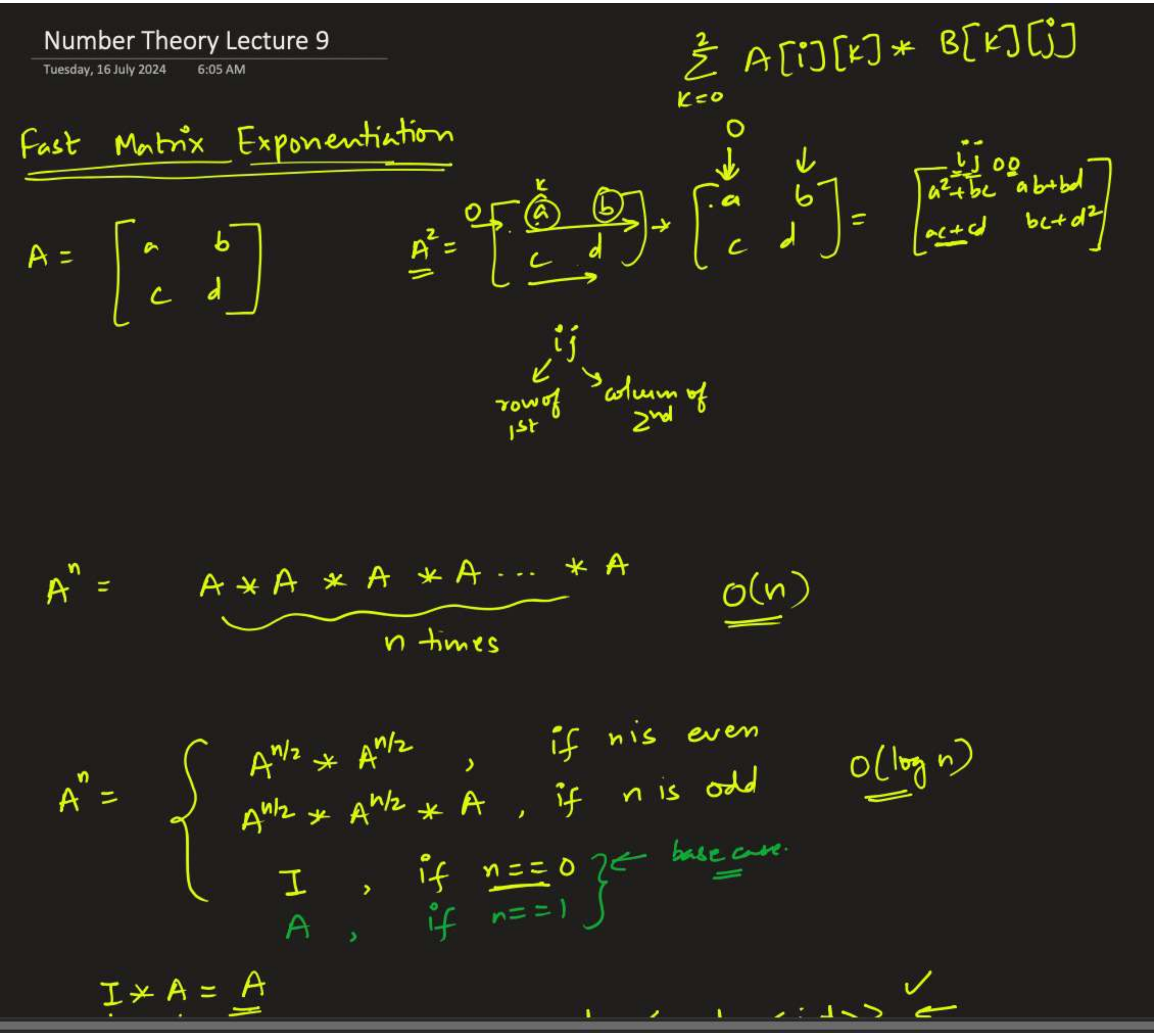


Number Theory Lecture 9

Tuesday, 16 July 2024 6:05 AM

$$\sum_{k=0}^{2} A[i][k] * B[k][j]$$

Fast Matrix Exponentiation

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\underline{\underline{A^2}} = 0 \rightarrow \begin{bmatrix} \overset{k}{a} & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} \overset{\overset{0}{\downarrow}}{a} & \overset{\downarrow}{b} \\ c & d \end{bmatrix} = \begin{bmatrix} \overset{ij\;00}{a^2+bc} & ab+bd \\ \underline{ac}+cd & bc+d^2 \end{bmatrix}$$

ij

↙ row of 1st

↘ column of 2nd

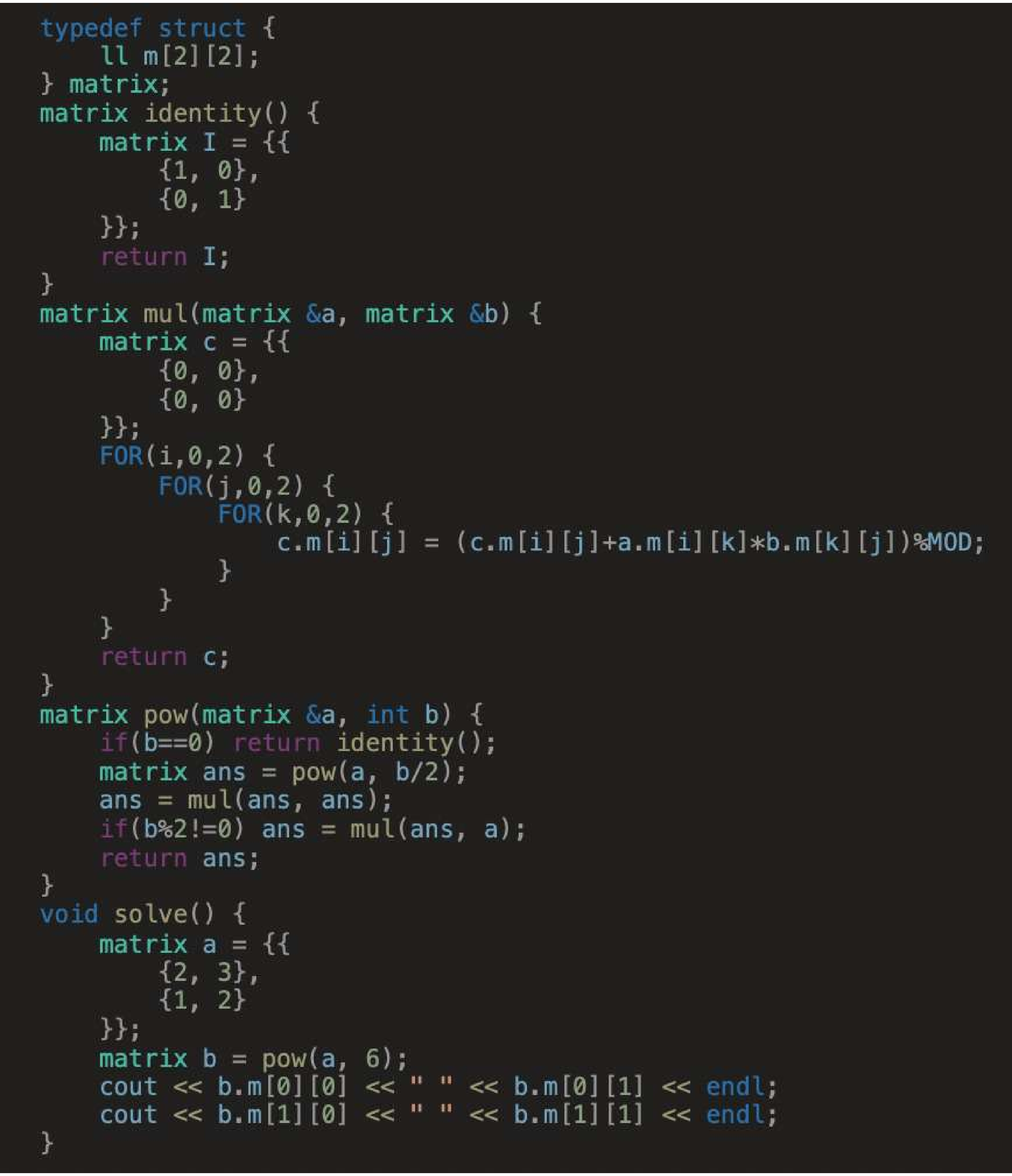
$$A^n = \underbrace{A * A * A * A \cdots * A}_{n \text{ times}} \qquad \underline{\underline{O(n)}}$$

$$A^n = \begin{cases} A^{n/2} * A^{n/2}, & \text{if } n \text{ is even} \\ A^{n/2} * A^{n/2} * A, & \text{if } n \text{ is odd} \\ I, & \text{if } \underline{n == 0} \\ A, & \text{if } n == 1 \end{cases} \qquad O(\underline{\underline{\log}}\, n)$$

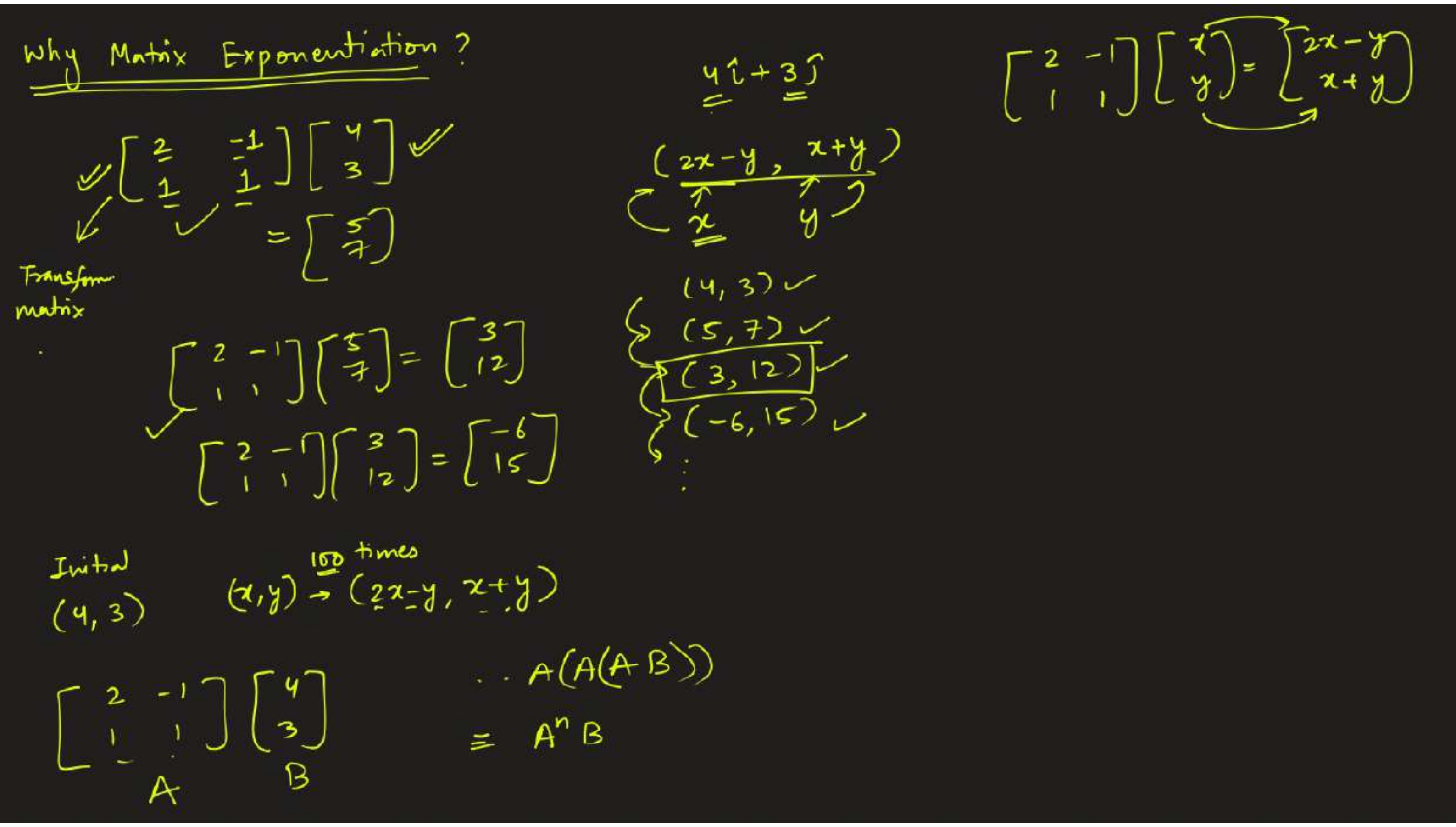
← base case.

$$I * A = \underline{\underline{A}}$$

✓



```
typedef struct {
    ll m[2][2];
} matrix;
matrix identity() {
    matrix I = {{
        {1, 0},
        {0, 1}
    }};
    return I;
}
matrix mul(matrix &a, matrix &b) {
    matrix c = {{
        {0, 0},
        {0, 0}
    }};
    FOR(i,0,2) {
        FOR(j,0,2) {
            FOR(k,0,2) {
                c.m[i][j] = (c.m[i][j]+a.m[i][k]*b.m[k][j])%MOD;
            }
        }
    }
    return c;
}
matrix pow(matrix &a, int b) {
    if(b==0) return identity();
    matrix ans = pow(a, b/2);
    ans = mul(ans, ans);
    if(b%2!=0) ans = mul(ans, a);
    return ans;
}
void solve() {
    matrix a = {{
        {2, 3},
        {1, 2}
    }};
    matrix b = pow(a, 6);
    cout << b.m[0][0] << " " << b.m[0][1] << endl;
    cout << b.m[1][0] << " " << b.m[1][1] << endl;
}
```



Why Matrix Exponentiation ?

$$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}4\\ 3\end{bmatrix} ✓ = \begin{bmatrix}5\\ 7\end{bmatrix}$$

Transform matrix

$$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}5\\ 7\end{bmatrix} = \begin{bmatrix}3\\ 12\end{bmatrix}$$

$$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3\\ 12\end{bmatrix} = \begin{bmatrix}-6\\ 15\end{bmatrix}$$

$4\hat{i} + 3\hat{j}$

$$\begin{array}{cc}(2x-y, & x+y)\\ \uparrow & \uparrow\\ x & y\end{array}$$

(4, 3) ✓

(5, 7) ✓

$\boxed{(3, 12)}$ ✓

(−6, 15) ✓

⋮

$$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}2x-y\\ x+y\end{bmatrix}$$

Initial

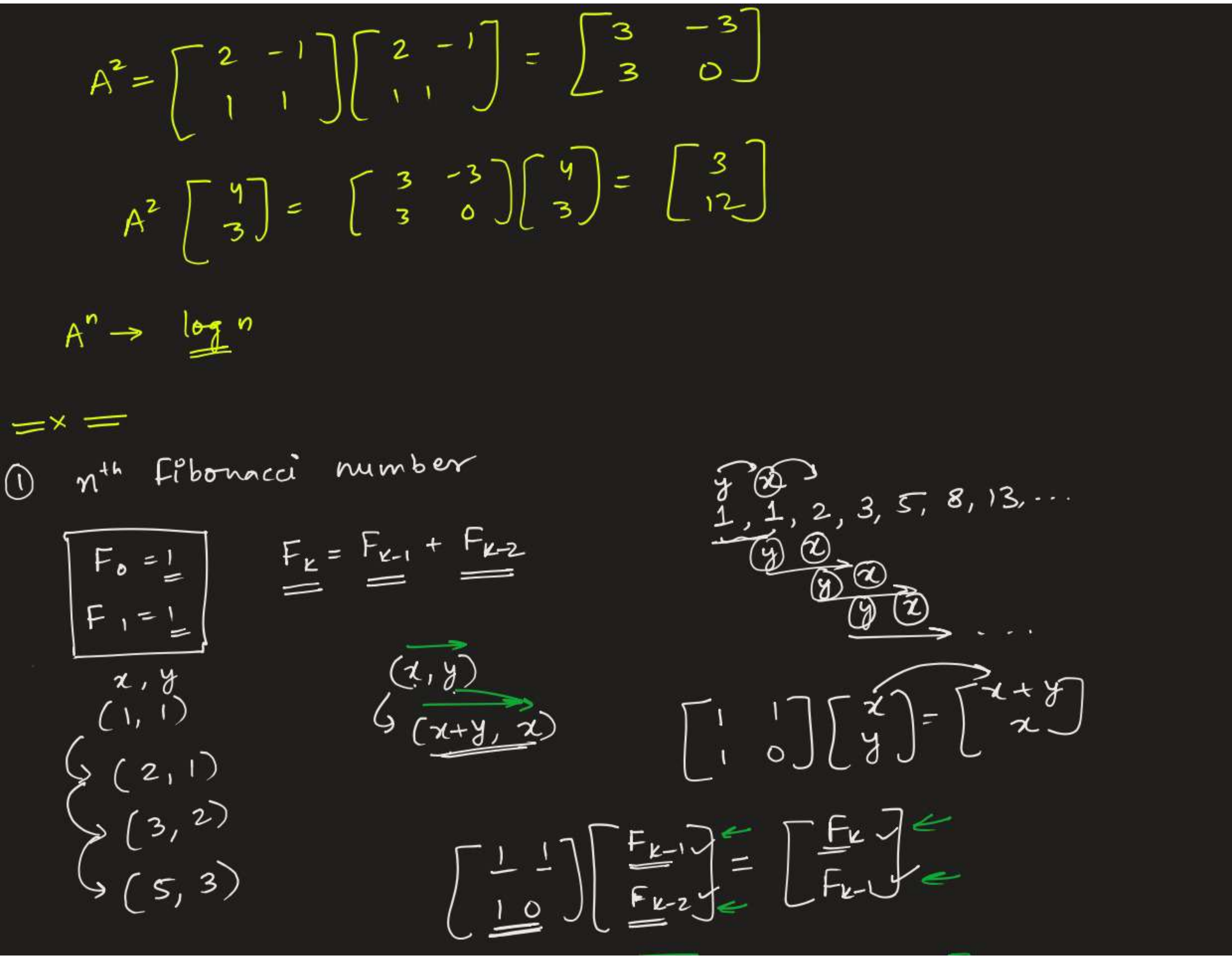
(4, 3)

$$(x, y) \xrightarrow{100 \text{ times}} (2x-y,\ x+y)$$

$$\underbrace{\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}4\\ 3\end{bmatrix}}_{B}$$

$$\ldots A(A(A\,B))$$

$$= A^n B$$



$$A^2 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}$$

$$A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$$

$A^n \rightarrow \underline{\underline{\log}}\, n$

=x=

① n^{th} fibonacci number

$$\boxed{\begin{matrix} F_0 = \underline{\underline{1}} \\ F_1 = \underline{\underline{1}} \end{matrix}} \qquad \underline{\underline{F_k}} = \underline{\underline{F_{k-1}}} + \underline{\underline{F_{k-2}}}$$

$y \quad x$

$1, 1, 2, 3, 5, 8, 13, \ldots$

x, y

$(1, 1)$

$\hookrightarrow (2, 1)$

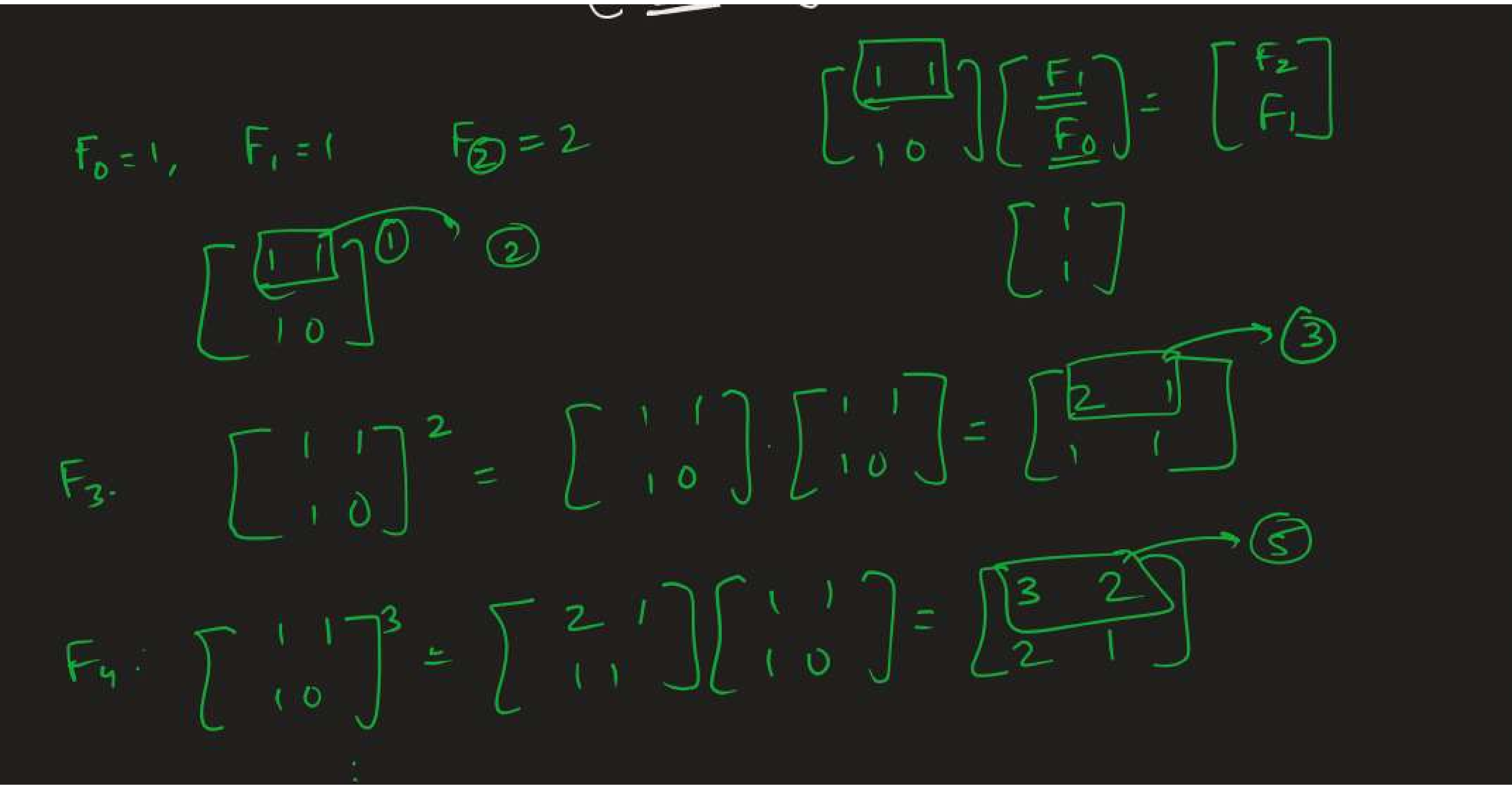
$\hookrightarrow (3, 2)$

$\hookrightarrow (5, 3)$

$$\begin{tikzcd} (x, y) \arrow[d] \\ \underline{\underline{(x+y,\ x)}} \end{tikzcd}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ \underline{\underline{1}} & \underline{\underline{0}} \end{bmatrix}\begin{bmatrix} \underline{\underline{F_{k-1}}} \\ \underline{\underline{F_{k-2}}} \end{bmatrix} = \begin{bmatrix} \underline{F_k} \\ F_{k-1} \end{bmatrix}$$



$F_0 = 1, \quad F_1 = 1 \qquad F_{\textcircled{2}} = 2$

$$\begin{bmatrix} \boxed{1 \quad 1} \\ 1 \quad 0 \end{bmatrix} \begin{bmatrix} \underline{\underline{F_1}} \\ \underline{\underline{F_0}} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

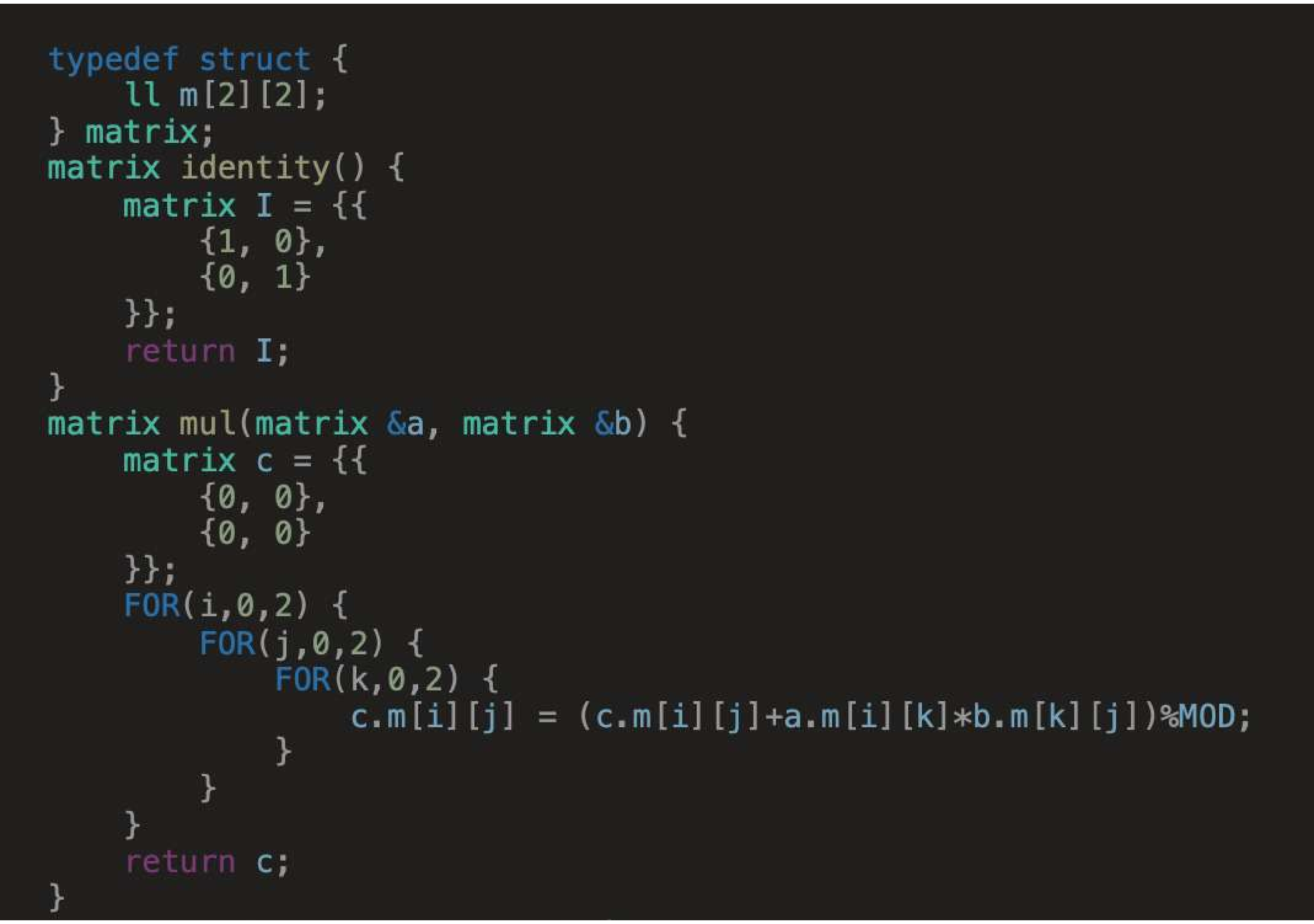
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1 \quad 1} \\ 1 \quad 0 \end{bmatrix}^{\textcircled{1}} \rightarrow \textcircled{2}$$

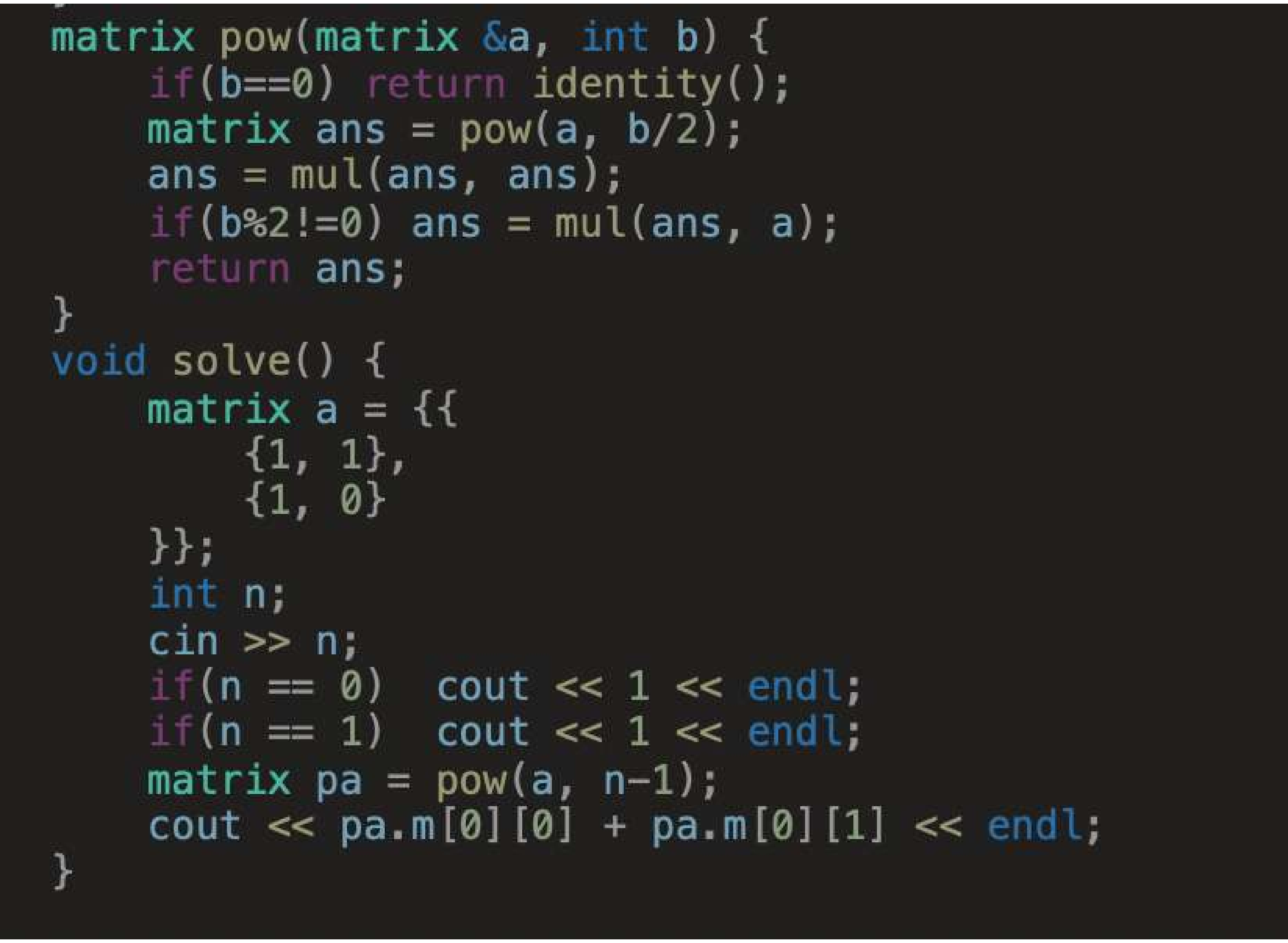
$$F_3. \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boxed{2 \quad 1} \\ 1 \quad 1 \end{bmatrix} \rightarrow \textcircled{3}$$

$$F_4: \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boxed{3 \quad 2} \\ 2 \quad 1 \end{bmatrix} \rightarrow \textcircled{5}$$

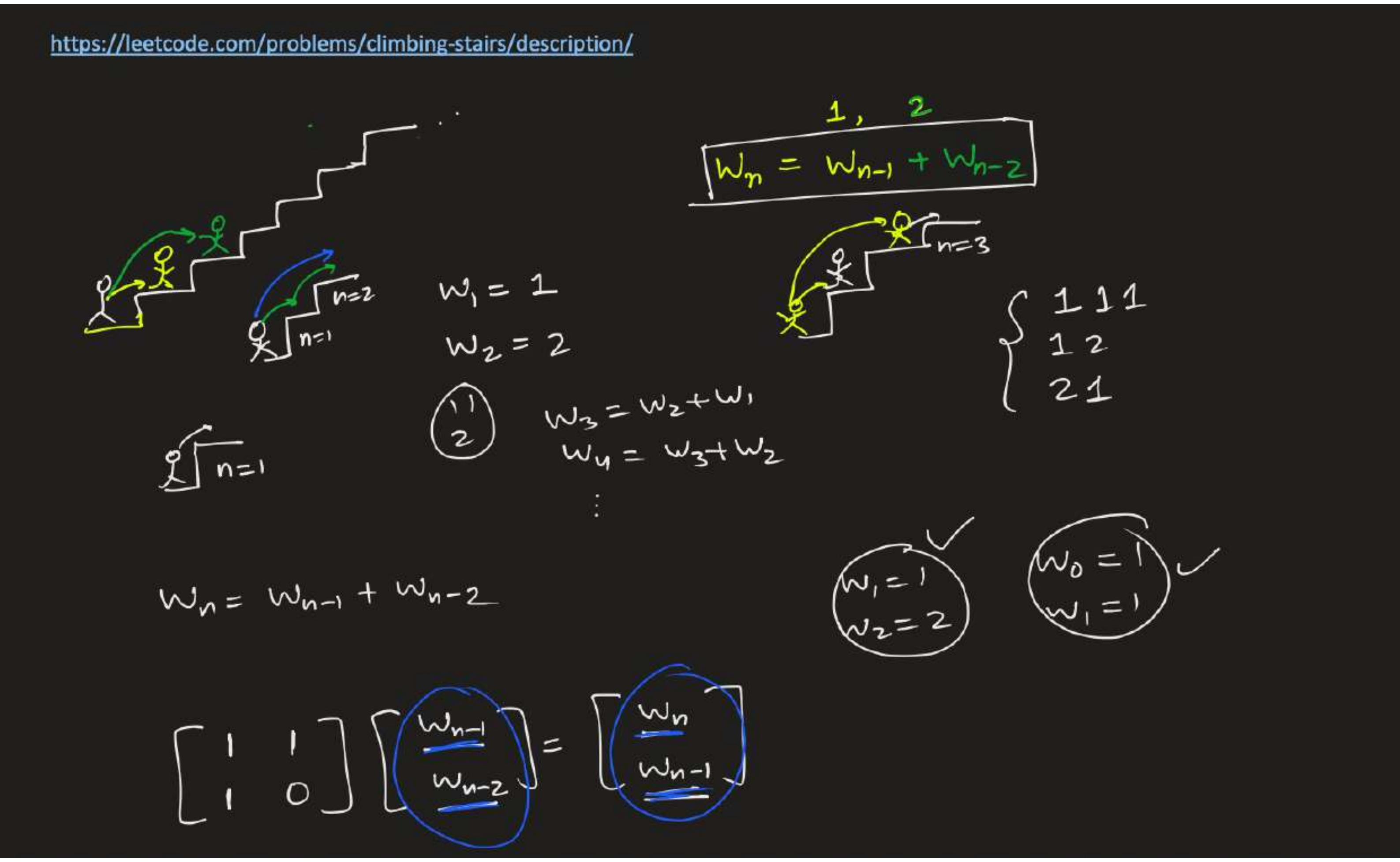
$$\vdots$$



```
typedef struct {
    ll m[2][2];
} matrix;
matrix identity() {
    matrix I = {{
        {1, 0},
        {0, 1}
    }};
    return I;
}
matrix mul(matrix &a, matrix &b) {
    matrix c = {{
        {0, 0},
        {0, 0}
    }};
    FOR(i,0,2) {
        FOR(j,0,2) {
            FOR(k,0,2) {
                c.m[i][j] = (c.m[i][j]+a.m[i][k]*b.m[k][j])%MOD;
            }
        }
    }
    return c;
}
```



```
matrix pow(matrix &a, int b) {
    if(b==0) return identity();
    matrix ans = pow(a, b/2);
    ans = mul(ans, ans);
    if(b%2!=0) ans = mul(ans, a);
    return ans;
}
void solve() {
    matrix a = {{
        {1, 1},
        {1, 0}
    }};
    int n;
    cin >> n;
    if(n == 0)  cout << 1 << endl;
    if(n == 1)  cout << 1 << endl;
    matrix pa = pow(a, n-1);
    cout << pa.m[0][0] + pa.m[0][1] << endl;
}
```



https://leetcode.com/problems/climbing-stairs/description/

1, 2

$$\boxed{W_n = W_{n-1} + W_{n-2}}$$

$W_1 = 1$

$W_2 = 2$

$\bigcirc\!\!\begin{matrix}1\,1\\2\end{matrix}$

$W_3 = W_2 + W_1$

$W_4 = W_3 + W_2$

$\vdots$

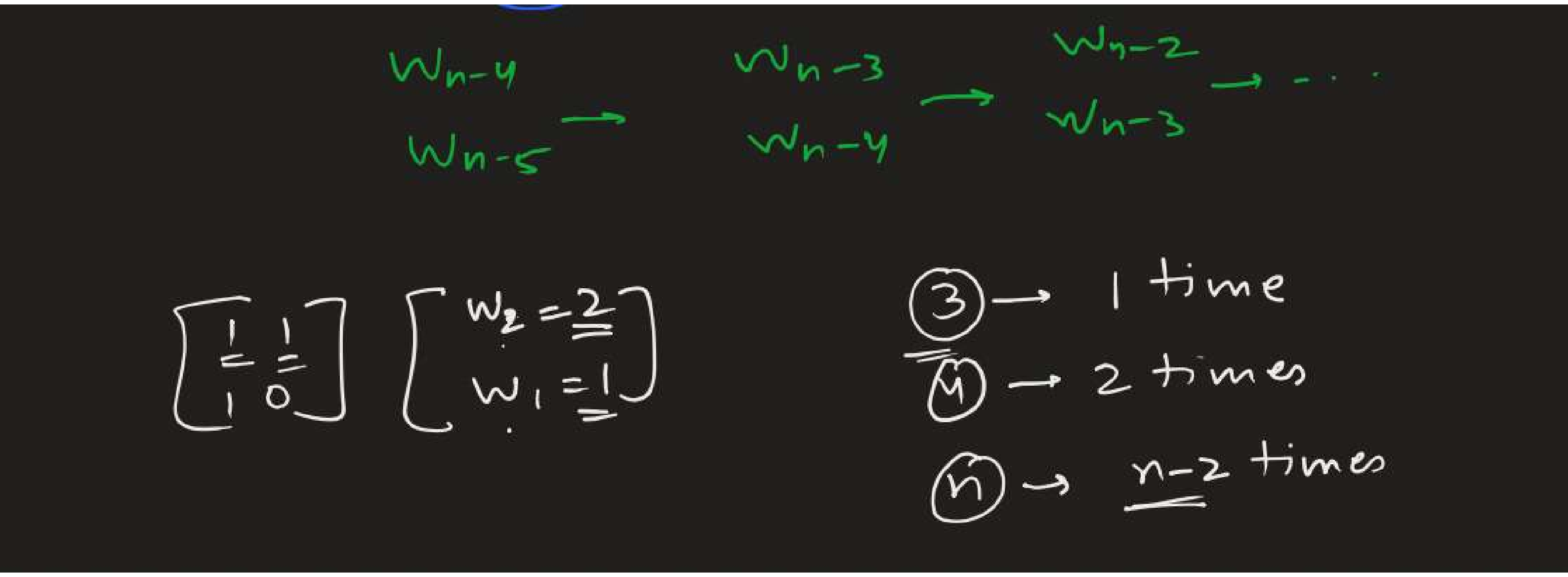
$\left\{\begin{matrix}1\,1\,1\\1\,2\\2\,1\end{matrix}\right.$

$W_n = W_{n-1} + W_{n-2}$

$W_1 = 1$, $W_2 = 2$ ✓

$W_0 = 1$, $W_1 = 1$ ✓

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}W_{n-1}\\W_{n-2}\end{bmatrix} = \begin{bmatrix}W_n\\W_{n-1}\end{bmatrix}$$



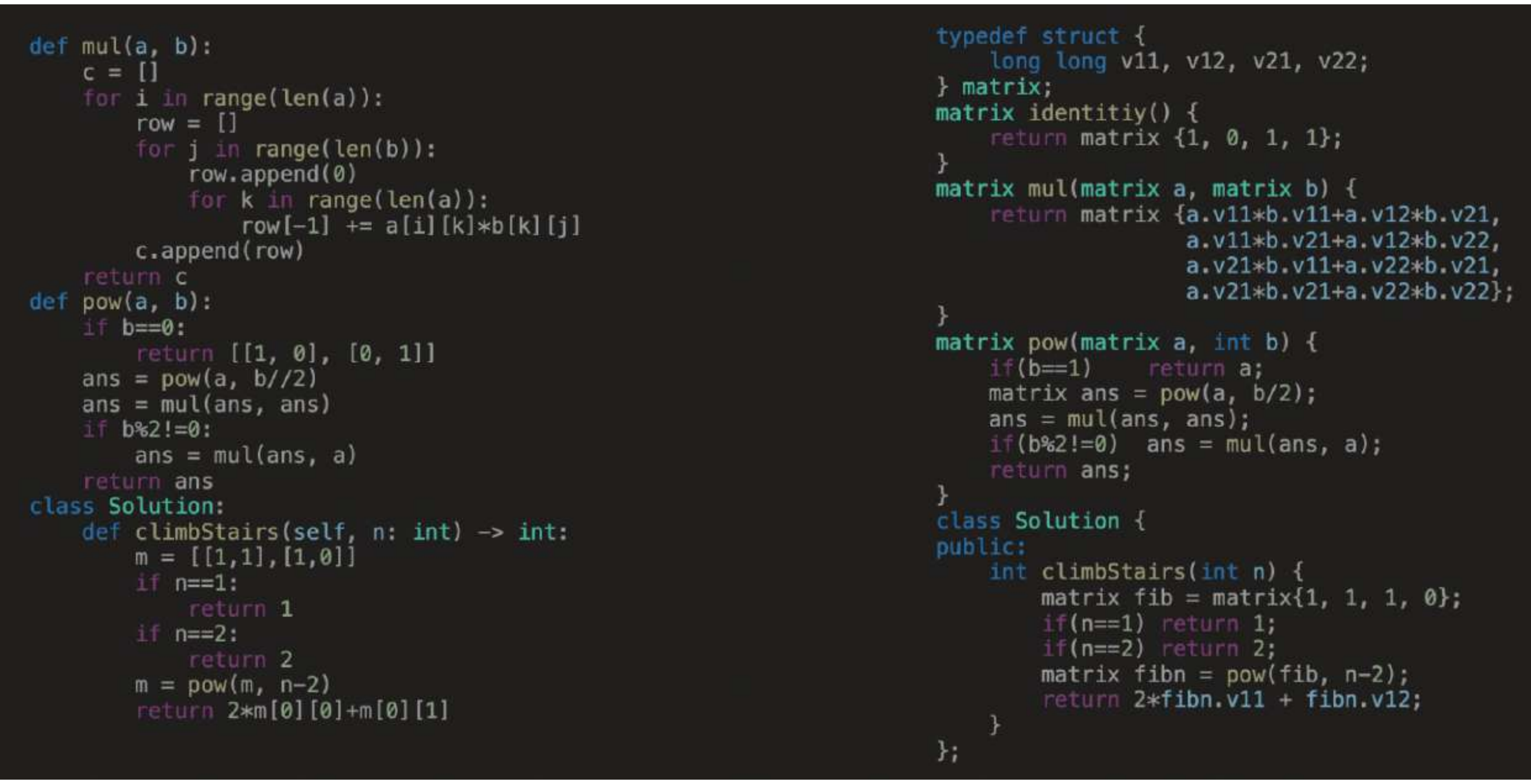
$$\begin{matrix} W_{n-4} \\ W_{n-5} \end{matrix} \longrightarrow \begin{matrix} W_{n-3} \\ W_{n-4} \end{matrix} \longrightarrow \begin{matrix} W_{n-2} \\ W_{n-3} \end{matrix} \longrightarrow \cdots$$

$$\begin{bmatrix} \underline{1} & \underline{1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} W_2 = \underline{\underline{2}} \\ W_1 = \underline{\underline{1}} \end{bmatrix}$$

③ → 1 time

④ → 2 times

ⓝ → $\underline{n-2}$ times



```python
def mul(a, b):
    c = []
    for i in range(len(a)):
        row = []
        for j in range(len(b)):
            row.append(0)
            for k in range(len(a)):
                row[-1] += a[i][k]*b[k][j]
        c.append(row)
    return c
def pow(a, b):
    if b==0:
        return [[1, 0], [0, 1]]
    ans = pow(a, b//2)
    ans = mul(ans, ans)
    if b%2!=0:
        ans = mul(ans, a)
    return ans
class Solution:
    def climbStairs(self, n: int) -> int:
        m = [[1,1],[1,0]]
        if n==1:
            return 1
        if n==2:
            return 2
        m = pow(m, n-2)
        return 2*m[0][0]+m[0][1]
```

```cpp
typedef struct {
    long long v11, v12, v21, v22;
} matrix;
matrix identitiy() {
    return matrix {1, 0, 1, 1};
}
matrix mul(matrix a, matrix b) {
    return matrix {a.v11*b.v11+a.v12*b.v21,
                   a.v11*b.v21+a.v12*b.v22,
                   a.v21*b.v11+a.v22*b.v21,
                   a.v21*b.v21+a.v22*b.v22};
}
matrix pow(matrix a, int b) {
    if(b==1)    return a;
    matrix ans = pow(a, b/2);
    ans = mul(ans, ans);
    if(b%2!=0)  ans = mul(ans, a);
    return ans;
}
class Solution {
public:
    int climbStairs(int n) {
        matrix fib = matrix{1, 1, 1, 0};
        if(n==1) return 1;
        if(n==2) return 2;
        matrix fibn = pow(fib, n-2);
        return 2*fibn.v11 + fibn.v12;
    }
};
```