PHY 623

NON-LINEAR DYNAMICS AND CHAOS

Analysis of a Drinking Epidemic Model

2020-21 Semester I

Authors

Krishna Prahlaadh R (17129)

Siddharth Bachoti (17274)

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Introduction

The goal of this project is to summarize the work of Swarnali Sharma and GP Samanta: formulating a mathematical model describing the dynamics of people who are victims of alcohol abuse.

Earlier work in this direction, carried out by Sanchez et al. as well as by Mulone and Straughan has the following drawbacks:

1. Has only 3 classes – moderate and occasional drinkers susceptible to drinking epidemic, heavy drinkers and recovered population
2. Does not describe the possibility or dynamics of recovered population relapsing back into the abuse cycle
3. Does not incorporate additional drinking-related death rates of heavy drinkers
4. Does not involve treatment, control, etc.

In order to overcome these limitations, the population is divided into four classes - moderate and occasional drinkers, heavy drinkers, drinkers in treatment and temporarily recovered class. A treatment programme is also introduced into the system, and the possibility of relapse is taken into account, in order to make the study more realistic.

In this report, we analyze the dynamical system, its equilibrium points and its bifurcation characteristics via qualitative and quantitative reasoning supported by numerical simulations. The GitHub repository for this project can be found here.

Epidemic Model without Control

The mathematical model for our drinking epidemic is:

$$\frac{dS}{dt} = \Lambda - \beta_1 S(t)\, D(t) - \mu\, S(t) + \eta\, R(t) \tag{1a}$$

$$\frac{dD}{dt} = \beta_1 S(t)\, D(t) + \beta_2 T(t)\, D(t) - (\mu + \delta_1 + \gamma)\, D(t) \tag{1b}$$

$$\frac{dT}{dt} = \gamma\, D(t) - \beta_2 T(t)\, D(t) - (\mu + \delta_2 + \sigma)\, T(t) \tag{1c}$$

$$\frac{dR}{dt} = \sigma\, T(t) - (\mu + \eta)\, R(t) \tag{1d}$$

Terms and Definitions:

- Dynamical Variables (Population Densities)
  $S(t)$ – Moderate and occasional drinkers
  $D(t)$ – Heavy drinkers
  $T(t)$ – Drinkers in treatment
  $R(t)$ – Temporarily recovered drinkers
- Rates and Control Parameters
  Λ - Recruitment rate of moderate and occasional drinkers

β_1 - Transmission coefficient from moderate & occasional drinkers to heavy drinkers
β_2 - Transmission coefficient from drinkers in treatment to heavy drinkers
μ - Natural death rate of all population classes
δ_1 - Drinking-related death rate of heavy drinkers
δ_2 - Drinking-related death rate of drinkers in treatment
γ - The population of drinkers who enter into treatment
σ - Recovery rate of drinkers in treatment
η - The proportion of the recovered class that re-enters into the moderate & occasional drinkers class.

The total population N is hence given by $N = S(t) + D(t) + T(t) + R(t)$.

The model can be represented by the flow diagram 1:

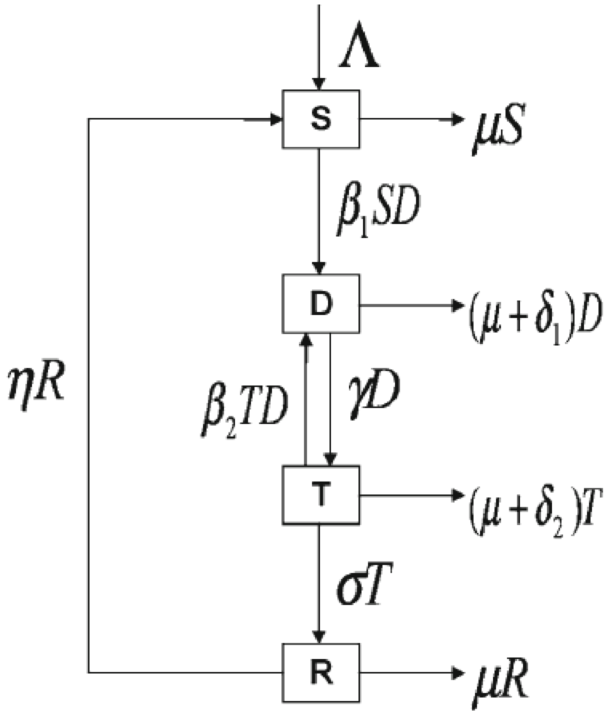


Figure 1: Image from the original paper by Sharma and Samanta

Assumptions and a Theorem

Within our model, the following assumptions hold:

1. All the four populations mix homogeneously,
2. Contact with heavy drinkers NOT in treatment induces heavy drinking in moderate and occasional drinkers
3. Heavy drinkers NOT in treatment are also infectious to drinkers in treatment
4. Drinkers in treatment are NOT infectious to moderate drinkers

5. Contact with heavy drinkers not in treatment causes relapse of heavy drinkers who are in treatment
6. Those who complete the treatment either recover temporarily, or re-enter into the moderate drinkers' class.

There is another obvious assumption to make - $S(t) > 0$ and $D(t), T(t), R(t) \geq 0$ for all $t \geq 0$. This is an intuitive assumption - however, the authors have mathematically proved it as a theorem in their paper.

Now, consider the rate of change of the entire population. Adding all the equations from the system (1), we obtain:

$$\frac{dN}{dt} = \Lambda - \mu\, N - \delta_1\, D(t) - \delta_2\, T(t) \tag{2}$$

Hence,

$$\frac{dN}{dt} \leq \Lambda - \mu N$$

since $D(t)$, $T(t) \geq 0$.

The solution to this equation is:

$$0 < N(t) \leq N(0)\, e^{-\mu t} + \frac{\Lambda}{\mu}\left(1 - e^{-\mu t}\right)$$

As $t \to \infty$, $N \to \frac{\Lambda}{\mu}$ for any positive initial $N(0)$. Hence, we have the theorem:

Theorem 1. The region defined by

$$G = \{(S(t), D(t), T(t), R(t)) \in \mathrm{R}_+^4 : 0 < N \leq \frac{\Lambda}{\mu}\}$$

is positively invariant.

By *positively invariant*, we mean that the solutions or the trajectories of the system corresponding to N either remain in G or approach it asymptotically as $t \to \infty$. This goes to say that within this well-defined region G, the problem we have is well-posed. Hence, it will suffice for us to study the dynamics of the system within the region G.

Basic reproduction number R_0

The number of heavy drinkers produced when a single heavy drinker is introduced into moderate and occasional drinkers' population. It is defined in terms of the coefficients of our model as:

$$R_0 = \frac{\Lambda \beta_1}{\mu(\mu + \delta_1 + \gamma)}$$

R_0 is an extremely important parameter for our model. Here, we examine the sensitivity of R_0 to the parameters it is defined in terms of. The sensitivity indices are given by:

$$A_{\beta_1} = \frac{\partial R_0}{\partial \beta_1}\frac{\beta_1}{R_0} = 1 \tag{3a}$$

$$A_{\delta_1} = \frac{\delta_1}{R_0}\frac{\partial R_0}{\partial \delta_1} = -\frac{\delta_1}{\mu + \delta_1 + \gamma} \tag{3b}$$

$$A_{\gamma} = \frac{\gamma}{R_0}\frac{\partial R_0}{\partial \gamma} = -\frac{-\gamma}{\mu + \delta_1 + \gamma} \tag{3c}$$

Hence, we see that $|A_{\beta_1}| = 1$, and $|A_{\delta_1}|, |A_{\gamma}| < 1$. We see that R_0 is most sensitive to changes in β_1. We can also conclude from this, that R_0 is directly proportional to β_1 and inversely proportional to γ and δ_1.

We can argue that δ_1, being the drinking-related death rate of heavy drinkers, is unlikely to be susceptible to much change - hence, the dependence of R_0 on this quantity can be ignored for our practical purposes.

Among the other two quantities that remain, we see that R_0 shows greater sensitivity towards β_1 than towards γ. Neglecting the latter, we can hence conclude that reducing β_1 (transmission rate from moderate drinkers to heavy drinkers) is the most effective strategy to combat a drinking epidemic.

Equilibrium Points: Existence and Stability

There are two principal kinds of equilibria that we find. These are:

- **Drinking/Disease-free Equilibirum:** $E_0 \equiv (\frac{\Lambda}{\mu}, 0, 0, 0)$
  This is an equilibrium where the only existing section of the population is the moderate and occasional drinkers. As we saw earlier, $N(t) \to \frac{\Lambda}{\mu}$ as $t \to \infty$. In this case, $N(t) = S(t) \to \infty$ and all others $D(t), T(t), R(t) \to 0$ as $t \to \infty$. This constitutes a somewhat trivial and uninteresting case.
- **Endemic Equilibrium:** $E^* = (S^*, D^*, T^*, R^*)$
  These are non-trivial equilibirum points. In the below section, we review the analysis of these endemic equilibrium points.

At endemic equilibrium points, the following hold

$$S, D, T, R \ > 0$$
$$\dot{S} = \dot{D} = \dot{T} = \dot{R} = 0$$

We obtain expressions for S^*, T^* and R^* in terms of D^*

$$
\begin{aligned}
S^* &= \frac{(\mu+\delta_1)(\beta_2\, D^* + \mu + \delta_2 + \sigma) + \gamma(\mu+\delta_2+\sigma)}{\beta_1(\beta_2\, D^* + \mu + \delta_2 + \sigma)} \\
T^* &= \frac{\gamma\, D^*}{\beta_2\, D^* + \mu + \delta_2 + \sigma} \\
R^* &= \frac{\sigma\gamma D^*}{(\mu+\eta)(\beta_2\, D^* + \mu + \delta_2 + \sigma)}
\end{aligned}
$$

We obtain the fixed points in terms of D^* in the form

$$a_1\,(D^*)^2 + a_2\, D^* + a_3 = 0 \tag{4}$$

where

$$
\begin{aligned}
a_1 =& \; -(\mu+\eta)(\mu+\delta_1)\beta_1\beta_2 && \text{(5a)}\\
a_2 =& \; \Lambda\beta_1(\mu+\eta)\beta_2 - [\beta_1(\mu+\delta_2+\sigma) + \beta_2\mu](\mu+\eta)(\mu+\delta_1) && \text{(5b)}\\
& - \gamma(\mu+\eta)(\mu+\delta_2+\sigma)\beta_1 + \beta_1\eta\sigma\gamma && \text{(5c)}\\
a_3 =& \; \mu(\mu+\eta)(\mu+\delta_2+\sigma)(\mu+\delta_1+\gamma)[R_0 - 1] && \text{(5d)}
\end{aligned}
$$

From the above, we can see that a_1 is always negative. a_3 is positive if $R_0 > 1$ and negative if $R_0 < 1$.

We can conclude the following about the fixed points for various values of a_1, a_2 and a_3:

1. $R_0 > 1$ ($a_3 > 0$), one fixed point exists, regardless of the sign of a_2.
2. If $R_0 < 1$ and $a_2 < 0$, then the system has no physically feasible fixed point (apart from drinking-free point)
3. If $R_0 < 1$ and $a_2 > 0$, then the system may have two or no fixed points (apart from the drinking-free point) - with the possibility for a *backward bifurcation* occurring between them. A backward bifurcation constitutes the case when a stable drinking-free equilibrium coexists with two endemic equilibria - one stable with a large number of heavy drinkers and one unstable with a smaller number of heavy drinkers.

Note that the drinking-free equilibrium exists for all parameter values. In the above points, we have described the existence of the endemic equilibrium points.

Local Stability of Drinking-Free Equilibrium E_0

The local stability is determined by finding the eigenvalues of the Jacobian of the system at E_0. These eigenvalues are given by

$$
\begin{aligned}
\lambda_1 &= -\mu \\
\lambda_2 &= \beta_1\frac{\Lambda}{\mu} - (\mu+\delta_1+\gamma) \\
\lambda_3 &= -(\mu+\sigma+\delta_2) \\
\lambda_4 &= -(\mu+\eta)
\end{aligned}
$$

E_0 would be stable when $\lambda_2 < 0$, that is, when $R_0 < 1$. Hence, $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$ *is locally asymptotically stable when* $R_0 < 1$.

Global Stability of E_0

Defining a Lyapunov function

$$L_1(D, T, R) = \sigma D + \sigma T + (\mu + \delta_2 + \sigma)R$$

For global asymptotic stability, $\frac{dL_1}{dt}$ must be globally negative definite. On taking the time derivative and after some algebra, we obtain:

$$\frac{dL_1}{dt} \leq \sigma \left[\beta_1 \frac{\Lambda}{\mu} - (\mu + \delta_1)\right] D$$

We can see that $\frac{dL_1}{dt} < 0$ iff $\beta_1 \frac{\Lambda}{\mu} < \mu + \delta_1$ (since it is not possible for D or σ to be negative. Also, $\frac{dL_1}{dt} = 0$ iff $D = 0$. This is possible for only one point, and that is E_0.

Hence, E_0 is globally asymptotically stable when $\beta_1 \frac{\Lambda}{\mu} < \mu + \delta_1$.

Bifurcation Analysis, setting $R_0 = 1$

For our bifurcation analysis, we use $\beta_1^* = \beta_1$ as the bifurcation parameter, since we have seen that R_0 is most sensitive to changes in β_1. Keeping R_0 constant at $R_0 = 1$, we have

$$\beta_1^* = \frac{\mu(\mu + \delta_1 + \gamma)}{\Lambda}$$

The Jacobian matrix of the dynamical system is found and evaluated at the drinking-free equilibrium E_0, at $\beta_1^* = \beta_1$. We have:

$$\begin{bmatrix} -\mu & -\beta_1^* \frac{\Lambda}{\mu} & 0 & \eta \\ 0 & 0 & 0 & 0 \\ 0 & \eta & -(\mu + \delta_2 + \sigma) & 0 \\ 0 & 0 & \sigma & -(\mu + \eta) \end{bmatrix}$$

The eigenvalues are 0, $-\mu$, $-(\mu + \delta_2 + \sigma)$ and $-(\mu + \eta)$.

The authors of the paper use *Center Manifold Theory* to study the dynamics of the system and the bifurcations. There is a theorem concerning the stability of a general system of ODEs that follows, which is quite involved. We shall skip the theorem here for sake of brevity.

After calculation of various quantities that follows from the theorem, we have the following two quantities:

$$X = \gamma \left[\eta \, \sigma \, (\mu + \delta_1 + \gamma) + \Lambda \, \beta_2 \, (\mu + \eta) \right] \tag{6}$$

$$\Gamma = (\mu + \eta)(\mu + \delta_1 + \gamma)^2 (\mu + \delta_2 + \sigma) \tag{7}$$

In summary, we have:

If $X > \Gamma$, then the system (1) has a *backward bifurcation* at $R_0 = 1$. Otherwise, if $X < \Gamma$, the system undergoes a *forward bifurcation* and the endemic equilibrium point is locally asymptotically stable for $R_0 > 1$ and close to 1.

Numerical Simulations

Numerical Integration

We solved the system of ODEs - (1a - 1d) in Python using SciPy's ODE integrating tool that performs fast, accurate numerical integration (up to order 12) for a system of ODEs. Let us look at the solutions (S(t), D(t), T(t), R(t)). We have replicated the three sets of simulations carried out by authors of the paper to show – the disease-free equilibrium, the endemic equilibrium following the forward bifurcation, and the endemic equilibrium followed by the backward bifurcation.

The first set of parameters are given in the table below. Here, the value of R_o is 0.56. In addition, the parameter values satisfy the theoretical condition for a disease-free equilibrium which is at (1.6,0,0,0) the 4D space. We have randomly generated 10 initial conditions for this set of parameters and performed the integration for each of them. The plots for the 4 population classes and the total population (N(t)) is given in figure 2. Numerically the equilibrium points are – (1.600, 2.255e-16, 5.067e-14, -9.219e-13), matching with the theoretical prediction. We also see that the total population asymptotically approaches 1.599. This is in accordance with theorem 1, which states that in the feasible region for the solutions, N(t) cannot go beyond 1.6 ($\frac{\lambda}{\mu}$), thereby giving greater verifying the correctness of the code.

Parameter	Value (set 1)	Value (set 2)	Value (set 3)
Λ	0.4 population/year	0.7 population/year	0.5 population/year
β_1	0.35/population/year	0.8/population/year	0.04/population/year
β_2	0.3/population/year	0.3/population/year	0.99/population/year
μ	0.25/year	0.25/year	0.025/year
σ	0.1/year	0.1/year	0.01/year
η	0.1/year	0.1/year	0.1/year
γ	0.4/year	0.4/year	0.9/year
δ_1	0.35/year	0.35/year	0.035/year
δ_2	0.3/year	0.3/year	0.03/year
R_o	0.56	2.24	0.83

Table 1: Parameter Values

Similarly, the second set of parameters are given in the table above. Here, the value of R_o is 2.24 (> 1). In addition, the parameter values satisfy the theoretical condition for an endemic equilibrium at – (1.16469, 0.445752, 0.227504, 0.0650011). As in the first case, integration for 10 random initial conditions were performed and the concerned plots are given in figure 3. Numerically the equilibrium points are – (1.1647, 0.4458, 0.2275, 0.0651), matching with the theoretical prediction. We also see that the total population asymptotically approaches 1.902; again, in accordance with theorem 1 according to which the total population can't cross 2.8.

In the third parameter set, the value of R_o is 0.83. Theorem 1 is satisfied once again. However, the parameter set has been chosen such that there are two possible stable equilibria for the same value of R_o (i.e. the disease-free equilibrium and the endemic equilibrium). We can see this in the plot for D(t) in figure 3 depicting the solutions for the third parameter set. For some initial conditions, the solution approaches the endemic equilibrium while for others, it approaches the DFE.

Parameter Set 1:

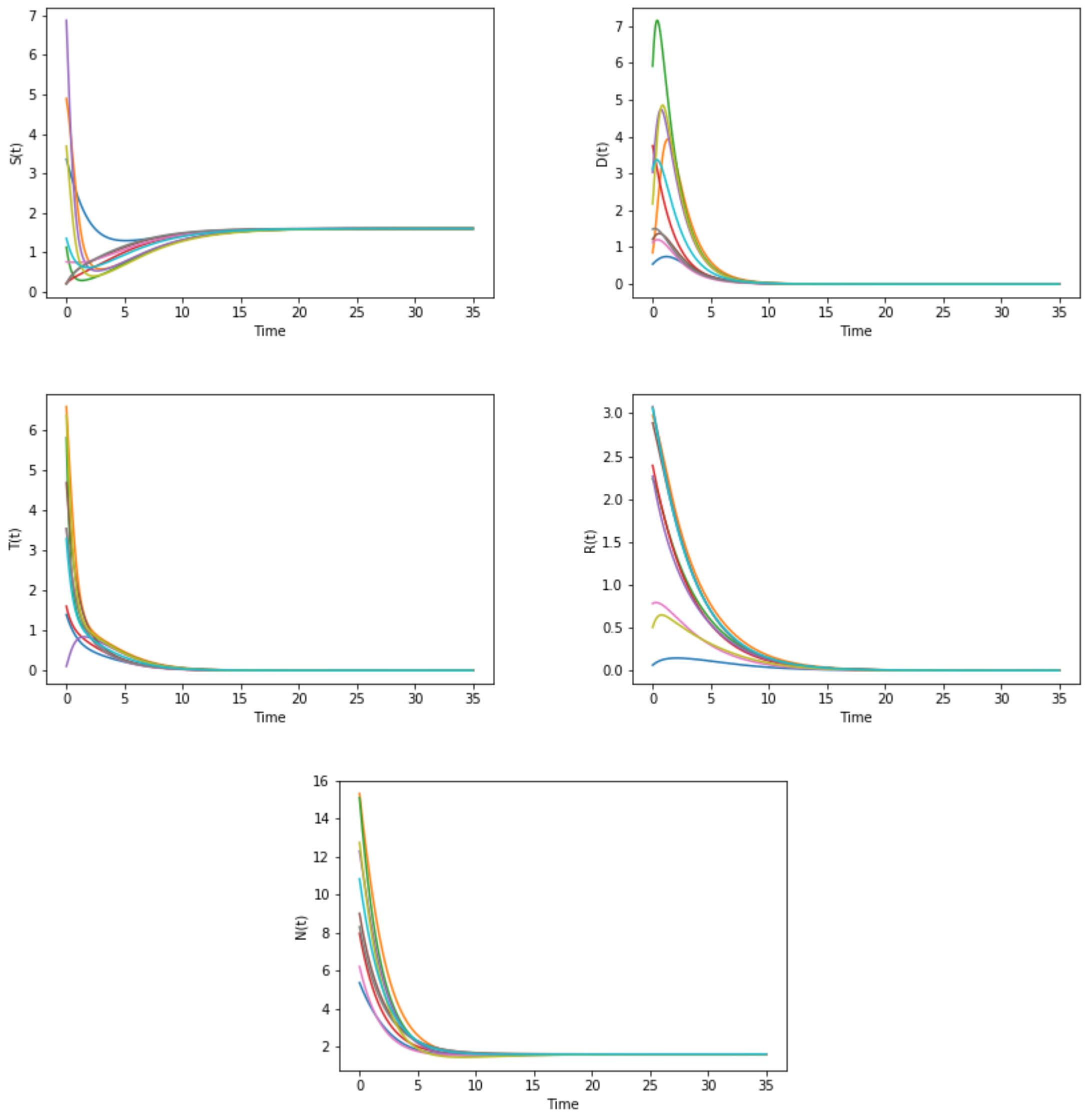


Figure 2: Parameter Set 1

Parameter Set 2:

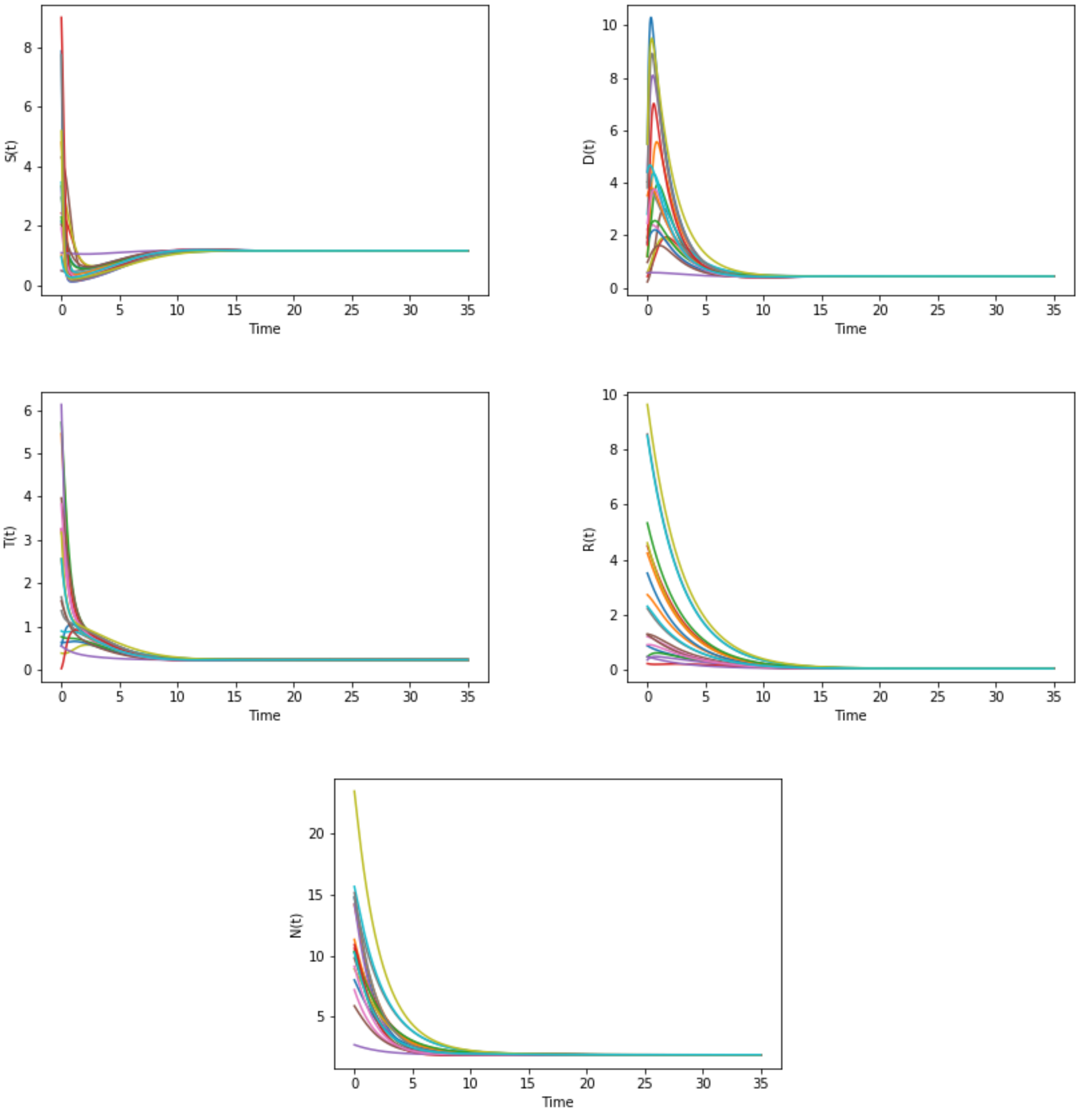


Figure 3: Parameter Set 2

Parameter Set 3:

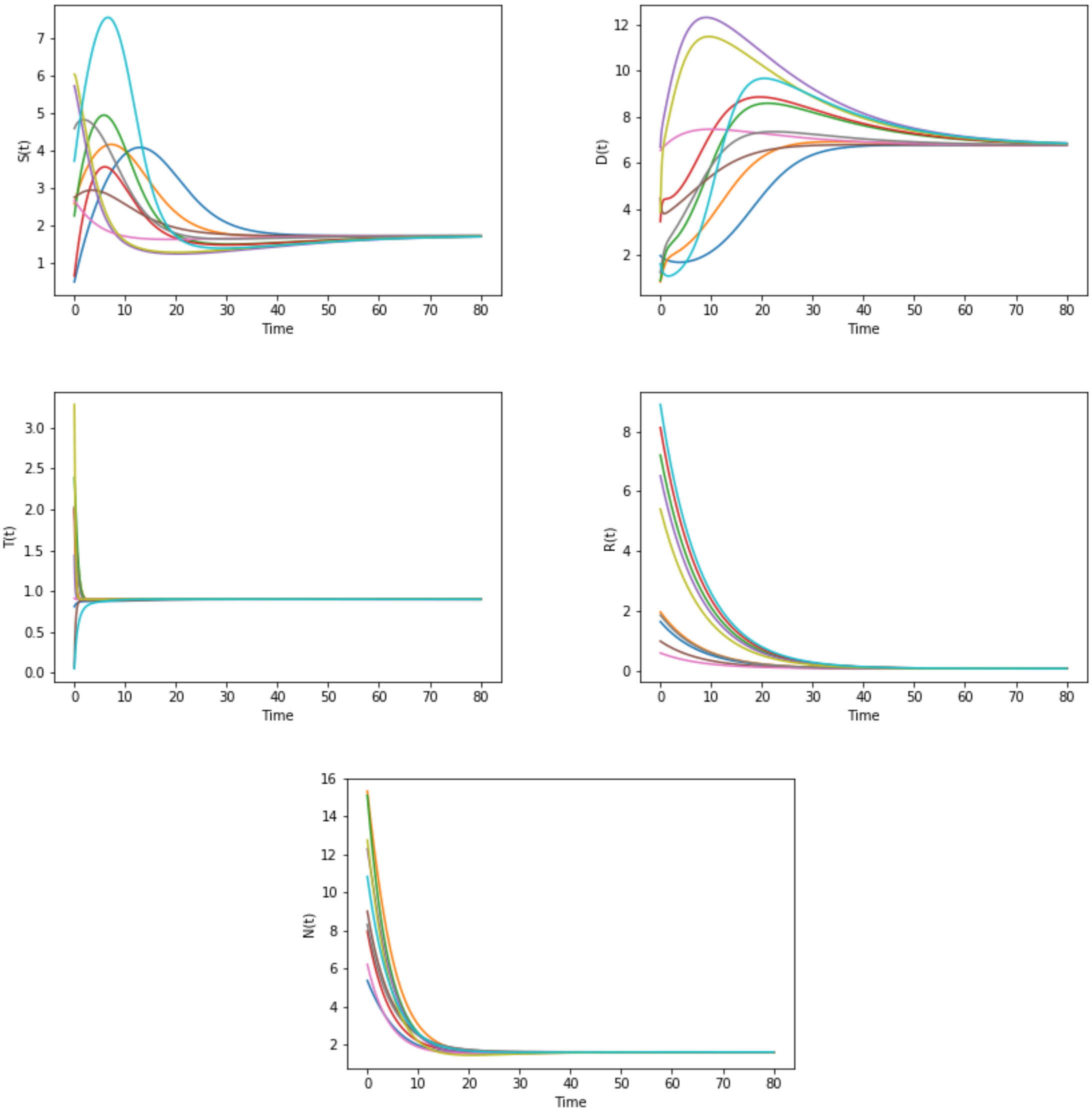
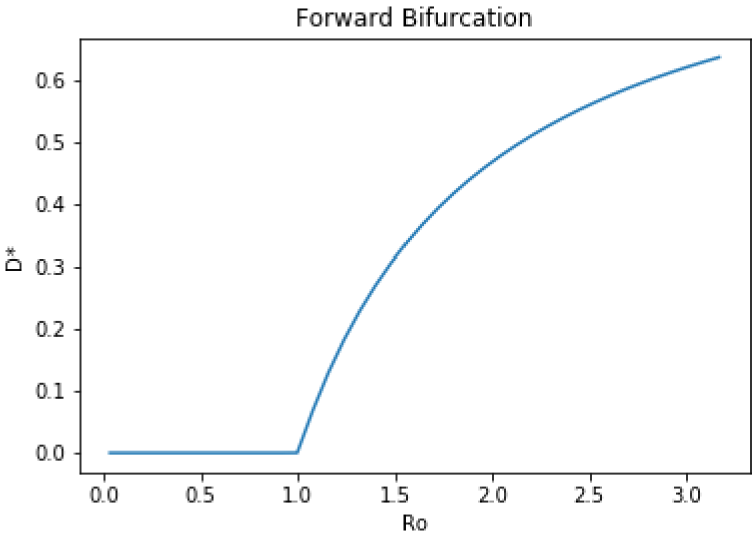
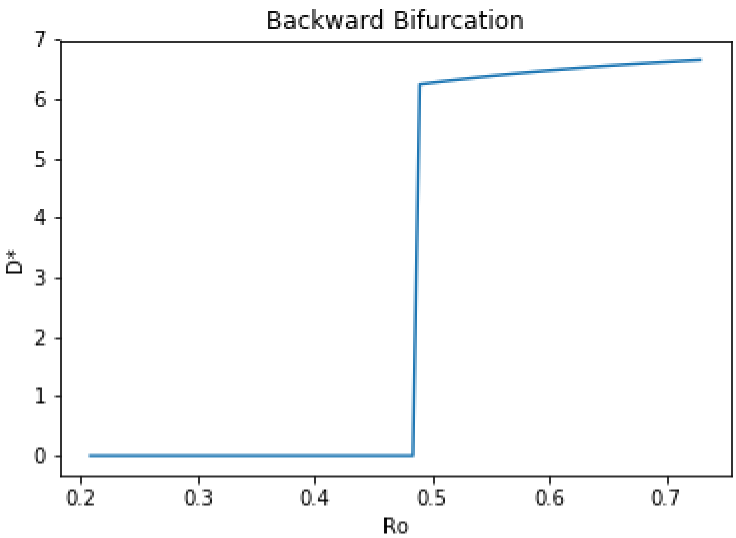


Figure 4: Parameter Set 3

Bifurcation Diagrams

We have numerically generated the forward and bifurcation diagrams (depicting only the stable equilibria) by finding the equilibrium points for various R_o values. We can see that these agree with the qualitative bifurcation diagrams. In case of the forward bifurcation diagram, a stable endemic equilibrium appears at $R_o = 1$. In case of the backward bifurcation diagram, we see that a stable endemic equilibrium appears at a critical R_o value of 0.48.





Conclusion

We have discussed nearly all the dynamical properties of the given system of an alcohol epidemic, and find that the most sensitive parameter is β_1, the transmission rate from moderate drinkers to heavy drinkers. We have analyzed the stability of various fixed points of the system, and we have observed the bifurcation characteristics of the system.

In summary, we see that the crucial difference of this theory from others is due to the fact that a relapse into drinking is taken into account, which causes a backward bifurcation between the fixed points. We note that the assumptions made in this model, although somewhat of a simplification, are still valid in most situations. To conclude, we note that the model is quite complete in itself and is an excellent improvement over its predecessors.