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Ans 1

Expected adjacency matrix M = $\mathbb{E}$[A] computed over all possible realizations of $\mathcal{B}$ is calculated as follows:

$\mathbb{E}$[B(p)] = 1 × p + 0 × (1 - p) = p and $\mathbb{E}$[B(q)] = 1 × q + 0 × (1 - q) = q.

Given T is a random permutation matrix,

$$T = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$T^T = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

Therefore, $A = T\mathcal{B}T^T = \mathcal{B}$. By substituting the Bernoulli random variable values, we get,

$$M = \begin{bmatrix} p & \dots & p & q & \dots & q \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ p & \dots & p & q & \dots & q \\ q & \dots & q & p & \dots & p \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ q & \dots & q & p & \dots & p \end{bmatrix}$$

Ans 2

The degree matrix is defined as the diagonal matrix with entries $d_i = \sum_{j=1}^{n} A_{i,j} = \frac{1}{2}(n \times p + n \times q) = \frac{n}{2}(p+q)$. Therefore, the expected degree matrix is,

$$\mathbb{E}[D] = \begin{bmatrix} \frac{n}{2}(p+q) & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \frac{n}{2}(p+q) & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{n}{2}(q+p) & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{n}{2}(q+p) \end{bmatrix}$$

Ans 3

If w_1 is an eigenvector of M, then $Mw_1 = \lambda w_1$. Given $w_1 = \frac{1}{\sqrt{n}}\mathcal{I}$ where $\mathcal{I}$ =1 $\forall$ i = 1, ... , n.

$$w_1 = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$$

Then,

$$Mw_1 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \\ \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \end{bmatrix}$$

Since Mw_1 can be expressed as λw_1, we can say that w_1 is an eigenvector of M.

Also the eigenvalue μ_1 is given by,

$$Mw_1 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \\ \frac{n}{2\sqrt{n}}(p+q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p+q) \end{bmatrix} = \lambda w_1 = \begin{bmatrix} \frac{\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{\lambda}{\sqrt{n}} \\ \frac{\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{\lambda}{\sqrt{n}} \end{bmatrix}$$

Therefore, we have,

$\frac{n}{2\sqrt{n}}(p+q) = \frac{\lambda}{\sqrt{n}}$

$\lambda = \frac{n}{2}(p+q)$

Ans 4

To prove that w_2 is an eigenvector of M. We should be able to prove that $Mw_2 = \lambda w_2$.

Let

$$w_2 = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \frac{-1}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{-1}{\sqrt{n}} \end{bmatrix}$$

Then,

$$Mw_2 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p-q) \\ \frac{n}{2\sqrt{n}}(q-p) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(q-p) \end{bmatrix} = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p-q) \\ \frac{-n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{-n}{2\sqrt{n}}(p-q) \end{bmatrix}$$

Since Mw_2 can be expressed as λw_2, we can say that w_2 is an eigenvector of M.

Also the eigenvalue μ_2 is given by,

$$Mw_2 = \begin{bmatrix} \frac{n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{n}{2\sqrt{n}}(p-q) \\ \frac{-n}{2\sqrt{n}}(p-q) \\ \vdots \\ \vdots \\ \frac{-n}{2\sqrt{n}}(p-q) \end{bmatrix} = \lambda w_2 = \begin{bmatrix} \frac{\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{\lambda}{\sqrt{n}} \\ \frac{-\lambda}{\sqrt{n}} \\ \vdots \\ \vdots \\ \frac{-\lambda}{\sqrt{n}} \end{bmatrix}$$

Therefore, we have,

$\frac{n}{2\sqrt{n}}(p-q) = \frac{\lambda}{\sqrt{n}}$

$\lambda = \frac{n}{2}(p-q)$

Ans 5

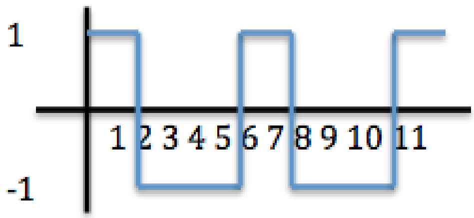

Figure 1: Graph of w_3

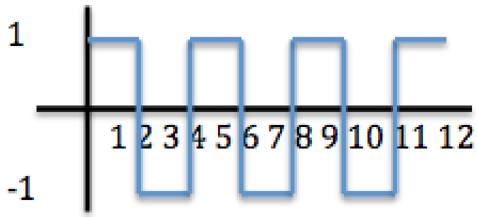

Figure 2: Graph of w_4

Ans 6