~~Euclidean Algorithm~~

The Diophantine Equation $ax+by=c$.

The simplest type of Diophantine equation that we shall consider is the linear Diophantine equation in two unknowns:

$$ax+by=c,$$

where a, b, c are given integers and a, b are not both zero.

A solution of this equation is a pair of integers x_0, y_0, i.e. $ax_0+by_0=c$.

A given linear Diophantine equation can have a number of solutions, as the case with $3x+6y=18$, where

$$3\cdot 4+6\cdot 1=18$$

$$3\cdot(-6)+6\cdot 6=18$$

$$3\cdot 10+6(-2)=18.$$

By contrast, there is no solution to the equation $2x+10y=17$.

Indeed, the L.H.S is an even integer whatever the choice of x and y, whereas the RHS is not.

So it is reasonable to enquire about the circumstances under which a solution is possible, and when a solution does exist., whether we can determine all solutions explicitly.

Condition for solvability:- The linear Diophantine equation $ax+by=c$ admits a solution if and only if d/c, where $d=\gcd(a,b)$.

We know that there are integers r and s for which $a = dr$ and $b = ds$.

If a solution of $ax+by=c$ exists, so that $ax_0+by_0=c$ for suitable x_0 and y_0, then

$$c = ax_0 + by_0 = drx_0 + dsy_0 = d(rx_0 + sy_0)$$

$$\Rightarrow d/c.$$

Conversely, assume that d/c, say $c = dt$.

By previously Thm (3), $\exists$ integers x_0 & y_0 s.t $d = ax_0 + by_0$.

Now,
$$c = dt = (ax_0+by_0)t = a(tx_0) + b(ty_0).$$

Hence, the Diophantine equation $ax+by=c$ has $x = tx_0$ and $y = ty_0$ as a particular solution.

Thm (9). The linear Diophantine equation $ax+by=c$ has a solution if and only if d/c, where $d = \gcd(a,b)$. If x_0, y_0 is any particular solution of this equation, then all other solutions are given by—

$$x = x_0 + \left(\frac{b}{d}\right)t, \quad y = y_0 - \left(\frac{a}{d}\right)t,$$ where t is an arbitrary integer.

Proof :- 2nd Part,

Let us suppose that a solution x_0, y_0 of the given equation is known. If x', y' is any other solution, then

$$ax_0 + by_0 = c = ax' + by'$$

$$\Rightarrow a(x' - x_0) = b(y_0 - y').$$

By the corollary to Thm (4), $\exists$ relatively prime integers r and s such that $a = dr$, $b = ds$.

So, we get
$$r(x'-x_0) = s(y_0-y').$$
$$\Rightarrow r \mid s(y_0-y') \text{ with } \gcd(r,s)=1$$
Using Euclid's lemma,
$$r \mid y_0-y'$$
$$\Rightarrow y_0-y' = rt, \text{ for some integer } t.$$
So, we get,-
$$x'-x_0 = st.$$
$$x' = x_0 + st = x_0 + \left(\frac{b}{d}\right)t$$
$$y' = y_0 - rt = y_0 - \left(\frac{a}{d}\right)t.$$
Now,
$$ax'+by' = a\left[x_0 + \left(\frac{b}{d}\right)t\right] + b\left[y_0 - \left(\frac{a}{d}\right)t\right]$$
$$= (ax_0+by_0) + \left(\frac{ab}{d} - \frac{ab}{d}\right)t.$$
$$= c + 0 \cdot t$$
$$= c.$$
$\Rightarrow x'$ & y' satisfy Diophantine equation, regardless of the choice of integer 't';.
Thus, there are an infinite number of solutions of the given equation, one for each value of t.

Ex Consider the linear Diophantine equation
$$172x + 20y = 1000$$
Soln. Apply Euclidean's Algorithm to the evaluation of gcd (172, 20), we get —

$$172 = 8\cdot 20 + 12$$
$$20 = 1\cdot 12 + 8$$
$$12 = 1\cdot 8 + 4$$
$$8 = 2\cdot 4.$$

$\therefore \gcd(172, 20) = 4.$

Since $4 | 1000$, a solution to this equation exists.

Now, to get integer 4 as a l.c of 172 & 20, we proceed from backward calculations:

$$\begin{aligned} 4 &= 12 - 8 \\ &= 12 - (20 - 12) \\ &= 2\cdot 12 - 20 \\ &= 2(172 - 8\cdot 20) - 20 \\ &= 2\cdot 172 + (-17)20. \end{aligned}$$

Now, multiply by 250, –

$$\begin{aligned} 1000 = 250\cdot 4 &= 250[2\cdot 172 + (-17)20] \\ &= 500\cdot 172 + (-4250)20 \end{aligned}$$

so that $x = 500$ and $y = -4250$ gives one solution to the Diophantine equation.

All other solutions are expressed by

$$x = 500 + \left(\frac{20}{4}\right)t = 500 + 5t$$
$$y = -4250 - \left(\frac{172}{4}\right)t = -4250 - 43t., \text{ for some integer } 't'.$$

<u>To get the solution in positive integers, if exist.</u>

t must be chosen to satisfy simultaneously the inequalities $5t + 500 > 0$ $\quad -43t - 4250 > 0$

or, $\quad -98\frac{36}{43} > t > -100.$

Since t is an integer, we conclude that $t=-99$.
Thus, our Diophantine equation has a unique positive solution $x=5$, $y=7$ corresponding to value $t=-99$.

<u>Corollary</u>: If $\gcd(a,b)=1$ and if x_0, y_0 is a particular solution of the linear Diophantine equation $ax+by=c$, then all solutions are given by

$$x=x_0+bt, \quad y=y_0-at.$$

for integral values of t.

eg:- $5x+22y=18$ has $x_0=8$, $y_0=-1$ as one solution; By the corollary, a complete solution is given by

$$x=8+22t, \quad y=-1-5t \text{ for arbitrary } t.$$

Eg A customer bought a dozen pieces of fruit, apples and oranges, for \$1.32. If an apple costs 3 cents more than an orange and more apples than oranges were purchased, how many pieces of each kind were bought?

Soln:- <u>Set up the problem as a Diophantine equation</u>

Let x = no. of apples purchased
y = no. of oranges purchased.

& in addition, let
z = the cost (in cents) of an orange.

Then the conditions lead to.

$$(z+3)x+zy=132$$

$$\Rightarrow 3x+(x+y)z=132.$$

Now, $x+y=12$ gives $3x+12z=132$.

$$\Rightarrow x+4z=44.$$

To find integers x and z satisfying the Diophantine equation

$$x + 4z = 44 \longrightarrow (1).$$

Now, $\gcd(1,4) = 1$ & $1 \mid 44$

$\Rightarrow$ there is a sol^n to (1).

Now $1 = 1(-3) + 4 \cdot 1$.

$\Rightarrow 44 = 1(-132) + 4 \cdot 44$ (Multiply by 44)

It follows that $x_0 = -132$ and $z_0 = 44$ serves as one sol^n.

All other solutions of (1) are of the form

$$x = -132 + 4t, \quad z = 44 - t, \quad t \text{ an integer}.$$

But not for all 't', we have solutions to the problem.

Only values of t that ensure $12 \geq x > 6$ should be considered.

So, $12 \geq -132 + 4t > 6$.

Now, $12 > -132 + 4t$ imply $t \leq 36$.

whereas $-132 + 4t > 6$ imply $t > 34\frac{2}{4}$.

$\frac{138}{4}$ $\quad 4)138(34$ $\quad 34\frac{2}{4}$

Only integral values of t to satisfy both inequalities are $t = 35$ and $t = 36$.

Thus there are two possible purchases:

(a) a dozen apples costing 11 cents a piece ($t = 36$)

(b) 8 apples at 12 cents each and 4 oranges at 9 cents each. ($t = 35$).