Graph:

A *linear*† *graph* (or simply a *graph*) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called *vertices*, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.

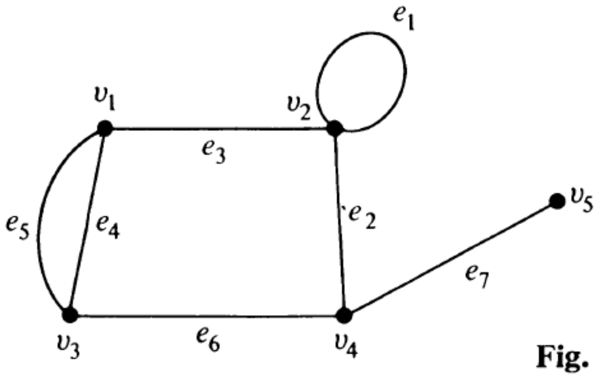


Fig. 1-1 Graph with five vertices and seven edges.

Self Loop:

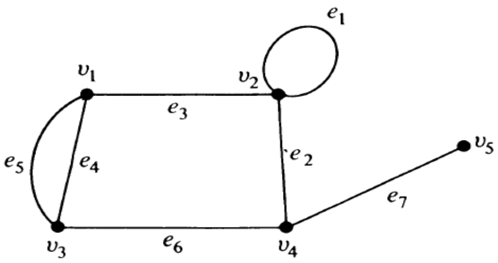
An edge of a graph that joins a vertex to itself or an edge having same vertex as its end vertices is called self loop.

Ex: e_1 is self loop.

Parallel Edges:

More than one edges associated with a given pair of vertices are called parallel edges.

Ex: e_4 and e_5 are parallel edges.



Incidence:

When a vertex v_i is end vertex of some edge e_j then v_i and e_j are said to be incident with (on or to) each other.

For example edges $e_{1,}e_{2,}e_3$ are incident on vertex v_2. Similarly $e_{4,}e_{5,}e_6$ are incident on vertex v_3.

Adjacency:

- **Adjacent Edges**:

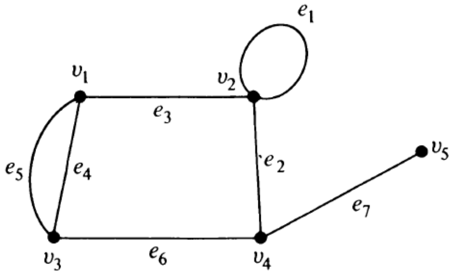
  Two non-parallel edges are said to be adjacent if they are incident on same vertex.

  Ex: $e_{2,}e_{7,}e_6$ are adjacent edges.

- **Adjacent Vertices**:

  Two vertices are said to be adjacent vertices if they are end vertices of same edge.

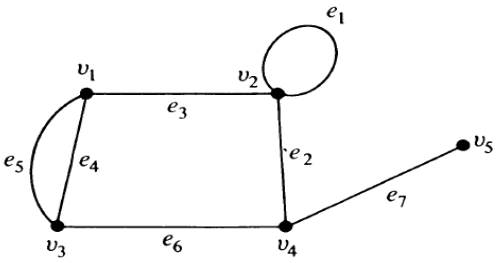
  Ex: v_2 and v_4 are adjacent but v_2 and v_3 are not adjacent.



Degree of a Vertex:

Number of edges incident on a vertex v_i with self loop counted twice is called degree of a vertex denoted by $d(v_i)$.

Ex: $d(v_1) = 3, d(v_3) = 3, d(v_2) = 4.$



.

Theorem: (The handshaking Theorem)

The sum of the degrees of all vertices in graph G is twice the number of edges in G.

i.e. $\sum_{i=0}^{n} d(v_i) = 2\,e$

Proof: In a graph every edge is incident with exactly two vertices, i.e. every edge contributes 2 to the sum of degree of the vertices.

All the e edges contribute 2e to the sum of the degrees of the vertices

i.e. $\sum_{i=0}^{n} d(v_i) = 2\,e$

Theorem: The number of vertices of odd degree in an graph is even.

Proof: Let G be a graph, V=$\{v_1, v_2, \ldots.., v_n\}$ be the set of vertices and e be the number of edges.
Let V_1 and V_2 be the set of vertices of even and odd degrees respectively

then $\sum_{i=0}^{n} d(v_i) = \sum_{even} d(\mathrm{v_i}) + \sum_{odd} d(\mathrm{v}_j)$

L.H.S is always even as we know $\sum_{i=0}^{n} d(v_i) = 2\,e$

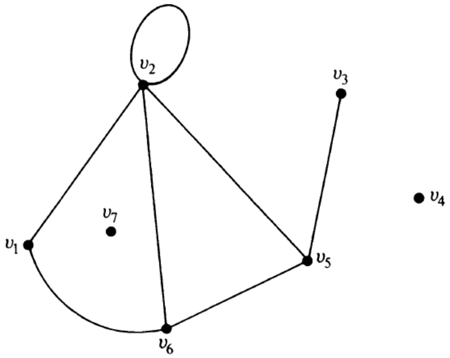
The first expression on RHS is always even(as being a sum of even numbers) i.e. the second expression must also be even.

i.e. $\sum_{odd} d(\mathrm{v}_j)$= even number

i.e. the number of vertices of odd degree in an graph is always even.

Isolated Vertex:

A node of a graph which is not adjacent to any other node is called isolated node or A vertex having no edge incidence edge is called isolated vertex.
Ex: $v_4 \ \ and \ \ v_7$ are isolated vertices.



Edges in Series:

Two edges are said to be in series if their common vertex is of degree 2
Ex: Two edges incident on v_1 are in series.

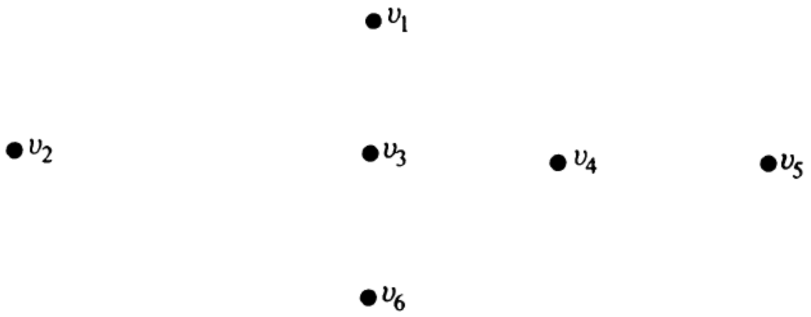
Pendant Vertex:

A vertex of degree one is called pendant vertex.

Ex: v_3 is pendant vertex.

Null Graph:

A graph containing only isolated node i.e. no edges is called null graph.



Null graph of six vertices.

It should also be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short. what is important is the incidence between the edges and vertices.

For example, the two graphs drawn in Figs. 1-2(a) and (b) are the same, because incidence between edges and vertices is the same in both cases.

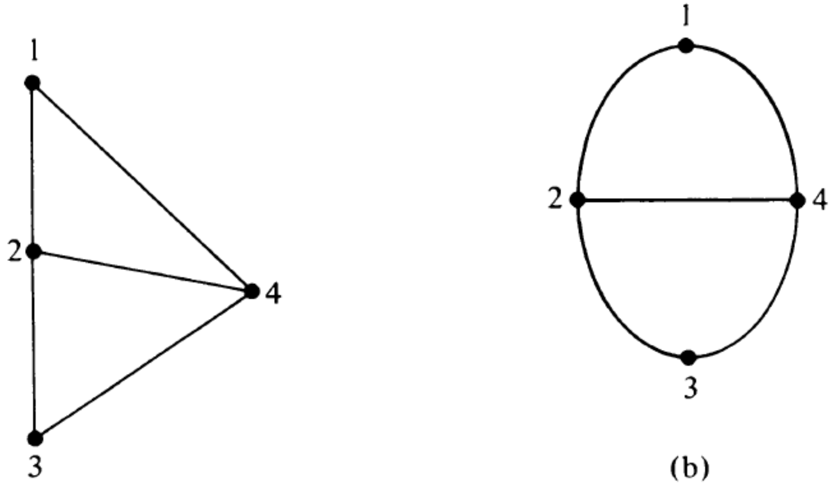
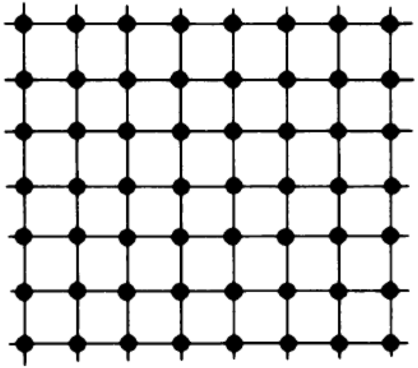
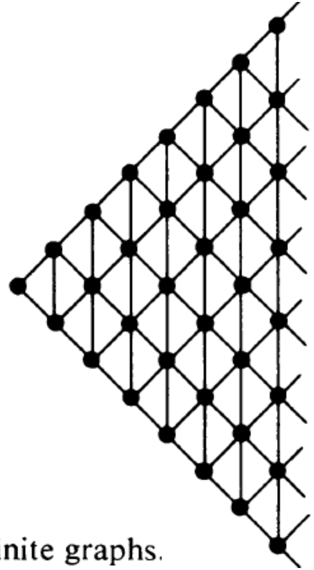


Fig. 1-2 Same graph drawn differently.

Finite and Infinite Graphs

A graph with a finite number of vertices as well as a finite number of edges is called a *finite graph;* otherwise, it is an *infinite graph*.

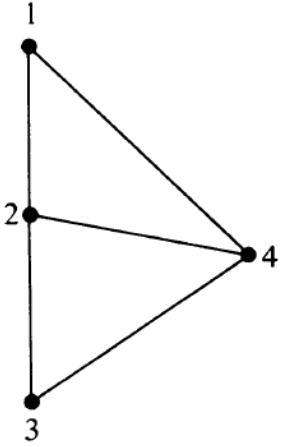




Portions of two infinite graphs.

Simple Graph:

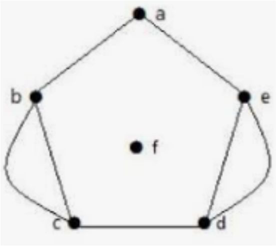
A graph that has neither self-loops nor parallel edges is called a *simple graph.*



Simple Graph:

Multigraph:

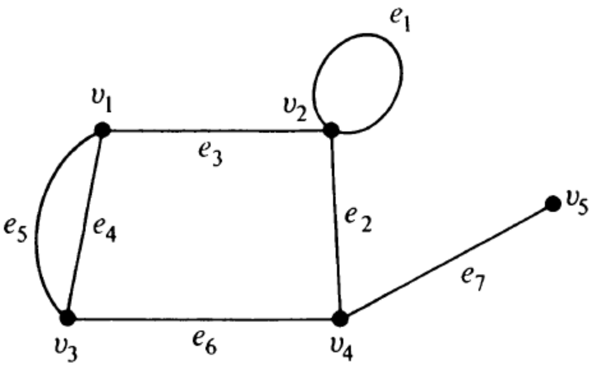
A graph which contains some parallel edges is called multigraph.



Multigraphs

Pseudograph:

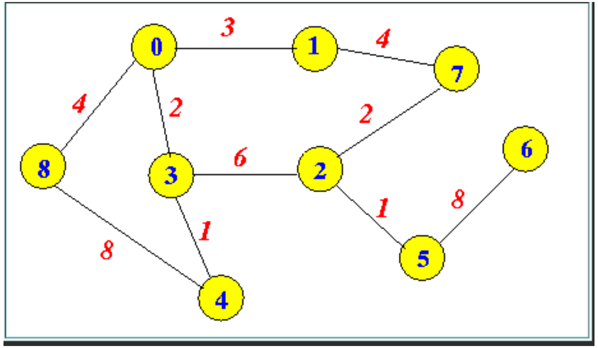
A graph in which loops and parallel edges are allowed is called a pseudograph.



Pseudograph

Weighted Graph:

A graph in which number (weight) is assigned to each edge is called weighted graph.



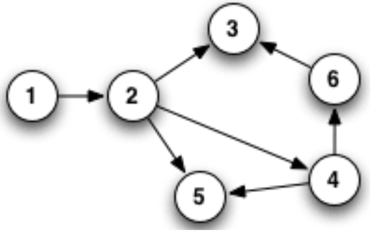
Directed and Undirected Graph

- A digraph or directed graph is a graph in which each edge of the graph has a direction. Such edge is called directed edge.

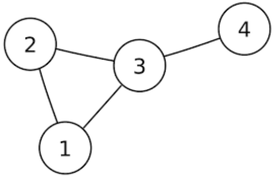
Ex: In the given directed graph
The vertices set V={1,2,3,4,5,6} Edges set E={ (1,2),(2,3),(2,5),(2,4),(4,5),(4,6),(6,3)}
In the directed graph the order of vertices in the pairs in edges set matters.

- An undirected graph is a graph in which edges do not have direction.
  The order of vertices in the pairs in edges set doesn't matters.

Ex: In the given graph undirected graph
The vertices set V={1,2,3,4} Edges set E={ (1,2),(2,3),(4,3),(1,3)}



Directed Graph



Undirected Graph

Complete Graph:

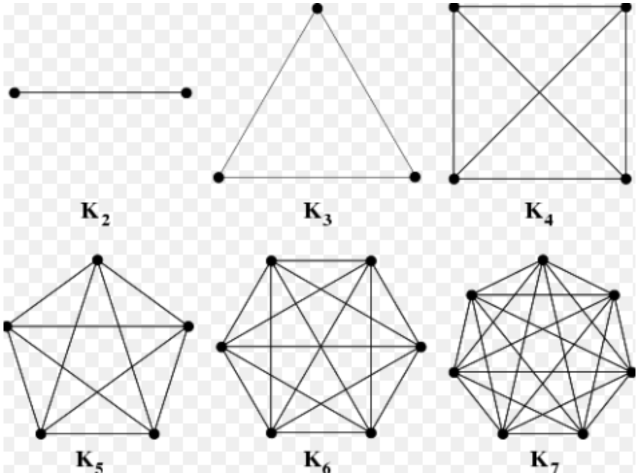
A simple graph in which every pair of distinct vertices are adjacent.
in other words a simple graph in which there is exactly one edge between each pair of distinct vertices is called Complete graph.

The complete graph of n vertices is denoted by K_n.

The number of edges in K_n are $n_{C_2} = \frac{n(n-1)}{2}$

Hence maximum number of edges in simple graph with n vertices are $\frac{n(n-1)}{2}$.

In K_n degree of each vertex is $(n-1)$.



Theorem: Show that a complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.

Proof: Let G be a simple complete graph with n vertices $v_1, v_2, \ldots\ldots, v_n$.
We can start from any of the vertex v_1, then number of edges drawn from v_1 to all other vertices are $(n-1)$.

Similarly, the number of edges drawn from v_2 to other vertices (except v_1) are $(n-2)$.

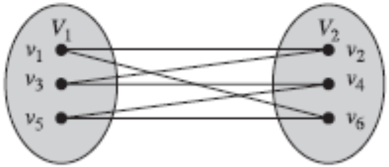
In the same way the number of edges drawn from v_3(except v_1, v_2) are $(n-3)$, from v_4(except $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$) are $(n-4)$, from $\mathrm{v}_{\mathrm{n}-1}$(except $\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3\ldots\ldots\mathrm{v}_{\mathrm{n}-2}$) to v_n is 1.

Hence total number of edges are

$$(n-1)+(n-2)+(n-3)+\cdots\ldots.+2+1$$
$$=\sum(n-1)$$
$$=\frac{n(n-1)}{2}$$

Bipartite Graphs

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.



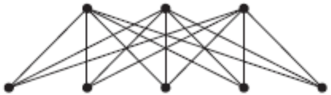
Bipartite.

Complete Bi-Partite Graph:

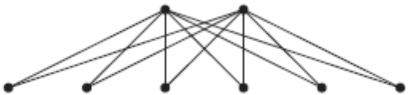
If each vertex of V_1 is connected with every vertex of V_2 by an edge, then graph G is called complete bipartite graph. If V_1 contains m vertices and V_2 contains n vertices, the completely bipartite graph is denoted by $K_{m,n}$

$K_{2,3}$

$K_{3,3}$



$K_{3,5}$



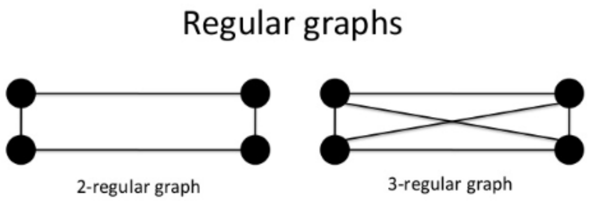
$K_{2,6}$

Some Complete Bipartite Graphs.

Regular Graph:

If every vertex of simple graph has the same degree, then the graph is called regular graph.

A graph is said to be k regular if every vertex is of degree k.



- Every complete graph K_n is $(n-1)$ regular.

- The complete graph of five vertices K_5 is the <u>first graph of Kuratowski</u>.

- <u>The second graph of Kuratowski</u> is a regular connected graph with six vertices and nine edges. ex. $K_{3,3.}$

PLANAR GRAPHS

A graph G is said to be *planar* if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.

A graph that cannot be drawn on a plane without a crossover between its edges is called *nonplanar*.

Embedding:

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called *embedding*. Thus, to declare that a graph G is nonplanar, we have to show that of all possible geometric representations of G none can be embedded in a plane.

Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane. Otherwise, G is nonplanar. An embedding of a planar graph G on a plane is called a *plane representation* of G.

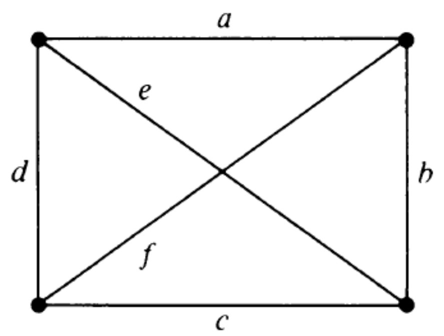
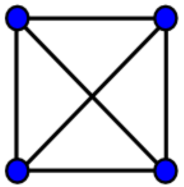
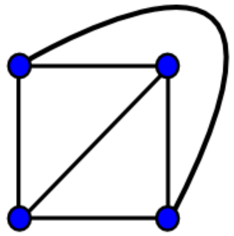


Fig. 1-3

The geometric representation shown in Fig. 1-3 clearly is not embedded in a plane, because the edges e and f are intersecting. But if we redraw edge f outside the quadrilateral, leaving the other edges unchanged, we have embedded the new geometric graph in the plane, thus showing that the graph which is being represented by Fig. 1-3 is planar.



Given Graph G(K_4)



Plane Representation of G or Embedding

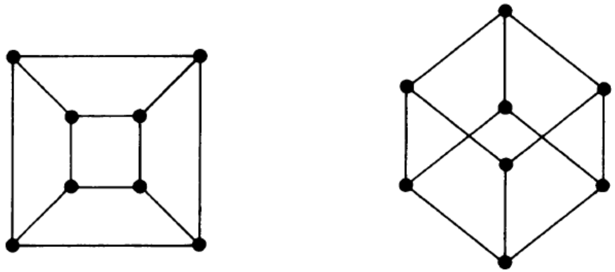
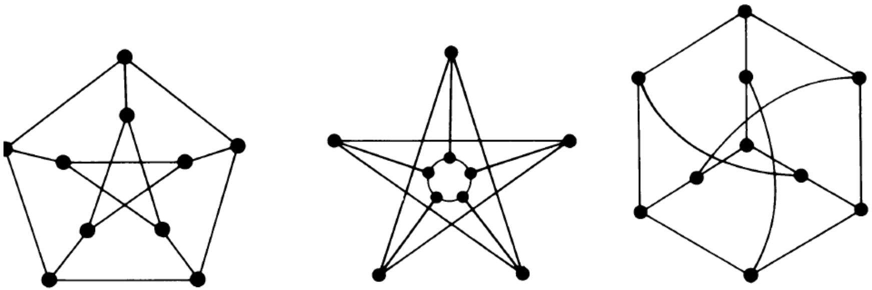


Fig. 2-2 Isomorphic graphs.

The two isomorphic diagrams in Fig. 2-2 are different geometric representations of one and the same graph.

One of the diagrams is a plane representation; the other one is not. The graph, of course, is planar.



(a) (b) (c)

Fig. 2-3 Isomorphic graphs.

It is not possible to draw any of the three configurations in Fig. 2-3 on a plane without edges intersecting.

The reason is that the graph which these three different diagrams in Fig. 2-3 represent is nonplanar.

EXAMPLE 1 Is K_4 planar?

Solution: K_4 is planar because it can be drawn without crossings, as shown in Figure .

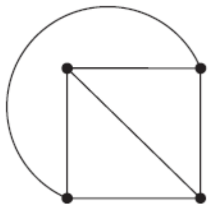


FIGURE K_4 **Drawn with No Crossings.**

Regions or Faces or meshes of planer graph:

A plane representation of a graph divides the graphs into regions or faces. A region is characterised by the set of edges(or vertices) forming its boundary.
Regions or faces are not defined for the nonplanar graph.

Euler Formula: A connected planer graph with n vertices and e edges has $e - n + 2$ regions(or faces).

i.e. $\boldsymbol{f = e - n + 2}$

where $\boldsymbol{f}$=no. of faces, $\boldsymbol{e}$= no. of edges and $\boldsymbol{n}$ = no. of vertices

For example in given plane representation of graph,

$$f = 6, e = 11, n = 7$$

(1,2,3,4,5 are interior regions and 6 is exterior region.)

We can verify that $f = e - n + 2$

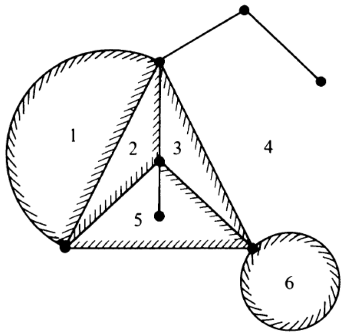


Fig. Plane representation (the numbers stand for regions).

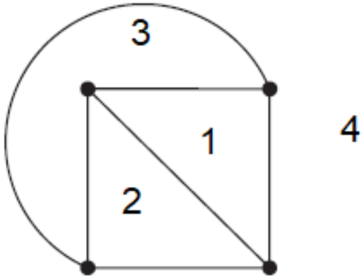
EX:

Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution: This graph has 20 vertices, each of degree 3, so $v = 20$. Because the sum of the degrees of the vertices, $3v = 3 \cdot 20 = 60$, is equal to twice the number of edges, $2e$, we have $2e = 60$, or $e = 30$. Consequently, from Euler's formula, the number of regions is

$$r = e - v + 2 = 30 - 20 + 2 = 12.$$

Ex:



The plane representation of the above graph divides the graph into four regions.
Here $f = 4, e = 6, n = 4$

We can verify that $f = e - n + 2$.

Condition of planarity:

In any simple, connected planar graph with f regions, n vertices, and e edges ($e > 2$), the following inequalities must hold:

$$e \geq \frac{3}{2}f,$$

$f = e - n + 2$

i.e. $$e \leq 3n - 6.$$

Proof: Since each region is bounded by at least three edges and each edge belongs to exactly two regions,

$$2e \geq 3f$$

or $$e \geq \frac{3}{2}f.$$

Substituting for f from Euler's formula in inequality (5-5),

$$e \geq \frac{3}{2}(e - n + 2)$$

or $$e \leq 3n - 6. \quad \blacksquare$$

Ex: To check whether K_5, the first graph of Kuratowski is planer or not.

No of edges in K_5 are $\frac{5(5-1)}{2}$ =10.

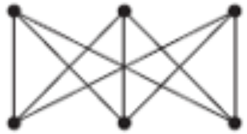
$e = 10, n = 5$

$3n - 6 = 9$

Which violates the inequality $e \leq 3n - 6$

i.e. K_5 is nonplanar.

Ex: $K_{3,3}$(the second graph of Kuratowski's) is nonplanar.



No. of edges $e = 9$,
No. of vertices $n = 6$

$$3n - 6 = 12$$

It is satisfying the inequality $e \leq 3n - 6$, still it is nonplanar.

To prove the nonplanarity of Kuratowski's second graph, we make use of the additional fact that no region in this graph can be bounded with fewer than four edges. Hence, if this graph were planar, we would have

$$2e \geq 4f,$$

and, substituting for f from Euler's formula,

$$2e \geq 4(e - n + 2),$$

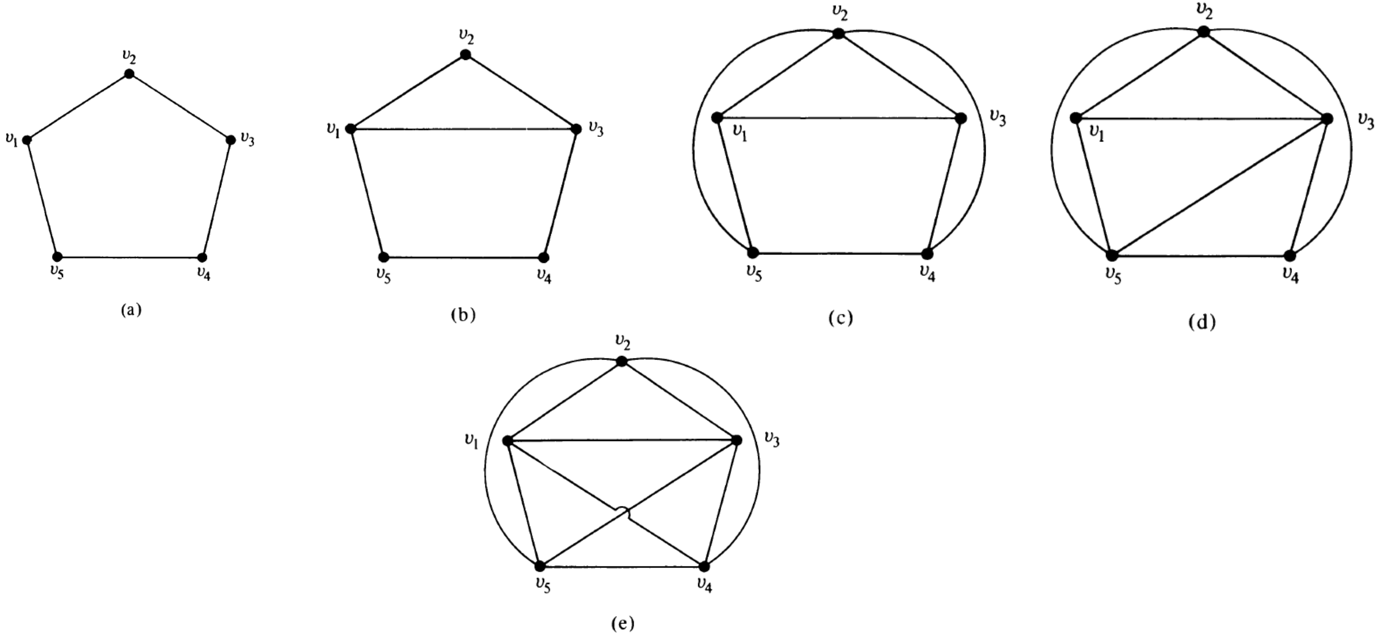
or $$2 \cdot 9 \geq 4(9 - 6 + 2),$$

or $$18 \geq 20, \qquad \text{a contradiction.}$$

Hence the graph cannot be planar.

THEOREM

The complete graph of five vertices is nonplanar.



Building up of the five-vertex complete graph.

Proof: Let the five vertices in the complete graph be named v_1, v_2, v_3, v_4, and v_5. A complete graph, is a simple graph in which every vertex is joined to every other vertex by means of an edge. This being the case, we must have a circuit from v_1 to v_2 to v_3 to v_4 to v_5 to v_1—that is, a pentagon. See Fig. 5-1(a). This pentagon must divide the plane of the paper into two regions, one *inside* and the other *outside* (Jordan curve theorem).

Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Suppose that we choose to draw a line from v_1 to v_3 inside the pentagon. See Fig. 5-1(b).

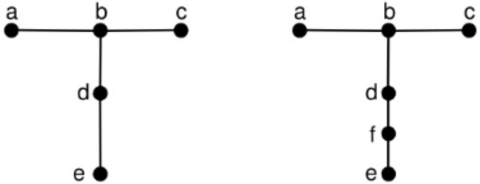
Now we have to draw an edge from v_2 to v_4 and another one from v_2 to v_5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon. See Fig. 5-1(c). The edge connecting v_3 and v_5 cannot be drawn outside the pentagon without crossing the edge between v_2 and v_4. Therefore, v_3 and v_5 have to be connected with an edge inside the pentagon. See Fig. 5-1(d).

Now we have yet to draw an edge between v_1 and v_4. This edge cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane. See Fig. 5-1(e).

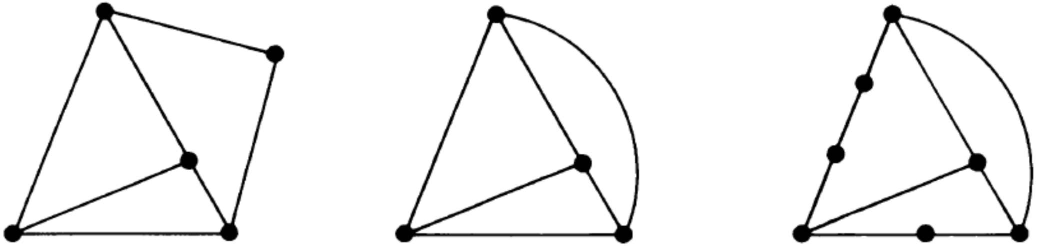
Homoeomorphic Graphs:

Two graphs are said to be homoeomorphic if one can be obtain from the other graph by creation of edges in series or by merger of edges in series.(in other words if one graph can be obtain from other graph by inserting or removing vertex of degree 2.)

Ex 1: Given two graphs are homoeomorphic to each other.



Ex 2: Given graphs are homoeomorphic to each other.



Three graphs homeomorphic to each other.

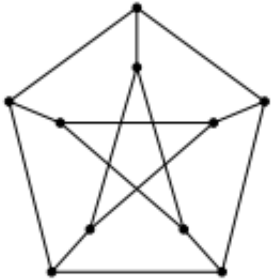
Kuratowski's Theorem:

A necessary and sufficient condition for a graph G to be planer is that G does not contain either of Kuratowski's two graphs as a subgraph.(or any graph homoeomorphic to either of them.)

Or

A graph G is said to be planer if and only if it has no subgraph homoeomorphic to k_5 or $k_{3,3}$.

EX: Prove that the Petersen graph (shown below) is not planar by finding a subgraph that is homeomorphic to $K_{3,3}$ or K_5.

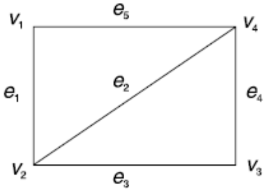


<u>Matrix representation of Graphs:</u>

A graph can be represented by a matrix.

- <u>Incidence Matrix</u>: Let G be a graph with n vertices and e edges. We define n × e matrix B=$[a_{ij}]$, whose n rows corresponds to n vertices and e column corresponds to e edges as follows:

$$
\begin{aligned}
a_{ij} &= 1, \quad \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i, \text{ and} \\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$



$$
X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1
\end{pmatrix}
\end{array}
$$

Graph and its incidence matrix

Basic Properties of Incidence Matrix:

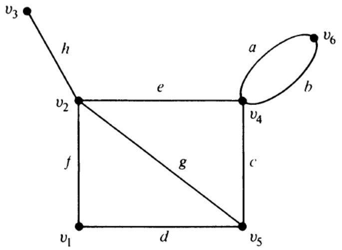
The incidence matrix contains only two elements 0 and 1.

Each column in the incidence matrix of a graph has exactly two 1's appearing in that column.

The sum of 1's in each row represents the degree of a vertex corresponding to the row.

A row with all 0's in the incidence matrix represents an isolated vertex.

Two identical columns in an incidence matrix correspond to parallel edges in the graphs G.



$$
\begin{array}{cc}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

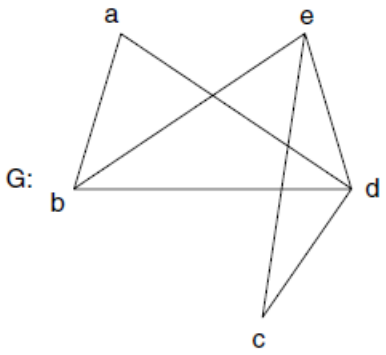
Graph and its incidence matrix.

- **Adjacency Matrix**:

Let G is a simple graph with n vertices $v_1, v_2, \ldots\ldots v_n$, the matrix A(G)=$[a_{ij}]$,where

$$a_{ij} = \begin{cases} 1, & if\ v_i\ v_j\ is\ an\ edge \\ 0, & otherwise \end{cases}$$

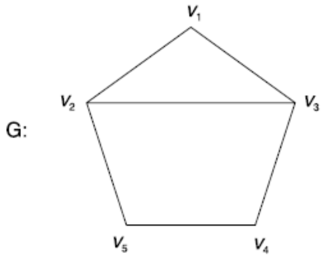
is called adjacency matrix of G.



(*i*)

$$\text{Adjacency matrix } x;\ A\ (G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

(*ii*)



(a)

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b)

The adjacency matrix has following properties:

- Since a simple graph has no loops, each diagonal entry of A, a_{ij}=0, for $i = 1,2,3, \dots..n$.
- The adjacency matrix of simple graph is symmetric. i.e. a_{ij}=a_{ji} .
  both of these entries are 1 when v_i and v_j are adjacent and 0 otherwise.
- Degree of v_i is equal to the number of 1's in the i^{th} row or i^{th} column.

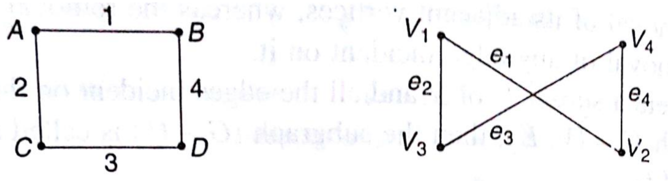
Isomorphism:

Two Graph G and G' are said to be isomorphic if there is one to one correspondence between their vertices and edges such that incidence relation is preserved.
In other words if an edge e in incident on v_1 and v_2 $in\ G$, then their corresponding edge $e'in\ G'$must be incident on the vertices v_1' and v_2'.

The isomorphic graph has

1. Same number of vertices
2. Same number of edges
3. The corresponding vertices with same degree.

If any of these conditions is not satisfied in two graphs, they cannot be isomorphic. However these conditions are not sufficient for isomorphism.



(a)

Graphs in Figure (a) are isomorphic to each other as

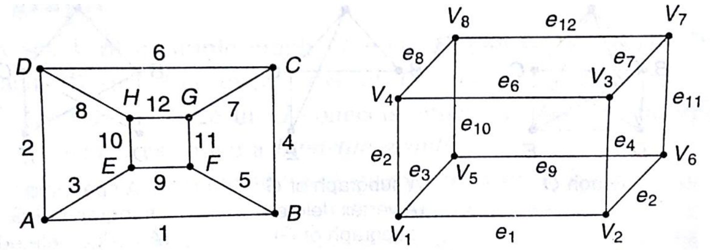
No of edges in both the graphs=4

No of vertices= 4

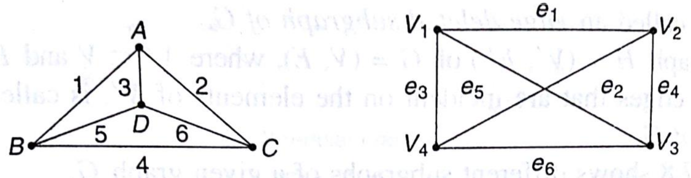
$\deg(\mathrm{A}) = \deg(V_1) = 2, \deg(\mathrm{B}) = \deg(V_2) = 2,$

$\deg(\mathrm{C}) = \deg(V_3) = 2, \deg(\mathrm{D}) = \deg(V_4) = 2,$

Let A corresponds to V_1, B corresponds to V_2, C corresponds to V_3, D corresponds to V_4 then we can see that incidence relation is also preserved.

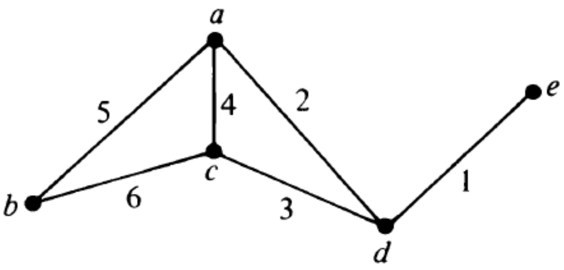


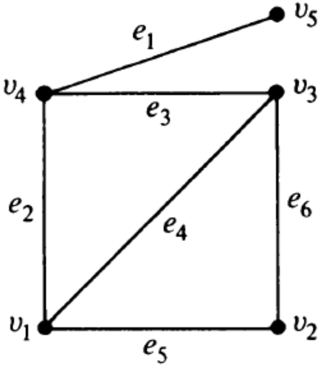
(c)



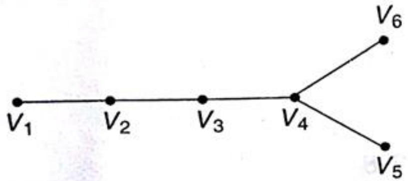
(b)

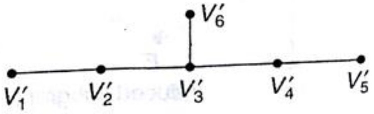
Similarly Graphs in figures (b) and (c) are isomorphic.





Isomorphic graphs.





Here V_1 , V_2, V_3 , V_4 , V_5 , V_6 corresponds to $V_1{}'$, $V_2{}'$, $V_4{}'$, $V_3{}'$, $V_5{}'$ $V_6{}'$.

Number of edges, no of vertices and degree of corresponding vertices in two graphs are same.
But these two graphs are not isomorphic as incidence relation is not preserved.

In first graph V_2 and V_3 are adjacent but in second graph the corresponding vertices $V_2{}'$ and $V_4{}'$ are not adjacent.

Hence these two graphs are not isomorphic to each other.

Determining isomorphism by adjacency matrix:

Theorem: Two graphs are isomorphic if and only if their vertices can be labeled in such a way that their adjacency matrices are equal.

Theorem: Two labeled graphs G_1 and G_2 with adjacency matrices A_1 and A_2 respectively are isomorphic, if and only if, there exist a permutation matrix P such that
$PA_1P^T = A_2$.

- *A matrix whose rows are the rows of unit matrix, but not necessarily in their natural order, is called permutation matrix.*

Example: let us consider the two graphs shown in Fig.

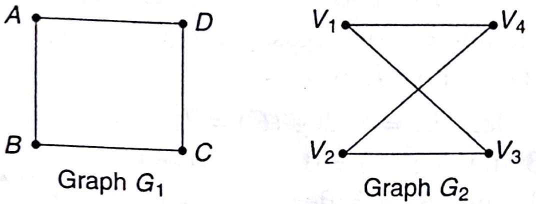


Fig.

Now $A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

If we assume that $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we can see that $PA_1P^T = A_2$. Hence,

the two graphs G_1 and G_2 are isomorphic such that $A \to V_1$, $B \to V_3$, $C \to V_2$ and $D \to V_4$.

Example: Establish the isomorphism of the two graphs given in Fig. by considering their adjacency matrices.

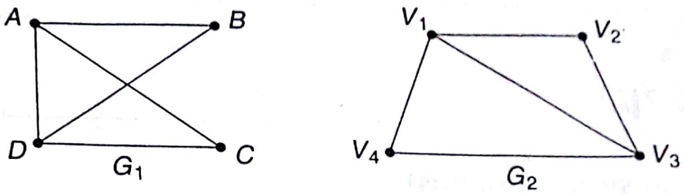


Fig.

The adjacency matrices A_1 and A_2 of G_1 and G_2 respectively are given below:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The matrices A_1 and A_2 are not the same.

To establish isomorphism between G_1 and G_2, we have to find a permutation matrix P such that $PA_1P^T = A_2$.

Since A_1 and A_2 are fourth order matrices, P is a 4^{th} order matrix got by permuting the rows of the unit matrix I_4. Thus, there are $4! = 24$ different forms for P. It is difficult to find the appropriate P from among the 24 matrices by trial that will satisfy $PA_1P^T = A_2$.

To find the appropriate P, we proceed as follows, using the degree of the vertices of G_1 and G_2:

$$\text{Deg}(A) = 3 \text{ and } \text{Deg}(V_1) = 3$$

Hence, the first row of I_4 can be taken as the first row of P

$$\text{Deg}(D) = 3 \text{ and } \text{Deg}(V_3) = 3$$

i.e., the 4^{th} vertex of G_1 corresponds to the 3^{rd} vertex of G_2.

Hence, the 4^{th} row of I_4 may be taken as the 3^{rd} row of P.

Hence, the 2^{nd} and 3^{rd} rows of I_4 may be taken either as the 2^{nd} and 4^{th} rows of P or as the 4^{th} and 2^{nd} rows of P.

Thus, there are 2 possible forms for P, namely

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

For both the forms of P, it is easily verified that $PA_1P^T = A_2$.
Hence, the two graphs G_1 and G_2 are isomorphic.

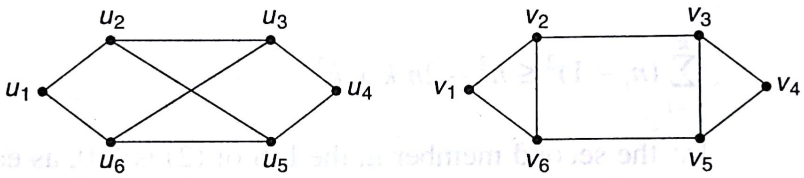
Circuits and Isomorphism

Apart from the three invariants of two isomorphic graphs already discussed, namely number of vertices, number of edges and degrees of corresponding vertices, we have one more invariant of isomorphic graphs.

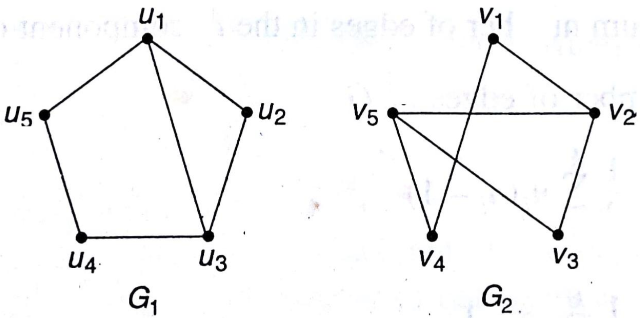
If two graphs are isomorphic, they will contain circuits of the same length k, where $k > 2$.

If this invariant condition is not satisfied then the two graphs will not be isomorphic.

For example, the two graphs G_1 and G_2 given in Fig. have 6 vertices each, 8 edges each, 8 edges each and 4 vertices of degree 3 and 2 vertices of degrees 2. still they are not isomorphic, because G_2 has a circuit of length 3, namely, $v_1 - v_2 - v_\tau - v_1$, whereas G_1 has no circuit of length 3.



G_1 G_2



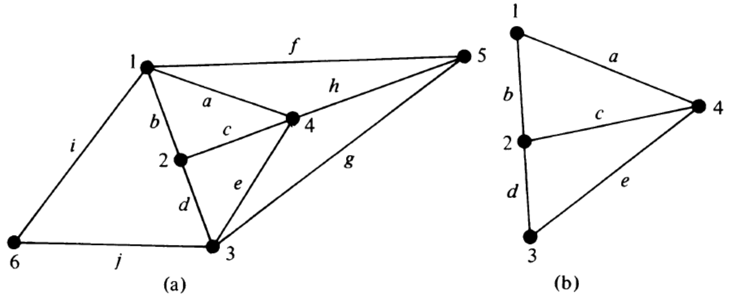
The two graphs G_1 and G_2 given in Fig. satisfy the usual (three) invariant conditions. We also note they have got circuits of length 5 which pass through all vertices, namely, $u_1 - u_2 - u_3 - u_4 - u_5 - u_1$ and $v_5 - v_3 - v_2 - v_1 - v_4 - v_5$.

Also we can verify that their adjacency matrix is same.

$$A_{G_1} \equiv \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} ; \quad A_{G_2} \equiv \begin{array}{c} \\ v_5 \\ v_3 \\ v_2 \\ v_1 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} v_5 & v_3 & v_2 & v_1 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

SUBGRAPHS

A graph g is said to be a *subgraph* of a graph G if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G.

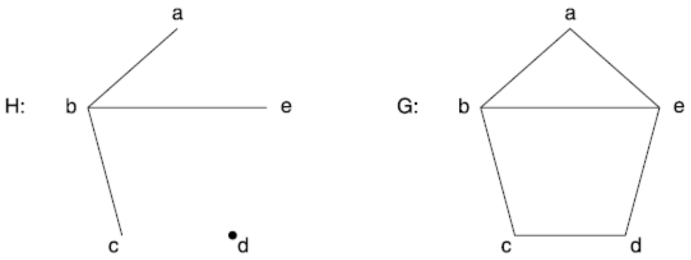


Graph (a) and one of its subgraphs (b).

The following observations can be made immediately:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.

H is a subgraph of G.



Walk:

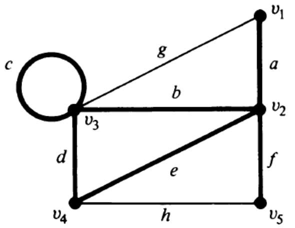
A walk in a graph is finite alternating sequence of vertices and edges beginning and ending with the vertices, such that edge is incident on the vertices preceding and following it.

No edge appears (covered or traversed) more than once in a walk.
A vertex may appear more than once.

$v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a walk shown with heavy lines.

Vertices with which a walk begins and ends are called its *terminal vertices*.

It is possible for a walk to begin and end at the same vertex. Such a walk is called a *closed walk*. A walk that is not closed is called an *open walk*



(a) An Open Walk

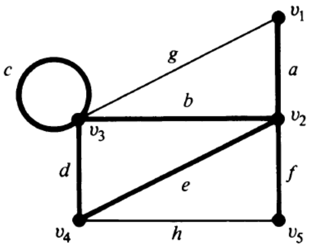
Path: An open walk in which no vertex appears more than once is called a *path*

In Fig. $v_1\ a\ v_2\ b\ v_3\ d\ v_4$ is a path, whereas $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is not a path.

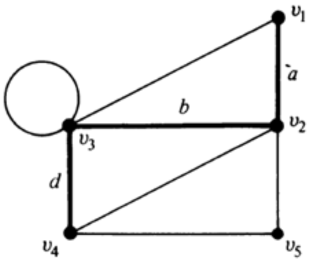
The number of edges in a path is called the *length of a path.*

An edge which is not a self-loop is a path of length one.

It should also be noted that a self-loop can be included in a walk but not in a path.



(a) An Open Walk



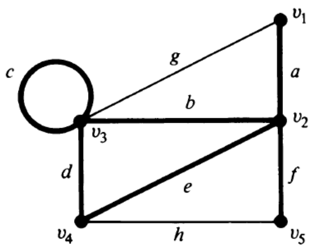
(b) A Path of Length Three

Circuit:

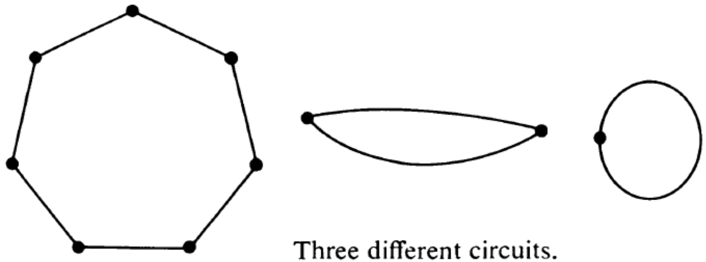
A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a *circuit*.

$v_2\ b\ v_3\ d\ v_4\ e\ v_2$ is a circuit.

A circuit is also called a *cycle*, *elementary cycle*, *circular path*, and *polygon*.



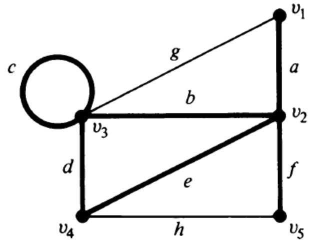
(a) An Open Walk



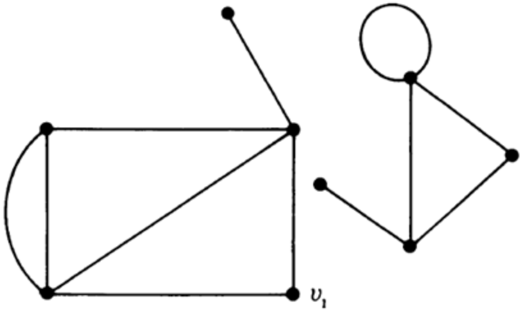
Three different circuits.

CONNECTED GRAPHS

A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. Otherwise, G is *disconnected*.



connected graph



A disconnected graph with two components.

- A null graph of more than one vertex is disconnected.
- A disconnected graph consist of two or more connected graph each of the connected subgraph is called component.

Theorem: A graph G is disconnected if and only if its vertex set V can be partitioned in to two nonempty subsets V_1 and V_2 such that there exist no edge in G whose one vertex is in subset V_1 and other is in V_2.

Theorem: A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

Proof: Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \ldots, n_k$. Thus we have

$$n_1 + n_2 + \cdots + n_k = n, \quad n_i \geq 1.$$

$$\sum_{i=1}^{k} n_i = n$$

$$\sum_{i=1}^{k} n_i - k = n - k$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$$\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i \neq j}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 + k^2 - 2nk$$ as $\sum_{i \neq j}(n_i - 1)(n_j - 1) \geq 0 \; as \; each \; n_i \geq 1.$

$$\left(\sum_{i=1}^{k} {n_i}^2\right) - k + 2nk \leq n^2 + k^2 - 2nk$$

$$\left(\sum_{i=1}^{k} {n_i}^2\right) \leq n^2 - (2n - k)(k - 1) \qquad \text{.........(1)}$$

We know maximum number of edges in ith component is $\frac{n_i(n_i - 1)}{2}$

Maximum no. of edges in graph G are

$$\sum_{i=1}^{k}\frac{n_i(n_i - 1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

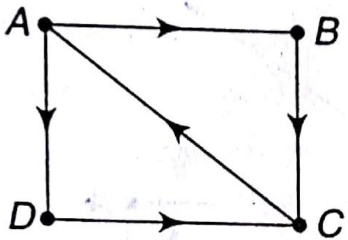
$$\leq \frac{1}{2}\left(n^2 - (2n - k)(k - 1)\right) - \frac{n}{2}$$

$$= \frac{1}{2}(n - k)(n - k + 1)$$

Connectedness in directed graph:

A directed graph is said to be *strongly connected*, if there is a path from V_i to V_j and from V_j to V_i where V_i and V_j are any pair of vertices of the graph.

For a directed graph to be strongly connected, there must be a sequence of directed edges from any vertex in the graph to any other vertex.



G_1

G_1 is a strongly connected graph, as the possible pairs of vertices in G_1 are (A, B), (A, C), (A, D), (B, C), (B, D) and (C, D) and there is a path from the first vertex to the second and from the second vertex to the first in all the pairs.

DELETION OF VERTICES AND EDGES FROM THE GRAPH:

If v_i is a vertex in graph G, then $G - v_i$ denotes a subgraph of G obtained by deleting (i.e., removing) v_i from G.

Deletion of a vertex always implies the deletion of all edges incident on that vertex. If e_j is an edge in G, then $G - e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not imply deletion of its end vertices.

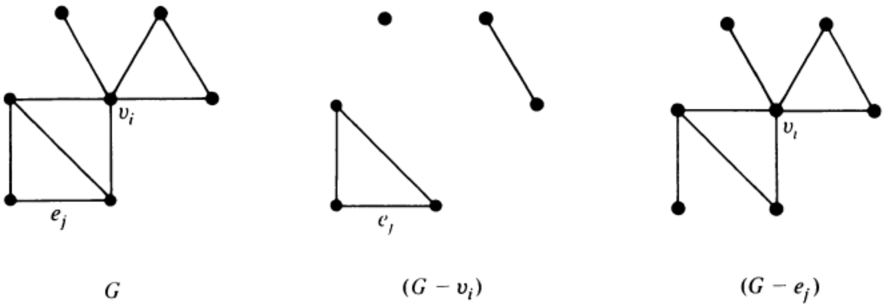


Fig. 2-15 Vertex deletion and edge deletion.

Cut Set: In a connected graph G, a *cut-set* is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.

For instance, in Fig. 4-1 the set of edges $\{a, c, d, f\}$ is a cut-set.
There are many other cut-sets, such as $\{a, b, g\}$, $\{a, b, e, f\}$, and $\{d, h, f\}$. Edge $\{k\}$ alone is also a cut-set. The set of edges $\{a, c, h, d\}$, on the other hand, is *not* a cut-set, because one of its proper subsets, $\{a, c, h\}$, is a cut-set.

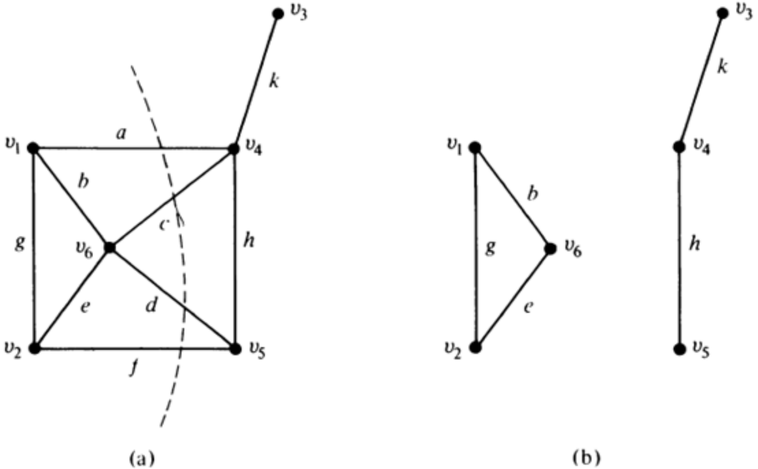


Fig. 4-1 Removal of a cut-set $\{a, c, d, f\}$ from a graph "cuts" it into two.

CONNECTIVITY

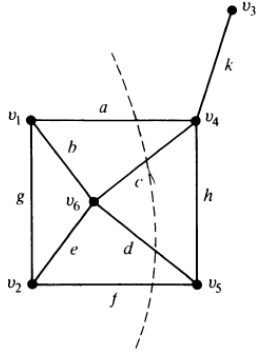
Edge Connectivity

Definition . Let G be a connected graph. The edge connectivity of G is the minimum number of lines (edges) whose removal results in a disconnected graph.

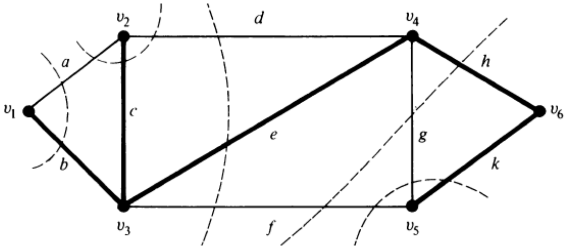
In other words

The number of edges in the smallest cut-set (i.e., cut- set with fewest number of edges) is defined as the *edge connectivity* of G.

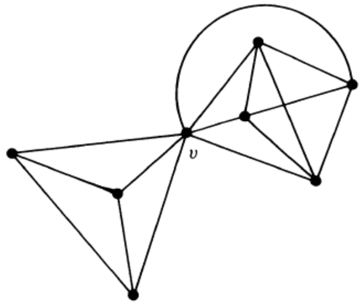
The edge connectivity of a connected graph G is denoted by $\lambda(G)$. If G is a disconnected graph, then $\lambda(G) = 0$.



Edge Connectivity - One



Edge Connectivity-Two



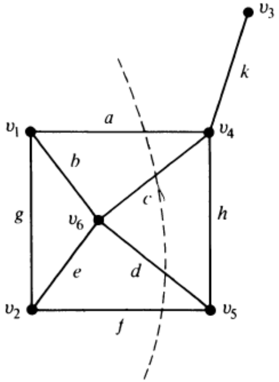
Edge Connectivity-Three

Vertex Connectivity

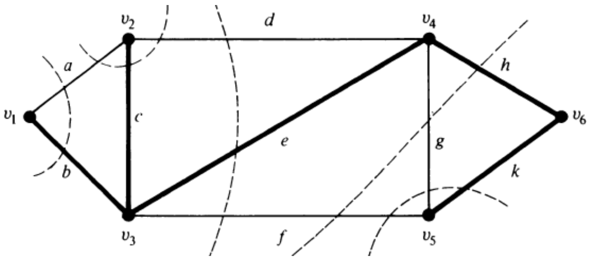
Definition Let G be a connected graph. The minimum number of vertices whose removal results in a disconnected or trivial graph is called the vertex connectivity of G.

The vertex connectivity of G is denoted by $k(G)$. If $k(G) = 1$, then G has a vertex v such that $G - v$ is not connected and the vertex v is called a cut vertex .

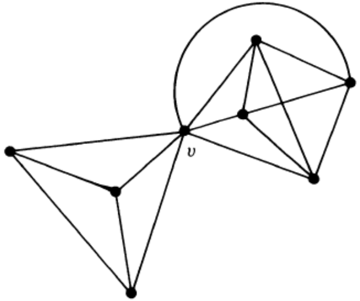
If $G = K_n$ the complete graph with n vertices then $k(G)$ is $n - 1$.



Vertex Connectivity - One



Vertex Connectivity - Two



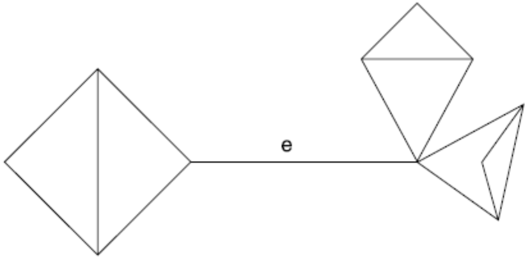
Vertex Connectivity - One

Theorem The vertex connectivity of a graph G is always less then or equal to the edge connectivity of G i.e., $k(G) \leq \lambda(G)$.

Theorem The edge connectivity of a connected graph G cannot exceed the minimum degree of G, i.e., $\lambda(G) \leq \delta(G)$.

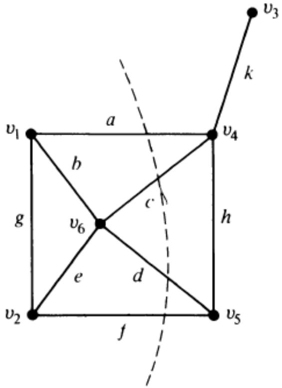
Theorem : For any graph G, $k(G) \leq \lambda(G) \leq \delta(G)$

Cut edge(Bridge): Let G be a connected graph. If e is an edge of G, such that G-e is not connected, the edge e is called a cut edge (or bridge). In the graph the edge e is a cut edge.



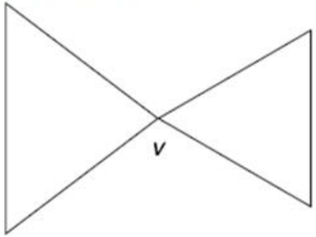
In other words if edge connectivity of a graph is one then that edge is called cut edge.

Ex: In the given graph edge connectivity is one and edge k is cut edge.



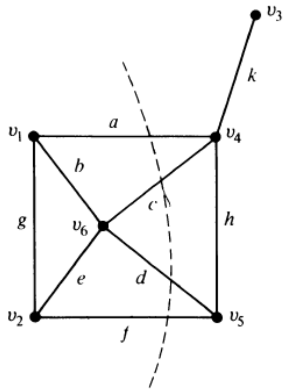
CUT VERTEX

Let G be a connected graph. If v is a vertex of G such that $G - v$ is not connected then, the vertex v is called a cut vertex.



If v is a cut vertex of G, then the removal of the vertex v increase the number of components in G. A cut vertex is also called a cut point.

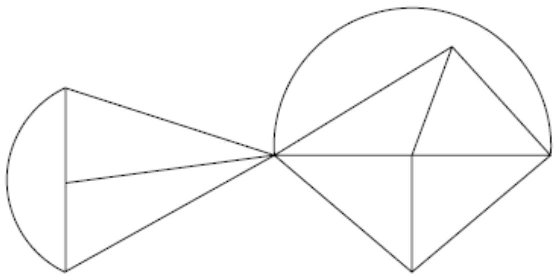
In other words if vertex connectivity is one then that vertex is called cut vertex.
Ex: In the given graph vertex connectivity is one and vertex v_4 is cut vertex.



Separable Graph:

A graph G is said to be separable if its vertex connectivity is one.

Example Find the edge connectivity and the vertex connectivity of the graph

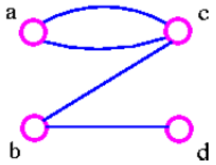


Solution: The minimum number of edges removal disconnects the graph is 3 and the minimum number of vertices required to disconnect the graph is 1.

$\therefore$ Edge connectivity $\lambda(G) = 3$

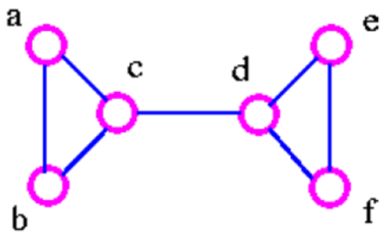
Vertex connectivity $k\,(G) = 1$.

Let us consider some examples:



The above graph can be split up into two components by removing one of the edges bc or bd. Therefore, edge bc or bd is a bridge.

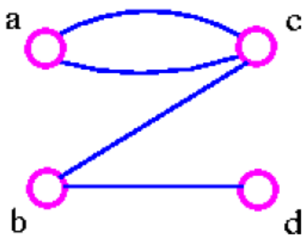
Thus the above graph has edge connectivity 1.



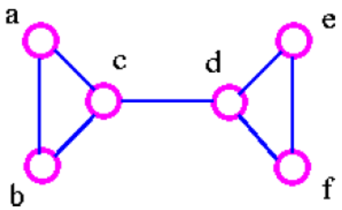
The above graph can be disconnected by removing a single edge, cd. Therefore, edge cd is a bridge.

Thus the above graph has edge connectivity 1.

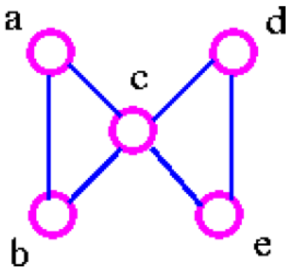
Let us consider some examples:



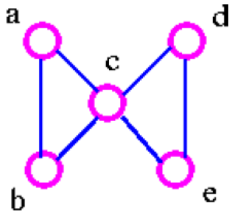
The above graph can be disconnected by removal of single vertex (either b or c). It has connectivity 1.



The above graph can be disconnected by removal of single vertex (either c or d). The vertex c or d is a cut-vertex. It has connectivity 1.



The above graph can be disconnected by removing just one vertex i.e., vertex c. The vertex c is the cut-vertex. It has connectivity 1.



The above graph cannot be disconnected by removing a single edge, but the removal of two edges (such as ac and bc) disconnects it.

Thus the edge connectivity for the above graph is 2.

EULER GRAPHS

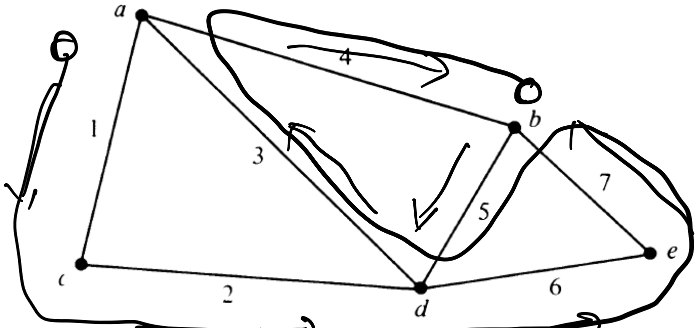
Eulerian line(or circuit): A closed walk of a graph G which includes every edge of the graph exactly once is called Eulerian line(or circuit) .

Eulerian Graph: A graph that contains Euler line is called Euler Graph.

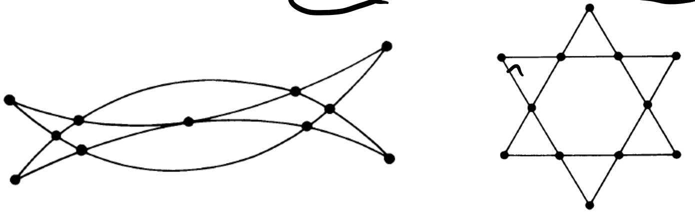
- Since the Euler circuit (or cycle)contains all the edges of the graph, therefore Euler graph is always connected.

- **Open Euler line**: An open walk that includes all edges of the graph G exactly once, is called open Euler line or unicursal line.

  Unicursal graph: A graph that has unicursal line is called unicursal graph.

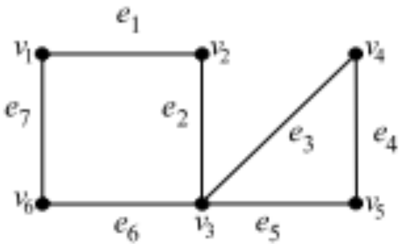


Unicursal graph.



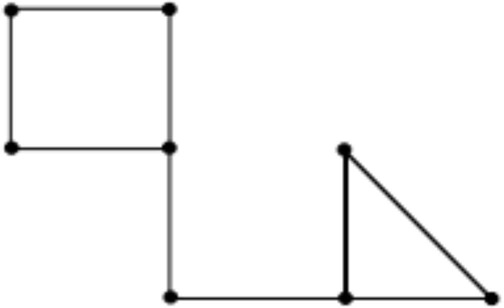
Two Euler graphs.

Example Consider the graph shown in Figure . Clearly, $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_5\ e_5\ v_3\ v_6\ e_7\ v_1$ in (a) is an Euler line, whereas the graph shown in (b) is non-Eulerian.



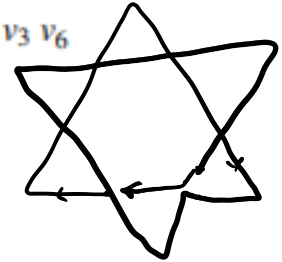
(a)

Eulerian Graph

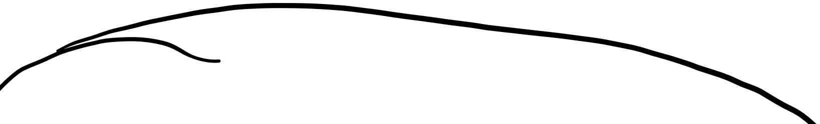


(b)

Non-Eulerian Graph



Theorem: A given connected graph G is an Euler graph if and only if all the vertices of G are of even degree.

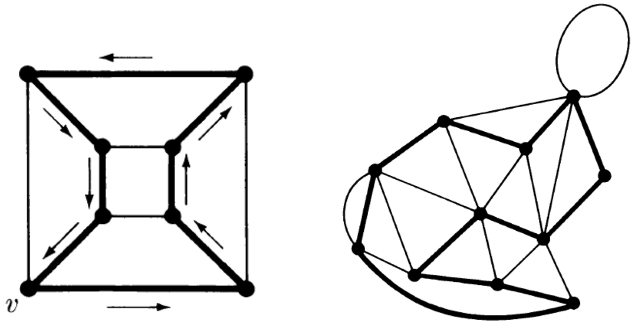


HAMILTONIAN PATHS AND CIRCUITS

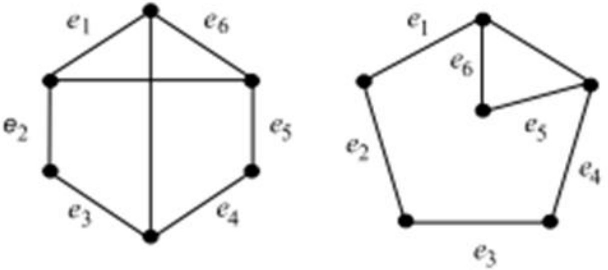
Hamiltonian Circuit:

A Hamiltonian circuit in connected graph G is defined as a closed walk that traverses every vertex of G exactly once, except the starting vertex at which walk terminates.

Hamiltonian circuit in a graph of n vertices consist of exactly n edges.



Hamiltonian circuits.



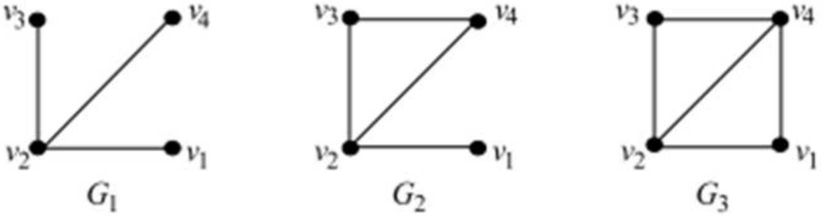
Hamiltonian Graphs

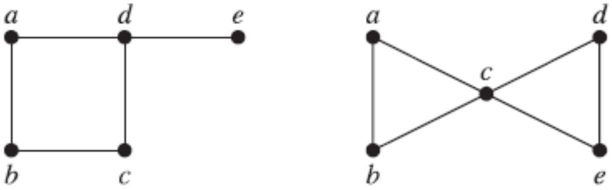
Hamiltonian Path: If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called a *Hamiltonian path*. Clearly, a Hamiltonian path in a graph G traverses every vertex of G.

every graph that has a Hamiltonian circuit also has a Hamiltonian path. There are, however, many graphs with Hamiltonian paths that have no Hamiltonian circuits .

The length of a Hamiltonian path (if it exists) in a connected graph of n vertices is $n - 1$.

For example, in Figure , G_1 has no Hamiltonian path, and so no Hamiltonian cycle; G_2 has the Hamiltonian path $v_1v_2v_3v_4$, but has no Hamiltonian cycle, while G_3 has the Hamiltonian cycle $v_1v_2v_3v_4v_1$.





G *H*

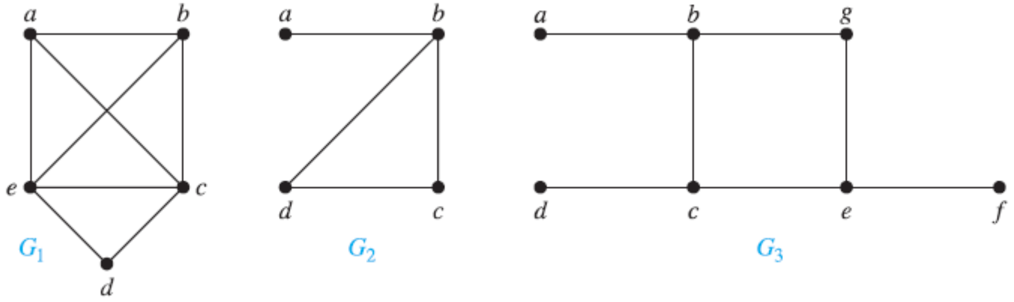
Two Graphs That Do Not Have a Hamilton Circuit.

Few Properties of Hamiltonian Graphs:

- Path obtain by deleting any edge from a Hamiltonian circuit is called Hamiltonian path.
- Hamiltonian circuit contains a Hamiltonian path, but a graph containing a Hamiltonian path may not have Hamiltonian circuit.
- A complete graph K_n will always have a Hamiltonian circuit when, $n \geq 3$.
- A given graph may contain more than one Hamiltonian circuit.

EXAMPLE Which of the simple graphs in Figure have a Hamilton circuit or, if not, a Hamilton path?

Solution: G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but G_2 does have a Hamilton path, namely, a, b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once. ◀



Three Simple Graphs.

THEOREM

In a complete graph with n vertices there are $(n - 1)/2$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

Proof: A complete graph G of n vertices has $n(n - 1)/2$ edges, and a Hamiltonian circuit in G consists of n edges. Therefore, the number of edge-disjoint Hamiltonian circuits in G cannot exceed $(n - 1)/2$. That there are $(n - 1)/2$ edge-disjoint Hamiltonian circuits, when n is odd, can be shown as follows:

The subgraph (of a complete graph of n vertices) in Fig. is a Hamiltonian circuit. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise

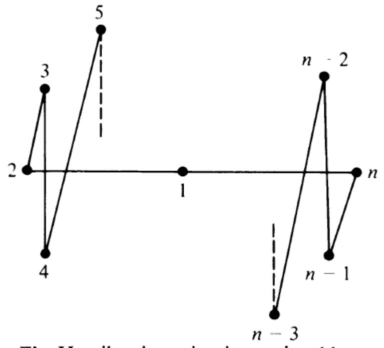


Fig. Hamiltonian circuit; n is odd.

by $360/(n-1), 2 \cdot 360/(n-1), 3 \cdot 360/(n-1), \ldots, (n-3)/2 \cdot 360/(n-1)$ degrees. Observe that each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones. Thus we have $(n-3)/2$ new Hamiltonian circuits, all edge disjoint from the one in Fig. and also edge disjoint among themselves. Hence the theorem.

Weighted Graph:

A graph in which each edge e is assigned a non-negative real number w(e) is called weighted graph. Weight of the edge e may represent distance, time, cost etc in

A simple weighted digraph G of n vertices is described by an n by n matrix $\mathbf{D} = [d_{ij}]$, where

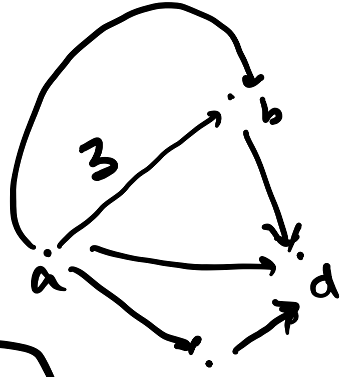
d_{ij} = length (or distance or weight) of the directed edge from vertex i to vertex j, $d_{ij} \geqslant 0$,

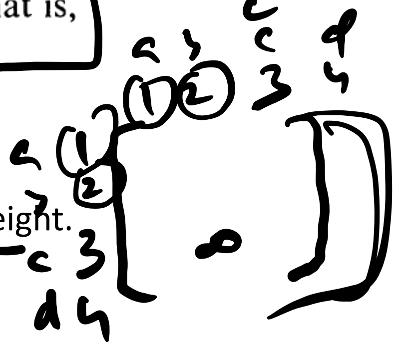
$d_{ii} = 0$,

$d_{ij} = \infty$, if there is no edge from i to j.

In general, $d_{ij} \neq d_{ji}$, and the triangle inequality need not be satisfied. That is, $d_{ij} + d_{jk}$ may be less than d_{ik}.

$[d_{ij}]$





Shortest path:

A shortest path between two vertices in a weighted graph is a path of least weight. In a unweighted graph a shortest path means a path with least no. of edges.

Dijkstra's Algorithm:

To find the length of the shortest path between two vertices, we use Dijkstra's algorithm.
This algorithm begins by assigning a label 0 to the starting vertex s and a temporary level ∞ to the remaining $(\boldsymbol{n-1})$ vertices. In each Iteration other vertex gets a **permanent label** , according to the following rule.

1. Each vertex j that is not yet permanently labeled gets a new temporary label, whose value is given by **min [old label of $\boldsymbol{j}$, (old label of $\boldsymbol{i}$ + $\boldsymbol{d_{ij}}$)],**
   where i is the latest vertex permanently labeled, in the previous iteration,
   and d_{ij} is direct distance between vertices i and j.

   If i and j are not joined by an edge then $\boldsymbol{d_{ij}}$ **= ∞.**

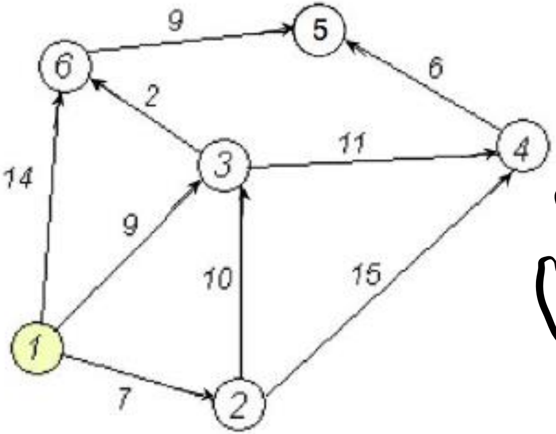
2. The smallest value among all the temporary labels is found, and this becomes the permanent level of corresponding vertex.

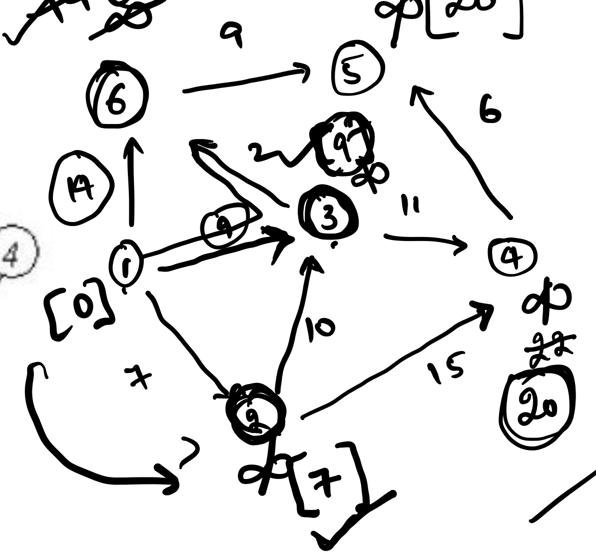
$\min\{14, 9+2\}$
$= 11$

In case of tie we can select any one for permanent labeling.
Repeat steps 1 and 2 until destination vertex (say d) gets a permanent label.

[11]

$\min\{\infty, 0+7\} = 7$



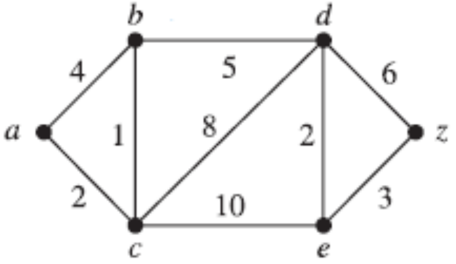


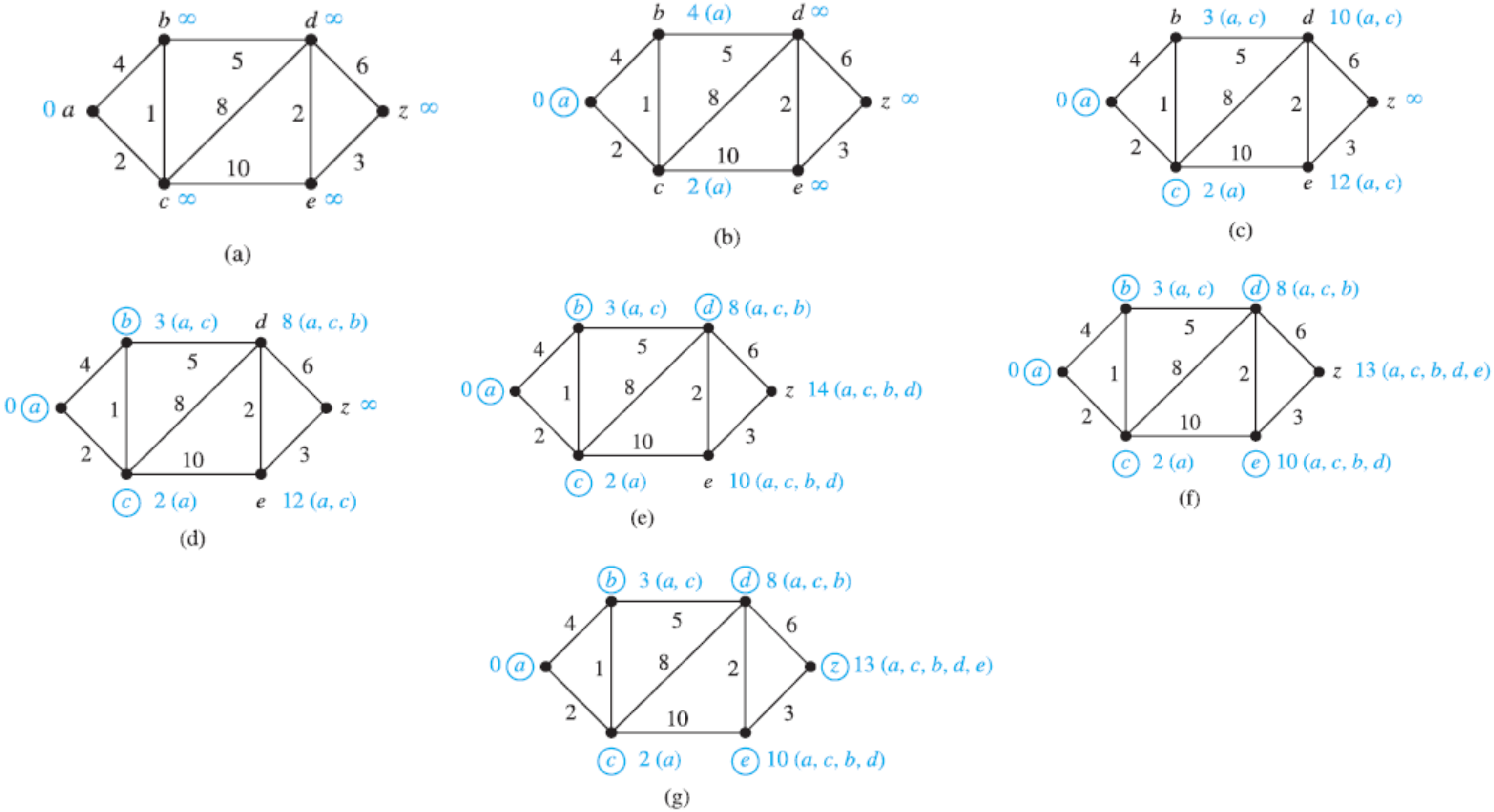
We want to find the shortest path from node 1 to all other nodes using Dijkstra's algorithm.

1	2	3	4	5	6	
[0]	∞	∞	∞	∞	∞	Starting vertex is labeled as 0 i.e. permanent vertex.
[0]	7	9	∞	∞	14	Adjacent vertices of 1 get labelled.
[0]	[7]	9	22	∞	14	Smallest label get permanent and adjacent vertices of 2 get labeled.
[0]	[7]	[9]	20	∞	11	
[0]	[7]	[9]	20	20	[11]	
[0]	[7]	[9]	[20]	20	[11]	

This gives shortest distance between any two vertices.

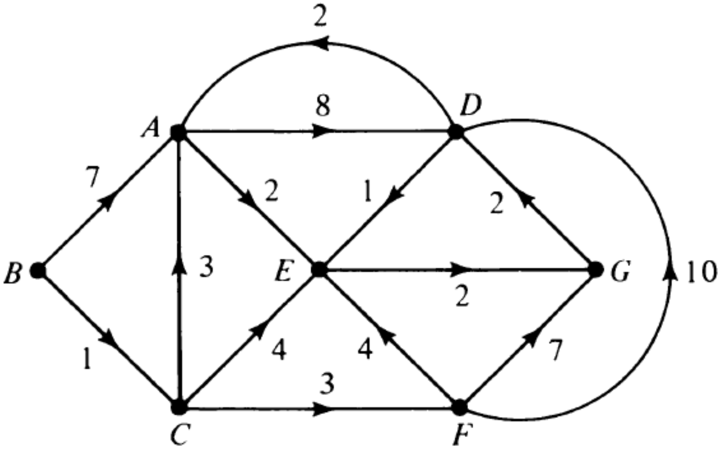
Ex: Find shortest distance between a to z by Dijkstra's algorithm.





0 a
b ∞
c ∞
d ∞
e ∞
z ∞
4
2
1
5
8
10
2
6
3
(a)
0 a
b 4 (a)
c 2 (a)
d ∞
e ∞
z ∞
(b)
0 a
b 3 (a, c)
c 2 (a)
d 10 (a, c)
e 12 (a, c)
z ∞
(c)
0 a
b 3 (a, c)
c 2 (a)
d 8 (a, c, b)
e 12 (a, c)
z ∞
(d)
0 a
b 3 (a, c)
c 2 (a)
d 8 (a, c, b)
e 10 (a, c, b, d)
z 14 (a, c, b, d)
(e)
0 a
b 3 (a, c)
c 2 (a)
d 8 (a, c, b)
e 10 (a, c, b, d)
z 13 (a, c, b, d, e)
(f)
0 a
b 3 (a, c)
c 2 (a)
d 8 (a, c, b)
e 10 (a, c, b, d)
z 13 (a, c, b, d, e)
(g)

EX: Use Dijkstra's Algorithm to find shortest distance between B and G



Simple weighted digraph

A	B	C	D	E	F	G		
∞	$\boxed{0}$	∞	∞	∞	∞	∞	:	Starting Vertex B is labeled 0.
7	$\boxed{0}$	1	∞	∞	∞	∞	:	All successors of B get labeled.
7	$\boxed{0}$	$\boxed{1}$	∞	∞	∞	∞	:	Smallest label becomes permanent.
4	$\boxed{0}$	$\boxed{1}$	∞	5	4	∞	:	Successors of C get labeled.
4	$\boxed{0}$	$\boxed{1}$	∞	5	$\boxed{4}$	∞		
4	$\boxed{0}$	$\boxed{1}$	14	5	$\boxed{4}$	11		
$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	14	5	$\boxed{4}$	11		
$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	12	5	$\boxed{4}$	11		
$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	12	$\boxed{5}$	$\boxed{4}$	11		
$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	12	$\boxed{5}$	$\boxed{4}$	7		
$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	12	$\boxed{5}$	$\boxed{4}$	$\boxed{7}$	:	Destination vertex gets permanently labeled.

Warshall's Algorithm:

Warshall's algorithm determines the shortest distance between all pair of vertices in a graph. It can be applied to a directed graph too.

First form the weight matrix W=(w_{ij}),

Where $w_{ij} = \begin{cases} w(ij), & if\ there\ is\ an\ edge\ between\ v_i\ and\ v_j. \\ 0, & otherwise \end{cases}$

Let $v_1, v_2, \ldots v_n$ *be the n vertices.*

Now sequence of matrices $L_0, L_1, \ldots L_n$ are formed, where $L_r = \{l_r(i,j)\}$.

$l_r(ij)$ the ij^{th} entry of L_r is computed by using following rule.

$l_r(i,j) = \min[l_{r-1}(i,j); l_{r-1}(i,k)+l_{r-1}(k,j)]$

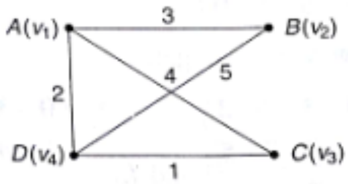
Where k takes the values 1,2 ,…,n in first ,second,…… n^{th} iteration respectively.

In L_3: (i,j)

$(i,3)+(3,j)$

- The initial matrix L_0 is same as the weight matrix W except that each non diagonal 0 in W is replaced by ∞.
- The final matrix L_n is the required shortest distance matrix between the vertex v_i *and* v_j .

Ex. Find shortest distance matrix and the corresponding shortest path matrix for all pair of vrtices in the given graph, using Warshall's algorithm.



Weight matrix

$$W = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 0 & 5 \\ 4 & 0 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix} \end{array}$$

$$L_0 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & \infty & 5 \\ 4 & \infty & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix} \end{array}$$

L_1, L_2, L_3, L_4

$i \to j$

$i \to v_1 \to j$

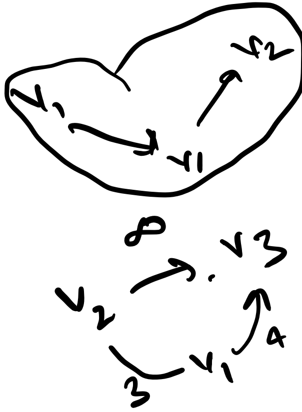
By Warshall's algorithm,

$$\boxed{l_1(1, 2)} = \min\{l_0(1, 2); \boxed{l_0(1, 1) + l_0(1, 2)}\}$$
$$= \min\{3; 0 + 3\} = 3$$
$$l_1(1, 3) = \min\{l_0(1, 3); l_0(1, 1) + l_0(1, 3)\}$$
$$= \min\{4; 0 + 4\} = 4$$
$$l_1(1, 4) = \min\{l_0(1, 4); l_0(1, 1) + l_0(1, 4)\}$$
$$= \min\{2; 0 + 2\} = 2$$
$$l_1(2, 3) = \min\{l_0(2, 3); l_0(2, 1) + l_0(1, 3)\}$$
$$= \min\{\infty; 3 + 4\} = 7$$
$$l_1(2, 4) = \min\{l_0(2, 4); l_0(2, 1) + l_0(1, 4)\}$$
$$= \min\{5; 3 + 2\} = 5$$
$$l_1(3, 4) = \min\{l_0(3, 4); l_0(3, 1) + l_0(1, 4)\}$$
$$= \min\{1; 4 + 2\} = 1$$

L_0

$$L_0 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & \infty & 5 \\ 4 & \infty & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix} \end{array}$$

$v_1 \to v_2$



Since L_0 is symmetric matrix, the subsequent matrices , L_1 , L_2 , L_3, L_4 will also be symmetric.

$$L_1 = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 5 \\ 4 & 7 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix} \to \begin{bmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 5 \\ 4 & 7 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{bmatrix}$$

Now $\quad l_2(1, 2) = \min\{l_1(1, 2); l_1(1, 2) + l_1(2, 2)\}$
$\quad\quad = \min\{3; 3 + 0\} = 3$

Similarly proceeding, we get,

$$l_2(1, 3) = 4;\ l_2(1, 4) = 2;\ l_2(2, 3) = 7;\ l_2(2, 4) = 5;\ l_2(3, 4) = 1$$

Hence, $\quad L_2 = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 5 \\ 4 & 7 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix}$ ✓

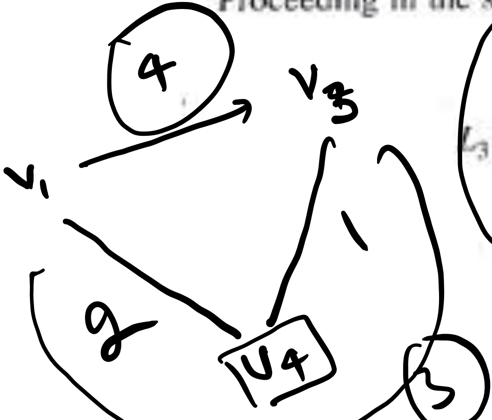
(1,3) (1,4) + (4,3)

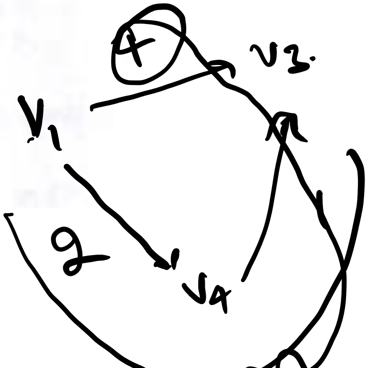
min {4, 2 + 1}

= 3.

Proceeding in the same way, we can get,

$$L_3 = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 5 \\ 4 & 7 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix} \text{ and } L_4 = \begin{pmatrix} 0 & 3 & 3 & 2 \\ 3 & 0 & 6 & 5 \\ 3 & 6 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix}$$

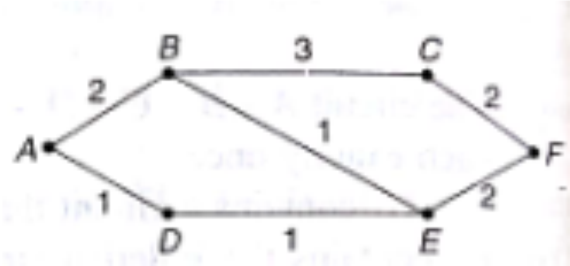




L_4 gives the shortest distance matrix which gives shortest distance between all pair of vertices.
The shortest path matrix is given by

$$
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
A & - & AB & ADC & AD \\
B & BA & - & BADC & BD \\
C & CDA & CDAB & - & CD \\
D & DA & DB & DC & -
\end{array}
$$

Ex. Find shortest distance matrix and the corresponding shortest path matrix for all pair of vertices in the given graph, using Warshall's algorithm.



The weight matrix of the given graph is given by

$$W = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 & 2 & 0 \end{pmatrix} \end{array}$$

The initial distance (length) matrix L_0 is got from W by replacing all the non-diagonal 0's by ∞ each. Thus

$$L_0 = \begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & 0 & 2 & \infty & 1 & \infty & \infty \\ B & 2 & 0 & 3 & \infty & 1 & \infty \\ C & \infty & 3 & 0 & \infty & \infty & 2 \\ D & 1 & \infty & \infty & 0 & 1 & \infty \\ E & \infty & 1 & \infty & 1 & 0 & 2 \\ F & \infty & \infty & 2 & \infty & 2 & 0 \end{array}$$

Since all he L_r matrices are symmetric with zero diagonal element, we need only compute the following elements

$$l_{12}, l_{13}, l_{14}, l_{15}, l_{16}, \qquad l_{23}, l_{24}, l_{25}, l_{26}, \qquad l_{34}, l_{35}, l_{36}, \qquad l_{45}, l_{46}, \qquad and\ l_{56}$$

$$l_{ij} = \min [l_{ij}; l_{i1} + l_{ij} \text{ of the } L_0 \text{ matrix}]$$

$$l_{12} \text{ of } L_1 = \min [l_{12}; l_{11} + l_{12} \text{ of } L_0]$$
$$= \min [2; 0 + 2] = 2 \text{ and so on.}$$

$$l_{23} \text{ of } L_1 = \min [l_{23}; l_{21} + l_{13} \text{ of } L_0]$$
$$= \min [3; 2 + \infty] = 3 \text{ and so on.}$$

$$l_{34} \text{ of } L_1 = \min [l_{34}; l_{31} + l_{14} \text{ of } L_0]$$
$$= \min [\infty; \infty + 1] = \infty \text{ and so on.}$$

$$l_{45} \text{ of } L_1 = \min [l_{45}; l_{41} + l_{15} \text{ of } L_0]$$
$$= \min [1; 1 + \infty] = 1 \text{ and so on.}$$

$$l_{56} \text{ of } L_1 = \min [l_{56}; l_{51} + l_{16} \text{ of } L_0]$$
$$= \min [2; \infty + \infty] = 2 \text{ and so on.}$$

$$L_1 = \begin{pmatrix} 0 & 2 & \infty & 1 & \infty & \infty \\ 2 & 0 & 3 & 3 & 1 & \infty \\ \infty & 3 & 0 & \infty & \infty & 2 \\ 1 & 3 & \infty & 0 & 1 & \infty \\ \infty & 1 & \infty & 1 & 0 & 2 \\ \infty & \infty & 2 & \infty & 2 & 0 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 0 & 2 & 5 & 1 & 3 & \infty \\ 2 & 0 & 3 & 3 & 1 & \infty \\ 5 & 3 & 0 & 6 & 4 & 2 \\ 1 & 3 & 6 & 0 & 1 & \infty \\ 3 & 1 & 4 & 1 & 0 & 2 \\ \infty & \infty & 2 & \infty & 2 & 0 \end{pmatrix}; \quad L_3 = \begin{pmatrix} 0 & 2 & 5 & 1 & 3 & 7 \\ 2 & 0 & 3 & 3 & 1 & 5 \\ 5 & 3 & 0 & 6 & 4 & 2 \\ 1 & 3 & 6 & 0 & 1 & 8 \\ 3 & 1 & 4 & 1 & 0 & 2 \\ 7 & 5 & 2 & 8 & 2 & 0 \end{pmatrix}$$

$$L_4 = \begin{pmatrix} 0 & 2 & 5 & 1 & 2 & 7 \\ 2 & 0 & 3 & 3 & 1 & 5 \\ 5 & 3 & 0 & 6 & 4 & 2 \\ 1 & 3 & 6 & 0 & 1 & 8 \\ 2 & 1 & 4 & 1 & 0 & 2 \\ 7 & 5 & 2 & 8 & 2 & 0 \end{pmatrix}; \quad L_5 = \begin{pmatrix} 0 & 2 & 5 & 1 & 2 & 4 \\ 2 & 0 & 3 & 2 & 1 & 3 \\ 5 & 3 & 0 & 5 & 4 & 2 \\ 1 & 2 & 5 & 0 & 1 & 3 \\ 2 & 1 & 4 & 1 & 0 & 2 \\ 4 & 3 & 2 & 3 & 2 & 0 \end{pmatrix}$$

$$L_6 = \begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & 0 & 2 & 5 & 1 & 2 & 4 \\ B & 2 & 0 & 3 & 2 & 1 & 3 \\ C & 5 & 3 & 0 & 5 & 4 & 2 \\ D & 1 & 2 & 5 & 0 & 1 & 3 \\ E & 2 & 1 & 4 & 1 & 0 & 2 \\ F & 4 & 3 & 2 & 3 & 2 & 0 \end{array}$$

L_6 is the required shortest distance matrix that gives the shortest distances between all pairs of vertices of the given graph. The corresponding shortest path matrix is as follows:

$$\begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & — & AB & ABC & AD & ADE & ADEF \\ B & BA & — & BC & BED & BE & BEF \\ C & CBA & CB & — & CFED & CFE & CF \\ D & DA & DEB & DEFC & — & DE & DEF \\ E & EDA & EB & EFC & ED & — & EF \\ F & FEDA & FEB & FC & FED & FE & — \end{array}$$