Euler's Totient Function

It is also known as phi function, $\phi(n)$. It counts the number of integers between 1 and n (inclusive) which are relatively prime to n.

7, 3 are relatively prime

4, 9 are coprime

a, b are coprime iff. $\gcd(a, b) = 1$

- 1 is coprime with every positive integer.
- $\phi(1) = 1$ $\{1\}$
- $\phi(2) = 1$ $\{1\}$
- $\phi(3) = 2$ $\{1, 2\}$

- $\gcd(a, a) = 1$ then $a = 1$ ($\because \gcd(a, a) = a$)

✓ prime $2, 3, 5, 7, 11 \ldots$

✓ Coprime a, b are coprime iff their gcd $= 1$

$\gcd(a, b) = 1$

eg. $\gcd(\underset{2^2}{4}, \underset{3^2}{9}) = 1$.

$\gcd(4, 6) = 2$

$\gcd(2, 6) = 2$

$\{\overset{✓}{1}, \overset{\times}{2}, \overset{\times}{3}, \overset{\times}{4}, \overset{✓}{5}, \overset{\times}{6}\}$

$U_6 = \{1, 5\}$

$\phi(6) = \boxed{2}$

✓ Phi $\phi(n) = \{1, 2, 3 \ldots, n\}$

$\phi(p) = p - 1 \quad \{\overset{✓}{1}, \overset{✓}{2}, \overset{✓}{3}, \ldots, \overset{✓}{p-1}, \overset{\times}{p}\}$

- If p is prime, $\phi(p) = p-1$

  eg. $\phi(7) = 6$, $\{1,2,3,4,5,6\}$

- $\phi(n) < n$, if $n > 1$

- $\phi(4) = 2$ $\{1, 3\}$

- $\phi(5) = 4$, • $\phi(6) = 2$ $\{1, 5\}$

- $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1) = p^k(1 - 1/p)$, p is prime

Proof: $\phi(3^7) = 3^7 - 3^6$

Not relatively prime with 3^7

$\{1, 2, 3, \ldots, p, p+1, \ldots, 2p, 2p+1 \ldots, \ldots, p^k\}$

eg. $\phi(3^3)$

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,$
$22, 23, 24, 25, 26, 27\}$

• All the multiples of p are NOT coprime with p^k

p^k - Multiples of p

$|p, 2p, 3p \dots, p^k|$

$$= \frac{p^k}{p} = p^{k-1}$$

$$\therefore \ \phi(p^k) = p^k - p^{k-1}$$

$$= p^{k-1}(p-1)$$

$$= p^k(1 - 1/p) \quad \boxtimes$$

• $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ if $\gcd(a,b) = 1$ ✓

> ϕ is multiplicative for coprime numbers.

eg. $\phi(36) = \phi(4 \cdot 9) = \phi(4) \cdot \phi(9) = \phi(2^2) \cdot \phi(3^2)$

$= 2 \times 6 = \underline{\underline{12}}$

- The prime factorization of n is $p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

then $\phi(n) = \phi(p_1^{k_1}) \cdot \phi(p_2^{k_2}) \cdots \phi(p_r^{k_r})$

$$= \underline{p_1}^{k_1}\left(1-\frac{1}{p_1}\right) \cdot \underline{p_2}^{k_2}\left(1-\frac{1}{p_2}\right) \cdots \underline{p_r}^{k_r}\left(1-\frac{1}{p_r}\right)$$

$$= \boxed{n \cdot \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right) \cdots \left(1-\frac{1}{p_r}\right)}$$

$$\therefore \phi(n) = n \cdot \prod_{\substack{p = \text{prime} \\ \text{factors} \\ \text{of } n}} \left(1-\frac{1}{p}\right)$$

- $\phi(a \cdot b) = \phi(a) \cdot \phi(b) \cdot \dfrac{g}{\phi(g)}$, $g = \gcd(a,b)$

[For any 2 integers a, b.

Sp. case :- if $\gcd(a,b) = 1$. $g = 1$, $\phi(g) = 1$

$\therefore \phi(a \cdot b) = \phi(a) \cdot \phi(b)$

Proof :-

$$\phi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

$$\phi(a \cdot b) = a \cdot b \prod_{p|a \cdot b} \left(1 - \frac{1}{p}\right)$$

$$= a\,b \; \frac{\prod_{p|a}\left(1 - \frac{1}{p}\right) \prod_{p|b}\left(1 - \frac{1}{p}\right)}{\prod_{p|gcd(a,b)}\left(1 - \frac{1}{p}\right)} \cdot \frac{gcd(a,b)}{gcd(a,b)}$$

$$= \frac{\phi(a) \cdot \phi(b) \cdot gcd(a,b)}{\phi(gcd(a,b))}$$

$$= \frac{\phi(a) \cdot \phi(b) \cdot g}{\phi(g)}$$

eg: $\phi(10 \times 15) = \phi(150)$

$$\phi(150) = 150 \cdot \frac{\prod_{p|10}\left(1 - \frac{1}{p}\right) \prod_{p|15}\left(1 - \frac{1}{p}\right)}{\prod_{p|5}\left(1 - \frac{1}{p}\right)}$$

2, 3, 5

2, 5 ✓ 3, 5 ✓

5

Ex =

Q. Given an integer n, find $\phi(n)$

```
int phi(n) {
    ans ← 0
    for (i: 1→n) {
        if gcd(i,n)==1
            ans++
    }
    return ans
}
```

$$\phi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

Brute force

$T = O(n \log n)$

```
{
 float
 ans = n
 for (int i=2; i*i<=n; i++){   <- O(√n)
     if (n%i==0){
        while (n%i==0){
            n=n/i;
        }
     ans= ans*(1-1.0/i); -> ans = ans-ans/i;
     }
 if (n>1)   ans= ans*(1-1.0/n); -> ans=ans-ans/n;
}
```

$\boxed{O(\sqrt{n})}$

2

$$n = \boxed{2^3 \times 3^2 \times 5^2}$$
$$\boxed{2^2 \times 3^2 \times 5^2}$$
$$2 \times 3^2 \times 5^2$$
$$3^2 \times 5^2$$
$$3 \times 5^2$$
$$5^2$$
$$5$$
$$1$$

$$n = 2^3 \times 3^2 \times 5$$
$$3^2 \times 5$$
$$\boxed{5}$$
$$5^2$$

}

$$ans = n \times \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$ns = \boxed{n \cdot \prod_{p \mid n} \left(1-\frac{1}{p}\right)}$$

$$n = 2^3 \times 5 \times 7^2 \times 11^2$$

$$\to 5 \times 7^2 \times 11^2$$

$$\to 7^2 \times 11^2$$

$$\to 11^2$$

$$\to 1$$

$i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

```cpp
int phi(int n) {
    int ans = n;
    for(int i=2; i*i<=n; i++) {
        if(n%i == 0) {
            while(n%i == 0)
                n /= i;
            ans = ans - ans/i;
        }
    }
    if(n>1) ans = ans-ans/n;
    return ans;
}
```

$\cancel{2} \times 5 = 20$

$\{1, \cancel{2}, 3, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, 9, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, 15, \cancel{16}, 17, \cancel{18}, 19, \cancel{20}\}$

$\{1, 3, \cancel{5}, 7, 9, 11, 13, \cancel{15}, 17, 19\}$

$ans = 10 \quad 8$

$\{1, 3, 7, 9, 11, 13, 17, 19\}$

$\cancel{2} \times 3^2$

$\{1, \ldots\ 36\}$ $ans = 36$

$\{1, 3, 5, \ldots, 35\}$ $ans = 18$

$-\{3, 9, 15, 21, 27, 33\}$

$\{1, 5, \ldots, 35\}$ $ans = \boxed{12}$

$\boxed{1 \le B \le A}$

$\{1, 5, \dots, 35\}$ ans = $\boxed{12}$

$\boxed{1 \le B \le A}$

gcd(A,B) * $\boxed{lcm(A,B) = A * B}$

$\Rightarrow gcd(A,B) = 1$

https://www.codechef.com/problems/COZIE

N = 8

l = 3 r = 6

$\{30, 20, 37, 20, 19, 2, 4, 10\}$

Q Queries

Sweetness factor = $\phi(A)$

$\phi(30)$ $\phi(20)$ $\phi(37)$ - - - -

prime

$\{30, 20, 37, 20, 19, 2, 4, 10\} \leftarrow N$

$\boxed{N \times \sqrt{A}}$

$\{\phi(30), \phi(20), \phi(37), \phi(20), \phi(19), \phi(2), \phi(4), \phi(10)\}$

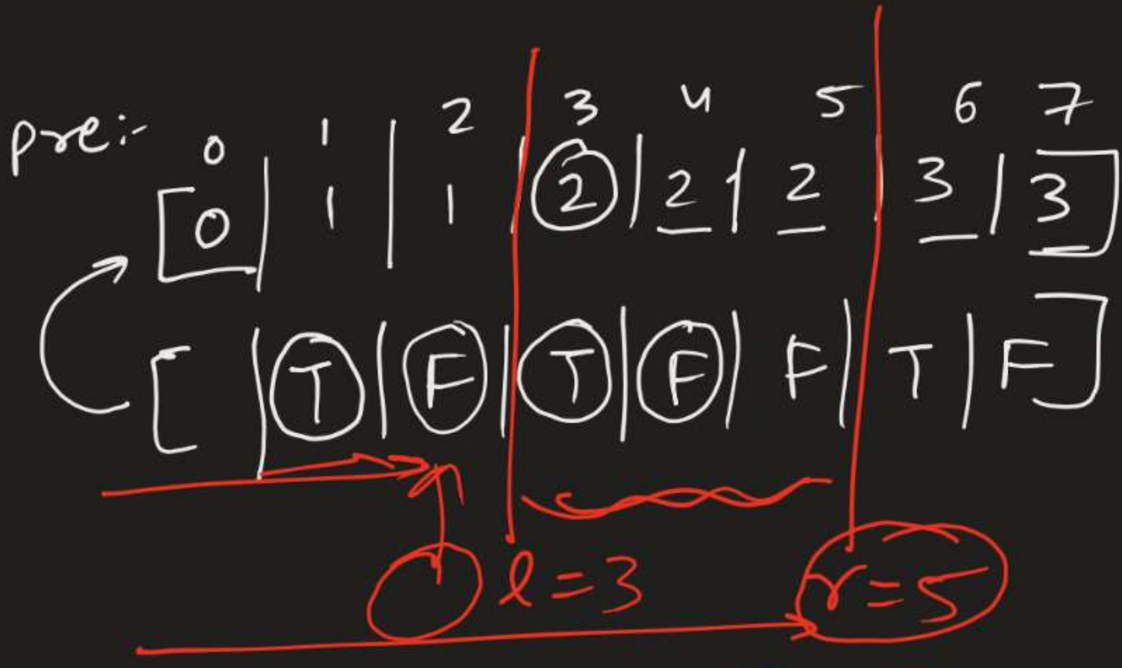
$\{8, 8, 36, 8, 18, 1, 2, 4\} \leftarrow O(N * \sqrt{A})$

$\{F, F, F, F, F, F, T, F\} \leftarrow O(N * \sqrt{A})$

ϕ Prime or not

$\{0, 0, 0, 0, 0, 0, 1, 1\}$

Prefix sum



pre[r] − pre[l−1]

```cpp
#include<bits/stdc++.h>
using namespace std;
#define endl '\n'
#define FOR(i,a,b) for(int i=(a); i<(b); i++)
#define FORk(i,a,b,k) for(int i=(a); i<(b); i+=k)
#define RFOR(i,a,b) for(int i=(a); i>=(b); i--)
#define RFORk(i,a,b,k) for(int i=(a); i>=(b); i-=k)
#define pb push_back
typedef vector<int> vi;
typedef vector<string> vs;
typedef long long int ll;
typedef unsigned long long int  ull;
typedef vector<ll> vll;
typedef vector<ull> vull;
int phi(int n) {
    int ans = n;
    for(int i=2; i*i<=n; i++) {
        if(n%i == 0) {
            while(n%i == 0)
                n /= i;
            ans = ans - ans/i;
        }
    }
    if(n>1) ans = ans-ans/n;
    return ans;
}
```

```cpp
bool is_prime(int n) {
    if(n==1) return false;
    if(n==2) return true;
    for(int i=2; i*i<=n; i++)
        if(n%i == 0)
            return false;
    return true;
}
void solve() {
    int n, q;
    cin >> n >> q;
    vi a(n+1);
    FOR(i,1,n+1)  cin >> a[i];
    vi pre(n+1, 0);
    FOR(i,1,n+1) {
        pre[i] = pre[i-1];
        if(is_prime(phi(a[i])))
            pre[i]++;
    }
    int l,r;
    FOR(i,0,q) {
        cin >> l >> r;
        cout << pre[r] - pre[l-1] << endl;
    }
}
int main() {
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    cout.tie(NULL);
    int t = 1;
    // cin >> t;
    while(t--) {
        solve();
    }
    return 0;
}
```

H/W :- Find $\phi(n)$ for all $1 \le n \le N$