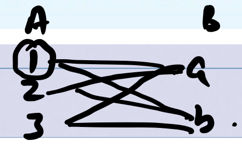
Counting Principles

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

$\underline{n_1} \times \underline{n_2}$



A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

$$\underset{S}{\underline{12}} \times \underset{P}{\underline{11}} = \underline{\underline{132}}$$

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

$$\underline{26}\ ,\ \underline{100} = \underline{2600}$$

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

$32 \times 24.$

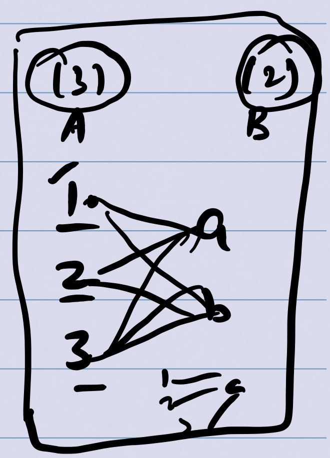
How many different bit strings of length seven are there?

$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$

$= 128$

Counting Functions How many functions are there from a set with m elements to a set with n elements?

$$f: \underset{}{\overset{(m)}{A}} \longrightarrow \overset{(n)}{B}$$



n^m $\quad 2 \times 2 \times 2$ (n times)

$= 2^3$

Counting One-to-One Functions How many one-to-one functions are there from a set with m elements to one with n elements?

$$f: \overset{(m)}{A} \longrightarrow \overset{(n)}{B}$$

(one-one)

3 5

$\boxed{m > n} \longrightarrow$ No one-one func

$5 \times 4 \times 3 = 60$

$(m \le n)$ $\quad \underline{n} \times \underline{(n-1)}$ _ _ _ _ _ _ $\underline{(n-(m-1))}$

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

```
k := 0
for i1 := 1 to n1
    for i2 := 1 to n2
        .
        .
        .
            for im := 1 to nm
                k := k + 1
```

1, 2, 3, ... n_m

$K = n_1 n_2 \ldots n_m$

$n_m \times$

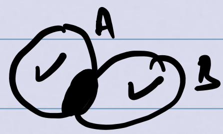
Counting Subsets of a Finite Set Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

$|S| = n$

$\underline{2} \times \underline{2} \times \underline{2}$ _ _ _ _ _ $\underline{2} = 2^n$ $\quad |P(S)| = 2^n$

$P(S) := \{A : A \subseteq S\}$

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.



A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

$23+15+19=$

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

```
k := 0
for i1 := 1 to n1
    k := k + 1 ;
for i2 := 1 to n2
    k := k + 1 ;
      .
      .
      .
for im := 1 to nm
    k := k + 1 ;
```

$k = n_1 \ldots n_m$

$K = n_1 + n_2 + \cdots n_m.$

In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

$V_1 \to 26$

$V = V_1 + V_2$

v_2

v_1

$26 + (26 \cdot 36) - 5$

1

26×36

5

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

$P_6 + P_7 + P_8$

$36 \times 36 \times 36 \times 36 \times 36 \times 36$

$\boxed{36^6 - 26^6}$

n(atleast one)

No digit

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

1 _ _ _ _ _ _ 0 0

2^8 =

$2^7 + 2^6 - ?$

2^5.

THE DIVISION RULE There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

$\frac{n}{d}$

1 — 2
2 — 3
3 — 1

(1 2 3)

2 — 3
3 — 1
1 — 2

(2 3 1)

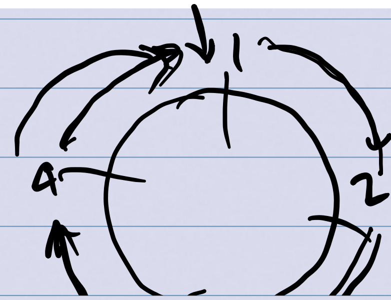
(3 1 2)

How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

4 × 3 × 2 × 1

= 24

1234
1324
1,2,3
2,3,1



S_n

1234

234

24 = 6

Tree Diagrams

How many bit strings of length four do not have two consecutive 1s?

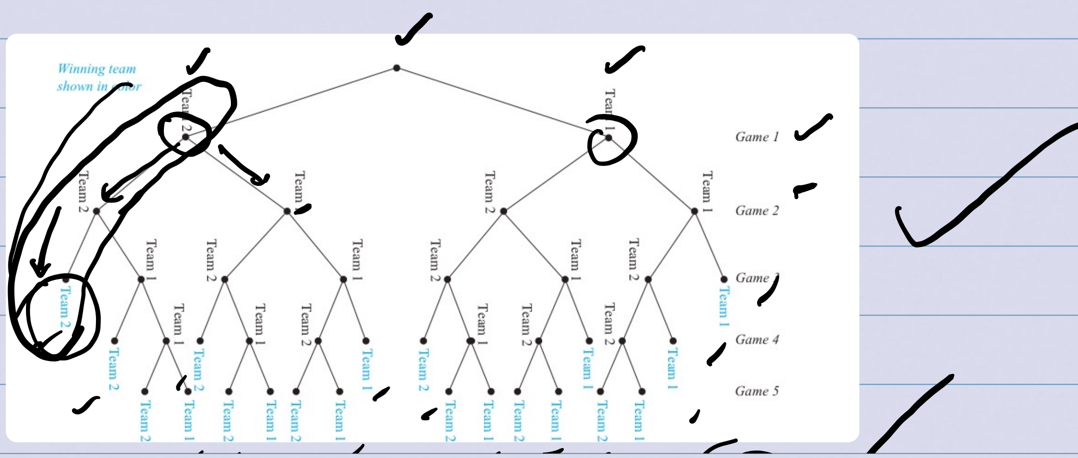
16

1010

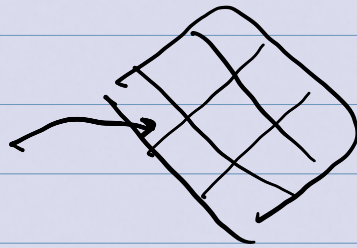
1001

1000

A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?



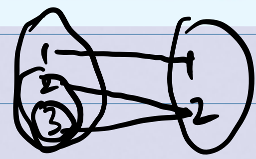
20



The Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.



Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

26

In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Im

Ans: = 102

Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

1, 11, - - - 11111, (n+1),

x, y

$x = n_1 n + r_1$

$y = n_2 n + r_2$

$x - y = (n_1 - n_2) n$

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

5 $\lceil \frac{5}{2} \rceil = 3$ 2

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

60 $\boxed{12}$ $\lceil \frac{100}{12} \rceil = 9$ $\boxed{12}$

$\lceil \frac{60}{12} \rceil = \boxed{5}$ 100 $\rightarrow$

12, 2

During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \le a_j \le 45$. Moreover, $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ is also an increasing sequence of distinct positive integers, with $15 \le a_j + 14 \le 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers a_j, $j = 1, 2, \ldots, 30$ are all distinct and the integers $a_j + 14$, $j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i. ◀

$a_1 \le 45$

$a_2 \le 44$

$\le 15 \le$ [illegible]

$a_j + 14$

a_i

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Permutations and Combinations

$^{n}P_{r} \rightarrow \frac{\lfloor n}{\lfloor n-r}$

$^{n}C_{r} \rightarrow \frac{\lfloor n}{\lfloor n-r \cdot \lfloor r}$

4

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

How many permutations of the letters $ABCDEFGH$ contain the string ABC?

How many different committees of three students can be formed from a group of four students?

Let S be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S. (Note that $\{4, 1, 3\}$ is the same 3-combination as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.)

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

How many bit strings of length n contain exactly r 1s?