

Lecture 14: 2^{k-p} Fractional Factorial Design

Montgomery: Chapter 8

Fundamental Principles Regarding Factorial Effects

Suppose there are k factors (A, B, \dots, J, K) in an experiment. All possible factorial effects include

effects of order 1: A, B, \dots, K (main effects)

effects of order 2: AB, AC, \dots, JK (2-factor interactions)

.....

- Hierarchical Ordering principle
 - Lower order effects are more likely to be important than higher order effects.
 - Effects of the same order are equally likely to be important
- Effect Sparsity Principle (Pareto principle)
 - The number of relatively important effects in a factorial experiment is small
- Effect Heredity Principle
 - In order for an interaction to be significant, at least one of its parent factors should be significant.

Fractional Factorials

- May not have sources (time, money, etc) for full factorial design
- Number of runs required for full factorial grows quickly
 - Consider 2^k design
 - If $k = 7 \rightarrow 128$ runs required
 - Can estimate 127 effects
 - Only 7 df for main effects, 21 for 2-factor interactions
 - the remaining 99 df are for interactions of order ≥ 3
- Often only lower order effects are important
- Full factorial design may not be necessary according to
 - Hierarchical ordering principle
 - Effect Sparsity Principle
- A fraction of the full factorial design (i.e. a subset of all possible level combinations) is sufficient \implies **Fractional Factorial Design**

Example 1

- Suppose you were designing a new car
- Wanted to consider the following nine factors each with 2 levels
 1. Engine Size;
 2. Number of cylinders;
 3. Drag;
 4. Weight;
 5. Automatic vs Manual;
 6. Shape;
 7. Tires;
 8. Suspension;
 9. Gas Tank Size;
- Only have resources for conduct $2^6 = 64$ runs
 - If you drop three factors for a 2^6 full factorial design, those factor and their interactions with other factors cannot be investigated.
 - Want investigate all nine factors in the experiment
 - A fraction of 2^9 full factorial design will be used.
 - Confounding (aliasing) will happen because using a subset
- **How to choose (or construct) the fraction?**

Example 2: Filtration rate experiment

- Recall that there are four factors in the experiment (A , B , C and D), each of 2 levels. Suppose the available resource is enough for conducting 8 runs. 2^4 full factorial design consists of all the 16 level combinations of the four factors. We need to choose half of them.
- The chosen half is called 2^{4-1} fractional factorial design.

Which half we should select (construct)?

factor			
<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
–	–	–	–
+	–	–	–
–	+	–	–
+	+	–	–
–	–	+	–
+	–	+	–
–	+	+	–
+	+	+	–
–	–	–	+
+	–	–	+
–	+	–	+
+	+	–	+
–	–	+	+
+	–	+	+
–	+	+	+
+	+	+	+

2^{4-1} Fractional Factorial Design

- The number of factors: $k = 4$
- The fraction index: $p = 1$
- The number of runs (level combinations): $N = \frac{2^4}{2^1} = 8$
- Construct 2^{4-1} designs via “confounding” (**aliasing**)
 - Select 3 factors (e.g. A, B, C) to form a 2^3 full factorial (basic design)
 - Confound (**alias**) D with a high order interaction of A, B and C . For example,

$$D = ABC$$

factorial effects (contrasts)							
I	A	B	C	AB	AC	BC	ABC=D
1	-1	-1	-1	1	1	1	-1
1	1	-1	-1	-1	-1	1	1
1	-1	1	-1	-1	1	-1	1
1	1	1	-1	1	-1	-1	-1
1	-1	-1	1	1	-1	-1	1
1	1	-1	1	-1	1	-1	-1
1	-1	1	1	-1	-1	1	-1
1	1	1	1	1	1	1	1

- Therefore, the chosen fraction includes the following 8 level combinations:
- $(-, -, -, -), (+, -, -, +), (-, +, -, +), (+, +, -, -), (-, -, +, +),$
 $(+, -, +, -), (-, +, +, -), (+, +, +, +)$
- Note: 1 corresponds to $+$ and -1 corresponds to $-$.

Verify:

1. The chosen level combinations form a half of the 2^4 design.
2. the product of columns A , B , C and D equals 1, i.e.,

$$I = ABCD$$

which is called the **defining relation**, or $ABCD$ is called a **defining word** (contrast).

Aliasing in 2^{4-1} Design

- Four factors: A , B , C and D
- $2^4 - 1$ effects: A , B , C , D , AB , AC , AD , BC , BD , CD , ABC , ABD , ACD , BCD , $ABCD$

Response	I	A	B	C	D	AB	..	CD	ABC	BCD	...	ABCD
y_1	1	-1	-1	-1	-1	1	..	1	-1	-1	...	1
y_2	1	1	-1	-1	1	-1	..	-1	1	1	...	1
y_3	1	-1	1	-1	1	-1	..	-1	1	-1	...	1
y_4	1	1	1	-1	-1	1	..	1	-1	1	...	1
y_5	1	-1	-1	1	1	1	..	1	1	-1	...	1
y_6	1	1	-1	1	-1	-1	..	-1	-1	1	...	1
y_7	1	-1	1	1	-1	-1	..	-1	-1	-1	...	1
y_8	1	1	1	1	1	1	..	1	1	1	...	1

Contrasts for main effects by converting $-$ to -1 and $+$ to 1; contrasts for other effects obtained by multiplication.

- A, BCD are aliases or aliased. The contrast is for $A + BCD$. A and BCD are not distinguishable.

$$\hat{A} = \bar{y}_{A_+} - \bar{y}_{A_-} = \frac{1}{4}(-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8)$$

$$\widehat{BCD} = \frac{1}{4}(-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8)$$

- AB, CD are aliases or aliased. The contrast is for $AB + CD$. AB and CD are not distinguishable.

$$\widehat{AB} = \bar{y}_{AB_+} - \bar{y}_{AB_-} = \frac{1}{4}(y_1 - y_2 - y_3 + y_4 + y_5 - y_6 - y_7 + y_8)$$

$$\widehat{CD} = \bar{y}_{CD_+} - \bar{y}_{CD_-} = \frac{1}{4}(y_1 - y_2 - y_3 + y_4 + y_5 - y_6 - y_7 + y_8)$$

- There are other 5 pairs. They are caused by the defining relation

$$I = ABCD,$$

that is, I (the intercept) and 4-factor interaction $ABCD$ are aliased.

Alias Structure for 2^{4-1} with $I = ABCD$ (Denoted by d_1)

- Alias Structure:

$$I = ABCD$$

$$A = A * I = A * ABCD = BCD$$

$$B = \dots\dots\dots = ACD$$

$$C = \dots\dots\dots = ABD$$

$$D = \dots\dots\dots = ABC$$

$$AB = AB * I = AB * ABCD = CD$$

$$AC = \dots\dots\dots = BD$$

$$AD = \dots\dots\dots = BC$$

all 16 factorial effects for A , B , C and D are partitioned into 8 groups each with 2 aliased effects.

A Different 2^{4-1} Fractional Factorial Design

- the defining relation $I = ABD$ generates another 2^{4-1} fractional factorial design, denoted by d_2 . Its alias structure is given below.

$$I = ABD$$

$$A = BD$$

$$B = AD$$

$$C = ABCD$$

$$D = AB$$

$$ABC = CD$$

$$ACD = BC$$

$$BCD = AC$$

- Recall d_1 is defined by $I = ABCD$. Comparing d_1 and d_2 , which one we should choose or which one is better?
 - Length** of a defining word is defined to be the number of the involved factors.
 - Resolution** of a fractional factorial design is defined to be the minimum length of the defining words, usually denoted by Roman numbers, III, IV, V, ...

Resolution and Maximum Resolution Criterion

- $d_1: I = ABCD$ is a resolution IV design denoted by 2_{IV}^{4-1} .
- $d_2: I = ABD$ is a resolution III design denoted by 2_{III}^{4-1} .
- If a design is of resolution R , then none of the i -factor interactions is aliased with any other interaction of order less than $R - i$.

d_1 : main effects are not aliased with other main effects and 2-factor interactions

d_2 : main effects are not aliased with main effects

- d_1 is better, because d_1 has higher resolution than d_2 . In fact, d_1 is optimal among all the possible fractional factorial 2^{4-1} designs
- **Maximum Resolution Criterion**
fractional factorial design with maximum resolution is optimal

Analysis for 2^{4-1} Design: Filtration Experiment

Recall that the filtration rate experiment was originally a 2^4 full factorial experiment. We pretend that only half of the combinations were run. The chosen half is defined by $I = ABCD$. So it is now a 2^{4-1} design. We keep the original responses.

basic design				
A	B	C	$D = ABC$	filtration rate
—	—	—	—	45
+	—	—	+	100
—	+	—	+	45
+	+	—	—	65
—	—	+	+	75
+	—	+	—	60
—	+	+	—	80
+	+	+	+	96

Let $\mathcal{L}_{\text{effect}}$ denote the estimate of effect (based on the corresponding contrast). Because of aliasing,

$$\mathcal{L}_I \rightarrow I + ABCD$$

$$\mathcal{L}_A \rightarrow A + BCD$$

$$\mathcal{L}_B \rightarrow B + ACD$$

$$\mathcal{L}_C \rightarrow C + ABD$$

$$\mathcal{L}_D \rightarrow D + ABC$$

$$\mathcal{L}_{AB} \rightarrow AB + CD$$

$$\mathcal{L}_{AC} \rightarrow AC + BD$$

$$\mathcal{L}_{AD} \rightarrow AD + BC$$

2^{4-1} Filtration Experiment: Using SAS

```
data filter;
  do C = -1 to 1 by 2; do B = -1 to 1 by 2; do A = -1 to 1 by 2;
    D=A*B*C; input y @@; output;
  end; end; end;
  datalines;
45 100 45 65 75 60 80 96
;

/* Define Interaction Terms */
data inter; set filter;
  AB=A*B; AC=A*C; AD=A*D;

/* GLM Proc to Obtain Effects */
proc glm data=inter;
  class A B C D AB AC AD;
  model y=A B C D AB AC AD;
  estimate 'A' A -1 1; estimate 'B' B -1 1; estimate 'C' C -1 1;
  estimate 'D' D -1 1; estimate 'AB' AB -1 1; estimate 'AC' AC -1 1;
  estimate 'AD' AD -1 1; run;
```

```
/* REG Proc to Obtain Effects */
proc reg outest=effects data=inter;
    model y=A B C D AB AC AD;

data effect2; set effects;
    drop y intercept _RMSE_;

proc transpose data=effect2 out=effect3;
data effect4; set effect3; effect=col1*2;

proc sort data=effect4; by effect;
proc print data=effect4;

proc rank data=effect4 normal=blom out=effect5;
    var effect; ranks neff;

symbol1 v=circle;
proc gplot data=effect5;
    plot effect*neff=_NAME_;
run; quit;
```

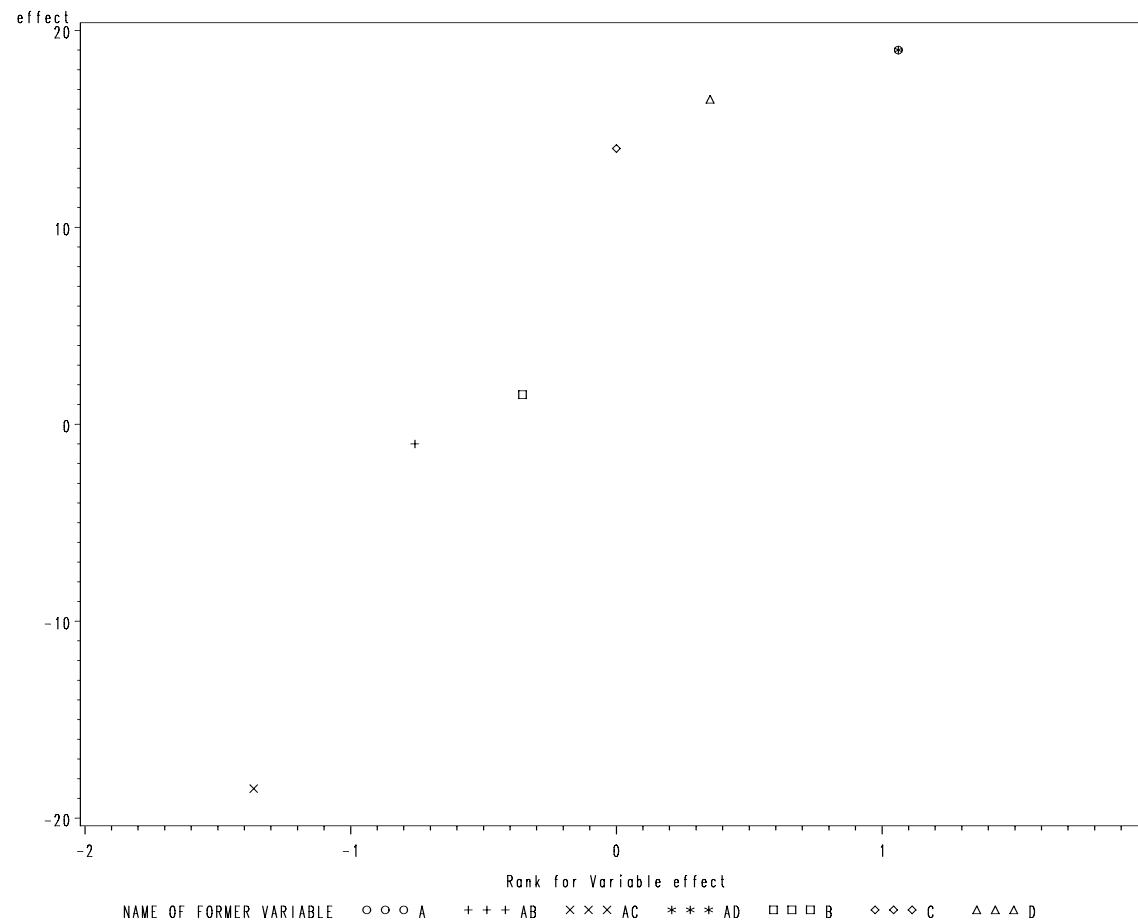
SAS Output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	3071.500000	438.785714	.	.
Error	0	0.000000	.		
CoTotal	7	3071.500000			

Source	DF	Type I SS	Mean Square	F Value	Pr > F
A	1	722.0000000	722.0000000	.	.
B	1	4.5000000	4.5000000	.	.
C	1	392.0000000	392.0000000	.	.
D	1	544.5000000	544.5000000	.	.
AB	1	2.0000000	2.0000000	.	.
AC	1	684.5000000	684.5000000	.	.
AD	1	722.0000000	722.0000000		

Obs	_NAME_	COL1	effect
1	AC	-9.25	-18.5
2	AB	-0.50	-1.0
3	B	0.75	1.5
4	C	7.00	14.0
5	D	8.25	16.5
6	A	9.50	19.0
7	AD	9.50	19.0

QQ plot to Identify Important Effects



- Potentially important effects: A , C , D , AC and AD .

Regression Model

- Let x_1, x_3, x_4 be the variables for factor A, C and D . The model is

$$y = 70.75 + 9.50x_1 + 7.00x_3 + 8.25x_4 - 9.25x_1x_3 + 9.50x_1x_4$$

- In Lecture 10, the regression model based on all the data (2^4) is

$$y = 70.06 + 10.81x_1 + 4.94x_3 + 7.31x_4 - 9.06x_1x_3 + 8.31x_1x_4$$

It appears that the model based on 2^{4-1} is as good as the original one.

– **Is this really true?**

- The answer is **NO**, because the chosen effects are aliased with other effects, so we have to resolve the ambiguities between the aliased effects first.

Aliased effects and Techniques for Resolving the Ambiguities

The estimates are for the sum of aliased factorial effects.

$$\mathcal{L}_I = 70.75 \rightarrow I + ABCD$$

$$\mathcal{L}_A = 19.0 \rightarrow A + BCD$$

$$\mathcal{L}_B = 1.5 \rightarrow B + ACD$$

$$\mathcal{L}_C = 14.0 \rightarrow C + ABD$$

$$\mathcal{L}_D = 16.5 \rightarrow D + ABC$$

$$\mathcal{L}_{AB} = -1.0 \rightarrow AB + CD$$

$$\mathcal{L}_{AC} = -18.5 \rightarrow AC + BD$$

$$\mathcal{L}_{AD} = 19.0 \rightarrow AD + BC$$

Techniques for resolving the ambiguities in aliased effects

- Use the fundamental principles (Slide 1)
- Follow-up Experiment
 - add orthogonal runs, or optimal design approach, or fold-over design

Sequential Experiment from a Fold Over Design

If it is necessary, the remaining 8 runs of the original 2^4 design can be conducted.

- Recall that the 8 runs we have used are defined by $I = ABCD$. The remaining 8 runs are indeed defined by the following relationship

$$D = -ABC, \text{ or } I = -ABCD$$

basic design				
A	B	C	$D = -ABC$	filtration rate
—	—	—	+	43
+	—	—	—	71
—	+	—	—	48
+	+	—	+	104
—	—	+	—	68
+	—	+	+	86
—	+	+	+	70
+	+	+	—	65

$I = -ABCD$ implies that: $A = -BCD$, $B = -ACD$, ..., $AB = -CD$, ...

Similarly, we can derive the following estimates ($\tilde{\mathcal{L}}_{\text{effect}}$) and alias structure

$$\tilde{\mathcal{L}}_I = 69.375 \quad \rightarrow \quad I - ABCD$$

$$\tilde{\mathcal{L}}_A = 24.25 \quad \rightarrow \quad A - BCD$$

$$\tilde{\mathcal{L}}_B = 4.75 \quad \rightarrow \quad B - ACD$$

$$\tilde{\mathcal{L}}_C = 5.75 \quad \rightarrow \quad C - ABD$$

$$\tilde{\mathcal{L}}_D = 12.75 \quad \rightarrow \quad D - ABC$$

$$\tilde{\mathcal{L}}_{AB} = 1.25 \quad \rightarrow \quad AB - CD$$

$$\tilde{\mathcal{L}}_{AC} = -17.75 \quad \rightarrow \quad AC - BD$$

$$\tilde{\mathcal{L}}_{AD} = 14.25 \quad \rightarrow \quad AD - BC$$

Combine Sequential Experiments

- Combining two experiments $\Rightarrow 2^4$ full factorial experiment
- Combining the estimates from these two experiments \Rightarrow estimates based on the full experiment

$$\mathcal{L}_A = 19.0 \rightarrow A + BCD$$

$$\tilde{\mathcal{L}}_A = 24.25 \rightarrow A - BCD$$

$$A = \frac{1}{2}(\mathcal{L}_A + \tilde{\mathcal{L}}_A) = 21.63$$

$$\widehat{BCD} = \frac{1}{2}(\mathcal{L}_A - \tilde{\mathcal{L}}_A) = -2.63$$

- Other effects are summarized in the following table

i	$\frac{1}{2}(\mathcal{L}_i + \tilde{\mathcal{L}}_i)$	$\frac{1}{2}(\mathcal{L}_i - \tilde{\mathcal{L}}_i)$
A	$21.63 \rightarrow A$	$-2.63 \rightarrow BCD$
B	$3.13 \rightarrow B$	$-1.63 \rightarrow ACD$
C	$9.88 \rightarrow C$	$4.13 \rightarrow ABD$
D	$14.63 \rightarrow D$	$1.88 \rightarrow ABC$
AB	$.13 \rightarrow AB$	$-1.13 \rightarrow CD$
AC	$-18.13 \rightarrow AC$	$-0.38 \rightarrow BD$
AD	$16.63 \rightarrow AD$	$2.38 \rightarrow BC$

- We know the combined experiment is not a completely randomized experiment.
 - Is there any underlying factor we need consider?
 - What is it?

General 2^{k-1} Design

- k factors: A, B, \dots, K
- Can only afford half of all the combinations (2^{k-1})
- Basic design: a 2^{k-1} full factorial for $k - 1$ factors: A, B, \dots, J .
- The setting of k th factor is determined by aliasing K with the $ABC \dots J$, i.e.,
 $K = ABC \dots J$
- Defining relation: $I = ABCD \dots K$. Resolution=k
- 2^k factorial effects are partitioned into 2^{k-1} groups each with two aliased effects.
- Only one effect from each group (the representative) should be included in ANOVA or regression model.
- Use fundamental principles, domain knowledge, follow-up experiment to de-alias.

One Quarter Fraction: 2^{k-2} Design

- Parts manufactured in an injection molding process show excessive shrinkage. A quality improvement team has decided to use a designed experiment to study the injection molding process so that shrinkage can be reduced. The team decides to investigate six factors

A : mold temperature

B : screw speed

C : holding time

D : cycle time

E : gate size

F : holding pressure

- Each at two levels, with the objective of learning about main effects and interactions.
- They decide to use 16-run fractional factorial design.
 - A full factorial has $2^6=64$ runs.
 - 16-run is one quarter of the full factorial
 - How to construct the fraction?

Injection Molding Experiment: 2^{6-2} Design

basic design						
<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E = ABC</i>	<i>F = BCD</i>	shrinkage
—	—	—	—	—	—	6
+	—	—	—	+	—	10
—	+	—	—	+	+	32
+	+	—	—	—	+	60
—	—	+	—	+	+	4
+	—	+	—	—	+	15
—	+	+	—	—	—	26
+	+	+	—	+	—	60
—	—	—	+	—	+	8
+	—	—	+	+	+	12
—	+	—	+	+	—	34
+	+	—	+	—	—	60
—	—	+	+	+	—	16
+	—	+	+	—	—	5
—	+	+	+	—	+	37
+	+	+	+	+	+	52

- Two defining relations are used to generate the columns for E and F .

$$I = ABCE, \text{ and } I = BCDF$$

- They induce another defining relation:

$$I = ABCE * BCDF = AB^2C^2DEF = ADEF$$

- The complete defining relation:

$$I = ABCE = BCDF = ADEF$$

- Defining contrasts subgroup: $\{I, ABCE, BCDF, ADEF\}$

Alias Structure for 2^{6-2} with $I = ABCE = BCDF = ADEF$

- $I = ABCE = BCDF = ADEF$ implies

$$A = BCE = ABCDF = DEF$$

- Similarly, we can derive the other groups of aliased effects.

$$A = BCE = DEF = ABCDF \quad AB = CE = ACDF = BDEF$$

$$B = ACE = CDF = ABDEF \quad AC = BE = ABDF = CDEF$$

$$C = ABE = BDF = ACDEF \quad AD = EF = BCDE = ABCF$$

$$D = BCF = AEF = ABCDE \quad AE = BC = DF = ABCDEF$$

$$E = ABC = ADF = BCDEF \quad AF = DE = BCEF = ABCD$$

$$F = BCD = ADE = ABCEF \quad BD = CF = ACDE = ABEF$$

$$BF = CD = ACEF = ABDE$$

$$ABD = CDE = ACF = BEF$$

$$ACD = BDE = ABF = CEF$$

- **Wordlength pattern** $W = (W_0, W_1, \dots, W_6)$, where W_i is the number of defining words of length i (i.e., involving i factors)

$$W = (1, 0, 0, 0, 3, 0, 0)$$

- Resolution is the smallest i such that $i > 0$ and $W_i > 0$. Hence it is a 2_{IV}^{6-2} design

2^{6-2} Design: an Alternative

- Basic Design: A, B, C, D
- $E = ABCD, F = ABC$, i.e., $I = ABCDE$, and $I = ABCF$
- which induces: $I = DEF$
- complete defining relation: $I = ABCDE = ABCF = DEF$
- wordlength pattern: $W = (1, 0, 0, 1, 1, 1, 0)$

- Alias structure (ignore effects of order 3 or higher)

$A = ..$	$AB = CF = ..$
$B = ..$	$AC = BF = ..$
$C = ..$	$AD = ..$
$D = EF = ..$	$AE = ..$
$E = DF = ..$	$AF = BC = ..$
$F = DE = ..$	$BD = ..$
	$BE = ..$
	$CD = ..$
	$CE = ..$

- an effect is said to be **clearly estimable** if it is not aliased with main effect or two-factor interactions.
- Which design is better d_1 or d_2 ? d_1 has six clearly estimable main effects while d_2 has three clearly estimable main effects and six clearly estimable two-factor interactions.

Injection Molding Experiment Analysis

```

data molding;
  do D = -1 to 1 by 2; do C = -1 to 1 by 2;
    do B = -1 to 1 by 2; do A = -1 to 1 by 2;
      E=A*B*C; F=B*C*D;  input y @@;  output;
    end; end;
  end; end;  datalines;
6 10 32 60 4 15 26 60 8 12 34 60 16 5 37 52
;

/* Define Interaction Terms */
data inter; set molding;
  AB=A*B; AC=A*C; AD=A*D; AE=A*E; AF=A*F; BD=B*D; BF=B*F;
  ABD=A*B*D; ACD=A*C*D;

/* GLM Proc to Obtain Effects */
proc glm data=inter;
  class A B C D E F AB AC AD AE AF BD BF ABD ACD;
  model y=A B C D E F AB AC AD AE AF BD BF ABD ACD;
  estimate 'A' A -1 1;      estimate 'B' B -1 1;      estimate 'C' C -1 1;
  estimate 'D' D -1 1;      estimate 'E' E -1 1;      estimate 'F' F -1 1;
  estimate 'AB' AB -1 1;    estimate 'AC' AC -1 1;    estimate 'AD' AD -1 1;
  estimate 'AE' AE -1 1;    estimate 'AF' AF -1 1;    estimate 'BD' BD -1 1;
  estimate 'BF' BF -1 1;    estimate 'ABD' ABD -1 1; estimate 'ACD' ACD -1 1;

```

```
/* REG Proc to Obtain Effects */
proc reg outest=effects data=inter;
    model y=A B C D E F AB AC AD AE AF BD BF ABD ACD;

data effect2;
    set effects; drop y intercept _RMSE_;

proc transpose data=effect2 out=effect3;
data effect4;
    set effect3; effect=col1*2;

proc sort data=effect4; by effect;
proc print data=effect4;

proc rank data=effect4 normal=blom out=effect5;
    var effect;
    ranks neff;
    symbol1 v=circle;

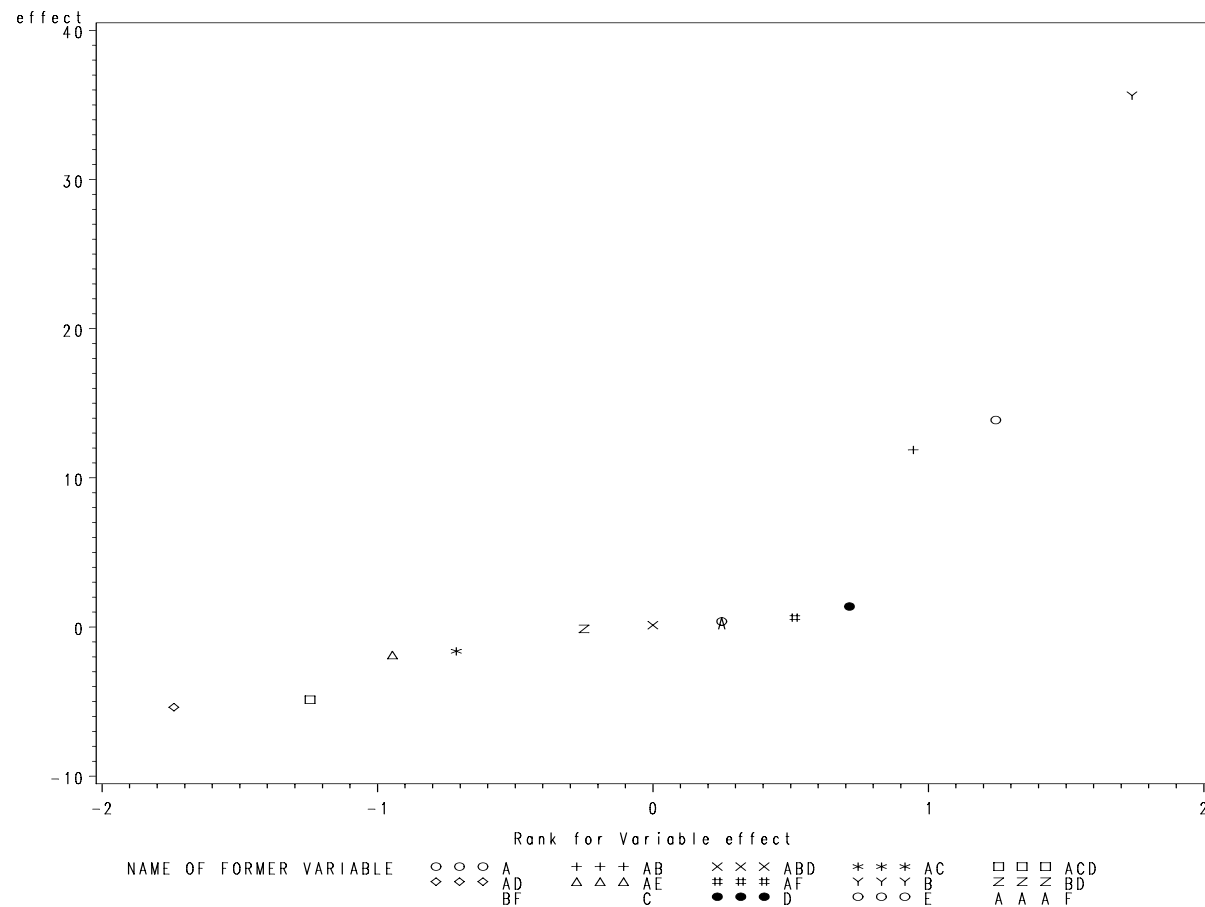
proc gplot data=effect5;
    plot effect*neff=_NAME_;
run; quit;
```

Estimates of Factorial Effects

Obs	_NAME_	COL1	effect	aliases
1	AD	-2.6875	-5.375	AD+EF
2	ACD	-2.4375	-4.875	
3	AE	-0.9375	-1.875	AE+BC+DF
4	AC	-0.8125	-1.625	AC+BE
5	C	-0.4375	-0.875	
6	BD	-0.0625	-0.125	BD+CF
7	BF	-0.0625	-0.125	BF+CD
8	ABD	0.0625	0.125	
9	E	0.1875	0.375	
10	F	0.1875	0.375	
11	AF	0.3125	0.625	AF+DE
12	D	0.6875	1.375	
13	AB	5.9375	11.875	AB+CE
14	A	6.9375	13.875	
15	B	17.8125	35.625	

- Effects B , A , AB , AD , ACD , are large.

QQ plot to Identify Important Effects



- Effects B , A , AB appear to be important; effects AD and ACD are suspicious.

De-aliasing and Model Selection

- Model 1:

```
proc reg data=inter;
  model y=A B AB AD ACD; run; quit;
```

Root MSE	1.95256	R-Square	0.9943
Dependent Mean	27.31250	Adj R-Sq	0.9914
Coeff Var	7.14897		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > t
Intercept	1	27.31250	0.48814	55.95	<.0001
A	1	6.93750	0.48814	14.21	<.0001
B	1	17.81250	0.48814	36.49	<.0001
AB	1	5.93750	0.48814	12.16	<.0001
AD	1	-2.68750	0.48814	-5.51	0.0003
ACD	1	-2.43750	0.48814	-4.99	0.0005

- Model 2:

```
proc reg data=inter;
  model y=A B AB; run; quit;
```

Root MSE	4.55293	R-Square	0.9626
Dependent Mean	27.31250	Adj R-Sq	0.9533
Coeff Var	16.66976		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > t
Intercept	1	27.31250	1.13823	24.00	<.0001
A	1	6.93750	1.13823	6.09	<.0001
B	1	17.81250	1.13823	15.65	<.0001
AB	1	5.93750	1.13823	5.22	0.0002

Three-Factor Interaction: Using SAS

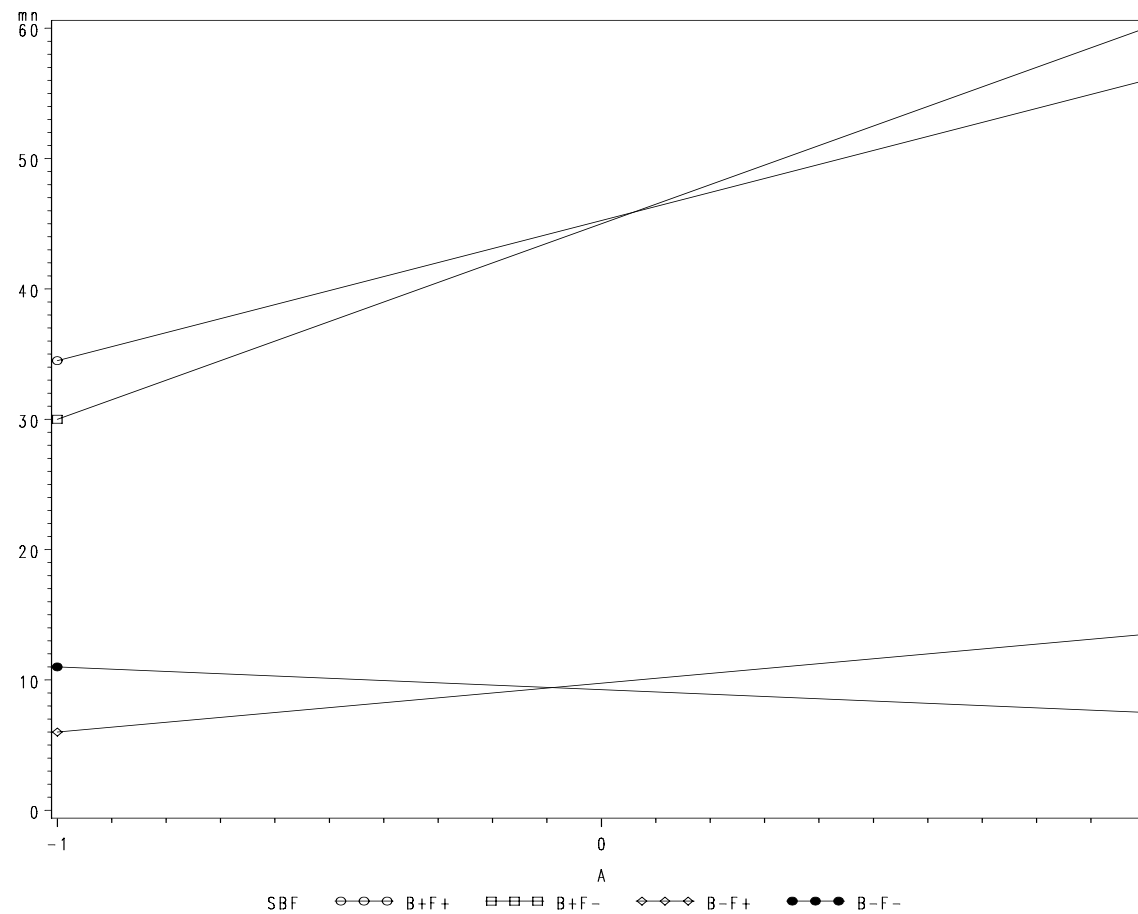
```
/* Define Interaction Terms */
data inter; set molding;
  AB=A*B; AC=A*C; AD=A*D; AE=A*E; AF=A*F; BD=B*D; BF=B*F;
  ABD=A*B*D; ACD=A*C*D;
  if B=-1 and F=-1 then SBF='B-F-';
  if B=-1 and F=1 then SBF='B-F+';
  if B=1 and F=-1 then SBF='B+F-';
  if B=1 and F=1 then SBF='B+F+';

proc sort data=inter; by A SBF;
proc means noprint;
  var y; by A SBF;
  output out=ymeanabf mean=mn;

/* Note ACD=BDE=ABF=CEF */
symbol1 v=circle i=join; symbol2 v=square i=join;
symbol3 v=diamond i=join; symbol4 v=dot i=join;
proc gplot data=ymeanabf;
  plot mn*A=SBF; run; quit;
```

3-Factor Interaction Plot

$$ACD=BDE=ABF=CEF$$



General 2^{k-p} Fractional Factorial Designs

- k factors, 2^k level combinations, but want to run a $1/2^p$ fraction only.
- Select the first $k - p$ factors to form a full factorial design (basic design).
- Alias the remaining p factors with some high order interactions of the basic design.
- There are p defining relation, which induces other $2^p - p - 1$ defining relations. The complete defining relation is $I = \dots = \dots = \dots$
- Defining contrasts subgroup: $G = \{ \text{defining words} \}$
- Wordlength pattern: $W = (W_i)$, W_i =the number of defining words of length i .
- Alias structure: 2^k factorial effects are partitioned into 2^{k-p} groups of effects, each of which contains 2^p effects. Effects in the same group are aliased (aliases).
- Use maximum resolution and minimum aberration to choose the optimal design.
- In analysis, only select one effect from each group to be included in the full model.
- Choose important effect to form models, pool unimportant effects into error component
- De-aliasing and model selection.

Minimum Aberration Criterion

- Recall 2^{k-p} with maximum resolution should be preferred.
 - But, it is possible that there are two designs that attain the maximum resolution.
 - How should we further distinguish them?
- For example, consider 2^{7-2} fractional factorial design
 - d_1 : basic design: A, B, C, D, E ; $F = ABC, G = ABDE$
 - * Complete defining relation: $I = ABCF = ABDEG = CDEFG$
 - * Wordlength pattern: $W = (1, 0, 0, 0, 1, 2, 0, 0)$
 - * Resolution: IV
 - d_2 : basic design: A, B, C, D, E ; $F = ABC, G = ADE$
 - * Complete defining relation: $I = ABCF = ADEG = BCDEFG$
 - * Wordlength pattern: $W = (1, 0, 0, 0, 2, 0, 1, 0)$
 - * Resolution: IV
- d_1 and d_2 , which is better?

Minimum Aberration Criterion

- Definition:
 - Let d_1 and d_2 be two 2^{k-p} designs, let r be the smallest **positive** integer such that $W_r(d_1) \neq W_r(d_2)$.
 - If $W_r(d_1) < W_r(d_2)$, then d_1 is said to have less aberration than d_2 .
 - If there does not exist any other design that has less aberration than d_1 , then d_1 has minimum aberration.
- Small Minimum Aberration Designs are used a lot in practice. They are tabulated in most design books.
 - See Table 8.14 in Montgomery.
 - For the most comprehensive table, consult Wu & Hamada.