Q: An amount of 10,000 Rs. is deposited in a bank with annual interest of 4%.

Determine the balance of account after 15 years.

$b_n \rightarrow$ balance after n^{th} year

$$b_n = b_{n-1} + b_{n-1} \times \frac{4}{100}$$

$$P\left(1+\frac{R}{100}\right)^T$$

$$b_n = b_{n-1}\left(1+\frac{4}{100}\right)$$

$$\boxed{b_n = b_{n-1}(1.04)}$$

$$b_{n-1} = b_{n-2}(1.04)$$

$$b_{15} = (1.04)^{15} \cdot b_0$$

$$b_n = (1.04)^2 \, b_{n-2}$$

$$b_0 = 10{,}000$$

$$\boxed{b_n = (1.04)^n \, b_0}$$

Recurrence Relations

a rule for determining subsequent terms from those that precede them.

A Sequence is called a Solution of a recurrence relation if its terms satisfy the recurrence relation

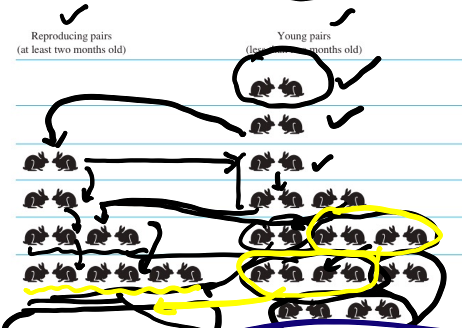
For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours? To solve this problem, let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines a_n for all nonnegative integers n. We can find a formula for a_n using the iterative approach followed in Chapter 2, namely that $a_n = 5 \cdot 2^n$ for all nonnegative integers n.

a_1
a_2
a_3
⋮

$a_n = 2 . a_{n-1}$
$a_n = 2^n . a_0$
$= 5 \cdot 2^n$

EXAMPLE 1 **Rabbits and the Fibonacci Numbers** Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 1. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Links



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

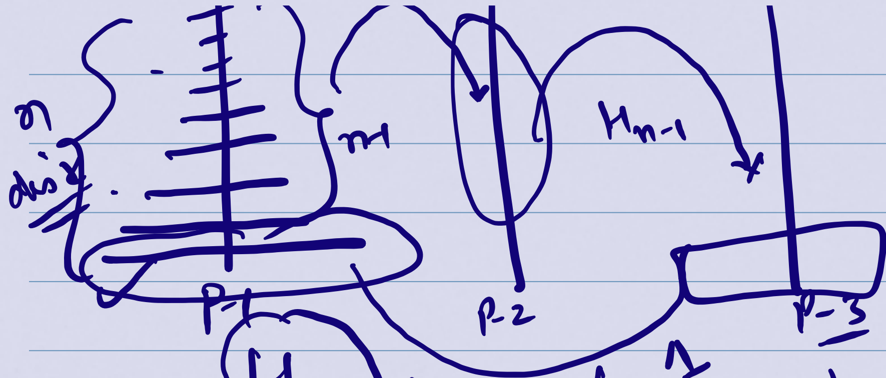
FIGURE 1 Rabbits on an Island.

1
1
2
3
5
8
⋮

$$a_n = a_{n-1} + a_{n-2}$$

$a_0 = 1$
$a_1 = 1$
$a_2 = 2$

#Tower of Hanoi



$H_n \to$ no of moves to place n disks from Peg-1 to Peg 3. (using 2)

$$H_n = H_{n-1} + 1 + H_{n-1}$$

$$H_n = 2H_{n-1} + 1$$

$$H_1 = 1$$

$$H_n = 2H_{n-1} + 1$$

$$H_{n-1} = 2H_{n-2} + 1$$

$$H_n = 2(2H_{n-2} + 1) + 1 = 2^n - 1$$

$$= 2^2 H_{n-2} + 2 + 1$$

$$= 2^3 H_{n-3} + 2^2 + 2 + 1$$

$$= 2^{n-1} + \dots + 1 = 1\,\frac{2^n - 1}{2 - 1}$$

Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

Hint:

End with a 1:	Any bit string of length $n-1$ with no two consecutive 0s	1
End with a 0:	Any bit string of length $n-2$ with no two consecutive 0s	1 0

— — — — — — — 1 | 1

$n-1$ n

a_{n-1}

$a_n \rightarrow$ no of strin without two consecutive zero

$a_1 = 2$

$a_2 = 3$

$a_3 = 5$

$$a_n = a_{n-1} + a_{n-2}$$

Codeword Enumeration A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

$a_n \rightarrow$ no of valid Codewords.

9 — — —

$a_n = 9 \times a_{n-1} + 10^{n-1} - a_{n-1}$

$a_1 = 9$

$a_n = 10^{n-1} + 8a_{n-1}$

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

r_1 $a(r_1)^n + b(r_2)^n$ $\boxed{a(r_1)^n}$ $(r_1)^n$

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \rightarrow r^n = r^{n-1} + 2r^{n-2}$$

$$\Rightarrow r^2 = r + 2.$$

with $a_0 = 2$ and $a_1 = 7$?

$(r_1)^n$

A.E $\rightarrow r^2 - r - 2 = 0$

$\Rightarrow \quad r = 2, -1.$

$$\therefore \boxed{a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n}$$

$\boxed{2^n + (-1)^n}$

$a_0 = \alpha_1 + \alpha_2$

$2 = \alpha_1 + \alpha_2$

$a_1 = 2\alpha_1 - \alpha_2$

$7 = 2\alpha_1 - \alpha_2$

$\alpha_1 = 3, \quad \alpha_2 = -1$

$$\boxed{a_n = 3 \cdot 2^n - (-1)^n}$$

Find an explicit formula for the Fibonacci numbers.

$$\boxed{a_n = a_{n-2} + a_{n-2}}$$

0
1
1
2
3
5
8
⋮

A.E $\rightarrow$ $r^2 - r - 1 = 0$

$$\Rightarrow r = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2}$$

$$= \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$a_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\underline{\underline{n=0}}$

$$0 = \alpha_1 + \alpha_2$$

$\underline{\underline{n=1}}$

$$1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\alpha_1 = \frac{1}{\sqrt{5}}; \quad \alpha_2 = -\frac{1}{\sqrt{5}}.$$

$$\alpha_1(\sqrt{5}) = 1$$

$$\alpha_1 = 1/\sqrt{5}.$$

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

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Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $\boxed{a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n}$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

$r = r_0, r_0$

$(r_0)^n$, $n(r_0)^n$

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

AE $\Rightarrow$ $\quad r^3 - 6r^2 + 11r - 6 = 0$

$(r-1)(r^2 - 5r + 6) = 0$

1	1	−6	11	−6
	x	1	−5	6
	1	−5	6	0

$r = 1; 2, 3.$

$$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 3^n$$

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$a_n = 6a_{n-1} - 9a_{n-2} \Rightarrow r^2 - 6r + 9 = 0$

$\Rightarrow 3, 3.$

$a_n = \alpha_1 3^n + \alpha_2 n \cdot 3^n$

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = \; & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

①

AE $\rightarrow$ $r^3 + 3r^2 + 3r + 1 = 0$

$\Rightarrow (r+1)^3 = 0$

$r = -1, -1, -1$

$$\boxed{a_n = \alpha_1 (-1)^n + \alpha_2 (-1)^n \cdot n + \alpha_3 (-1)^n n^2}$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a **linear nonhomogeneous recurrence relation with constant coefficients**, that is, a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

$a_n^{(p)} + a_n^{(h)}$

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}.$$

Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

$H_n = 2H_{n-1} + 1$

$F(n) = 2n$

$r - 3 = 0$

$r = 3$

$a_n^{(h)} = \alpha_1 3^n$

$p_n = an + b.$

$a = -1, \quad b = -3/2$

$p_n = -n - 3/2$

$+$

$$a_n = -n - 3/2 + \alpha_1 3^n$$

$\alpha_1 = \frac{11}{6}$

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

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Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

Solution: The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$. Its characteristic equation, $r^2 - 6r + 9 = (r - 3)^2 = 0$, has a single root, 3, of multiplicity two. To apply Theorem 6, with $F(n)$ of the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to ask whether s is a root of this characteristic equation.

Because $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, Theorem 6 tells us that a particular solution has the form $p_0 n^2 3^n$ if $F(n) = 3^n$, the form $n^2(p_1 n + p_0)3^n$ if $F(n) = n3^n$, the form $(p_2 n^2 + p_1 n + p_0)2^n$ if $F(n) = n^2 2^n$, and the form $n^2(p_2 n^2 + p_1 n + p_0)3^n$ if $F(n) = (n^2 + 1)3^n$. ◀

$$a_n = a_{n-1} + n.$$

Sum of frist n naturel no's.

The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k.$$

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Solution: Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_kx^k$. First note that

$$xG(x) = \sum_{k=0}^{\infty} a_kx^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k.$$

Using the recurrence relation, we see that

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_kx^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_0 + \sum_{k=1}^{\infty}(a_k - 3a_{k-1})x^k \\ &= 2, \end{aligned}$$

because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus,

$$G(x) - 3xG(x) = (1 - 3x)G(x) = 2.$$

Solving for $G(x)$ shows that $G(x) = 2/(1 - 3x)$. Using the identity $1/(1 - ax) = \sum_{k=0}^{\infty} a^kx^k$, from Table 1, we have

$$G(x) = 2\sum_{k=0}^{\infty} 3^kx^k = \sum_{k=0}^{\infty} 2 \cdot 3^kx^k.$$

Consequently, $a_k = 2 \cdot 3^k$. ◀

Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0s. Let a_n denote the number of valid codewords of length n. In Example 4 of Section 8.1 we showed that the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1}$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: To make our work with generating functions simpler, we extend this sequence by setting $a_0 = 1$; when we assign this value to a_0 and use the recurrence relation, we have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$, which is consistent with our original initial condition. (It also makes sense because there is one code word of length 0—the empty string.)

We multiply both sides of the recurrence relation by x^n to obtain

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. We sum both sides of the last equation starting with $n = 1$, to find that

$$\begin{aligned} G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\ &= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\ &= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\ &= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\ &= 8xG(x) + x/(1-10x), \end{aligned}$$

where we have used Example 5 to evaluate the second summation. Therefore, we have

$$G(x) - 1 = 8xG(x) + x/(1-10x).$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}.$$

Expanding the right-hand side of this equation into partial fractions (as is done in the integration of rational functions studied in calculus) gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

Using Example 5 twice (once with $a = 8$ and once with $a = 10$) gives

$$\begin{aligned} G(x) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n. \end{aligned}$$

Consequently, we have shown that

$$a_n = \frac{1}{2}(8^n + 10^n).$$ ◀