Number Theory Lecture 7

Saturday, 13 July 2024 8:15 PM

Problem:- Find a^b given a, b

$$a^b = \underbrace{a * a * a \cdots * a}_{b \text{ times}}$$

Brute Force:-

```
ans = 1;
for (i 1 -> b) {
    ans = ans * a;
}
```

→ Time = $O(b)$

eg: $2^{10} = ?$

$$2^{10} = \boxed{2^5 \times 2^5} \leftarrow O(1)$$

$$2^5 = \boxed{2^2 \times 2^2 \times 2} \leftarrow O(1)$$

$$2^2 = \boxed{2^1 \times 2^1} \leftarrow O(1)$$

3 operations

a^b

$$pow(a, b) = \begin{cases} pow(a, b/2) * pow(a, b/2) & \text{if b is even} \\ pow(a, b/2) * pow(a, b/2) * a & \text{if b is odd} \\ 1, & \text{if b is 0} \end{cases}$$

```
int pow(a, b) {
    if (b==0) return 1;
    if (b%2==0)
    → return pow(a, b/2) * pow(a, b/2);
    else
    → return pow(a, b/2) * pow(a, b/2) * a;
}
```

??

$\rightarrow T(b) = 2 * T(b/2) + c$

$\boxed{T(b) = O(b)}$

$2^1 *$

$2^0 * 2^0 *$

1 1

① will this code work? Yes ✓

② Time complexity = O(b)

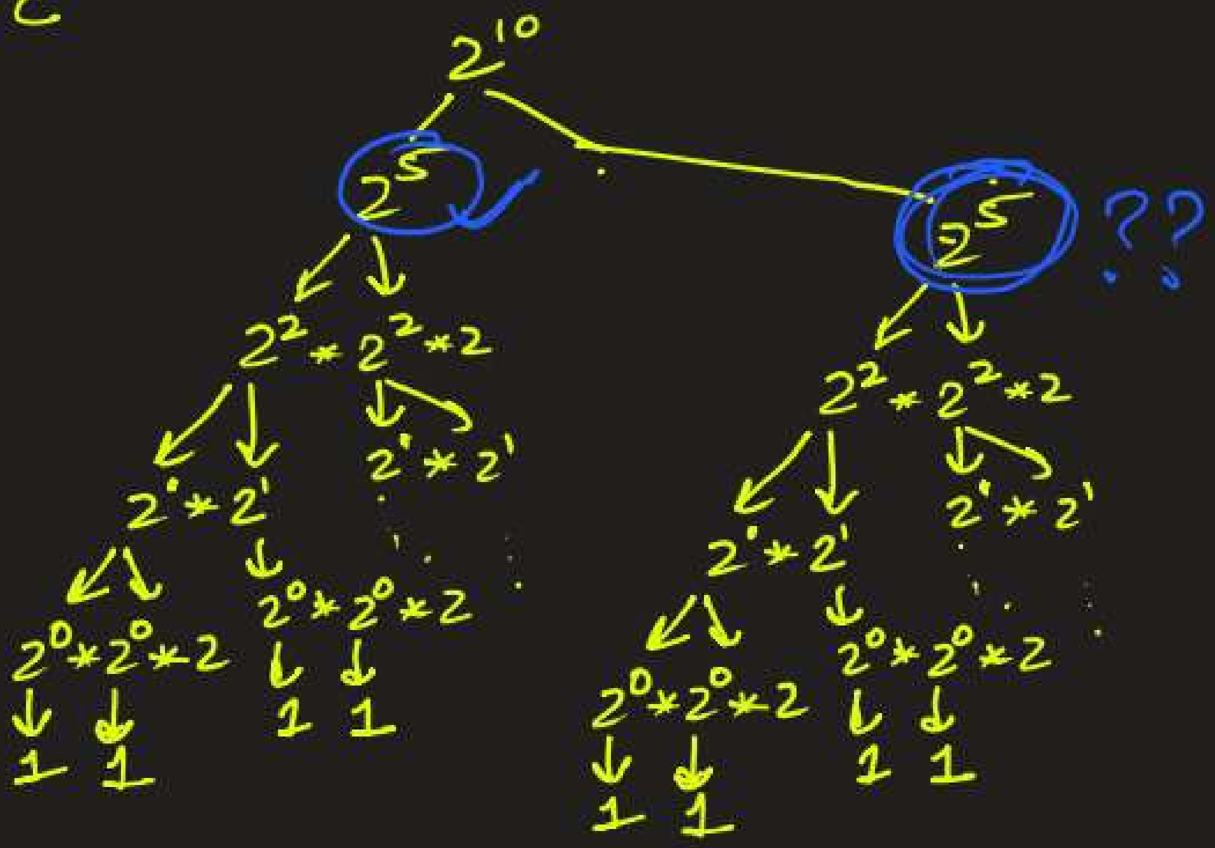
$T(n) = 2$

$= 4$

$T(b/2) + c$

$\boxed{O(b)}$

n;



$$T(n) = 2T(n/2) + c$$

$$= 4T(n/4) + c$$

$$= 8T(n/8) + c$$

$$\vdots$$

$$= n \cdot T(1) + c$$

$$= O(n)$$

= 8T(

⋮

= n. T

= O(

optimal :-

```
int pow( int a, int b) {
    if (b==0) return 1;
    int ans = pow(a, b/2);
    if (b%2==0)
            ans = ans * ans;
    else
            ans = ans * ans * a;
    return ans;
}
```

$T(b) = T(b/2) + c$

$\boxed{O(\log b)}$

$\boxed{S = O(\log b)}$

$T = O(\log b)$

$S = O(1)$

```
int ans = 1;
    while(b) {
        if(b%2 == 1) ans = ans*a;
        b >>= 1;
        a *= a;
    }
    cout << ans << endl;
```

2^{10}

ans = 1, b = 10, a = 2

ans = 1, b = 5, a = 4

ans = 4, b = 2, a = 16

ans = 4, b = 1, a = 256

$\boxed{\text{ans} = 1024}$, b = 0, a = 256 * 256;

$1007^{1007} \% \, p$ $\rightarrow$ prime number

$< p$

$a \% b < b$

int ~ $10^9 + 7$

1000000007

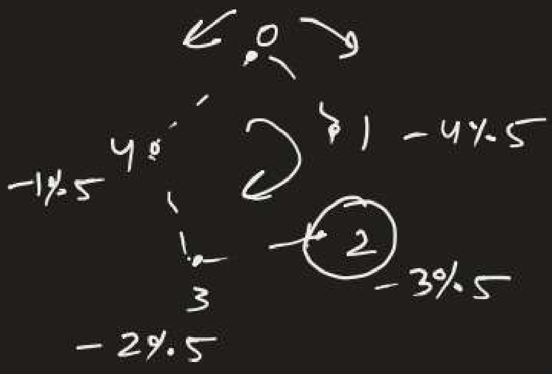
Modular Arithmetic

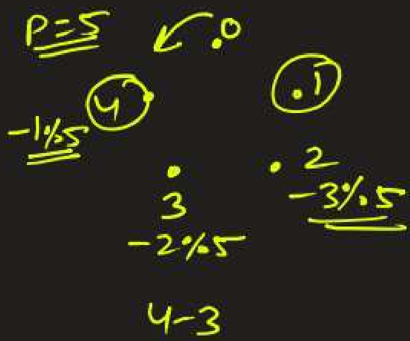
- $(a+b) \% p = ((a \% p) + (b \% p)) \% p$ ✓
- $(a-b) \% p = ((a \% p) - (b \% p)) \% p$ ✓
- $(a * b) \% p = ((a \% p) * (b \% p)) \% p$ ✓
- ~~$(a/b) \% p = ((a \% p) / (b \% p)) \% p$~~ Doesn't work

$p = 5$

$$(13+24)\%5 = (13\%5 + 24\%5)\%5$$
$$= (3+4)\%5$$
$$= 7\%5$$
$$= \boxed{2}$$

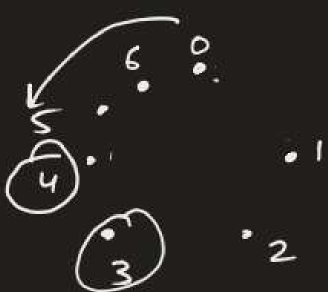
$$-3\%5 = \boxed{2}$$





$$(\underbrace{a * a * a * a \dots * a}_{b \text{ times}})\ \%\ p$$

$$= ((((a*a)\%p * a)\%p)\%p \dots * a)\%p.$$



$$(p+a)\%p = a\%p$$

$$(7+3)\%7 = 3\%7$$

$(-3) \% 7$

$= (-3+7) \% 7$

$= 4 \% 7 = 4$

$= \times =$

How to define $\left(\frac{a}{b}\right) \bmod p$?

$\frac{1}{b} \bmod p$??

modular inverse of b mod p

$\frac{1}{5} \bmod 7$

$= 5^{-1} \bmod 7$

Modular inverse of 5 mod 7

$b \times \frac{1}{b} = 1$

$(b * b^{-1}) \bmod p \equiv 1$

$(5 * 5^{-1}) \bmod 7 = 1$

$5^{-1} \bmod 7 = 3$

$(5 * 3) \bmod 7 = 1$

$\frac{1}{5} \bmod 7 = 3$

$(5 \times 1) \bmod 7 = 5$

$(5 \times 2) \bmod 7 = 3$

$(5 \times 3) \bmod 7 = \boxed{1}$

$(5 \times 4) \bmod 7 = 6$

$(5 \times 5) \bmod 7 = 4$

$(5 \times 6) \bmod 7 = 2$

$(5 \times 7) \bmod 7 = 0$

Fermat's Little Theorem

- $(a^{p-1}) \bmod p = 1$ if p is prime
  a is not a multiple of p

$p = 7 \qquad a = 4$

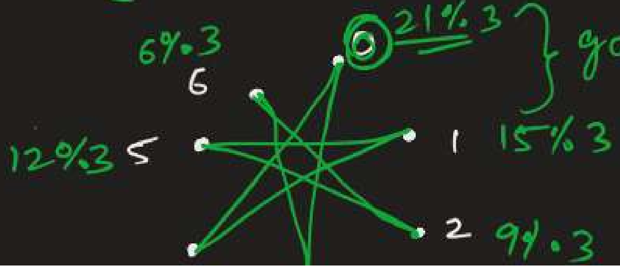
$$\left.\begin{array}{l} 5^6 \% 7 = 1 \\ 4^6 \% 7 = 1 \\ 3^6 \% 7 = 1 \\ 2^6 \% 7 = 1 \\ 1^6 \% 7 = 1 \\ 6^6 \% 7 = 1 \end{array}\right\}$$

Proof: $\Rightarrow \underbrace{a\%p,\ 2a\%p,\ 3a\%p,\ 4a\%p\ \cdots\ (p-1)a\%p}$

$\equiv \{\underbrace{1, 2, 3 \cdots, p-1}\}$

$\rightarrow$ $5\%7,\quad 10\%7,\quad 15\%7,\quad 20\%7,\quad 25\%7,\quad 30\%7$

$\{5 \quad 3 \quad 1 \quad 6 \quad 4 \quad 2\}$



} gcd = 1 $\quad p \leftarrow a$

$(a * 2a * 3a * 4a * \cdots * (p-1)a) \% p = (1 * 2 * 3 \cdots * p-1) \% p$

$= (a^{p-1} (p-1)!) \% p$

$= (p-1)! \% p$

$= [a^{p-1} \% p] [(p-1)! \% p]$

$(a^{p-1} \% p) \cdot ((p-1)! \% p) = (p-1)! \% p$

$\Rightarrow \boxed{a^{p-1} \% p = 1}$

Given that $a^{p-1} \% p = 1$

Can you calculate the modulo inverse of a mod p?

$a^{-1} \bmod p$

$X = \frac{1}{a} \bmod p$

$\Rightarrow (X \cdot a) \bmod p = 1$

$\boxed{a^{p-1} \% p = 1}$

$\Rightarrow a^{p-2} \% p = \frac{1}{a} \bmod p$

$\Rightarrow a^{-1} \bmod p = (a^{p-2}) \bmod p$

* $a^{-1} \bmod p = \boxed{a^{p-2} \bmod p}$ ✓

$5^{-1} \bmod 7 = 3$

$5^{-1} \bmod 7 = 5^{5} \bmod 7 = 3$

$3^{-1} \bmod 7 = 5$

$(3 * 5) \bmod 7 = 1$

```cpp
#include<bits/stdc++.h>
using namespace std;

const int MOD = 7;

ll pow(int a, int b) {
    if(b==0)    return 1;
    ll ans = pow(a, b/2);
    ans = (ans*ans)%MOD;
    if(b%2 == 1)  ans = (ans*a)%MOD;
    return ans;
}

ll mod_inv(int a) {
    return pow(a, MOD-2);
}

void solve() {
    for(int i=1; i<7; i++) {
        cout << mod_inv(i) << endl;
    }
}
```

H.W :- Find ${}^nC_r \% p$ given n and r , $p = 10^9+7$

limits :- T testcases

$$\boxed{\begin{array}{l} 1 \le T \le 10^5 \\ 1 \le r \le n \le 10^5 \end{array}}$$

Hint :- $${}^nC_r = \frac{n!}{(n-r)!\ r!}$$

$$\Rightarrow {}^nC_r \% p = (n! \% p) \cdot \text{mod\_inv}\,(n-r)! \% p \cdot \text{mod\_inv}\,(r! \% p)$$

$$\boxed{\frac{a}{b} \% p = \big((a \% p) \cdot (b^{-1} \% p)\big) \% p}$$