Number Theory Lecture 6

Friday, 12 July 2024 8:14 PM

$$\phi(n) = n - \sum_{p|n} \left(\frac{n}{p}\right) \leftarrow O(\sqrt{n})$$

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

H/W:- https://www.spoj.com/problems/ETF/

Find Euler's totient function of n

$1 \le n \le 10^6$

$1 \le T \le 2 \times 10^4$

$O(T \cdot \sqrt{n})$

$2 \times 10^4 \times 10^3 \sim 2 \times 10^7$ ✓

like Sieve

$n \log \log n$

$10^6 \times \log \log 10^6$

$5 \times 10^6 + 2 \times 10^7$

$\log 6 \cdot \log 10 \le \log 6 \cdot 4 < 5$

```
int phi(int n) {
    int ans = n;
    for(int i=2; i*i<=n; i++) {
        if(n%i == 0) {
            while(n%i == 0)
                n /= i;
            ans = ans - ans/i;
        }
    }
    if(n>1) ans = ans-ans/n;
    return ans;
}
```

$n = 2^3 \times 3^2 \times 7^4 \times 11^2 \ldots$

ans = n

for every prime factor p of n, ans = ans − ans/p

$p \rightarrow$ ans = ans − ans/p ✓

```
N = 1000000
phi = [i for i in range(N+1)]
for i in range(2, N+1):
    if phi[i] == i:
        for j in range(i, N+1, i):
            phi[j] -= phi[j]//i
```

$\boxed{O(N \log \log N)}$

$$\text{phi} = [0, 1, \overset{1}{\cancel{2}}, \overset{2}{\cancel{3}}, \overset{2}{\cancel{4}}, \overset{4}{\cancel{5}}, \overset{\overset{2}{\cancel{3}}}{\cancel{6}}, \overset{6}{\cancel{7}}, \overset{4}{\cancel{8}}, \overset{6}{\cancel{9}}, \overset{\overset{4}{\cancel{5}}}{\cancel{10}}]$$

```python
N = 1000000
phi = [i for i in range(N+1)]
for i in range(2, N+1):
    if phi[i] == i:
        for j in range(i, N+1, i):
            phi[j] -= phi[j]//i
t = int(input())
for _ in range(t):
    n = int(input())
    print(phi[n])
```

Gauss Theorem ⊛

For each integer $n \geq 1$, $\sum_{d|n} \phi(d) = n$

$\phi(10) = 10$

$\phi(1)$

① $\phi(2)$

② $\phi(5)$

eg. $n = 10$ 1, 2, 5, 10

$\phi(1) + \phi(2) + \phi(5) + \phi(10) = 10$

$1 + 1 + 4 + 4 = 10$

Proof: $F(n) = \sum_{d|n} \phi(d)$

$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ if $\gcd(a,b) = 1$

$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r}$ (prime factorization of n)

$F(n) = F(p_1^{k_1}) \cdot F(p_2^{k_2}) \cdots F(p_r^{k_r})$

$$F(p_1^{k_1}) = \sum_{d|p_1^{k_1}} \phi(d)$$

$$= \phi(1) + \phi(p_1) + \phi(p_1^2) + \phi(p_1^3) + \cdots + \phi(p_1^{k_1})$$

$$= \cancel{1} + (\cancel{p_1} - \cancel{1}) + (\cancel{p_1^2} - \cancel{p_1}) + (\cancel{p_1^3} - \cancel{p_1^2}) + \cdots + (p_1^{k_1} - \cancel{p_1^{k_1-1}})$$

$$= p_1^{k_1}$$

$$\therefore\ F(p_1^{k_1}) = p_1^{k_1}$$

$$\therefore\ F(n) = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r} = n$$

```
#include<bits/stdc++.h>
using namespace std;
void phi_1_to_n(int n) {
    vector<int> phi(n+1);
    for(int i=0; i<=n; i++) phi[i] = i;
    for(int i=1; i<=n; i++) {
        for(int j=2*i; j<=n; j+=i) {
            phi[j] -= phi[i];
        }
    }
}
```

$T = ?$

$i=1$	n
$i=2$	$n/2$
$i=3$	$n/3$
$\vdots$	
$i=n$	n/n

2 4 6 4

$$\text{phi} = 0,\ 1,\ \cancel{2}\ 1,\ \cancel{3}\ 2,\ \cancel{4}\ \cancel{3}\ 2,\ \cancel{5}\ 4,\ \cancel{6}\ \cancel{5}\ \cancel{4}\ 2,\ \cancel{7}\ 6,\ \cancel{8}\ \cancel{7}\ \cancel{6}\ 4,\ \cancel{9}\ \cancel{8}\ 6,\ \cancel{10}\ \cancel{9}\ \cancel{8}\ 4$$

$$T = n + \frac{n}{2} + \frac{n}{3} + \frac{n}{4} + \dots + \frac{n}{n}$$

$$= n\left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right]$$

$$= n \cdot \sum_{i=1}^{n} \frac{1}{i}$$

$$\approx n \cdot \int_1^n \frac{1}{i}\, di$$

$$= n \cdot \left[\log i\right]_1^n$$

$$= n \cdot \log n$$

$$\boxed{O(n \log n)}$$

$$n + \frac{n}{2} + \frac{n}{3} + \frac{n}{5} + \frac{n}{7} \dots$$

$$\boxed{O(n \log \log n)}$$

—×—

① Find $\sum_{k=1}^{n} \gcd(n,k)$

$n=5$: $\gcd(1,5)+\gcd(2,5)+\gcd(3,5)+\gcd(4,5)+\gcd(5,5)$

assuming $T(\gcd(a,b)) = O(\log b)$

Brute force $T = O(n \cdot \log n)$

$$\boxed{\sum_{k=1}^{n} \gcd(n,k) = \sum_{d|n} d \cdot \phi(n/d)}$$

Proof: $N = 12$, divisors of 12:- $\boxed{1, 2, 3, 4, 6, 12}$

$\gcd(1,12) + \gcd(2,12) + \gcd(3,12) + \dots + \gcd(12,12)$

$\gcd(k,12) = g$ $\quad g|k, \; g|12$

1 - $\gcd(k,12) = 1$ $\quad k \in U(12)$ $\quad \phi(12) = 4$ $\quad \phi\left(\frac{12}{1}\right)$

$\{1, 5, 7, 11\}$

② $\gcd(k,12)=2 \Rightarrow \gcd\left(\frac{k}{2},6\right)=1$ $\qquad \phi(6)=2 \quad \phi\left(\frac{12}{2}\right)$

$\frac{k}{2} \in U(6) \quad \phi(6)$

$\frac{k}{2} \in \{1,5\}$

$k \in \{2,10\}$

③ $\gcd(k,12)=3 \Rightarrow \gcd\left(\frac{k}{3},4\right)=1$ $\qquad \phi(4)=2 \quad \phi\left(\frac{12}{3}\right)$

$\frac{k}{3} \in U_4 = \{1,3\}$

$\Rightarrow k \in \{3,9\}$

ⓓ $\gcd(k,n)=d \Rightarrow \gcd\left(\frac{k}{d},\frac{n}{d}\right)=1$

$\hookrightarrow \phi\left(\frac{n}{d}\right)$

$$\therefore \boxed{\sum_{k=1}^{n} \gcd(n,k) = \sum_{d \mid n} d \cdot \phi\left(\frac{n}{d}\right)} \quad \square$$

$\sqrt{12}$

12: d :- → 1, 2, 3, 4, 6, 12

n/d :- → 12, 6, 4, 3, 2, 1

$\boxed{2\sqrt{n}}$

$2 \cdot \sqrt{n} \cdot \sqrt{n}$

$\boxed{O(n)}$

```
for (i -> 1 to √n) {
    if (n % i == 0) }
        i ✓    n/i ✓
}
```

② $$\sum_{k=1}^{n} \operatorname{lcm}(n, k) = \frac{n}{2}\left[1 + \sum_{d \mid n} d \cdot \phi(d)\right]$$

Proof :- Hints ::

① $\operatorname{lcm}(a, b) = \dfrac{a * b}{\gcd(a, b)}$

② $\gcd(a, n) = \gcd(n-a, n)$ ← Eucledian

③ $\operatorname{lcm}(a, n) + \operatorname{lcm}(n-a, n) = \dfrac{n^2}{\gcd(a, n)}$

Lemma 1: $\mathrm{lcm}(a,n)+\mathrm{lcm}(n-a,n)=\frac{an}{\gcd(a,n)}+\frac{(n-a)n}{\gcd(n-a,n)}=\frac{n\times n}{\gcd(a,n)}$.

Lemma 2: $\sum \frac{n}{\gcd(a,n)}=\sum_{f\mid n}\frac{n}{f}\times\phi(\frac{n}{f})=\sum_{d\mid n}d\phi(d)$,

Proof: consider what happens if $\gcd(a,n)=f\mid n$. It appears $\phi(\frac{n}{f})$ times on the LHS, and each time it has value of $\frac{n}{f}$. Now substitute $d=\frac{n}{f}$, which is also a divisor of n.

Now, to your problem, pull out $\mathrm{lcm}(n,n)=n$.

We have $2\sum_{a=1}^{n-1}\mathrm{lcm}(a,n)=\sum\,[\mathrm{lcm}(a,n)+\mathrm{lcm}(n-a,n)]=n\sum\frac{n}{\gcd(a,n)}=n\times\sum_{d\mid n}d\phi(d)$.

Add back $\mathrm{lcm}(n,n)=n$, and you get the formula in OEIS.