Parallel line through trapezium

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***Abstract*—This document proves that two chords of a circle, which when extended, intersect at an external point, the angle between them is equal to half the difference of angles subtended by the corresponding ends of the chords at the centre.**

Download python codes from

```
svn co https://github.com/krishnajakodali/
    Summer2020/trunk/circle/codes
```

and latex-tikz codes from

```
svn co https://github.com/krishnajakodali/
    Summer2020/trunk/circle/figs
```

1 Problem

Let the vertex of $\angle ABC$ be located outside a circle and let the sides of angle intersect equal chords AD and CE with the circle.
Prove that $\angle ABC$ is equal to half the difference of the angles subtended by chords AC and DE at the centre

2 Construction

2.1. The two chords AD and CE of circle with centre **O** and radius r intersect at **B** as shown in the figure 2.1

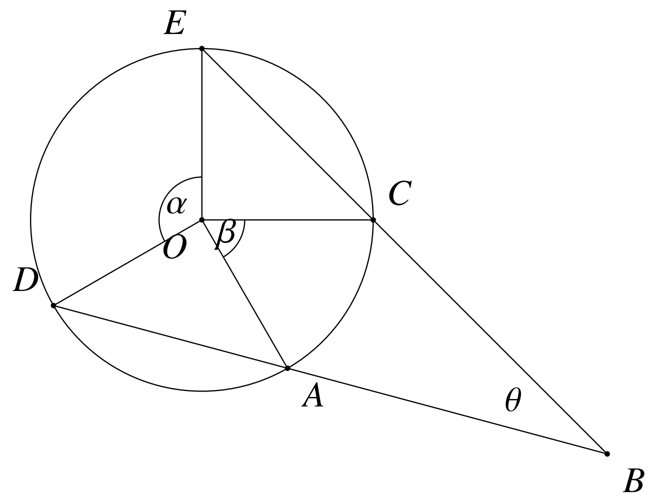

Fig. 2.1: Chords of equal length intersecting at B by Latex-Tikz

2.2. List the design parameters for construction
Solution: See Table. 2.2

Parameter	Description	Value
Radius of the circle	r	4
$\angle EOD$	α	120
$\angle COA$	β	60

TABLE 2.2: To construct circle with chords intersecting at an external point

2.3. Find the coordinates of the various points in Fig. 2.1
Solution: **O** is taken to be the origin and OC is taken to be along the x-axis.

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.3.1}$$

$$\mathbf{C} = \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{2.3.2}$$

2.4. Any point X lying on the circle satisfies the following equation

$$r = \|\mathbf{X} - \mathbf{O}\| \tag{2.4.1}$$

$$r = \|\mathbf{X}\| \tag{2.4.2}$$

$$\mathbf{X} = r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.4.3}$$

Where θ is taken to be the angle measured in anti-clockwise direction from x-axis.

2.5. Using 2.4.3

$$\implies \mathbf{A} = r\begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} \tag{2.5.1}$$

2.6. In $\triangle AOD$ and $\triangle COE$

$$OD = OE = r \tag{2.6.1}$$
$$OC = OA = r \tag{2.6.2}$$
$$AD = EC \tag{2.6.3}$$

So by SSS criteria $\triangle AOD \cong \triangle COE$

$$\angle EOC = \angle DOA \tag{2.6.4}$$
$$\angle EOC = \angle DOA = \frac{360° - (\alpha + \beta)}{2} \tag{2.6.5}$$
$$\angle EOC = \angle DOA = 180° - \frac{\alpha + \beta}{2} \tag{2.6.6}$$

2.7. Using 2.4.3 and 2.6.6

$$\mathbf{D} = r\begin{pmatrix} \cos\left(180° - \frac{\alpha+\beta}{2} + \alpha\right) \\ \sin\left(180° - \frac{\alpha+\beta}{2} + \alpha\right) \end{pmatrix} \tag{2.7.1}$$
$$\mathbf{D} = r\begin{pmatrix} \cos\left(180° + \frac{\alpha-\beta}{2}\right) \\ \sin\left(180° + \frac{\alpha-\beta}{2}\right) \end{pmatrix} \tag{2.7.2}$$
$$\mathbf{D} = r\begin{pmatrix} -\cos\frac{\alpha-\beta}{2} \\ -\sin\frac{\alpha-\beta}{2} \end{pmatrix} \tag{2.7.3}$$

$$\mathbf{E} = r\begin{pmatrix} \cos\left(180° - \frac{\alpha+\beta}{2}\right) \\ \sin\left(180° - \frac{\alpha+\beta}{2}\right) \end{pmatrix} \tag{2.7.4}$$
$$\mathbf{E} = r\begin{pmatrix} -\cos\frac{\alpha+\beta}{2} \\ \sin\frac{\alpha+\beta}{2} \end{pmatrix} \tag{2.7.5}$$

2.8. To find **B**
If BT is tangent from point B to the circle, we know AD = CE then,

$$BT^2 = BC.BE = BA.BD \tag{2.8.1}$$
$$\implies BC.(BC + CE) = BA.(BA + AD) \tag{2.8.2}$$
$$\implies BC = BA \tag{2.8.3}$$

Since BC = BA and CE =AD , **B** divides DA and EC in equal ratio say k+1:1 externally. By section formulae

$$\mathbf{B} = (k+1)\,\mathbf{A} - \mathbf{D} \tag{2.8.4}$$
$$\mathbf{B} = (k+1)\,\mathbf{C} - \mathbf{E} \tag{2.8.5}$$

equating equations 2.8.4 and 2.8.5 We get

$$k\mathbf{A} + \mathbf{A} - \mathbf{D} = k\mathbf{C} + \mathbf{C} - \mathbf{E} \tag{2.8.6}$$
$$k = \frac{\|\mathbf{C} - \mathbf{E} - \mathbf{A} + \mathbf{D}\|}{\|\mathbf{A} - \mathbf{C}\|} \tag{2.8.7}$$

the value of k is substituted in equation 2.8.4 to get **B**
The derived coordinates are listed in table 2.8.

Vector	Coordinates
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\mathbf{C}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
$\mathbf{A}$	$\begin{pmatrix} 2 \\ -2\sqrt{3} \end{pmatrix}$
$\mathbf{D}$	$\begin{pmatrix} -2\sqrt{3} \\ -2 \end{pmatrix}$
$\mathbf{B}$	$\begin{pmatrix} 9.5 \\ -5.5 \end{pmatrix}$
$\mathbf{E}$	$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

TABLE 2.8: Derived coordinates

2.9. Draw Fig. 2.1 using python
Solution: The following Python code generates Fig. 2.9

```
codes/circle.py
```

and the equivalent latex-tikz code generating Fig. 2.1 is

```
figs/circle.tex
```

The above latex code can be compiled as a standalone document as

```
figs/circle_alone.tex
```

3 Solution

3.1. Join CD as shown in the figure 3.1
3.2. Show that angle subtended by a chord at the centre is twice the angle subtended by it at any point on the circumference
Solution:
In the figure 3.2

$$OC = OD = OE = r \tag{3.2.1}$$

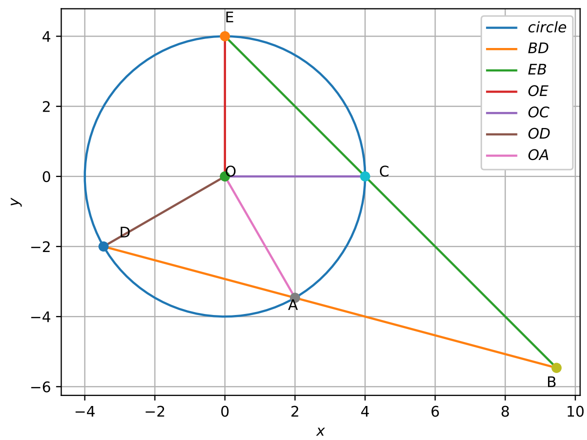

Fig. 2.9: Circle with intersecting chords generated using python

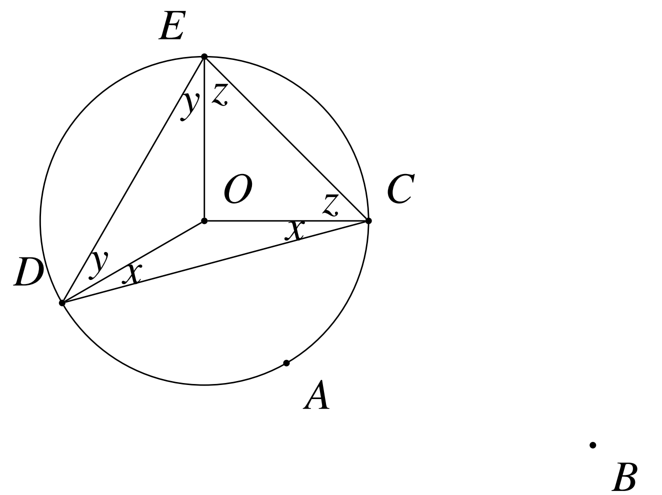

Fig. 3.2: Angles subtended by chord ED at centre O and point C by Latex-Tikz

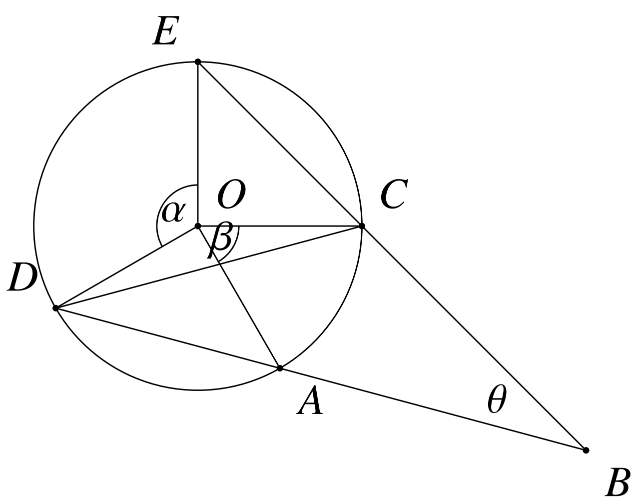

Fig. 3.1: Chords of the circle intersecting at point B by Latex-Tikz

$\triangle DOE, \triangle DOC, \triangle COE$ are isosceles triangles. Hence their base angles are equal as indicated in figure 3.2

3.3. In $\triangle EDC$ in figure 3.2 , We know sum of the angles in a triangle is $180°$

$$2(x + y + z) = 180° \tag{3.3.1}$$
$$180° - 2y = 2(x + z) \tag{3.3.2}$$

3.4. In $\triangle EOD$ in figure 3.2 as sum of the angles in a triangle is $180°$,

$$\angle EOD = 180° - 2y \tag{3.4.1}$$

Substituting equation 3.3.2 in equation 3.4.1 we get

$$\angle EOD = 2(x + z) \tag{3.4.2}$$
$$\angle EOD = 2\angle ECD \tag{3.4.3}$$
$$\angle ECD = \frac{\angle EOD}{2} = \frac{\alpha}{2} \tag{3.4.4}$$

Hence angle subtended by the chord at the centre is twice the angle subtended by it at a point on the circumference.

3.5. From 3.1 AC subtends $\angle AOC = \beta$ at the centre and $\angle ADC$ at the circumference.

$$\implies \angle ADC = \frac{\angle AOC}{2} \tag{3.5.1}$$
$$\implies \angle ADC = \frac{\beta}{2} \tag{3.5.2}$$

3.6. In $\triangle DBC$ from 3.4.4 and 3.5.2

$$\angle BCD = 180° - \angle ECD \tag{3.6.1}$$
$$\angle BCD = 180° - \frac{\alpha}{2} \tag{3.6.2}$$
$$\angle BDC = \angle ADC = \frac{\beta}{2} \tag{3.6.3}$$
$$\angle CBD = \angle CBA = \theta \tag{3.6.4}$$

Sum of the angles in a triangle is $180°$. So In $\triangle DBC$

$$\angle CBD + \angle BDC + \angle BCD = 180° \tag{3.6.5}$$

Substituting equations 3.6.4,3.6.4,3.6.4 in 3.6.5

$$\theta + \frac{\beta}{2} + 180^\circ - \frac{\alpha}{2} = 180^\circ \tag{3.6.6}$$

$$\theta = \frac{\alpha}{2} - \frac{\beta}{2} \tag{3.6.7}$$

$$\theta = \frac{\alpha - \beta}{2} \tag{3.6.8}$$

Hence proved.