The **girth** of a graph is the length of a shortest cycle, if any, in G.

The **circumference** of a graph G is the length of any longest cycle, if any, in G.

For example, the girth of G in Fig. 6.13 is 3 and the circumference of G is 5.

Directed trails, directed paths, and directed cycles are directed walks that satisfy conditions analogous to those of their undirected counterparts. For example, a directed trail is a directed walk in which no directed edge appears more than once.

The connectedness of a graph can be defined in terms of paths.

A graph G is **connected** if every pair of vertices are joined by a path.

A maximal connected subgraph of G is called a **connected component** or a component of G. Thus a **disconnected** graph has atleast two components. For example, the graph G in Fig. 6.13 is a connected graph and graph H is a disconnected graph and has 4 components.

A digraph is **connected** if its underlying graph is connected.

The **distance** between two vertices u and v in a graph, denoted $d(u, v)$, is the length of a shortest path joining them, if any. If there is no such path, we write $(u, v) = \infty$, *i.e.*, u and v are in different components of a disconnected graph.

The shortest $u - v$ path is usually called a **geodesic** and the length of any longest geodesic in a connected graph G is the **diameter** of G. For example, the diameter of G in Fig. 6.13 is 4.

Theorem 6.5 *A graph g is bipartite if and only if it has no cycles of odd length.*

Proof. Let G be bipartite and its vertex set has a bipartition such that each edge of G has one endvertex in one partition and other endvertex in another partition. So it requires an even number of steps (edge traversals) for a walk to return to the side from which it started. Thus, a cycle must have even length.

For the converse, let G be a graph with no cycles of odd length, and suppose that G is connected, for otherwise we can consider the components of G separately. Take any vertex u of G, and define a partition (V_1, V_2) of the vertex set $V(G)$ as follows,

$$V_1 = \{x \in G \mid d(u, x) \text{ is even}\}$$

$$V_2 = \{y \in G \mid d(u, y) \text{ is odd}\}.$$

Now (V_1, V_2) is a bipartition of $V(G)$ such that every edge of G joins a vertex of V_1 with a vertex of V_2, since all cycles of G are even. If (V_1, V_2) is not such a bipartition then suppose there is an edge wv joining two vertices w and v of V_1(or V_2). Then the geodesics from u to w and from u to v together with edge wv contains an odd cycle, contradicting the hypothesis. Hence (V_1, V_2) is a bipartition and G is a bipartite graph. □

Theorem 6.6 *Let G be a simple graph with n vertices and k components. Then it can have atmost* $\frac{1}{2}(n-k)(n-k+1)$ *edges*

Proof. Let n_i, $i = 1, 2, \ldots, k$, be number of vertices in each component of G, so

$$n_1 + n_2 + \ldots + n_k = n,$$

and $$n_i \geq 1, \quad i = 1, 2, \ldots, n. \qquad \ldots(1)$$

Now the maximum number of edges in the ith component of G is

$$^{n_i}C_2 = \frac{1}{2} n_i (n_i - 1)$$

Hence, the maximum number of edges in G is

$$\frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1) = \frac{1}{2} \Sigma n_i^2 - \frac{1}{2} \Sigma n_i$$

$$= \frac{1}{2} \Sigma n_i^2 - \frac{n}{2} \qquad \ldots(2)$$

Now, we have $$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$$\Rightarrow \qquad \left(\sum_{i=1}^{k} (n_i - 1) \right)^2 = (n-k)^2$$

$$\Rightarrow \qquad \sum_{i=1}^{k} (n_i^2 - 2n_i) + k + \text{nonnegative terms} = (n-k)^2, \text{ since } (n_i - 1) \geq 0, \text{ for all } i$$

So, $$\sum_{i=1}^{k} n_i^2 + \text{nonnegative terms} = (n-k)^2 + 2n - k$$

i.e., $$\sum_{i=1}^{k} n_i^2 \leq (n-k)^2 + 2n - k \qquad \ldots(3)$$

Hence from (2) and (3),

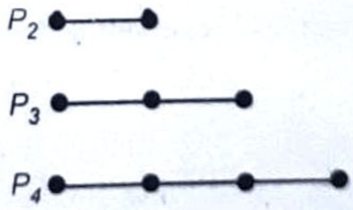
$$\frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1) \leq \frac{1}{2} [(n-k)^2 + 2n - k] - \frac{n}{2}$$

$$= \frac{1}{2} (n-k)(n-k+1).$$

□

More Graphs

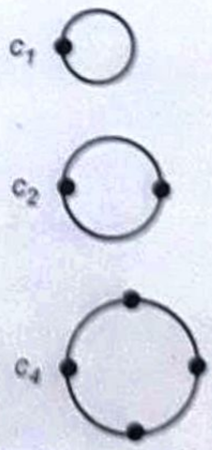
Path Graphs



A **path graph** P is a simple connected graph with $|V(P)| = |E(P)| + 1$.

An n-vertex path graph is denoted by P_n. For example,

Cycle Graphs



A **cycle graph** is a single vertex with a self-loop or a simple connected graph C with $|V(C)| = |E(C)|$ that can be drawn so that all of its vertices and edges lie on a circle.

An n-vertex cycle graph is denoted by C_n. For example,

Hypercube Graphs

The **hypercube graph** Q_n is the n-regular graph whose vertex set is the set of bit strings of length n, and such that two vertices of Q_n are adjacent if any only if their binary representations differ at exactly one place, that is they differ in exactly one bit.

For example, hyper cube Q_3 is shown in Fig. 6.5.

Circular Ladder Graphs

The **circular ladder graph** CL_n is represented as two concentric n-cycles in which each of the n pairs of corresponding vertices is joined by an edge.

For example, the circular ladder graph CL_4 is shown in Fig. 6.5.

6.8 Operations on Graphs

A graph is often regarded as a variable by computer scientists and adding and deleting a vertex or edge from a graph G are considered to be operations which give new value to G. It is sometimes convenient to be able to express the structure of a given graph in terms of smaller and simpler graphs.

Deleting vertices or Edges

If v_i is a vertex of a graph G, then **deletion** or **removal of v_i** from G results in a sub graph, denoted $G - v_i$, consisting of all vertices of G except v_i and all edges not incident with v_i. That is, $G - v_i$ is the sub graph of G induced by the vertex set $V(G) - \{v_i\}$.

On the other hand, the **deletion** or **removal of an edge e_i** from G results in the spanning sub graph $G - e_i$ consisting of all edges of G except e_i.

Adding vertices of Edges

Adding a vertex v_i to a graph G, where v_i is a new vertex not already in $V(G)$, means creating a supergraph, denoted $G \cup \{v_i\}$, with vertex set $V(G) \cup \{v_i\}$ and edge set $E(G)$.

On the other hand, if v_i and v_j are not adjacent in G, the **addition of edge** v_iv_j results in the smallest supergraph of G containing the edge v_iv_j, we denote $G + v_iv_j$.

Fusion

Let G be a graph and v_i and v_j be two distinct vertices of G. Then a new graph G_1 can be constructed by **fusing** or **identifying** these two vertices and replacing them by a single new vertex u such that every edge that was incident with either v_i or v_j in G is now incident with u in G_1, *i.e.*, the end v_i and the end v_j become end u.

Suspension of a Graph

Let G be a graph and v be any new vertex not in $V(G)$. If this vertex v is joined to each of the existing vertices of G, then the resulting graph is called the join of G and v or the suspension of G from v, and is denoted by $G + v$.

Above operations are illustrated in Fig. 6.14.

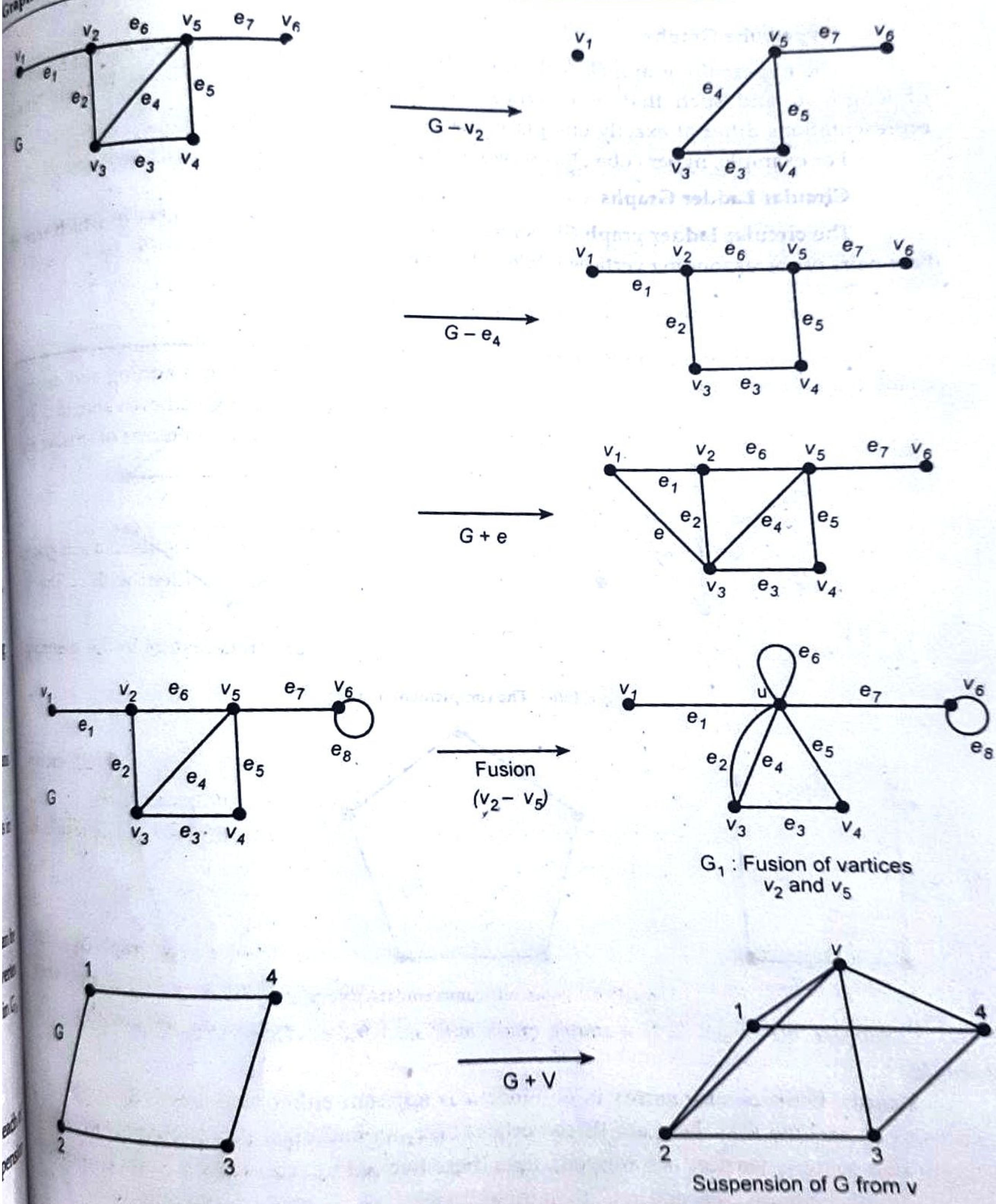
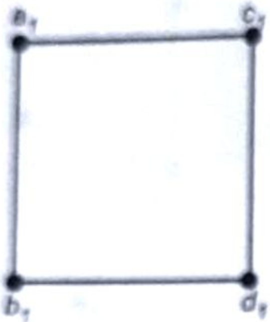


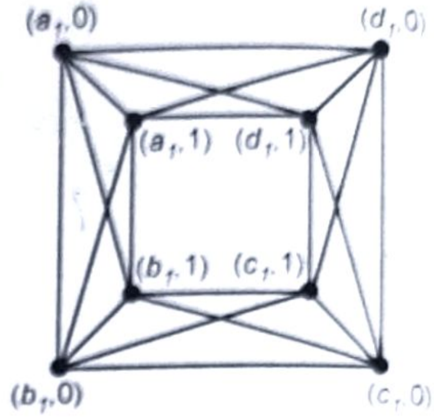
Fig. 6.14 : Some graph operations

For example,



C_4

k_2



$C_4[k_2]$

6.9 Matrix Representation of Graphs

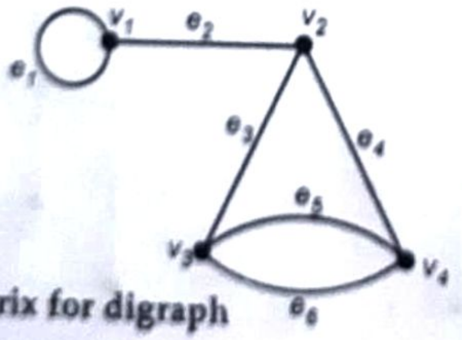
There are two ways of representing a graph in computer application, namely by using the adjacency matrix or the incidence matrix of a graph.

Incidence matrix

Suppose that a graph G has n vertices, as $v_1, v_2, \ldots, v_n$ and k edges as $e_1, e_2, \ldots, e_k$. The incidence matrix of G is the $n \times k$ matrix $I(G) = (a_{ij})$ such that

$$\begin{aligned} a_{ij} &= 1, \text{ if } e_j \text{ in incident on } v_i. \\ &= 0, \text{ if } v_i \text{ is not an end of } e_j, \\ &= 2, \text{ if } e_j \text{ is a self-loop at } v_i. \end{aligned}$$

Observe that changing the orderings of vertices and edges permutes the rows and columns of $I(G)$. Also, it is immediate from the definition of the incidence matrix that the sum of the entries in any row of an incidence matrix is the degree of the corresponding vertex, and the sum of the entries in any column is equal to 2. For example,



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

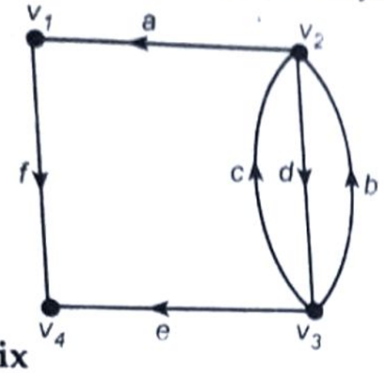
Incidence matrix for digraph

The incidence matrix, for a digraph G with n vertices and k edges, is analogously defined as $I(G) = (a_{ij})$, such that

$$\begin{aligned} a_{ij} &= 1, \text{ if } e_j \text{ is incident out of } v_i, \\ &= -1, \text{ if } e_j \text{ is incident into } v_i, \\ &= 0, \text{ if } v_i \text{ is not an endpoint of } e_j, \\ &= 2, \text{ if } e_j \text{ is a self-loop at } v_i \end{aligned}$$

Using a "2" to indicate a self-loop in a digraph is some what artificial. Sometimes the incidence matrix is defined for loopless digraph only.

For example,

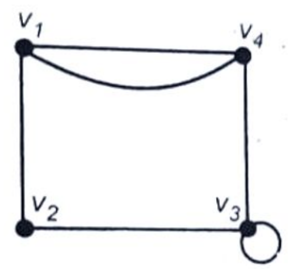


$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Adjacency matrix

Let G be a graph with n vertices. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where the (i, j)th entry a_{ij} is the number of edges joining the vertex v_i to the vertex v_j.

For example,



$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Note that if G has no loops then all the entries of the main diagonal of $A(G)$ are 0, and $a_{ij} = a_{ji}$ for each i and j, in $A(G)$.

So $A(G)$ is a symmetric matrix.

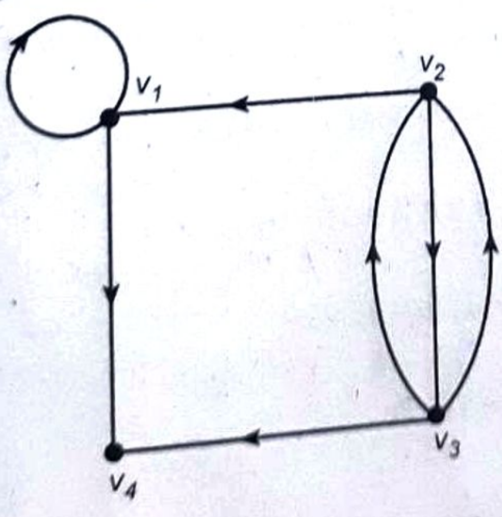
Adjacency matrix for digraph

Let G be a digraph with n vertices. Then the adjacency matrix $A(G) = (a_{ij})$ is an $n \times n$ matrix such that

$$a_{ij} = \text{the number of edges directed from } i\text{th vertex to } j\text{th vertex, and}$$
$$= 0, \text{ otherwise.}$$

If it has no parallel edges then it is (0, 1)-matrix.

For example,



$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step 2. Set $i = 1$.

Step 3. Set $j = i + 2 = 1 + 2 = 3$.

Step 4. Set $C_{13} : (v_1v_3v_2v_4v_5v_6v_1) = (1523461)$, so that

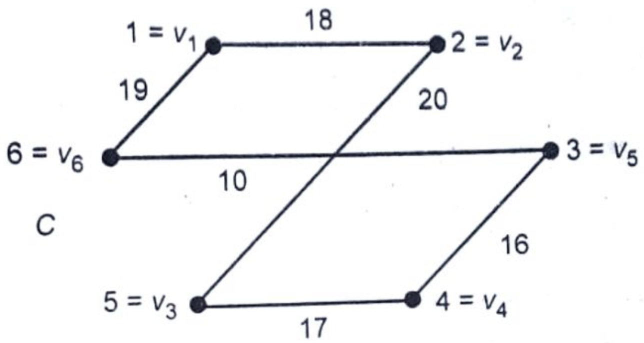
$$w_{13} = 12 + 20 + 27 + 16 + 19 + 19 = 113.$$

In a similar way we can check that the next change occurs when $i = 3$ and $j = 5$, so that

$$C_{ij} : C_{35} : (v_1v_2v_3v_4v_5V_6v_1) = (1254361), \text{ so that}$$

$$w_{ij} = w_{35} = 18 + 20 + 17 + 16 + 10 + 19 = 100$$

that gives a new cycle C as shown in figure :



Optimal Hamiltonian cycle C.

At this stage, we stop the algorithm since we observe that now there is no further modification and thus the best possible Hamiltonian cycle is obtained. □

6.12 Planar Graphs

Plane and Planar Graphs

A **planar graph** is a graph drawn in the plane such that no two of its edges intersect, *i.e.*, any pair of edges meet only at their end vertices.

A **planar graph** is a graph which is isomorphic to a plane graph, *i.e.*, it can b drawn or redrawn as a plane graph without a crossover between its edges.

A graph that cannot be drawn on a plane without a crossover between its edges is called **nonplanar.**

We shall use the Jordan Curve Theorem to prove that there are nonplanar graphs.

A **Jordan Curve** is a continuous nonself intersecting curve in the plane starting and terminal points coincide. It is a closed curve.

A Jordan curve in the plane separates it into regions, one interior region and other exterior. The **Jordan Curve Theorem** states that if x is a point in the interior region and y is a point in the exterior, then any line joining x to y must intersect the Jordan curve at some point.

Some planar and nonplanar graphs are shown in Fig. 6.32. First five G_1, G_2, G_3, G_4, G_5 are all planar graphs. G_1 and G_2 are not plane graphs. G_6 and G_7 are nonplanar.

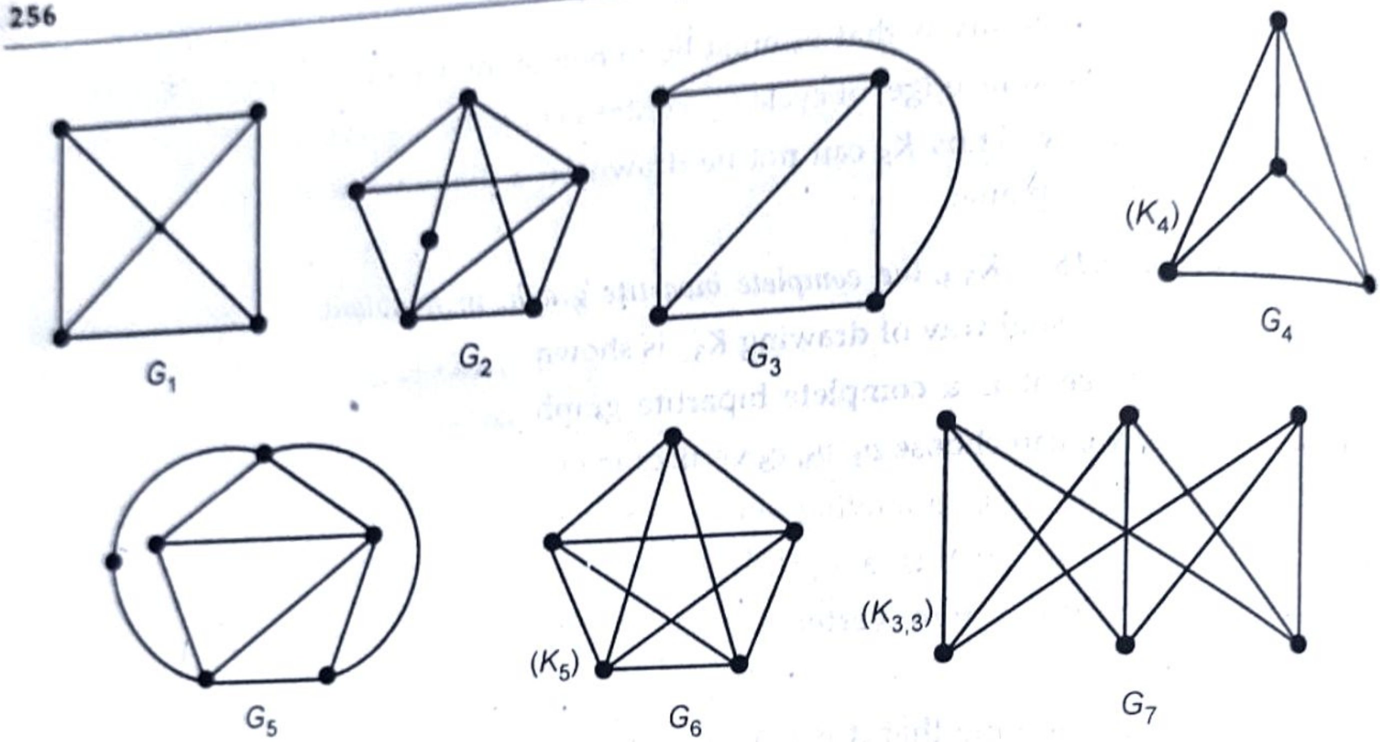
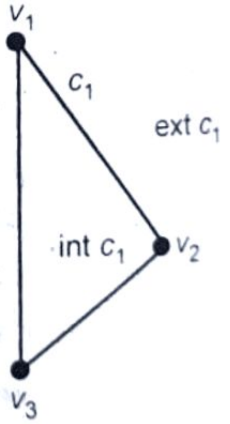
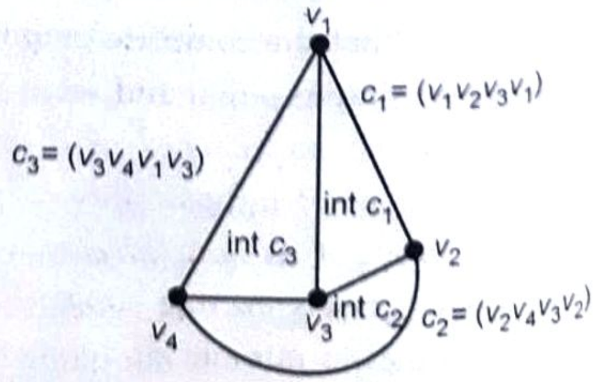


Fig. 6.32 : Some planar and nonplanar graphs

Theorem 6.19 *K_5,the complete graph on five vertices, is nonplanar.*



Proof. The usual way of drawing K_5 is shown in Fig. 6.32. Assume that K_5 is planar and we shall derive a contradiction from this assumption. Let us denote the vertices of K_5 by v_1, v_2, v_3, v_4, v_5. Since K_5 is a complete graph, any pair of distinct vertices are joined by an edge. So there is a cycle joining v_1, v_2 and v_3. Let $C_1 = v_1v_2v_3v_1$ be a cycle in K_5, which forms a Jordan curve in the plane when drawing it. This closed curve separates the plane into two regions, interior of C_1 and exterior of C_1 by Jordan Curve Theorem.



Now, obviously, v_4 does not lie on C_1, so it must lie in int. C_1 or in ext. C_1. Let us suppose that v_4 is in ext. C_1 (the other possibility has a similar argument). Then v_4 can be joined to v_1, v_2 and v_3 in the nonintersecting way, forming two more cycles in the plane as $C_2 = v_2v_4v_3v_2$ and $C_3 = v_3v_4v_1v_3$.

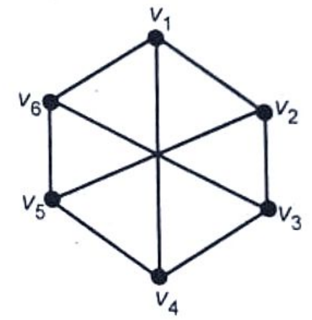
Now, the remaining vertex v_5 must lie in one of the four regions, int. C_1, int. C_2, int. C_3 and ext. C, where $C = v_1v_2v_4v_1$.

If v_5 is in ext. C then, because v_3 is in int. C, the edge joining v_5 and v_3 must intersect the curve C at some point, by Jordan Curve Theorem, and therefore the edge v_3v_5 must crossover one of the three edges, v_1v_4, v_1v_2 and v_2v_4. This contradicts our assumption.

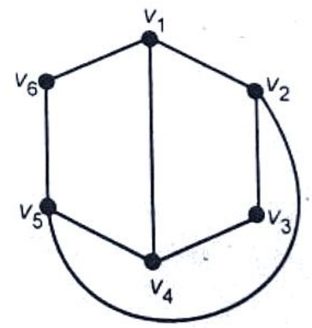
The other possibility is that v_5 must lie in one of int. C_1, int. C_2, int. C_3. If v_5 lies in int. C_1 then v_5v_4 intersects some edge of cycle C_1, contradicting the assumption. The two other cases being treated similarly. Thus K_5 can not be drawn on a plane without crossover between its edges. Hence K_5 is nonplanar. □

Theorem 6.18 *$K_{3,3}$, the complete bipartite graph, in nonplanar.*

Proof. The usual way of drawing $K_{3,3}$ is shown in Fig. 6.32, or since it is a complete bipartite graph with six vertices, we can choose v_1, v_3, v_5 vertices in one vertex subset and v_2, v_4, v_6 in another vertex subset, so that $C = (v_1v_2v_3v_4v_5v_6v_1)$ is a cycle whose edges running between these two vertex subsets. So its alternate drawing is.



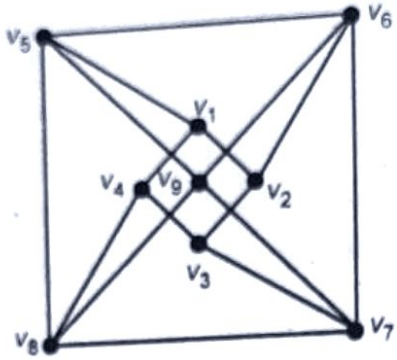
Now, if we assume that it is planar then it has a plane drawing without cross over between its edges. So first draw a cycle C, joining v_1 to v_2, v_2 to v_3, v_3 to v_4, v_4 to v_5, v_5 to v_6 and v_6 to v_1. It separates the plane by Jordan Curve Theorem into two regions, int. C and ext. C. So the edge v_1v_4 can be drawn either in inside region or in outside region. Suppose we join v_1 to v_4 in int. C, then v_2 to v_5 can be joined from outside only in a nonintersecting way.



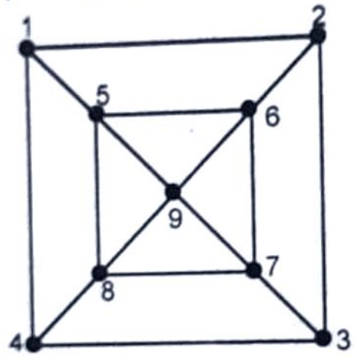
Now, we have to join v_3 to v_6. If we join in int C then the edge v_3v_6 intersects v_1v_4. On the other hand if we join v_3 to v_6 from outside then the edge v_3v_6 intersects v_2v_5. Thus $K_{3,3}$ can not be drawn on a plane without crossover between its edges. Hence $K_{3,3}$ is nonplanar. □

Note that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are called the **Kuratowski graphs**—first and second respectively. We have seen that a systematic way of all the conceivable approaches of drawing these graphs without edge-crossings leads to a geometric proof of impossibility. However, for each of the Kuratowski graphs there is an algebraic proof of its nonplanarity, which follows from the Euler polyhedral equation. Both Kuratowski graphs are the simplest nonplanar graphs, where the first graph is nonplanar with the smallest number of vertices, and the second graph is nonplanar with the smallest number of edges. It is clear that any supergraphs of a nonplanar graph is also nonplanar, and any subgraph of a planar graph is planar while proving Theorem 6.17 and 6.18, we have seen that it e is any edge of K_5 then $K_5 - e$ is planar, and similarly if e is any edge of $K_{3,3}$ then $K_{3,3} - e$ is planar.

Example 6.20 *The following graph,*

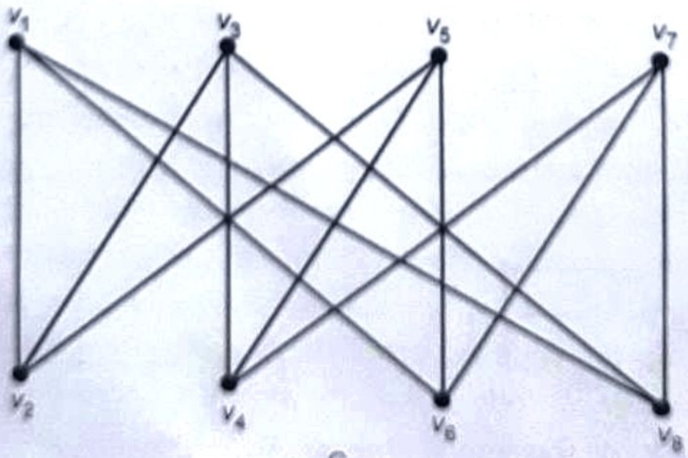


is planar since it is isomorphic to the plane graph,



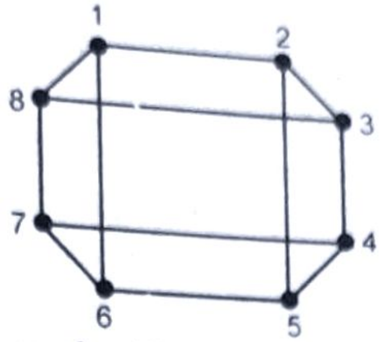
without edge-crossings. □

Example 6.21 *By redrawing as plane graph show that the following graph* G *is planar.*

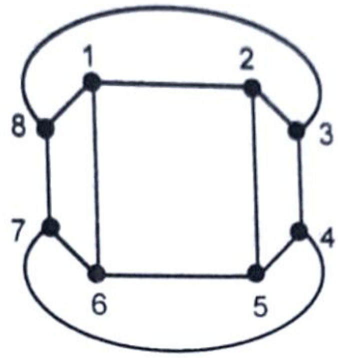


G

Sol. *G* is isomorphic to the following graph :



It can be redrawn as a plane graph without edge-crossings as follows :

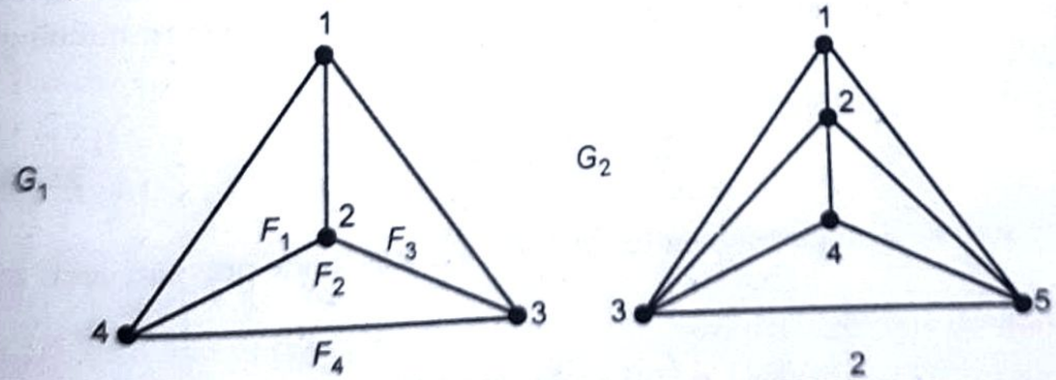


Hence the graph G is planar. □

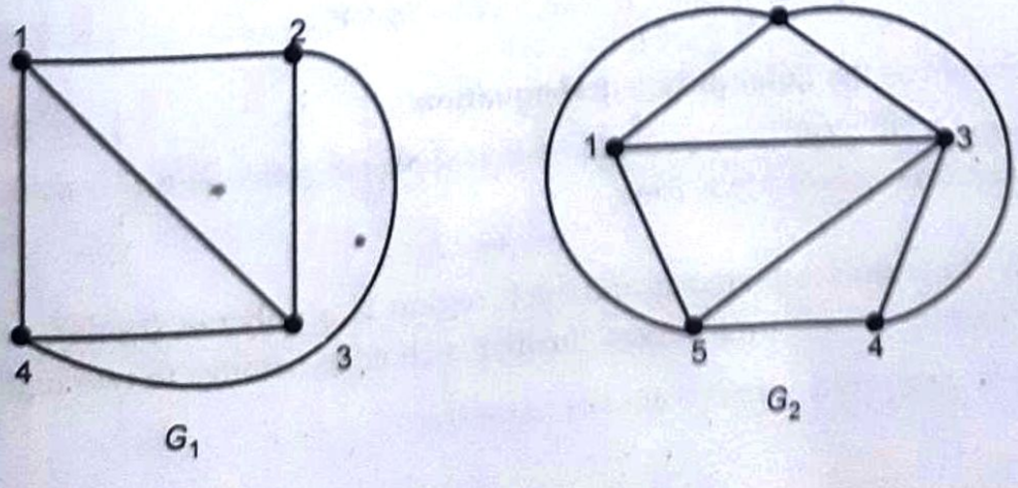
Euler Polyhedral Equation

A drawing of a graph on any surface such that no edges intersect is called **embedding**. Thus a graph G is **planar** if there exists a graph isomorphic to G that is embedded in a plane.

Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment. For example, the straight line representations of the following graphs,



are drawn as



A plane representation of a planar graph (drawn without edge-crossings) divides the plane into regions (faces) as shown above, where F_1, F_2, F_3, F_4 are regions (faces) of the graph G_1. We shall refer to the **regions** defined by a plane graph as its **faces**, specified by the set of edges forming its boundary. The boundary of a face of a plane graph is a cycle. The unbounded region is called the **exterior face** or **infinite region**. Note that region is a property of the specified plane representation of a graph. A region is not defined in a nonplanar graph. Further, a nonplanar graph remains nonplanar if a self-loop is deleted or if one edge of a multi-edge is deleted, and a planar graph remains planar if a multiple edge or self-loop is added to it.

Theorem 6.19 *Let $G = (V, E)$ be a simple connected planar graph with F regions, then* $F = |E| - |V| + 2$.

Proof. The simple planar graph G can be drawn on a plane such that each edge is a straight-line and each region is a polygon (cycle). So regions F are polygons and G is a polygon net. Let m_k be the number of k-sided regions, where $k \geq 3$ and r-sided regions are with maximum edges.

Thus total regions are

$$m_3 + m_4 + m_5 + \ldots + m_r = F \qquad \ldots(1)$$

Again, since each edge is on the boundary of exactly two regions, total edges are

$$\frac{1}{2}(3m_3 + 4m_4 + \ldots + rm_r) = |E|$$

i.e.,
$$3m_3 + 4m_4 + \ldots + rm_r = 2|E| \qquad \ldots(2)$$

Now, we know that the sum of all interior angles of a k-sided polygon is $\pi(k-2)$ and the sum of the exterior angles is $\pi(k+2)$, therefore compute the grand sum of all interior angles of $F - 1$ finite regions plus the sum of the exterior angles of the polygon defining the infinite region, as follows :

$$\pi(3-2)m_3 + \pi(4-2)m_4 + \ldots + \pi(r-2)m_r + 4\pi$$
$$= \pi(2|E| - 2F) + 4\pi, \qquad \text{[using (1) and (2)]}$$
$$= 2\pi(|E| - F) + 4\pi \qquad \ldots(3)$$

Also, the sum of all angles subtended at $|V|$ vertices in the polygon net G is given by

$$2\pi|V| \qquad \ldots(4)$$

Thus, from (3) and (4), we have

$$2\pi(|E| - F) + 4\pi + 2\pi|V|$$

i.e.,
$$F = |E| - |V| + 2$$

It is known as the **Euler polyhedral equation.** □

Theorem 6.20 *Let G be a simple connected planar graph with F regions, $|E| = e$ edges, and $|V| = n$ vertices, where $n \geq 3$. Then*

$$e \leq 3n - 6.$$

Proof. In a simple planar graph each region is a polygon (cycle) and therefore each region is bounded by at least three edges. Further each edge belongs to two regions of the planar graph. So

$$2e \geq 3F \quad .....(1)$$

The result (1) is also known as the **Edge-Face in equality.**

Now, using Euler polyhedral equation

$$F = e - n + 2$$

in (1) and simplifying, we obtain

$$2e \geq 3(e - n + 2)$$

i.e.,

$$2e \geq 3e + (6 - 3n)$$

Hence

$$e \leq 3n - 6.$$

□

Corollary

Let G be any connected simple graph such that

$$e > 3n - 6.$$

Then G is nonplanar graph.

Proof. The above Theorem 6.20 is a necessary condition for planar graph. So if it is not satisfied by any graph, that is $e > 3n - 6$, then the graph must be nonplanar. □

For example, the complete graph K_5 is nonplanar, since it has $|E| = e = 10$, $|V| = n = 5$. So $3n - 6 = 9$ and $e = 10$ and thus the inequality $e > 3n - 6$ is satisfied.

Example 6.22 *Let G be any simple connected graph with 8 vertices and $|E| \geq 19$. Then G is nonplanar.*

Sol. We have $|V| = n = 8$ and $|E| = e \geq 19$

So that $\quad 3n - 6 = 3(8) - 6 = 18.$

Thus, $\quad e \geq 19$ implies that

$$e > 3n - 6.$$

Hence by the above corollary, G is nonplanar. □

Note that, the inequality in Theorem 6.20 is not a sufficient condition for the planarity of a graph. For example, the complete bipartite graph $K_{3,3}$ does satisfy this inequality, yet the graph $K_{3,3}$ is nonplanar.

Nonplanarity of Bipartite Graphs

For a bipartite graph, the same algebraic method used in Theorem 6.20 yields a corresponding inequality by making use of the fact that no region in a bipartite graph can be bounded with fewer than four edges.

Theorem 6.21 *Let G be a connected bipartite simple planar graph. Then*

$$e \leq 2n - 4,$$

where $|E| = e$ and $|V| = n$.

Proof. In a bipartite graph each region is bounded by at least four edges. So the corresponding Face-edge inequality becomes

$$2e \geq 4F$$

Then using Euler polyhedral equation we obtain

$$2e \geq 4(e - n + 2)$$

i.e.,

$$e \leq 2n - 4.$$

Corollary

Let G be a connected bipartite simple graph such that

$$e > 2n - 4.$$

Then G is nonplanar.

Proof. The proof is obvious because the inequality in Theorem 6.21 is a necessary condition for a bipartite planar graph.

For example, the complete bipartite graph $K_{3,3}$ is nonplanar since $2 \mid V \mid - 4 = 8$ and $\mid E \mid = 9$, that is the inequality $e > 2n - 4$ is satisfied. □

Example 6.23 *Let G be a connected simple planar graph with n vertices and e edges, embedded in a plane. If each region of G is bounded by exactly r edges, then*

$$e = \frac{r(n-2)}{(r-2)}.$$

Sol. Since each region of G is bounded exactly by r edges and each edge of G belongs to exactly two regions, we have

$$2e = rf \qquad ....(1)$$

But Euler polyhedral equation for planar graph with f regions, gives

$$f = e - n + 2 \qquad ....(2)$$

Thus form (1) and (2)

$$2e = r(e - n + 2)$$

i.e.,

$$e = \frac{r(n-2)}{(r-2)}.$$

□

Theorem 6.22 *A simple planar graph must have at least one vertex with degree five or less.*

Proof. The result is obviously true if the simple planar graph G has vertices 1, 2, 3, 4, 5 or 6. So let vertices $n > 6$. Let edges in G be e, then total $2e$ degrees is divided among n vertices. Therefore, there must be at least one vertex in G whose degree is equal to or less than the number $2e/n$.

But G is simple planar graph, so by Theorem 6.20,

$$e \leq 3n - 6$$

i.e.,

$$\frac{2e}{n} \leq \frac{2}{n}(3n - 6)$$

i.e., the number

$$\left[\frac{2e}{n}\right] \leq 6 - \frac{12}{n}$$

is always less than or equal to 5.

Hence the planar graph G must have at least one vertex with degree five or less. □

Subdivision and Homeomorphism

Subdividing an edge in a graph G is an operation that inserts a new vertex of degree two into the edge, thereby creating edges in series.