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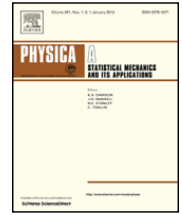
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# Dynamical generalized Hurst exponent as a tool to monitor unstable periods in financial time series

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## ABSTRACT

We investigate the use of the Hurst exponent, dynamically computed over a weighted moving time-window, to evaluate the level of stability/instability of financial firms. Financial firms bailed-out as a consequence of the 2007–2008 credit crisis show a neat increase with time of the generalized Hurst exponent in the period preceding the unfolding of the crisis. Conversely, firms belonging to other market sectors, which suffered the least throughout the crisis, show opposite behaviors. We find that the multifractality of the bailed-out firms increase at the crisis suggesting that the multi fractal properties of the time series are changing. These findings suggest the possibility of using the scaling behavior as a tool to track the level of stability of a firm. In this paper, we introduce a method to compute the generalized Hurst exponent which assigns larger weights to more recent events with respect to older ones. In this way large fluctuations in the remote past are less likely to influence the recent past. We also investigate the scaling associated with the tails of the log-returns distributions and compare this scaling with the scaling associated with the Hurst exponent, observing that the processes underlying the price dynamics of these firms are truly multi-scaling.

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## 1. Introduction

The search for scaling behaviors in financial markets is nowadays a very rich discipline [1–11] where the growing amount of empirical data is continuously advancing the understanding of markets behaviors. Two types of scaling [12,13] are observed and studied in the finance literature: the first one is associated with any volatility measure and its scaling in time (e.g. moments of the returns distribution), while the second one reflects the behavior of the tails of the distribution of returns. In this paper we look at both of them and at the relationship between the two by using the generalized Hurst exponent (GHE) approach. Previous works [14,15] have highlighted that the value of the GHE allows one to characterize the stage of development of a market, with values of the GHE greater than 0.5 indicating a low stage of development, typical of the emerging markets, while values of the GHE lower than 0.5 correspond to an advanced stage of development. Here we study whether the same paradigm can be applied to characterize the level of stability of a firm. To this purpose we introduce a weighted average to compute the dynamical generalized Hurst exponent obtaining a finer differentiation in the historical time series by smoothing the propagation of large fluctuations from the remote past to the near present. Although multi-scaling analysis based on the GHE has been already extensively pursued in the literature [9–11,14–20], the dynamics of the GHE has been scarcely investigated [21]. From a technical perspective a dynamical study of multi-scaling properties is very

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challenging because one must analyze a long enough time-period to obtain a reliable statistics but simultaneously the time-period must be short enough to catch changes occurring at given times. In this work we have used an exponential smoothing on a moving time-window and studied the best combination of window sizes and exponential weights that satisfies both these requirements. In particular, our analyses have been focused in determining whether the GHE may be used to track the stability of firms from several market sectors. The data are from 395 stock prices of companies listed in the New York Stock Exchange (NYSE) and have been provided by Reuters. We have analyzed several companies belonging to different market sectors but we have focused our attention on the companies most severely involved in the unfolding of the 2007–2008 “credit crunch” crisis. The scaling analysis based on the estimation of the GHE is also compared to the one associated with the behavior of the tails of the distribution.

This paper is structured as follows: Section 2 recalls the definition of the GHE; Section 3 describes the weighted-average algorithm; in Section 4 the empirical analysis is performed and a proper choice of the parameters of the system is discussed; Section 5 introduces the scaling of the distributions of the returns whose relations to the GHE is reported in Section 6; in Section 7 we provide significance tests to assess the validity of our conclusions; a summary and outlook are drawn in Section 8.

## 2. Generalized Hurst exponent

The generalized Hurst exponent is a tool to study directly the scaling properties of the data via the  $q$ th-order moments of the distribution of the increments and it is associated with the long-term statistical dependence of a certain time series  $S(t)$ , with  $t = (1, 2, \dots, k, \dots, \Delta t)$ , defined over a time-window  $\Delta t$  with unitary time-steps.<sup>1</sup> Being a measure of correlation persistence, it is necessarily related to fundamental statistical quantities which turn out to be the  $q$ th-order moments of the distribution of the increments, defined as Refs. [14,22]

$$K_q(\tau) = \frac{\langle |S(t+\tau) - S(t)|^q \rangle}{\langle |S(t)|^q \rangle}, \quad (1)$$

where  $\tau$  can vary between 1 and  $\tau_{\max}$  and  $\langle \cdot \rangle$  denotes the sample average over the time-window. Note that for  $q = 2$ ,  $K_q(\tau)$  is proportional to the autocorrelation function:  $C(t, \tau) = \langle S(t+\tau)S(t) \rangle$ . The generalized Hurst exponent is then defined from the scaling behavior of  $K_q(\tau)$  when the following relation holds:

$$K_q(\tau) \propto \tau^{qH(q)}. \quad (2)$$

Processes exhibiting this scaling behavior can be divided into two classes: (i) Processes with  $H(q) = H$ , i.e. independent of  $q$ . These processes are unscaling (or unifractal) and their scaling behavior is uniquely determined by the constant  $H$  (Hurst exponent or self-affine index [14]). (ii) Processes with  $H(q)$  not constant. These processes are called multiscaling (or multifractal) and each moment scales with a different exponent. Previous works have pointed out how financial time series exhibit scaling behaviors which are not simply fractal, rather multi-fractal, or multiscaling [14,15]. The GHE is computed from an average over a set of values corresponding to different values of  $\tau_{\max}$  in Eq. (1) [15,16,23]. The analysis based on the generalized Hurst exponent is very simple as all the information about the scaling properties of a signal is enclosed in the scaling exponent  $H(q)$ .

## 3. Weighted exponential smoothing

To take into account the fact that the recent past is more important than the remote past we can assume that the informational relevance of observations decays exponentially. This ‘exponential smoothing’ is attained by defining weights as

$$w_s = w_0 \exp\left(-\frac{s}{\theta}\right), \quad \forall s \in \{0, 1, 2, \dots, \Delta t - 1\} \quad (3)$$

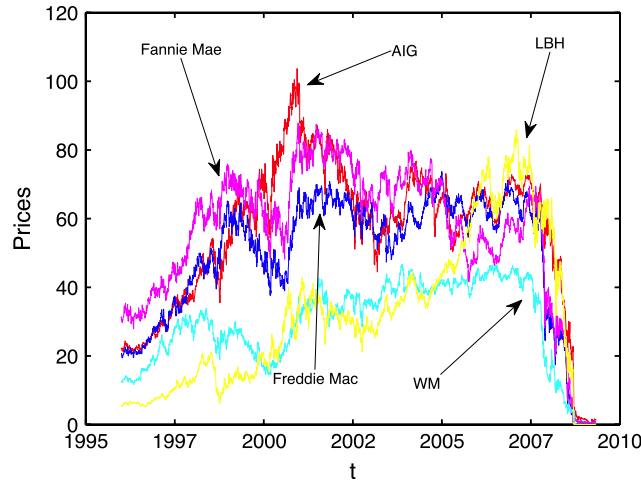
where  $\theta$  is the weights’ characteristic time and its inverse is the exponential decay factor  $\alpha = \frac{1}{\theta}$ . The parameter  $w_0$  is given by Ref. [24]

$$w_0(\alpha) = \frac{1 - e^{-\alpha}}{1 - e^{-\alpha\Delta t}}. \quad (4)$$

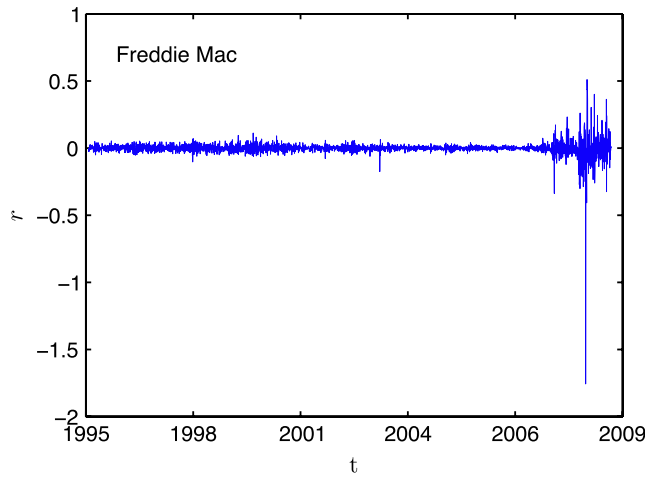
The weighted average over the time-window  $[t - \Delta t + 1, t]$  for a general quantity  $f(x_t)$  is thus

$$\langle f \rangle_w(t) = \sum_{s=0}^{\Delta t-1} w_s f(x_{t-s}) \quad (5)$$

<sup>1</sup> Here, to simplify notation, we use unitary time-steps; generalization to arbitrary time-steps is straightforward.



**Fig. 1.** Prices of four bailed-out companies Fannie Mae, American International Group (AIG), Freddie Mac, Washington Mutual (WM) plus Lehman Brothers (LBH) as function of time.



**Fig. 2.** Logarithmic returns,  $r$ , for the time series of the Freddie Mac stock prices as function of time  $t$  in the period between 1 January 1996 and 30 April 2009. The large fluctuations corresponding to the unfolding of the 2007–2008 crisis are clearly visible.

and the weighted GHE (wGHE) is therefore obtained by substituting the normal averages in Eq. (1) with weighted averages:

$$K_q^w(t, \tau) = \frac{\langle |S(t + \tau) - S(t)|^q \rangle_w(t)}{\langle |S(t)|^q \rangle_w(t)}. \quad (6)$$

From the scaling law in Eq. (2) this leads to the linear relation

$$\ln(K_q^w(t, \tau)) = qH^w(q) \ln(\tau) + \text{const} \quad (7)$$

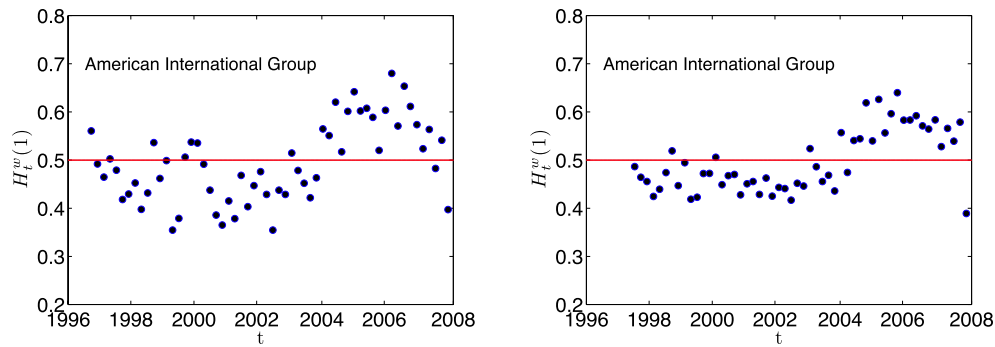
from which the wGHE can be computed. In the next section we apply this tool to the empirical time series.

#### 4. Empirical analysis

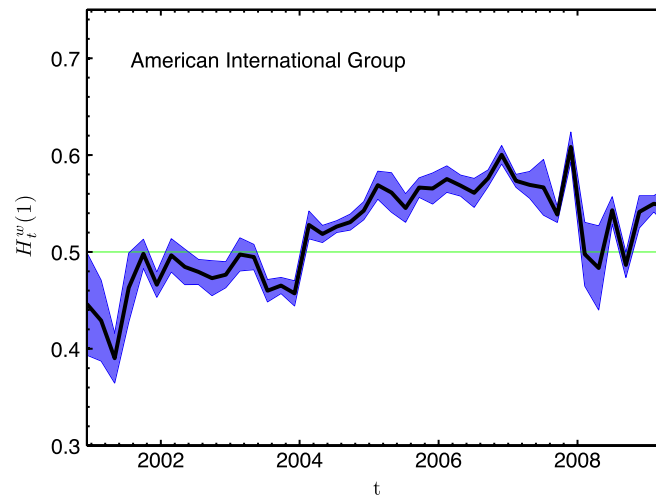
The empirical time series here analyzed include daily stock prices from 1 January 1996 to 30 April 2009 (see Fig. 1 where we plot the prices of Fannie Mae, American International Group (AIG), Freddie Mac, Washington Mutual Corp (WM) and Lehman Brothers Holdings (LBH)). From these prices we define a new time series of the daily log-returns

$$r(t) = \ln(P(t + 1)) - \ln(P(t)) \quad (8)$$

where  $P(t)$  is the daily price. In Fig. 2 an example of log-returns for the Freddie Mac stock price is shown. Not surprisingly, these returns exhibit large fluctuations in the crisis period. From the log-returns we have then computed the wGHE by using Eq. (7). In analogy with Refs. [16,15,23] we have estimated the  $H^w(q)$  as an average of several linear fits of Eq. (7) with  $\tau \in [1, \tau_{\max}]$  and varying  $\tau_{\max}$  between 5 and 19 days. As proxy of the statistical uncertainty of the scaling law we have computed the standard deviation of the  $H^w(q)$  over this range of  $\tau_{\max}$ . To track the evolution of the stage of development of a certain company, we have studied the dynamics in time of the wGHE on overlapping time-windows with a constant 50 days shift between any two successive windows.



**Fig. 3.** Weighted generalized Hurst exponent  $H^w(q = 1)$  as a function of time for American International Group (AIG). Left panel:  $\Delta t = 200$  days time-window. Right panel:  $\Delta t = 400$  days time-window. The characteristic time is kept constant at  $\theta = 300$  days in both plots. The points are reported in correspondence of the end of the time-window.

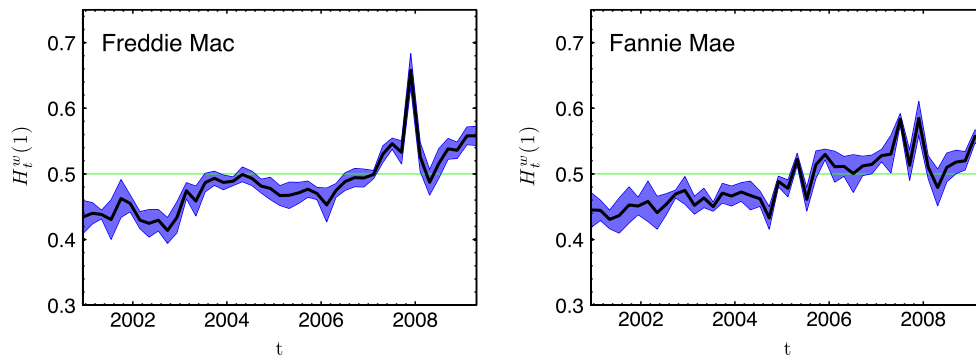


**Fig. 4.** Weighted generalized Hurst exponent  $H^w(q = 1)$  as a function of time for American International Group. The overlapping time-windows are  $\Delta t = 1250$  days, with  $\theta = 250$  days. The values are plotted in correspondence of the end of the time-window (solid black line). The shaded areas around the tick-line plot represent the sizes of the standard deviations.

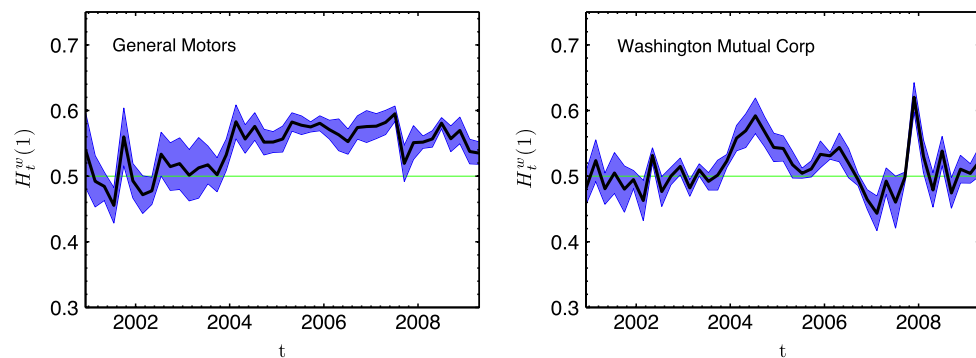
First of all, to fully capture the advantages of the weighted average method, a choice of the parameters  $\theta$  and  $\Delta t$ , namely the characteristic time and the width of the time-window, has to be made. In particular the time-window  $\Delta t$  must be large enough to provide good statistical significance but it should not be too large in order to retain sensitivity to changes in the scaling properties occurring over time. In order to satisfy both these requirements we take a rather long time-window  $\Delta t$  combined with a relatively short characteristic time  $\theta$ . For instance, in Fig. 3 we show how the manipulation of the parameters  $\theta$  and  $\Delta t$  affects the dynamics of the Hurst exponent of the company AIG. As it can be appreciated in the figure, which shows plots for AIG with time-windows of respectively 200 days (left panel) and 400 days (right panel) while keeping  $\theta = 300$  days, the shape of the outline shrinks and gets neater as the time-window is increased. The left panel of Fig. 3 shows more noisy dynamics when  $\Delta t$  is smaller. Conversely, in the right panel we can appreciate that a slimmer outline is achieved by increasing the statistics, but duly weighting it. We find the best match of the two parameters to be  $\Delta t = 1250$  days (five years of trading time) and  $\theta = 250$  days (one year of trading time). The result of this is shown in Fig. 4 for AIG where the thick lines are the average  $H^w(1)$  and the bands are the standard deviations over  $\tau_{\max}$  between 5 and 19 days [16,15,23]. This choice of the parameters has been optimized in order to obtain a sufficiently large statistics, while still allowing to perform the analysis on the moving window. At the same time the events are weighted such that not all the information present in the time series is given the same importance. One can see that the firm shows a well-defined increasing trend, with a transition from values  $< 0.5$  to values  $> 0.5$ .

We focus on  $H^w(1)$  even though, especially for the bailed-out companies, it would also be interesting to look at  $H^w(2)$ , which, as we said, is associated to the scaling of the auto-correlation function of the time series. However, in spite of the behavior being very similar to that observed for  $H^w(1)$ , the second moment is not defined (the tails of the returns distributions of these firms have  $\alpha < 2$  and hence the variance diverges) and thus it is difficult to interpret the real meaning of  $H^w(2)$ .

In Fig. 5 the dynamics in time of  $H^w(1)$  for the companies Freddie Mac and Fannie Mac is reported. These are public government sponsored enterprises which in September 2008 had to be put into conservatorship by the US Treasury; namely the huge debts of these companies were purchased by the US government. After playing a central role in the market during the mortgages's boost both firms defaulted. Their fate is pretty well pictured by the dynamical wGHE. Indeed, there is a



**Fig. 5.** Weighted generalized Hurst exponent  $H^w(q = 1)$  as a function of time. Left panel: Freddie Mac. Right panel: Fannie Mae. The increasing trend over the whole period highlights a transition from values of  $H^w(1) < 0.5$  to values of  $H^w(1) > 0.5$ . This suggests a progressive change in the stability of the companies under study.



**Fig. 6.** Weighed generalized Hurst exponent  $H^w(q = 1)$  as a function of time. Left panel: General Motors, a company that went bankrupt following Chrysler in June 2009. Its bankruptcy was classified as the fourth largest in US history. Right panel: Washington Mutual Corp. The increasing trend over the whole period highlights a transition from values of  $H^w(1) < 0.5$  to values of  $H^w(1) > 0.5$ . This suggests a progressive change in the stability of the companies under study.

clearly visible trend in these plots showing how the value of  $H^w(1)$  for these companies has been increasing since 1996 until 2009. This is particularly interesting if we compare the two very similar panels. According to Refs. [14,15] these trends might suggest a transition from a stable stage of the companies to an unstable one.

Other bailed-out companies which show similar trends are shown in Fig. 6. Again the trend is increasing and crossing over the value of 0.5 towards the end of the time-period when the crisis fully unfolded.

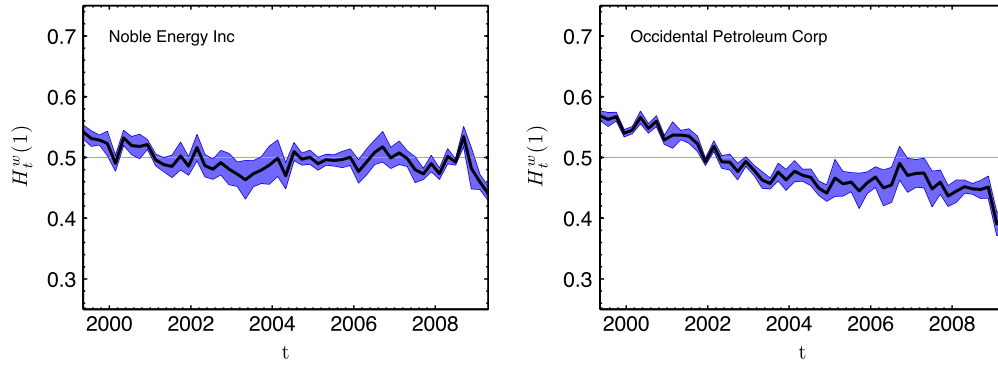
We have compared these results with those obtained by looking at other companies either from the financial sector or belonging to other market sectors to test the significance of these results. For example, in the Basic Materials sector, we find many companies whose dynamical wGHE decreases in time, thus exhibiting an opposite behavior to that shown by the bailed-out companies in the financial sector. An example is reported in Fig. 7 where the dynamical wGHE's for two companies belonging to the sector of Basic Materials are shown. We notice a very definite overall decreasing trend, as if the companies' securities gained persistence in going through the period of crisis. This is in agreement with what has been considered as the boost of the commodities market during the crisis, where investors were turning away from the financial sector.

There are other sectors that have revealed instead no particular trend in the dynamical wGHE. We stress that even in the Financial sector itself, the increasing trend found for the bailed-out companies is not common to others; for instance, many companies, like American Express Co and Morgan Stanley show stable behaviors, with wGHE values steadily fluctuating about 0.5. We will see in the next paragraph that the sectors exhibiting a defining trend in the dynamical wGHE are also those showing extreme values in the tail exponents of their distributions of returns. Although the increase or decrease of the wGHE is not simply related with the return statistics only, both behaviors are associated with the fluctuations of the log-returns distributions.

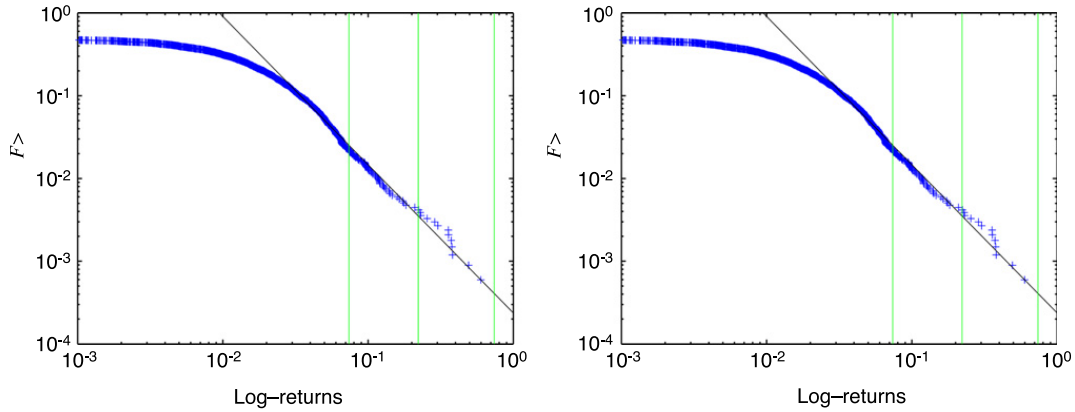
## 5. Fat-tails and extreme events

The unfolding of the 2007–2008 “credit crunch” financial crisis has made all of us again aware that very large fluctuations can happen with finite probability in financial markets. Indeed large fluctuations are very unlikely, say impossible, in a normal statistics frame but are instead rather common in complex systems and they are properly accounted by non-normal statistics. In order to quantitatively catch these large fluctuations we have investigated the scaling of the tails of the distributions of the log-returns. In Fig. 8 we report the complementary cumulative distribution for the stock prices





**Fig. 7.** Weighed generalized Hurst exponent  $H^w(q = 1)$  as a function of time for: Left panel–Noble Energy Inc.; Right panel–Occidental Petroleum. The time-window is taken to be  $\Delta t = 1250$  days and  $\theta = 250$  days.



**Fig. 8.** (Color online) Complementary cumulative distributions of the log-returns for the stock prices of Lehman Brothers (left panel) and American International Group (right panel). The vertical green lines mark respectively one, three and ten standard deviations (from left to right). The black line is the best fit of the tail region with the power law function  $\mathcal{F}_>(r) \propto r^{-\alpha}$ . The estimated best-fit exponent is  $\alpha \sim 1.7$  for both companies.

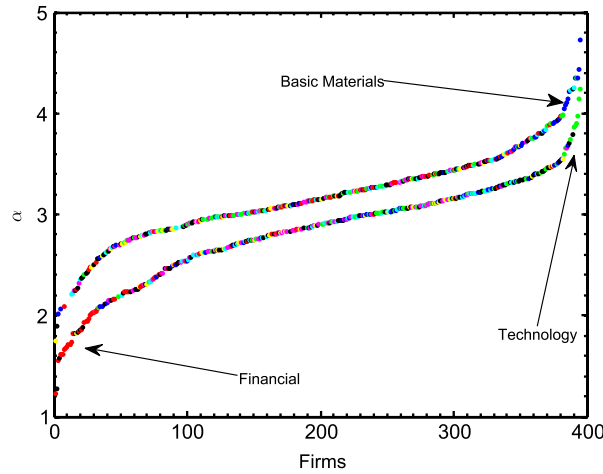
of the same companies studied in the previous section. Let us recall that, given a probability density function  $F(x)$ , its complementary cumulative distribution<sup>2</sup> is defined by

$$\mathcal{F}_>(x) = 1 - \mathcal{F}_<(x) = 1 - \int_{-\infty}^x F(s)ds. \quad (9)$$

We can see from Fig. 8 that fluctuations above  $3\sigma$  have frequencies above  $10^{-2}$  and therefore are occurring on average several times a year. We can also observe that the tails decrease linearly in log–log scale. Indeed, we find, in the tail region, good fits with the power law function  $\mathcal{F}_>(r) \propto r^{-\alpha}$  with  $\alpha \sim 1.7$ . Although the linear decrease of large fluctuations in log–log scale is not necessarily a proof for power-law behavior, in this case the power law hypothesis is enforced by the  $p$ -value test ( $p = 0.43$  for AIG and  $p = 0.48$  for LBH) [25]. However we stress that by excluding the recent unstable period from the same dataset, i.e. taking off the years 2007–2009, a slightly different picture emerges with the scaling exponents exhibiting larger values and the frequency of very large fluctuations becoming an order of magnitude smaller. Fig. 9 shows the exponents for all the firms, computed both over the entire period and over the period excluding the crisis. As one can appreciate, excluding the crisis period, the exponent increases for all firms and the occurrence of extreme events is much lower than that observed when the crisis is included. In particular Fig. 9 shows how the financial sector forms a cluster at the bottom end of the sorted companies, when the crisis period is included. It is also interesting to note that the firms belonging to the Technology sector appear to be the most stable.

Values of the scaling exponents  $\alpha$  between 2 and 4 are commonly observed in these systems [26,27]. These distributions typically have finite second moment  $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$  but diverging larger moments and this explains in turn why we find very large values for the excess kurtosis (139 for AIG and 761 for LBH). The fact that the tail exponents change by including or excluding in the statistics data referring to some extreme events is not a surprise though [28,29]. It is not a surprise either, the fact that stock prices do not obey normal statistics. Nonetheless these large fluctuations over the last time-period when

<sup>2</sup> The plot of  $\mathcal{F}_>$  in Fig. 8 is a so-called rank-frequency plot. This is a very convenient and simple method to analyze the tail region of the distribution without any loss of information which would instead derive from gathering together data points with an artificial binning. In order to make this plot from a given set of observations  $\{x_1, x_2, \dots, x_T\}$ , one first sorts the  $T$  observed values in ascending order and then plots them against the vector  $[1, (T-1)/T, (T-2)/T, \dots, 1/T]$ . Indeed, we have that  $\text{Rank}(x_i) = 1 - \mathcal{F}_>(x_i)$ .



**Fig. 9.** (Color online) The tail exponents for all the companies analyzed including (lower curve) and excluding (upper curve) the time-period from December 2007 to April 2009, when the crisis occurred. We notice a clustering of the financial sector (red) at very low values of  $\alpha$ , with many points lying in the region  $\alpha < 2$ . The other end of the curve, at high values of  $\alpha$ , is instead mostly populated by the Technology (green), which has been the less affected by the crisis. This is in agreement with the fact that the financial sector was the one most profoundly affected by the crisis and whose fluctuations were the largest. Instead, before the crisis, the sector of Basic Material (blue) appears to be the most stable.

the crisis was unfolding may be somehow the cause for the increase of the wgHE, and this is what we are going to discuss in the next section.

## 6. Discussion

In order to understand the link between the two types of scaling, let us investigate the simple ideal case where the underlying process is a random walk with  $x(t) = x(t-1) + \eta(t-1)$  where  $x(t) = \ln(P(t))$ . In this case, for an arbitrary  $\tau$ , the log-returns  $r(t, \tau) = x(t+\tau) - x(t)$  can be written as a sum of  $n = \tau$  random variables:

$$r(t, \tau) = \sum_{s=0}^{\tau-1} \eta(s+t). \quad (10)$$

If the  $\eta(t)$  are i.i.d., the Central Limit Theorem applies to  $r(t, \tau)$  and there are two cases: (1) the probability distribution function of  $\eta(t)$  has finite variance and therefore the distribution of  $r(t, \tau)$  converges to a normal distribution for large  $\tau$ ; (2) the variance is not defined and the asymptotic distribution of  $r(t, \tau)$  converges to a Levy Stable distribution. For distributions well approximated by power-law functions in the tail region, the parameter that distinguishes between these two cases is the tail index  $\alpha$ . Namely  $\alpha \geq 2$  leads to normal distributions, while  $\alpha < 2$  leads to Levy Stable distributions. Moreover, given that  $r(t, \tau)$  is a sum of random variables and given that both cases (1) and (2) lead to stable distributions,<sup>3</sup> the probability distribution  $p_\tau(r)$ , of the log-returns must scale with  $\tau$  as [26,27]

$$p_\tau(r) = \begin{cases} \frac{1}{\tau^{1/\alpha}} p\left(\frac{r}{\tau^{1/\alpha}}\right) & \text{if } \alpha < 2 \\ \frac{1}{\tau^{1/2}} p\left(\frac{r}{\tau^{1/2}}\right) & \text{if } \alpha \geq 2. \end{cases} \quad (12)$$

Accordingly, the  $q$ -moments scale as

$$E(|r(t, \tau)|^q) = \begin{cases} \tau^{q/\alpha} E(|r(t, 1)|^q) & \text{if } \alpha < 2 \\ \tau^{q/2} E(|r(t, 1)|^q) & \text{if } \alpha \geq 2. \end{cases} \quad (13)$$

Here  $E(\dots)$  denotes the expectation value. Finally, if we restrict ourselves to the class of self-affine processes, i.e. those processes  $x(t)$  where the probability distribution of  $\{x(ct)\}$  is equal to the probability of  $\{c^H x(t)\}$ , for any positive  $c$ , and we consider stationary increments, the  $q$ -moments must scale as

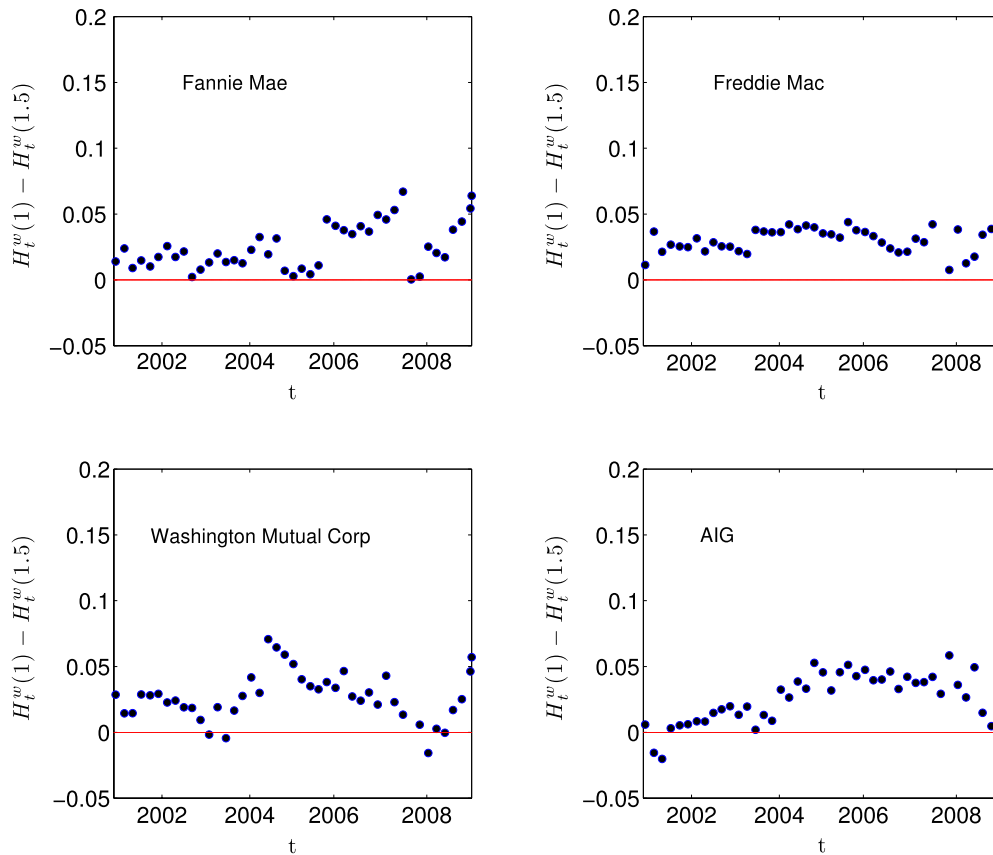
$$E(|r(t, \tau)|^q) = c(q) \tau^{qH}. \quad (14)$$

<sup>3</sup> A distribution is stable if and only if, for any  $n > 1$ , the distribution of  $y = x_1 + x_2 + \dots + x_n$  is equal to the distribution of  $n^{1/\alpha} x + d$ , with  $d \in \mathbb{R}$ . This implies

$$p_n(y) = \frac{1}{n^{1/\alpha}} p\left(\frac{y-d}{n^{1/\alpha}}\right) \quad (11)$$

where  $p_n(y)$  is the aggregate distribution of the sum of the i.i.d. variables and  $p(x)$  is the distribution of the  $x_i$ .





**Fig. 10.** The difference  $H_t^w(1) - H_t^w(1.5)$  as function of time. This quantity is a simple measure of multifractality of the system, as it quantifies the departure of the wGHE from the unifractal value  $H \sim 0.5$ . From the top left, clockwise: Fannie Mae, Freddie Mac, Washington Mutual Corp and American International Group. The parameters for the weighted mean are  $\theta = 250$  days and  $\Delta t = 1250$  days.

By comparing Eq. (13) with Eq. (14) we get

$$H = \begin{cases} 1/\alpha & \text{if } \alpha < 2 \\ 1/2 & \text{if } \alpha \geq 2. \end{cases} \quad (15)$$

Eq. (14) holds also for the moments computed using the weighted average, by substituting  $H$  with  $H^w$  and the expectation values  $E(\cdot \cdot \cdot)$  with weighted averages. Processes with the property in Eq. (14) are deemed unscaling. For  $\alpha \geq 2$  we retrieve  $H \sim 0.5$  and the process scales as a Brownian motion.

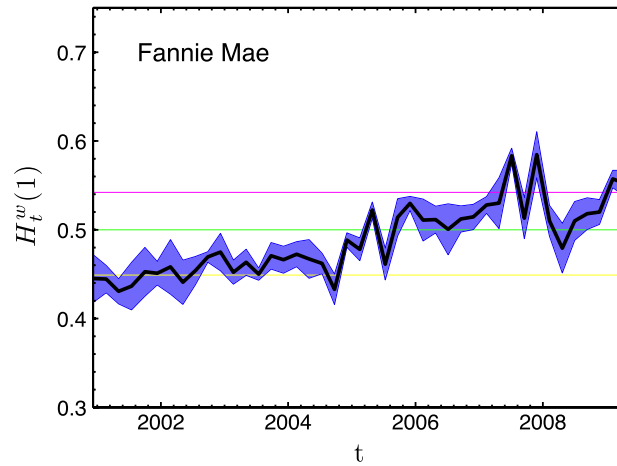
Let us here stress that the result in Eq. (15) is only valid for a random-walk type i.i.d. process with defined noise distribution and it is well known that financial time series cannot be described within this framework. However, Eq. (15) is a valuable reference which can be used as a tool to compare the relation between the tail exponent and the Hurst exponent in more complex signals.

Multifractality can be measured by tracking the difference  $H^w(q) - H^w(q')$  (with  $q \neq q'$ ) over the time-windows (see Fig. 10). Intriguingly, this difference remains stable for most of the time for all the companies reported in the figure but, instead, it increases as soon as the unstable period is reached, suggesting that the scaling properties of the time series change with the unfolding of the crisis. We stress that the behavior observed in the empirical data is not necessarily related to a change of the stochastic process underlying the financial time series. The increase in the multifractality of these kinds of signals is likely to occur in the presence of large price fluctuations. In this case indeed, the attitude of the investors, and thus the prices' movements, in the short period, are very rarely reflecting the price behavior over larger periods.

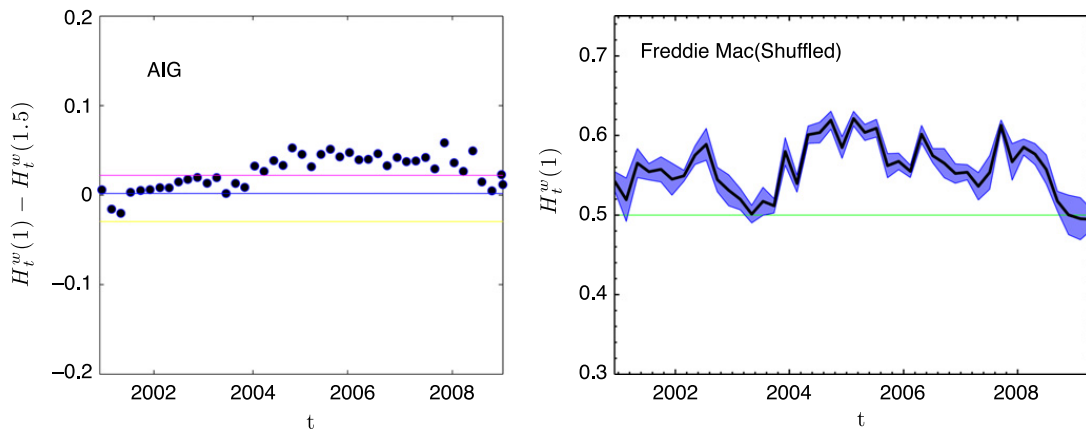
## 7. Significance tests

The main problem one has to face when interpreting these results is the finite size of our sample. Especially for what concerns the changes in the multifractality, one needs to verify that the fluctuations of the quantity  $H(q) - H(q')$  (with  $q \neq q'$ ) are indeed larger than those expected for a unifractal process, whose fluctuations may be due only to the finite size of the time series.

First of all, to test the significance of the raise of the wGHE we have simulated 1000 random normally distributed series with  $\Delta t = 1250$  and computed the quantiles corresponding to the {2.5%, 50%, 97.5%} confidence interval. The values obtained are {0.4490, 0.4973, 0.5421}. Empirical values of  $H^w(1)$  falling in the interval (0.4490, 0.5421) are hardly



**Fig. 11.** (Color online) The three horizontal lines correspond to the {2.5%(yellow), 50%(green), 97.5%(magenta)} quantiles obtained from the distribution of 1000  $H^w(1)$  from random series. Empirical values at the two extremes are outside the confidence interval.



**Fig. 12.** (Color online) The empirical multifractality tested against the FBM quantiles. Points lying beyond the magenta line confirm multifractality is changing. Right panel: the trend of wGHE is destroyed by shuffling the time series.

distinguishable from a pure random walk. Nonetheless, one finds that values at the two extremes of the whole time period are truly crossing over the random regime and have thus more than 97.5% chances not to be originated from a pure random walk process (see Fig. 11). This holds for the other bailed-out firms as well. This also suggests that a raise is taking place over the whole period and the firm is switching between two different stability regimes. Concerning the variation of the multifractality over the crisis, caution has to be paid for the same reason as above. Sometimes even unifractal processes exhibit fluctuating  $H(q)$  just because of the short size of the sample. To test that the change of multifractality that we observe is really significant, we compare it with the case of Fractional Brownian motion (FBM), a unifractal process  $B_H(t)$  whose covariance is given by

$$\text{Cov}(B_H(t)B_H(t')) = \frac{1}{2}(|t|^{2H} + |t'|^{2H} - |t - t'|^{2H}), \quad (16)$$

where  $H > 0.5$  corresponds to long-range dependence. We set  $H = 0.75$  and simulate 100 FBM's. For every simulation we compute the  $H^w(1)$  and obtain mean and standard deviation  $H^w(1) = 0.7368 \pm 0.0397$ . We also verify that our algorithm returns sound values for the shuffled FBM series. After shuffling each of the simulated series we obtain  $H^w(1) = 0.4871 \pm 0.0402$ , compatible with the expected  $H = 0.5$ . Then we test the multifractal behavior by computing  $H^w(1) - H^w(1.5)$  over the FBM's and obtaining the {2.5%, 50%, 97.5%} quantiles:  $\{-0.0294, 0.0018, 0.022\}$ . The comparison is reported in Fig. 12 (left panel), where we note that the points corresponding to the crisis are really beyond the quantiles and correspond thus to a true change in multifractality.

Let us also mention that, although the time series of the wGHE's is dependent, as the time windows are overlapping, the trends are significant. This is demonstrated in the right panel of Fig. 12, where the trend is completely destroyed by shuffling the time series. The statistics of dependent observations goes beyond the purpose of this work but we plan to address it in future work (see Ref. [30] for a very interesting recent discussion).

## 8. Conclusions

We have studied the scaling behavior in time of log-returns of the companies more severely affected by the ‘credit-crunch’ crisis. The results obtained for these companies have been compared to those obtained for companies belonging to different market sectors, showing persistent differences. To allow a reasonable differentiation in the time series we have introduced a weighting procedure which renders recent events more significant than remote ones. With this exponential smoothing method we have computed the weighted generalized Hurst exponent for overlapping time-windows spanning a period of 13 years (1996–2009). The bailed-out companies reveal an increasing trend which crosses 0.5 hinting therefore to a transition between different stages of development. This behavior, not observed for many other companies, including others belonging to the financial sector itself, might suggest that the wGHE is conveying important information about the stability of a company and that by tracking its value in time one could have a further tool to assess risk. A comparison with the scaling of the distributions of the log-returns confirms that large fluctuations are related to the increase of the wGHE. We have also looked at the multifractal behavior in time of these companies revealing an increase of multifractality when the crisis occurred. These empirical facts will be the basis of future work aiming to realistically model the price formation and evolution in financial markets [10,11,31]. Interesting perspectives are to extend this analysis to high frequency data and to look at the statistics of the dependent wGHE time series.

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## Further reading

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