Linear Algebra

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***Abstract*—This document has solutions of Problem 3 of each section of the Linear Algebra manual.**

Download python codes from

```
svn co https://github.com/krishnajakodali/
    Summer2020/trunk/geometry/linalg/codes
```

1 Triangle

1.1 Problem

Draw graphs of equations

$$5x - y = 5 \tag{1.1.1}$$

$$3x - y = 3 \tag{1.1.2}$$

Determine the coordinates of vertices of triangle formed by these lines and y axis.

1.2 Solution

1.1. Line $5x - y = 5$ can be represented in vector form as,

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{X} = 5 \tag{1.1.1}$$

1.2. Line $3x - y = 3$ can be represented in vector form as,

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{X} = 3 \tag{1.2.1}$$

Also the equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{X} = 0 \tag{1.2.2}$$

Let line 1.1.1 and line 1.2.1 meet at point $\mathbf{A}$.Then,

$$\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.2.3}$$

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.2.4}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.5}$$

Let line 1.1.1 and line 1.2.2 meet at point $\mathbf{B}$. Then,

$$\begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{1.2.6}$$

$$\mathbf{B} = \begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{1.2.7}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.2.8}$$

Let line 1.2.1 and line 1.2.2 meet at point $\mathbf{C}$. Then,

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.2.9}$$

$$\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.2.10}$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \tag{1.2.11}$$

So, $\triangle ABC$ is formed by intersection of 1.1.1,1.2.1 and 1.2.2. The following Python code generates Fig. 1.2 The lines 1.1.1 and 1.2.1 and the triangle ABC formed by the two lines and y-axis are plotted in the figure below

```
codes/triangle/linesandtri.py
```

2 Quadrilateral

2.1 Problem

Draw Quadrilateral in cartesian plane whose vertices are $\begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix}$.
Also Find its area.

2.2 Solution

2.1. Vertices of the quadrilateral are,

$$\mathbf{A} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \mathbf{D} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{2.1.1}$$

Quadrilateral ABCD is drawn by joining its vertices $\mathbf{A}$ and $\mathbf{B}$,$\mathbf{B}$ and $\mathbf{C}$, $\mathbf{C}$ and $\mathbf{D}$, $\mathbf{D}$ and

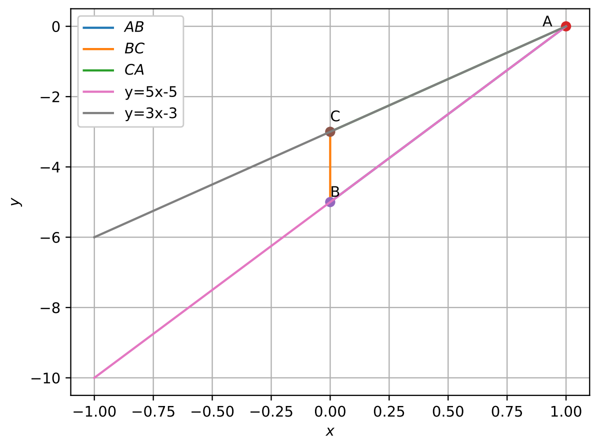

Fig. 1.2: Plot of lines and the Triangle ABC

A. The following Python code generates Fig. 2.1

```
codes/quad/quad.py
```

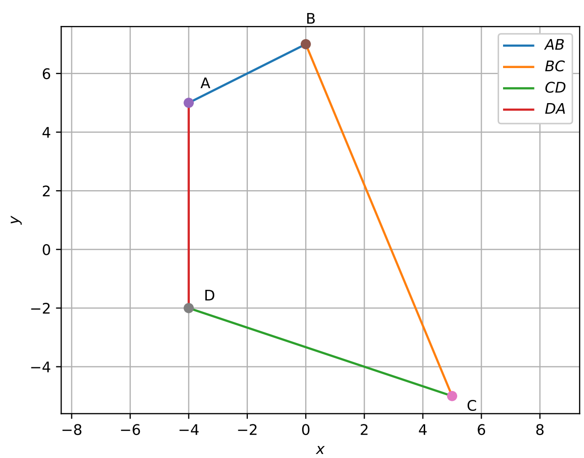

Fig. 2.1: Quadrilateral ABCD

2.2. From Figure 2.1 Area of the Quadrilateral ABCD can be given as

$$Ar\left(\triangle ABC\right) + Ar\left(\triangle BCD\right) \tag{2.2.1}$$

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| + \frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| \tag{2.2.2}$$

For two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\left\|\mathbf{a}\times\mathbf{b}\right\| = \left|a_1 b_2 - a_2 b_1\right| \tag{2.2.3}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{2.2.4}$$

$$\mathbf{A}-\mathbf{D} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{2.2.5}$$

$$\mathbf{C}-\mathbf{B} = \begin{pmatrix} 5 \\ -12 \end{pmatrix} \tag{2.2.6}$$

$$\mathbf{C}-\mathbf{D} = \begin{pmatrix} 9 \\ -3 \end{pmatrix} \tag{2.2.7}$$

Using 2.2.3

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| = \frac{1}{2}\left|(-28)\right| = 14 \tag{2.2.8}$$

$$\frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| = \frac{1}{2}\left|(-15+108)\right| = 46.5 \tag{2.2.9}$$

Substituting the above values in equation 2.2.2,We get

$$Area = 14 + 46.5 = 60.5 sq.units \tag{2.2.10}$$

3 Line

3.1 Complex numbers

3.1.1 Problem: Find multiplicative inverse of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

3.1.2 Solution:

3.1. Complex number $\begin{pmatrix} a \\ b \end{pmatrix}$ is expressed as a matrix as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{3.1.1}$$

If complex number $\begin{pmatrix} x \\ y \end{pmatrix}$ is the multiplicative inverse of $\begin{pmatrix} a \\ b \end{pmatrix}$, Then their product must be complex number $\begin{pmatrix}1\\0\end{pmatrix}$

$$\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}x & -y\\ y & x\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad (3.1.2)$$

$$\implies \begin{pmatrix}ax-by & -(bx+ay)\\ bx+ay & ax-by\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad (3.1.3)$$

$$\implies ax-by = 1 \quad (3.1.4)$$

$$bx+ay = 0 \quad (3.1.5)$$

From equations 3.1.4 and 3.1.5

$$\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \quad (3.1.6)$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}\begin{pmatrix}a & -b\\ b & a\end{pmatrix}^{-1} \quad (3.1.7)$$

Using 3.1.7 The multiplicative inverse of complex number $\begin{pmatrix}2\\-3\end{pmatrix}$ is $\begin{pmatrix}0.15384615\\0.23076923\end{pmatrix}$
The python code for above problem is

```
codes/line/comp/comp.py
```

3.2 *Points and vectors*

3.2.1 Problem: A town B is located 36km east and 15 km north of town A. How would you find the distance from town A to town B without actually measuring it.

3.2.2 Solution:

3.1. We represent town A by **A** and town B by **B**. If **A** is taken to be the origin and East and North directions are considered to be +ve x-axis and +ve y-axis respectively, Then **B** is given as

$$\mathbf{B} = \begin{pmatrix}36\\15\end{pmatrix} \quad (3.1.1)$$

The distance d between A and B is given by

$$d = \|\mathbf{A}-\mathbf{B}\| \quad (3.1.2)$$

$$d = 39km \quad (3.1.3)$$

The following Python code generates Fig. 3.1

```
codes/line/towns/towns.py
```

So the distance between Town A and Town B is 39km.

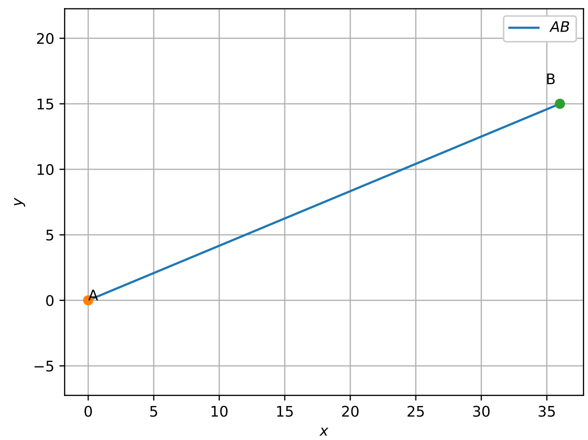

Fig. 3.1: Position of Towns A and B

3.3 *Points on a Line*

3.4 *Problem*

Find the ratio in which line segment joining points $\begin{pmatrix}-3\\10\end{pmatrix}\begin{pmatrix}6\\-8\end{pmatrix}$ is divide by $\begin{pmatrix}-1\\6\end{pmatrix}$

3.4.1 Solution:

3.1.

$$\mathbf{A} = \begin{pmatrix}-3\\10\end{pmatrix}\mathbf{B} = \begin{pmatrix}6\\-8\end{pmatrix}\mathbf{C} = \begin{pmatrix}-1\\6\end{pmatrix} \quad (3.1.1)$$

Let **C** divide AB in ratio k:1.Then by section formulae,

$$\mathbf{C} = \frac{k\mathbf{B}+\mathbf{A}}{k+1} \quad (3.1.2)$$

$$\begin{pmatrix}-1\\6\end{pmatrix} = \frac{1}{k+1}\begin{pmatrix}6k-3\\-8k+10\end{pmatrix} \quad (3.1.3)$$

$$k = \frac{2}{7} \quad (3.1.4)$$

So **C** divides AB in ratio 2:7
The following Python code generates Fig. 3.1

```
codes/line/section/section.py
```

3.5 *Lines and planes*

3.5.1 Problem: Find two solutions for each of the following equations.

$$(a)\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \quad (1.1)$$

$$(b)\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 0 \quad (1.2)$$

$$(c)\begin{pmatrix}0 & 3\end{pmatrix}\mathbf{x} = 4 \quad (1.3)$$

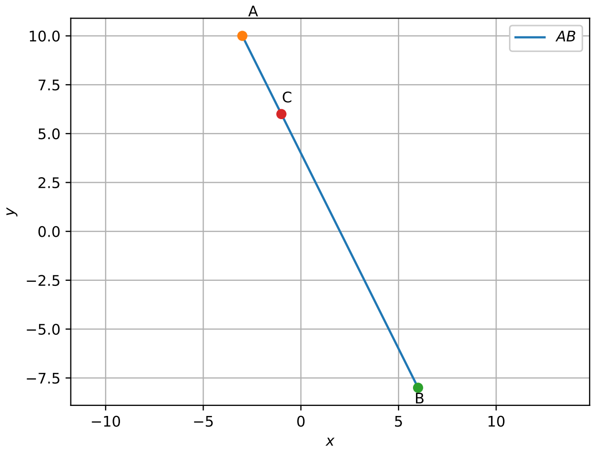

Fig. 3.1: C divides AB in ratio k:1

3.5.2 Solution:

3.1. A point $\mathbf{c}$ lying on the line

$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} = d \tag{3.1.1}$$

at a distance λ from point $\mathbf{x}$ lying on the same line is given as

$$\mathbf{c} = \mathbf{x} + \frac{\lambda}{\sqrt{a^2+b^2}}\begin{pmatrix} b \\ -a \end{pmatrix} \tag{3.1.2}$$

$$\lambda = \sqrt{a^2+b^2} \implies \mathbf{c} = \mathbf{x} + \begin{pmatrix} b \\ -a \end{pmatrix} \tag{3.1.3}$$

3.2. Equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{3.2.1}$$

3.3.

$$(a)\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \tag{3.3.1}$$

The line meets y-axis at point $\mathbf{y_1}$ given using 3.2.1 as,

3.4.

$$\begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}\mathbf{y_1} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{3.4.1}$$

$$\mathbf{y_1} = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{3.4.2}$$

$$\mathbf{y_1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{3.4.3}$$

Another point $\mathbf{c_1}$ on the line is found using equation 3.1.3

$$\mathbf{c_1} = \mathbf{y_1} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{3.4.4}$$

$$\implies \mathbf{c_1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.4.5}$$

$$(b)\begin{pmatrix} 2 & 5 \end{pmatrix}\mathbf{x} = 0 \tag{3.4.6}$$

The line meets y-axis at point $\mathbf{y_2}$ given using 3.2.1 as,

3.5.

$$\begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}\mathbf{y_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.5.1}$$

$$\mathbf{y_1} = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.5.2}$$

$$\mathbf{y_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.5.3}$$

Another point $\mathbf{c_2}$ on the line is found using equation 3.1.3

$$\mathbf{c_2} = \mathbf{y_2} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{3.5.4}$$

$$\implies \mathbf{c_2} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{3.5.5}$$

$$(c)\begin{pmatrix} 0 & 3 \end{pmatrix}\mathbf{x} = 4 \tag{3.5.6}$$

The line meets y-axis at point $\mathbf{y_2}$ given using 3.2.1 as,

3.6.

$$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}\mathbf{y_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{3.6.1}$$

$$\mathbf{y_1} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{3.6.2}$$

$$\mathbf{y_1} = \begin{pmatrix} 0 \\ \frac{4}{3} \end{pmatrix} \tag{3.6.3}$$

Another point $\mathbf{c_2}$ on the line is found using equation 3.1.3

$$\mathbf{c_2} = \mathbf{y_2} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.6.4}$$

$$\implies \mathbf{c_2} = \begin{pmatrix} 3 \\ \frac{4}{3} \end{pmatrix} \tag{3.6.5}$$

The python code for the above problem , plotting the figure 3.6 is available at

```
codes/line/pointonline2/pointonline.py
```

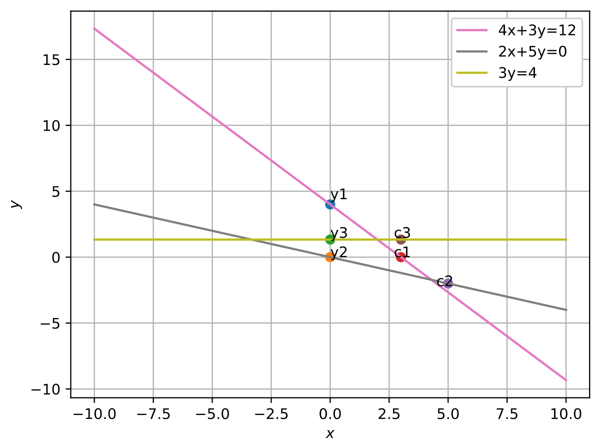

Fig. 3.6: Plot of the three lines and the points on them

3.6 Motion in a plane

3.6.1 Problem: Rain is falling vertically with a speed of 35 m/s.A woman rides a bicycle with a speed of 12 m/s in east to west direction .What is the direction in which she should hold the umbrella?

3.6.2 Solution:

3.1. Let us take vertically upward direction as +ve y-axis and west to east direction as +ve x-axis.The woman experiences rain in the direction of the relative velocity of the rain wrt her own velocity. The velocity of rain = $\mathbf{v_r} = \begin{pmatrix} 0 \\ -35 \end{pmatrix}$
The velocity of woman = $\mathbf{v_w} = \begin{pmatrix} -12 \\ 0 \end{pmatrix}$
relative velocity of rain wrt woman is given as

$$\mathbf{v_{r_w}} = \mathbf{v_r} - \mathbf{v_w} \implies \begin{pmatrix} 12 \\ -35 \end{pmatrix} \tag{3.1.1}$$

She must hold the umbrella opposite to the direction of rain she experiences.So the woman must hold the umbrella along the direction of $-\mathbf{v_{r_w}}$ So the woman must hold the umbrella forward at an angle θ to the vertical where

$$\theta = \tan^{-1}\left(\frac{12}{35}\right) \tag{3.1.2}$$

The following python code generates figure 3.1 which illustrates the velocities $\mathbf{v_r}, \mathbf{v_w}, \mathbf{v_{r_w}}$

```
codes/line/rain/rain.py
```

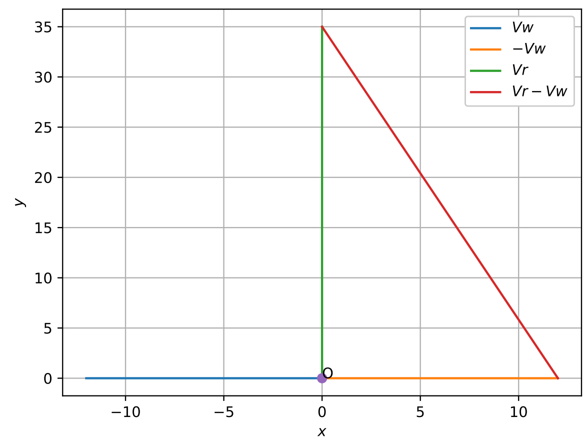

Fig. 3.1: Direction of umbrella

3.7 Matrix

3.7.1 Problem: If a matrix has 18 elements ,what are the possible orders it can have. What if it has 5 elements?

3.7.2 Solution:

3.1. A matrix with n elements can be represented as a matrix of order $(r \times c)$ if and only if n,r and c are all natural numbers(Here r is the number of rows and c is the number of coloumns in the matrix.).This is possible only if r is a divisor of n.
3.2. So the total possible orders a matrix with n elements can have is equal to the total number of divisors of n.
The following python code finds the total possible orders (d) for a matrix of n elements.

```
codes/line/matrix/matrix.py
```

So a matrix of 18 elements has 6 possible orders and a matrix of 5 elements can have 2 possible orders.

3.8 Determinants

3.8.1 Problem: If $\mathbf{A} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$ Show that:

$$|2\mathbf{A}| = 4|\mathbf{A}| \tag{2.1}$$

3.8.2 Solution:

3.1. Determinant of a (2×2) matrix is calculated as follows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{3.1.1}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \tag{3.1.2}$$

$$\mathbf{2A} = \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix} \tag{3.1.3}$$

Using 3.1.1 ,3.1.2, 3.1.3

$$|A| = 2 - 8 = -6 \implies 4|A| = -24 \tag{3.1.4}$$

$$|2A| = 8 - 32 = -24 \tag{3.1.5}$$

$$\implies |2A| = 4|A| \tag{3.1.6}$$

3.9 Linear Inequalities

3.9.1 Problem: Solve

$$\frac{3x-4}{2} \geq \frac{x+1}{4} \tag{1.1}$$

Show graph of solutions on numberline.

3.9.2 Solution:

3.1.

$$\frac{3x-4}{2} \geq \frac{x+1}{4} - 1 \tag{3.1.1}$$

$$\frac{3x-4}{2} - \frac{x+1}{4} \geq -1 \tag{3.1.2}$$

3.2. Make RHS positive by multiplying with -1 on both sides , Inequality changes.

$$-\frac{3x-4}{2} + \frac{x+1}{4} \leq 1 \tag{3.2.1}$$

3.3. Convert $\leq$ sign to $=$ sign by adding slack variable s on the LHS such that s is non-negative , That is

$$s \geq 0 \implies -\frac{3x-4}{2} + \frac{x+1}{4} + s = 1 \tag{3.3.1}$$

$$\implies -5x + 9 + 4s = 4 \tag{3.3.2}$$

$$\implies x = 1 + \frac{4s}{5} \tag{3.3.3}$$

$$\implies x \geq 1 \tag{3.3.4}$$

The following code marks the solution of inequality on numberline as shown in figure 3.3

```
codes/line/ineq/ineq.py
```

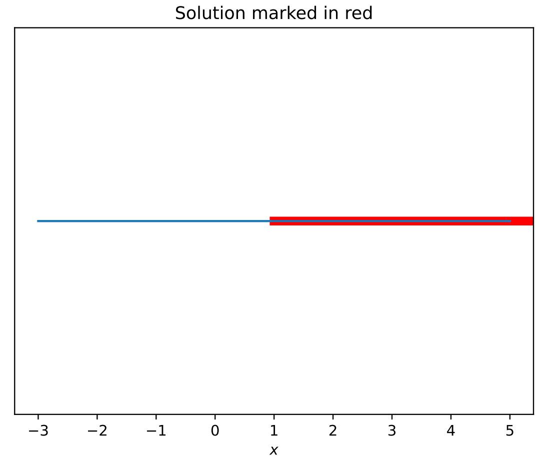

Fig. 3.3: Solution of the inequality

3.10 Miscellaneous

3.10.1 Problem: In $\triangle ABC$ Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{3.1}$$

3.10.2 Solution:

3.1. The vertices of $\triangle ABC$ are taken as follows: **Solution:** See Table. 3.1

vertex	coordinates
A	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
B	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
C	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

TABLE 3.1: To construct triangle ABC

3.2. The midpoints of sides are given as

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{3.2.1}$$

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{3.2.2}$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{3.2.3}$$

3.3. Also Centroid divides median in ratio 2:1.So by section formulae $\mathbf{O}$ is given as

$$\mathbf{O} = \frac{2\mathbf{D} + \mathbf{A}}{3} \tag{3.3.1}$$

The derived coordinates are listed in 3.3.

Vector	Coordinates
$\mathbf{D}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
$\mathbf{E}$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$\mathbf{F}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\mathbf{O}$	$\begin{pmatrix} 3 \\ \frac{2}{3} \end{pmatrix}$

TABLE 3.3: Derived coordinates of triangle ABC

3.10.3 Proof:

3.1. Substituting 3.2.1 in 3.3.1 we get

$$\implies \mathbf{O} = \frac{\mathbf{B}+\mathbf{C}+\mathbf{A}}{3} \tag{3.1.1}$$

The following code plots figure 3.1

```
codes/line/median/median.py
```

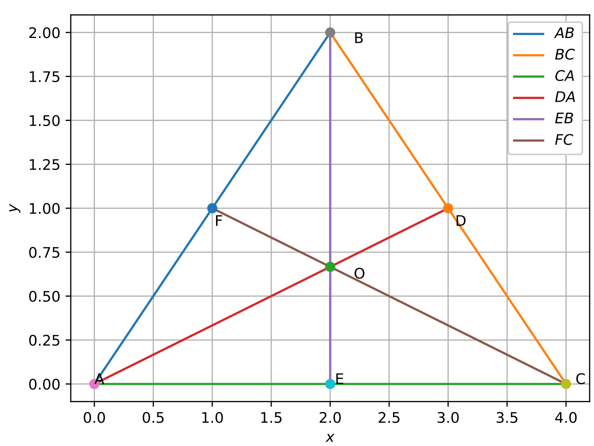

Fig. 3.1: Triangle ABC with centroid O

4 Circle

4.1 Example

4.1.1 Problem: Find the equation of circle which passes through point $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and whose centre lies on the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = 2 \tag{1.1}$$

4.1.2 Solution:

4.1. Let $\mathbf{O}$ be the centre , r be the radius and $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ be the points lying on the circle.
Any point on the circle is equidistant from its centre

$$\implies \|\mathbf{A}-\mathbf{O}\| = \|\mathbf{B}-\mathbf{O}\| = r \tag{4.1.1}$$
$$\implies \|\mathbf{A}-\mathbf{O}\|^2 - \|\mathbf{B}-\mathbf{O}\|^2 = 0 \tag{4.1.2}$$
$$\implies (\mathbf{A}-\mathbf{O})^T (\mathbf{A}-\mathbf{O}) \tag{4.1.3}$$
$$-(\mathbf{B}-\mathbf{O})^T (\mathbf{B}-\mathbf{O}) = 0 \tag{4.1.4}$$
$$\implies (\mathbf{A}-\mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{4.1.5}$$

Also centre O lies on the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = 2 \tag{4.1.6}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{O} = 2 \tag{4.1.7}$$

(4.1.5) and (4.1.7), can be combined to form the matrix equation

$$\mathbf{NO} = \mathbf{c} \tag{4.1.8}$$
$$\implies \mathbf{O} = \mathbf{N}^{-1}\mathbf{c} \tag{4.1.9}$$

where

$$\mathbf{N} = \begin{pmatrix} (\mathbf{A}-\mathbf{B})^T \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{pmatrix} \tag{4.1.10}$$
$$\mathbf{c} = \begin{pmatrix} \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \\ 2 \end{pmatrix} \tag{4.1.11}$$

radius r of the circle is given as

$$r = \|\mathbf{A}-\mathbf{O}\| \tag{4.1.12}$$

Therefore the equation of the circle is

$$\|\mathbf{X}-\mathbf{O}\| = r \tag{4.1.13}$$

Where O and r are derived using equations 4.1.9 and 4.1.12
The following code calculates centre and radius and plots figure 4.1

```
codes/circle1/circle1.py.py
```

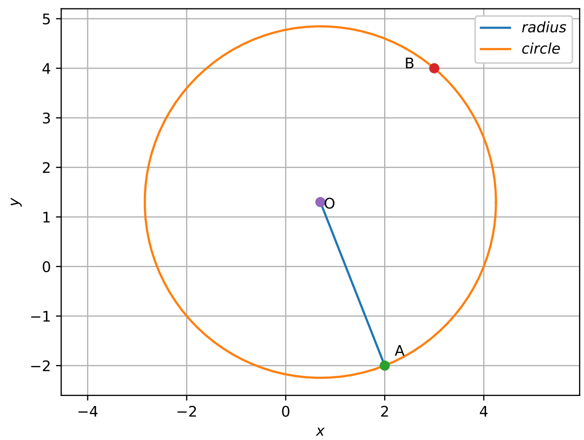

Fig. 4.1: Circle with centre at **O** and radius r

4.2 Exercise

4.2.1 Problem:

Sketch the circles with

$$(a)\, centre \begin{pmatrix} 0 \\ 2 \end{pmatrix}, radius 2 \tag{1.1}$$

$$(b)\, centre \begin{pmatrix} -2 \\ 32 \end{pmatrix}, radius 4 \tag{1.2}$$

$$(c)\, centre \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, radius \frac{1}{12} \tag{1.3}$$

$$(d)\, centre \begin{pmatrix} 1 \\ 1 \end{pmatrix}, radius\ \sqrt{2} \tag{1.4}$$

$$(e)\, centre \begin{pmatrix} -a \\ -b \end{pmatrix}, radius\ \sqrt{a^2 - b^2} \tag{1.5}$$

4.2.2 Solution:

4.1. Let **O** be the centre , r be the radius of the circle.Any point **X** lying on the circle is at a distance r from **O**.
Therefore the equation of the circle is

$$\|\mathbf{X} - \mathbf{O}\| = r \tag{4.1.1}$$

4.2.

$$(a)\, \mathbf{O} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, r = 2 \tag{4.2.1}$$

The following code sketches the circle 4.2.1 in figure 4.2 using the equation 4.1.1

```
codes/circle2/circle2a.py
```

4.3.

$$(b)\, \mathbf{O} = \begin{pmatrix} -2 \\ 32 \end{pmatrix}, r = 4 \tag{4.3.1}$$

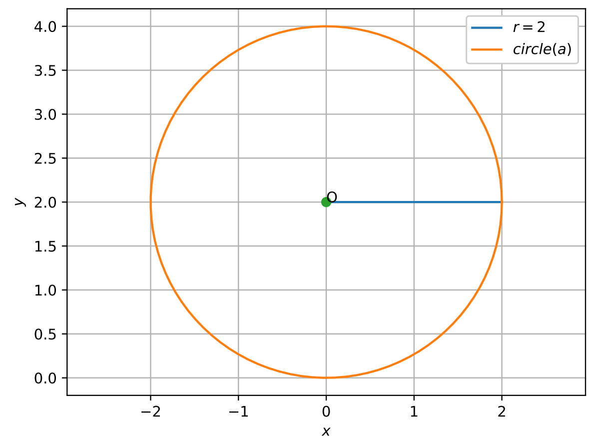

Fig. 4.2: Circle with centre at $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and radius 2

The following code sketches the circle 4.3.1 in figure 4.3 using the equation 4.1.1

```
codes/circle2/circle2b.py
```

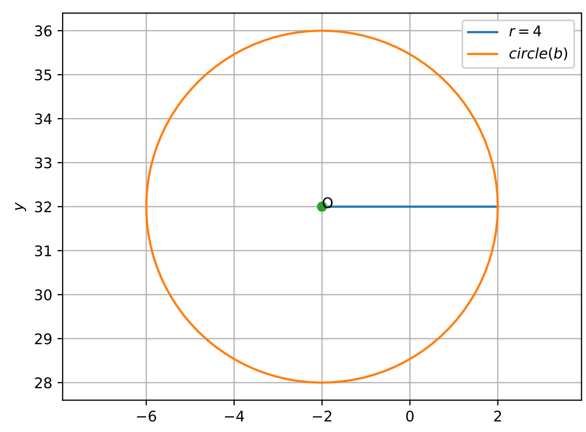

Fig. 4.3: Circle with centre at $\begin{pmatrix} -2 \\ 32 \end{pmatrix}$ and radius 4

4.4.

$$(c)\, \mathbf{O} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, r = \frac{1}{12} \tag{4.4.1}$$

The following code sketches the circle 4.4.1 in figure 4.4 using the equation 4.1.1

```
codes/circle2/circle2c.py
```

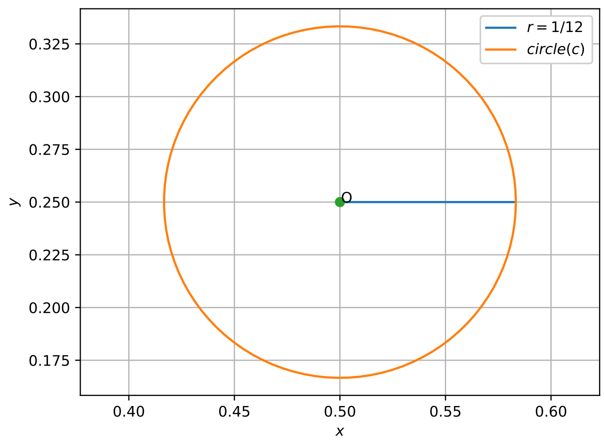

Fig. 4.4: Circle with centre at $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ and radius $\frac{1}{12}$

4.5.

$$(d)\,\mathbf{O} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, r = \sqrt{2} \tag{4.5.1}$$

The following code sketches the circle 4.5.1 in figure 4.5 using the equation 4.1.1

```
codes/circle2/circle2d.py
```

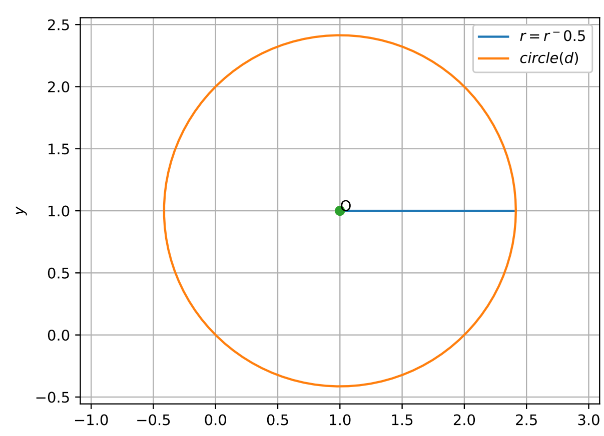

Fig. 4.5: Circle with centre at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and radius $\sqrt{2}$

4.6.

$$(e)\,\mathbf{O} = \begin{pmatrix} -a \\ -b \end{pmatrix}, r = \sqrt{a^2 - b^2} \tag{4.6.1}$$

The parameters used to sketch the circle are taken as

$$a = 5, b = 4 \implies \mathbf{O} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{4.6.2}$$

$$r = \sqrt{5^2 - 4^2} = 3 \tag{4.6.3}$$

The following code sketches the circle 4.6.3 in figure 4.6 using the equation 4.1.1

```
codes/circle2/circle2e.py
```

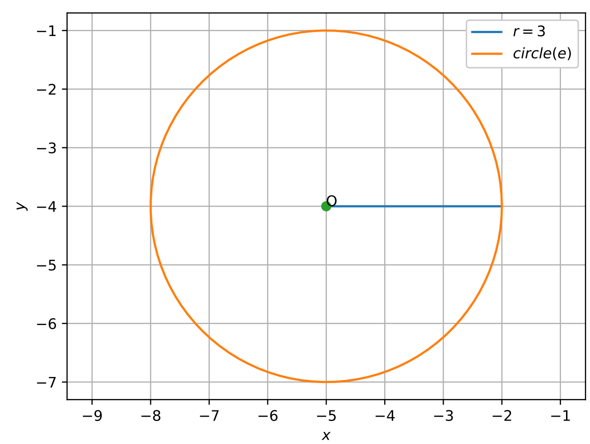

Fig. 4.6: Circle with centre at $\begin{pmatrix} -5 \\ -4 \end{pmatrix}$ and radius 3

5 Conics

5.1 Example

5.1.1 Problem: Find p(0), p(1), p(2) for each of the following polynomials

$$(a)\, p(y) = y^2\ (b)\, p(x) = (x-1)(x+1) \tag{6.1}$$

5.1.2 Solution:

5.1. (a) To find p(0) we substitute 0 in place of the variable y in p(y). Similarly we find p(1) and p(2)

$$p(y) = y^2 \tag{5.1.1}$$
$$\implies p(0) = 0 \tag{5.1.2}$$
$$p(1) = 1 \tag{5.1.3}$$
$$p(2) = 4 \tag{5.1.4}$$

The following code sketches the graph of 5.1.1 in figure 5.1

```
codes/conic1/conic1a.py
```

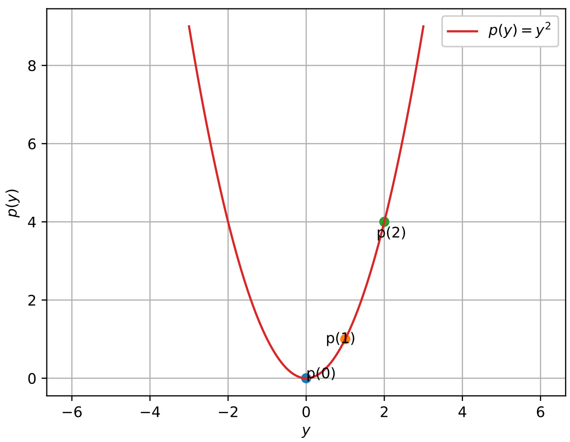

Fig. 5.1: Graph of p(y)

5.2. (b)Similarly we find p(0), p(1) and p(2) of p(x) by replacing x

$$p(x) = (x-1)(x+1) \quad (5.2.1)$$
$$\implies p(0) = -1 \quad (5.2.2)$$
$$p(1) = 0 \quad (5.2.3)$$
$$p(2) = 3 \quad (5.2.4)$$

The following code sketches the graph of 5.2.1 in figure 5.2

```
codes/conic1/conic1b.py
```

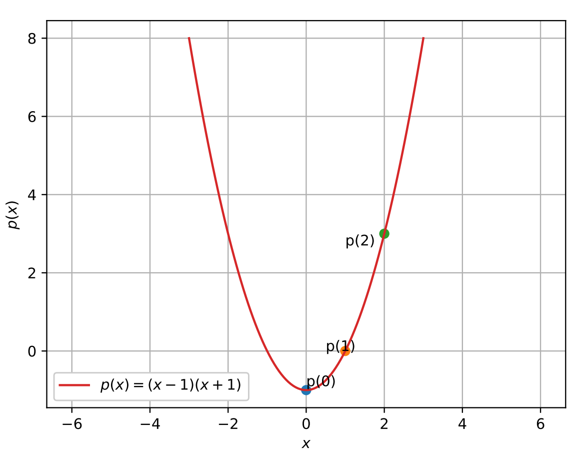

Fig. 5.2: Graph of p(x)

5.2 *Exercise*

5.2.1 Problem: Factorise:

$$(a)\, 12x^2 - 7x + 1 \quad (2.1)$$
$$(b)\, 6x^2 + 5x - 6 \quad (2.2)$$
$$(c)\, 2x^2 + 7x + 3 \quad (2.3)$$
$$(d)\, 3x^2 - x - 4 \quad (2.4)$$

5.2.2 Solution:

5.1. A conic section has the following equation5.1.1

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (5.1.1)$$

The equation is expressed in vector form is as follows The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (5.1.2)$$

If x_1 is root of an equation ,Then $(x - x_1)$ is a factor

5.2.

$$(a)\, 12x^2 - 7x + 1 \quad (5.2.1)$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 12 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (5.2.2)$$

To find roots using 5.2.2, substitute

$$y = 0 \quad (5.2.3)$$
$$\implies 12x^2 - 7x + 1 = 0 \quad (5.2.4)$$
$$x = \frac{1}{3}, \frac{1}{4} \quad (5.2.5)$$

Hence $\left(x - \frac{1}{3}\right)$ and $\left(x - \frac{1}{4}\right)$ are the factors

$$\implies (3x-1)(4x-1) = 12x^2 - 7x + 1 \quad (5.2.6)$$

The following code sketches the graph of 5.2.1 in figure 5.2

```
codes/conic2/conic2a.py
```

5.3.

$$(b)\, 6x^2 + 5x - 6 \quad (5.3.1)$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{x} - 6 = 0 \quad (5.3.2)$$

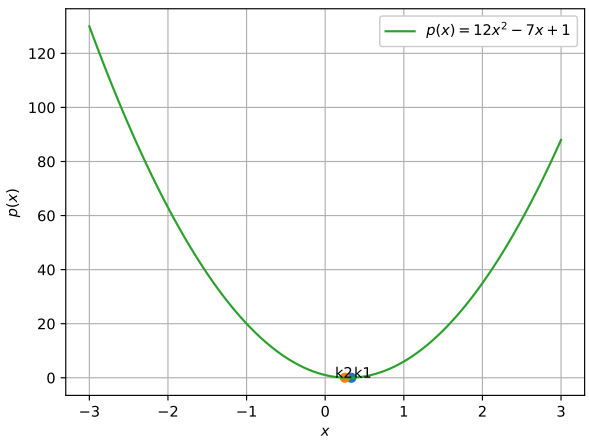

Fig. 5.2: Graph of $12x^2 - 7x + 1$

Substituting $y = 0$ in equation 5.3.2 to find roots,

$$\implies 6x^2 + 5x - 6 = 0 \tag{5.3.3}$$

$$x = \frac{-3}{2}, \frac{2}{3} \tag{5.3.4}$$

$$(2x + 3)(3x - 2) = 6x^2 + 5x - 6 \tag{5.3.5}$$

The following code sketches the graph of 5.3.1 in figure 5.3

```
codes/conic2/conic2b.py
```

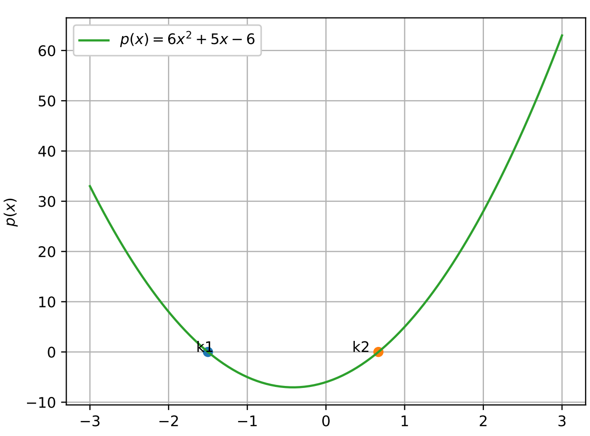

Fig. 5.3: Graph of $6x^2 + 5x - 6$

5.4.

$$(c)\, 2x^2 + 7x + 3 \tag{5.4.1}$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{5.4.2}$$

Substituting $y = 0$ in equation 5.4.2,

$$\implies 2x^2 + 7x + 3 = 0 \tag{5.4.3}$$

$$x = \frac{-1}{2}, -3 \tag{5.4.4}$$

$$(2x + 1)(x + 3) = 2x^2 + 7x + 3 \tag{5.4.5}$$

The following code sketches the graph of 5.4.1 in figure 5.4

```
codes/conic2/conic2c.py
```

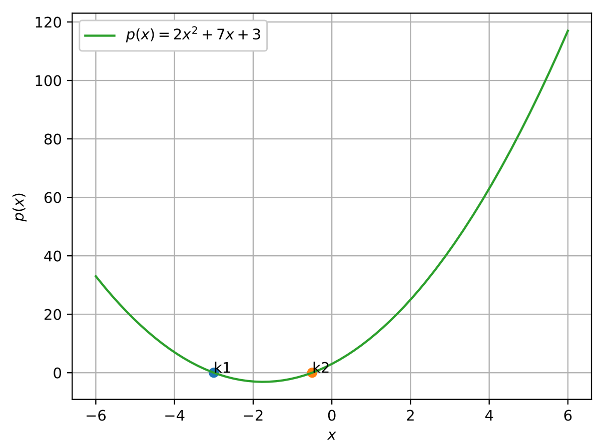

Fig. 5.4: Graph of $2x^2 + 7x + 3$

5.5.

$$(d)\, 3x^2 - x - 4 \tag{5.5.1}$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{5.5.2}$$

Substituting $y = 0$ in equation 5.4.2,

$$\implies 3x^2 - x - 4 = 0 \tag{5.5.3}$$

$$x = \frac{4}{3}, -1 \tag{5.5.4}$$

$$(3x - 4)(x + 1) = 3x^2 - x - 4 \tag{5.5.5}$$

The following code sketches the graph of 5.5.1 in figure 5.5

```
codes/conic2/conic2d.py
```

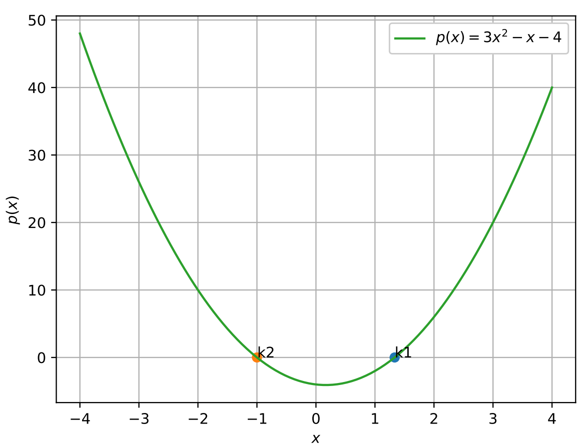

Fig. 5.5: Graph of $3x^2 - x - 4$