MIRI
Data Mining & Business Intelligence

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Session 2:
Canonical Correlation Analysis

Sessions on Multivariate Modeling.

Course on Kernel Based Learning and Multivariate Modeling

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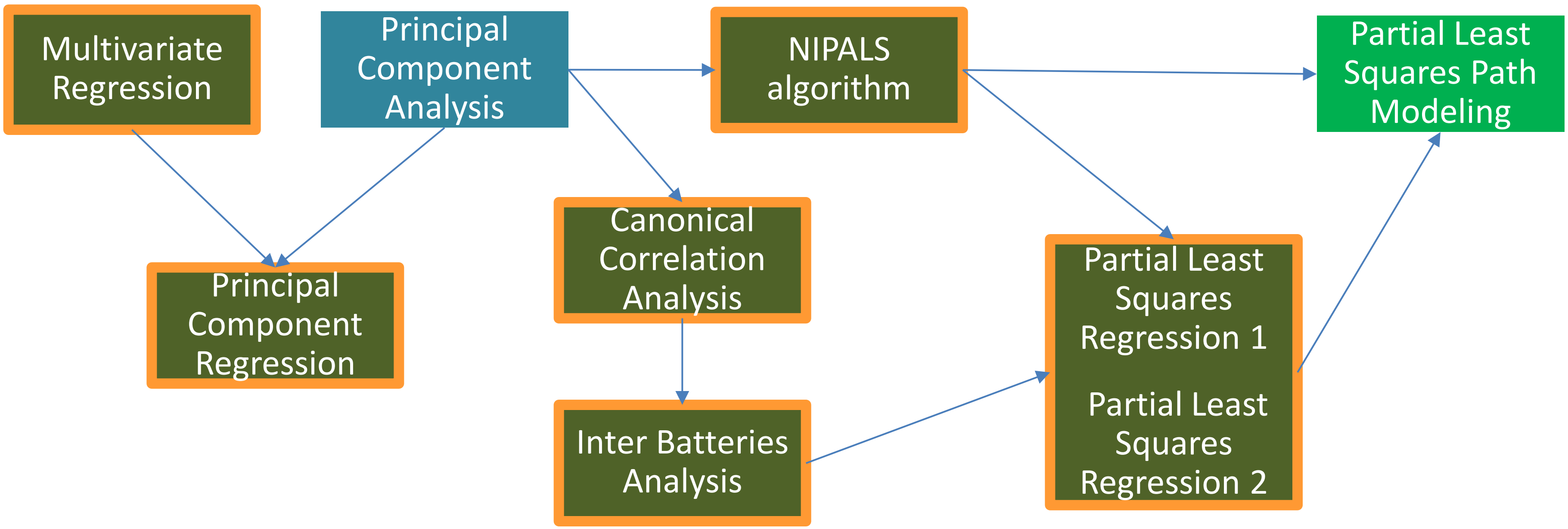
INFORMATION MODELING & PROCESSING

The plan of the course



Multivariate Descriptive techniques

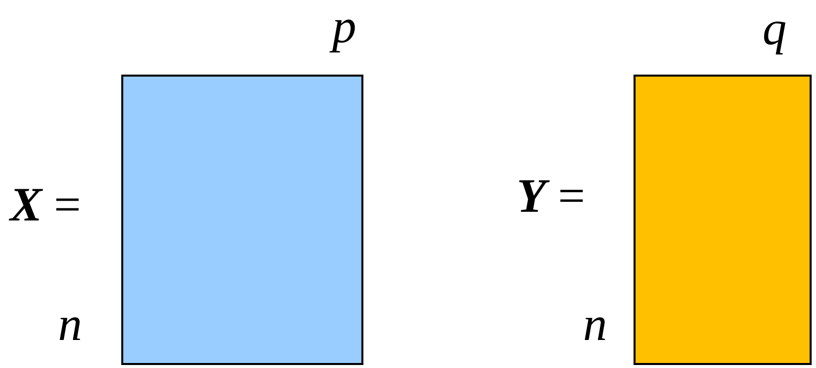
Criterion to optimize:	- in PCA, component with max. variance $\max \operatorname{var}(t_h)$ - in CCA, components with max. correlation - in …

Restrictions:	on R^p → normalized weights $\|w_h\| = 1$ on R^n → normalized components $\|t_h\| = 1$

The CCA problem

We have two vectors of variables measured on n individuals: $(x_1, x_2, \cdots, x_p)$

$(y_1, y_2, \cdots, y_q)$



The goal is to measure the relationship between both multivariate vectors

We assume that we can each group can be summarized by a set of components

We compute these components from each group x and y as ***the ones most related with the components of the other group***

In CCA we treat both groups symmetrically, but in fact group x is the explanatory group and group y is the response one.

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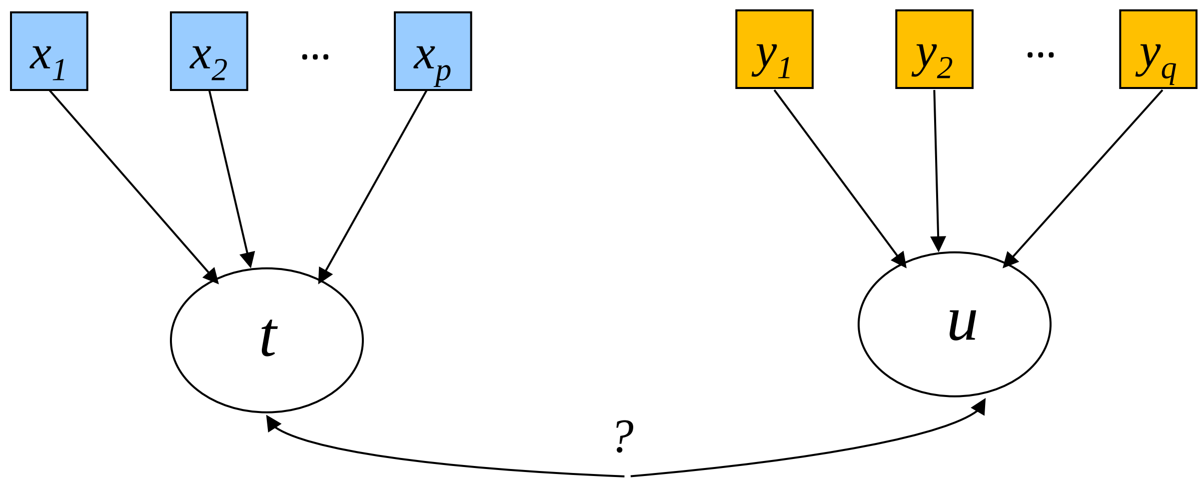
The CCA graphical model



t is a component for the X group,
u is a component for the Y group

- How to obtain these components?
- How to measure their relationship?
- How many components do we need per each group?

The Canonical Correlation Analysis Approach

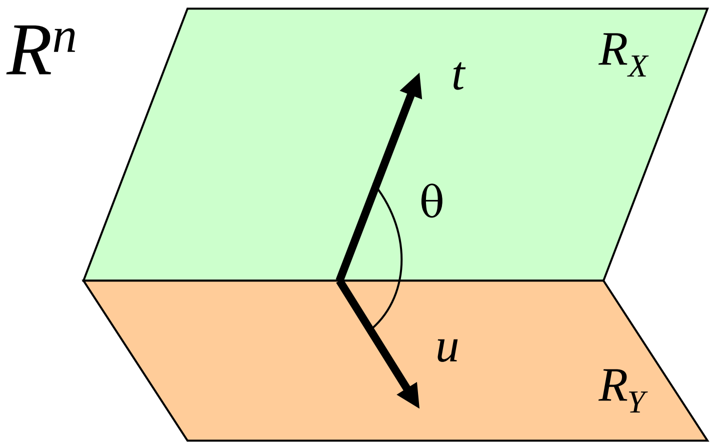
We want to find pairs of vectors t_h and u_h:

$$\max cor(t_h, u_h)$$

$$t_h = Xa_h \quad u_h = Yb_h$$

$$a_h \in \mathbb{R}^p \quad b_h \in \mathbb{R}^q$$

We assume X and Y centered and standardized (not necessary but very often assumed) and of full rank.



Since we are interested in the correlation we take standardized components:

$$\|t_h\| = \|u_h\| = 1$$

$$t'_h N t_{h'} = 0 \quad u'_h N u_{h'} = 0$$ and orthogonal

Solution in R^p and R^q

Taking N as R^n metric

$$\max\ cor(t_h, u_h) = a_h' X' N Y b_h$$
$$t_h' N t_h = a_h' X' N X a_h = 1$$
$$u_h' N u_h = b_h' Y' N Y b_h = 1$$

$$Max \quad a_h' R_{XY} b_h$$
$$a_h' R_X a_h = 1$$
$$b_h' R_Y b_h = 1$$

Assuming standardized data, otherwise we would have covariance matrices

Mahalanobis metric in R^p and R^q

$$\ell = a_h' R_{XY} b_h - \lambda/2 (a_h' R_X a_h - 1) - \beta/2 (b_h' R_Y b_h - 1)$$

$$\frac{\partial \ell}{\partial a_h} \rightarrow \quad R_{XY} b_h = \lambda R_X a_h$$
$$\frac{\partial \ell}{\partial b_h} \rightarrow \quad R_{YX} a_h = \beta R_Y b_h$$

*

$$a_h' R_{XY} b_h = \lambda\, a_h' R_X a_h = \lambda$$
$$b_h' R_{YX} a_h = \beta\, b_h' R_Y b_h = \beta$$

$$\lambda = \beta$$

$$\lambda a_h = R_X^{-1} R_{XY} b_h$$
$$\lambda b_h = R_Y^{-1} R_{YX} a_h$$

$$\lambda = cor(t_h, u_h) = r_h$$

$$\Rightarrow \quad R_X^{-1} R_{XY} R_Y^{-1} R_{YX} a_h = \lambda^2 a_h$$
$$R_Y^{-1} R_{YX} R_X^{-1} R_{XY} b_h = \lambda^2 b_h$$

*

$$t_h = X a_h$$
$$u_h = Y b_h$$

Canonical components of X and Y

***rank(R_{XY})=min{rank(R_X),rank(R_Y)}* components**

CCA in practice

usually $q<p$

* $$R_Y^{-1} R_{YX} R_X^{-1} R_{XY} b_h = r_h^2 b_h$$
$$b_h' R_Y^{1} b_h = 1$$

symmetrisation

$$R_Y^{-\frac{1}{2}} R_{YX} R_X^{-1} R_{XY} R_Y^{-\frac{1}{2}} R_Y^{\frac{1}{2}} b_h = r_h^2 R_Y^{\frac{1}{2}} b_h$$

$$R_Y^{-\frac{1}{2}} R_{YX} R_X^{-1} R_{XY} R_Y^{-\frac{1}{2}} \dot{b}_h = r_h^2 R_Y^{\frac{1}{2}} \dot{b}_h$$

$$\dot{b}_h' \dot{b}_h = 1 \qquad \dot{b}_h = R_Y^{\frac{1}{2}} b_h$$

* $$r_h R_X a_h = R_{XY} b_h$$

$$a_h = \frac{1}{r_h} R_X^{-1} R_{XY} R_Y^{-\frac{1}{2}} R_Y^{\frac{1}{2}} b_h = \frac{1}{r_h} R_X^{-1} R_{XY} R_Y^{-\frac{1}{2}} \dot{b}_h$$

$$t_h = X a_h \qquad t_h^{test} = X^{test} a_h$$

$$u_h = Y b_h = Y R_Y^{-\frac{1}{2}} \dot{b}_h$$

power of a matrix by SVD

$$A = UDV' \qquad A^s = UD^sV'$$

Dual solution in R^n

Solution in R^p and R^q

$$R_Y^{-1} R_{YX} R_X^{-1} R_{XY} b_h = r_h^2 b_h$$
$$R_X^{-1} R_{XY} R_Y^{-1} R_{YX} a_h = r_h^2 a_h$$

$$Y(Y'NY)^{-1} Y'N X(X'NX)^{-1} X'NYb_h = r_h^2 Yb_h$$
$$X(X'NX)^{-1} X'NY(Y'NY)^{-1} Y'NXa_h = r_h^2 Xa_h$$

Solution in R^n

$$t_h = Xa_h \qquad \Pi_X = X(X'NX)^{-1} X'N$$
$$u_h = Yb_h \qquad \Pi_Y = Y(Y'NY)^{-1} Y'N$$

Orthogonal projectors

$$\Pi_Y \Pi_X u_h = r_h^2 u_h$$
$$\Pi_X \Pi_Y t_h = r_h^2 t_h$$

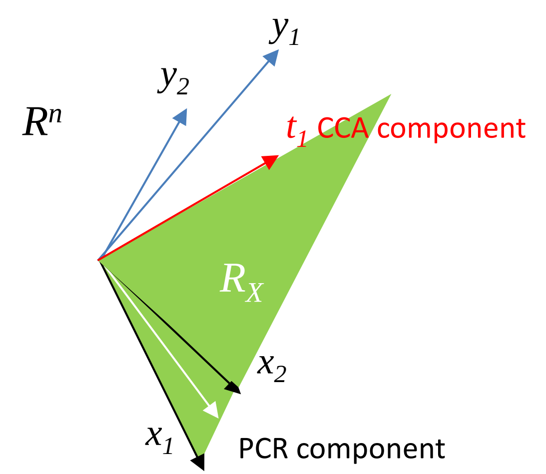
$$a_h' R_X a_h = a_h' X' N X a_h = t_h' N t_h = 1$$
$$b_h' R_Y b_h = b_h' Y' N Y b_h = u_h' N u_h = 1$$

Properties of the canonical components

CCA components are different from PCRs'

$$cor(Xa_h, Yb_h) = \mathrm{cov}(Xa_h, Yb_h) \Big/ \sqrt{\mathrm{var}(Xa_h)}\sqrt{\mathrm{var}(Yb_h)}$$

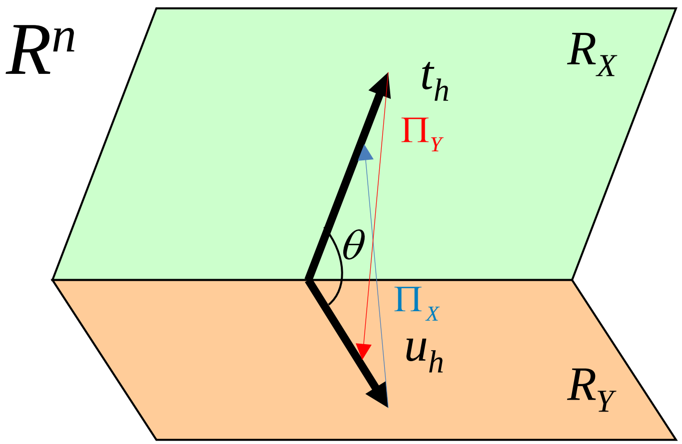
PCA criterion



Maximising the correlation implies that we don't take care of the variances (PCA criterion), hence canonical variates don't need to explain each own group.

Properties of the canonical components

t_h and u_h need to be collinear



$r = \lambda = \cos(\theta)$ canonical correlation

$$* \; R_{XY} b_h = r_h R_X a_h \qquad X'NYb_h = r_h X'NXa_h$$

$$R_{YX} a_h = r_h R_Y b_h \qquad Y'NXa_h = r_h Y'NYb_h$$

$$X\left(X'NX\right)^{-1} X'NYb_h = r_h X a_h$$

$$Y\left(Y'NY\right)^{-1} Y'NXa_h = r_h Y b_h$$

$$\Pi_X = X(X'NX)^{-1} X'N$$

$$\Pi_Y = Y(Y'NY)^{-1} Y'N$$

$$\boxed{\begin{aligned}\Pi_X u_h &= r_h t_h \\ \Pi_Y t_h &= r_h u_h\end{aligned}}$$

We have: $$cor^2(t_h, u_h) = R^2(u_h, X) = R^2(t_h, Y) = \cos^2\theta = r_h^2$$

Properties of the canonical components

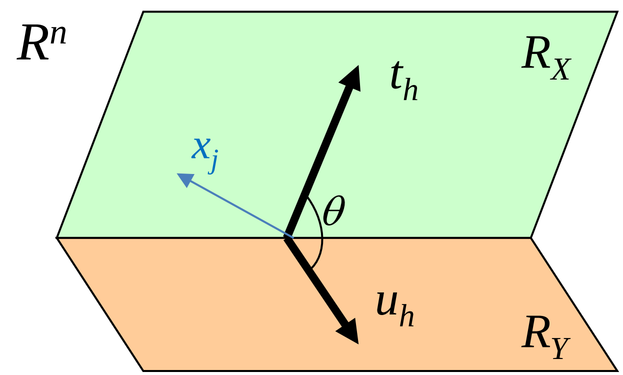
y_1

Relationships with correlations

* $X'NYb_h = r_h X'NXa_h \qquad X'Nu_h = r_h X'Nt_h$

$Y'NXa_h = r_h Y'NYb_h \qquad Y'Nt_h = r_h Y'Nu_h$

$$\begin{cases} cor(x_j, u_h) = r_h cor(x_j, t_h) \\ cor(y_k, t_h) = r_h cor(y_k, u_h) \end{cases}$$



Orthogonality of canonical variates

$$t'_h N t_l = a_h X'NXa_l = a_h R_X a_l = 0$$

$$t'_h N u_l = a_h X'Nu_l = a_h (r_l X'Nt_l) = r_l t'_h N t_l = 0$$

Graphical displays of individuals

of individuals

2 displays (*not optimal, not interesting*)

$$Xa_h = t_h \qquad (t_1, t_2), \ldots$$

$$Yb_h = u_h \qquad (u_1, u_2), \ldots$$

Display of individuals in the (t_h, u_h) basis.
Just to reveal the strength of the liaison

$$(t_1, u_1), (t_2, u_2), \ldots$$

Graphical displays

of Variables

As correlation with the canonical components

- on $(t_h\ t_{h'})$ basis

$$X'Nt_h = \begin{pmatrix} \vdots \\ cor(x_j, t_h) \\ \vdots \end{pmatrix}, \quad Y'Nt_h = \begin{pmatrix} \vdots \\ cor(y_j, t_h) \\ \vdots \end{pmatrix}$$

- on the $(u_h\ u_{h'})$ canonical components

$$X'Nu_h = \begin{pmatrix} \vdots \\ cor(x_j, u_h) \\ \vdots \end{pmatrix}, \quad Y'Nu_h = \begin{pmatrix} \vdots \\ cor(y_j, u_h) \\ \vdots \end{pmatrix}$$

Singular Value Decomposition and CCA Biplot

equivalence svd – eigen:

$$X = U\Lambda V' \begin{cases} X'X = V\Lambda^2 V' & X'XV = V\Lambda^2 \\ XX' = U\Lambda^2 U' & XX'U = U\Lambda^2 \end{cases}$$

$$V'V = I$$

$$U'U = I$$

$$* \quad R_X^{-1/2} R_{XY} R_Y^{-1} R_{YX} R_X^{-1/2} R_X^{1/2} a_h = r_h^2 R_X^{1/2} a_h$$

$$R_Y^{-1/2} R_{YX} R_X^{-1} R_{XY} R_Y^{-1/2} R_Y^{1/2} b_h = r_h^2 R_Y^{1/2} b_h$$

$$A = \begin{array}{c} a_h \\ \left(\vdots \quad \vdots \quad \vdots \right) \end{array} \qquad B = \begin{array}{c} b_h \\ \left(\vdots \quad \vdots \quad \vdots \right) \end{array}$$

$$R_X^{-1/2} R_{XY} R_Y^{-1/2} = R_X^{1/2} A \Lambda B' R_Y^{1/2}$$

$$R_{XY} = R_X A \overbrace{\begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_s \end{pmatrix}}^{\Lambda} B' R_Y = X'NXA\Lambda B'Y'NY = \sum_h r_h X'N t_h u_h' NY$$

$$cor(x_j, y_k) = \sum_{h=1}^{s} cor(x_j, t_h) \times cor(y_k, t_h) = \sum_{h=1}^{s} cor(x_j, u_h) \times cor(y_k, u_h)$$

Biplot of vars. in (t_1,t_2) basis

Biplot of vars. in (u_1,u_2) basis

Interpreting the results

- Loadings

OLS coef. of *X* respect to the components

correlations

$$\begin{pmatrix} \vdots \\ cor(x_j, t_h) \\ \vdots \end{pmatrix} = R_X a_h$$

$$\begin{pmatrix} \vdots \\ cor(y_j, u_h) \\ \vdots \end{pmatrix} = R_Y b_h$$

- Communality (part of explained variance of variables from their own component(s))

$$R^2(x_j, t_h) = cor^2(x_j, t_h) \qquad R^2(y_k, u_h) = cor^2(y_k, u_h)$$

$$R^2(x_j; t_1, \cdots, t_s) = \sum_{h=1}^{s} cor^2(x_j, t_h) \qquad R^2(y_k; u_1, \cdots, u_s) = \sum_{h=1}^{s} cor^2(y_k, u_h)$$

$$R^2(X; t_h) = \frac{1}{p} \sum_{j=1}^{p} cor^2(x_j, t_h) \qquad R^2(Y; u_h) = \frac{1}{q} \sum_{k=1}^{q} cor^2(y_k, u_h)$$

Interpreting the results

Redundancy.

Part of the variance of one group explained by the canonical components of the other group.

$$Rd(y_k, t_h) = cor^2(y_k, t_h) = r_h^2 R^2(y_k, u_h)$$

$$Rd(Y; t_h) = \frac{1}{q}\sum_{k=1}^{q} cor^2(y_k, t_h) = \frac{r_h^2}{q}\sum_{k=1}^{q} R^2(y_k, u_h)$$

$$Rd(Y; t_1, \cdots, t_s) = \frac{1}{q}\sum_{h=1}^{s}\sum_{k=1}^{q} cor^2(y_k, t_h) = \sum_{h=1}^{s} r_h^2 R^2(Y, u_h)$$

Number of significant canonical variates

- By crossvalidation, computing the R2cv

for each k fold

X,Y
k-1
folds

CCA

$T_{(k\text{-}1)}$

for $r=1:rank(RXY)$

$$\hat{Y}_{(k-1)} = T_{(k-1).r}\tilde{B}_{(k-1).r} = X_{(k-1)}B_{(k-1).r}$$

k-fold

positioned as supplementary

T_k

$$\hat{Y}_k = T_{k.r}\tilde{B}_{(k-1).r} = X_k B_{(k-1).r}$$

Approximation

X,Y

CCA

T

for each k fold

$T_{(k\text{-}1)}$

T_k

for $r=1:rank(RXY)$

$$\hat{Y}_{(k-1)} = T_{(k-1.r)}\tilde{B}_{(k-1.r)}$$

$$\hat{Y}_k = T_{k.r}\tilde{B}_{(k-1.r)}$$

Number of significant canonical variates

- Assuming multivariate normality

$$(x,y) \sim N\left(\begin{pmatrix}\mu_X \\ \mu_Y\end{pmatrix}, \begin{pmatrix}\Sigma_X & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_Y\end{pmatrix}\right)$$

We can test the hypothesis about the number canonical correlations r_h are zero. Correlations of the Σ_{XY} matrix

Test of all canonical correlations are zero

$$H_0 : \rho_1 = 0, \ldots, \rho_s = 0$$

LRT:

$$H_0 : \Sigma_{XY} = 0$$
$$H_1 : \Sigma_{XY} \neq 0$$

$$\Sigma_0 = \begin{bmatrix}\Sigma_X & 0 \\ 0 & \Sigma_Y\end{bmatrix}$$

Likelihood Ratio Test

$$H_0 : \Sigma_{XY} = 0$$
$$H_1 : \Sigma_{XY} \neq 0$$

⇨ MLE estimators

$$\hat{\Sigma}_0 = \begin{bmatrix} V_X & 0 \\ 0 & V_Y \end{bmatrix}$$

$$\hat{\Sigma} = V = \begin{bmatrix} V_X & V_{XY} \\ V_{YX} & V_Y \end{bmatrix}$$

Likelihood function maximums

$$L_0(\hat{\Sigma}_0) = \frac{1}{\left|\hat{\Sigma}_0\right|^{n/2}} \mathrm{e}^{-\frac{1}{2}np} \qquad L(\hat{\Sigma}) = \frac{1}{\left|V\right|^{n/2}} \mathrm{e}^{-\frac{1}{2}np}$$

LRT:

$$-2\ln\frac{\max L_0(\Sigma_0)}{\max L(V)} = -n\ln\left|\hat{\Sigma}_0^{-1}V\right| = -n\ln\frac{|V|}{|V_X||V_Y|}$$

$$|V| = |V_x|\left|V_Y - V_{YX}V_X^{-1}V_{XY}\right|$$

$$-n\ln\left(\left|V_Y - V_{YX}V_X^{-1}V_{XY}\right| \Big/ \left|V_Y\right|\right) = -n\ln(\left|I - V_Y^{-1}V_{YX}V_X^{-1}V_{XY}\right|) = -n\ln\prod(1-r_h^2) \sim \chi^2_{p*q}$$

Finding the significant canonical correlations

Sequential tests for a decreasing non zero canonical correlations

from $h=0,\ldots,rang(V_{XY})-1$

$$H_0 : \rho_1 \neq 0, \ldots, \rho_h \neq 0, \rho_{h+1} = 0, \ldots, \rho_s = 0$$

$$\boxed{-n \ln \prod_{j=h+1}^{rang(V_{XY})} (1 - r_j^2) \sim \chi^2_{(p-h)(q-h)}}$$

CCA Regression

CCA Regression in practice:
We use the t_1, t_2, … significant canonical components as explanatory latent components of the y_j variables.

$$Y = T_{(r)}^{CCA}\tilde{B}_{(r)} + \varepsilon_Y \qquad \tilde{B}_{(r)} = (T_{(r)}^{CCA\prime} N T_{(r)}^{CCA})^{-1} T_{(r)}^{CCA\prime} NY = T_{(r)}^{CCA\prime} NY$$

$$Y = XA_{(r)}\tilde{B}_{(r)} + \varepsilon_Y = XB + \varepsilon_Y \qquad B = A_{(r)} T_{(r)}^{CCA\prime} NY$$

Limitations of CCA

- X and Y have to be of full rank
- Unstable solution if X or Y are ill conditioned (high collinearity)
- Components not representatives of the own group
- Number max. of components = $min(p,q)$
- $n > max(p,q)$

Advantages of CCA

- Optimal solution for components (max. correlation)
- Orthogonal components
- Joint representations of x and y variables are biplots

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CCA of the Linnerud case

```
> library(calibrate)
> cc <- canocor(X,Y)
```
(Jan Graffelman)

```
$ccor
        [,1]   [,2]    [,3]
[1,] 0.7956 0.0000 0.00000
[2,] 0.0000 0.2006 0.00000
[3,] 0.0000 0.0000 0.07257
```

Canonical correlations cor(t_h, u_h)

```
$A
          [,1]    [,2]   [,3]
[1,]  0.77540 -1.8844  0.191
[2,] -1.57935  1.1806 -0.506
[3,]  0.05912 -0.2311 -1.051
```

Coefficients of R^p

```
$B
         [,1]    [,2]    [,3]
[1,]  0.3495 -0.3755  1.2966
[2,]  1.0540  0.1235 -1.2368
[3,] -0.7164  1.0622  0.4188
```

Coefficients of R^q

The canonical components

$U $=t_h$

	[,1]	[,2]	[,3]
1	0.043457	-0.52961	0.89006
2	-0.496195	-0.07235	0.42509
3	-0.814622	-0.20122	-0.57639
4	-0.275645	0.93031	-0.92501
5	0.441092	-0.61749	1.61555
6	-0.189989	-0.03505	-0.05395
7	-0.265736	-1.51087	-0.14570
8	0.358221	0.24409	-0.43684
9	2.235379	-1.99768	-1.93337
10	0.410405	0.99573	0.20357
11	0.339038	0.41197	1.03596
12	0.754464	0.20810	0.87932
13	-0.017242	1.10804	-1.12032
14	-3.130297	-1.11627	-0.25711
15	0.073469	-0.55404	1.48846
16	-0.005941	-1.38503	-0.93167
17	-0.888057	0.85570	0.03307
18	0.965063	0.52626	0.96774
19	0.456816	0.90719	0.51051
20	0.006322	1.83222	-1.66898

$V $=u_h$

	[,1]	[,2]	[,3]
1	0.12682	0.13525	-1.50078
2	-0.94753	0.24574	-1.20869
3	-1.01084	0.36684	1.75684
4	-0.04927	-0.95097	1.15505
5	0.56575	-0.48833	0.58346
6	-0.71543	-0.28696	-0.68724
7	-0.39508	-0.65399	0.26119
8	-0.15094	-0.42311	-0.68745
9	1.70755	-0.91445	0.03746
10	-0.23510	3.39410	1.23503
11	0.52002	-1.25586	2.09308
12	0.69592	0.80093	-0.03821
13	0.98598	0.53262	0.02655
14	-1.88470	-0.00879	-0.34958
15	-0.95174	-0.71809	0.32626
16	0.55994	0.97554	-0.24265
17	-1.16860	-0.72003	-0.01562
18	1.38962	0.25750	-1.20998
19	1.66764	-0.18154	-0.18721
20	-0.71001	-0.10640	-1.34753

The correlations

```
$Fs
            [,1]     [,2]     [,3]
Weight -0.6206 -0.77239  0.13496
Waist  -0.9254 -0.37766  0.03099
Pulse   0.3328  0.04148 -0.94207
```

= cor(X,T) (X loadings)

```
$Gs
              [,1]   [,2]     [,3]
Tractions 0.7276 0.2370  0.64375
Push.ups  0.8177 0.5730 -0.05445
Jumps     0.1622 0.9586  0.23394
```

= cor(Y,U) (Y loadings)

```
$Fp
            [,1]     [,2]      [,3]
Weight -0.4938 -0.15491  0.009794
Waist  -0.7363 -0.07574  0.002249
Pulse   0.2648  0.00832 -0.068366
```

= cor(X,U)

```
$Gp
              [,1]    [,2]      [,3]
Tractions 0.5789 0.04752  0.046717
Push.ups  0.6506 0.11492 -0.003951
Jumps     0.1290 0.19226  0.016977
```

= cor(Y,T)

The quality of the fit

```
$fitRxy
      [,1]    [,2]     [,3]
lamb 0.633 0.04022 0.005266
frac 0.933 0.05928 0.007762
cumu 0.933 0.99224 1.000000
```

$$lamb_h = r_h^2$$

$$frac = lamb_h \Big/ \sum_h lamb_h$$

```
$fitXs
        [,1]   [,2]   [,3]
AdeX  0.4508 0.2470 0.3022
cAdeX 0.4508 0.6978 1.0000
```

X Communality $R^2(X;t_h)$

```
$fitXp
        [,1]     [,2]     [,3]
RedX  0.2854 0.009934 0.001592
cRedX 0.2854 0.295286 0.296878
```

X Redundancy $Rd^2(X;u_h)$

```
$fitYs
        [,1]   [,2]   [,3]
AdeY  0.4081 0.4345 0.1574
cAdeY 0.4081 0.8426 1.0000
```

Y Communality $R^2(Y;u_h)$

```
$fitYp
        [,1]    [,2]      [,3]
RedY  0.2584 0.01748 0.0008288
cRedY 0.2584 0.27583 0.2766555
```

Y Redundancy $Rd^2(Y;t_h)$

Number of significant dimensions

$$H_0: \rho_1 = 0, \rho_2 = 0, \rho_3 = 0$$

$$H_0: \rho_1 \neq 0, \rho_2 = 0, \rho_3 = 0$$

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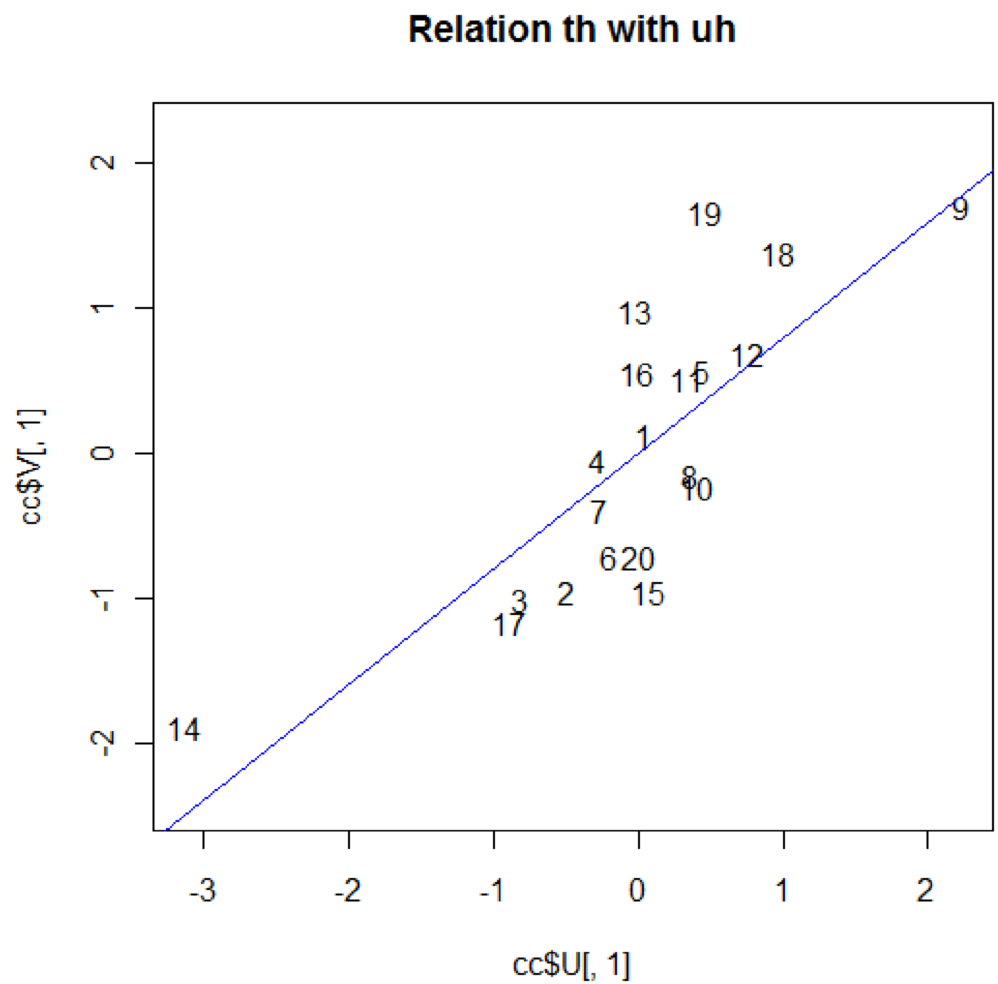
	corca2	stat.val	gd	p.val
1	0.632992	20.9741	9	0.01277
2	0.040223	0.9267	4	0.92070
3	0.005266	0.1056	1	0.74520

Just the first component is significant

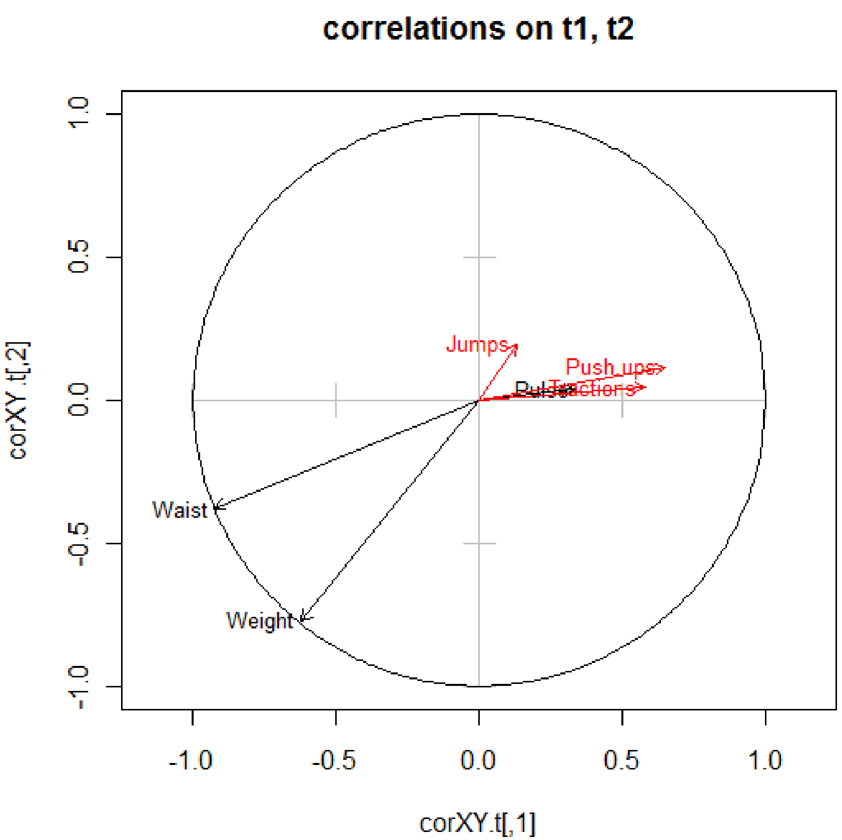
Crossvalidation of $\hat{Y} = T_{(h)}B$

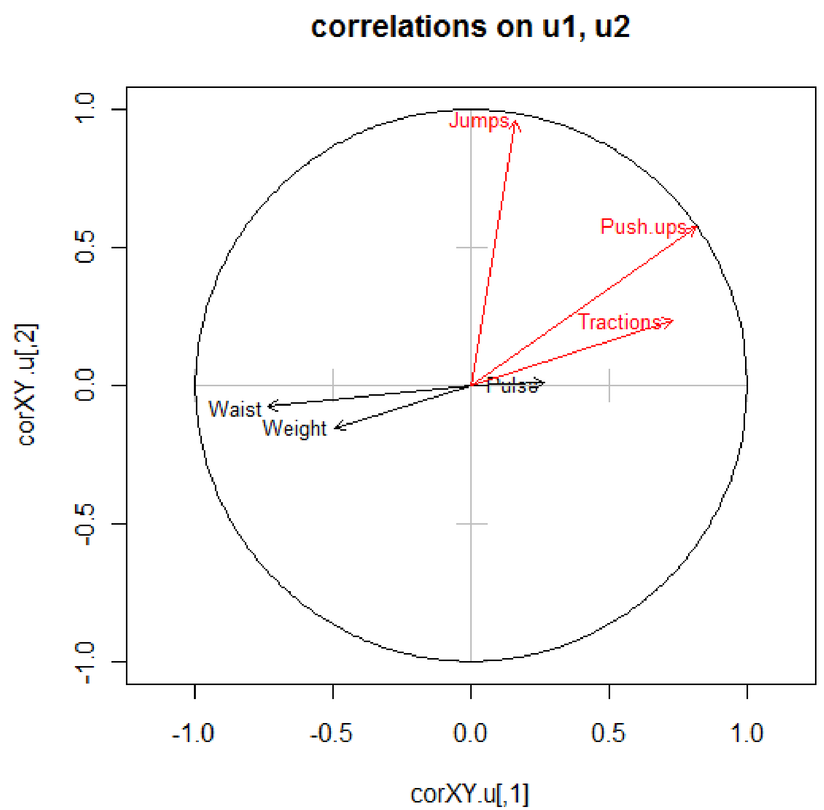
```
"Num. components: 1"
"RMPRESS“ Tractions  Push.ups     Jumps
             0.8076    0.7841    0.9924
R2cv“       0.28932   0.33007  -0.07298
"Num. components: 2"
"RMPRESS“ Tractions  Push.ups     Jumps
             0.8599    0.8488    1.0351
"R2cv“       0.1943    0.2150   -0.1674
"Num. components: 3"
"RMPRESS“ Tractions  Push.ups     Jumps
             0.9200    0.9124    1.0898
"R2cv“      0.07789   0.09289  -0.29409
```

Relation between components t_1 and u_1



Circle of correlations (biplots)



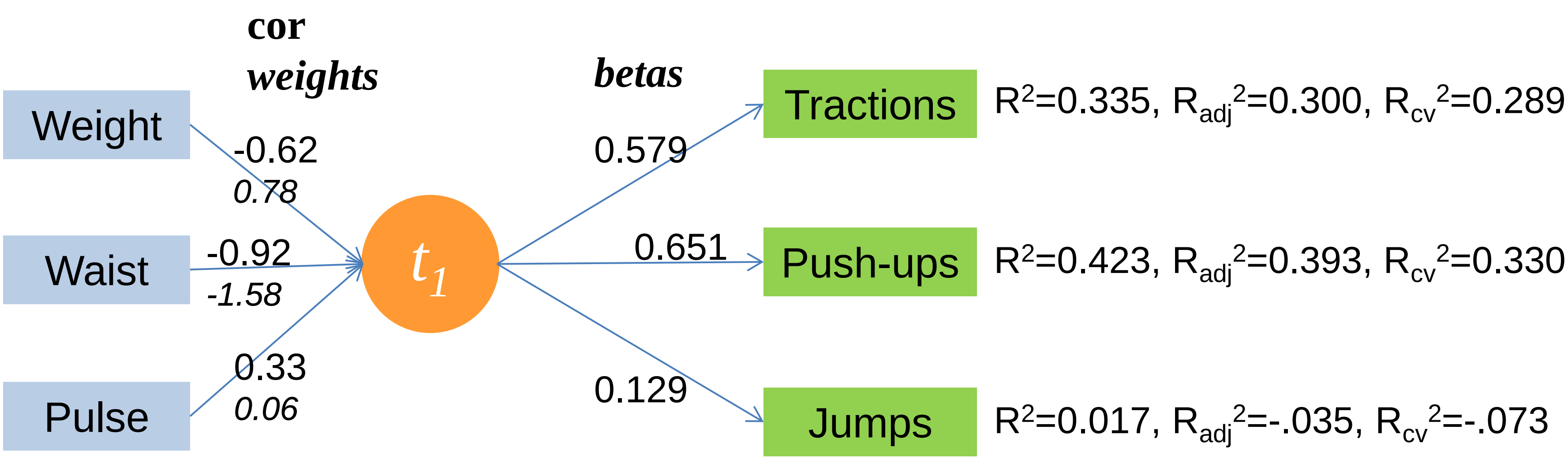


The CCA model

```
> lmY <- lm(Ys~cc$U[,1]-1)
> summary(lmY)
Response Tractions :
Coefficients:
          Estimate Std. Error t value Pr(>|t|)
cc$U[, 1]    0.579      0.187    3.09    0.006 **
Residual standard error: 0.815 on 19 degrees of freedom
Multiple R-squared:  0.335,     Adjusted R-squared:   0.3
F-statistic: 9.58 on 1 and 19 DF,  p-value: 0.00597

Response Push.ups :
Coefficients:
          Estimate Std. Error t value Pr(>|t|)
cc$U[, 1]    0.651      0.174    3.73   0.0014 **
Residual standard error: 0.759 on 19 degrees of freedom
Multiple R-squared:  0.423,     Adjusted R-squared:  0.393
F-statistic: 13.9 on 1 and 19 DF,  p-value: 0.00141

Response Jumps :
Coefficients:
          Estimate Std. Error t value Pr(>|t|)
cc$U[, 1]    0.129      0.227    0.57     0.58
Residual standard error: 0.992 on 19 degrees of freedom
Multiple R-squared:  0.0167,    Adjusted R-squared:  -0.0351
F-statistic: 0.322 on 1 and 19 DF,  p-value: 0.577

> summary(manova(lmY))
          Df Pillai approx F num Df den Df  Pr(>F)
cc$U[, 1]  1  0.633     9.77      3     17 0.00056 ***
Residuals 19
```

```
# Coefficients as functions of the
original variables

> b.coef <- cc$A[,1:nd] %*%
lmY$coefficients

> print(b.coef,digits=4)
       Tractions Push.ups     Jumps
Weight   0.44888  0.50447  0.100057
Waist   -0.91429 -1.02751 -0.203799
Pulse    0.03422  0.03846  0.007629
```

Summary of results of CCA on Linnerud case



Com = 0.451 Red = 0.258 = mean(R^2) mean(R_{cv}^2)=0.182