The purpose of this homework is to prove that the sphere can be identified with (is homeomorphic to) the quotient group $SO(n)/SO(n-1)$.

We recall the following definitions and facts

Definition 1. *A group is a nonempty set G with an operation $*$ with the following properties:*

1. *for any g, h in G, $h * g \in G$.*
2. *for any $f, g, h \in G$, $f * (g * h) = (f * g) * h$ (associativity)*
3. *there exists an element $e \in G$ such that for any $g \in G$, $g * e = e * g = g$ (neutral element)*
4. *for any $g \in G$, there exists $g^{-1} \in G$, such that $g * g^{-1} = g^{-1} * g = e$ (existence of an inverse).*

We have seen in class several examples of groups:

1. $\mathbb{Z}$ equipped with the addition,
2. the orthogonal group ($O(n)$ equipped with the matrix multiplication,
3. the unit circle, equipped with the multiplication.

Definition 2. *A subset H of a group G is a subgroup if*

1. *for any g, h in H, $h * g \in H$ (H is closed under the operation $*$)*
2. *for any $g \in H$, there exists $g^{-1} \in H$, such that $g * g^{-1} = g^{-1} * g = e$ (existence of an inverse in H).*

An example of a subgroup is the special orthogonal group $SO(n)$ formed by the elements of $O(n)$ with determinant equal to 1. This is the group of rotations.

Definition 3. *The orthogonal group $O(n)$ is the subset of $n \times n$ real matrices U such that $U^t U = UU^t = I_n$, where I_n is the $n \times n$ identity matrix.*

Definition 4. *The special orthogonal group $SO(n)$ consists of the elements of $O(n)$ with determinant 1, i.e., the rotations. It is a subgroup of $O(n)$.*

Definition 5. *A function ϕ between the group G with group operation $\bigstar$, and the group F with group operation $\square$ is called an homomorphism if φ commutes with the group operations,*

$$\forall G_1, G_2 \in G, \quad \varphi(g_1 \bigstar g_2) = \varphi(g1) \square \varphi(g_2) \tag{1}$$

Examples:

1. The exponential map from the set of reals equipped with the addition to the set of positive numbers equipped with the multiplication,

$$\begin{aligned}(\mathbb{R}, +) &\longrightarrow (\mathbb{R}_+^*, \times)\\ x &\longmapsto e^x\end{aligned}$$

2. The determinant of a matrix from the group of non singular matrices of size n, the general linear group $GL(n)$, to the set of reals with zero removed, equipped with the multiplication.

$$\begin{aligned} GL(n) &\longrightarrow (\mathbb{R}^*, \times) \\ M &\longmapsto \det(M) \end{aligned}$$

Definition 6. *Let $f : G \longrightarrow F$ be a homomorphism. f is an isomorphism if f is bijective. In this case its inverse is also a homomorphism.*

Examples:

1. The exponential map from the set of reals equipped with the addition to the set of positive numbers equipped with the multiplication is an isomorphism. The inverse is the logarithm.

2. The determinant of a matrix from the general linear group $GL(n)$ to the set of real with zero removed, equipped with the multiplication is not an isomorphism. Indeed, the map is an homomorphism, it is surjective, but not injective: if two matrices have the same determinant they need not be equal.

We would like to identify every point x on the unit sphere S^{n-1} in $\mathbb{R}^n$ with the first vector of a rotation matrix $U \in SO(n)$. The problem is that there are very many rotations U with the first column equal to x. We need therefore to lump all these matrices together: this is the process of taking the quotient of the group $SO(n)$ by $SO(n-1)$. The quotient group is similar to the concept of $\mathbb{Z}/p\mathbb{Z}$ where all integers that have the same remainder by the division by p are lumped together in the same equivalence class.

1. Prove that $O(n)$ is a group when equipped with the matrix multiplication.

2. Prove that
$$U \in O(n) \implies \det(M) = \pm 1 \tag{2}$$
where $\det(M)$ is the determinant of M.

3. Prove that $SO(n)$ is a subgroup of $O(n)$.

4. We consider the subset G of $SO(n)$ defined by
$$\mathcal{G} = \{U \in SO(n),\ Ue_1 = e_1\}, \tag{3}$$
where
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$
is the first element of the canonical basis. Prove that $\mathcal{G}$ is a subgroup of $SO(n)$.

5. Prove that every element U in $\mathcal{G}$ can be written as follows

$$U = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} \tag{5}$$

where W is $n-1 \times n-1$ matrix that belong to $SO(n-1)$.

6. Prove that the mapping Ψ defined by

$$\Psi : \mathcal{G} \longrightarrow SO(n-1) \tag{6}$$
$$U \longmapsto \Psi(U) = W \tag{7}$$
$$\tag{8}$$

where W is the $n-1 \times n-1$ matrix defined in the equation (5), is an isomorphism between $\mathcal{G}$ and $SO(n-1)$. In other words, show that

$$\text{for any } U_,U_2 \in \mathcal{G}, \quad \Psi(U_1U_2) = \Psi(U_1)\Psi(U_2), \tag{9}$$

and Ψ is bijective,

$$\text{for any} U, V \in \mathcal{G}, \quad \Psi(U) = \Psi(V) \implies U = V \qquad \text{(injective)}, \tag{10}$$
$$\text{for any} Q \in SO(n-1), \quad \text{there exists } U \in \mathcal{G}, \text{such that} \quad \Psi(U) = Q \qquad \text{(surjective)}. \tag{11}$$

We say that $\mathcal{G}$ is isomorphic[1] to $SO(n-1)$, and later we will identify G with $SO(n-1)$.

7. Prove that for any $x \in S^{n-1}$ there exists a rotation matrix $U \in SO(n)$ such that $Ue_1 = x$. Is U unique?

8. Prove that if U and V are two $n \times n$ matrices in $SO(n)$ such that $Ue_1 = Ve_1 = x$, then there exists $W \in SO(n-1)$ such that

$$U = V \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix}. \tag{12}$$

9. Equation (12) defines an equivalence class: two matrices U and V are equivalent under this relation if $Ue_1 = Ve_1 = x$, or equivalently, there exits $Q \in \mathcal{G}$ such that

$$U = VQ. \tag{13}$$

Let U be an element of $SO(n)$, the set of all elements V of the form UQ, where $Q \in \mathcal{G}$ is called a left coset. We denote it by $U\mathcal{G}$. The set of left cosets is denoted by

$$SO(n)/\mathcal{G}, \tag{14}$$

and is called the *quotient group*.

Any element in the coset $U\mathcal{G}$ is called a coset representative. Prove that if U_1 and U_2 are two coset representatives for the same coset, then

$$U_1\mathcal{G} = U_2\mathcal{G}. \tag{15}$$

In other words, when we write $U\mathcal{G}$ to describe a coset, we can pick any coset representative U in that coset.

[1]Greek: has the same structure

10. We define the map φ

$$\varphi : SO(n)/\mathcal{G} \longrightarrow S^{n-1} \tag{16}$$
$$U \longmapsto \varphi(U) = Ue_1 \tag{17}$$
$$\tag{18}$$

Prove that φ respects the action of $SO(n)$,

$$\text{for any } \Omega \in SO(n), \quad \text{and for any} U \in SO(n)/\mathcal{G}, \quad \varphi(\Omega U) = \Omega\varphi(U) \tag{19}$$

11. Prove that the map φ is bijective and continuous.

Conclusion: we can identify $\mathcal{G}$ with $SO(n-1)$, since $\mathcal{G}$ and $SO(n-1)$ are isomorphic. We can therefore identity $SO(n)/\mathcal{G}$ with $SO(n)/SO(n-1)$. Finally, we just proved that S^{n-1} is homeomorphic to $SO(n)/\mathcal{G}$: there exists a bijective continuous map that allows us to identify S^{n-1} with $SO(n)/\mathcal{G}$.

We conclude that we can identify S^{n-1} with $SO(n)/SO(n-1)$. This is interesting because it endows the sphere with a group structure that it would not have otherwise.