$$y = \frac{1}{D}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2$$

$$\& \ z = \frac{1}{D}\begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1.$$

Hence $x = 1$, $y = 2$, $z = -1$

~~Existence~~

Solution of $n \times n$ Linear system of equations.

Consider the system of n equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$- \quad - \quad - \quad -$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.$$

In matrix form, we can write the system of eq^{n}s as -

$$Ax = b.$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

$\Downarrow$ coefficient matrix, $\Downarrow$ column vector, $\downarrow$ solution vector.

If $b \neq 0$, i.e., at least one of the elements $b_1, b_2, \ldots b_n$ is not zero, then the system of eq^{n}s is called non-homogeneous

If $b = 0$, then the system of eq^{n}s is called homogeneous.

→ The system of eqⁿs is called consistent if it has at least one solⁿ & inconsistent if it has no solution.

Non-homogeneous system of eqⁿs

$$AX = b.$$

↓

Solved by following methods.

Matrix method.

Let $A \rightarrow$ non-singular.

$$Ax = b.$$

Pre-multiplying by A^{-1}, we obtain

$$x = A^{-1} b.$$

The system of eqⁿs is consistent & has a unique solution. If $b=0$, then $x=0$ (trivial solⁿ) is the only solⁿ.

Cramer's rule Let $A \rightarrow$ non-singular.

$$x_i = \frac{|A_i|}{|A|}, \; i = 1, 2 \ldots, n; \longrightarrow ①$$

where $|A_i| \rightarrow$ is the det. of the matrix A_i obtained by replacing ith column of A by the R-H-S column vector b.

Following cases arises:

Case 1 :- When $|A| \neq 0$, the system of eqⁿs is consistent and the unique solⁿ is obtained by using ①.

Case 2: When $|A|=0$ & one or more of $|A_i|$, $i=1,2,\dots,n$ are not zero, then the system of eqn's has no soln, i.e., system is inconsistent.

Case 3: When $|A|=0$ & all $|A_i|=0$, $i=1,2,\dots,n$, then system of eqn's is consistent & has infinite no. of soln's.

Homogeneous system of eqn's.

$$AX=0.$$

Trivial soln $X=0$ is always a soln of this system.

If $A \to$ non-singular, then again $X=A^{-1}0=0$ is the soln.

→ A homogeneous system of eqn's is always consistent.

→ Non-trivial soln's for $AX=0$ exist if and only if A is singular. In this case, the homogeneous system of eqn's has infinite no. of soln's.

Ex S.T the system of eqn's

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}.$$

has infinite no. of soln's. Hence, find the soln's.

Soln: We find

$$|A|=\begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix}=0, \quad |A_1|=\begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix}=0$$

$$=\begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 2+1 & 3-1 & 1+3 \end{vmatrix}=\begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix}+\begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 3 \end{vmatrix}=0.$$

$$|A_2|=\begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix}=0, \quad |A_3|=\begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix}=0.$$

∴ The system of eqⁿ's has infinite no. of

Using first two eqⁿ's

$$x_1 - x_2 = 3 - 3x_2$$

$$2x_1 + 3x_2 = 2 - x_3$$

$3x_1 + 2x_2 + 4x_3 = 5$

$$x_1 = \frac{(11 - 10x_3)}{5}$$

& $$x_2 = \frac{-4 + 5x_3}{5}$$

→ This solⁿ satisfies the third eqⁿ.

[<u>Procedure to test the consistency of a system of eqⁿ's in n unknowns</u>:

Find the ranks of the coeff. matrix A & the augmented matrix K, by reducing A to the triangular form by elementary row operations.

Let the rank of A be r & that of K be r'

(i) If $r \neq r'$, the eqⁿ's are inconsistent, i.e no solⁿ.

(ii) If $r = r' = n$, the eqⁿ's are consistent & there is a unique solⁿ

(iii) If $r = r' < n$, the eqⁿ's are consistent & there are infinite no. of solⁿ's.]

<u>Ex</u> Test for consistency and solve

$5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$.

Vector Spaces

Let $V \rightarrow$ non-empty set of certain objects, which may be vectors, matrices, functions or some other objects.

Each object is an elt. of V and is called a vector.

Elts. of V are a, b, c, u, v etc.

Assume that the two algebraic operations (i) vector addition and (ii) scalar multiplication. are defined on elts. of V.

Vector addition

$$a+b = (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1+b_1, a_2+b_2, \dots, a_n+b_n).$$

Scalar multiplication:

$$\alpha a = \alpha(a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n).$$

The set V defines a V.S if any elements a, b, c in V and any scalars α, β the following properties are satisfied..

Prop. w.r.t vector addition.

1. $a+b \in V$
2. $a+b = b+a$ (commutative law)
3. $(a+b)+c = a+(b+c)$ (associative law)
4. $a+0 = 0+a = a$ (existence of a unique zero elt. in V)
5. $a+(-a) = 0$. (existence of additive inverse or (-)ve vector in V)

Prop. w.r.t. scalar multiplication

6. $\alpha a \in V$
7. $(\alpha+\beta)a = \alpha a + \beta a$ (left distributive law)
8. $(\alpha\beta)a = \alpha(\beta a)$
9. $\alpha(a+b) = \alpha a + \alpha b$ (right distributive law)
10. $1a = a$ (existence of multiplicative inverse)

The properties 1 & 6 are called the closure properties.

→ When these two properties are satisfied, we say that the V.S is closed under vector addition & scalar multiplication.

→ If the elts. of V are real, then it is called a real V.S when the scalars α, β are real nos., whereas V is a complex V.S if the elts. of V are complex & the scalars α, β may be real or complex nos. Or if the elts. of V are real & the scalars α, β are complex nos.

Remark:

(a) If even one of the above prop. is not satisfied, then V is not a V.S. We usually check the closure properties first before checking the other properties.

(b) The set of real nos. & complex nos., are called fields of scalars. We shall consider V.S only on the fields of scalars.

(c) V.S $V = \{0\}$ is called a trivial V.S.

Following are examples of V.S under usual operations of vector addition & scalar multiplication:-

1. V of real or complex nos.
2. Set of real valued continuous functions f on any closed interval $[a, b]$. The '0' vector defined in prop. (4) is the zero function.
3. Set V of n-tuples in R^n or C^n.
4. Set V of all $m \times n$ matrices. The elt. '0' defined in prop (4) is the null matrix of order $m \times n$.

Following are some examples which are not vector spaces

①. Set V of all polynomials of degree n.

Let P_n & Q_n be two polynomials of degree n in V.

Then $\alpha P_n + \beta Q_n$ need not be a polynomial of degree n

& thus may not be in V.

For eg: if $P_n = x^n + a_1 x^{n-1} + \dots + a_n$

$\quad$ & $Q_n = -x^n + b_1 x^{n-1} + \dots + b_n$.

then $P_n + Q_n$ is a polynomial of degree $(n-1)$.

Ex $V \rightarrow$ set of all polynomials, with real coefficients of degree n, where addition is defined by $a+b = ab$ & under usual scalar multiplication. S.T V is not a V.S.

Solⁿ Let P_n & Q_n be two elts. in V.

Now $P_n + Q_n = (P_n)(Q_n) \rightarrow$ a polynomial of degree $2n$ is not in V.

$\therefore V \rightarrow$ not a V.S.

Ex Let V be the set of all ordered pairs (x, y), x, y are real no. Let $a = (x_1, y_1)$ & $b = (x_2, y_2)$ be two elts. in V. Define the addition as $a + b = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$

& scalar multiplication as $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$

S.T V is not a V.S. Which of the props. are not satisfied?

Solⁿ $(1, 1)$ is an elt. of V. From the given defⁿ of V.A, we find $(x_1, y_1) + (1, 1) = (x_1, y_1)$ & true only for elt. $(1, 1)$.

$\therefore$ elt. $(1,1)$ plays the role of '0' elt. as defined in prop. ④.

Now, there is no elt. in V for which $a + (-a) = 0 = (1,1)$

since

$$(x_1, y_1) + (-x_1, -y_1) = (-x_1^2, -y_1^2) \neq (1,1)$$

$\therefore$ Prop. ⑤ is not satisfied (existence of add. the inverse not satisfied).

Now let, $\alpha = 1$, $\beta = 2$ be any two scalars. We have

$$(\alpha+\beta)(x_1, y_1) = 3(x_1, y_1) = (3x_1, 3y_1)$$

& $\alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1)$

$$= (2x_1^2, 2y_1^2)$$

$\therefore$ $(\alpha+\beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$ & prop ⑦ is not satisfied.

Similarly, it can be shown that prop. ⑨ is not satisfied.

i.e. $\alpha(a+b) = \alpha a + \alpha b$ is not satisfied.

Hence, V is not a V.S.