OPTION PRICING STRATEGIES FOR ONE CARD POKER

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by

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Abstract

One Card Poker is a two-person, zero-sum, three-card-deck game. An opener and a dealer play a series of one card poker games. A voyeur privy to the game intends to bet upon the amount won by one of the players. The amount won by a player is a discrete random variable (rv) subject to a probability mass function (pmf). The voyeur can equivalently bet upon the value of this rv. We construct a stock whose share price intentionally mimics the amount won by the player. Such a stock is a continuous rv subject to a lognormal probability density function (pdf). We thus obtain a mapping from the pmf to a similar pdf. Betting upon the amount won by the player is equivalent to betting upon the share price of the stock. In order to execute this bet, Euro-style derivatives are priced on the stock using Black-Scholes option pricing models. The voyeur holds a basket of such derivatives in his portfolio. The voyeur's optimal bet is then equivalent to maximizing the returns of the portfolio, as the share price tracks the player winnings.

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I would like to thank Dr. John George for his constant support and encouragement. Dr. George and Dr. Kathleen Salter were instrumental in devising coin-toss simulations to test various scenarios of the game. Dr. George painstakingly walked me through the combinatorics underlying the game as well as proofs involving the moment generating function, and I thank him for his infinite patience. Thanks are due to Dr. Jason Swanson for devising a simplified game-theoretic version of poker amenable to probabilistic analysis and generalization relevant to the derivatives marketplace, and to Dr. Tom Brown for introducing Swanson's One Card Poker in a lively exposition at the weekly Math seminar, which resulted in the inception of this thesis.

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Chapter I. **Definitions pertinent to the game of 1-card poker**

1. Dealer: One of the two players in the game. The dealer shuffles the deck, deals one card to the opener, one to self, and places the third card aside, in that order. All three cards are dealt face down. The dealer always plays after the opener.

2. Opener: One of the two players. The opener plays the first hand.

3. Deck: A set of 3 cards – ace, deuce and trey, with trey the high card and ace the low.

4. Zero Sum Game: One player's loss is the other player's gain. 1-card poker is strictly a zero-sum game. The sum of the opener's stake and the dealer's stake at the beginning of the game equals the sum of their winnings at the end of the game.

5. Showdown: A show of cards. If one of the players does not fold, the game always ends in a showdown. The player with the high card wins.

6. Pot: The sum of money at stake in one round of a poker game.

7. Ante: A poker stake put up before the deal to build the pot.

8. Check: To waive the right to bet. Check and bet are mutually exclusive ie. if a player does not check, he must bet. A check by the opener is followed by either a check or a bet by the dealer. The dealer and the opener must always check with the deuce. Further, the dealer never can check with the trey. Constraints such as these are imposed upon the players' strategies to ensure neither player makes plays that are dominated from a game theory standpoint.

9. Bet: To stake on the outcome of the game. Check and bet are mutually exclusive. A bet by a player is always followed by a call or a fold by the other player. The dealer always bets the trey. Neither player can bet with a deuce.

10. Call: To challenge a bet. Call and fold are mutually exclusive. If a player calls a bet and ends up with the low card, he loses $2. If the player instead has the high card, he gains $2. Neither player can call with an ace. Both players must call with a trey.

11. Fold: To concede defeat in response to a bet. The player who folds always loses $1. Call and fold are mutually exclusive. Both players must fold with an ace. Neither player folds the trey.

12. Strategy: A player's strategy is a set of probabilities for each action associated with a game. Since there are 4 actions (check, bet, call, fold) associated with a 3 card deck, we have 12 probabilities per player. The player's strategy i.e. the set of 12 probabilities for a game is captured in a 4x3 probability matrix. When a series of games are played, the players cannot change their strategy midstream.

13. One Card Poker: A simplified version of poker devised by Dr. Jason Swanson to illustrate game-theoretic principles. 1-card poker is a zero sum, two-person, three-card-deck game. To play 1 game, each player must have a minimum bankroll of $2. Assume the players bring exactly $2 each to the table. Each player pays a $1 ante to the pot. The three card deck comprises an ace, a deuce and a trey. The dealer deals one card to the opener, one to self, and the third card is placed aside, in that order. All three cards are dealt face down. The third card has no role other than contributing an element of randomness to the game.
There are exactly four actions each player may perform – check, bet, call and fold. Each player has a strategy i.e. a set of 12 probabilities for each action. Some of these actions are mutually exclusive. Furthermore, each action by one player leads to a fixed set of actions by the other. To illustrate, the opener may either check or bet. Say the opener checks, the dealer may then check or bet. If the opener bets, the dealer can call the bet or fold. The opener may similarly call the dealer's bet or fold. Unless

one of the players folds, the game ends in a showdown, and the player with the high card takes the pot. The player gains an additional dollar from his opponent if he calls the opponent's bet and wins. Thus if both players check, there is a show of cards, and the winner ends up with $3 while the other has $1. If one of the players bets and the other calls the bet, there is a show of cards and the winner ends up with $4 while the other has $0. On the other hand, if one of the players folds, he loses the pot to the opponent as there is no showdown. In this case, one of the players ends up with $3 and the other with $1. The cases outlined so far are mutually exclusive and exhaustive. Hence, from the point of view of a player in 1-card poker, each game has exactly one of four possible outcomes – the player wins $1, wins $2, loses $1, or loses $2. At the end of each game, the players switch positions i.e. the dealer in the previous game becomes the opener in the next. If the dealer stays the dealer during the course of a series of games, it can be shown that the dealer has an advantage from a game theory standpoint. To neutralize this advantage, we mandate switching positions after each game. Furthermore, to simplify the analysis, we require each player to stick to his playing strategy throughout the duration of the series of games.

An example is in order. Assume the dealer has a deuce and the opener a trey. Each player comes to the table with $4 (typically, each player has a much larger, potentially unlimited bankroll). Thus the total amount at stake is $8. They ante $1 to the pot. The opener checks with the trey.

The dealer cannot bet with the deuce and checks as well. The game ends in a show of cards. The dealer loses \$1 to the opener. They play a second round where the dealer gets the trey and the opener the ace. The opener attempts to bluff the dealer by betting the ace. The dealer calls the bet and wins \$2 from the opener. At the end of two rounds, the dealer has \$5, the opener \$3. Notice the zero sum nature of the series of games i.e. Cash before games 4+4 = Cash after 1st game 3+5 = Cash after 2nd = 5+3.

14. Voyeur: A non-player who is privy to the result of each 1-card poker game played by the opener and the dealer. The voyeur seeks to bet upon the amount won by one of the players. If the amount won were regarded as a share price, we demonstrate that the voyeur may instead purchase options associated with that share. Betting upon the amount won is equivalent to buying said options.

15. Random Variable (rv): A random variable is a set function. A random experiment has a set of outcomes, called the sample space. The random variable X assigns to each element c of the sample space a real number x, such that x = X(c). The space of X is the set of reals. In a coin-toss, the sample space is the set {Heads, Tails}, the random variable X may map Heads to 1 and Tails to 0. Thus, 0 = X(Tails). A finite sample space indicates a discrete rv, otherwise the rv is continuous.

16. Probability Mass Function (pmf): $f_X(x) = Pr(X=x)$ is a function that assigns to each real number x in the range space of the random variable X, a real number p in [0,1], where p denotes the probability that the random experiment has an outcome c in the sample space, and x = X(c). pmf's apply to discrete rv's only.

17. Probability Density Function (pdf): Feller's definition "A never-decreasing-function F(x) that tends to 0 as x goes to negative infinity, and tends to 1 as x goes to infinity" suffices in this context. pdf differs from pmf in that pdf applies to continuous rv's only. The probability of an rv falling within an interval [a,x] is given by the integral of the pdf over (a,x].

18. Stock/Share: For the purposes of 1-card poker, a stock/share is a financial instrument constructed such that its price moves in tandem with the amount won by a player. The share price is a continuous rv subject to a lognormal pdf.

19. Derivative/Option: For the purposes of 1-card poker, an option is a financial instrument whose value depends on the share price of the underlying stock. We are interested in 2 kinds of options: Puts and Calls.

20. Call Option/Call: A Euro-style call option is the right, but not the obligation, to buy the underlying share for a fixed price (called the strike

price) at a specified later date. The time between the purchase of the call option and the later date is termed time to Expiry. The unique price of a call option, given by the solution to the Black-Scholes partial differential equation, is a stochastic variable dependent on the price of the underlying, the strike price, the time to expiry, the volatility of the underlying, and a risk-free rate of interest.

21. Put Option/Put: An inverse-call i.e. a Euro-style put option is the right, but not the obligation, to sell the underlying share for a fixed price (called the strike price) at a specified later date.

22. Arbitrage: The notion that money is made with no risk taken. The notion of making money with no risk is in comparison with investing said money in a risk-free interest-bearing government bond. No arbitrage, called the “no free lunch” principle, stipulates it is impossible to make money without taking the risk of losing money invested in a portfolio comprised of stocks and options. Formally then, an arbitrage is a trading strategy in a portfolio whose initial value is zero and future value is positive with a non-zero probability. No dynamic arbitrage, the idea that continuously trading securities does not result in an arbitrage, is the central notion that underpins Black-Scholes derivative pricing.

23. Black-Scholes option pricing model: The great breakthrough in derivatives pricing was made by Black and Scholes in 1973, after which "the price of a derivative ceased to be an economic problem and became a mathematical problem" (Financial Mathematician Mark Joshi quoted in The Princeton Companion to Mathematics). Black and Scholes showed that under the assumptions that there exists no dynamic arbitrage, no transaction costs, that trading in a share does not affect its price, and that it is possible to trade continuously, the price of a call option is unique. This unique call price is the solution of the Black-Scholes partial differential equation. There are two crucial insights in the Black-Scholes model. First, if a call option is bought for less or sold for more than the unique Black-Scholes price, a risk-less profit can be made. Second, the drift does not appear in the Black-Scholes formula! Option prices are thus unaffected by the expected behavior of the share's future mean price, denoted by its drift (ie. upward/downward drift).

Chapter II **Introduction**

This work is an exercise in mapping a discrete pmf to a continuous pdf that exhibits similar characteristics. The motivations for finding such a mapping are outlined below.

The opener and the dealer play a series of one-card poker games. A voyeur seeks to bet upon the total amount won by one of the players at the end of the series. Each one-card poker game is a stochastic experiment with four possible outcomes. Thus each experiment corresponds to four discrete rv's occurring with fixed probabilities such that these probabilities sum to unity. The series of n games is equivalent to n trials of the experiment. The amount won by a player at the end of n trials is a linear combination of the four discrete rv's. This sum is a discrete rv subject to a pmf. We seek a mapping from this pmf to a continuous lognormal pdf that exhibits similar behavior. To ensure that the discrete rv pertaining to the winnings of the player and the continuous rv pertaining to the lognormal pdf exhibit similar behavior, the drift of the lognormal is obtained from the mean of the pmf. Furthermore, the variances of the rv's are identical. Under these circumstances, we demonstrate that the rv's behave similarly.

The lognormal rv can be regarded as a stock, whose share price fluctuates in accordance with the strategy adopted by the players in each game. The voyeur bets upon the final price attained by the stock. Such a bet can be executed by owning a portfolio of European options. A one month European call at a given strike, for example, benefits on the upside when the price of the stock exceeds the strike at the expiry of one month. European puts exhibit the opposite behavior. Optimally betting upon the outcome of the game is equivalent to owning an optimal mix of puts and calls in the voyeur's portfolio.

Thus, we have converted a problem of optimally betting upon the outcome of a two-person game, to that of finding the optimal mix of derivatives. Such a mapping is fruitful since we can use our knowledge of trading equities to inform game theory and vice-versa. For instance, in the publication "Probability and Finance – It's only a game!" Glenn Shafer and Vladimir Vovk explore the inverse problem, that of treating a stock as a two-person game.

Chapter III **Results**

We make the following game-theoretic observations:

a. 3 cards (ace, deuce, and trey) and 4 associated actions (bet, check, fold, call) give rise to 3*4 = 12 distinct probabilities (prob.) per player.

b. prob. of player betting a card + prob. of checking a card = 1

c. prob. of player folding + prob. of calling a bet = 1

d. prob. of calling with an ace = 0

e. prob. of dealer betting with a trey = 1

f. prob. of betting with a deuce = 0

g. prob. of folding with a trey = 0

Subject to a-g, each player has the following strategy/probability matrix:

Action/Card	Ace	Deuce	Trey
Checks	probability that the player checks with an ace	probability that the player checks with a deuce	probability that the player checks with trey
Bets	probability that the player bets with an ace	probability that the player bets with a deuce	probability that the player bets with trey
Calls	probability that the player calls with an ace	probability that the player calls with a deuce	probability that the player calls with trey
Folds	probability that the player folds with an ace	probability that the player folds with a deuce	Probability that the player folds with trey

Ex Sample probabilities for the two players are as shown:

```
> Opener := <<0.3,0.7,0,1>|<1,0,2/3,1/3>|<1/2,1/2,1,0>>:
  Dealer := <<0.7,0.3,0,1>|<1,0,1/5,4/5>|<0,1,1,0>>:
  print('Opener'= Opener,'Dealer'= Dealer);
```

$$Opener = \begin{bmatrix} 0.3 & 1 & \frac{1}{2} \\ 0.7 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & 1 \\ 1 & \frac{1}{3} & 0 \end{bmatrix}, \quad Dealer = \begin{bmatrix} 0.7 & 1 & 0 \\ 0.3 & 0 & 1 \\ 0 & \frac{1}{5} & 1 \\ 1 & \frac{4}{5} & 0 \end{bmatrix}$$

In the example above, probability that dealer folds with a deuce = 4/5.

In game theory, two-player games may be represented by a digraph, whose nodes represent the possible state of play. Play begins at the root, and flows through the tree along a path determined by the players until a leaf is reached, whereupon play ends and payoffs are assigned to all players. Edges indicate the probabilities of the action. We can get to a leaf with a probability equal to the product of the edges that connect the root to the leaf. Assign 1 card to each player from a deck of 3 cards; there are 6 possible ways to perform this assignment. We construct digraphs pertaining to each of the 6 assignments.

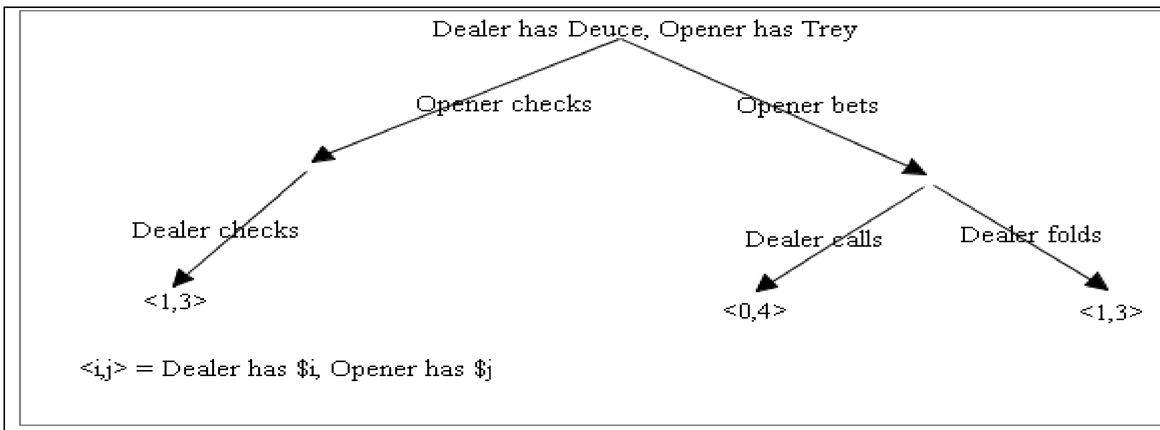
	Ace	Deuce	Trey
Checks	prob. checks with ace Op[1,1], De[1,1]	prob. checks with deuce Op[1,2] = 1, De[1,2] = 1	prob. checks with a trey Op[1,3], De[1,3]=0
Bets	prob. bets with ace Op[2,1], De[2,1]	prob. bets with deuce Op[2,2] = 0, De[2,2] = 0	prob. bets with a trey Op[2,3], De[1,3]=1
Calls	prob. calls with ace Op[3,1]=0, De[3,1]=0	prob. calls with deuce Op[3,2], De[3,2]	prob. calls with a trey Op[3,3] = 1, De[3,3]=1
Folds	prob. folds with ace Op[4,1]=1, De[4,1]=1	prob. folds with deuce Op[4,2], De[4,2]	prob. folds with a trey Op[4,3]=0, De[4.3]=0

From the point-of-view of a player, there is exactly one of four possible outcomes – the player wins \$1, wins \$2, loses \$1, or loses \$2. Let Op[x,y] denote entries in the opener's probability matrix,

De[x,y] denote entries in the dealer's probability matrix.

p[i] = prob. that dealer makes \$i.

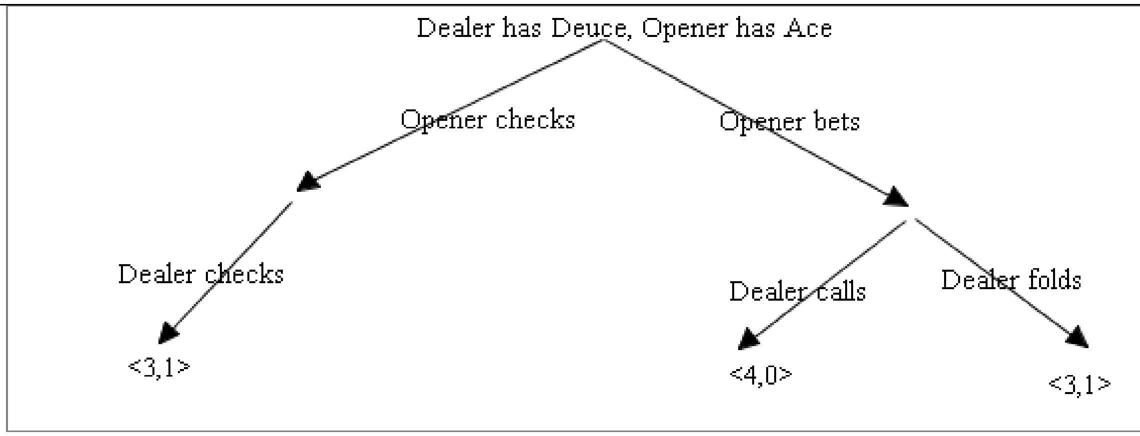
Let each player possess \$2; we list outcomes for each of the 6 digraphs:



p[-1] = Op[1,3] +Op[2,3]*De[4,2]

p[-2] = Op[2,3]*De[3,2]

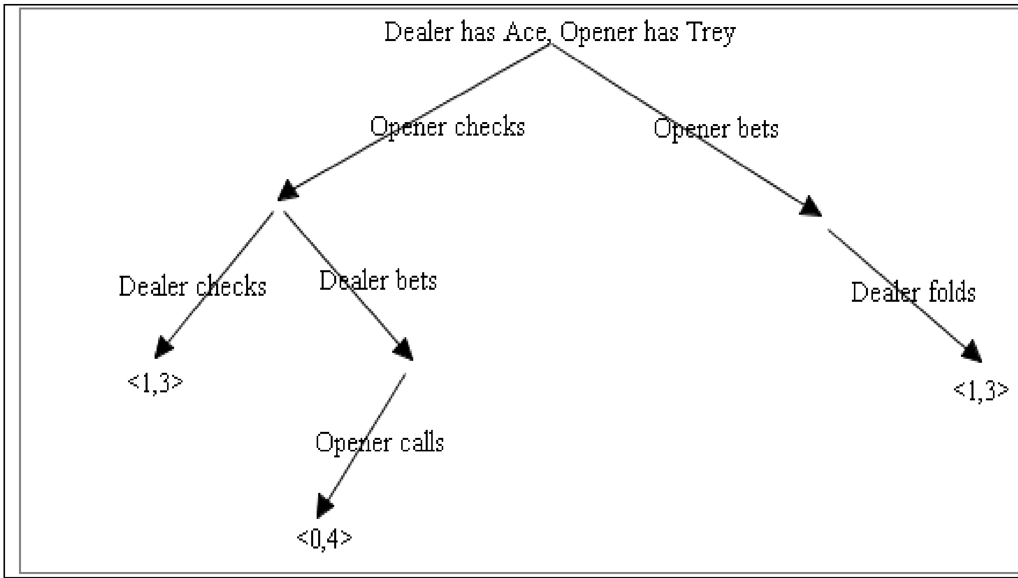
p[1] = p[2] = 0



p[-1] = p[-2]=0

p[1] = Op[1,1] + Op[2,1]*De[4,2]

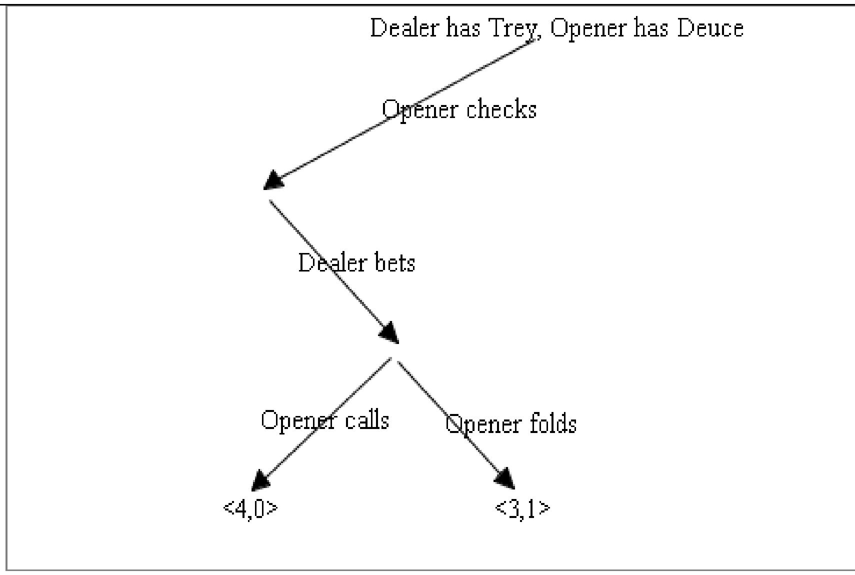
p[2] = Op[2,1]*De[3,2]



p[-1] = Op[1,3]*De[1,1]+Op[2,3]

p[-2] = Op[1,3]*De[2,1]

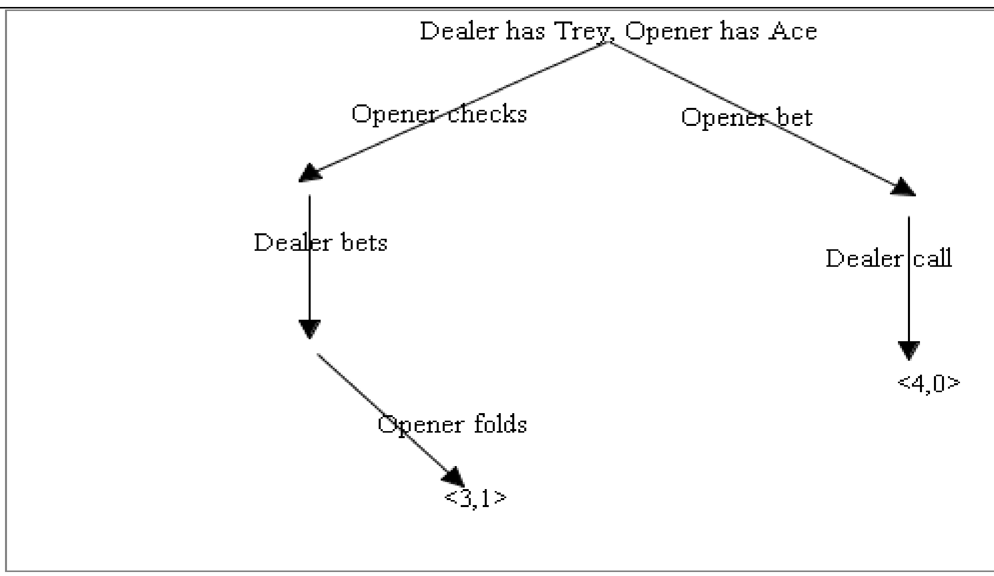
p[1] = p[2]=0



p[-1] = p[-2]=0

p[1] =Op[3,2]

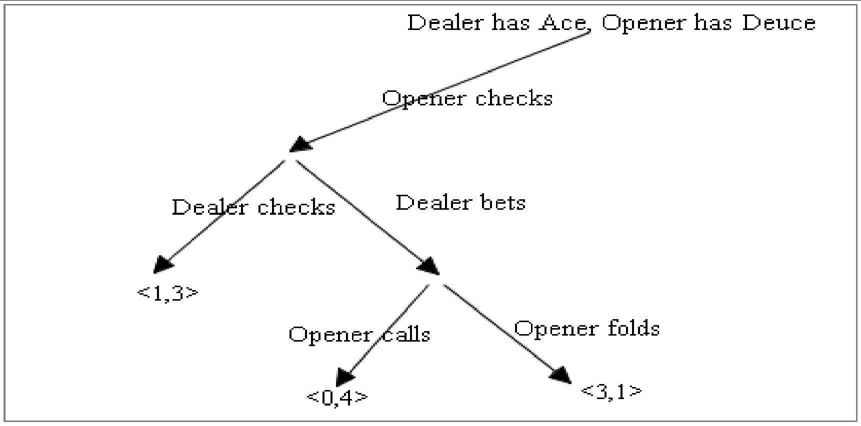
p[2]= Op[4,2]



p[-1] = p[-2]=0

p[1] = Op[1,1];

p[2] = Op[2,1]



p[-1] = De[1,1]

p[-2] = De[2,1]*Op[3,2]

p[1] = De[2,1]*Op[4,2]

p[2] = 0

The probabilities p[-2], p[-1], p[1] and p[2] corresponding to each of the 6 digraphs are mutually exclusive and exhaustive. Since the game has 6 hands, probability of playing any one hand is 1/6. Thus the probability p[i] of winning $i is obtained by summing over the p[i]'s for the 6 digraphs and multiplying by 1/6, resulting in:

p[-1] = (Op[1,3] +Op[2,3]*De[4,2] + Op[1,3]*De[1,1]+Op[2,3]+ De[1,1])/6

p[-2] = (Op[2,3]*De[3,2]+ Op[1,3]*De[2,1]+ De[2,1]*Op[3,2])/6

p[1] = (Op[1,1]+ Op[2,1]*De[4,2]+ Op[3,2]+ Op[1,1]+ De[2,1]*Op[4,2])/6

p[2] = (Op[2,1]*De[3,2]+ Op[4,2]+ Op[2,1])/6

Furthermore we have the identities

Op[1,i] +Op[2,i] = 1, De[1,i] +De[2,i] = 1.

Op[3,i] +Op[4,i] = 1, De[3,i] +De[4,i] = 1.

We require p[-1]+p[-2]+p[1]+p[2]=1, which we shall now show.

```
> for i from 1 to 4 do
   Op[1, i] := 1 - Op[2, i]:
   De[1, i] := 1 - De[2, i]:
   Op[3, i] := 1 - Op[4, i]:
   De[3, i] := 1 - De[4, i]:
   od:
  p[-1] := ( Op[1, 3] + Op[2, 3]*De[4, 2] + Op[1, 3]*De[1, 1] + Op[2, 3] + De[1, 1]) / 6 :
  p[-2] := ( Op[2, 3]*De[3, 2] + Op[1, 3]*De[2, 1] + De[2, 1]*Op[3, 2]) / 6 :
  p[1] := ( Op[1, 1] + Op[2, 1]*De[4, 2] + Op[3, 2] + Op[1, 1] + De[2, 1]*Op[4, 2]) / 6 :
  p[2] := ( Op[2, 1]*De[3, 2] + Op[4, 2] + Op[2, 1]) / 6 :
  simplify(eval(p[-1] + p[-2] + p[1] + p[2] ));

1
```

The Maple snippet above establishes p[-1]+p[-2]+p[1]+p[2] = 1.

We may now treat each game as a black box that simply evolves in a well known fashion – with exactly one of 4 possible outcomes.

Consider n trials of the game. The four-tuple rv <x,y,z,w> hereafter called fourple, denotes the number of occurrences of each of the four possible outcomes in n games/trials.

x = # of occurrences of $-2 (in n games)

y = # of occurrences of $-1

z = # of occurrences of $1

w = # of occurrences of $2

Since there are n trials, with x, y, z, w being the counts of mutually exclusive events, we have **A. x + y + z + w = n**

Furthermore, the amount k won by the dealer is given by **B. k = -2x – y + z + 2w**

In order to obtain the pmf of a fourple, we wish to compute the set of all fourples <x,y,z,w> and their linear combination k, for a fixed n. Such a computation would normally be O(n^4) since we seek 4 positive integers x,y,z,w in the interval [0,n]. However, we use the two constraints listed above and some clever arithmetic to obtain an original algorithm that is computationally efficient of the order of O(n^2) and substantially faster.

```
> fourples2 := proc(n)
  local i, c1, c2, c3, c4, c3positive, mylist, cash, k, howmanyways;
  cash := 2·n;

  for k from -cash to cash do
  i := 0; unassign(mylist);
   for c1 from 0 to n do
     c3positive := true;
      for c2 from 0 to n - c1 while c3positive do
         c3 := 2·n - 4·c1 - 3·c2 -k;
          if c3 ≥ 0 then
               c4 := n - c1 - c2 - c3;
               if c4 ≥ 0 then
                  i := i + 1;
                  mylist[i] := [c1, c2, c3, c4];
               fi;
          else
               c3positive := false;
          fi;
     od;
  od;
  howmanyways[k] := [i, [seq(mylist[j], j = 1..i)]];
  od;

   return howmanyways;
  end proc:
```

We examine a portion of the results of this computation for a fixed n = 21.

```
1 ways to reach $-42: [21,0,0,0]

8 ways to reach $-32: [11,10,0,0] [13,7,1,0] [14,6,0,1] [15,4,2,0]
[16,3,1,1] [17,1,3,0] [17,2,0,2] [18,0,2,1]

24 ways to reach $-22: [1,20,0,0] [3,17,1,0] [4,16,0,1] [5,14,2,0]
[6,13,1,1] [7,11,3,0] [7,12,0,2] [8,10,2,1] [9,8,4,0] [9,9,1,2]
[10,7,3,1] [10,8,0,3] [11,5,5,0] [11,6,2,2] [12,4,4,1] [12,5,1,3]
[13,2,6,0] [13,3,3,2] [13,4,0,4] [14,1,5,1] [14,2,2,3] [15,0,4,2]
[15,1,1,4] [16,0,0,5]

38 ways to reach $-12: [0,17,3,1] [0,18,0,3] [1,15,5,0] [1,16,2,2]
[2,14,4,1] [2,15,1,3] [3,12,6,0] [3,13,3,2] [3,14,0,4] [4,11,5,1]
[4,12,2,3] [5,9,7,0] [5,10,4,2] [5,11,1,4] [6,8,6,1] [6,9,3,3] [6,
10,0,5] [7,6,8,0] [7,7,5,2] [7,8,2,4] [8,5,7,1] [8,6,4,3] [8,7,1,
5] [9,3,9,0] [9,4,6,2] [9,5,3,4] [9,6,0,6] [10,2,8,1] [10,3,5,3]
[10,4,2,5] [11,0,10,0] [11,1,7,2] [11,2,4,4] [11,3,1,6] [12,0,6,3]
[12,1,3,5] [12,2,0,7] [13,0,2,6]

44 ways to reach $-2: [0,12,8,1] [0,13,5,3] [0,14,2,5] [1,10,10,0]
[1,11,7,2] [1,12,4,4] [1,13,1,6] [2,9,9,1] [2,10,6,3] [2,11,3,5]
[2,12,0,7] [3,7,11,0] [3,8,8,2] [3,9,5,4] [3,10,2,6] [4,6,10,1]
[4,7,7,3] [4,8,4,5] [4,9,1,7] [5,4,12,0] [5,5,9,2] [5,6,6,4] [5,7,
3,6] [5,8,0,8] [6,3,11,1] [6,4,8,3] [6,5,5,5] [6,6,2,7] [7,1,13,0]
[7,2,10,2] [7,3,7,4] [7,4,4,6] [7,5,1,8] [8,0,12,1] [8,1,9,3] [8,
2,6,5] [8,3,3,7] [8,4,0,9] [9,0,8,4] [9,1,5,6] [9,2,2,8] [10,0,4,
7] [10,1,1,9] [11,0,0,10]

41 ways to reach $8: [0,7,13,1] [0,8,10,3] [0,9,7,5] [0,10,4,7]
[0,11,1,9] [1,5,15,0] [1,6,12,2] [1,7,9,4] [1,8,6,6] [1,9,3,8] [1,
10,0,10] [2,4,14,1] [2,5,11,3] [2,6,8,5] [2,7,5,7] [2,8,2,9] [3,2,
16,0] [3,3,13,2] [3,4,10,4] [3,5,7,6] [3,6,4,8] [3,7,1,10] [4,1,
15,1] [4,2,12,3] [4,3,9,5] [4,4,6,7] [4,5,3,9] [4,6,0,11] [5,0,14,
2] [5,1,11,4] [5,2,8,6] [5,3,5,8] [5,4,2,10] [6,0,10,5] [6,1,7,7]
[6,2,4,9] [6,3,1,11] [7,0,6,8] [7,1,3,10] [7,2,0,12] [8,0,2,11]

31 ways to reach $18: [0,2,18,1] [0,3,15,3] [0,4,12,5] [0,5,9,7]
[0,6,6,9] [0,7,3,11] [0,8,0,13] [1,0,20,0] [1,1,17,2] [1,2,14,4]
[1,3,11,6] [1,4,8,8] [1,5,5,10] [1,6,2,12] [2,0,16,3] [2,1,13,5]
[2,2,10,7] [2,3,7,9] [2,4,4,11] [2,5,1,13] [3,0,12,6] [3,1,9,8]
[3,2,6,10] [3,3,3,12] [3,4,0,14] [4,0,8,9] [4,1,5,11] [4,2,2,13]
[5,0,4,12] [5,1,1,14] [6,0,0,15]

13 ways to reach $28: [0,0,14,7] [0,1,11,9] [0,2,8,11] [0,3,5,13]
[0,4,2,15] [1,0,10,10] [1,1,7,12] [1,2,4,14] [1,3,1,16] [2,0,6,13]
[2,1,3,15] [2,2,0,17] [3,0,2,16]
```

Ex <2, 8, 2, 9> at n=21, k=8: 2+8+2+9=21 , -2*2-1*8+1*2+2*9 = 8

The assumption that k lies in the interval [-2n, 2n] for n>1, was vital to the fast computation of the fourples with the O(n^2) algorithm. A rigorous proof of this assumption follows.

Thm: *Prove that a sequence of* n *games of* $1-$ *card poker with* $n > 1$, *results* in *the gain* or *loss of an amount* k, $k \in [-2n, 2n]$

Proof: *Let* x = # *of games where player loses \$2*
y = # *of games where player loses \$1*
z = # *of games where player wins \$1*
w = # *of games where player wins \$2*

Constraints:
$x, y, z, w \geq 0$
A. $x + y + z + w = n$
B. $-2x - y + z + 2w = k$

NTS $f(x, y, w, z) = -2x - y + z + 2w$, *is onto the integer set* $[-2n, 2n]$

i. *Symmetry of the linear system*
$-2x - y + z + 2w = k \quad (a)$
$\Rightarrow 2x + y - z - 2w = -k \quad (b)$
If (x, y, z, w) *satisfies* (a), then (w, z, y, x) *satisfies* (b).

Thus, proving $k \in [0, 2n]$ *suffices.*

ii. *Case* $k = 2n$
The only solution to *the linear system is* $x = y = z = 0$, $w = n$
Check A. $x + y + z + w = 0 + 0 + 0 + n = n$
Check B. $-2x - y + z + 2w = 0 + 0 + 0 + 2n = 2n = k$

iii. *Case* $k = n$
A solution is $x = y = w = 0$, $z = n$
Check A. $x + y + z + w = 0 + 0 + n + 0 = n$
Check B. $-2x - y + z + 2w = 0 + 0 + n + 0 = n = k$.

iv. Case $k = 0$, *n even*

Highest number of solutions to *this system. A solution is* $x = w = 0,\ y = z = \frac{n}{2}$

Check A. $x + y + z + w = 0 + \frac{n}{2} + \frac{n}{2} + 0 = n$

Check B. $-2x - y + z + 2w = 0 - \frac{n}{2} + \frac{n}{2} + 0 = 0 = k$

v. Case $k = 0$, $n =$ *odd*

A solution is $x = 0,\ y = \frac{(n+1)}{2},\ z = \frac{(n-1)}{2} - 1,\ w = 1$

Check A. $x + y + z + w = 0 + \frac{(n+1)}{2} + \frac{(n-1)}{2} - 1 + 1 = n$

Check B. $-2x - y + z + 2w = 0 - \frac{(n+1)}{2} + \frac{(n-1)}{2} - 1 + 2 = 0 = k$

vi. Case $0 < k < n \Rightarrow (n - k) > 0$, *with* $(n - k)$ *even.*

A solution is $x = w = 0,\ y = \frac{(n-k)}{2},\ z = k + \frac{(n-k)}{2}$

Check A. $x + y + z + w = 0 + \frac{(n-k)}{2} + k + \frac{(n-k)}{2} + 0 = n$

Check B. $-2x - y + z + 2w = 0 - \frac{(n-k)}{2} + k + \frac{(n-k)}{2} + 0 = k$

vii. Case $0 < k < n \Rightarrow (n - k) > 0$, *with* $(n - k)$ *odd* $\Rightarrow (n - k + 1)$ *is even*

A solution is $x = 0,\ w = 1,\ y = \frac{(n-k+1)}{2},\ z = n - \frac{(n-k+1)}{2} - 1$

Check A. $x + y + z + w = 0 + \frac{(n-k+1)}{2} + n - \frac{(n-k+1).}{2} - 1 + 1 = n$

Check B. $-2x - y + z + 2w = 0 - \frac{(n-k+1)}{2} + n - \frac{(n-k+1).}{2} - 1 + 2 = k$

viii. Case $n < k < 2n$

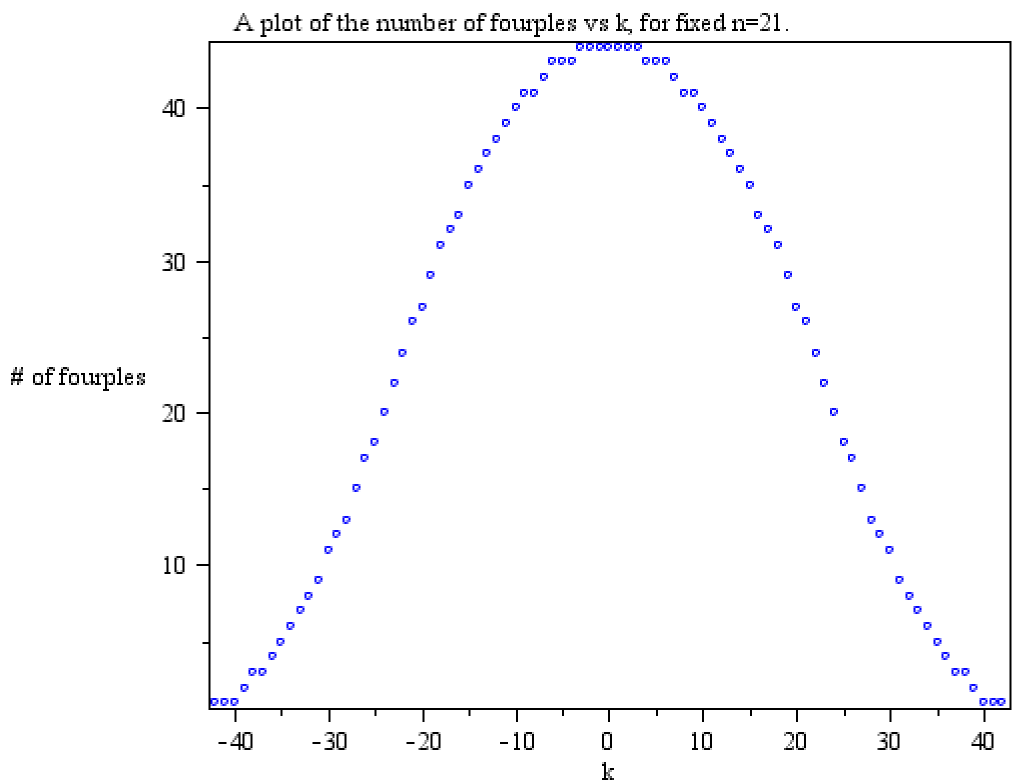
A solution is $x = y = 0,\ w = k - n,\ z = 2n - k$

Check A. $x + y + z + w = 0 + 0 + 2n - k + k - n = n$

Check B. $-2x - y + z + 2w = 0 - 0 + 2n - k + 2k - 2n = k$

QED.

A plot of the total number of all fourples <x,y,z,w> that satisfy the two constraints i.e. A. x + y + z + w = n, and B. k = -2x-y+z+2w, versus k, is shown below.



From the plot we see that the maximum number of solutions to the linear system above happens at k=0, as was claimed in the onto-function proof (case iv, v). There are 44 solutions to the system at n =21, k=0. As k moves away from 0, the number of solutions progressively decreases, until at k = 2n = 42, there is only 1 solution to the system. Due to the combinatorial properties of the fourple, not just k=42, rather, k = 40, 41, and 42 have exactly 1 solution. Similarly there are 44 solutions when k=0, 1, 2 and 3. The fourples that satisfy a given k for a fixed n are needed to compute the pmf of the dealer's winnings. However, finding the total number of fourples for k =0 i.e. number of ways to make \$0 in n games, is an interesting self-contained problem, addressed in the Maple snippet below.

Aside #1: Any closed form expression to compute # of solutions that satisfy the system
x+y+z+w=n,
-2x-y+z+2w=k=0,
for fixed n ?

```
> N := 50 :
  for n from 1 to N do
  a := fourples2(n) :
  points[n] := [n, a[0][1]];
  od:
```

```
> seq( points[n], n = 1 ..N);
  with(plots) : pointplot({seq(points[n], n = 1 ..N)}, symbol = circle, symbolsize = 8, col
      = blue, axes = boxed);
```

[1, 0], [2, 2], [3, 2], [4, 3], [5, 4], [6, 6], [7, 6], [8, 9], [9, 10], [10, 12], [11, 14], [12, 17], [13, 18], [14, 22], [15, 24], [16, 27], [17, 30], [18, 34], [19, 36], [20, 41], [21, 44], [22, 48], [23, 52], [24, 57], [25, 60], [26, 66], [27, 70], [28, 75], [29, 80], [30, 86], [31, 90], [32, 97], [33, 102], [34, 108], [35, 114], [36, 121], [37, 126], [38, 134], [39, 140], [40, 147], [41, 154], [42, 162], [43, 168], [44, 177], [45, 184], [46, 192], [47, 200], [48, 209], [49, 216], [50, 226]

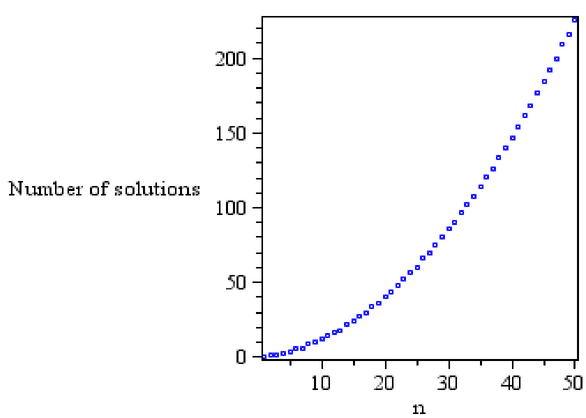
$> f := x \to (1+x^3)/((1-x^2)^2 \cdot (1-x^3));$

$$f := x \to \frac{1+x^3}{(1-x^2)^2\,(1-x^3)}$$

$series(f(x), x, 20)$

$$1 + 2x^2 + 2x^3 + 3x^4 + 4x^5 + 6x^6 + 6x^7 + 9x^8 + 10x^9 + 12x^{10} + 14x^{11} + 17x^{12} + 18x^{13} + 22x^{14} + 24x^{15} + 27x^{16} + 30x^{17} + 34x^{18} + 36x^{19} + O(x^{20})$$

>



The coefficient of f(x) above yield the number of ways we seek. For example, the coefficient of x^11 is 14, hence there are 14 ways to win \$0 in 11 games!

Rudimentary Combinatorial analysis of a fourple:

Lemma 1. *Given* n_1 *elements* $a_1, a_2, \ldots a_{n_1}$,

n_2 *elements* $b_1, b_2, \ldots b_{n_2}$,

n_3 *elements* $c_1, c_2, \ldots c_{n_3}$,

n_4 *elements* $d_1, d_2, \ldots d_{n_4}$,

there are precisely $n_1 n_2 n_3 n_4$ *distinct fourples,* $\left\langle a_{i_1}, b_{i_2}, c_{i_3}, d_{i_4} \right\rangle$,

containing one element of each kind.

Proof: *Represent the elements of the first kind* by *points on the* x *axis.*
Represent the elements of the 2 *nd kind* by *points on the* y *axis.*
Then the possible ordered pairs (a_i, b_j) *are points of a rectangular lattice* in *the* xy *plane.*
It is obvious there are exactly $n_1 n_2$ *such ordered pairs.*

By the same reasoning, each fourple is simply a point in *a lattice* in $\mathbb{R}^4$.
The total number of such points is clearly $n_1 n_2 n_3 n_4$.

Lemma 2. *Given a population of n elements,*

with positive integers $x, y, z, w \ni x + y + z + w = n$,
there are precisely $N = \left(\dfrac{n!}{x!\, y!\, z!\, w!} \right)$ *ways of partitioning n into subpopulations of sizes* x, y, z and w.

Proof: *Form a group of x elements* from *the original population.*
This can be done in $C(n, x) = n$ *choose* $x = \dfrac{n!}{x!\,(n-x)!}$ *ways.*
Form a group of y elements from *the remaining* $(n-x)$ *elements.*
This can be done in $C(n-x, y) = \dfrac{(n-x)!}{y!\,(n-x-y)!}$ *ways.*
Form a group of z elements from *the remaining* $(n-x-y)$ *elements.*
This can be done in $C(n-x-y, z) = \dfrac{(n-x-y)!}{z!\,(n-x-y-z)!}$ *ways.*
Form a group of w elements from *the remaining* $(n-x-y-z)$ *elements.*
This can be done in $C(n-x-y-z, w) = \dfrac{(n-x-y-z)!}{w!\,(n-x-y-y-z-w)!}$ *ways.*
Note that $(n-x-y-z-w)! = 0! = 1$.

Using Lemma 1, $N = \dfrac{n!}{x!\,(n-x)!} \cdot \dfrac{(n-x)!}{y!\,(n-x-y)!} \cdot \dfrac{(n-x-y)!}{z!\,(n-x-y-z)!} \cdot \dfrac{(n-x-y-z)!}{w! \cdot 1}$

which simplifies to $\dfrac{n!}{x!\,y!\,z!\,w!}$, *also called the multinomial coefficient*

Lemma 3. Given a fourple $X = \langle x, y, z, w\rangle$ with probabilities
$p[-2]$ = probability of event x
$p[-1]$ = probability of event y
$p[1]$ = probability of event z
$p[2]$ = probability of event w
the discrete probability mass function of the fourple is

$$p(\, X = \langle x, y, z, w\rangle\,) = \frac{n!}{x!\, y!\, z!\, w!} \cdot (p[-2])^{x} \cdot (p[-1])^{y} \cdot (p[1])^{z} \cdot (p[2])^{w}$$

Proof: *If the probability of x is* $p[-2]$ and *the event happens x times,*
then *the total probability of x is* $p[-2] \cdot p[-2] \cdot \ldots . p[-2]$, *$x$ times*
$= (p[-2])^{x}$

Similarly, the other events have a prob. $(p[-1])^{y}$, $(p[1])^{z}$, $(p[2])^{w}$.

For a fourple to *exist,* x, y, z and w *must happen simultaneously.*
Thus the probability of a fourple is
$(p[-2])^{x} \cdot (p[-1])^{y} \cdot (p[-1])^{z} \cdot (p[2])^{w}$

However, from *Lemma* 2, *we find there are a total of* $\left(\dfrac{n!}{x!\, y!\, z!\, w!} \right)$ *such fourples.*
Hence, *the probability mass function of a fourple is given* by

$$p(X = \langle x, y, z, w\rangle\,) = \left(\frac{n!}{x!\, y!\, z!\, w!} \right) \cdot (p[-2])^{x} \cdot (p[-1])^{y} \cdot (p[-1])^{z} \cdot (p[2])^{w}$$

The fourple <x,y,z,w> is thus a discrete rv that takes on values in the sample space <[0,n],[0,n],[0,n],[0,n]>, subject to the two constraints. We have derived an expression for the pmf of the fourple above. A fourple follows a multinomial distribution. The pmf of the amount won by the dealer i.e. k, can be obtained by simply summing over the pmf of the fourple subject to the two constraints i.e. an original expression given by

$$\boldsymbol{p(X = k) = \sum \left(\frac{n!}{x!\, y!\, z!\, w!} \right) \cdot (p[-2])^{x} \cdot (p[-1\,])^{y} \cdot (p[-1\,])^{z} \cdot (p[2])^{w},}$$
$$\boldsymbol{\forall\ (x,\ y,\ z,\ w) \ni x + y + z + w = n,\ \ -2\,x - y + z + 2\,w = k}$$

The expression above is the discrete pmf of X=k, and can be used to find the average value attained by X after n trials, as well as how far X strays away from its average. These are respectively given by:

Factorial Moment Generating function that completely describes the distribution of X.

$$\text{MGF} = \Sigma\, t^{k} \cdot p(X = k)$$
$$\forall\, k \in [\,-2\,n + 1, 2\,n + 1\,]$$

The first derivative of M w.r.t t, at t=1 is the mean:

$$\text{Mean} = \text{subs}\left(t = 1,\ t \rightarrow \frac{\partial}{\partial t}\,\text{MGF}(t, p)\right).$$

The variance is a function of the second derivative and the first derivative :

$$\text{Variance} := p \rightarrow \text{subs}\left(t = 1,\ \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\,\text{MGF}(t, p)\right)\right) + \text{Mean}(p) - \text{Mean}(p)^{2}$$

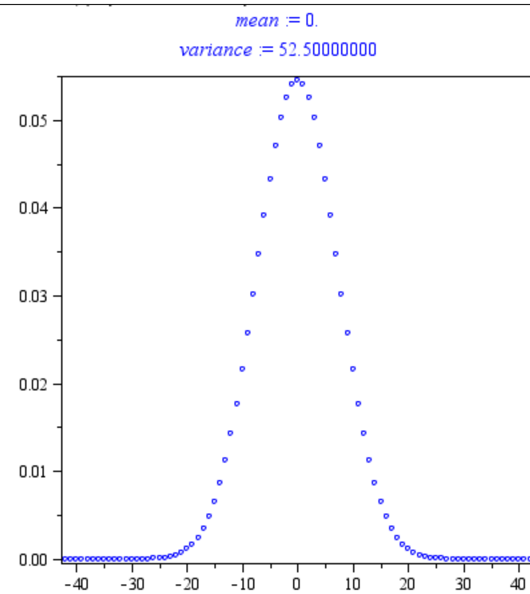
Since the mean and variance are derivatives of the moment generating function, which in turn is dependent on the probability matrices of the players, we can ensure an average positive or negative payoff (positive/negative mean) for the dealer by changing the values in the probability matrices which in turn affect p[-1], p[-2], p[1] and p[2]. We now compute the mean and variance for a fixed n = 21 and a fixed set of probabilities p[i]. If each of the four probabilities was equally likely, we should expect to win \$0 i.e. mean = 0. In this situation, the maximum variance of 52.5 is observed. However, if there is a higher probability of losing money, say p[-1] = p[-2] = (1/3), p[1] = p[2] = (1/6), then we expect a negative mean, and we observe mean = -10.5, variance = 47.25. The pmfs for these two scenarios are shown below.

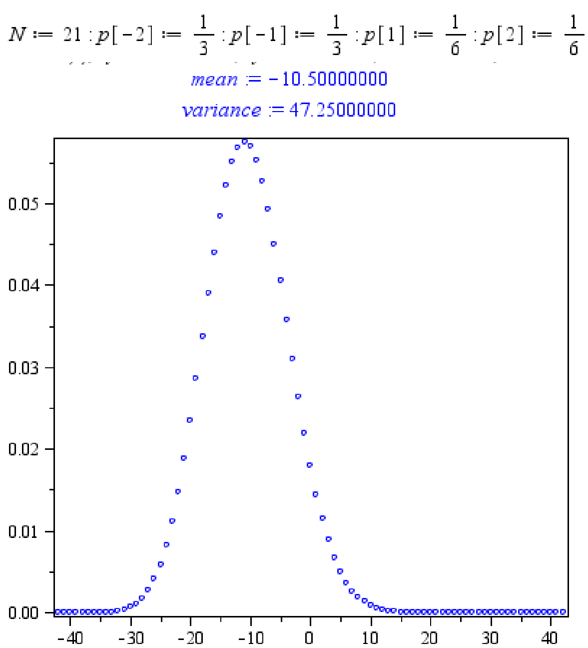
At n=21,

p[-2]=p[-1]=p[1]=p[2]=(1/4),

mean = 0

variance = 52.5





At n=21,

p[-2]=p[-1]=1/3,

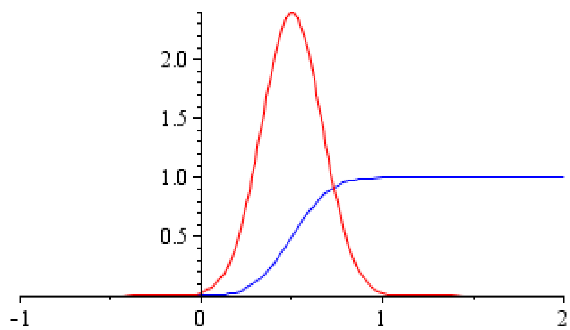
p[1]=p[2]=1/6,

the dealer stands to lose an average of $10.50 after 21 games.

Furthermore, we can verify our results experimentally for any given number of trials. We do so by performing a random walk i.e. constructing stock charts. In order to construct such a stock chart, we pick a random number from a normal distribution with a mean of ½, standard deviation of (1/6).

The pdf of the distribution, and the area under the curve i.e. the cumulative distribution function (cdf), are shown in the graph below.

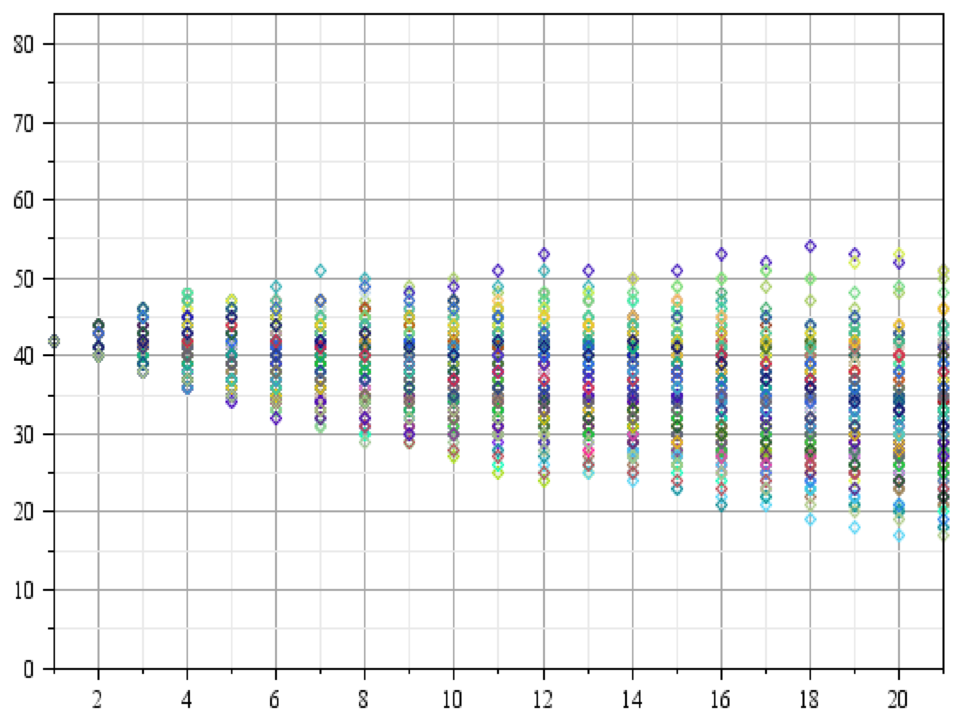
$$
\begin{aligned}
&plots[display]\Bigg(\Bigg\{\\
&\quad plot\left(statevalf\left[pdf,\, normald\left[\frac{1}{2}, \frac{1}{6}\right]\right], -1..2, colour = red\right),\\
&\quad plot\left(statevalf\left[cdf,\, normald\left[\frac{1}{2}, \frac{1}{6}\right]\right], -1..2, colour = blue\right),\\
&\qquad\Bigg\}\Bigg);
\end{aligned}
$$



Notice that such a distribution will yield normally distributed random numbers r, almost all of which will lie in the [0,1] interval. For each r, we compute the cdf. The cdf is a real number between 0 and 1. If cdf < p[-2], we decrease the share price of the stock by $2. If cdf < p[-2]+p[-1], we decrease the price by $1. cdf < p[-2]+p[-1]+p[1] results in a price gain of $1, and if none of these three cases occur, we increase the share price by $2. Picking 21 random numbers gives us one stock chart, whose prices are shown by the circles in the chart. Shown below are 200 such stock charts. Each chart starts at $42, and a majority finish at the mean = $31.50 after n = 21 trials, confirming our intuition about the mean.

```
> normalStockchart :=proc(p, N)
      local data, i, r, area, profit, normalPoints, randomnums;
      randomnums := [stats[random, normald[1/2, 1/6]](N)]:

      data[1] := 42;
      for i from 2 to N do
          r := randomnums[i];
          area := statevalf[cdf, normald[1/2, 1/6]](r);
          profit := piecewise(area <=p[-2], -2, area <=p[-2] + p[-1], -1, area <=p[-2]
           + p[-1] + p[1], 1, 2);
          data[i] := data[i-1] + profit
      end do;
      return data;
  end proc:
> unassign(stock) :
  N := 21 : p[-2] := 1/3 : p[-1] := 1/3 : p[1] := 1/6 : p[2] := 1/6 :
   for j  from 1 to 200 do
  data := normalStockchart(p, N) :
  winnings[j] := data[N];
  normalPoints := {seq([i, data[i]], i = 1 ..N)} :
  stock[j] := pointplot(normalPoints, color = COLOR(RGB, rand()/10^12, rand()/10^12, rand()/10^12),
      legendstyle = [location = left], view = [1 ..21, 0 ..84], connect = false) :
  od:
  display(convert(stock, list), axes = boxed, gridlines = true);
```



We must now find a mapping from the space of the discrete rv X (the dealer's winnings) to the space of a fictional one-card poker stock whose share price mimics X. Stocks are typically modeled using a continuous lognormal process. We shall briefly explain the rationale for the lognormal model of a stock.

Let price of a stock at time 0 = S(0)

Let price of a stock at time 1 = S(1)

We divide the time interval into 1000 slices, thus S_{1000} denotes S(1).

$$> \quad S(1) = S_{1000} = \frac{S_{1000}}{S_{999}} \cdot \frac{S_{999}}{S_{998}} \cdot \frac{S_{998}}{S_{997}} \cdots \frac{S_2}{S_1} \cdot \frac{S_1}{S_0} \cdot S_0;$$

We now denote each of the 1000 ratios (S_k/S_{k-1}) by an rv, R_k.

To simplify the model, we assume the 1000 rv's are independent.

Taking the natural log on both sides,

$\ln S_{1000} = \ln R_{1000} + \ln R_{999} + \ldots + \ln R_2 + \ln R_1 + \ln S_0$

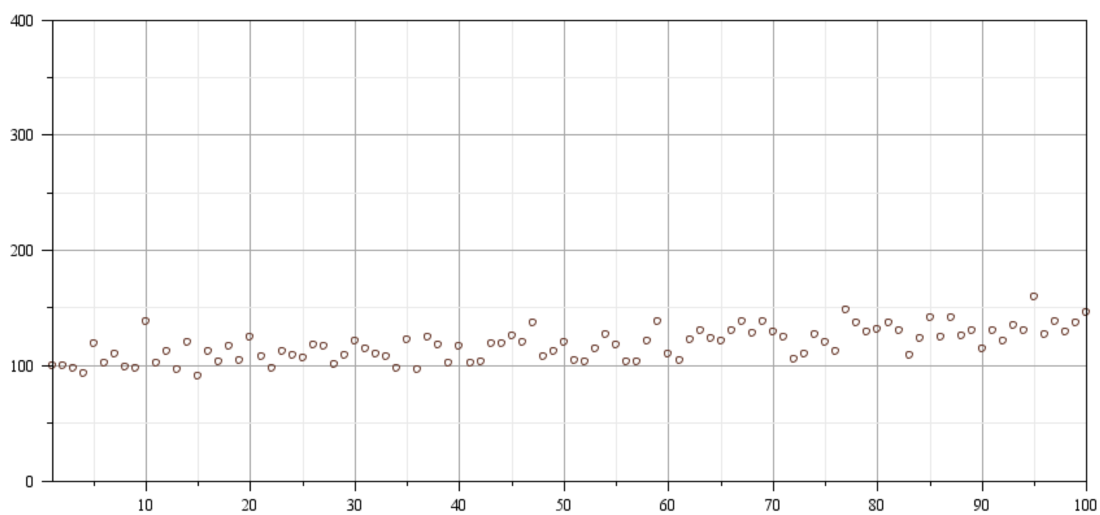
The Central Limit Theorem asserts that the sum of a large number of independent rv's can be approximated by a Normal distribution with the same mean and variance as the sum. Assuming all the 1000 rv's belong to the same distribution with a mean m, positive variance s (on account of the logarithm), we conclude $\ln S_{1000} = \ln S_0 + N(1000m, 1000s)$, where N denotes a normal distribution with mean 1000m, variance 1000s. Hence, it seems plausible to model stocks thusly –

$$\ln S(t) = \ln S(0) + N\left(\mu \cdot t, \sigma^2, t\right)$$

$$S(t) = S(0) \cdot e^{N\left(\mu \cdot t, \sigma^2 t\right)}$$

The lognormal models are examples of Brownian motion, of which there are three kinds – Standard, Arithmetic and Geometric. In practice, stocks and derivatives on stocks are modeled using Geometric Brownian Motion, $S(t) = S_0 \cdot e^{\mu \cdot t + \sigma \cdot N(0,t)}$ with a fixed drift and volatility.

The drift of the discrete rv X is the slope of the line joining the initial value of X at time 0, to the mean of X after n trials. The volatility of X is its standard deviation about this mean. Equating these parameters of the discrete rv X to those of the continuous lognormal S(t), we obtain a fictional one-card poker stock whose share price mimics X.



Shown above is the stock chart pertaining to the dealer's winnings denoted by the discrete rv X, with a mean 48.3 and a variance 0.01, which maps to a lognormal stock S(t) with 30% drift, 10% volatility. We may now price derivatives on this stock using the Black-Scholes option pricing model, and construct the voyeur's optimal portfolio of options.

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