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## Ans 1

To prove $O(n)$ is a group when equipped with the matrix multiplication, we need to prove that the 4 points mentioned in Definition 1 hold good.

**Point 1:** We need to prove that on multiplying two orthogonal matrices in $O(n)$, we get back an orthogonal matrix thus meaning the resultant matrix is also in $O(n)$.
Let us consider two orthogonal matrices G and H, then,

$(GG^T)(HH^T) = (GH)(G^TH^T) = (HG)(HG)^T = I$. This means that the resultant matrix is in the group $O(n)$.

**Proof by example:** $A_{n\times n}$ is an orthogonal matrix if, $AA^T = A^TA = I$. Consider the multiplication of following orthogonal matrices,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

The transpose of this matrix is ,

$$A^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$AA^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Also,

$$A^TA = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

The resultant matrix also belongs to group $O(n)$.

**Point 2:** We need to prove that matrix multiplication of 3 matrices that belong to $O(n)$ is associative.
Let us consider three orthogonal matrices F, G and H, then,

$\sum_p \sum_q F_{ip}G_{pq}H_{qj} = \sum_p F_{ip}(\sum_q G_{pq}H_{qj}) = \sum_p F_{ip}(GH)_{pj} = F(GH)$
$\sum_p \sum_q F_{ip}G_{pq}H_{qj} = \sum_q(\sum_p F_{ip}G_{pq})H_{qj} = \sum_q (FG)_{iq}H_{qj} = (FG)H$

**Point 3:** Let the element that belongs to the group $O(n)$ be an identity matrix (as Identity matrix is also orthogonal), then for another orthogonal matrix G $\in O(n)$, we get,

$G(I) = I(G) = G$. This is because matrix multiplication of an identity matrix and any other matrix is commutative in nature and returns the original matrix.

**Point 4:** We need to prove that the multiplication of an inverse and the matrix itself is an identity matrix which also belongs to the group $O(n)$.
Since, inverse of an orthogonal matrix is also orthogonal, it follows the Point 1 stated above where we check for the multiplication of two orthogonal matrices.
In this case, we get,
$G(G^{-1}) = I$ and $G^{-1}(G) = I$. Thus we can prove that the identity matrix also belongs to $O(n)$, we can say an inverse exists in $O(n)$.

## Ans 2

Since $U \in O(n)$, this means that U is in the group of orthogonal matrices, so for any matrix M that belongs to this group, we know that,

$M^T = M^{-1}$ (The transpose of M is the inverse of M).

Also, $MM^{-1} = I$

Therefore,

$MM^T = I$

Now taking a determinant on both sides, $det(MM^T) = detI$

We know that $detM * detM^T = 1$ (since $detI = 1$).

Now $detM * detM = 1$ (since $detM^T = detM$)

Meaning $(detM)^2 = 1$

Therefore, det(M) = ± 1.

## Ans 3

To prove $SO(n)$ is a subgroup of $O(n)$, we need to prove that the 2 points mentioned in Definition 2 hold good.

**Point 1:** To prove this, let us consider two orthogonal matrices G, H $\in SO(n)$, this means that the product of two orthogonal matrices should belong to the $SO(n)$.

**Proof by example:** Consider the multiplication of the following orthogonal matrices that below to $SO(n)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of the resultant matrix = 1 which belongs to $SO(n)$ which is an special orthogonal matrix. Thus the resultant matrix also belongs to $SO(n)$.

**Point 2:** We need to prove that the inverse of a matrix and the multiplication of the same matrix is an identity matrix.

Since, inverse of an orthogonal matrix is also orthogonal, it follows the Point 1 stated above where we check for the multiplication of two orthogonal matrices.
In this case, we get,
$G(G^{-1}) = I$ and $G^{-1}(G) = I$. Thus we can prove that the identity matrix also belongs to $SO(n)$, we can say an inverse exists in $SO(n)$.

## Ans 4

Given that

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $\mathcal{G} = \{U \in SO(n), Ue_1 = e_1\}$

Let two elements $U_1, U_2 \in \mathcal{G}$, then $U_1e_1 = e_1$ and $U_2e_1 = e_1$.

(a) Property 1 Proof: Since every element in $\mathcal{G} \in SO(n)$, we have,

$U_1U_2(U_1U_2)^T = U_1U_2U_2^TU_1^T = I * I = I$ and

$(U_1U_2)e_1 = U_1U_2e_1 = U_1e_1 = e_1$.

Therefore $\mathcal{G}$ is closed under the operator *.

(b) Property 2 Proof: Since $U \in SO(n)$, we have ,

$U^{-1} = U^T$ and $U * U^T = I$. Therefore, $U * U^{-1} = U^{-1} * U = I$. Therefore $U^{-1} \in SO(n)$.

Hence $\mathcal{G}$ is a subgroup of SO(n).

## Ans 5

Given that W is $n-1 \times n-1$ matrix that belongs to $SO(n-1)$. That means, det(W) =1 and $WW^T = I$.

Then

$$UU^T = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & W^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & WW^T \end{bmatrix}$$

$WW^T = I$ only if W $\in SO(n-1)$ and since it is given, $UU^T = I$. Property 1 holds true.

And,

$$Ue_1 = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since W is $n-1 \times n-1$ matrix, the result is $e_1$. Thus, property 2 holds true.

We can say every element U in $\mathcal{G}$ is of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix}$$

## Ans 6

(a) Since $\mathcal{G} \longrightarrow SO(n-1)$ and $U \longmapsto \Psi(U) = W$ and if $U_1U_2 \in \mathcal{G}$, we have,

$$\Psi(U_1U_2) = \begin{bmatrix} 1 & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & W_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & W_1W_2 \end{bmatrix}$$

$$\Psi(U_1) = \begin{bmatrix} 1 & 0 \\ 0 & W_1 \end{bmatrix}$$

$$\Psi(U_2) = \begin{bmatrix} 1 & 0 \\ 0 & W_2 \end{bmatrix}$$

$$\Psi(U_1)\Psi(U_2) = \begin{bmatrix} 1 & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & W_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & W_1W_2 \end{bmatrix}$$

(b) To prove $\Psi$ is bijective, it needs to be injective and surjective.

(i) Since $\Psi(U) = \Psi(V)$ and U, V $\in \mathcal{G}$, we must prove that U = V. Since for any output of the $\Psi$ function gives the same result which is 'W'. That means that the inputs are also equal for those outputs i.e., U = V.

(ii) Since $\Psi(U) = W$ and $\Psi(U) = Q$, we get, W = Q and since Q $\in$ SO(n-1), W $\in$ SO(n-1), we get that,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix}$$

which states U $\in \mathcal{G}$. Hence proved it is surjective as well. Thus function $\Psi$ is bijective.

## Ans 7

For any $x \in S^{n-1}$, there exists a $U \in SO(n)$. For the condition $Ue_1 = x$ to be satisfied, U can be constructed in such a way that the first column of U is similar to x and all other columns in U are both orthogonal to each other and the first column. Only then , the condition can be satisfied. Thus this makes U not to be unique as you can have multiple columns in U where they are orthogonal to each other and the first column since $U \in SO(n)$.

## Ans 8

Given U and V are two n $\times$ n matrices in SO(n) such that $Ue_1 = Ve_1 = x$. For the condition

$$U = V \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix}$$

to be satisfied, to get a n $\times$ n matrix after multiplying an n$\times$n matrix with another n$\times$n matrix that contains W, it is necessary that W must $\in$ SO(n-1) so that resulting U can be orthogonal and also matrix multiplication V and

$$\begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix}$$

will be possible.

## Ans 9

## Ans 10

Since $\Omega \in SO(n)$ and $U \in \frac{SO(n)}{\mathcal{G}}$, we have $\varphi(\Omega U) = \Omega U e_1$ and since $\varphi(U) = Ue_1$, we finally get $\varphi(\Omega U) = \Omega\varphi(U)$. Also we can state that multiplication of $\Omega U$ which result in a mapping that respects the action of $SO(n)$ since $\Omega$ and $U$ already $\in SO(n)$. Hence proved.

# Ans 11

(a) To prove that $\varphi$ is bijective, we need to prove that:

1. for any U,V $\in \mathcal{G}$, $\varphi(U) = \varphi(V) \Longrightarrow U = V$.
2. for any $x \in S^{n-1}$, there exists $U \in \frac{SO(n)}{\mathcal{G}}$, such that $\varphi(U) = x$.

To prove that the map $\varphi$ is injective, we know that $\varphi(U) = Ue_1$ and $\varphi(V) = Ve_1$. Since $\varphi(U) = \varphi(V)$, we have $Ue_1 = Ve_1$. Since U $\in \mathcal{G}$ and V $\in \mathcal{G}$, from this we get $U = V$. Hence map $\varphi$ is injective.

To prove that the map $\varphi$ is surjective, since $\varphi(U) = Ue_1$. We get that $Ue_1 = x$. This proves that $U \in SO(n)$. Since $\mathcal{G}$ is a subgroup of SO(n), we can state that $U$ also $\in \frac{SO(n)}{\mathcal{G}}$. Hence proved.

Thus $\varphi$ is bijective.

(b) The map $\varphi$ is continuous at a point 'a' $\in S^{n-1}$ if,

$\forall \epsilon > 0$, $\exists \delta > 0$, such that $|x - a| < \delta \Longrightarrow |\varphi(x) - \varphi(a)| < \epsilon$.

Since $\varphi(U) = Ue_1$, we need to prove that:

$\forall \epsilon > 0$, $\exists \delta > 0$, such that $|x - a| < \delta \Longrightarrow |xe_1 - ae_1| < \epsilon$.