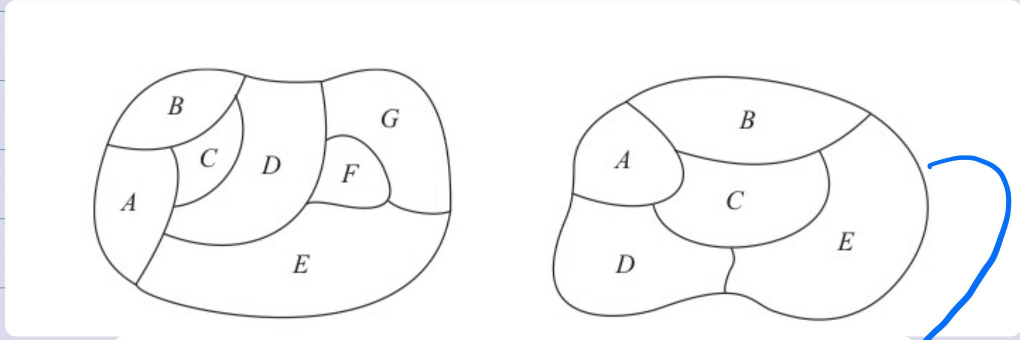
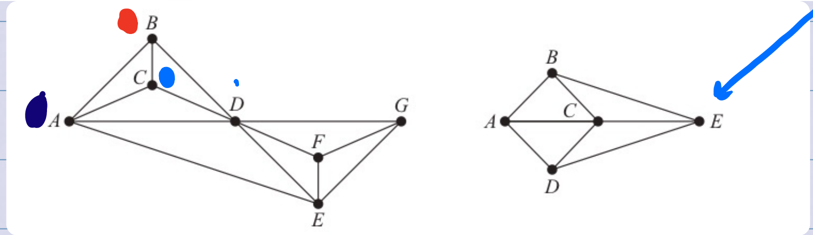
Graph Coloring





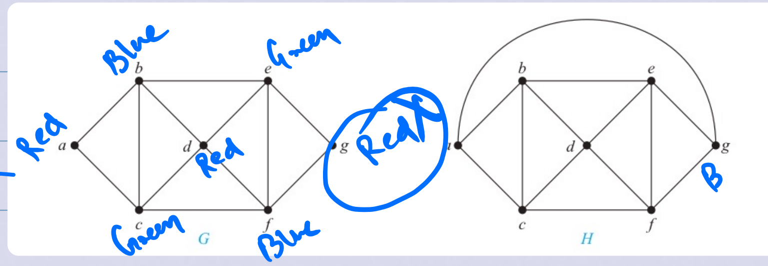
A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

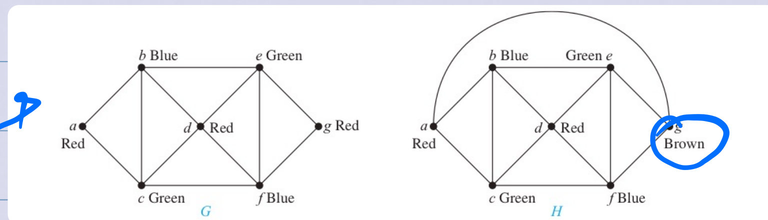
A graph can be colored by assigning a different color to each of its vertices. However, for most graphs a coloring can be found that uses fewer colors than the number of vertices in the graph. What is the least number of colors necessary?

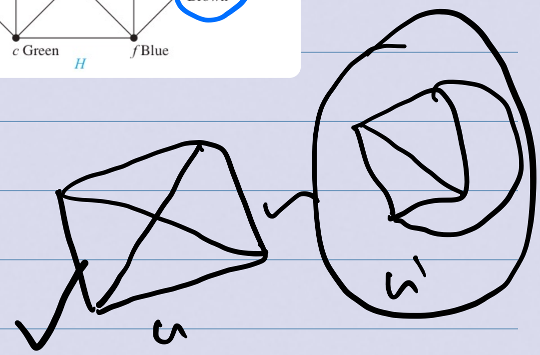
The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by $\chi(G)$. (Here χ is the Greek letter *chi*.)

$\boxed{\chi(G)}$

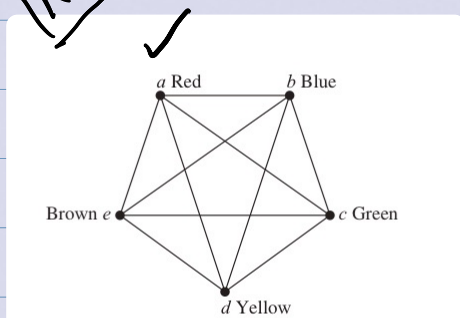
THE FOUR COLOR THEOREM The chromatic number of a planar graph is no greater than four.



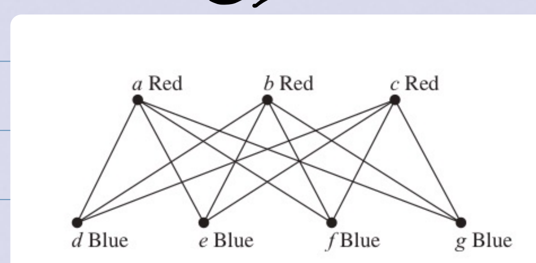




K5



K3,4

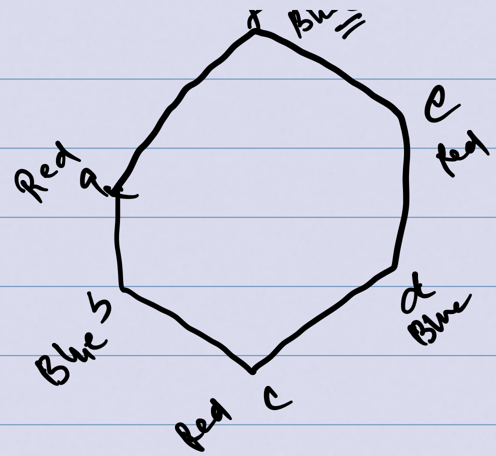


What is the chromatic number of K_n?

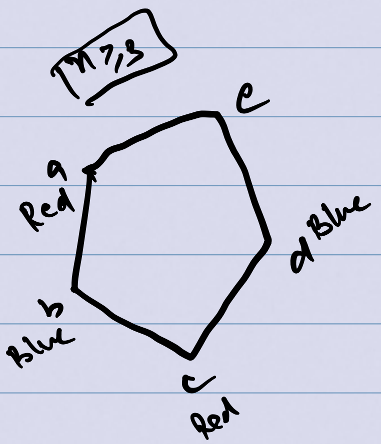
What is the chromatic number of the complete bipartite graph $K_{m,n}$, where m and n are positive integers?

$G(V, E)$

What is the chromatic number of the graph C_n, where $n \geq 3$? (Recall that C_n is the cycle with n vertices.)



f Blue
e
Red
Red a
Blue b
Red c
d
Blue



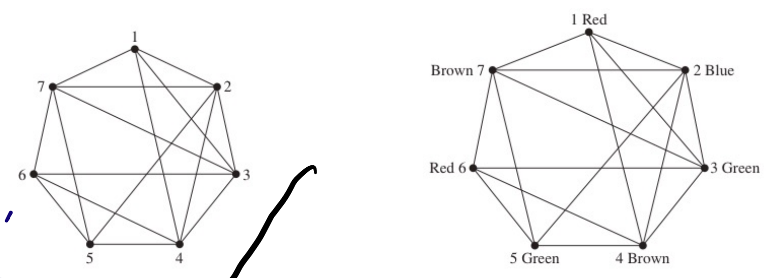
m ≥ 3
e
a
Red
d Blue
b
Blue
c
Red

Applications of Graph Colorings

Scheduling Final Exams How can the final exams at a university be scheduled so that no student has two exams at the same time?

For instance, suppose there are seven finals to be scheduled. Suppose the courses are numbered 1 through 7. Suppose that the following pairs of courses have common students: 1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, and 6 and 7. In Figure 8 the graph associated with this set of classes is shown. A scheduling consists of a coloring of this graph.

Because the chromatic number of this graph is 4 (the reader should verify this), four time slots are needed. A coloring of the graph using four colors and the associated schedule are shown in Figure 9.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Chromatic polynomial

A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of graph can be expressed by means of a polynomial. This polynomial is called as the chromatic polynomial of G and is defined as follows:

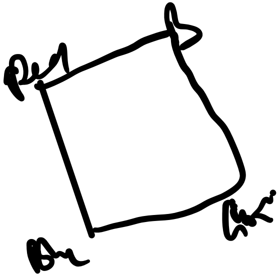
The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring G using λ or fewer colors.

$P_n(\boldsymbol{\lambda}) = The\ number\ of\ ways\ of\ proper\ coloring\ of\ a\ graph\ with\ n\ vertices, using\ \boldsymbol{\lambda}$ colors

Let c_i be the different ways of properly coloring graph G using exactly i colors.

Since i colors can be choosen out of λ colors in $\binom{\lambda}{i}$ different ways

No. of ways of proper coloring of G exactly i colors out of $\lambda = c_i \binom{\lambda}{i}$



Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic number is the sum of these terms, that is

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_2 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots$$

$$+c_n \frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)}{n!}$$

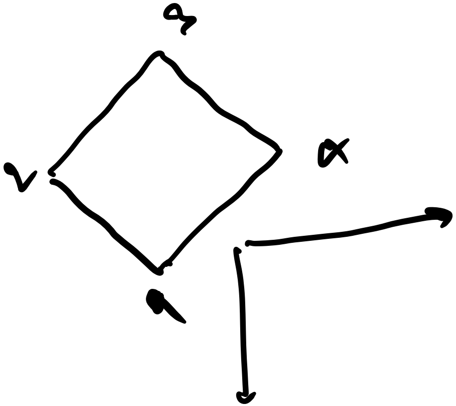
Each c_i has to be evaluated individually for the given graph.

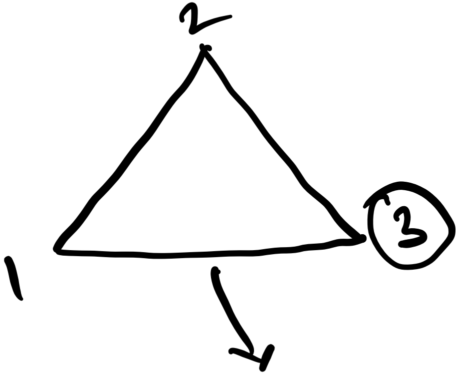
For example, any graph with one edge requires atleast two colors for proper coloring and therefore

$$c_1 = 0.$$

A graph with n vertices and using n different colors can be properly colored in $n!$ ways, i.e.,

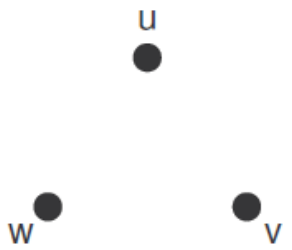
$$c_n = n!$$





13.

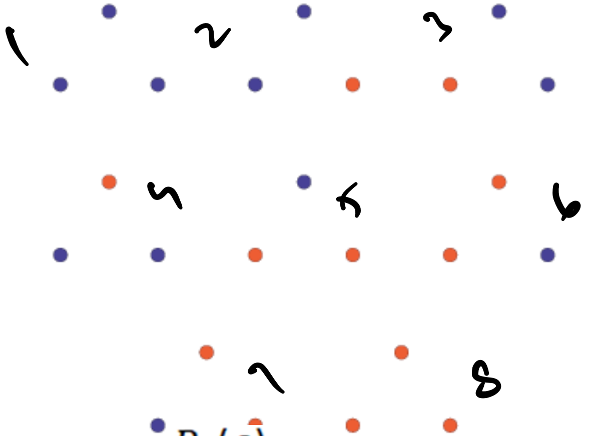
3 Vertices and 0 Edges



⋆ $\boldsymbol{\lambda}$ ways to color independently each of the vertices u, v, w.

$$P_3(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^3$$

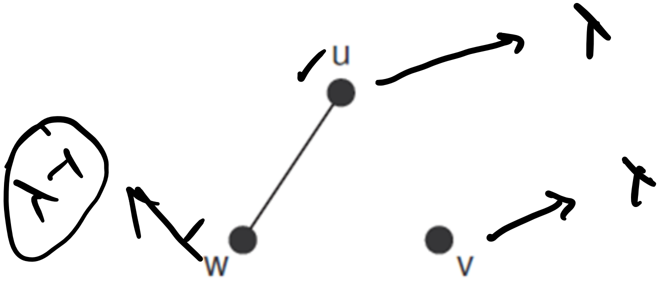
3 Vertices, 0 Edges, and 2 colors



$P_3(2)$

$$P_G(2) = 2^3 = 8$$

3 Vertices and 1 Edge

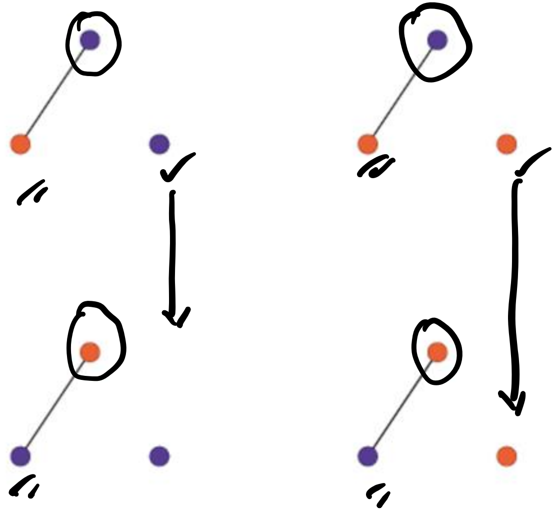


$\boxed{\lambda \times \lambda \times (\lambda - 1)}$

- ⋆ $\boldsymbol{\lambda}$ ways to color v; $\boldsymbol{\lambda}$ ways to color u; $\boldsymbol{\lambda} - 1$ ways to color w that cannot get the color of u.

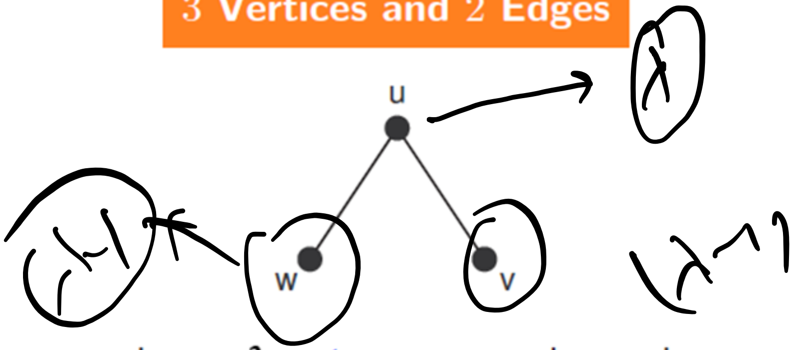
$$\begin{aligned} P_3(\boldsymbol{\lambda}) &= \boldsymbol{\lambda}^2(\boldsymbol{\lambda} - 1) \\ &= \boldsymbol{\lambda}^3 - \boldsymbol{\lambda}^2 \end{aligned}$$

3 Vertices, 1 Edge, and 2 colors



$$P_3(2) = 2^3 - 2^2 = 4$$

3 Vertices and 2 Edges



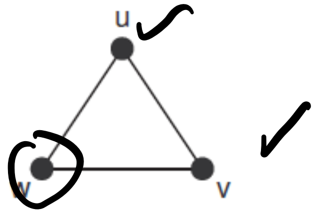
$\boldsymbol{\lambda}$ ways to color u; $\boldsymbol{\lambda} - 1$ ways to color v that cannot get the color of u; $\boldsymbol{\lambda} - 1$ ways to color w that cannot get the color of u.

$$\begin{aligned} P_3(\boldsymbol{\lambda}) &= \boldsymbol{\lambda}(\boldsymbol{\lambda} - 1)^2 \\ &= \boldsymbol{\lambda}^3 - 2\boldsymbol{\lambda}^2 + \boldsymbol{\lambda} \end{aligned}$$

3 Vertices, 2 Edges, and 2 colors

$$P_3(2) = 2^3 - 2 \cdot 2^2 + 2 = 2$$

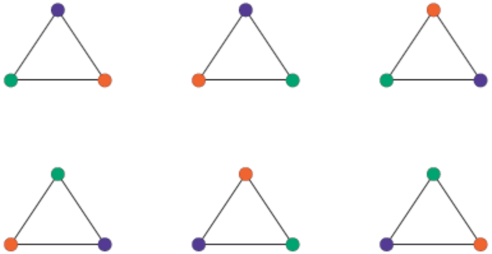
3 Vertices and 3 Edges



$\boldsymbol{\lambda}$ ways to color u; $\boldsymbol{\lambda}-1$ ways to color v that cannot get the color of u; $\boldsymbol{\lambda}-2$ ways to color w that cannot get the colors of u and v.

$$\begin{aligned} P_3(\boldsymbol{\lambda}) &= \boldsymbol{\lambda}(\boldsymbol{\lambda}-1)(\boldsymbol{\lambda}-2) \\ &= \boldsymbol{\lambda}^3 - 3\boldsymbol{\lambda}^2 + 2\boldsymbol{\lambda} \\ &= (\boldsymbol{\lambda}-1)^3 - (\boldsymbol{\lambda}-1) \end{aligned}$$

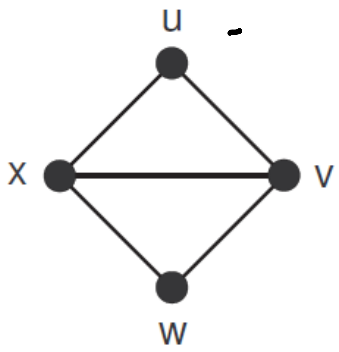
3 Vertices, 3 Edges, and 3 colors



$P_3(3) = \ 3 \cdot 2 \cdot 1 = 6$

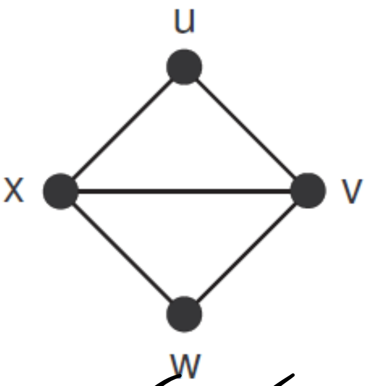
4 Vertices and 5 Edges

$u \to \lambda$
$v \to \lambda - 1$
$x \to \lambda - 2$
$w \to \lambda - 2$



- ⋆ $\boldsymbol{\lambda}$ ways to color v; $\boldsymbol{\lambda} - 1$ ways to color x that cannot get the color of v; $\boldsymbol{\lambda} - 2$ ways to color u that cannot get the colors of v and x; $\boldsymbol{\lambda} - 2$ ways to color w that cannot get the colors of v and x.

4 Vertices and 5 Edges



$$P_4(\lambda) = \lambda(\lambda-1)(\lambda-2)^2$$

$$= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda + ?$$

$|\text{Coef of } \lambda^{n-1}|$ = no. of edge

Null Graphs – N_n

- ⋆ The null graph N_n has n vertices and no edges.
- ⋆ Each vertex can be colored independently with k colors.

$$\begin{aligned} P_n(\boldsymbol{\lambda}) &= \boldsymbol{\lambda}^n \\ &= \boldsymbol{\lambda}^n - 0 \cdot \boldsymbol{\lambda}^{n-1} + 0 \cdot \boldsymbol{\lambda}^{n-2} - \cdots +- 0 \end{aligned}$$

Stars – S_n

- ⋆ The Star graph S_n has $n-1$ edges. A root vertex is connected to the rest of the $n-1$ vertices each connected only to the root.
- ⋆ The root can be colored with $\boldsymbol{\lambda}$ colors and each of the other $n-1$ vertices can be colored with $\boldsymbol{\lambda}-1$ colors.

$$\begin{aligned} P_n(\boldsymbol{\lambda}) &= \boldsymbol{\lambda}(\boldsymbol{\lambda}-1)^{n-1} \\ &= \boldsymbol{\lambda}^n - (n-1)\boldsymbol{\lambda}^{n-1} + \cdots +- 0 \end{aligned}$$

Theorem: A graph of n vertices is complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots\ldots\ldots(\lambda-(n-1))$$

$C_{ni} = 0 \ \forall \ i < n$

Proof: Let given graph is complete graph.
Choose any vertex and paint with colors. The number of ways of different coloring for the selected vertex is λ.
Now choose second vertex which is adjacent to first vertex and it can be properly colored in $(\lambda-1)$ ways.
Same way the n^{th} vertex will be coloured in $(\lambda-(n-1))$ ways if and only if every vertex is adjacent to every other. That is if graph is complete.

And chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots\ldots\ldots(\lambda-(n-1))$$

$\lambda \longrightarrow \lambda-1$

$(\lambda-2)$

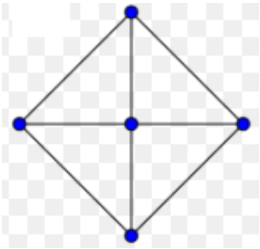
λ

Ex: Find Chromatic Polynomial of Given graph.

1.

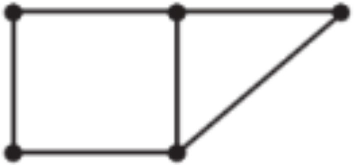
$\lambda(\lambda-1)^2(\lambda-2)$

2.



$\rightarrow \boxed{\lambda(\lambda-1)(\lambda-2)(\lambda-2)(\lambda-3)}$

3.



4.

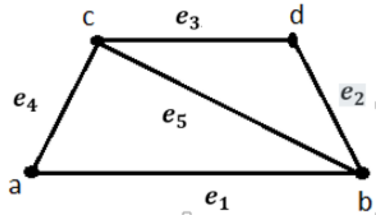
Matching:

Matching in a graph is subset of edges in which no two edges are ~~adjacent~~. incident at same vertex.

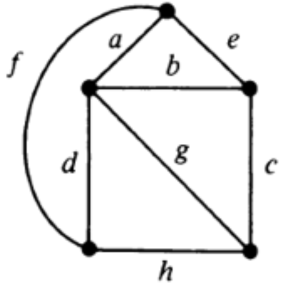
Remark: A single edge in a graph is obviously a matching.

Example:
Matching are $\{\boldsymbol{e_1}\}$, $\{\boldsymbol{e_2}\}$, $\{\boldsymbol{e_3}\}$, $\{\boldsymbol{e_4}\}$, $\{\boldsymbol{e_5}\}$,

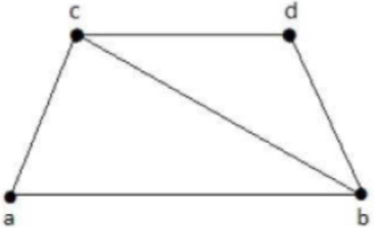
$\{\boldsymbol{e_1}, \boldsymbol{e_3}\}, \{\boldsymbol{e_2}, \boldsymbol{e_3}\}$

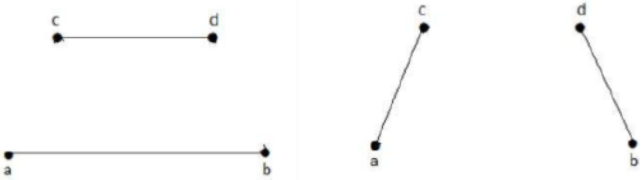


Example: In the given graph the matchings are
$M_1 = \{h, b\}, M_2 = \{a, c \quad\}, M_3 = \{e, g \qquad\}, M_4 = \{c, d\}, M_5 = \{a, c\}$ etc .



Example:



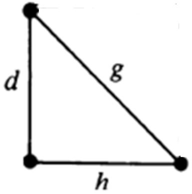


M_1 M_2

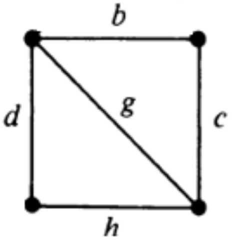
Maximal Matching:

A matching of the graph is said to be maximal in which no other edges can be added.

Ex1 : In a complete graph of three vertices any single edge is a matching.
i.e. maximal matching are $\{d\}, \{h\}, \{g\}$.



Ex 2 : In the given graph the maximal matching are $\{b, h\}, \{d, c\}, \{g\}$ is the maximal matching.

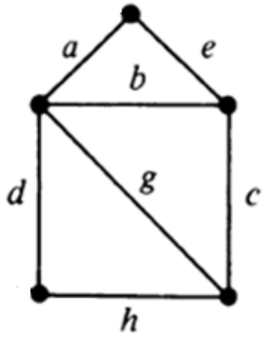


Ex 3 : Graph and its two maximal matching are

(a) (b)

Graph and two of its maximal matchings.

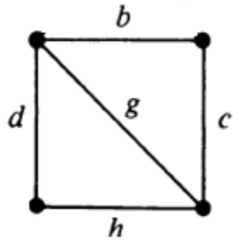
Ex: $\{h, b\}, \{d, c\}, \{d, e\}, \{c, a\}, \{g, e\}, \{h, a\}, \{h, e\}$ are maximal matchings of given graph.



Largest Maximal Matching or Maximum Matching:

A maximal matching of the graph with largest number of edges is defined as largest matching.

Ex: In the given graph the largest matchings are $\{b, h\}, \{d, c\}$.



Matching Number:

The number of edges in largest matching of G is called matching number.

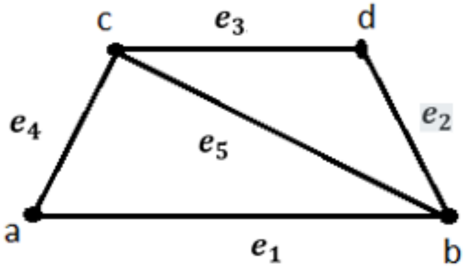
Ex: In above example the matching number is 2.

Perfect Matching or Complete Matching:

A matching M of a graph G is said to be a perfect match if every vertex of G is incident to exactly one edge of the matching M.

Example: $M_1 = \{e_1,\ e_3\}, M_2 = \{e_2, e_4\}$ are

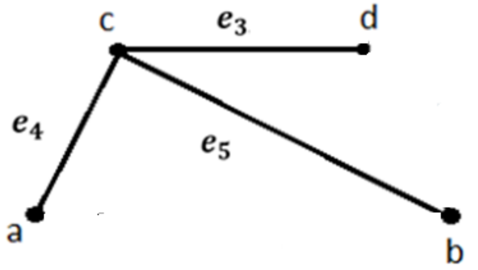
are perfect matching of G.



Note: Every perfect matching of graph is also a maximum matching of a graph, because there is no chance of adding one more edge in a perfect matching graph. But a maximum matching need not to be a perfect matching.

Note: If a graph G has perfect matching then number of vertices is always even but the converse need not be true.

$M_1 = \{e_4\}, M_2=\{e_5\}, M_3 = \{e_3\}$ are maximal matching of given graph but not perfect matching Though it has even number of vertices.

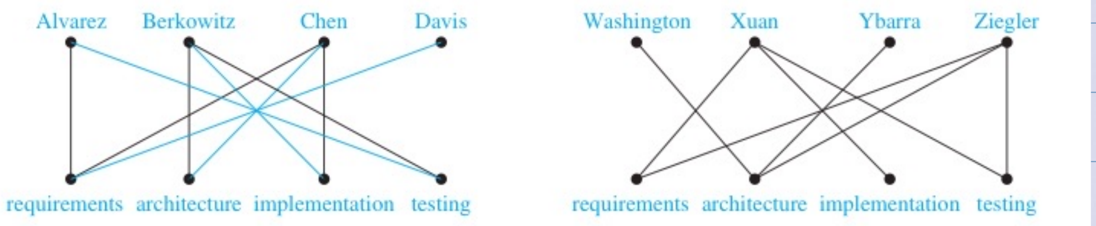


Note: If the number of vertices in the graph is odd then perfect matching does not exist but the converse need not to be true.

Matching :⇒ $deg(v_i) \leq 1 \quad \forall i$.

Perfect :⇒ $deg(v_i) = 1 \quad \forall i$

Bipartite Graphs and Matchings



To complete Project 1, we must assign an employee to each job so that every job has an employee assigned to it, and so that no employee is assigned more than one job. We can do this by assigning Alvarez to testing, Berkowitz to implementation, Chen to architecture, and Davis to requirements, as shown in Figure 10(a) (where blue lines show this assignment of jobs).

A **maximum matching** is a matching with the largest number of edges. We say that a matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a **complete matching from V_1 to V_2** if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

Marriages on an Island Suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse. A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair. A maximum matching is a largest possible set of married couples, and a complete matching of V_1 is a set of married couples where every man is married, but possibly not all women. ◀

HALL'S MARRIAGE THEOREM The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

For a subset W of X, let $N_G(W)$ denote the neighborhood of W in G, i.e., the set of all vertices in Y adjacent to some element of W. The marriage theorem in this formulation states that there is an X-perfect matching if and only if for every subset W of X:

$$|W| \leq |N_G(W)|.$$

$G=(X+Y,E)$

In other words: every subset W of X has sufficiently many adjacent vertices in Y.

Proof: We first prove the *only if* part of the theorem. To do so, suppose that there is a complete matching M from V_1 to V_2. Then, if $A \subseteq V_1$, for every vertex $v \in A$, there is an edge in M connecting v to a vertex in V_2. Consequently, there are at least as many vertices in V_2 that are neighbors of vertices in V_1 as there are vertices in V_1. It follows that $|N(A)| \geq |A|$.

To prove the *if* part of the theorem, the more difficult part, we need to show that if $|N(A)| \geq |A|$ for all $A \subseteq V_1$, then there is a complete matching M from V_1 to V_2. We will use strong induction on $|V_1|$ to prove this.

Basis step: If $|V_1| = 1$, then V_1 contains a single vertex v_0. Because $|N(\{v_0\})| \geq |\{v_0\}| = 1$, there is at least one edge connecting v_0 and a vertex $w_0 \in V_2$. Any such edge forms a complete matching from V_1 to V_2.

Inductive step: We first state the inductive hypothesis.

Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2), and $|V_1| = j \leq k$, then there is a complete matching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is met.

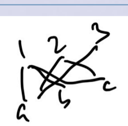
Now suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$. We will prove that the inductive holds using a proof by cases, using two case. Case *(i)* applies when for all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2. Case *(ii)* applies when for some j with $1 \leq j \leq k$ there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2. Because either Case *(i)* or Case *(ii)* holds, we need only consider these cases to complete the inductive step.

Case (i): Suppose that for all integers j with $1 \leq j \leq k$, the vertices in every subset of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2. Then, we select a vertex $v \in W_1$ and an element $w \in N(\{v\})$, which must exist by our assumption that $|N(\{v\}| \geq |\{v\}| = 1$. We delete v and w and all edges incident to them from H. This produces a bipartite graph H' with bipartition $(W_1 - \{v\}, W_2 - \{w\})$. Because $|W_1 - \{v\}| = k$, the inductive hypothesis tells us there is a complete matching from $W_1 - \{v\}$ to $W_2 - \{w\}$. Adding the edge from v to w to this complete matching produces a complete matching from W_1 to W_2.

Case (ii): Suppose that for some j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2. Let W_2' be the set of these neighbors. Then, by the inductive hypothesis there is a complete matching from W_1' to W_2'. Remove these $2j$ vertices from W_1 and W_2 and all incident edges to produce a bipartite graph K with bipartition $(W_1 - W_1', W_2 - W_2')$.

We will show that the graph K satisfies the condition $|N(A)| \geq |A|$ for all subsets A of $W_1 - W_1'$. If not, there would be a subset of t vertices of $W_1 - W_1'$ where $1 \leq t \leq k + 1 - j$ such that the vertices in this subset have fewer than t vertices of $W_2 - W_2'$ as neighbors. Then, the set of $j + t$ vertices of W_1 consisting of these t vertices together with the j vertices we removed from W_1 has fewer than $j + t$ neighbors in W_2, contradicting the hypothesis that $|N(A)| \geq |A|$ for all $A \subseteq W_1$.

1. Initialize $M := \{\}$. // Empty matching.
2. Assert: M is a matching in G.
3. If M saturates all vertices of X, then **return the X-perfect matching M.**
4. Let x_0 be an unmatched vertex (a vertex in $X \setminus V(M)$).
5. Using the Hall violator algorithm, find either a Hall violator or an M-augmenting path.
6. In the first case, **return the Hall violator**.
7. In the second case, use the M-augmenting path to increase the size of M (by one edge), and **go back to step 2**.



$|N(A)| \geq |A|$

or

$|N(A)| < |A|$ ✓