Topic 29 - Analysis of Covariance

STAT 525 - Fall 2013

Outline

- One-way analysis of covariance
  - Data
  - Model
  - Inference
- Multifactor analysis of covariance
- Diagnostics and remedies

Background

- Consider a variable X that is available prior to, or at the start of, an experiment. In other words, X is unaffected by treatment
- Suppose it is expected that variable X is correlated with the response variable Y
- Also, suppose you can measure X but can't control it
- Nuisance variable X is called a **covariate**
- Analysis of covariance (ANCOVA) considers adjusting Y for differences in X prior to comparing treament levels

ANCOVA

- Can be considered a hybrid of regression and ANOVA but really is just a linear model combining indicator variables (trt levels) and predictor variables (covariates)
- Without adjustment, effect of X may
  - Inflate σ^2 – unexplained variation goes into error
  - Alter treatment comparisons – if there are differences in X across trts then Y may naturally vary without trt level differences

Data for One-Way ANCOVA

- Y_{ij} is j^{th} observation of the response in the i^{th} level of the factor
- X_{ij} is j^{th} observation of the covariate in the i^{th} level of the factor
- $i = 1, 2, ..., r$
- $j = 1, 2, ..., n_i$

Examples

- **Pretest/Posttest score analysis**: The change in score Y may be associated with current GPA X. Also the posttest score Y may be associated with the pretest score X. Analysis of covariance provides a way to "handicap" students. That way, one does not need to find a groups or pairs of students with similar GPAs and randomly assign them to a control and treatment group.
- **Weight gain experiments in animals**: When comparing different feeds, the weight gain Y may be associated with the dominance of the animal. Again hard to control for dominance but can measure it.

One-Way ANCOVA

- Statistical model is

$$Y_{ij} = \mu + \tau_i + \beta(X_{ij} - \overline{X}_{..}) + \epsilon_{ij} \qquad \begin{cases} i = 1, 2, \dots, r \\ j = 1, 2, \dots n_i \end{cases}$$

- Additional assumptions

  X_{ij} not affected by treatment

  X and Y are linearly related

  Constant slope (can be relaxed)

Estimation

- General Procedure:

  Fit one-way model (Y = trt)

  Fit one-way model (X = trt)

  Regress residuals ($\text{resid}_Y = \text{resid}_X$)

  Model estimates are

$$\hat{\mu} = \overline{Y}_{..}$$
$$\hat{\beta} = \sum\sum e_{Yij} e_{Xij} / \sum\sum e^2_{Xij}$$
$$\hat{\tau}_i = \overline{Y}_{i.} - \overline{Y}_{..} - \hat{\beta}(\overline{X}_{i.} - \overline{X}_{..})$$

Hypotheses

- Test $H_0 : \tau_1 = \tau_2 = \ldots = \tau_r = 0$
  - Compare treatment means **after adjusting for differences among treatments due to differences in covariate levels**

$$F_0 = \frac{\mathrm{SSTR|X/(r-1)}}{\mathrm{SSE/(n_T - r - 1)}}$$

- Test: $\beta = 0$
  - SS regression (SSX): $\hat{\beta}^2 \sum\sum (X_{ij} - \overline{X}_{i.})^2$

$$F_0 = \frac{\mathrm{SSX/1}}{\mathrm{SSE/(n_T - r - 1)}}$$

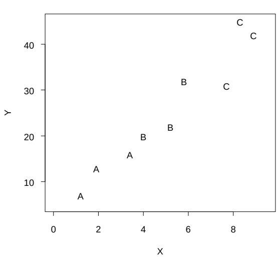
Mean Estimates

- Adjusted treatment means
  - Estimate: $\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i = \overline{Y}_{i.} - \hat{\beta}(\overline{X}_{i.} - \overline{X}_{..})$
  - Expected value of Y when X is equal to the average covariate value
  - Can use any value of X. Make sure it is reasonable for all factor levels
  - Variance: $\hat{\sigma}^2 \left(1/n_i + (\overline{X}_{i.} - \overline{X}_{..})^2 / \sum\sum (X_{ij} - \overline{X}_{i.})^2\right)$
- Pairwise differences
  - Estimate: $\hat{\tau}_i - \hat{\tau}_{i*} = \overline{Y}_{i.} - \overline{Y}_{i*.} - \hat{\beta}(\overline{X}_{i.} - \overline{X}_{i*.})$
  - Variance: $\hat{\sigma}^2 \left(1/n_i + 1/n_{i*} + (\overline{X}_{i.} - \overline{X}_{i*.})^2 / \sum\sum (X_{ij} - \overline{X}_{i.})^2\right)$

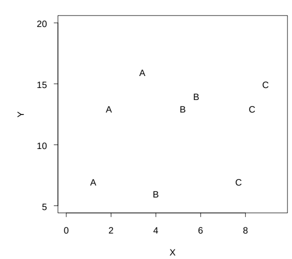
Analysis of Covariance

Two Examples : Both of which emphasize how a covariate can change the treatment comparisons. Be wary of this in practice because one is comparing Y at a common value $\overline{X}$, which may not be common in all treatment populations. Usually ANCOVA just reduces the MSE.

1 No treatment differences
  - Positive linear relationship
  - Covariate larger in each group
  - Thus, appears to be treatment difference



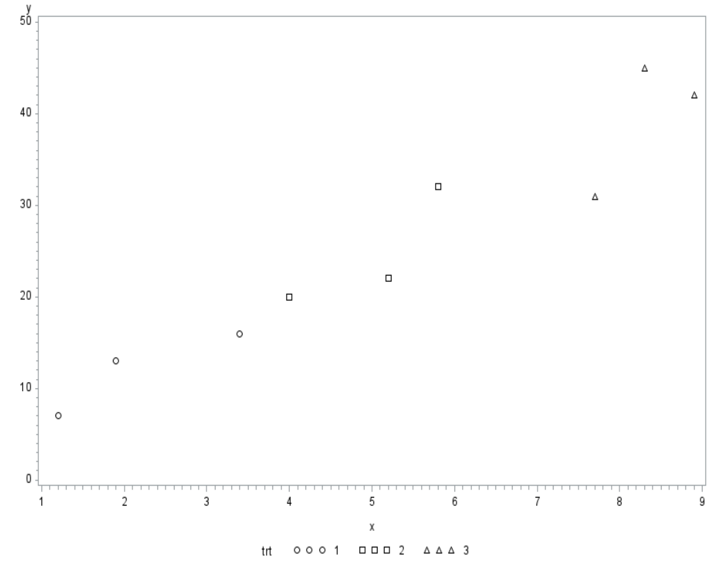
2 Treatment differences exist
  - Positive linear relationship
  - Covariate larger in each group
  - Thus, no apparent treatment difference



Using SAS

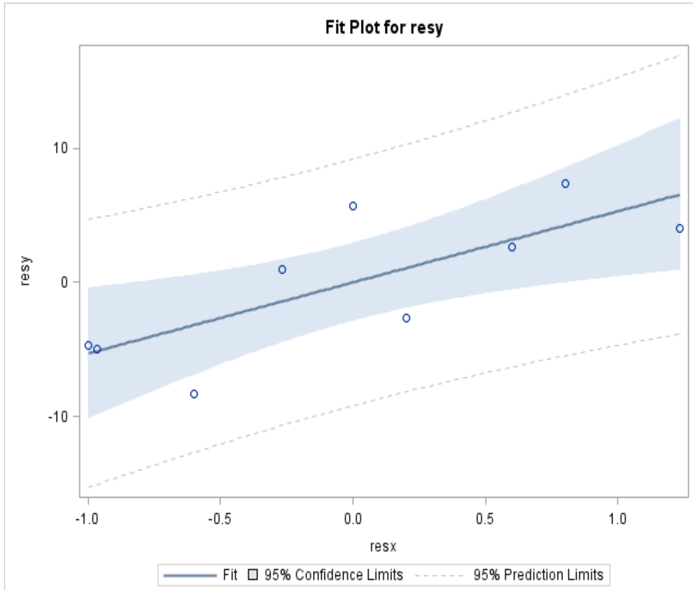
```
data example1;
  input trt x y @@;
  cards;
  1 1.2 7 1 1.9 13 1 3.4 16
  2 4.0 20 2 5.2 22 2 5.8 32
  3 7.7 31 3 8.3 45 3 8.9 42
  ;

proc sort; by trt;
symbol1 v=circle i=  c=black; symbol2 v=square i=  c=black; symbol3 v=triangle i=  c=black;
proc gplot; plot y*x=trt;
run;

proc glm; class trt;
 model y=trt; output out=resid r=resy;
proc glm; class trt;
 model x=trt; output out=resid1 r=resx;
proc glm; model resy=resx;
symbol1 v=circle i=rl;
proc gplot; plot resy*resx;

proc glm data=example1;
 class trt; model y=trt x / solution;
 means trt /lines lsd;
 lsmeans trt / tdiff adjust=t;
run;
```


Scatterplot of X vs Y




```
                REGRESSION RES Y vs RES X TO GET SLOPE

Dependent Variable: resy
                                        Sum of
Source                     DF          Squares     Mean Square    F Value    Pr > F
Model                       1      138.2699594     138.2699594      10.18    0.0153
Error                       7       95.0633739      13.5804820
Corrected Total             8      233.3333333

R-Square     Coeff Var      Root MSE     resy Mean
0.592586    1.03728E17      3.685171    3.5527E-15

Source                     DF        Type I SS     Mean Square    F Value    Pr > F
resx                        1      138.2699594     138.2699594      10.18    0.0153

                                    Standard
Parameter         Estimate             Error    t Value    Pr > |t|
Intercept      0.000000000        1.22839018       0.00      1.0000
resx           5.297699594        1.66027872       3.19      0.0153
```


Scatterplot of RES X vs RES Y



```
               ANALYSIS USING GLM - ADDING COVARIATE
Dependent Variable: y
                                      Sum of
Source                     DF        Squares    Mean Square   F Value   Pr > F
Model                       3    1260.936626     420.312209     22.11   0.0026
Error                       5      95.063374      19.012675
Corrected Total             8    1356.000000

Source                     DF      Type I SS    Mean Square   F Value   Pr > F
trt                         2    1122.666667     561.333333     29.52   0.0017
x                           1     138.269959     138.269959      7.27   0.0430

Source                     DF    Type III SS    Mean Square   F Value   Pr > F
trt                         2      3.2122606      1.6061303      0.08   0.9203
x                           1    138.2699594    138.2699594      7.27   0.0430

                                      Standard
Parameter           Estimate             Error    t Value    Pr > |t|
Intercept       -4.637573297 B     16.49828508      -0.28      0.7899
trt       1      5.159224177 B     12.56372645       0.41      0.6983
trt       2      2.815741994 B      7.39601943       0.38      0.7191
trt       3      0.000000000 B       .                .         .
x                5.297699594        1.96446828       2.70      0.0430
```



```
***** WARNING : DO NOT USE MEANS STATEMENT ******
t Tests (LSD) for y
Alpha                              0.05
Error Degrees of Freedom              5
Error Mean Square              19.01267
Critical Value of t             2.57058
Least Significant Difference    9.1518
Means with the same letter are not significantly different.
           Mean      N    trt
A        39.333      3    3
B        24.667      3    2
C        12.000      3    1

****** USE LSMEANS WHICH GIVES YOU THE ADJUSTED MEANS *****
trt        y LSMEAN     LSMEAN Number
1        27.8342355           1
2        25.4907533           2
3        22.6750113           3

      Least Squares Means for Effect trt
   t for H0: LSMean(i)=LSMean(j) / Pr > |t|
            Dependent Variable: y
i/j              1              2              3
   1                     0.354685       0.410644
                           0.7373         0.6983
   2      -0.35468                       0.38071
            0.7373                        0.7191
   3      -0.41064       -0.38071
            0.6983         0.7191
```



```
data example2;
  input trt x y @@;
  cards;
  1 1.2 7 1 1.9 13 1 3.4 16 2 4.0 6 2 5.2 13 2 5.8 14 3 7.7 7 3 8.3 13 3 8.9 15
  ;
proc glm data=example2;
 class trt; model y=trt x / solution;
 lsmeans trt / tdiff; lsmeans trt / tdiff adjust=bon;
run;
------------------------------------------------------------------------------
Source                     DF  Sum of Squares   Mean Square   F Value   Pr > F
Model                       3    100.6915501     33.5638500     10.81   0.0126
Error                       5     15.5306721      3.1061344
Corrected Total             8    116.2222222

Source                     DF      Type I SS    Mean Square   F Value   Pr > F
trt                         2     1.55555556     0.77777778      0.25   0.7877
x                           1    99.13599459    99.13599459     31.92   0.0024

Source                     DF    Type III SS    Mean Square   F Value   Pr > F
trt                         2    94.55407736    47.27703868     15.22   0.0075
x                           1    99.13599459    99.13599459     31.92   0.0024

Parameter           Estimate         Std Error    t Value    Pr > |t|
Intercept      -25.56540370 B      6.66848712      -3.83      0.0122
trt       1     27.84618854 B      5.07816707       5.48      0.0028
trt       2     14.13644565 B      2.98941739       4.73      0.0052
trt       3      0.00000000 B       .                .         .
x                4.48579161        0.79402382       5.65      0.0024
```



```
                         LSMEAN
trt        y LSMEAN      Number
1        25.4075327           1
2        11.6977898           2
3        -2.4386558           3

      Least Squares Means for Effect trt
   t for H0: LSMean(i)=LSMean(j) / Pr > |t|
            Dependent Variable: y
i/j              1              2              3
   1                     5.133597       5.483512
                           0.0037         0.0028
   2       -5.1336                       4.72883
            0.0037                        0.0052
   3      -5.48351       -4.72883
            0.0028         0.0052

Adjustment for Multiple Comparisons: Bonferroni
      Least Squares Means for Effect trt
   t for H0: LSMean(i)=LSMean(j) / Pr > |t|
            Dependent Variable: y
i/j              1              2              3
   1                     5.133597       5.483512
                           0.0110         0.0083
   2       -5.1336                       4.72883
            0.0110                        0.0156
   3      -5.48351       -4.72883
            0.0083         0.0156
```

Nonconstant Slope

- Can allow for different slopes by including interaction

$$y_{ij} = \mu + \tau_i + (\beta + (\beta\tau)_i)(x_{ij} - \overline{x}_{..}) + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots n_i \end{cases}$$

- Provides joint test for constant slope
- Can also build model for other relationships between X and Y (e.g., quadratic)

Using SAS

```
data example1;
 input trt x y @@;
 cards;
1 1.2 7 1 1.9 13 1 3.4 16 2 4.0 20 2 5.2 22 2 5.8 32 3 7.7 31 3 8.3 45 3 8.9 42
;

proc sort; by trt;
symbol1 v=circle i=  c=black; symbol2 v=square i=  c=black; symbol3 v=triangle i=  c=black;
proc gplot;
  plot y*x=trt;
run;

proc glm;
 class trt;
 model y=trt x / solution;
 lsmeans trt / tdiff;

proc glm;
 class trt;
 model y=trt x trt*x / solution;
 lsmeans trt / tdiff;
run;
```



```
Source                    DF Sum of Squares  Mean Square  F Value  Pr > F
Model                      3   1260.936626    420.312209    22.11  0.0026
Error                      5     95.063374     19.012675
Corrected Total            8   1356.000000

Source                    DF   Type III SS   Mean Square  F Value  Pr > F
trt                        2     3.2122606     1.6061303     0.08  0.9203
x                          1   138.2699594   138.2699594     7.27  0.0430

--------------------------------------------------------------------------

Source                    DF Sum of Squares  Mean Square  F Value  Pr > F
Model                      5   1278.409474    255.681895     9.89  0.0441
Error                      3     77.590526     25.863509
Corrected Total            8   1356.000000

Source                    DF   Type III SS   Mean Square  F Value  Pr > F
trt                        2    20.5146998    10.2573499     0.40  0.7034
x                          1   149.7599282   149.7599282     5.79  0.0953
x*trt                      2    17.4728475     8.7364237     0.34  0.7374

Parameter           Estimate         Std Error    t Value    Pr > |t|
Intercept       -36.75000000 B     49.83227932      -0.74      0.5143
trt       1      40.60356201 B     50.39772400       0.81      0.4794
trt       2      31.65476190 B     53.63535098       0.59      0.5966
x                 9.16666667 B      5.99345810       1.53      0.2236
x*trt     1      -5.40677221 B      6.79395005      -0.80      0.4843
x*trt     2      -3.21428571 B      7.16355259      -0.45      0.6841
```



```
                          LSMEAN
trt        y LSMEAN      Number
1        27.8342355           1
2        25.4907533           2
3        22.6750113           3
      Least Squares Means for Effect trt
   t for H0: LSMean(i)=LSMean(j) / Pr > |t|
i/j             1             2             3
   1                   0.354685      0.410644
                         0.7373        0.6983
   2      -0.35468                    0.38071
            0.7373                     0.7191
   3      -0.41064      -0.38071
            0.6983        0.7191
--------------------------------------------------------------------
                          LSMEAN
trt        y LSMEAN      Number
1        23.2379068           1
2        25.5925926           2
3        10.5092593           3
      Least Squares Means for Effect trt
   t for H0: LSMean(i)=LSMean(j) / Pr > |t|
i/j             1             2             3
   1                   -0.22548         0.591
                         0.8361        0.5961
   2      0.225476                   0.781205
            0.8361                     0.4917
   3        -0.591      -0.78121
            0.5961        0.4917
```

Multifactor ANCOVA

- Can incorporate covariate into any model
- For two factor model

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + b(X_{ijk} - \overline{X}_{...}) + \epsilon_{ijk}$$

- Constant slope **for each** ij **combination**
- Can include interaction terms to vary slopes
- Plot y vs x for each combination to visually assess

Background Reading

- KNNL Chapter 22
- knnl926.sas
- KNNL Chapter 14