} & \color{blue}{5} \end{vmatrix} = 5 - 12 = -7 \neq 0.$$

The rank is 2.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We pick an element which is not 0.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We calculate order 2 minors containing this element.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \text{ (because it has two equal rows)}$$

Every other order 2 minor is 0 because it's the same as the others. In this case, the rank of the matrix is 1.

Eigen Values and Eigen Vectors

Eigen vector of a matrix A is a vector represented by a matrix X such that when X is multiplied with matrix A, then the direction of the resultant matrix remains same as vector X.

Mathematically, above statement can be represented as:

$AX = \lambda X$

where A is any arbitrary matrix, λ are eigen values and X is an eigen vector corresponding to each eigen value.

Here, we can see that AX is parallel to X. So, X is an eigen vector.

Method to find eigen vectors and eigen values of any square matrix A
We know that,

$AX = \lambda X$

$\Rightarrow AX - \lambda X = 0$

$\Rightarrow (A - \lambda I) X = 0$ …..(1)

Above condition will be true only if $(A - \lambda I)$ is singular. That means,

$|A - \lambda I| = 0$ …..(2)

(2) is known as characteristic equation of the matrix.
null

The roots of the characteristic equation are the eigen values of the matrix A.

Now, to find the eigen vectors, we simply put each eigen value into (1) and solve it by Gaussian elimination, that is, convert the augmented matrix $(A - \lambda I) = 0$ to row echelon form and solve the linear system of equations thus obtained.

Some important properties of eigen values

- Eigen values of real symmetric and hermitian matrices are real
- Eigen values of real skew symmetric and skew hermitian matrices are either pure imaginary or zero
- Eigen values of unitary and orthogonal matrices are of unit modulus $|\lambda| = 1$

- If $\lambda_{1,} \lambda_2 \ldots\ldots\lambda_n$ are the eigen values of A, then $k\lambda_1, k\lambda_2 \ldots\ldots k\lambda_n$ are eigen values of kA
- If $\lambda_{1,} \lambda_2 \ldots\ldots\lambda_n$ are the eigen values of A, then $1/\lambda_1, 1/\lambda_2 \ldots\ldots 1/\lambda_n$ are eigen values of A^{-1}
- If $\lambda_{1,} \lambda_2 \ldots\ldots\lambda_n$ are the eigen values of A, then $\lambda_1^k, \lambda_2^k \ldots\ldots \lambda_n^k$ are eigen values of A^k
  null
- Eigen values of A = Eigen Values of A^T (Transpose)
- Sum of Eigen Values = Trace of A (Sum of diagonal elements of A)
- Product of Eigen Values = $|A|$
- Maximum number of distinct eigen values of A = Size of A
- If A and B are two matrices of same order then, Eigen values of AB = Eigen values of BA

Note –Eigenvalues and eigenvectors are only for square matrices.

Eigenvectors are *by definition nonzero*. Eigenvalues may be equal to zero.

EXAMPLE 1: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

SOLUTION:

- In such problems, we first find the **eigenvalues** of the matrix.

FINDING EIGENVALUES

- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A, namely those values of λ for which

  $$\det(A - \lambda I) = 0,$$

  where I is the 3×3 **identity matrix**.

- Form the matrix $A - \lambda I$:

  $$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}.$$

  Notice that this matrix is just equal to A with λ subtracted from each entry on the main diagonal.

- Calculate $\det(A - \lambda I)$:

  $$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} \\ &= (1-\lambda)\left((-5-\lambda)(4-\lambda) - (3)(-6)\right) + 3(3(4-\lambda) - 3\times 6) + 3(3\times(-6) - (-5-\lambda)6) \\ &= (1-\lambda)(-20 + 5\lambda - 4\lambda + \lambda^2 + 18) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) \\ &= (1-\lambda)(-2 + \lambda + \lambda^2) + 3(-6 - 3\lambda) + 3(12 + 6\lambda) \\ &= -2 + \lambda + \lambda^2 + 2\lambda - \lambda^2 - \lambda^3 - 18 - 9\lambda + 36 + 18\lambda \\ &= 16 + 12\lambda - \lambda^3. \end{aligned}$$

- Therefore

  $$\det(A - \lambda I) = -\lambda^3 + 12\lambda + 16.$$

  REQUIRED: To find solutions to $\det(A - \lambda I) = 0$ i.e., to solve

  $$\lambda^3 - 12\lambda - 16 = 0. \tag{1}$$

  * Look for **integer** valued solutions.

* Such solutions **divide** the **constant** term (-16). The list of possible integer solutions is

$$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$

* Taking $\lambda = 4$, we find that $4^3 - 12.4 - 16 = 0$.
* Now factor out $\lambda - 4$:

$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = \lambda^3 - 12\lambda^2 + 16.$$

* Solving $\lambda^2 + 4\lambda + 4$ by formula[1] gives

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4.1.4}}{2} = \frac{-4 \pm 0}{2},$$

  and so $\lambda = -2$ (a repeated root).

- Therefore, the eigenvalues of A are $\lambda = 4, -2$. ($\lambda = -2$ is a repeated root of the **characteristic equation**.)

FINDING EIGENVECTORS

- Once the **eigenvalues** of a matrix (A) have been found, we can find the **eigenvectors** by Gaussian Elimination.
- **STEP 1**: For each eigenvalue λ, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$

  where x is the **eigenvector** associated with **eigenvalue** λ.

- **STEP 2**: Find $\mathbf{x}$ by Gaussian elimination. That is, convert the augmented matrix

$$\left(A - \lambda I \vdots \mathbf{0}\right)$$

  to row echelon form, and solve the resulting linear system by back substitution.

We find the **eigenvectors** associated with each of the **eigenvalues**

- **Case 1:** $\lambda = 4$

  - We must find vectors $\mathbf{x}$ which satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

[1] To find the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ (with $a \neq 0$) first compute $\Delta = b^2 - 4ac$, then if $\Delta \geq 0$ the roots exist and are equal to $x = \frac{-b-\sqrt{\Delta}}{2a}$ and $x = \frac{-b+\sqrt{\Delta}}{2a}$.

- First, form the matrix $A - 4I$:

$$A - 4I = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}.$$

- Construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{array}{c} \begin{pmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \xrightarrow{R1 \to -1/3 \times R3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow[R3 \to R3 - 6 \times R1]{R2 \to R2 - 3 \times R1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow{R2 \to -1/12 \times R2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow{R3 \to R3 + 12 \times R2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow{R1 \to R1 - R2} \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \end{array}$$

- Rewriting this augmented matrix as a linear system gives

$$\begin{aligned} x_1 - 1/2x_3 &= 0 \\ x_2 - 1/2x_3 &= 0 \end{aligned}$$

So the eigenvector $\mathbf{x}$ is given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = \frac{x_3}{2} \\ x_2 = \frac{x_3}{2} \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

For any real number $x_3 \neq 0$. Those are the **eigenvectors of A associated with the eigenvalue** $\lambda = 4$.

- **Case 2**: $\lambda = -2$

  - We seek vectors $\mathbf{x}$ for which $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
  - Form the matrix $A - (-2)I = A + 2I$

$$A + 2I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}.$$

- Now we construct the augmented matrix $\left(A - \lambda I \vdots \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{pmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \xrightarrow{R1 \to 1/3 \times R3} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow{\substack{R2 \to R2 - 3 \times R1 \\ R3 \to R3 - 6 \times R1}} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

- When this augmented matrix is rewritten as a linear system, we obtain

$$x_1 + x_2 - x_3 = 0,$$

so the eigenvectors $\mathbf{x}$ associated with the eigenvalue $\lambda = -2$ are given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$$

- Thus

$$\mathbf{x} = \begin{pmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for any } x_2, x_3 \in \mathbb{R}\backslash\{0\}$$

are the **eigenvectors of** A **associated with the eigenvalue** $\lambda = -2$.

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