$\boxed{\text{Mod-4}}$ Logic

logic, rules of inference

logic → Symbolic

Object language

Two valued logic

Definition 1. (Proposition)
A *statement* or *proposition* is a declarative sentence that is either true or false (but not both).

For instance, the following are propositions:

1. $3 > 1$ (true).
2. $2 < 4$ (true).
3. $4 = 7$ (false)

$\boxed{p, q, r, s.}$

However the following are a not propositions:

1. what is your name?.
2. x is an even number.

Definition 2. (Atomic statements)
Declarative sentences which cannot be further split into simple sentences are called *atomic statements* (also called *primary statements* or *primitive statements*).

Example: p is a prime number

Definition 3. (Compound statements)
New statements can be formed from atomic statements through the use of connectives such as "and, but, or etc..." The resulting statement are called *molecular* or *compound* (composite) statements.

Example: If p is a prime number then, the divisors are p and 1 itself

P(x)

Definition 4. (truth value)
The truth or falsehood of a proposition is called its *truth value*.
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Definition 5. (Truth Table)
A table, giving the truth values of a compound statement interms of its component parts, is called a *Truth Table*.

Definition 6. *(Connectives)*

Connectives are used for making compound propositions. The main ones are the following (p and q represent given two propositions):

Not in a proper place

wedge vee Statement

$\therefore p$

Table 1. Logic Connectives

Name	Represented	Meaning
Negation	$\neg p$	not in p
Conjunction	$p \wedge q$	p and q
Disjunction	$p \vee q$	p or q (or both)
Implication	$p \rightarrow q$	if p then q
Biconditional	$p \leftrightarrow q$	p if and only if q

$\bar{p}$

T v T = T
T v F = T.

(It is not the case:)

Canada is a Country

The truth value of a compound proposition depends only on the value of its components. Writing *F* for false and *T* for true, we can summarize the meaning of the connectives in the following way:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$q \rightarrow p$
T	T	F	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	T	F	F
F	F	T	F	F	T	T	T

Definition 7. (Tautology)
A proposition is said to be a *tautology* if its truth value is *T* for any assignment of truth values to its components.

Example: The proposition $p \vee \neg p$ is a tautology. T

Definition 8. (Contradiction)
A proposition is said to be a *contradiction* if its truth value is *F* for any assignment of truth values to its components.

Example: The proposition $p \wedge \neg p$ is a contradiction.

Definition 9.(Contingency)
A proposition that is neither a tautology nor a contradiction is called a *contingency*.

I. Construct the truth table for the following statements:

(i) $(p \to q) \longleftrightarrow (\neg p \vee q)$

(ii) $p \wedge (p \vee q)$

(iii) $(p \to q) \to p$

(iv) $\neg(p \wedge q) \longleftrightarrow (\neg p \vee \neg q)$

(v) $(p \vee \neg q) \to q$

Solution:

$(\neg p \vee q)$ $p \to q$

$(p \to q)$ $(\neg p \vee q)$

(i) Let $S = \boxed{(p \to q) \longleftrightarrow (\neg p \vee q)}$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$	S
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Logical Equivalence

The compound propositions $p \to q$ and $\neg p \vee q$ have the same truth values:

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

When two compound propositions have the same truth value they are called logically equivalent.

For instance $p \to q$ and $\neg p \lor q$ are logically equivalent, and it is denoted by

$A \Leftrightarrow B$

$(A \to B) \land (B \to A)$

$$p \to q \Leftrightarrow \neg p \lor q$$

T T
T ↔ F
F T
F ↔ F

Definition 10. (Logically Equivalent)
Two propositions A and B are *logically equivalent* precisely when $A \Leftrightarrow B$ is a tautology.

Example: The following propositions are logically equivalent:

$$p \leftrightarrow q \Leftrightarrow (p \to q) \land (q \to p)$$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$	S
T	T	T	T	T	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

F ∧ T = F
T ∧ T = T

Table 2. Logic equivalences

$\lor \to$ or
$\land \to$ and.

Equivalences	Name
$p \land T \Leftrightarrow p$ $p \lor F \Leftrightarrow p$	Identity law
$p \lor T \Leftrightarrow T$ $p \land F \Leftrightarrow F$	Dominent law
$p \lor p \Leftrightarrow p$ $p \land p \Leftrightarrow p$	Idempotent law
$p \lor q \Leftrightarrow q \lor p$ $p \land q \Leftrightarrow q \land p$	Commutative law
$(p \lor q) \lor r \Leftrightarrow p \lor (q \lor r)$ $(p \land q) \land r \Leftrightarrow p \land (q \land r)$	Associative law

$p^2 = p$
$p + p = p$
$p \cdot p = p$.

$a \cdot (b + c)$
$a \cdot b + a \cdot c$

$\wedge \to \cap$
$\vee \to \cup$

Table 2. Logic equivalences (Continued...)

Equivalences	Name
$(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$ $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$	Distributive law
$(p \vee q) \wedge p \Leftrightarrow p$ $(p \wedge q) \vee p \Leftrightarrow p$	Absorbtion law
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De morgan's law
$\neg p \wedge p \Leftrightarrow F$ $\neg p \vee p \Leftrightarrow T$ $\neg(\neg p) \Leftrightarrow p$	Negation law

$=$

$(p \cap Q) \vee p$
$\subseteq P, Q$

$\neg(p \wedge Q)$

Table 3. Logic equivalences involving implications

Implications
$p \to q \Leftrightarrow \neg p \vee q$
$p \to q \Leftrightarrow \neg q \to \neg p$
$\neg(p \to q) \Leftrightarrow p \wedge \neg q$
$p \vee q \Leftrightarrow (\neg p \to q)$
$p \wedge q \Leftrightarrow \neg(p \to \neg q)$
$(p \to q) \wedge (p \to r) \Leftrightarrow p \to (q \wedge r)$
$(p \to q) \vee (p \to r) \Leftrightarrow p \to (q \vee r)$
$(p \to r) \wedge (q \to r) \Leftrightarrow (p \vee q) \to r$
$(p \to r) \vee (q \to r) \Leftrightarrow (p \wedge q) \to r$

$p \to q$
$\neg q$

$\neg(p \to q)$
$\neg(\neg p \vee q)$
$p \wedge \neg q$

$\neg(\neg p) \vee q$
$p \vee q$

Table 4. Logic equivalences involving Biconditions

Biconditions
$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$

$\Leftrightarrow$
$\leftrightarrow$

$p \to q$

$\neg p \vee q$

Definition 11. (Converse)
The *converse* of a conditional proposition $p \to q$ is the proposition $q \to p$

Definition 12. (Inverse)
The *inverse* of a conditional proposition $p \to q$ is the proposition $\neg p \to \neg q$

Definition 13. (Contrapositive)
The *contrapositive* of a conditional proposition $p \to q$ is the proposition $\neg q \to \neg p$.

Let us consider the statement,
"The crops will be destroyed, if there is a flood."
Let F : there is a flood & C : The crops will be destroyed
The symbolic form is, $F \to C$.
Converse $(C \to F)$
i.e., "if the crops will be destroyed then there is flood."
Inverse $(\neg F \to \neg C)$
i.e.,"if there is no flood then the crops won't be destroyed, ."
Contrapositive $(\neg C \to \neg F)$
i.e., "if the crops won't be destroyed then there is no flood."

$p \to q$
$q \to p$

Without using truth table

(i) Show that $(p \vee q) \wedge \neg p \Leftrightarrow (\neg p \wedge q)$

(ii) Show that $(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p$

(iii) Show that $\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg q$

(iv) Show that $\neg(\neg((p \vee q) \wedge r) \vee \neg q) \Leftrightarrow q \wedge r$

(v) Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

(vi) Show that $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$.

(i)

$S \Leftrightarrow (p \vee q) \wedge \neg p$	Reasons
$\Leftrightarrow (p \vee q) \wedge \neg p$	Given
$\Leftrightarrow (p \wedge \neg p) \vee (q \wedge \neg p)$	Distributive law
$\Leftrightarrow F \vee (\neg p \wedge q)$	Negation law, Commutative law
$\Leftrightarrow \neg p \wedge q$	Identity law

(ii)

$S \Leftrightarrow (p \vee q) \wedge \neg(\neg p \wedge q)$	Reasons
$\Leftrightarrow (p \vee q) \wedge \neg(\neg p \wedge q)$	Given
$\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	De Morgan's law
$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Negation law
$\Leftrightarrow p \vee (q \wedge \neg q)$	Distributive law
$\Leftrightarrow p \vee F$	Negation law
$\Leftrightarrow p$	Identity law

(iii)

$S \Leftrightarrow \neg(p \vee (\neg p \wedge q))$	Reasons
$\Leftrightarrow \neg(p \vee (\neg p \wedge q))$	Given
$\Leftrightarrow \neg p \wedge \neg(\neg p \wedge q))$	De Morgan's law
$\Leftrightarrow \neg p \wedge (p \vee \neg q))$	De Morgan's law
$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive law
$\Leftrightarrow F \vee (\neg p \wedge \neg q)$	Negation law
$\Leftrightarrow \neg p \wedge \neg q$	Identity law

(iv)

$S \Leftrightarrow \neg(\neg((p \vee q) \wedge r) \vee \neg q) \Leftrightarrow q \wedge r$	Reasons
$\Leftrightarrow \neg(\neg((p \vee q) \wedge r) \vee \neg q)$	Given
$\Leftrightarrow ((p \vee q) \wedge r) \wedge q$	De Morgan's law
$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$	Associative law
$\Leftrightarrow (p \wedge (r \wedge q)) \vee (q \wedge (r \wedge q))$	Distributive law
$\Leftrightarrow (p \wedge (r \wedge q)) \vee (r \wedge q)$	Idempotent law
$\Leftrightarrow r \wedge q$	Absorption law

$\vee, \wedge, \neg$

Definition 14. (Duality)
The *dual* of a compound proposition that contains only the logical operators $\vee$, $\wedge$ and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T. The dual of proposition A is denoted by A^*.

Example. The dual of $(T \wedge p) \vee q$ is $(F \vee p) \wedge q$

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Definition 15. (Functionally complete set of connectives)
Any set of connectives in which every formula can be expressed as another equivalent formula containing connectives from this set is called *functionally complete set of connectives*.

Example. The set of connectives
$\{\vee, \neg\}$ and $\{\wedge, \neg\}$ are functionally complete.
$\{\neg\}$,$\{\vee\}$ or $\{\vee, \wedge\}$ are not functionally complete.

Problem. Prove that the set $\{\vee, \neg\}$ is functionally complete.

Solution:

To prove $\{\vee, \neg\}$ is functionally complete.
We have to show that for all formulas with other connectives their exists a equivalent formula which contains $\neg$ and $\vee$ only.

$p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$

$(p \to q) \wedge (q \to p)$

$(\neg p \vee q) \wedge (\neg q \vee p)$

$p \to q \Leftrightarrow (\neg p \vee q)$

$p \wedge q \Leftrightarrow \neg(\neg p \vee \neg q)$

The resultant is free from biconditional, conditional and conjunction.
Hence $\{\vee, \neg\}$ is functionally complete.

An argument is a sequence of propositions $H_1, H_2, \ldots, H_n$ called premises (or hypotheses) followed by a proposition C called conclusion. An argument is usually written:

$\wedge$

$$\begin{array}{l} H_1 \\ H_2 \\ \vdots \\ H_n \\ \text{implies} \quad C \end{array}$$

or

$$H_1 \wedge H_2 \wedge \ldots \wedge H_n \Rightarrow C$$

The argument is valid if C is true whenever $H_1, H_2, \ldots, H_n$ are true; otherwise it is invalid.

$H_1 \wedge H_2 \Rightarrow ?$

Example: $H_1 : p$ and $H_2 : p \to q$ then $C : q$ (Modus Ponens)

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

p

$p \to q$

$\therefore q$

Notice that $(p \wedge (p \to q)) \to q$ is a tautology. Therefore it is valid.

Thus $\boxed{p,\ p \to q \Rightarrow q.}$

Implication Table:

S.No	Formula	Name
1	$p \wedge q \Rightarrow p$	simplification
	$p \wedge q \Rightarrow q$	
2	$p \Rightarrow p \vee q$	addition
	$q \Rightarrow p \vee q$	
3	$p,\ q \Rightarrow p \wedge q$	
4	$p,\ p \to q \Rightarrow q$	modus ponens
5	$\neg p,\ p \vee q \Rightarrow q$	disjunctive syllogism
6	$\neg q,\ p \to q \Rightarrow \neg p$	modus tollens
7	$p \to q\,, q \to r \Rightarrow p \to r$	

	Rule of Inference	Tautology	Name
1)	P $P \to Q$ $\therefore Q$	$(P \wedge (P \to Q)) \to Q$	Modus ponens
2)	$\neg Q$ $P \to Q$ $\therefore \neg P$	$[\neg Q \wedge (P \to Q)] \to \neg P$	Modus tollens
3)	$P \to Q$ $Q \to R$ $\therefore P \to R$	$[(P \to Q) \wedge (Q \to R)] \to (P \to$	Hypothetical syllogism
4)	$P \vee Q$ $\neg P$ $\therefore Q$	$[(P \vee Q) \wedge \neg P] \to Q$	Disjunctive syllogism

OR $\neg P \vee Q$

I $\neg p \vee Q$

$\neg Q$

$\neg p$

Rules of inference:

Rule P: A premises can be introduced at any step of derivation.

Rule T: A formula can be introduced provided it is Tautologically implied by previously introduced formulas in the derivation.

Rule CP: If the conclusion is of the form $r \rightarrow s$ then we include r as an additional premises and derive s.

Indirect method: We use negation of the conclusion as an additional premise and try to arrive a contradiction.

Inconsistent: A set of premises are inconsistent provided their conjunction implies a contradiction.

$H_1 \wedge H_2$ A H_3

$p \wedge q$

$\Downarrow$

p

$p \rightarrow q$

$p, q \rightarrow r$

$p, q, \neg r$

Example: Show that $\neg q$ and $p \to q$ implies $\neg p$.

① $\sim q$,
② $p \to q$
$\therefore \sim p$.

$\sim q$
$p \to q$

Solution: (A formal proof is as follows):

Step 1. $p \to q$ — Rule P

Step 2. $\neg q \to \neg p$ — Rule T

Step 3. $\neg q$ — Rule P

Step 4. $\neg p$ — Combined $\{2, 3\}$ and apply Modus Ponens

MP(2,3)

$(p \to q) \Rightarrow\ ?$

$\Rightarrow$

$p \wedge q \to r$

Example: Show that r is a valid inference from the premises $p \to q$, $q \to r$ and p.

Solution:

Step	Derivation	Rule
1	p	P
2	$p \to q$	P
3	q	$\{1,2\}$, $\cancel{\mathcal{I}_4}$
4	$q \to r$	P
5	r	$\{3,4\}$, $\mathcal{I}_4$

MP(1,2)

MP(3,4)

$p,\ p \to q$
$\Downarrow$
q

Example: Show that $s \vee r$ is tautologically implied by $p \vee q$, $p \to r$ and $q \to s$.

Solution:

Step	Derivation	Rule
1	$p \vee q$	P
2	$\neg p \to q$	T
3	$q \to s$	P
4	$\neg p \to s$	$\{2, 3\}$, T.
5	$\neg s \to p$	T
6	$p \to r$	P
7	$\neg s \to r$	$\{5, 6\}$, $\mathcal{I}_7$
8	$s \vee r$	T

$\boxed{a \vee b}$
$\Leftrightarrow$
$\boxed{\sim a \to b}$

\# Indirect Proof := Proof by Contradiction

To prove $P, Q \to R$: Consider $P, Q, \sim R$ and prove a contradiction

Example: Prove by indirect method that $p \to q$, $p \vee r$, $\neg q$ implies r.
Solution: The desired result is r. Include $\neg r$ as a new premise.

$P \vee r$

$\neg p \to r$
$\equiv \neg r \to p$

Step	Derivation	Rule
1	$p \vee r$	P
2	$\neg r \to p$	T
3	$\neg r$	P(additional premise)
4	p	$\{2,3\}$, ~~$\mathcal{I}_4$~~ T
5	$p \to q$	P
6	q	$\{4,5\}$, $\mathcal{I}_4$
7	$\neg q$	P
8	$q \wedge \neg q$	$\{6,7\}$, $\mathcal{I}_3$

The new premise together with the given premises, leads to a contradiction
Thus $p \to q$, $p \vee r$, $\neg q$ implies r.

$H_1 \wedge H_2 \wedge H_3 \wedge H_4 \Rightarrow F$

Example: Prove that $p \to q$, $p \to r$, $q \to \neg r$ and p are inconsistent.
Solution: The desired result is false.

Step	Derivation	Rule
1	p	P
2	$p \to q$	P
3	q	$\{1,2\}$, ~~$\mathcal{I}$~~ [illegible] MP
4	$q \to \neg r$	P
5	$\neg r$	$\{3,4\}$, ~~$\mathcal{I}$~~ [illegible] MP
6	$p \to r$	P
7	$\boxed{\neg p}$	$\{5,6\}$, ~~$\mathcal{I}$~~ [illegible] MT
8	F	$\{1,7\}$, $\mathcal{I}$ [illegible]

$P \wedge \neg P$

(i) Show that $r \vee s$ is tautologically implied by $c \vee d$, $(c \vee d) \to \neg h$, $\neg h \to (a \wedge \neg b)$ and $(a \wedge \neg b) \to (r \vee s)$.

(ii) Show that $r \wedge (p \vee q)$ is tautologically implied by $p \vee q$, $q \to r$, $p \to m$ and $\neg m$.

(iii) Show that $r \to s$ is tautologically implied by $\neg r \vee p$, $p \to (q \to s)$ and q.

(iv) Show that $p \to s$ is tautologically implied by $\neg p \vee q$, $\neg q \vee r$ and $r \to s$.

(v) Show that $p \to (q \to s)$ is tautologically implied by $p \to (q \to r)$ and $q \to (r \to s)$ using CP rule.

(vi) Show that the following premises are inconsistent. $v \to l$, $l \to b$, $m \to \neg b$ and $v \wedge m$.

(vii) Show that $p \to \neg s$ logically follows from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$ and p by indirect method.

(viii) Show that r logically follows from the premises $p \to q$, $\neg q$ and $p \vee r$ by indirect method.

Example: Consider the following statements: 'I take the bus or I walk. If I walk I get tired. I do not get tired. Therefore I take the bus.' We can formalize this by calling $B=$ I take the bus, $W=$ I walk and $J=$ I get tired. The premises are $B \vee W$, $W \to J$ and $\neg J$, and the conclusion is B. The argument can be described in the following steps:

step	statement	reason
1	$W \to J$	P
2	$\neg J$	P
3	$\neg W$	$\{1, 2\}$, Modus Tollens
4	$B \vee W$	P
5	B	$\{3, 4\}$, Disjunctive Syllogism

$(B \vee W)$
$\wedge (W \to J)$ and
$\wedge (\neg J)$ and

$\therefore B$

$\neg B \to W$
$\neg W \to B$

(i) Show that the following set of premises is inconsistent.
1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books.

$H_1: M \to F$
$H_2: F \to U$
$H_3: R \to \neg U$
$H_4: M \wedge R$

$\Rightarrow M \to U$
$R \to \neg U$
$U \to \neg R$

$\Rightarrow M \to \neg R$
$\neg M \vee \neg R$
$\neg R$

$M \to \neg R$

$\neg M \vee \neg R$

$M \wedge R$

M

R

$\neg R$

$R \wedge \neg R$

F

$M \wedge R$

M

R

Let us consider,
E : Jack misses many classes through illness
S : Jack fails high school
A : Jack reads a lot of books
H : Jack is uneducated.
The premises are,

- $E \rightarrow S$
- $S \rightarrow H$
- $A \rightarrow \neg H$ and
- $E \wedge A$

Step	Derivation	Rule
1	$E \wedge A$	P
2	E	T
3	A	T
4	$E \rightarrow S$	P
5	S	$\{2,4\}$, $\mathcal{I}_4$
6	$S \rightarrow H$	P
7	H	$\{5,6\}$, $\mathcal{I}_4$
8	$A \rightarrow \neg H$	P
9	$\neg H$	$\{3,8\}$, $\mathcal{I}_4$
10	$H \wedge \neg H$	$\{7,9\}$, $\mathcal{I}_3$

$pr \rightarrow p$; $S \vee \neg p$; $SH \rightarrow \neg S \Rightarrow pr \rightarrow \neg SH$

p: proud
pr: praise
SH: Study hard
S: well in sports.

HS

Step Rule

$pr \to p$ P

$S \vee \sim p$ P

$p \to S$ T

$\sim S \to \sim p$ T

$SH \to \sim S$ P

$SH \to \sim p$ HS

$p \to \sim SH$ T

$pr \to \sim SH$ T

(i) Show that the following argument is valid.
My father praises me only if i can be proud of myself. Either I do well in sports
or I cannot be proud of myself. If study hard, then I cannot do well in sports.
Therefore, if father praises me, then I do not study well.
(ii) Show that the following set of premises is inconsistent.
If the contract is valid, then John is liable for penalty. If John is liable for penalty,
he will go bankrupt. If the bank will loan him money, he will not go bankrupt.
As a matter of fact, the contract is valid and the bank will loan him money.

p only if q

i. $\sim q \to \sim p$

ii. $p \to q$

$q \not\to p$

$V \to L$

$L \to B$

$M \to \sim B$

$\therefore V \wedge M$

$V \to B$

$B \to \sim M$

$V \to \sim M$

$V \vee \sim M$

$V \wedge M$

V

M

$\sim M$

Consider the statement

p : x is a prime number (the statement is not a proposition)

The truth value of p depends on the value of x.
p is true when $x = 3$, and false when $x = 10$.

In this section we extend the system of logic to include such an above statements.

propositional function

describes the property of variable x.

Definition 1. (predicates).
A *predicate* refers to a property that the subject of the statement can have.
A predicate is a sentence that contains a finite number of specific values are substituted for the variables.

That is, let $P(x)$ be a statement involving variable x and a set D. We call P as a propositional function if for each x in D, $P(x)$ is a proposition.

$P(x)$

eg: x is a cat can be denoted as: $C(x)$: cat

$x \in D$ set of animals

$C(x) :=$ x is a cat.

$x = a$

$C(a)$:

Definition 2. (universe of discourse)
The set D is called the *domain of discourse* (or*universe of discourse*) of P. It is the set of all possible values which can be assigned to variables in statements involving predicates.

Example: Let $p(x)$ denote the statement $x \geq 4$. What are the truth values of $p(5)$ (T) and $p(2)$ (F).

$P(4) \to T$ $4 \geq 4$

Example: Let $g(x, y)$ denote the statement $g.c.d(x, y) = 1$. What are the truth values of $g(3, 5)$ (T) and $g(2, 8)$ (F)

$P(3) \leftrightarrow 3 \geq 4 \to F$

Quanifier

Definition 3. (universal quantifier)
Consider the proposition
All odd prime numbers are greater than 2. The word all in this proposition is a logical quantifier. The proposition can be translated as follows:
For every x, if x is an odd prime then x is greater than 2

$\forall x\left(O(x) \to G(x)\right)$

Similarly, the proposition:

Every rational number is a real number may be translated as.
For every x, if x is a rational number, then x is a real number.

The phrase for every x is called a *universal quantifier*.

In symbols it is denoted by $(\forall x)$ or (x).

The phrases for every x, for all x and for each x have the same meaning and we can symbolize each by (x).

If $P(x)$ denotes a predicate (propositional function), then the universal quantification for $P(x)$, is the statement.

$(x)\ P(x)$ is true.

Example :
(a) Let $A = \{x : \text{x is a natural number less than } 9\}$
Here $P(x)$ is the sentence x is a natural number less than 9. The common property *is a natural number less than* 9. $P(1)$ is true, therefore, $1 \in A$ and $P(12)$ is not true, therefore $12 \notin A$.

(b) The proposition $(\forall N)\,(n + 4 > 3)$ is true.
Since $\{n | n + 4 > 3\} = \{1, 2, 3, \ldots\} = N$.

(c) The proposition $(\forall N)\,(n + 2 > 8)$ is false.
Since $\{n | n + 2 > 8\} = \{7, 8, 9, \ldots\} \neq N$.

(Quantifier + Predicate (Propositional function) = Statement (Proposition))

Definition 4. (existential quantifier).
In some situations we only require that there be at least one value for each the predicate is true. This can be done by prefixing $P(x)$ with the phrase *there exists an*. The phrase there exists an is called an *existential quantifier*.

The existential quantification for a predicate is the statement *There exists a value of* x for which $P(x)$.

The symbol, $\exists$ is used to denote the logical quantifier there exists. The phrases There exists an x, There is a x, for some x and for at least one x have the same meaning.

The existential quantifier for $P(x)$ is denoted by $(\exists x)\ P(x)$

Example :
(a) The proposition there is an integer between 1 and 3 may be written as ($\exists$ an integer) (the integer is between 1 and 3)

(b) The proposition $(\exists N)\,(n+4<7)$ is true.
Since $\{n|n+4<7\} = \{1,2\} \neq \phi$.

(c) The proposition $(\exists N)\,(n+6<4)$ is false.
Since $\{n|n+6<4\} = \phi$.

Rule of Inference	Name
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{\text{P(c) for an arbitrary c}}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c)\ for\ some\ c}$	Existential instantiation
$\dfrac{\text{P(c) for some c}}{\therefore \exists x P(x)}$	Existential generalization

Eg: All rock music is loud music.

Some rock music exists.

Therefore some loud music exists.

Let $L(x)$: x is loud $\qquad x \in D \rightarrow$ all Musics.

$R(x)$: x is rock

$$(\forall x)(R(x) \rightarrow L(x))$$
$$(\exists x)(R(x))$$
$$\therefore \exists x(L(x)).$$

Steps

		Rule
1)	$(\forall x)(R(x) \rightarrow L(x))$	P
2)	$R(a) \rightarrow L(a)$	US
3)	$\exists x(R(x))$	P
4)	$R(a)$	ES
5)	$L(a)$	MP(2,4)
6)	$\exists x(L(x))$	EG.

(i) Show that $(x)(H(x) \longrightarrow M(x)) \wedge H(a) \Longrightarrow M(a)$.

Solution:

Step 1	$(x)(H(x) \longrightarrow M(x))$	Rule P $\rightarrow$ Premises
Step 2	$H(a) \longrightarrow M(a)$	Rule US $\rightarrow$ Universal Specification
Step 3	$H(a)$	Rule P
Step 4	$M(a)$	$\{2,3\}$ and apply Modus Ponens

$$\begin{array}{l} \neg P \\ P \vee Q \\ \hline \therefore Q \end{array} \qquad \begin{array}{l} a \rightarrow b \\ a \\ \hline \therefore b. \end{array}$$

(ii) Show that $(x)(P(x) \longrightarrow Q(x)) \wedge (x)(Q(x) \longrightarrow R(x)) \Longrightarrow (x)(P(x) \longrightarrow R(x))$.
Solution:

Step 1	$(x)(P(x) \longrightarrow Q(x))$	Rule P
Step 2	$P(a) \longrightarrow Q(a)$	Rule US
Step 3	$(x)(Q(x) \longrightarrow R(x))$	Rule P
Step 4	$Q(a) \longrightarrow R(a)$	Rule US
Step 5	$P(a) \longrightarrow R(a)$	$\{2,4\},\mathcal{I}_7$
Step 6	$(x)P(x) \longrightarrow R(x)$	Rule UG

a is arbitrary.

$\forall a \in D$.

(iii) Show that $(\exists x)(P(x) \wedge Q(x)) \Longrightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$.
Solution:

Step 1	$(\exists x)(P(x) \wedge Q(x))$	Rule P
Step 2	$P(a) \wedge Q(a)$	Rule ES
Step 3	$P(a)$	$\mathcal{I}_1$
Step 4	$Q(a)$	$\mathcal{I}_1$
Step 5	$(\exists x)P(x)$	$\{3\}$,EG
Step 6	$(\exists x)Q(x)$	$\{4\}$,EG
Step 7	$(\exists x)P(x) \wedge (\exists x)Q(x)$	$\{5,6\}$, $\mathcal{I}_3$

(iv) Show that $(x)(P(x) \vee Q(x)) \Longrightarrow (x)P(x) \vee (\exists x)Q(x)$.
Solution: Proof by indirect method

Step 1	$\neg((x)(P(x) \vee Q(x)))$	Rule P ✓
Step 2	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	Rule T ✓
Step 3	$\neg(x)P(x)$	$\mathcal{I}_1$ ✓
Step 4	$\neg(\exists x)Q(x)$	$\mathcal{I}_1$ ✓
Step 5	$(\exists x)\neg P(x)$	3,Rule T
Step 6	$(x)\neg Q(x)$	4,Rule T
Step 7	$\neg P(a)$	5,ES → a is particular
Step 8	$\neg Q(a)$	6,US → a is arbitrary

(i) $\sim[\exists x\, Q(x)] \Leftrightarrow \forall x(\sim Q(x))$

and

(ii) $\sim(\forall x\, Q(x)) \Leftrightarrow \exists x(\sim Q(x))$

Step 9	$\neg P(a) \wedge \neg Q(a)$ → a is particular	$\{7,8\}, \mathcal{I}_3$
Step 10	$\neg(P(a) \vee Q(a))$	Rule T
Step 11	$(x)(P(x) \vee Q(x))$ → given	Rule P
Step 12	$P(a) \vee Q(a)$ → a is arbitrary	US
Step 13	$\neg(P(a) \vee Q(a)) \wedge (P(a) \vee Q(a))$	$\{10,12\}, \mathcal{I}_3$
Step 14	F	Rule T

Contradiction by 10, 12 //.

(v) Show that from

(a) $(\exists x)(F(x) \wedge S(x)) \longrightarrow (y)(M(y) \longrightarrow W(y))$

(b) $(\exists y)(M(y) \wedge \neg W(y))$

the conclusion $(x)(F(x) \longrightarrow \neg S(x))$.

Solution:

Step 1	$(\exists y)(M(y) \wedge \neg W(y))$	Rule P
Step 2	$(M(a) \wedge \neg W(a))$	ES $\quad \sim[\sim M(a) \vee W(a)] \Leftrightarrow \sim[M(a) \rightarrow W(a)]$
Step 3	$\neg(M(a) \longrightarrow W(a))$	Rule T
Step 4	$(\exists y)\neg(M(y) \longrightarrow W(y))$	EG
Step 5	$\neg(y)(M(y) \longrightarrow W(y))$	Rule T

[There exists a value for which P is not true is equivalent to P is not true for all values.]

Step 6	$(\exists x)(F(x) \wedge S(x)) \longrightarrow (y)(M(y) \longrightarrow W(y))$	Rule P
Step 7	$\neg(\exists x)(F(x) \wedge S(x))$ (MT)	$\{5,6\}, \mathcal{I}_6$
Step 8	$(x)\neg(F(x) \wedge S(x))$	Rule T
Step 9	$\neg(F(a) \wedge S(a))$	US
Step 10	$F(a) \longrightarrow \neg S(a)$	Rule T
Step 11	$(x)(F(x) \longrightarrow \neg S(x))$	UG

Predicate can be generalized to two variables.

eg:

Every crocodile is bigger than every alligator.
Sam is a crocodile.
there is a snake and sam is not bigger than that snake.
therefore something is not an alligator.

$x \in D :=$ set of Creatures.

Sol:

let $C(x)$: x is a crocodile.
$A(x)$: x is an alligator.
$B(x,y)$: x is bigger than y.
$x = s$ for Sam.
$S(x)$: x is a snake.

Given:

$$(\forall x)(\forall y)\left[C(x) \wedge A(y) \rightarrow B(x,y)\right]$$

and $C(s)$

and $(\exists x)\left[S(x) \wedge \sim B(s,x)\right]$

$$\therefore \quad (\exists x)(\sim A(x))$$

Pf:

Steps	Rule.
1) $(\forall x)(\forall y)(C(x) \wedge A(y) \rightarrow B(x,y))$	P
2) $(\forall y)[C(s) \wedge A(y) \rightarrow B(s,y)]$	US (1)
3) $(\exists x)[S(x) \wedge \sim B(s,x)]$	P
4) $S(a) \wedge \sim B(s,a)$	ES
5) $C(s) \wedge A(a) \rightarrow B(s,a)$	US (2)
6) $S(a)$	LI (4)
7) $\sim B(s,a)$	LI (4)
8) $\sim [C(s) \wedge A(a)]$	MT (5,7)
9) $\sim C(s) \vee \sim A(a)$	T
10) $C(s) \rightarrow \sim A(a)$	T
11) $C(s)$	P
12) $\sim A(a)$	MP
13) $\exists x(\sim A(x))$	EG.

Prove or disprove

- $(\forall x)[P(x) \vee Q(x)] \rightarrow (\exists x)P(x) \vee (\forall x)Q(x)$
- $(\exists x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)[P(x) \wedge Q(x)]$
- $(\exists x)P(x) \wedge [(\exists x)(P(x) \wedge Q(x))]' \rightarrow (\exists x)[Q(x)]'$