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Ans 1

Expected adjacency matrix M = $\mathbb{E}$[A] computed over all possible realizations of $\mathcal{B}$ is calculated as follows:
$\mathbb{E}$[B(p)] = 1 × p + 0 × (1 - p) = p and $\mathbb{E}$[B(q)] = 1 × q + 0 × (1 - q) = q.
Given T is a random permutation matrix,

$$T = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$T^T = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

Therefore, $A = T\mathcal{B}T^T = \mathcal{B}$. By substituting the Bernoulli random variable values, we get,

$$M = \begin{bmatrix} p & \dots & p & q & \dots & q \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ p & \dots & p & q & \dots & q \\ q & \dots & q & p & \dots & p \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ q & \dots & q & p & \dots & p \end{bmatrix}$$

Ans 2

The degree matrix is defined as the diagonal matrix with entries $d_i = \sum_{j=1}^{n} A_{i,j} = \frac{1}{2}(n \times p + n \times q) = \frac{n}{2}(p+q)$.
Therefore, the expected degree matrix is,

$$\mathbb{E}[D] = \begin{bmatrix} \frac{n}{2}(p+q) & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \frac{n}{2}(p+q) & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{n}{2}(q+p) & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{n}{2}(q+p) \end{bmatrix}$$

Ans 3

If w_1 is an eigenvector of M, then $Mw_1 = \lambda w_1$. Given $w_1 = \frac{1}{\sqrt{n}}\mathcal{I}$ where $\mathcal{I}$ =1 $\forall$ i = 1, . . . , n.

$$w_1 = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$$

Then,

$$Mw_1 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \\ \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \end{bmatrix}$$

Since Mw_1 can be expressed as λw_1, we can say that w_1 is an eigenvector of M.

Also the eigenvalue μ_1 is given by,

$$Mw_1 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \\ \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \end{bmatrix} = \lambda w_1 = \begin{bmatrix} \frac{\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{\lambda}{\sqrt{n}} \\ \frac{\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{\lambda}{\sqrt{n}} \end{bmatrix}$$

Therefore, we have,

$\frac{n}{2\sqrt{n}}(p+q) = \frac{\lambda}{\sqrt{n}}$

$\lambda = \frac{n}{2}(p+q)$

Ans 4

To prove that w_2 is an eigenvector of M. We should be able to prove that $Mw_2 = \lambda w_2$.

Let

$$w_2 = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \frac{-1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{-1}{\sqrt{n}} \end{bmatrix}$$

Then,

$$Mw_2 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p-q) \\ \frac{n}{2\sqrt{n}}(q-p) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(q-p) \end{bmatrix} = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p-q) \\ \frac{-n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{-n}{2\sqrt{n}}(p-q) \end{bmatrix}$$

Since Mw_2 can be expressed as λw_2, we can say that w_2 is an eigenvector of M.

Also the eigenvalue μ_2 is given by,

$$Mw_2 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p-q) \\ \frac{-n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{-n}{2\sqrt{n}}(p-q) \end{bmatrix} = \lambda w_2 = \begin{bmatrix} \frac{\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{\lambda}{\sqrt{n}} \\ \frac{-\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{-\lambda}{\sqrt{n}} \end{bmatrix}$$

Therefore, we have,

$\frac{n}{2\sqrt{n}}(p-q) = \frac{\lambda}{\sqrt{n}}$

$\lambda = \frac{n}{2}(p-q)$

Ans 5

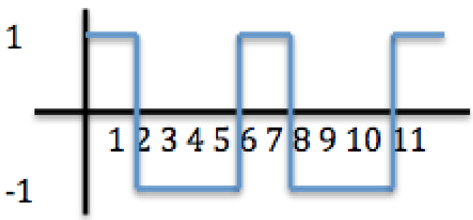

Figure 1: Graph of w_3

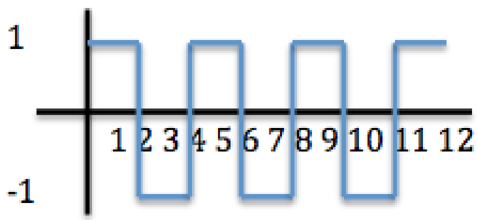

Figure 2: Graph of w_4

Ans 6

We need to prove that $Mw_3 = 0$ and $Mw_4 = 0$. We have

$$w_3 = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \frac{-1}{\sqrt{n}} \\ \vdots \\ \frac{-1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$$

Therefore,

$$Mw_3 = \begin{bmatrix} \frac{n}{4\sqrt{n}}(p+q) - \frac{n}{2\sqrt{n}}(p+q) + \frac{n}{4\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{4\sqrt{n}}(p+q) - \frac{n}{2\sqrt{n}}(p+q) + \frac{n}{4\sqrt{n}}(p+q) \\ \frac{n}{4\sqrt{n}}(p+q) - \frac{n}{2\sqrt{n}}(p+q) + \frac{n}{4\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{4\sqrt{n}}(p+q) - \frac{n}{2\sqrt{n}}(p+q) + \frac{n}{4\sqrt{n}}(p+q) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$$w_4 = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{-1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{-1}{\sqrt{n}} \\ \vdots \\ \frac{-1}{\sqrt{n}} \end{bmatrix}$$

Therefore,

$$Mw_4 = \begin{bmatrix} \frac{n}{4\sqrt{n}}(p) - \frac{n}{4\sqrt{n}}(q) + \frac{n}{4\sqrt{n}}(q) - \frac{n}{4\sqrt{n}}(p) \\ \vdots \\ \vdots \\ \frac{n}{4\sqrt{n}}(p) - \frac{n}{4\sqrt{n}}(q) + \frac{n}{4\sqrt{n}}(q) - \frac{n}{4\sqrt{n}}(p) \\ \frac{n}{4\sqrt{n}}(p) - \frac{n}{4\sqrt{n}}(q) + \frac{n}{4\sqrt{n}}(q) - \frac{n}{4\sqrt{n}}(p) \\ \vdots \\ \vdots \\ \frac{n}{4\sqrt{n}}(p) - \frac{n}{4\sqrt{n}}(q) + \frac{n}{4\sqrt{n}}(q) - \frac{n}{4\sqrt{n}}(p) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Ans 7

We have $\mu_1 = \frac{n(p+q)}{2}$ and $\mu_2 = \frac{n(p-q)}{2}$. To prove that $M = \mu_1 w_1 w_1^T + \mu_2 w_2 w_2^T$, lets multiply the corresponding matrices. By doing so, we get,

$$\begin{bmatrix} \frac{n(p+q)}{2\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{n(p+q)}{2\sqrt{n}} \\ \frac{n(p+q)}{2\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{n(p+q)}{2\sqrt{n}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} & \cdots & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix} + \begin{bmatrix} \frac{n(p-q)}{2\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{n(p-q)}{2\sqrt{n}} \\ \frac{n(q-p)}{2\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{n(q-p)}{2\sqrt{n}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} & \cdots & \cdots & \frac{1}{\sqrt{n}} & \frac{-1}{\sqrt{n}} & \cdots & \cdots & \frac{-1}{\sqrt{n}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{p+q}{2} & \dots & \frac{p+q}{2} & \frac{q+p}{2} & \dots & \frac{q+p}{2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{p+q}{2} & \dots & \frac{p+q}{2} & \frac{q+p}{2} & \dots & \frac{q+p}{2} \\ \frac{q+p}{2} & \dots & \frac{q+p}{2} & \frac{p+q}{2} & \dots & \frac{p+q}{2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{q+p}{2} & \dots & \frac{q+p}{2} & \frac{p+q}{2} & \dots & \frac{p+q}{2} \end{bmatrix} + \begin{bmatrix} \frac{p-q}{2} & \dots & \frac{p-q}{2} & \frac{q-p}{2} & \dots & \frac{q-p}{2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{p-q}{2} & \dots & \frac{p-q}{2} & \frac{q-p}{2} & \dots & \frac{q-p}{2} \\ \frac{q-p}{2} & \dots & \frac{q-p}{2} & \frac{p-q}{2} & \dots & \frac{p-q}{2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{q-p}{2} & \dots & \frac{q-p}{2} & \frac{p-q}{2} & \dots & \frac{p-q}{2} \end{bmatrix} = \begin{bmatrix} p & \dots & p & q & \dots & q \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ p & \dots & p & q & \dots & q \\ q & \dots & q & p & \dots & p \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ q & \dots & q & p & \dots & p \end{bmatrix} = M$$

Ans 8

The steps involved in the algorithm are:

1. Compute the eigenvalues μ_1 and μ_2.
2. Compute the eigenvectors w_1 and w_2.
3. To recover the two communities using the eigenvectors w_1 and w_2, calculate the matrix $\mu_1 w_1 w_1^T$ and $\mu_2 w_2 w_2^T$ (where w_1^T and w_2^T are the transpose of the eigenvectors w_1 and w_2 respectively).
4. The matrices $\mu_1 w_1 w_1^T$ and $\mu_2 w_2 w_2^T$ thus represent the two communities.

Ans 9

The upper triangular matrix of 'X' would be as follows:

Let

$$C = \begin{cases} \text{1-p} & \text{with probability p,} \\ \text{-p} & \text{with probability 1-p} \end{cases}$$

and

$$D = \begin{cases} \text{1-q} & \text{with probability q,} \\ \text{-q} & \text{with probability 1-q} \end{cases}$$

Then, the symmetric random matrix is defined as:

$$X = \begin{bmatrix} C & \dots & C & D & \dots & D \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ C & \dots & C & D & \dots & D \\ D & \dots & D & C & \dots & C \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ D & \dots & D & C & \dots & C \end{bmatrix}$$

$$\mathbb{E}[X] = \begin{bmatrix} (1-p)p+(-p)(1-p) & \dots & (1-p)p+(-p)(1-p) & (1-q)q+(-q)(1-q) & \dots & (1-q)q+(-q)(1-q) \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ (1-p)p+(-p)(1-p) & \dots & (1-p)p+(-p)(1-p) & (1-q)q+(-q)(1-q) & \dots & (1-q)q+(-q)(1-q) \\ (1-q)q+(-q)(1-q) & \dots & (1-q)q+(-q)(1-q) & (1-p)p+(-p)(1-p) & \dots & (1-p)p+(-p)(1-p) \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ (1-q)q+(-q)(1-q) & \dots & (1-q)q+(-q)(1-q) & (1-p)p+(-p)(1-p) & \dots & (1-p)p+(-p)(1-p) \end{bmatrix}$$

$$\mathbb{E}[X] = \begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

Thus $\mathbb{E}[X] = 0$

Ans 10

Given that $\lambda_2 = \frac{n(p-q)}{2} + \frac{p+q}{p-q}$. We know that the arithmetic mean $\geq$ geometric mean, using this, we get,

$\frac{1}{2}(\frac{n(p-q)}{2} + \frac{p+q}{p-q}) \geq \sqrt{\frac{n(p+q)}{2}}$

$\frac{n(p-q)}{2} + \frac{p+q}{p-q} \geq \sqrt{2n(p+q)}$

Given that $n(p-q) > \sqrt{2n(p+q)}$.

Using this equation, if we can prove $\frac{n(p-q)}{2} + \frac{p+q}{p-q} \neq \sqrt{2n(p+q)}$, then we can state that λ_2 can be separated from the continuous "semi-circle" bulk as λ_2 will no longer lie on the edge of the semi circle.

If the inequality $n(p-q) > \sqrt{2n(p+q)}$ strictly holds true, adding a terms $\frac{n(p-q)}{2}$ and $\frac{p+q}{p-q}$ will push the value of λ_2 away from the edge of the semi-circle making the two terms $\frac{n(p-q)}{2} + \frac{p+q}{p-q}$ and $\sqrt{2n(p+q)}$ not equal to each other. Thus, we can state that $\frac{n(p-q)}{2} + \frac{p+q}{p-q} \neq \sqrt{2n(p+q)}$ and hence λ_2 can be separated from the continuous "semi-circle" bulk.

Ans 11

Given $p = \alpha\frac{\log(n)}{n}$ and $q = \beta\frac{\log(n)}{n}$. Since the condition $n(p-q) > \sqrt{2n(p+q)}$ holds good, we have,

$n(\alpha\frac{\log(n)}{n} - \beta\frac{\log(n)}{n}) > \sqrt{2n(\alpha\frac{\log(n)}{n} + \beta\frac{\log(n)}{n})}$

$\log(n)(\alpha - \beta) > \sqrt{2\log(n)(\alpha+\beta)}$

$\alpha - \beta > \frac{\sqrt{2}\sqrt{\log(n)}\sqrt{\alpha+\beta}}{\log(n)}$

$\alpha - \beta > \frac{\sqrt{2}\sqrt{\alpha+\beta}}{\sqrt{\log(n)}}$

$\alpha - \beta > \frac{2\sqrt{\alpha+\beta}}{\sqrt{2}\sqrt{\log(n)}}$

Upon rearranging terms, we get,

$\alpha - \beta > \frac{2}{\sqrt{\log(n)}}\sqrt{\frac{\alpha+\beta}{2}}$

Ans 12

Given $p = \frac{a}{n}$ and $q = \frac{b}{n}$. Since the condition $n(p-q) > \sqrt{2n(p+q)}$ holds good, we have,

$n(\frac{a-b}{n}) > \sqrt{2n(\frac{a+b}{n})}$

$a - b > \sqrt{2(a+b)}$

$a - b > \sqrt{2}\sqrt{a+b}$

$a - b > \frac{2}{\sqrt{2}}\sqrt{a+b}$

Upon rearranging terms, we get,

$\frac{a-b}{2} > \sqrt{\frac{a+b}{2}}$