

Parallel line through trapezium

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Abstract—This document proves that a line passing through a trapezium which is parallel to the parallel sides of a trapezium cuts the non-parallel sides in equal ratios.

Download python codes from

svn co <https://github.com/krishnajakodali/Summer2020/trunk/geometry/codes>

and latex-tikz codes from

svn co <https://github.com/krishnajakodali/Summer2020/trunk/geometry/figs>

Parameter	Description	Value
Length of AB	a	4
Length of CD	c	9
x-coordinate of A	x_a	4
Height of trapezium	h	3
AE / ED ratio	k	1

TABLE 2.2: To construct trapezium ABCD

1 PROBLEM

ABCD is a trapezium with $AB \parallel DC$. E and F are points on the non-parallel sides AD and BC respectively such that EF is parallel to AB. Show that:

$$\frac{AE}{ED} = \frac{BF}{FC}$$

2 CONSTRUCTION

2.1. The trapezium ABCD looks like the Fig. 2.1. with $AB \parallel DC \parallel EF$

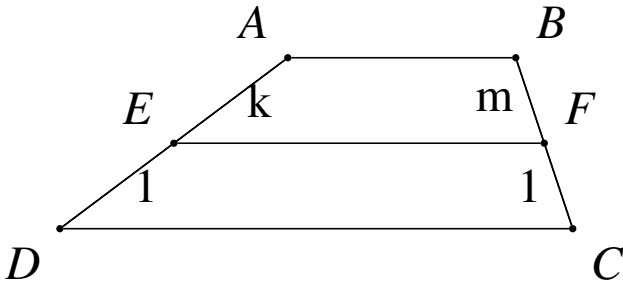


Fig. 2.1: Trapezium ABCD by Latex-Tikz

2.2. List the design parameters for construction

Solution: See Table. 2.2

2.3. Find the coordinates of the various points in Fig. 2.1

Solution: From the given information,

$$\mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.3.1)$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix} \quad (2.3.2)$$

\therefore A x coordinate is given, y coordinate is same as the height of the trapezium.

$$\Rightarrow \mathbf{A} = \begin{pmatrix} x_a \\ h \end{pmatrix} \quad (2.3.3)$$

Also, \mathbf{B} is at a distance from \mathbf{A} and AB is parallel to x axis.

From (2.3.3), $\mathbf{A} = \begin{pmatrix} x_a \\ h \end{pmatrix}$ Let $\mathbf{B} = \begin{pmatrix} x_b \\ h \end{pmatrix}$

$$a^2 = \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A} - \mathbf{B}\|^T \|\mathbf{A} - \mathbf{B}\| \quad (2.3.4)$$

$$= \mathbf{A}^T \mathbf{A} + \mathbf{B}^T \mathbf{B} - \mathbf{A}^T \mathbf{B} - \mathbf{B}^T \mathbf{A} \quad (2.3.5)$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T \mathbf{B} \quad (\because \mathbf{A}^T \mathbf{B} = \mathbf{B}^T \mathbf{A}) \quad (2.3.6)$$

$$= x_a^2 + h^2 + x_b^2 + h^2 - 2(x_a x_b + h^2) \quad (2.3.7)$$

$$= (x_a - x_b)^2 \quad (2.3.8)$$

$$x_b = x_a + a \quad (2.3.9)$$

$$(2.3.10)$$

E divides AD in the ratio k: 1.

$$\frac{AD}{ED} = k + 1 \quad (2.3.11)$$

$$\mathbf{E} = \frac{k\mathbf{D} + \mathbf{A}}{k + 1} \quad (2.3.12)$$

$$(2.3.13)$$

Let **F** divide BC in ratio m:1.

$$\mathbf{F} = \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \quad (2.3.14)$$

$$(2.3.15)$$

EF is parallel to DC and hence the x axis. So we equate y coordinates of E and F to find m. from 2.3.13 and 2.3.15

$$\frac{h}{k + 1} = \frac{h}{m + 1} \quad (2.3.16)$$

$$m = k \quad (2.3.17)$$

$$\mathbf{F} = \frac{k\mathbf{C} + \mathbf{B}}{k + 1} \quad (2.3.18)$$

$$(2.3.19)$$

The values are listed in 2.3.

D	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
C	$\begin{pmatrix} 9 \\ 0 \end{pmatrix}$
A	$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$
B	$\begin{pmatrix} 8 \\ 3 \end{pmatrix}$
E	$\begin{pmatrix} 2 \\ 1.5 \end{pmatrix}$
F	$\begin{pmatrix} 8.5 \\ 1.5 \end{pmatrix}$

TABLE 2.3: Derived coordinates trapezium ABCD

2.4. Draw Fig. 2.1 using python

Solution: The following Python code generates Fig. 2.1

```
codes/prob.py
```

and the equivalent latex-tikz code generating Fig. 2.1 is

```
figs/prob.tex
```

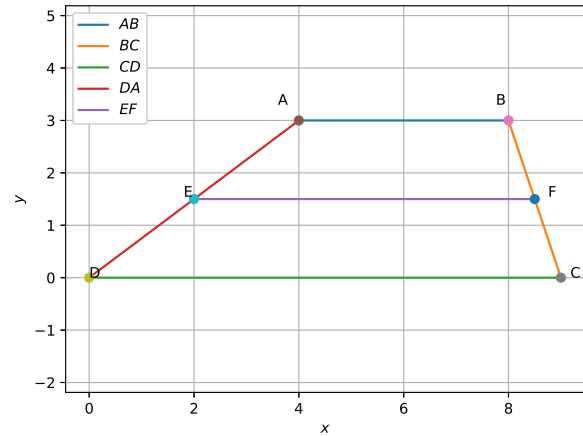


Fig. 2.4: Trapezium ABCD generated using python

The above latex code can be compiled as a standalone document as

```
figs/prob_alone.tex
```

3 SOLUTION

3.1. Let F divide CD in the ratio m:1. Then using section formulae,

$$\mathbf{F} = \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \quad (3.1.1)$$

Also using 2.3.13

$$\mathbf{E} = \frac{k\mathbf{D} + \mathbf{A}}{k + 1} \quad (3.1.2)$$

$$\Rightarrow \mathbf{E} - \mathbf{F} = \frac{k\mathbf{D} + \mathbf{A}}{k + 1} - \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \quad (3.1.3)$$

If D is taken as the origin then $\mathbf{D} = \mathbf{0}$. So 3.1.3 becomes

$$\Rightarrow \mathbf{E} - \mathbf{F} = \frac{\mathbf{A}}{k + 1} - \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \quad (3.1.4)$$

3.2. AB is parallel to DC. So for some constant l

$$\Rightarrow \mathbf{C} - \mathbf{D} = l(\mathbf{B} - \mathbf{A}) \quad (3.2.1)$$

$$\Rightarrow \mathbf{C} = l(\mathbf{B} - \mathbf{A}) \quad (3.2.2)$$

3.3. Substituting 3.2.2 in 3.1.4 We get

$$\Rightarrow \mathbf{E} - \mathbf{F} = \frac{\mathbf{A}}{k + 1} - \frac{ml(\mathbf{B} - \mathbf{A}) + \mathbf{B}}{m + 1} \quad (3.3.1)$$

$$\Rightarrow \mathbf{E} - \mathbf{F} = \mathbf{A} \left(\frac{1}{k + 1} + \frac{lm}{m + 1} \right) - \mathbf{B} \left(\frac{ml + 1}{m + 1} \right) \quad (3.3.2)$$

But $EF \parallel AB$. So, For some w

$$\implies \mathbf{E} - \mathbf{F} = w(\mathbf{A} - \mathbf{B}) \quad (3.3.3)$$

Comparing 3.3.3 and 3.3.2. We have

$$\frac{1}{k+1} + \frac{lm}{m+1} = \frac{ml+1}{m+1} = w \quad (3.3.4)$$

$$\implies m+1 + (k+1)lm = (lm+1)(k+1) \quad (3.3.5)$$

3.3.5 must hold true for all values of l, k and m . Thus $m = k$

Hence F divides BC in ratio $k:1$, same as the ratio in which E divides AD .