

# PROOFS IN MATHEMATICS

❖ *Proofs are to Mathematics what calligraphy is to poetry.  
 Mathematical works do consist of proofs just as  
 poems do consist of characters.*  
 — VLADIMIR ARNOLD ❖

## A.1.1 Introduction

In Classes IX, X and XI, we have learnt about the concepts of a statement, compound statement, negation, converse and contrapositive of a statement; axioms, conjectures, theorems and deductive reasoning.

Here, we will discuss various methods of proving mathematical propositions.

## A.1.2 What is a Proof?

Proof of a mathematical statement consists of sequence of statements, each statement being justified with a definition or an axiom or a proposition that is previously established by the method of deduction using only the allowed logical rules.

Thus, each proof is a chain of deductive arguments each of which has its premises and conclusions. Many a times, we prove a proposition directly from what is given in the proposition. But some times it is easier to prove an equivalent proposition rather than proving the proposition itself. This leads to, two ways of proving a proposition directly or indirectly and the proofs obtained are called direct proof and indirect proof and further each has three different ways of proving which is discussed below.

**Direct Proof** It is the proof of a proposition in which we directly start the proof with what is given in the proposition.

- (i) **Straight forward approach** It is a chain of arguments which leads directly from what is given or assumed, with the help of axioms, definitions or already proved theorems, to what is to be proved using rules of logic.

Consider the following example:

**Example 1** Show that if  $x^2 - 5x + 6 = 0$ , then  $x = 3$  or  $x = 2$ .

**Solution**  $x^2 - 5x + 6 = 0$  (given)

- $\Rightarrow (x - 3)(x - 2) = 0$  (replacing an expression by an equal/equivalent expression)  
 $\Rightarrow x - 3 = 0$  or  $x - 2 = 0$  (from the established theorem  $ab = 0 \Rightarrow$  either  $a = 0$  or  $b = 0$ , for  $a, b$  in  $\mathbf{R}$ )  
 $\Rightarrow x - 3 + 3 = 0 + 3$  or  $x - 2 + 2 = 0 + 2$  (adding equal quantities on either side of the equation does not alter the nature of the equation)  
 $\Rightarrow x + 0 = 3$  or  $x + 0 = 2$  (using the identity property of integers under addition)  
 $\Rightarrow x = 3$  or  $x = 2$  (using the identity property of integers under addition)

Hence,  $x^2 - 5x + 6 = 0$  implies  $x = 3$  or  $x = 2$ .

**Explanation** Let  $p$  be the given statement “ $x^2 - 5x + 6 = 0$ ” and  $q$  be the conclusion statement “ $x = 3$  or  $x = 2$ ”.

From the statement  $p$ , we deduced the statement  $r$ : “ $(x - 3)(x - 2) = 0$ ” by replacing the expression  $x^2 - 5x + 6$  in the statement  $p$  by another expression  $(x - 3)(x - 2)$  which is equal to  $x^2 - 5x + 6$ .

There arise two questions:

- How does the expression  $(x - 3)(x - 2)$  is equal to the expression  $x^2 - 5x + 6$ ?
- How can we replace an expression with another expression which is equal to the former?

The first one is proved in earlier classes by factorization, i.e.,

$$x^2 - 5x + 6 = x^2 - 3x - 2x + 6 = x(x - 3) - 2(x - 3) = (x - 3)(x - 2).$$

The second one is by valid form of argumentation (rules of logic)

Next this statement  $r$  becomes premises or given and deduce the statement  $s$  “ $x - 3 = 0$  or  $x - 2 = 0$ ” and the reasons are given in the brackets.

This process continues till we reach the conclusion.

The symbolic equivalent of the argument is to prove by deduction that  $p \Rightarrow q$  is true.

Starting with  $p$ , we deduce  $p \Rightarrow r \Rightarrow s \Rightarrow \dots \Rightarrow q$ . This implies that “ $p \Rightarrow q$ ” is true.

**Example 2** Prove that the function  $f: \mathbf{R} \rightarrow \mathbf{R}$

defined by  $f(x) = 2x + 5$  is one-one.

**Solution** Note that a function  $f$  is one-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ (definition of one-one function)}$$

Now, given that  $f(x_1) = f(x_2)$ , i.e.,  $2x_1 + 5 = 2x_2 + 5$

$$\Rightarrow 2x_1 + 5 - 5 = 2x_2 + 5 - 5 \text{ (adding the same quantity on both sides)}$$

$$\begin{aligned}
 \Rightarrow & 2x_1 + 0 = 2x_2 + 0 \\
 \Rightarrow & 2x_1 = 2x_2 \text{ (using additive identity of real number)} \\
 \Rightarrow & \frac{2}{2} x_1 = \frac{2}{2} x_2 \text{ (dividing by the same non zero quantity)} \\
 \Rightarrow & x_1 = x_2
 \end{aligned}$$

Hence, the given function is one-one.

## (ii) Mathematical Induction

Mathematical induction, is a strategy, of proving a proposition which is deductive in nature. The whole basis of proof of this method depends on the following axiom:

For a given subset  $S$  of  $\mathbb{N}$ , if

- (i) the natural number  $1 \in S$  and
- (ii) the natural number  $k + 1 \in S$  whenever  $k \in S$ , then  $S = \mathbb{N}$ .

According to the principle of mathematical induction, if a statement “ $S(n)$  is true for  $n = 1$ ” (or for some starting point  $j$ ), and if “ $S(n)$  is true for  $n = k$ ” implies that “ $S(n)$  is true for  $n = k + 1$ ” (whatever integer  $k \geq j$  may be), then the statement is true for any positive integer  $n$ , for all  $n \geq j$ .

We now consider some examples.

**Example 3** Show that if

$$A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} \cos n \theta & \sin n \theta \\ -\sin n \theta & \cos n \theta \end{bmatrix}$$

**Solution** We have

$$P(n) : A^n = \begin{bmatrix} \cos n \theta & \sin n \theta \\ -\sin n \theta & \cos n \theta \end{bmatrix}$$

We note that

$$P(1) : A^1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Therefore,  $P(1)$  is true.

Assume that  $P(k)$  is true, i.e.,

$$P(k) : A^k = \begin{bmatrix} \cos k \theta & \sin k \theta \\ -\sin k \theta & \cos k \theta \end{bmatrix}$$

We want to prove that  $P(k+1)$  is true whenever  $P(k)$  is true, i.e.,

$$P(k+1) : A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Now

$$A^{k+1} = A^k \cdot A$$

Since  $P(k)$  is true, we have

$$\begin{aligned} A^{k+1} &= \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos k\theta \cos\theta - \sin k\theta \sin\theta & \cos k\theta \sin\theta + \sin k\theta \cos\theta \\ -\sin k\theta \cos\theta - \cos k\theta \sin\theta & -\sin k\theta \sin\theta + \cos k\theta \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix} \quad (\text{by matrix multiplication}) \end{aligned}$$

Thus,  $P(k+1)$  is true whenever  $P(k)$  is true.

Hence,  $P(n)$  is true for all  $n \geq 1$  (by the principle of mathematical induction).

### (iii) Proof by cases or by exhaustion

This method of proving a statement  $p \Rightarrow q$  is possible only when  $p$  can be split into several cases,  $r, s, t$  (say) so that  $p = r \vee s \vee t$  (where “ $\vee$ ” is the symbol for “OR”).

If the conditionals  $r \Rightarrow q;$   
 $s \Rightarrow q;$

and  $t \Rightarrow q$

are proved, then  $(r \vee s \vee t) \Rightarrow q$ , is proved and so  $p \Rightarrow q$  is proved.

The method consists of examining every possible case of the hypothesis. It is practically convenient only when the number of possible cases are few.

**Example 4** Show that in any triangle ABC,

$$a = b \cos C + c \cos B$$

**Solution** Let  $p$  be the statement “ABC is any triangle” and  $q$  be the statement “ $a = b \cos C + c \cos B$ ”

Let ABC be a triangle. From A draw AD a perpendicular to BC (BC produced if necessary).

As we know that any triangle has to be either acute or obtuse or right angled, we can split  $p$  into three statements  $r, s$  and  $t$ , where

*r* : ABC is an acute angled triangle with  $\angle C$  is acute.

*s* : ABC is an obtuse angled triangle with  $\angle C$  is obtuse.

*t* : ABC is a right angled triangle with  $\angle C$  is right angle.

Hence, we prove the theorem by three cases.

**Case (i)** When  $\angle C$  is acute (Fig. A1.1).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e.

$$\begin{aligned} BD &= AB \cos B \\ &= c \cos B \end{aligned}$$

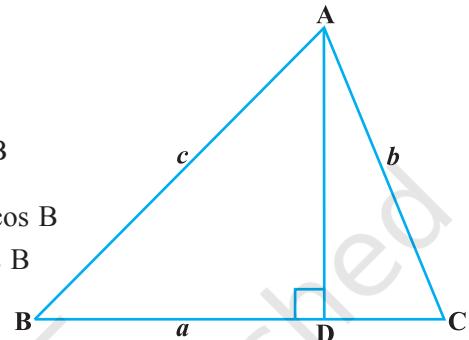


Fig A1.1

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos C$$

i.e.

$$\begin{aligned} CD &= AC \cos C \\ &= b \cos C \end{aligned}$$

Now

$$\begin{aligned} a &= BD + CD \\ &= c \cos B + b \cos C \end{aligned} \quad \dots (1)$$

**Case (ii)** When  $\angle C$  is obtuse (Fig A1.2).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e.

$$\begin{aligned} BD &= AB \cos B \\ &= c \cos B \end{aligned}$$

From the right angled triangle ADC,

$$\begin{aligned} \frac{CD}{AC} &= \cos \angle ACD \\ &= \cos (180^\circ - C) \\ &= -\cos C \\ \text{i.e. } CD &= -AC \cos C \\ &= -b \cos C \end{aligned}$$

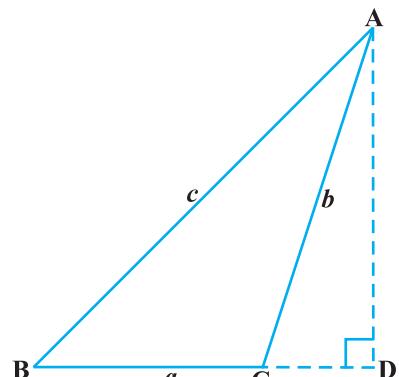


Fig A1.2

Now  $a = BC = BD - CD$   
 i.e.  $a = c \cos B - (-b \cos C)$   
 $a = c \cos B + b \cos C$  ... (2)

**Case (iii)** When  $\angle C$  is a right angle (Fig A1.3).

From the right angled triangle ACB,

i.e.  $\frac{BC}{AB} = \cos B$   
 $BC = AB \cos B$   
 $a = c \cos B$ ,

and  $b \cos C = b \cos 90^\circ = 0$ .

Thus, we may write  $a = 0 + c \cos B$   
 $= b \cos C + c \cos B$  ... (3)

From (1), (2) and (3). We assert that for any triangle ABC,

$$a = b \cos C + c \cos B$$

By case (i),  $r \Rightarrow q$  is proved.

By case (ii),  $s \Rightarrow q$  is proved.

By case (iii),  $t \Rightarrow q$  is proved.

Hence, from the proof by cases,  $(r \vee s \vee t) \Rightarrow q$  is proved, i.e.,  $p \Rightarrow q$  is proved.

**Indirect Proof** Instead of proving the given proposition directly, we establish the proof of the proposition through proving a proposition which is equivalent to the given proposition.

(i) **Proof by contradiction (Reductio Ad Absurdum)** : Here, we start with the assumption that the given statement is false. By rules of logic, we arrive at a conclusion contradicting the assumption and hence it is inferred that the assumption is wrong and hence the given statement is true.

Let us illustrate this method by an example.

**Example 5** Show that the set of all prime numbers is infinite.

**Solution** Let P be the set of all prime numbers. We take the negation of the statement “the set of all prime numbers is infinite”, i.e., we assume the set of all prime numbers to be finite. Hence, we can list all the prime numbers as  $P_1, P_2, P_3, \dots, P_k$  (say). Note that we have assumed that there is no prime number other than  $P_1, P_2, P_3, \dots, P_k$ .

Now consider  $N = (P_1 P_2 P_3 \dots P_k) + 1 \dots (1)$

N is not in the list as N is larger than any of the numbers in the list.

N is either prime or composite.

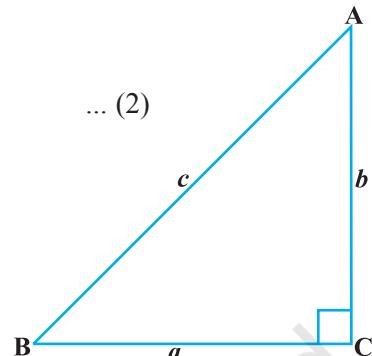


Fig A1.3

If  $N$  is a prime, then by (1), there exists a prime number which is not listed.

On the other hand, if  $N$  is composite, it should have a prime divisor. But none of the numbers in the list can divide  $N$ , because they all leave the remainder 1. Hence, the prime divisor should be other than the one in the list.

Thus, in both the cases whether  $N$  is a prime or a composite, we ended up with contradiction to the fact that we have listed all the prime numbers.

Hence, our assumption that set of all prime numbers is finite is false.

Thus, the set of all prime numbers is infinite.

 **Note** Observe that the above proof also uses the method of proof by cases.

### (ii) Proof by using contrapositive statement of the given statement

Instead of proving the conditional  $p \Rightarrow q$ , we prove its equivalent, i.e.,  $\sim q \Rightarrow \sim p$ . (students can verify).

The contrapositive of a conditional can be formed by interchanging the conclusion and the hypothesis and negating both.

**Example 6** Prove that the function  $f: \mathbf{R} \rightarrow \mathbf{R}$  defined by  $f(x) = 2x + 5$  is one-one.

**Solution** A function is one-one if  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Using this we have to show that " $2x_1 + 5 = 2x_2 + 5 \Rightarrow x_1 = x_2$ ". This is of the form  $p \Rightarrow q$ , where,  $p$  is  $2x_1 + 5 = 2x_2 + 5$  and  $q : x_1 = x_2$ . We have proved this in Example 2 of "direct method".

We can also prove the same by using contrapositive of the statement. Now contrapositive of this statement is  $\sim q \Rightarrow \sim p$ , i.e., contrapositive of "if  $f(x_1) = f(x_2)$ , then  $x_1 = x_2$ " is "if  $x_1 \neq x_2$ , then  $f(x_1) \neq f(x_2)$ ".

Now  $x_1 \neq x_2$

$\Rightarrow 2x_1 \neq 2x_2$

$\Rightarrow 2x_1 + 5 \neq 2x_2 + 5$

$\Rightarrow f(x_1) \neq f(x_2)$ .

Since " $\sim q \Rightarrow \sim p$ ", is equivalent to " $p \Rightarrow q$ " the proof is complete.

**Example 7** Show that "if a matrix  $A$  is invertible, then  $A$  is non singular".

**Solution** Writing the above statement in symbolic form, we have

$p \Rightarrow q$ , where,  $p$  is "matrix  $A$  is invertible" and  $q$  is " $A$  is non singular"

Instead of proving the given statement, we prove its contrapositive statement, i.e., if  $A$  is not a non singular matrix, then the matrix  $A$  is not invertible.

If  $A$  is not a non singular matrix, then it means the matrix  $A$  is singular, i.e.,

$$|A| = 0$$

Then  $A^{-1} = \frac{\text{adj } A}{|A|}$  does not exist as  $|A| = 0$

Hence,  $A$  is not invertible.

Thus, we have proved that if  $A$  is not a non singular matrix, then  $A$  is not invertible. i.e.,  $\sim q \Rightarrow \sim p$ .

Hence, if a matrix  $A$  is invertible, then  $A$  is non singular.

### (iii) Proof by a counter example

In the history of Mathematics, there are occasions when all attempts to find a valid proof of a statement fail and the uncertainty of the truth value of the statement remains unresolved.

In such a situation, it is beneficial, if we find an example to falsify the statement. The example to disprove the statement is called a *counter example*. Since the disproof of a proposition  $p \Rightarrow q$  is merely a proof of the proposition  $\sim(p \Rightarrow q)$ . Hence, this is also a method of proof.

**Example 8** For each  $n$ ,  $2^{2^n} + 1$  is a prime ( $n \in \mathbb{N}$ ).

This was once thought to be true on the basis that

$$2^{2^1} + 1 = 2^2 + 1 = 5 \text{ is a prime.}$$

$$2^{2^2} + 1 = 2^4 + 1 = 17 \text{ is a prime.}$$

$$2^{2^3} + 1 = 2^8 + 1 = 257 \text{ is a prime.}$$

However, at first sight the generalisation looks to be correct. But, eventually it was shown that  $2^{2^5} + 1 = 2^{32} + 1 = 4294967297$

which is not a prime since  $4294967297 = 641 \times 6700417$  (a product of two numbers).

So the generalisation “For each  $n$ ,  $2^{2^n} + 1$  is a prime ( $n \in \mathbb{N}$ )” is false.

Just this one example  $2^{2^5} + 1$  is sufficient to disprove the generalisation. This is the counter example.

Thus, we have proved that the generalisation “For each  $n$ ,  $2^{2^n} + 1$  is a prime ( $n \in \mathbb{N}$ )” is not true in general.

**Example 9** Every continuous function is differentiable.

**Proof** We consider some functions given by

- (i)  $f(x) = x^2$
- (ii)  $g(x) = e^x$
- (iii)  $h(x) = \sin x$

These functions are continuous for all values of  $x$ . If we check for their differentiability, we find that they are all differentiable for all the values of  $x$ . This makes us to believe that the generalisation “Every continuous function is differentiable” may be true. But if we check the differentiability of the function given by “ $\phi(x) = |x|$ ” which is continuous, we find that it is not differentiable at  $x = 0$ . This means that the statement “Every continuous function is differentiable” is false, in general. Just this one function “ $\phi(x) = |x|$ ” is sufficient to disprove the statement. Hence, “ $\phi(x) = |x|$ ” is called a counter example to disprove “Every continuous function is differentiable”.



# MATHEMATICAL MODELLING

## A.2.1 Introduction

In class XI, we have learnt about mathematical modelling as an attempt to study some part (or form) of some real-life problems in mathematical terms, i.e., the conversion of a physical situation into mathematics using some suitable conditions. Roughly speaking mathematical modelling is an activity in which we make models to describe the behaviour of various phenomenal activities of our interest in many ways using words, drawings or sketches, computer programs, mathematical formulae etc.

In earlier classes, we have observed that solutions to many problems, involving applications of various mathematical concepts, involve mathematical modelling in one way or the other. Therefore, it is important to study mathematical modelling as a separate topic.

In this chapter, we shall further study mathematical modelling of some real-life problems using techniques/results from matrix, calculus and linear programming.

## A.2.2 Why Mathematical Modelling?

Students are aware of the solution of word problems in arithmetic, algebra, trigonometry and linear programming etc. Sometimes we solve the problems without going into the physical insight of the situational problems. Situational problems need physical insight that is **introduction** of physical laws and some symbols to compare the mathematical results obtained with practical values. To solve many problems faced by us, we need a technique and this is what is known as *mathematical modelling*. Let us consider the following problems:

- (i) To find the width of a river (particularly, when it is difficult to cross the river).
- (ii) To find the optimal angle in case of shot-put (by considering the variables such as : the height of the thrower, resistance of the media, acceleration due to gravity etc.).
- (iii) To find the height of a tower (particularly, when it is not possible to reach the top of the tower).
- (iv) To find the temperature at the surface of the Sun.

- (v) Why heart patients are not allowed to use lift? (without knowing the physiology of a human being).
- (vi) To find the mass of the Earth.
- (vii) Estimate the yield of pulses in India from the standing crops (a person is not allowed to cut all of it).
- (viii) Find the volume of blood inside the body of a person (a person is not allowed to bleed completely).
- (ix) Estimate the population of India in the year 2020 (a person is not allowed to wait till then).

All of these problems can be solved and in fact have been solved with the help of Mathematics using mathematical modelling. In fact, you might have studied the methods for solving some of them in the present textbook itself. However, it will be instructive if you first try to solve them yourself and that too without the help of Mathematics, if possible, you will then appreciate the power of Mathematics and the need for mathematical modelling.

### **A.2.3 Principles of Mathematical Modelling**

Mathematical modelling is a principled activity and so it has some principles behind it. These principles are almost philosophical in nature. Some of the basic principles of mathematical modelling are listed below in terms of instructions:

- (i) Identify the need for the model. (for what we are looking for)
- (ii) List the parameters/variables which are required for the model.
- (iii) Identify the available relevant data. (what is given?)
- (iv) Identify the circumstances that can be applied (assumptions)
- (v) Identify the governing physical principles.
- (vi) Identify
  - (a) the equations that will be used.
  - (b) the calculations that will be made.
  - (c) the solution which will follow.
- (vii) Identify tests that can check the
  - (a) consistency of the model.
  - (b) utility of the model.
- (viii) Identify the parameter values that can improve the model.

The above principles of mathematical modelling lead to the following: steps for mathematical modelling.

**Step 1:** Identify the physical situation.

**Step 2:** Convert the physical situation into a mathematical model by introducing parameters / variables and using various known physical laws and symbols.

**Step 3:** Find the solution of the mathematical problem.

**Step 4:** Interpret the result in terms of the original problem and compare the result with observations or experiments.

**Step 5:** If the result is in good agreement, then accept the model. Otherwise modify the hypotheses / assumptions according to the physical situation and go to Step 2.

The above steps can also be viewed through the following diagram:

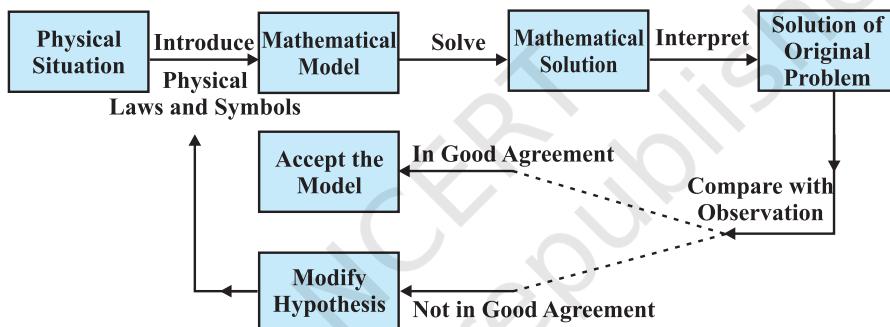


Fig A.2.1

**Example 1** Find the height of a given tower using mathematical modelling.

**Solution Step 1** Given physical situation is “to find the height of a given tower”.

**Step 2** Let AB be the given tower (Fig A.2.2). Let PQ be an observer measuring the height of the tower with his eye at P. Let  $PQ = h$  and let height of tower be H. Let  $\alpha$  be the angle of elevation from the eye of the observer to the top of the tower.

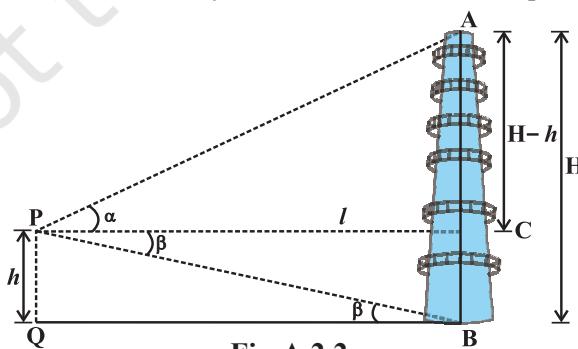


Fig A.2.2

Let

$$l = PC = QB$$

Now

$$\tan \alpha = \frac{AC}{PC} = \frac{H-h}{l}$$

or

$$H = h + l \tan \alpha \quad \dots (1)$$

**Step 3** Note that the values of the parameters  $h$ ,  $l$  and  $\alpha$  (using sextant) are known to the observer and so (1) gives the solution of the problem.

**Step 4** In case, if the foot of the tower is not accessible, i.e., when  $l$  is not known to the observer, let  $\beta$  be the angle of depression from  $P$  to the foot  $B$  of the tower. So from  $\Delta APQ$ , we have

$$\tan \beta = \frac{PQ}{QB} = \frac{h}{l} \text{ or } l = h \cot \beta$$

**Step 5** is not required in this situation as exact values of the parameters  $h$ ,  $l$ ,  $\alpha$  and  $\beta$  are known.

**Example 2** Let a business firm produces three types of products  $P_1$ ,  $P_2$  and  $P_3$  that uses three types of raw materials  $R_1$ ,  $R_2$  and  $R_3$ . Let the firm has purchase orders from two clients  $F_1$  and  $F_2$ . Considering the situation that the firm has a limited quantity of  $R_1$ ,  $R_2$  and  $R_3$ , respectively, prepare a model to determine the quantities of the raw material  $R_1$ ,  $R_2$  and  $R_3$  required to meet the purchase orders.

**Solution Step 1** The physical situation is well identified in the problem.

**Step 2** Let  $A$  be a matrix that represents purchase orders from the two clients  $F_1$  and  $F_2$ . Then,  $A$  is of the form

$$A = \begin{bmatrix} P_1 & P_2 & P_3 \\ F_1 & \bullet & \bullet & \bullet \\ F_2 & \bullet & \bullet & \bullet \end{bmatrix}$$

Let  $B$  be the matrix that represents the amount of raw materials  $R_1$ ,  $R_2$  and  $R_3$ , required to manufacture each unit of the products  $P_1$ ,  $P_2$  and  $P_3$ . Then,  $B$  is of the form

$$B = \begin{bmatrix} R_1 & R_2 & R_3 \\ P_1 & \bullet & \bullet & \bullet \\ P_2 & \bullet & \bullet & \bullet \\ P_3 & \bullet & \bullet & \bullet \end{bmatrix}$$

**Step 3** Note that the product (which in this case is well defined) of matrices A and B is given by the following matrix

$$AB = F_1 \begin{bmatrix} R_1 & R_2 & R_3 \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$$

which in fact gives the desired quantities of the raw materials  $R_1$ ,  $R_2$  and  $R_3$  to fulfill the purchase orders of the two clients  $F_1$  and  $F_2$ .

**Example 3** Interpret the model in Example 2, in case

$$A = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

and the available raw materials are 330 units of  $R_1$ , 455 units of  $R_2$  and 140 units of  $R_3$ .

**Solution** Note that

$$\begin{aligned} AB &= \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} \\ &= F_1 \begin{bmatrix} R_1 & R_2 & R_3 \\ 165 & 247 & 87 \\ 170 & 220 & 60 \end{bmatrix} \end{aligned}$$

This clearly shows that to meet the purchase order of  $F_1$  and  $F_2$ , the raw material required is 335 units of  $R_1$ , 467 units of  $R_2$  and 147 units of  $R_3$  which is much more than the available raw material. Since the amount of raw material required to manufacture each unit of the three products is fixed, we can either ask for an increase in the available raw material or we may ask the clients to reduce their orders.

**Remark** If we replace A in Example 3 by  $A_1$  given by

$$A_1 = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}$$

i.e., if the clients agree to reduce their purchase orders, then

$$A_1 B = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} = \begin{bmatrix} 141 & 216 & 78 \\ 170 & 220 & 60 \end{bmatrix}$$

This requires 311 units of  $R_1$ , 436 units of  $R_2$  and 138 units of  $R_3$  which are well below the available raw materials, i.e., 330 units of  $R_1$ , 455 units of  $R_2$  and 140 units of  $R_3$ . Thus, if the revised purchase orders of the clients are given by  $A_1$ , then the firm can easily supply the purchase orders of the two clients.

 **Note** One may further modify  $A$  so as to make full use of the available raw material.

**Query** Can we make a mathematical model with a given  $B$  and with fixed quantities of the available raw material that can help the firm owner to ask the clients to modify their orders in such a way that the firm makes the full use of its available raw material?

The answer to this query is given in the following example:

**Example 4** Suppose  $P_1, P_2, P_3$  and  $R_1, R_2, R_3$  are as in Example 2. Let the firm has 330 units of  $R_1$ , 455 units of  $R_2$  and 140 units of  $R_3$  available with it and let the amount of raw materials  $R_1, R_2$  and  $R_3$  required to manufacture each unit of the three products is given by

$$B = \begin{bmatrix} R_1 & R_2 & R_3 \\ P_1 & 3 & 4 & 0 \\ P_2 & 7 & 9 & 3 \\ P_3 & 5 & 12 & 7 \end{bmatrix}$$

How many units of each product is to be made so as to utilise the full available raw material?

**Solution Step 1** The situation is easily identifiable.

**Step 2** Suppose the firm produces  $x$  units of  $P_1$ ,  $y$  units of  $P_2$  and  $z$  units of  $P_3$ . Since product  $P_1$  requires 3 units of  $R_1$ ,  $P_2$  requires 7 units of  $R_1$  and  $P_3$  requires 5 units of  $R_1$  (observe matrix  $B$ ) and the total number of units, of  $R_1$ , available is 330, we have

$$3x + 7y + 5z = 330 \text{ (for raw material } R_1\text{)}$$

Similarly, we have

$$4x + 9y + 12z = 455 \text{ (for raw material } R_2\text{)}$$

and

$$3y + 7z = 140 \text{ (for raw material } R_3\text{)}$$

This system of equations can be expressed in matrix form as

$$\begin{bmatrix} 3 & 7 & 5 \\ 4 & 9 & 12 \\ 0 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 330 \\ 455 \\ 140 \end{bmatrix}$$

**Step 3** Using elementary row operations, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \\ 5 \end{bmatrix}$$

This gives  $x = 20$ ,  $y = 35$  and  $z = 5$ . Thus, the firm can produce 20 units of  $P_1$ , 35 units of  $P_2$  and 5 units of  $P_3$  to make full use of its available raw material.

**Remark** One may observe that if the manufacturer decides to manufacture according to the available raw material and not according to the purchase orders of the two clients  $F_1$  and  $F_2$  (as in Example 3), he/she is unable to meet these purchase orders as  $F_1$  demanded 6 units of  $P_3$  where as the manufacturer can make only 5 units of  $P_3$ .

**Example 5** A manufacturer of medicines is preparing a production plan of medicines  $M_1$  and  $M_2$ . There are sufficient raw materials available to make 20000 bottles of  $M_1$  and 40000 bottles of  $M_2$ , but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of  $M_1$ , it takes 1 hour to prepare enough material to fill 1000 bottles of  $M_2$  and there are 66 hours available for this operation. The profit is Rs 8 per bottle for  $M_1$  and Rs 7 per bottle for  $M_2$ . How should the manufacturer schedule his/her production in order to maximise profit?

**Solution Step 1** To find the number of bottles of  $M_1$  and  $M_2$  in order to maximise the profit under the given hypotheses.

**Step 2** Let  $x$  be the number of bottles of type  $M_1$  medicine and  $y$  be the number of bottles of type  $M_2$  medicine. Since profit is Rs 8 per bottle for  $M_1$  and Rs 7 per bottle for  $M_2$ , therefore the objective function (which is to be maximised) is given by

$$Z \equiv Z(x, y) = 8x + 7y$$

The objective function is to be maximised subject to the constraints (Refer Chapter 12 on Linear Programming)

$$\left. \begin{array}{l} x \leq 20000 \\ y \leq 40000 \\ x + y \leq 45000 \\ 3x + y \leq 66000 \\ x \geq 0, y \geq 0 \end{array} \right\} \dots (1)$$

**Step 3** The shaded region OPQRST is the feasible region for the constraints (1) (Fig A.2.3). The co-ordinates of vertices O, P, Q, R, S and T are (0, 0), (20000, 0), (20000, 6000), (10500, 34500), (5000, 40000) and (0, 40000), respectively.

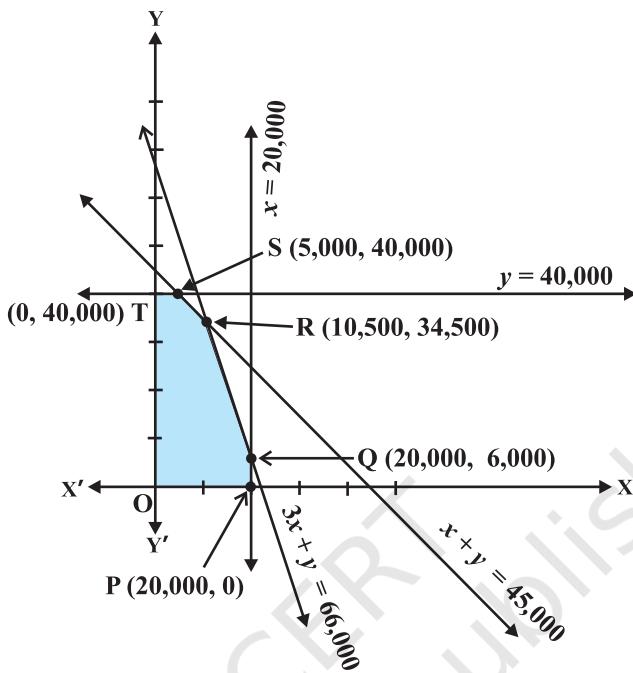


Fig A.2.3

Note that

$$Z \text{ at } P(0, 0) = 0$$

$$Z \text{ at } P(20000, 0) = 8 \times 20000 = 160000$$

$$Z \text{ at } Q(20000, 6000) = 8 \times 20000 + 7 \times 6000 = 202000$$

$$Z \text{ at } R(10500, 34500) = 8 \times 10500 + 7 \times 34500 = 325500$$

$$Z \text{ at } S(5000, 40000) = 8 \times 5000 + 7 \times 40000 = 320000$$

$$Z \text{ at } T(0, 40000) = 7 \times 40000 = 280000$$

Now observe that the profit is maximum at  $x = 10500$  and  $y = 34500$  and the maximum profit is ₹325500. Hence, the manufacturer should produce 10500 bottles of  $M_1$  medicine and 34500 bottles of  $M_2$  medicine in order to get maximum profit of ₹325500.

**Example 6** Suppose a company plans to produce a new product that incur some costs (fixed and variable) and let the company plans to sell the product at a fixed price. Prepare a mathematical model to examine the profitability.

**Solution Step 1** Situation is clearly identifiable.

**Step 2 Formulation:** We are given that the costs are of two types: fixed and variable. The fixed costs are independent of the number of units produced (e.g., rent and rates), while the variable costs increase with the number of units produced (e.g., material). Initially, we assume that the variable costs are directly proportional to the number of units produced — this should simplify our model. The company earn a certain amount of money by selling its products and wants to ensure that it is maximum. For convenience, we assume that all units produced are sold immediately.

### The mathematical model

Let  $x$  = number of units produced and sold  
 $C$  = total cost of production (in rupees)  
 $I$  = income from sales (in rupees)  
 $P$  = profit (in rupees)

Our assumptions above state that  $C$  consists of two parts:

- (i) fixed cost =  $a$  (in rupees),
- (ii) variable cost =  $b$  (rupees/unit produced).

Then  $C = a + bx$  ... (1)

Also, income  $I$  depends on selling price  $s$  (rupees/unit)

Thus  $I = sx$  ... (2)

The profit  $P$  is then the difference between income and costs. So

$$\begin{aligned} P &= I - C \\ &= sx - (a + bx) \\ &= (s - b)x - a \end{aligned} \quad \dots (3)$$

We now have a mathematical model of the relationships (1) to (3) between the variables  $x$ ,  $C$ ,  $I$ ,  $P$ ,  $a$ ,  $b$ ,  $s$ . These variables may be classified as:

|             |           |
|-------------|-----------|
| independent | $x$       |
| dependent   | $C, I, P$ |
| parameters  | $a, b, s$ |

The manufacturer, knowing  $x, a, b, s$  can determine  $P$ .

**Step 3** From (3), we can observe that for the break even point (i.e., make neither profit nor loss), he must have  $P = 0$ , i.e.,  $x = \frac{a}{s-b}$  units.

**Steps 4 and 5** In view of the break even point, one may conclude that if the company produces few units, i.e., less than  $x = \frac{a}{s-b}$  units, then the company will suffer loss

and if it produces large number of units, i.e., much more than  $\frac{a}{s-b}$  units, then it can make huge profit. Further, if the break even point proves to be unrealistic, then another model could be tried or the assumptions regarding cash flow may be modified.

**Remark** From (3), we also have

$$\frac{dP}{dx} = s - b$$

This means that rate of change of  $P$  with respect to  $x$  depends on the quantity  $s - b$ , which is the difference of selling price and the variable cost of each product. Thus, in order to gain profit, this should be positive and to get large gains, we need to produce large quantity of the product and at the same time try to reduce the variable cost.

**Example 7** Let a tank contains 1000 litres of brine which contains 250 g of salt per litre. Brine containing 200 g of salt per litre flows into the tank at the rate of 25 litres per minute and the mixture flows out at the same rate. Assume that the mixture is kept uniform all the time by stirring. What would be the amount of salt in the tank at any time  $t$ ?

**Solution Step 1** The situation is easily identifiable.

**Step 2** Let  $y = y(t)$  denote the amount of salt (in kg) in the tank at time  $t$  (in minutes) after the inflow, outflow starts. Further assume that  $y$  is a differentiable function.

When  $t = 0$ , i.e., before the inflow–outflow of the brine starts,

$$y = 250 \text{ g} \times 1000 = 250 \text{ kg}$$

Note that the change in  $y$  occurs due to the inflow, outflow of the mixture.

Now the inflow of brine brings salt into the tank at the rate of 5 kg per minute (as  $25 \times 200 \text{ g} = 5 \text{ kg}$ ) and the outflow of brine takes salt out of the tank at the rate of

$25\left(\frac{y}{1000}\right) = \frac{y}{40}$  kg per minute (as at time  $t$ , the salt in the tank is  $\frac{y}{1000}$  kg).

Thus, the rate of change of salt with respect to  $t$  is given by

$$\frac{dy}{dt} = 5 - \frac{y}{40} \quad (\text{Why?})$$

$$\text{or} \quad \frac{dy}{dt} + \frac{1}{40}y = 5 \quad \dots (1)$$

This gives a mathematical model for the given problem.

**Step 3** Equation (1) is a linear equation and can be easily solved. The solution of (1) is given by

$$ye^{\frac{t}{40}} = 200e^{\frac{t}{40}} + C \text{ or } y(t) = 200 + C e^{-\frac{t}{40}} \quad \dots (2)$$

where,  $c$  is the constant of integration.

Note that when  $t = 0$ ,  $y = 250$ . Therefore,  $250 = 200 + C$

or  $C = 50$

Then (2) reduces to

$$y = 200 + 50 e^{-\frac{t}{40}} \quad \dots (3)$$

or  $\frac{y-200}{50} = e^{-\frac{t}{40}}$

or  $e^{\frac{t}{40}} = \frac{50}{y-200}$

Therefore  $t = 40 \log_e \left( \frac{50}{y-200} \right) \quad \dots (4)$

Here, the equation (4) gives the time  $t$  at which the salt in tank is  $y$  kg.

**Step 4** Since  $e^{-\frac{t}{40}}$  is always positive, from (3), we conclude that  $y > 200$  at all times. Thus, the minimum amount of salt content in the tank is 200 kg.

Also, from (4), we conclude that  $t > 0$  if and only if  $0 < y - 200 < 50$  i.e., if and only if  $200 < y < 250$  i.e., the amount of salt content in the tank after the start of inflow and outflow of the brine is between 200 kg and 250 kg.

### ***Limitations of Mathematical Modelling***

Till today many mathematical models have been developed and applied successfully to understand and get an insight into thousands of situations. Some of the subjects like mathematical physics, mathematical economics, operations research, bio-mathematics etc. are almost synonymous with mathematical modelling.

But there are still a large number of situations which are yet to be modelled. The reason behind this is that either the situation are found to be very complex or the mathematical models formed are mathematically intractable.

The development of the powerful computers and super computers has enabled us to mathematically model a large number of situations (even complex situations). Due to these fast and advanced computers, it has been possible to prepare more realistic models which can obtain better agreements with observations.

However, we do not have good guidelines for choosing various parameters / variables and also for estimating the values of these parameters / variables used in a mathematical model. Infact, we can prepare reasonably accurate models to fit any data by choosing five or six parameters / variables. We require a minimal number of parameters / variables to be able to estimate them accurately.

Mathematical modelling of large or complex situations has its own special problems. These type of situations usually occur in the study of world models of environment, oceanography, pollution control etc. Mathematical modellers from all disciplines — mathematics, computer science, physics, engineering, social sciences, etc., are involved in meeting these challenges with courage.



# ANSWERS

## EXERCISE 1.1

1. (i) Neither reflexive nor symmetric nor transitive.  
(ii) Neither reflexive nor symmetric but transitive.  
(iii) Reflexive and transitive but not symmetric.  
(iv) Reflexive, symmetric and transitive.  
(v) (a) Reflexive, symmetric and transitive.  
(b) Reflexive, symmetric and transitive.  
(c) Neither reflexive nor symmetric nor transitive.  
(d) Neither reflexive nor symmetric but transitive.  
(e) Neither reflexive nor symmetric nor transitive.
3. Neither reflexive nor symmetric nor transitive.
5. Neither reflexive nor symmetric nor transitive.
9. (i)  $\{1, 5, 9\}$ , (ii)  $\{1\}$       12.  $T_1$  is related to  $T_3$ .
13. The set of all triangles      14. The set of all lines  $y = 2x + c, c \in \mathbf{R}$
15. B      16. C

## EXERCISE 1.2

1. No
2. (i) Injective but not surjective      (ii) Neither injective nor surjective  
(iii) Neither injective nor surjective      (iv) Injective but not surjective  
(v) Injective but not surjective
7. (i) One-one and onto      (ii) Neither one-one nor onto.
9. No      10. Yes      11. D      12. A

## Miscellaneous Exercise on Chapter 1

3. No      4.  $n!$       5. Yes      6. A      7. B

## EXERCISE 2.1

1.  $\frac{-\pi}{6}$       2.  $\frac{\pi}{6}$       3.  $\frac{\pi}{6}$       4.  $\frac{-\pi}{3}$   
 5.  $\frac{2\pi}{3}$       6.  $-\frac{\pi}{4}$       7.  $\frac{\pi}{6}$       8.  $\frac{\pi}{6}$   
 9.  $\frac{3\pi}{4}$       10.  $-\frac{\pi}{4}$       11.  $\frac{3\pi}{4}$       12.  $\frac{2\pi}{3}$   
 13. B      14. B

## EXERCISE 2.2

3.  $\frac{1}{2} \tan^{-1} x$       4.  $\frac{x}{2}$       5.  $\frac{\pi}{4} - x$   
 6.  $\sin^{-1} \frac{x}{a}$       7.  $3 \tan^{-1} \frac{x}{a}$       8.  $\frac{\pi}{4}$   
 9.  $\frac{x+y}{1-xy}$       10.  $\frac{\pi}{3}$   
 11.  $\frac{-\pi}{4}$       12.  $\frac{17}{6}$       13. B      14. D  
 15. B

## Miscellaneous Exercise on Chapter 2

1.  $\frac{\pi}{6}$       2.  $\frac{\pi}{6}$       11.  $x = n\pi + \frac{\pi}{4}$ ,  $n \in \mathbb{Z}$       12.  $x = \frac{1}{\sqrt{3}}$   
 13. D      14. C

## EXERCISE 3.1

1. (i)  $3 \times 4$       (ii) 12      (iii)  $19, 35, -5, 12, \frac{5}{2}$   
 2.  $1 \times 24, 2 \times 12, 3 \times 8, 4 \times 6, 6 \times 4, 8 \times 3, 12 \times 2, 24 \times 1; 1 \times 13, 13 \times 1$   
 3.  $1 \times 18, 2 \times 9, 3 \times 6, 6 \times 3, 9 \times 2, 18 \times 1; 1 \times 5, 5 \times 1$

4. (i) 
$$\begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$$
      (ii) 
$$\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$$
      (iii) 
$$\begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$$

5. (i) 
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$$
 (ii) 
$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$$

6. (i)  $x = 1, y = 4, z = 3$   
 (ii)  $x = 4, y = 2, z = 0$  or  $x = 2, y = 4, z = 0$   
 (iii)  $x = 2, y = 4, z = 3$

7.  $a = 1, b = 2, c = 3, d = 4$

8. C

9. B

10. D

### EXERCISE 3.2

1. (i)  $A + B = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$  (ii)  $A - B = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$   
 (iii)  $3A - C = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$  (iv)  $AB = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$  (v)  $BA = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

2. (i)  $\begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$  (ii)  $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$   
 (iii)  $\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$  (iv)  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. (i)  $\begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$  (ii)  $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$  (iii)  $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

(iv)  $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$  (v)  $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$  (vi)  $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

4.  $A + B = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$ ,  $B - C = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

5.  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7. (i)  $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ ,  $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$  (ii)  $X = \begin{bmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{bmatrix}$ ,  $Y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

8.  $X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$

9.  $x = 3, y = 3$

10.  $x = 3, y = 6, z = 9, t = 6$

11.  $x = 3, y = -4$

12.  $x = 2, y = 4, w = 3, z = 1$

15.  $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

17.  $k = 1$

19. (a) ₹15000, ₹15000

(b) ₹5000, ₹25000

20. ₹20160

21. A

22. B

### EXERCISE 3.3

1. (i)  $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$

(ii)  $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

(iii)  $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$

4.  $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$

9.  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. (i)  $A = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(ii)  $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$(iii) \quad A = \begin{bmatrix} 3 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & -2 & -2 \\ \frac{-5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ \frac{-5}{2} & 0 & 3 \\ \frac{-3}{2} & -3 & 0 \end{bmatrix} \quad (iv) \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

11. A

12. B

### EXERCISE 3.4

1. D

### Miscellaneous Exercise on Chapter 3

3.  $x = \pm \frac{1}{\sqrt{2}}, y = \pm \frac{1}{\sqrt{6}}, z = \pm \frac{1}{\sqrt{3}}$

4.  $x = -1 \quad 6. \quad x = \pm 4\sqrt{3}$

7. (a) Total revenue in the market - I = ₹ 46000

Total revenue in the market - II = ₹ 53000

(b) ₹ 15000, ₹ 17000

8.  $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \quad 9. \quad C \quad 10. \quad B \quad 11. \quad C$

### EXERCISE 4.1

1. (i) 18

2. (i) 1, (ii)  $x^3 - x^2 + 2$

5. (i) -12, (ii) 46, (iii) 0, (iv) 5

6. 0

7. (i)  $x = \pm \sqrt{3}, \quad$  (ii)  $x = 2$

8. (B)

### EXERCISE 4.2

1. (i)  $\frac{15}{2}, \quad$  (ii)  $\frac{47}{2}, \quad$  (iii) 15

3. (i) 0, 8, (ii) 0, 8

4. (i)  $y = 2x$ , (ii)  $x - 3y = 0$ 

5. (D)

**EXERCISE 4.3**1. (i)  $M_{11} = 3, M_{12} = 0, M_{21} = -4, M_{22} = 2, A_{11} = 3, A_{12} = 0, A_{21} = 4, A_{22} = 2$ (ii)  $M_{11} = d, M_{12} = b, M_{21} = c, M_{22} = a$  $A_{11} = d, A_{12} = -b, A_{21} = -c, A_{22} = a$ 2. (i)  $M_{11} = 1, M_{12} = 0, M_{13} = 0, M_{21} = 0, M_{22} = 1, M_{23} = 0, M_{31} = 0, M_{32} = 0, M_{33} = 1,$  $A_{11} = 1, A_{12} = 0, A_{13} = 0, A_{21} = 0, A_{22} = 1, A_{23} = 0, A_{31} = 0, A_{32} = 0, A_{33} = 1$ (ii)  $M_{11} = 11, M_{12} = 6, M_{13} = 3, M_{21} = -4, M_{22} = 2, M_{23} = 1, M_{31} = -20, M_{32} = -13, M_{33} = 5$  $A_{11} = 11, A_{12} = -6, A_{13} = 3, A_{21} = 4, A_{22} = 2, A_{23} = -1, A_{31} = -20, A_{32} = 13, A_{33} = 5$ 

3. 7

4.  $(x - y)(y - z)(z - x)$ 

5. (D)

**EXERCISE 4.4**

1. 
$$\begin{matrix} 4 & -2 \\ -3 & 1 \end{matrix}$$

2. 
$$\begin{matrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{matrix}$$

5. 
$$\frac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

6. 
$$\begin{matrix} \frac{1}{13} & 2 & -5 \\ 13 & 3 & -1 \end{matrix}$$

7. 
$$\frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

8. 
$$\frac{-1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

9. 
$$\frac{-1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10. 
$$\begin{matrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{matrix}$$

11. 
$$\begin{matrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{matrix}$$

13. 
$$\frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

14.  $a = -4, b = 1$

15.  $A^{-1} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

16. 
$$\frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

17. B

18. B

## EXERCISE 4.5

1. Consistent      2. Consistent      3. Inconsistent  
 4. Consistent      5. Inconsistent      6. Consistent  
 7.  $x = 2, y = -3$       8.  $x = \frac{-5}{11}, y = \frac{12}{11}$       9.  $x = \frac{-6}{11}, y = \frac{-19}{11}$   
 10.  $x = -1, y = 4$       11.  $x = 1, y = \frac{1}{2}, z = \frac{-3}{2}$   
 12.  $x = 2, y = -1, z = 1$       13.  $x = 1, y = 2, z = -1$   
 14.  $x = 2, y = 1, z = 3$
15. 
$$\begin{matrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{matrix}, x = 1, y = 2, z = 3$$
16. cost of onions per kg = ₹ 5  
 cost of wheat per kg = ₹ 8  
 cost of rice per kg = ₹ 8

*Miscellaneous Exercise on Chapter 4*

2. 1      3. 
$$\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$
5.  $-2(x^3 + y^3)$       6.  $xy$       7.  $x = 2, y = 3, z = 5$   
 8. A      9. D

## EXERCISE 5.1

2.  $f$  is continuous at  $x = 3$   
 3. (a), (b), (c) and (d) are all continuous functions  
 5.  $f$  is continuous at  $x = 0$  and  $x = 2$ ; Not continuous at  $x = 1$   
 6. Discontinuous at  $x = 2$       7. Discontinuous at  $x = 3$

8. Discontinuous at  $x = 0$       9. No point of discontinuity  
 10. No point of discontinuity      11. No point of discontinuity  
 12.  $f$  is discontinuous at  $x = 1$       13.  $f$  is not continuous at  $x = 1$   
 14.  $f$  is not continuous at  $x = 1$  and  $x = 3$   
 15.  $x = 1$  is the only point of discontinuity  
 16. Continuous      17.  $a = b + \frac{2}{3}$   
 18. For no value of  $\lambda$ ,  $f$  is continuous at  $x = 0$  but  $f$  is continuous at  $x = 1$  for any value of  $\lambda$ .  
 20.  $f$  is continuous at  $x = \pi$       21. (a), (b) and (c) are all continuous  
 22. Cosine function is continuous for all  $x \in \mathbf{R}$ ; cosecant is continuous except for  $x = n\pi$ ,  $n \in \mathbf{Z}$ ; secant is continuous except for  $x = (2n+1)\frac{\pi}{2}$ ,  $n \in \mathbf{Z}$  and cotangent function is continuous except for  $x = n\pi$ ,  $n \in \mathbf{Z}$   
 23. There is no point of discontinuity.  
 24. Yes,  $f$  is continuous for all  $x \in \mathbf{R}$       25.  $f$  is continuous for all  $x \in \mathbf{R}$   
 26.  $k = 6$       27.  $k = \frac{3}{4}$       28.  $k = \frac{-2}{\pi}$   
 29.  $k = \frac{9}{5}$       30.  $a = 2, b = 1$   
 34. There is no point of discontinuity.

### EXERCISE 5.2

1.  $2x \cos(x^2 + 5)$       2.  $-\cos x \sin(\sin x)$       3.  $a \cos(ax + b)$   
 4. 
$$\frac{\sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x}}{2\sqrt{x}}$$
  
 5.  $a \cos(ax + b) \sec(cx + d) + c \sin(ax + b) \tan(cx + d) \sec(cx + d)$   
 6.  $10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2 x^5$   
 7. 
$$\frac{-2\sqrt{2} x}{\sin x^2 \sqrt{\sin 2x^2}}$$
      8. 
$$-\frac{\sin \sqrt{x}}{2\sqrt{x}}$$

## EXERCISE 5.3

1.  $\frac{\cos x - 2}{3}$

2.  $\frac{2}{\cos y - 3}$

3.  $-\frac{a}{2by + \sin y}$

4.  $\frac{\sec^2 x - y}{x + 2y - 1}$

5.  $-\frac{(2x+y)}{(x+2y)}$

6.  $-\frac{(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$

7.  $\frac{y \sin xy}{\sin 2y - x \sin xy}$

8.  $\frac{\sin 2x}{\sin 2y}$

9.  $\frac{2}{1 + x^2}$

10.  $\frac{3}{1 + x^2}$

11.  $\frac{2}{1 + x^2}$

12.  $\frac{-2}{1 + x^2}$

13.  $\frac{-2}{1 + x^2}$

14.  $\frac{2}{\sqrt{1 - x^2}}$

15.  $-\frac{2}{\sqrt{1 - x^2}}$

## EXERCISE 5.4

1.  $\frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$  2.  $\frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}, x \in (-1, 1)$

3.  $3x^2 e^{x^3}$  4.  $-\frac{e^{-x} \cos(\tan^{-1} e^{-x})}{1 + e^{-2x}}$

5.  $-e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{N}$  6.  $e^x + 2x^{e^{x^2}} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$

7.  $\frac{e^{\sqrt{x}}}{4\sqrt{x}e^{\sqrt{x}}}, x > 0$  8.  $\frac{1}{x \log x}, x > 1$

9.  $-\frac{(x \sin x \cdot \log x + \cos x)}{x(\log x)^2}, x > 0$  10.  $-\frac{1}{x} + e^x \sin(\log x + e^x), x > 0$

## EXERCISE 5.5

1.  $-\cos x \cos 2x \cos 3x [\tan x + 2 \tan 2x + 3 \tan 3x]$

2.  $\frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right]$

3.  $(\log x)^{\cos x} \left[ \frac{\cos x}{x \log x} - \sin x \log(\log x) \right]$
4.  $x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$
5.  $(x + 3)(x + 4)^2(x + 5)^3(9x^2 + 70x + 133)$
6.  $\left( x + \frac{1}{x} \right)^x \left[ \frac{x^2 - 1}{x^2 + 1} + \log \left( x + \frac{1}{x} \right) \right] + x^{1 + \frac{1}{x}} \left( \frac{x + 1 - \log x}{x^2} \right)$
7.  $(\log x)^{x-1} [1 + \log x \cdot \log(\log x)] + 2x^{\log x - 1} \cdot \log x$
8.  $(\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2} \frac{1}{\sqrt{x-x^2}}$
9.  $x^{\sin x} \left[ \frac{\sin x}{x} + \cos x \log x \right] + (\sin x)^{\cos x} [\cos x \cot x - \sin x \log \sin x]$
10.  $x^{x \cos x} [\cos x \cdot (1 + \log x) - x \sin x \log x] - \frac{4x}{(x^2 - 1)^2}$
11.  $(x \cos x)^x [1 - x \tan x + \log(x \cos x)] + (x \sin x)^{\frac{1}{x}} \left[ \frac{x \cot x + 1 - \log(x \sin x)}{x^2} \right]$
12.  $-\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$
13.  $\frac{y}{x} \left( \frac{y - x \log y}{x - y \log x} \right)$
14.  $\frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$
15.  $\frac{y(x-1)}{x(y+1)}$
16.  $(1+x)(1+x^2)(1+x^4)(1+x^8) \left[ \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} \right]; f'(1) = 120$
17.  $5x^4 - 20x^3 + 45x^2 - 52x + 11$

### EXERCISE 5.6

1.  $t^2$
2.  $\frac{b}{a}$
3.  $-4 \sin t$
4.  $-\frac{1}{t^2}$
5.  $\frac{\cos \theta - 2 \cos 2\theta}{2 \sin 2\theta - \sin \theta}$
6.  $-\cot \frac{\theta}{2}$
7.  $-\cot 3t$
8.  $\tan t$
9.  $\frac{b}{a} \operatorname{cosec} \theta$
10.  $\tan \theta$

## EXERCISE 5.7

1. 2
2.  $380x^{18}$
3.  $-x \cos x - 2 \sin x$
4.  $-\frac{1}{x^2}$
5.  $x(5 + 6 \log x)$
6.  $2e^x(5 \cos 5x - 12 \sin 5x)$
7.  $9e^{6x}(3 \cos 3x - 4 \sin 3x)$
8.  $-\frac{2x}{(1+x^2)^2}$
9.  $-\frac{(1+\log x)}{(x \log x)^2}$
10.  $-\frac{\sin(\log x) + \cos(\log x)}{x^2}$
12.  $-\cot y \operatorname{cosec}^2 y$

*Miscellaneous Exercise on Chapter 5*

1.  $27(3x^2 - 9x + 5)^8(2x - 3)$
2.  $3\sin x \cos x (\sin x - 2 \cos^4 x)$
3.  $(5x)^{3\cos 2x} \left[ \frac{3\cos 2x}{x} - 6\sin 2x \log 5x \right]$
4.  $\frac{3}{2} \sqrt{\frac{x}{1-x^3}}$
5.  $-\left[ \frac{1}{\sqrt{4-x^2} \sqrt{2x+7}} + \frac{\cos^{-1} \frac{x}{2}}{(2x+7)^{\frac{3}{2}}} \right]$
6.  $\frac{1}{2}$
7.  $(\log x)^{\log x} \left[ \frac{1}{x} + \frac{\log(\log x)}{x} \right], x > 1$
8.  $(a \sin x - b \cos x) \sin(a \cos x + b \sin x)$
9.  $(\sin x - \cos x)^{\sin x - \cos x} (\cos x + \sin x) (1 + \log(\sin x - \cos x)), \sin x > \cos x$
10.  $x^x (1 + \log x) + ax^{a-1} + a^x \log a$
11.  $x^{x^2-3} \left[ \frac{x^2-3}{x} + 2x \log x \right] + (x-3)^{x^2} \left[ \frac{x^2}{x-3} + 2x \log(x-3) \right]$
12.  $\frac{6}{5} \cot \frac{t}{2}$
13. 0
17.  $\frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$

## EXERCISE 6.1

1. (a)  $6\pi \text{ cm}^2/\text{cm}$  (b)  $8\pi \text{ cm}^2/\text{cm}$
2.  $\frac{8}{3} \text{ cm}^2/\text{s}$  3.  $60\pi \text{ cm}^2/\text{s}$  4.  $900 \text{ cm}^3/\text{s}$
5.  $80\pi \text{ cm}^2/\text{s}$  6.  $1.4\pi \text{ cm/s}$
7. (a)  $-2 \text{ cm/min}$  (b)  $2 \text{ cm}^2/\text{min}$
8.  $\frac{1}{\pi} \text{ cm/s}$  9.  $400\pi \text{ cm}^3/\text{cm}$  10.  $\frac{8}{3} \text{ cm/s}$
11.  $(4, 11)$  and  $\left(-4, \frac{-31}{3}\right)$  12.  $2\pi \text{ cm}^3/\text{s}$
13.  $\frac{27}{8}\pi(2x+1)^2$  14.  $\frac{1}{48\pi} \text{ cm/s}$  15. ₹20.967
16. ₹208 17. B 18. D

## EXERCISE 6.2

4. (a)  $\left(\frac{3}{4}, \infty\right)$  (b)  $\left(-\infty, \frac{3}{4}\right)$
5. (a)  $(-\infty, -2)$  and  $(3, \infty)$  (b)  $(-2, 3)$
6. (a) decreasing for  $x < -1$  and increasing for  $x > -1$   
 (b) decreasing for  $x > -\frac{3}{2}$  and increasing for  $x < -\frac{3}{2}$   
 (c) increasing for  $-2 < x < -1$  and decreasing for  $x < -2$  and  $x > -1$   
 (d) increasing for  $x < -\frac{9}{2}$  and decreasing for  $x > -\frac{9}{2}$   
 (e) increasing in  $(1, 3)$  and  $(3, \infty)$ , decreasing in  $(-\infty, -1)$  and  $(-1, 1)$ .
8.  $0 < x < 1$  and  $x > 2$  12. A, B
13. D 14.  $a > -2$  19. D

## EXERCISE 6.3

1. (i) Minimum Value = 3      (ii) Minimum Value = - 2  
 (iii) Maximum Value = 10      (iv) Neither minimum nor maximum value
2. (i) Minimum Value = - 1; No maximum value  
 (ii) Maximum Value = 3; No minimum value  
 (iii) Minimum Value = 4; Maximum Value = 6  
 (iv) Minimum Value = 2; Maximum Value = 4  
 (v) Neither minimum nor Maximum Value
3. (i) local minimum at  $x = 0$ ,      local minimum value = 0  
 (ii) local minimum at  $x = 1$ ,      local minimum value = - 2  
 local maximum at  $x = - 1$ ,      local maximum value = 2  
 (iii) local maximum at  $x = \frac{\pi}{4}$ ,      local maximum value =  $\sqrt{2}$   
 (iv) local maximum at  $x = \frac{3\pi}{4}$ ,      local maximum value =  $\sqrt{2}$   
 local minimum at  $x = \frac{7\pi}{4}$ ,      local minimum value =  $-\sqrt{2}$   
 (v) local maximum at  $x = 1$ ,      local maximum value = 19  
 local minimum at  $x = 3$ ,      local minimum value = 15  
 (vi) local minimum at  $x = 2$ ,      local minimum value = 2  
 (vii) local maximum at  $x = 0$ ,      local maximum value =  $\frac{1}{2}$   
 (viii) local maximum at  $x = \frac{2}{3}$ ,      local maximum value =  $\frac{2\sqrt{3}}{9}$

5. (i) Absolute minimum value = -8, absolute maximum value = 8  
 (ii) Absolute minimum value = -1, absolute maximum value =  $\sqrt{2}$   
 (iii) Absolute minimum value = -10, absolute maximum value = 8  
 (iv) Absolute minimum value = 19, absolute maximum value = 3
6. Maximum profit = 113 unit.
7. Minima at  $x = 2$ , minimum value = -39, Maxima at  $x = 0$ , maximum value = 25.
8. At  $x = \frac{\pi}{4}$  and  $\frac{5\pi}{4}$  9. Maximum value =  $\sqrt{2}$
10. Maximum at  $x = 3$ , maximum value 89; maximum at  $x = -2$ , maximum value = 139
11.  $a = 120$
12. Maximum at  $x = 2\pi$ , maximum value =  $2\pi$ ; Minimum at  $x = 0$ , minimum value = 0
13. 12, 12 14. 45, 15 15. 25, 10 16. 8, 8
17. 3 cm 18.  $x = 5$  cm
21. radius =  $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$  cm and height =  $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$  cm
22.  $\frac{112}{\pi+4}$  cm,  $\frac{28\pi}{\pi+4}$  cm 27. A 28. D 29. C

### Miscellaneous Exercise on Chapter 6

2.  $b\sqrt{3}$  cm<sup>2</sup>/s
3. (i)  $0 \leq x \leq \frac{\pi}{2}$  and  $\frac{3\pi}{2} < x < 2\pi$  (ii)  $\frac{\pi}{2} < x < \frac{3\pi}{2}$
4. (i)  $x < -1$  and  $x > 1$  (ii)  $-1 < x < 1$



5.  $\frac{3\sqrt{3}}{4}ab$

6. Rs 1000

8. length =  $\frac{20}{\pi+4}$  m, breadth =  $\frac{10}{\pi+4}$  m

10. (i) local maxima at  $x = \frac{2}{7}$  (ii) local minima at  $x = 2$   
(iii) point of inflection at  $x = -1$

11. Absolute maximum =  $\frac{5}{4}$ , Absolute minimum = 1

14.  $\frac{4\pi R^3}{3\sqrt{3}}$

16. A



## SUPPLEMENTARY MATERIAL

### CHAPTER 5

*Theorem 5 (To be on page 129 under the heading Theorem 5)*

**(i) Derivative of Exponential Function  $f(x) = e^x$ .**

If  $f(x) = e^x$ , then

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{e^{x + \Delta x} - e^x}{\Delta x} \\ &= e^x \cdot \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x} - 1}{\Delta x} \\ &= e^x \cdot 1 \quad [\text{since } \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1] \end{aligned}$$

Thus,  $\frac{d}{dx}(e^x) = e^x$ .

**(ii) Derivative of logarithmic function  $f(x) = \log_e x$ .**

If  $f(x) = \log_e x$ , then

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\log_e(x + \Delta x) - \log_e x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\log_e \left(1 + \frac{\Delta x}{x}\right)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{x} \frac{\log_e \left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} \\ &= \frac{1}{x} \quad [\text{since } \lim_{h \rightarrow 0} \frac{\log_e(1 + h)}{h} = 1] \end{aligned}$$

Thus,  $\frac{d}{dx} \log_e x = \frac{1}{x}$ .

## NOTES

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not to be republished

# **MATHEMATICS**

*Textbook for Class XII*

**PART I**



|2079



**राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्**  
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## Foreword

The National Curriculum Framework 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

NCERT appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisor for this book, Professor P.K. Jain for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

New Delhi  
20 December 2005

*Director*  
National Council of Educational  
Research and Training

## **Rationalisation of Content in the Textbook**

In view of the COVID-19 pandemic, it is imperative to reduce content load on students. The National Education Policy 2020, also emphasises reducing the content load and providing opportunities for experiential learning with creative mindset. In this background, the NCERT has undertaken the exercise to rationalise the textbooks across all classes. Learning Outcomes already developed by the NCERT across classes have been taken into consideration in this exercise.

**Contents of the textbooks have been rationalised in view of the following:**

- Overlapping with similar content included in other subject areas in the same class
- Similar content included in the lower or higher class in the same subject
- Difficulty level
- Content, which is easily accessible to students without much interventions from teachers and can be learned by children through self-learning or peer-learning
- Content, which is irrelevant in the present context

**This present edition, is a reformatted version after carrying out the changes given above.**

# THE CONSTITUTION OF INDIA

## PREAMBLE

**WE, THE PEOPLE OF INDIA**, having solemnly resolved to constitute India into a **<sup>1</sup>[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

**JUSTICE**, social, economic and political;

**LIBERTY** of thought, expression, belief, faith and worship;

**EQUALITY** of status and of opportunity; and to promote among them all

**FRATERNITY** assuring the dignity of the individual and the **<sup>2</sup>[unity and integrity of the Nation]**;

**IN OUR CONSTITUENT ASSEMBLY** this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

## Preface

The National Council of Educational Research and Training (NCERT) had constituted 21 Focus Groups on Teaching of various subjects related to School Education, to review the National Curriculum Framework for School Education - 2000 (NCFSE - 2000) in face of new emerging challenges and transformations occurring in the fields of content and pedagogy under the contexts of National and International spectrum of school education. These Focus Groups made general and specific comments in their respective areas. Consequently, based on these reports of Focus Groups, National Curriculum Framework (NCF)-2005 was developed.

NCERT designed the new syllabi and constituted Textbook Development Teams for Classes XI and XII to prepare textbooks in mathematics under the new guidelines and new syllabi. The textbook for Class XI is already in use, which was brought in 2005.

The first draft of the present book (Class XII) was prepared by the team consisting of NCERT faculty, experts and practicing teachers. The draft was refined by the development team in different meetings. This draft of the book was exposed to a group of practicing teachers teaching mathematics at higher secondary stage in different parts of the country, in a review workshop organised by the NCERT at Delhi. The teachers made useful comments and suggestions which were incorporated in the draft textbook. The draft textbook was finalised by an editorial board constituted out of the development team. Finally, the Advisory Group in Science and Mathematics and the Monitoring Committee constituted by the HRD Ministry, Government of India have approved the draft of the textbook.

In the fitness of things, let us cite some of the essential features dominating the textbook. These characteristics have reflections in almost all the chapters. The existing textbook contain 13 main chapters and two appendices. Each Chapter contain the followings:

- Introduction: Highlighting the importance of the topic; connection with earlier studied topics; brief mention about the new concepts to be discussed in the chapter.
- Organisation of chapter into sections comprising one or more concepts/sub concepts.
- Motivating and introducing the concepts/sub concepts. Illustrations have been provided wherever possible.

- Proofs/problem solving involving deductive or inductive reasoning, multiplicity of approaches wherever possible have been inducted.
- Geometric viewing / visualisation of concepts have been emphasised whenever needed.
- Applications of mathematical concepts have also been integrated with allied subjects like science and social sciences.
- Adequate and variety of examples/exercises have been given in each section.
- For refocusing and strengthening the understanding and skill of problem solving and applicabilities, miscellaneous types of examples/exercises have been provided involving two or more sub concepts at a time at the end of the chapter. The scope of challenging problems to talented minority have been reflected conducive to the recommendation as reflected in NCF-2005.
- For more motivational purpose, brief historical background of topics have been provided at the end of the chapter and at the beginning of each chapter relevant quotation and photograph of eminent mathematician who have contributed significantly in the development of the topic undertaken, are also provided.
- Lastly, for direct recapitulation of main concepts, formulas and results, brief summary of the chapter has also been provided.

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# RELATIONS AND FUNCTIONS

❖ *There is no permanent place in the world for ugly mathematics . . . It may be very hard to define mathematical beauty but that is just as true of beauty of any kind, we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it. — G. H. HARDY* ❖

## 1.1 Introduction

Recall that the notion of relations and functions, domain, co-domain and range have been introduced in Class XI along with different types of specific real valued functions and their graphs. The concept of the term ‘relation’ in mathematics has been drawn from the meaning of relation in English language, according to which two objects or quantities are related if there is a recognisable connection or link between the two objects or quantities. Let  $A$  be the set of students of Class XII of a school and  $B$  be the set of students of Class XI of the same school. Then some of the examples of relations from  $A$  to  $B$  are

- (i)  $\{(a, b) \in A \times B : a$  is brother of  $b\}$ ,
- (ii)  $\{(a, b) \in A \times B : a$  is sister of  $b\}$ ,
- (iii)  $\{(a, b) \in A \times B : \text{age of } a \text{ is greater than age of } b\}$ ,
- (iv)  $\{(a, b) \in A \times B : \text{total marks obtained by } a \text{ in the final examination is less than the total marks obtained by } b \text{ in the final examination}\}$ ,
- (v)  $\{(a, b) \in A \times B : a \text{ lives in the same locality as } b\}$ . However, abstracting from this, we define mathematically a relation  $R$  from  $A$  to  $B$  as an arbitrary subset of  $A \times B$ .

If  $(a, b) \in R$ , we say that  $a$  is related to  $b$  under the relation  $R$  and we write as  $a R b$ . In general,  $(a, b) \in R$ , we do not bother whether there is a recognisable connection or link between  $a$  and  $b$ . As seen in Class XI, functions are special kind of relations.

In this chapter, we will study different types of relations and functions, composition of functions, invertible functions and binary operations.



Lejeune Dirichlet  
(1805-1859)

## 1.2 Types of Relations

In this section, we would like to study different types of relations. We know that a relation in a set  $A$  is a subset of  $A \times A$ . Thus, the empty set  $\phi$  and  $A \times A$  are two extreme relations. For illustration, consider a relation  $R$  in the set  $A = \{1, 2, 3, 4\}$  given by  $R = \{(a, b) : a - b = 10\}$ . This is the empty set, as no pair  $(a, b)$  satisfies the condition  $a - b = 10$ . Similarly,  $R' = \{(a, b) : |a - b| \geq 0\}$  is the whole set  $A \times A$ , as all pairs  $(a, b)$  in  $A \times A$  satisfy  $|a - b| \geq 0$ . These two extreme examples lead us to the following definitions.

**Definition 1** A relation  $R$  in a set  $A$  is called *empty relation*, if no element of  $A$  is related to any element of  $A$ , i.e.,  $R = \phi \subset A \times A$ .

**Definition 2** A relation  $R$  in a set  $A$  is called *universal relation*, if each element of  $A$  is related to every element of  $A$ , i.e.,  $R = A \times A$ .

Both the empty relation and the universal relation are sometimes called *trivial relations*.

**Example 1** Let  $A$  be the set of all students of a boys school. Show that the relation  $R$  in  $A$  given by  $R = \{(a, b) : a \text{ is sister of } b\}$  is the empty relation and  $R' = \{(a, b) : \text{the difference between heights of } a \text{ and } b \text{ is less than 3 meters}\}$  is the universal relation.

**Solution** Since the school is boys school, no student of the school can be sister of any student of the school. Hence,  $R = \phi$ , showing that  $R$  is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 meters. This shows that  $R' = A \times A$  is the universal relation.

**Remark** In Class XI, we have seen two ways of representing a relation, namely raster method and set builder method. However, a relation  $R$  in the set  $\{1, 2, 3, 4\}$  defined by  $R = \{(a, b) : b = a + 1\}$  is also expressed as  $a R b$  if and only if  $b = a + 1$  by many authors. We may also use this notation, as and when convenient.

If  $(a, b) \in R$ , we say that  $a$  is related to  $b$  and we denote it as  $a R b$ .

One of the most important relations, which plays a significant role in Mathematics, is an *equivalence relation*. To study equivalence relation, we first consider three types of relations, namely reflexive, symmetric and transitive.

**Definition 3** A relation  $R$  in a set  $A$  is called

- reflexive, if  $(a, a) \in R$ , for every  $a \in A$ ,
- symmetric, if  $(a_1, a_2) \in R$  implies that  $(a_2, a_1) \in R$ , for all  $a_1, a_2 \in A$ .
- transitive, if  $(a_1, a_2) \in R$  and  $(a_2, a_3) \in R$  implies that  $(a_1, a_3) \in R$ , for all  $a_1, a_2, a_3 \in A$ .

**Definition 4** A relation  $R$  in a set  $A$  is said to be an *equivalence relation* if  $R$  is reflexive, symmetric and transitive.

**Example 2** Let  $T$  be the set of all triangles in a plane with  $R$  a relation in  $T$  given by  $R = \{(T_1, T_2) : T_1 \text{ is congruent to } T_2\}$ . Show that  $R$  is an equivalence relation.

**Solution**  $R$  is reflexive, since every triangle is congruent to itself. Further,  $(T_1, T_2) \in R \Rightarrow T_1 \text{ is congruent to } T_2 \Rightarrow T_2 \text{ is congruent to } T_1 \Rightarrow (T_2, T_1) \in R$ . Hence,  $R$  is symmetric. Moreover,  $(T_1, T_2), (T_2, T_3) \in R \Rightarrow T_1 \text{ is congruent to } T_2 \text{ and } T_2 \text{ is congruent to } T_3 \Rightarrow T_1 \text{ is congruent to } T_3 \Rightarrow (T_1, T_3) \in R$ . Therefore,  $R$  is an equivalence relation.

**Example 3** Let  $L$  be the set of all lines in a plane and  $R$  be the relation in  $L$  defined as  $R = \{(L_1, L_2) : L_1 \text{ is perpendicular to } L_2\}$ . Show that  $R$  is symmetric but neither reflexive nor transitive.

**Solution**  $R$  is not reflexive, as a line  $L_1$  can not be perpendicular to itself, i.e.,  $(L_1, L_1) \notin R$ .  $R$  is symmetric as  $(L_1, L_2) \in R$

$$\begin{aligned} \Rightarrow & \quad L_1 \text{ is perpendicular to } L_2 \\ \Rightarrow & \quad L_2 \text{ is perpendicular to } L_1 \\ \Rightarrow & \quad (L_2, L_1) \in R. \end{aligned}$$

$R$  is not transitive. Indeed, if  $L_1$  is perpendicular to  $L_2$  and  $L_2$  is perpendicular to  $L_3$ , then  $L_1$  can never be perpendicular to  $L_3$ . In fact,  $L_1$  is parallel to  $L_3$ , i.e.,  $(L_1, L_2) \in R, (L_2, L_3) \in R$  but  $(L_1, L_3) \notin R$ .

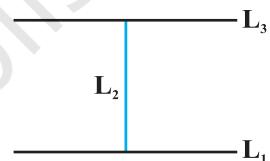


Fig 1.1

**Example 4** Show that the relation  $R$  in the set  $\{1, 2, 3\}$  given by  $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$  is reflexive but neither symmetric nor transitive.

**Solution**  $R$  is reflexive, since  $(1, 1), (2, 2)$  and  $(3, 3)$  lie in  $R$ . Also,  $R$  is not symmetric, as  $(1, 2) \in R$  but  $(2, 1) \notin R$ . Similarly,  $R$  is not transitive, as  $(1, 2) \in R$  and  $(2, 3) \in R$  but  $(1, 3) \notin R$ .

**Example 5** Show that the relation  $R$  in the set  $\mathbf{Z}$  of integers given by

$$R = \{(a, b) : 2 \text{ divides } a - b\}$$

is an equivalence relation.

**Solution**  $R$  is reflexive, as 2 divides  $(a - a)$  for all  $a \in \mathbf{Z}$ . Further, if  $(a, b) \in R$ , then 2 divides  $a - b$ . Therefore, 2 divides  $b - a$ . Hence,  $(b, a) \in R$ , which shows that  $R$  is symmetric. Similarly, if  $(a, b) \in R$  and  $(b, c) \in R$ , then  $a - b$  and  $b - c$  are divisible by 2. Now,  $a - c = (a - b) + (b - c)$  is even (Why?). So,  $(a - c)$  is divisible by 2. This shows that  $R$  is transitive. Thus,  $R$  is an equivalence relation in  $\mathbf{Z}$ .

In Example 5, note that all even integers are related to zero, as  $(0, \pm 2)$ ,  $(0, \pm 4)$  etc., lie in  $R$  and no odd integer is related to 0, as  $(0, \pm 1)$ ,  $(0, \pm 3)$  etc., do not lie in  $R$ . Similarly, all odd integers are related to one and no even integer is related to one. Therefore, the set  $E$  of all even integers and the set  $O$  of all odd integers are subsets of  $\mathbf{Z}$  satisfying following conditions:

- (i) All elements of  $E$  are related to each other and all elements of  $O$  are related to each other.
- (ii) No element of  $E$  is related to any element of  $O$  and vice-versa.
- (iii)  $E$  and  $O$  are disjoint and  $\mathbf{Z} = E \cup O$ .

The subset  $E$  is called the *equivalence class containing zero* and is denoted by  $[0]$ . Similarly,  $O$  is the equivalence class containing 1 and is denoted by  $[1]$ . Note that  $[0] \neq [1]$ ,  $[0] = [2r]$  and  $[1] = [2r + 1]$ ,  $r \in \mathbf{Z}$ . Infact, what we have seen above is true for an arbitrary equivalence relation  $R$  in a set  $X$ . Given an arbitrary equivalence relation  $R$  in an arbitrary set  $X$ ,  $R$  divides  $X$  into mutually disjoint subsets  $A_i$  called partitions or subdivisions of  $X$  satisfying:

- (i) all elements of  $A_i$  are related to each other, for all  $i$ .
- (ii) no element of  $A_i$  is related to any element of  $A_j$ ,  $i \neq j$ .
- (iii)  $\cup A_j = X$  and  $A_i \cap A_j = \emptyset$ ,  $i \neq j$ .

The subsets  $A_i$  are called *equivalence classes*. The interesting part of the situation is that we can go reverse also. For example, consider a subdivision of the set  $\mathbf{Z}$  given by three mutually disjoint subsets  $A_1$ ,  $A_2$  and  $A_3$  whose union is  $\mathbf{Z}$  with

$$A_1 = \{x \in \mathbf{Z} : x \text{ is a multiple of } 3\} = \{\dots, -6, -3, 0, 3, 6, \dots\}$$

$$A_2 = \{x \in \mathbf{Z} : x - 1 \text{ is a multiple of } 3\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$$

$$A_3 = \{x \in \mathbf{Z} : x - 2 \text{ is a multiple of } 3\} = \{\dots, -4, -1, 2, 5, 8, \dots\}$$

Define a relation  $R$  in  $\mathbf{Z}$  given by  $R = \{(a, b) : 3 \text{ divides } a - b\}$ . Following the arguments similar to those used in Example 5, we can show that  $R$  is an equivalence relation. Also,  $A_1$  coincides with the set of all integers in  $\mathbf{Z}$  which are related to zero,  $A_2$  coincides with the set of all integers which are related to 1 and  $A_3$  coincides with the set of all integers in  $\mathbf{Z}$  which are related to 2. Thus,  $A_1 = [0]$ ,  $A_2 = [1]$  and  $A_3 = [2]$ . In fact,  $A_1 = [3r]$ ,  $A_2 = [3r + 1]$  and  $A_3 = [3r + 2]$ , for all  $r \in \mathbf{Z}$ .

**Example 6** Let  $R$  be the relation defined in the set  $A = \{1, 2, 3, 4, 5, 6, 7\}$  by  $R = \{(a, b) : \text{both } a \text{ and } b \text{ are either odd or even}\}$ . Show that  $R$  is an equivalence relation. Further, show that all the elements of the subset  $\{1, 3, 5, 7\}$  are related to each other and all the elements of the subset  $\{2, 4, 6\}$  are related to each other, but no element of the subset  $\{1, 3, 5, 7\}$  is related to any element of the subset  $\{2, 4, 6\}$ .

**Solution** Given any element  $a$  in  $A$ , both  $a$  and  $a$  must be either odd or even, so that  $(a, a) \in R$ . Further,  $(a, b) \in R \Rightarrow$  both  $a$  and  $b$  must be either odd or even  $\Rightarrow (b, a) \in R$ . Similarly,  $(a, b) \in R$  and  $(b, c) \in R \Rightarrow$  all elements  $a, b, c$ , must be either even or odd simultaneously  $\Rightarrow (a, c) \in R$ . Hence,  $R$  is an equivalence relation. Further, all the elements of  $\{1, 3, 5, 7\}$  are related to each other, as all the elements of this subset are odd. Similarly, all the elements of the subset  $\{2, 4, 6\}$  are related to each other, as all of them are even. Also, no element of the subset  $\{1, 3, 5, 7\}$  can be related to any element of  $\{2, 4, 6\}$ , as elements of  $\{1, 3, 5, 7\}$  are odd, while elements of  $\{2, 4, 6\}$  are even.

### EXERCISE 1.1

- Determine whether each of the following relations are reflexive, symmetric and transitive:
  - Relation  $R$  in the set  $A = \{1, 2, 3, \dots, 13, 14\}$  defined as  

$$R = \{(x, y) : 3x - y = 0\}$$
  - Relation  $R$  in the set  $\mathbb{N}$  of natural numbers defined as  

$$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$
  - Relation  $R$  in the set  $A = \{1, 2, 3, 4, 5, 6\}$  as  

$$R = \{(x, y) : y \text{ is divisible by } x\}$$
  - Relation  $R$  in the set  $\mathbb{Z}$  of all integers defined as  

$$R = \{(x, y) : x - y \text{ is an integer}\}$$
  - Relation  $R$  in the set  $A$  of human beings in a town at a particular time given by
    - $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$
    - $R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$
    - $R = \{(x, y) : x \text{ is exactly } 7 \text{ cm taller than } y\}$
    - $R = \{(x, y) : x \text{ is wife of } y\}$
    - $R = \{(x, y) : x \text{ is father of } y\}$
- Show that the relation  $R$  in the set  $\mathbf{R}$  of real numbers, defined as  

$$R = \{(a, b) : a \leq b^2\}$$
 is neither reflexive nor symmetric nor transitive.
- Check whether the relation  $R$  defined in the set  $\{1, 2, 3, 4, 5, 6\}$  as  

$$R = \{(a, b) : b = a + 1\}$$
 is reflexive, symmetric or transitive.
- Show that the relation  $R$  in  $\mathbf{R}$  defined as  $R = \{(a, b) : a \leq b\}$ , is reflexive and transitive but not symmetric.
- Check whether the relation  $R$  in  $\mathbf{R}$  defined by  $R = \{(a, b) : a \leq b^3\}$  is reflexive, symmetric or transitive.

6. Show that the relation  $R$  in the set  $\{1, 2, 3\}$  given by  $R = \{(1, 2), (2, 1)\}$  is symmetric but neither reflexive nor transitive.
7. Show that the relation  $R$  in the set  $A$  of all the books in a library of a college, given by  $R = \{(x, y) : x \text{ and } y \text{ have same number of pages}\}$  is an equivalence relation.
8. Show that the relation  $R$  in the set  $A = \{1, 2, 3, 4, 5\}$  given by  $R = \{(a, b) : |a - b| \text{ is even}\}$ , is an equivalence relation. Show that all the elements of  $\{1, 3, 5\}$  are related to each other and all the elements of  $\{2, 4\}$  are related to each other. But no element of  $\{1, 3, 5\}$  is related to any element of  $\{2, 4\}$ .
9. Show that each of the relation  $R$  in the set  $A = \{x \in \mathbf{Z} : 0 \leq x \leq 12\}$ , given by
  - (i)  $R = \{(a, b) : |a - b| \text{ is a multiple of 4}\}$
  - (ii)  $R = \{(a, b) : a = b\}$
 is an equivalence relation. Find the set of all elements related to 1 in each case.
10. Give an example of a relation. Which is
  - (i) Symmetric but neither reflexive nor transitive.
  - (ii) Transitive but neither reflexive nor symmetric.
  - (iii) Reflexive and symmetric but not transitive.
  - (iv) Reflexive and transitive but not symmetric.
  - (v) Symmetric and transitive but not reflexive.
11. Show that the relation  $R$  in the set  $A$  of points in a plane given by  $R = \{(P, Q) : \text{distance of the point } P \text{ from the origin is same as the distance of the point } Q \text{ from the origin}\}$ , is an equivalence relation. Further, show that the set of all points related to a point  $P \neq (0, 0)$  is the circle passing through  $P$  with origin as centre.
12. Show that the relation  $R$  defined in the set  $A$  of all triangles as  $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$ , is equivalence relation. Consider three right angle triangles  $T_1$  with sides 3, 4, 5,  $T_2$  with sides 5, 12, 13 and  $T_3$  with sides 6, 8, 10. Which triangles among  $T_1$ ,  $T_2$  and  $T_3$  are related?
13. Show that the relation  $R$  defined in the set  $A$  of all polygons as  $R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$ , is an equivalence relation. What is the set of all elements in  $A$  related to the right angle triangle  $T$  with sides 3, 4 and 5?
14. Let  $L$  be the set of all lines in  $XY$  plane and  $R$  be the relation in  $L$  defined as  $R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$ . Show that  $R$  is an equivalence relation. Find the set of all lines related to the line  $y = 2x + 4$ .

15. Let  $R$  be the relation in the set  $\{1, 2, 3, 4\}$  given by  $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$ . Choose the correct answer.
- $R$  is reflexive and symmetric but not transitive.
  - $R$  is reflexive and transitive but not symmetric.
  - $R$  is symmetric and transitive but not reflexive.
  - $R$  is an equivalence relation.
16. Let  $R$  be the relation in the set  $N$  given by  $R = \{(a, b) : a = b - 2, b > 6\}$ . Choose the correct answer.
- $(2, 4) \in R$
  - $(3, 8) \in R$
  - $(6, 8) \in R$
  - $(8, 7) \in R$

### 1.3 Types of Functions

The notion of a function along with some special functions like identity function, constant function, polynomial function, rational function, modulus function, signum function etc. along with their graphs have been given in Class XI.

Addition, subtraction, multiplication and division of two functions have also been studied. As the concept of function is of paramount importance in mathematics and among other disciplines as well, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Consider the functions  $f_1, f_2, f_3$  and  $f_4$  given by the following diagrams.

In Fig 1.2, we observe that the images of distinct elements of  $X_1$  under the function  $f_1$  are distinct, but the image of two distinct elements 1 and 2 of  $X_1$  under  $f_2$  is same, namely  $b$ . Further, there are some elements like  $e$  and  $f$  in  $X_2$  which are not images of any element of  $X_1$  under  $f_1$ , while all elements of  $X_3$  are images of some elements of  $X_1$  under  $f_3$ . The above observations lead to the following definitions:

**Definition 5** A function  $f: X \rightarrow Y$  is defined to be *one-one* (or *injective*), if the images of distinct elements of  $X$  under  $f$  are distinct, i.e., for every  $x_1, x_2 \in X$ ,  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ . Otherwise,  $f$  is called *many-one*.

The function  $f_1$  and  $f_4$  in Fig 1.2 (i) and (iv) are one-one and the function  $f_2$  and  $f_3$  in Fig 1.2 (ii) and (iii) are many-one.

**Definition 6** A function  $f: X \rightarrow Y$  is said to be *onto* (or *surjective*), if every element of  $Y$  is the image of some element of  $X$  under  $f$ , i.e., for every  $y \in Y$ , there exists an element  $x$  in  $X$  such that  $f(x) = y$ .

The function  $f_3$  and  $f_4$  in Fig 1.2 (iii), (iv) are onto and the function  $f_1$  in Fig 1.2 (i) is not onto as elements  $e, f$  in  $X_2$  are not the image of any element in  $X_1$  under  $f_1$ .

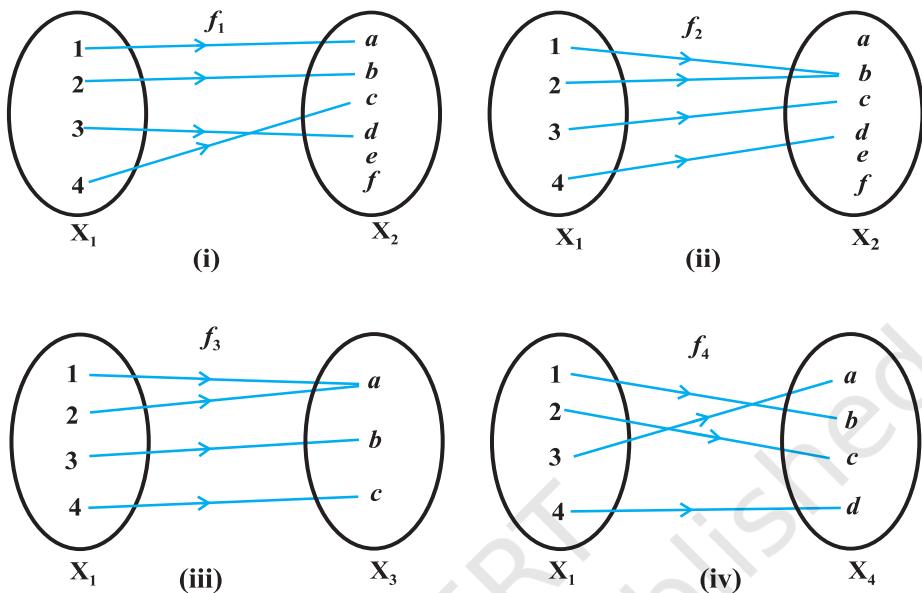


Fig 1.2 (i) to (iv)

**Remark**  $f: X \rightarrow Y$  is onto if and only if Range of  $f = Y$ .

**Definition 7** A function  $f: X \rightarrow Y$  is said to be *one-one* and *onto* (or *bijective*), if  $f$  is both one-one and onto.

The function  $f_4$  in Fig 1.2 (iv) is one-one and onto.

**Example 7** Let  $A$  be the set of all 50 students of Class  $X$  in a school. Let  $f: A \rightarrow \mathbb{N}$  be function defined by  $f(x) = \text{roll number of the student } x$ . Show that  $f$  is one-one but not onto.

**Solution** No two different students of the class can have same roll number. Therefore,  $f$  must be one-one. We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in  $\mathbb{N}$  is not roll number of any student of the class, so that 51 can not be image of any element of  $X$  under  $f$ . Hence,  $f$  is not onto.

**Example 8** Show that the function  $f: \mathbb{N} \rightarrow \mathbb{N}$ , given by  $f(x) = 2x$ , is one-one but not onto.

**Solution** The function  $f$  is one-one, for  $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$ . Further,  $f$  is not onto, as for  $1 \in \mathbb{N}$ , there does not exist any  $x$  in  $\mathbb{N}$  such that  $f(x) = 2x = 1$ .

**Example 9** Prove that the function  $f: \mathbf{R} \rightarrow \mathbf{R}$ , given by  $f(x) = 2x$ , is one-one and onto.

**Solution**  $f$  is one-one, as  $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$ . Also, given any real number  $y$  in  $\mathbf{R}$ , there exists  $\frac{y}{2}$  in  $\mathbf{R}$  such that  $f\left(\frac{y}{2}\right) = 2 \cdot \left(\frac{y}{2}\right) = y$ . Hence,  $f$  is onto.

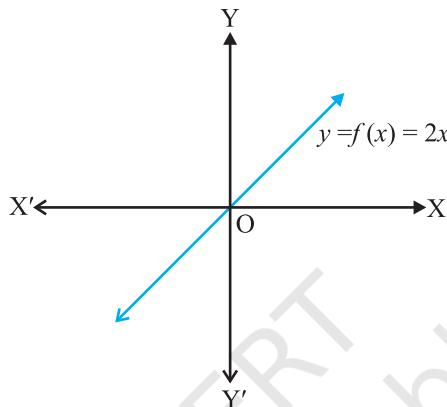


Fig 1.3

**Example 10** Show that the function  $f: \mathbf{N} \rightarrow \mathbf{N}$ , given by  $f(1) = f(2) = 1$  and  $f(x) = x - 1$ , for every  $x > 2$ , is onto but not one-one.

**Solution**  $f$  is not one-one, as  $f(1) = f(2) = 1$ . But  $f$  is onto, as given any  $y \in \mathbf{N}$ ,  $y \neq 1$ , we can choose  $x$  as  $y + 1$  such that  $f(y + 1) = y + 1 - 1 = y$ . Also for  $1 \in \mathbf{N}$ , we have  $f(1) = 1$ .

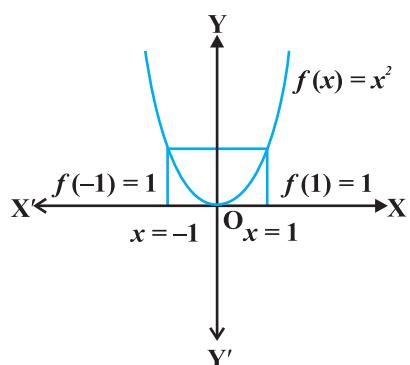
**Example 11** Show that the function  $f: \mathbf{R} \rightarrow \mathbf{R}$ , defined as  $f(x) = x^2$ , is neither one-one nor onto.

**Solution** Since  $f(-1) = 1 = f(1)$ ,  $f$  is not one-one. Also, the element  $-2$  in the co-domain  $\mathbf{R}$  is not image of any element  $x$  in the domain  $\mathbf{R}$  (Why?). Therefore  $f$  is not onto.

**Example 12** Show that  $f: \mathbf{N} \rightarrow \mathbf{N}$ , given by

$$f(x) = \begin{cases} x + 1, & \text{if } x \text{ is odd,} \\ x - 1, & \text{if } x \text{ is even} \end{cases}$$

is both one-one and onto.



The image of 1 and  $-1$  under  $f$  is 1.

Fig 1.4

**Solution** Suppose  $f(x_1) = f(x_2)$ . Note that if  $x_1$  is odd and  $x_2$  is even, then we will have  $x_1 + 1 = x_2 - 1$ , i.e.,  $x_2 - x_1 = 2$  which is impossible. Similarly, the possibility of  $x_1$  being even and  $x_2$  being odd can also be ruled out, using the similar argument. Therefore, both  $x_1$  and  $x_2$  must be either odd or even. Suppose both  $x_1$  and  $x_2$  are odd. Then  $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$ . Similarly, if both  $x_1$  and  $x_2$  are even, then also  $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$ . Thus,  $f$  is one-one. Also, any odd number  $2r + 1$  in the co-domain  $\mathbf{N}$  is the image of  $2r + 2$  in the domain  $\mathbf{N}$  and any even number  $2r$  in the co-domain  $\mathbf{N}$  is the image of  $2r - 1$  in the domain  $\mathbf{N}$ . Thus,  $f$  is onto.

**Example 13** Show that an onto function  $f: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$  is always one-one.

**Solution** Suppose  $f$  is not one-one. Then there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same. Also, the image of 3 under  $f$  can be only one element. Therefore, the range set can have at the most two elements of the co-domain  $\{1, 2, 3\}$ , showing that  $f$  is not onto, a contradiction. Hence,  $f$  must be one-one.

**Example 14** Show that a one-one function  $f: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$  must be onto.

**Solution** Since  $f$  is one-one, three elements of  $\{1, 2, 3\}$  must be taken to 3 different elements of the co-domain  $\{1, 2, 3\}$  under  $f$ . Hence,  $f$  has to be onto.

**Remark** The results mentioned in Examples 13 and 14 are also true for an arbitrary finite set  $X$ , i.e., a one-one function  $f: X \rightarrow X$  is necessarily onto and an onto map  $f: X \rightarrow X$  is necessarily one-one, for every finite set  $X$ . In contrast to this, Examples 8 and 10 show that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

### EXERCISE 1.2

1. Show that the function  $f: \mathbf{R}_* \rightarrow \mathbf{R}_*$  defined by  $f(x) = \frac{1}{x}$  is one-one and onto, where  $\mathbf{R}_*$  is the set of all non-zero real numbers. Is the result true, if the domain  $\mathbf{R}_*$  is replaced by  $\mathbf{N}$  with co-domain being same as  $\mathbf{R}_*$ ?
2. Check the injectivity and surjectivity of the following functions:
  - (i)  $f: \mathbf{N} \rightarrow \mathbf{N}$  given by  $f(x) = x^2$
  - (ii)  $f: \mathbf{Z} \rightarrow \mathbf{Z}$  given by  $f(x) = x^2$
  - (iii)  $f: \mathbf{R} \rightarrow \mathbf{R}$  given by  $f(x) = x^2$
  - (iv)  $f: \mathbf{N} \rightarrow \mathbf{N}$  given by  $f(x) = x^3$
  - (v)  $f: \mathbf{Z} \rightarrow \mathbf{Z}$  given by  $f(x) = x^3$
3. Prove that the Greatest Integer Function  $f: \mathbf{R} \rightarrow \mathbf{R}$ , given by  $f(x) = [x]$ , is neither one-one nor onto, where  $[x]$  denotes the greatest integer less than or equal to  $x$ .

4. Show that the Modulus Function  $f: \mathbf{R} \rightarrow \mathbf{R}$ , given by  $f(x) = |x|$ , is neither one-one nor onto, where  $|x|$  is  $x$ , if  $x$  is positive or 0 and  $|x|$  is  $-x$ , if  $x$  is negative.
5. Show that the Signum Function  $f: \mathbf{R} \rightarrow \mathbf{R}$ , given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

6. Let  $A = \{1, 2, 3\}$ ,  $B = \{4, 5, 6, 7\}$  and let  $f = \{(1, 4), (2, 5), (3, 6)\}$  be a function from  $A$  to  $B$ . Show that  $f$  is one-one.
7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.
- $f: \mathbf{R} \rightarrow \mathbf{R}$  defined by  $f(x) = 3 - 4x$
  - $f: \mathbf{R} \rightarrow \mathbf{R}$  defined by  $f(x) = 1 + x^2$
8. Let  $A$  and  $B$  be sets. Show that  $f: A \times B \rightarrow B \times A$  such that  $f(a, b) = (b, a)$  is bijective function.

9. Let  $f: \mathbf{N} \rightarrow \mathbf{N}$  be defined by  $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$  for all  $n \in \mathbf{N}$ .

State whether the function  $f$  is bijective. Justify your answer.

10. Let  $A = \mathbf{R} - \{3\}$  and  $B = \mathbf{R} - \{1\}$ . Consider the function  $f: A \rightarrow B$  defined by  $f(x) = \left(\frac{x-2}{x-3}\right)$ . Is  $f$  one-one and onto? Justify your answer.
11. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be defined as  $f(x) = x^4$ . Choose the correct answer.
- $f$  is one-one onto
  - $f$  is many-one onto
  - $f$  is one-one but not onto
  - $f$  is neither one-one nor onto.
12. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be defined as  $f(x) = 3x$ . Choose the correct answer.
- $f$  is one-one onto
  - $f$  is many-one onto
  - $f$  is one-one but not onto
  - $f$  is neither one-one nor onto.

## 1.4 Composition of Functions and Invertible Function

**Definition 8** Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two functions. Then the composition of  $f$  and  $g$ , denoted by  $gof$ , is defined as the function  $gof: A \rightarrow C$  given by

$$gof(x) = g(f(x)), \forall x \in A.$$

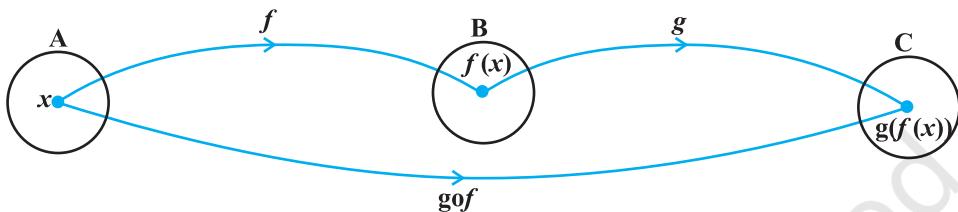


Fig 1.5

**Example 15** Let  $f: \{2, 3, 4, 5\} \rightarrow \{3, 4, 5, 9\}$  and  $g: \{3, 4, 5, 9\} \rightarrow \{7, 11, 15\}$  be functions defined as  $f(2) = 3, f(3) = 4, f(4) = f(5) = 5$  and  $g(3) = g(4) = 7$  and  $g(5) = g(9) = 11$ . Find  $gof$ .

**Solution** We have  $gof(2) = g(f(2)) = g(3) = 7, gof(3) = g(f(3)) = g(4) = 7, gof(4) = g(f(4)) = g(5) = 11$  and  $gof(5) = g(f(5)) = g(5) = 11$ .

**Example 16** Find  $gof$  and  $fog$ , if  $f: \mathbf{R} \rightarrow \mathbf{R}$  and  $g: \mathbf{R} \rightarrow \mathbf{R}$  are given by  $f(x) = \cos x$  and  $g(x) = 3x^2$ . Show that  $gof \neq fog$ .

**Solution** We have  $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3 \cos^2 x$ . Similarly,  $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$ . Note that  $3\cos^2 x \neq \cos 3x^2$ , for  $x = 0$ . Hence,  $gof \neq fog$ .

**Definition 9** A function  $f: X \rightarrow Y$  is defined to be *invertible*, if there exists a function  $g: Y \rightarrow X$  such that  $gof = I_X$  and  $fog = I_Y$ . The function  $g$  is called the *inverse of f* and is denoted by  $f^{-1}$ .

Thus, if  $f$  is invertible, then  $f$  must be one-one and onto and conversely, if  $f$  is one-one and onto, then  $f$  must be invertible. This fact significantly helps for proving a function  $f$  to be invertible by showing that  $f$  is one-one and onto, specially when the actual inverse of  $f$  is not to be determined.

**Example 17** Let  $f: \mathbf{N} \rightarrow \mathbf{Y}$  be a function defined as  $f(x) = 4x + 3$ , where,  $\mathbf{Y} = \{y \in \mathbf{N} : y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$ . Show that  $f$  is invertible. Find the inverse.

**Solution** Consider an arbitrary element  $y$  of  $\mathbf{Y}$ . By the definition of  $\mathbf{Y}$ ,  $y = 4x + 3$ ,

for some  $x$  in the domain  $\mathbf{N}$ . This shows that  $x = \frac{(y-3)}{4}$ . Define  $g: \mathbf{Y} \rightarrow \mathbf{N}$  by

$g(y) = \frac{(y-3)}{4}$ . Now,  $gof(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3-3)}{4} = x$  and

$fog(y) = f(g(y)) = f\left(\frac{(y-3)}{4}\right) = \frac{4(y-3)}{4} + 3 = y - 3 + 3 = y$ . This shows that  $gof = I_N$  and  $fog = I_Y$ , which implies that  $f$  is invertible and  $g$  is the inverse of  $f$ .

### Miscellaneous Examples

**Example 18** If  $R_1$  and  $R_2$  are equivalence relations in a set  $A$ , show that  $R_1 \cap R_2$  is also an equivalence relation.

**Solution** Since  $R_1$  and  $R_2$  are equivalence relations,  $(a, a) \in R_1$ , and  $(a, a) \in R_2 \forall a \in A$ . This implies that  $(a, a) \in R_1 \cap R_2 \forall a$ , showing  $R_1 \cap R_2$  is reflexive. Further,  $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$  and  $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$  and  $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2$ , hence,  $R_1 \cap R_2$  is symmetric. Similarly,  $(a, b) \in R_1 \cap R_2$  and  $(b, c) \in R_1 \cap R_2 \Rightarrow (a, c) \in R_1$  and  $(a, c) \in R_2 \Rightarrow (a, c) \in R_1 \cap R_2$ . This shows that  $R_1 \cap R_2$  is transitive. Thus,  $R_1 \cap R_2$  is an equivalence relation.

**Example 19** Let  $R$  be a relation on the set  $A$  of ordered pairs of positive integers defined by  $(x, y) R (u, v)$  if and only if  $xv = yu$ . Show that  $R$  is an equivalence relation.

**Solution** Clearly,  $(x, y) R (x, y) \forall (x, y) \in A$ , since  $xy = yx$ . This shows that  $R$  is reflexive. Further,  $(x, y) R (u, v) \Rightarrow xv = yu \Rightarrow uy = vx$  and hence  $(u, v) R (x, y)$ . This shows that  $R$  is symmetric. Similarly,  $(x, y) R (u, v)$  and  $(u, v) R (a, b) \Rightarrow xv = yu$  and

$ub = va \Rightarrow xv \frac{a}{u} = yu \frac{a}{u} \Rightarrow xv \frac{b}{v} = yu \frac{a}{u} \Rightarrow xb = ya$  and hence  $(x, y) R (a, b)$ . Thus,  $R$  is transitive. Thus,  $R$  is an equivalence relation.

**Example 20** Let  $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Let  $R_1$  be a relation in  $X$  given by  $R_1 = \{(x, y) : x - y \text{ is divisible by } 3\}$  and  $R_2$  be another relation on  $X$  given by  $R_2 = \{(x, y) : \{x, y\} \subset \{1, 4, 7\} \text{ or } \{x, y\} \subset \{2, 5, 8\} \text{ or } \{x, y\} \subset \{3, 6, 9\}\}$ . Show that  $R_1 = R_2$ .

**Solution** Note that the characteristic of sets  $\{1, 4, 7\}$ ,  $\{2, 5, 8\}$  and  $\{3, 6, 9\}$  is that difference between any two elements of these sets is a multiple of 3. Therefore,  $(x, y) \in R_1 \Rightarrow x - y$  is a multiple of 3  $\Rightarrow \{x, y\} \subset \{1, 4, 7\}$  or  $\{x, y\} \subset \{2, 5, 8\}$  or  $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$ . Hence,  $R_1 \subset R_2$ . Similarly,  $\{x, y\} \in R_2 \Rightarrow \{x, y\}$

$\subset \{1, 4, 7\}$  or  $\{x, y\} \subset \{2, 5, 8\}$  or  $\{x, y\} \subset \{3, 6, 9\} \Rightarrow x - y$  is divisible by 3  $\Rightarrow \{x, y\} \in R_1$ . This shows that  $R_2 \subset R_1$ . Hence,  $R_1 = R_2$ .

**Example 21** Let  $f: X \rightarrow Y$  be a function. Define a relation  $R$  in  $X$  given by  $R = \{(a, b): f(a) = f(b)\}$ . Examine whether  $R$  is an equivalence relation or not.

**Solution** For every  $a \in X$ ,  $(a, a) \in R$ , since  $f(a) = f(a)$ , showing that  $R$  is reflexive. Similarly,  $(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$ . Therefore,  $R$  is symmetric. Further,  $(a, b) \in R$  and  $(b, c) \in R \Rightarrow f(a) = f(b)$  and  $f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow (a, c) \in R$ , which implies that  $R$  is transitive. Hence,  $R$  is an equivalence relation.

**Example 22** Find the number of all one-one functions from set  $A = \{1, 2, 3\}$  to itself.

**Solution** One-one function from  $\{1, 2, 3\}$  to itself is simply a permutation on three symbols 1, 2, 3. Therefore, total number of one-one maps from  $\{1, 2, 3\}$  to itself is same as total number of permutations on three symbols 1, 2, 3 which is  $3! = 6$ .

**Example 23** Let  $A = \{1, 2, 3\}$ . Then show that the number of relations containing  $(1, 2)$  and  $(2, 3)$  which are reflexive and transitive but not symmetric is three.

**Solution** The smallest relation  $R_1$  containing  $(1, 2)$  and  $(2, 3)$  which is reflexive and transitive but not symmetric is  $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ . Now, if we add the pair  $(2, 1)$  to  $R_1$  to get  $R_2$ , then the relation  $R_2$  will be reflexive, transitive but not symmetric. Similarly, we can obtain  $R_3$  by adding  $(3, 2)$  to  $R_1$  to get the desired relation. However, we can not add two pairs  $(2, 1), (3, 2)$  or single pair  $(3, 1)$  to  $R_1$  at a time, as by doing so, we will be forced to add the remaining pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is three.

**Example 24** Show that the number of equivalence relation in the set  $\{1, 2, 3\}$  containing  $(1, 2)$  and  $(2, 1)$  is two.

**Solution** The smallest equivalence relation  $R_1$  containing  $(1, 2)$  and  $(2, 1)$  is  $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ . Now we are left with only 4 pairs namely  $(2, 3), (3, 2), (1, 3)$  and  $(3, 1)$ . If we add any one, say  $(2, 3)$  to  $R_1$ , then for symmetry we must add  $(3, 2)$  also and now for transitivity we are forced to add  $(1, 3)$  and  $(3, 1)$ . Thus, the only equivalence relation bigger than  $R_1$  is the universal relation. This shows that the total number of equivalence relations containing  $(1, 2)$  and  $(2, 1)$  is two.

**Example 25** Consider the identity function  $I_N: N \rightarrow N$  defined as  $I_N(x) = x \ \forall x \in N$ . Show that although  $I_N$  is onto but  $I_N + I_N: N \rightarrow N$  defined as

$$(I_N + I_N)(x) = I_N(x) + I_N(x) = x + x = 2x \text{ is not onto.}$$

**Solution** Clearly  $I_N$  is onto. But  $I_N + I_N$  is not onto, as we can find an element 3 in the co-domain  $N$  such that there does not exist any  $x$  in the domain  $N$  with  $(I_N + I_N)(x) = 2x = 3$ .

**Example 26** Consider a function  $f : \left[0, \frac{\pi}{2}\right] \rightarrow \mathbf{R}$  given by  $f(x) = \sin x$  and

$g : \left[0, \frac{\pi}{2}\right] \rightarrow \mathbf{R}$  given by  $g(x) = \cos x$ . Show that  $f$  and  $g$  are one-one, but  $f + g$  is not one-one.

**Solution** Since for any two distinct elements  $x_1$  and  $x_2$  in  $\left[0, \frac{\pi}{2}\right]$ ,  $\sin x_1 \neq \sin x_2$  and  $\cos x_1 \neq \cos x_2$ , both  $f$  and  $g$  must be one-one. But  $(f + g)(0) = \sin 0 + \cos 0 = 1$  and  $(f + g)\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1$ . Therefore,  $f + g$  is not one-one.

### Miscellaneous Exercise on Chapter 1

1. Show that the function  $f : \mathbf{R} \rightarrow \{x \in \mathbf{R} : -1 < x < 1\}$  defined by  $f(x) = \frac{x}{1+|x|}$ ,

$x \in \mathbf{R}$  is one one and onto function.

2. Show that the function  $f : \mathbf{R} \rightarrow \mathbf{R}$  given by  $f(x) = x^3$  is injective.

3. Given a non empty set  $X$ , consider  $P(X)$  which is the set of all subsets of  $X$ . Define the relation  $R$  in  $P(X)$  as follows:

For subsets  $A, B$  in  $P(X)$ ,  $ARB$  if and only if  $A \subset B$ . Is  $R$  an equivalence relation on  $P(X)$ ? Justify your answer.

4. Find the number of all onto functions from the set  $\{1, 2, 3, \dots, n\}$  to itself.

5. Let  $A = \{-1, 0, 1, 2\}$ ,  $B = \{-4, -2, 0, 2\}$  and  $f, g : A \rightarrow B$  be functions defined

by  $f(x) = x^2 - x$ ,  $x \in A$  and  $g(x) = 2\left|x - \frac{1}{2}\right| - 1$ ,  $x \in A$ . Are  $f$  and  $g$  equal?

Justify your answer. (Hint: One may note that two functions  $f : A \rightarrow B$  and  $g : A \rightarrow B$  such that  $f(a) = g(a) \forall a \in A$ , are called equal functions).

## Summary

In this chapter, we studied different types of relations and equivalence relation, composition of functions, invertible functions and binary operations. The main features of this chapter are as follows:

- ◆ *Empty relation* is the relation  $R$  in  $X$  given by  $R = \emptyset \subset X \times X$ .
  - ◆ *Universal relation* is the relation  $R$  in  $X$  given by  $R = X \times X$ .
  - ◆ *Reflexive relation*  $R$  in  $X$  is a relation with  $(a, a) \in R \quad \forall a \in X$ .
  - ◆ *Symmetric relation*  $R$  in  $X$  is a relation satisfying  $(a, b) \in R$  implies  $(b, a) \in R$ .
  - ◆ *Transitive relation*  $R$  in  $X$  is a relation satisfying  $(a, b) \in R$  and  $(b, c) \in R$  implies that  $(a, c) \in R$ .
  - ◆ *Equivalence relation*  $R$  in  $X$  is a relation which is reflexive, symmetric and transitive.
  - ◆ *Equivalence class*  $[a]$  containing  $a \in X$  for an equivalence relation  $R$  in  $X$  is the subset of  $X$  containing all elements  $b$  related to  $a$ .
  - ◆ A function  $f: X \rightarrow Y$  is *one-one* (or *injective*) if  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X$ .
  - ◆ A function  $f: X \rightarrow Y$  is *onto* (or *surjective*) if given any  $y \in Y, \exists x \in X$  such that  $f(x) = y$ .
  - ◆ A function  $f: X \rightarrow Y$  is *one-one and onto* (or *bijection*), if  $f$  is both one-one and onto.
  - ◆ Given a finite set  $X$ , a function  $f: X \rightarrow X$  is one-one (respectively onto) if and only if  $f$  is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set

### Historical Note

The concept of function has evolved over a long period of time starting from R. Descartes (1596-1650), who used the word ‘function’ in his manuscript “*Geometrie*” in 1637 to mean some positive integral power  $x^n$  of a variable  $x$  while studying geometrical curves like hyperbola, parabola and ellipse. James Gregory (1636-1675) in his work “*Vera Circuli et Hyperbolae Quadratura*” (1667) considered function as a quantity obtained from other quantities by successive use of algebraic operations or by any other operations. Later G. W. Leibnitz (1646-1716) in his manuscript “*Methodus tangentium inversa, seu de functionibus*” written in 1673 used the word ‘function’ to mean a quantity varying from point to point on a curve such as the coordinates of a point on the curve, the slope of the curve, the tangent and the normal to the curve at a point. However, in his manuscript “*Historia*” (1714), Leibnitz used the word ‘function’ to mean quantities that depend on a variable. He was the first to use the phrase ‘function of  $x$ ’. John Bernoulli (1667-1748) used the notation  $\phi x$  for the first time in 1718 to indicate a function of  $x$ . But the general adoption of symbols like  $f$ ,  $F$ ,  $\phi$ ,  $\psi$  ... to represent functions was made by Leonhard Euler (1707-1783) in 1734 in the first part of his manuscript “*Analysis Infinitorium*”. Later on, Joseph Louis Lagrange (1736-1813) published his manuscripts “*Theorie des functions analytiques*” in 1793, where he discussed about analytic function and used the notion  $f(x)$ ,  $F(x)$ ,  $\phi(x)$  etc. for different function of  $x$ . Subsequently, Lejeunne Dirichlet (1805-1859) gave the definition of function which was being used till the set theoretic definition of function presently used, was given after set theory was developed by Georg Cantor (1845-1918). The set theoretic definition of function known to us presently is simply an abstraction of the definition given by Dirichlet in a rigorous manner.





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Chapter 2

# INVERSE TRIGONOMETRIC FUNCTIONS

❖ *Mathematics, in general, is fundamentally the science of self-evident things. — FELIX KLEIN* ❖

## 2.1 Introduction

In Chapter 1, we have studied that the inverse of a function  $f$ , denoted by  $f^{-1}$ , exists iff  $f$  is one-one and onto. There are many functions which are not one-one, onto or both and hence we can not talk of their inverses. In Class XI, we studied that trigonometric functions are not one-one and onto over their natural domains and ranges and hence their inverses do not exist. In this chapter, we shall study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverses and observe their behaviour through graphical representations. Besides, some elementary properties will also be discussed.

The inverse trigonometric functions play an important role in calculus for they serve to define many integrals.

The concepts of inverse trigonometric functions is also used in science and engineering.

## 2.2 Basic Concepts

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e.,  $\sin : \mathbf{R} \rightarrow [-1, 1]$

cosine function, i.e.,  $\cos : \mathbf{R} \rightarrow [-1, 1]$

tangent function, i.e.,  $\tan : \mathbf{R} - \{x : x = (2n+1) \frac{\pi}{2}, n \in \mathbf{Z}\} \rightarrow \mathbf{R}$

cotangent function, i.e.,  $\cot : \mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \rightarrow \mathbf{R}$

secant function, i.e.,  $\sec : \mathbf{R} - \{x : x = (2n+1) \frac{\pi}{2}, n \in \mathbf{Z}\} \rightarrow \mathbf{R} - (-1, 1)$

cosecant function, i.e.,  $\csc : \mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \rightarrow \mathbf{R} - (-1, 1)$



Aryabhata  
(476-550 A. D.)

We have also learnt in Chapter 1 that if  $f: X \rightarrow Y$  such that  $f(x) = y$  is one-one and onto, then we can define a unique function  $g: Y \rightarrow X$  such that  $g(y) = x$ , where  $x \in X$  and  $y = f(x)$ ,  $y \in Y$ . Here, the domain of  $g$  = range of  $f$  and the range of  $g$  = domain of  $f$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$ . Further,  $g$  is also one-one and onto and inverse of  $g$  is  $f$ . Thus,  $g^{-1} = (f^{-1})^{-1} = f$ . We also have

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

and  $(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$

Since the domain of sine function is the set of all real numbers and range is the closed interval  $[-1, 1]$ . If we restrict its domain to  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , then it becomes one-one and onto with range  $[-1, 1]$ . Actually, sine function restricted to any of the intervals  $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$ ,  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ,  $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$  etc., is one-one and its range is  $[-1, 1]$ . We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by  $\sin^{-1}$  (arc sine function). Thus,  $\sin^{-1}$  is a function whose domain is  $[-1, 1]$  and range could be any of the intervals  $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$ ,  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  or  $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ , and so on. Corresponding to each such interval, we get a *branch* of the function  $\sin^{-1}$ . The branch with range  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  is called the *principal value branch*, whereas other intervals as range give different branches of  $\sin^{-1}$ . When we refer to the function  $\sin^{-1}$ , we take it as the function whose domain is  $[-1, 1]$  and range is  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ . We write  $\sin^{-1}: [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that  $\sin(\sin^{-1} x) = x$  if  $-1 \leq x \leq 1$  and  $\sin^{-1}(\sin x) = x$  if  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ . In other words, if  $y = \sin^{-1} x$ , then  $\sin y = x$ .

### Remarks

- We know from Chapter 1, that if  $y = f(x)$  is an invertible function, then  $x = f^{-1}(y)$ . Thus, the graph of  $\sin^{-1}$  function can be obtained from the graph of original function by interchanging  $x$  and  $y$  axes, i.e., if  $(a, b)$  is a point on the graph of sine function, then  $(b, a)$  becomes the corresponding point on the graph of inverse

of sine function. Thus, the graph of the function  $y = \sin^{-1} x$  can be obtained from the graph of  $y = \sin x$  by interchanging  $x$  and  $y$  axes. The graphs of  $y = \sin x$  and  $y = \sin^{-1} x$  are as given in Fig 2.1 (i), (ii), (iii). The dark portion of the graph of  $y = \sin^{-1} x$  represent the principal value branch.

- (ii) It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line  $y = x$ . This can be visualised by looking the graphs of  $y = \sin x$  and  $y = \sin^{-1} x$  as given in the same axes (Fig 2.1 (iii)).

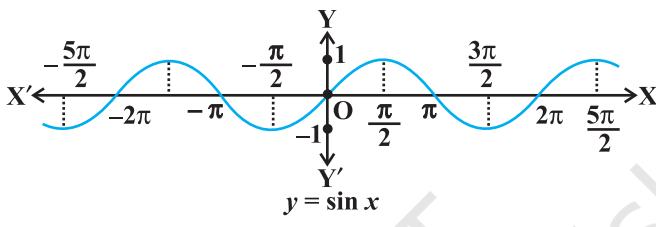


Fig 2.1 (i)

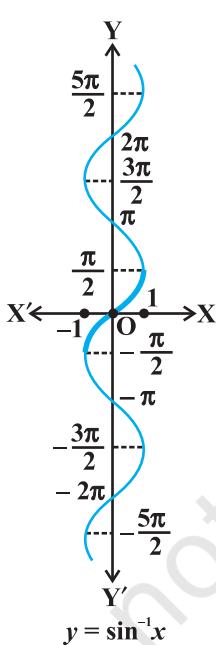


Fig 2.1 (ii)

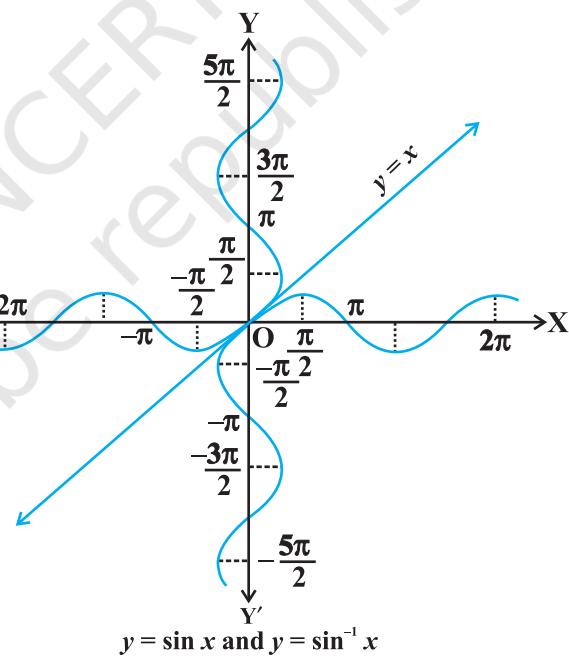


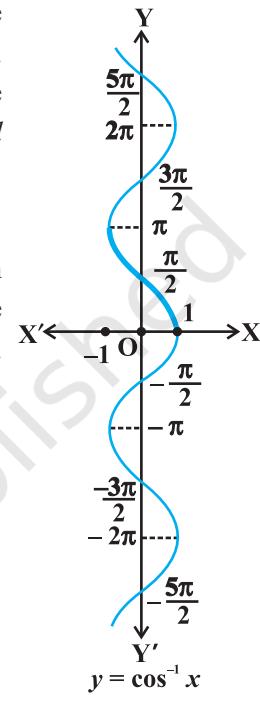
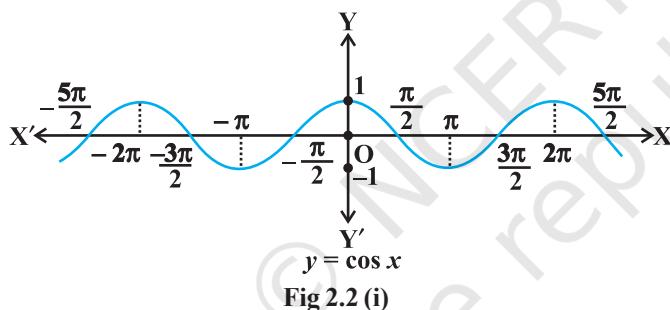
Fig 2.1 (iii)

Like sine function, the cosine function is a function whose domain is the set of all real numbers and range is the set  $[-1, 1]$ . If we restrict the domain of cosine function to  $[0, \pi]$ , then it becomes one-one and onto with range  $[-1, 1]$ . Actually, cosine function

restricted to any of the intervals  $[-\pi, 0]$ ,  $[0, \pi]$ ,  $[\pi, 2\pi]$  etc., is bijective with range as  $[-1, 1]$ . We can, therefore, define the inverse of cosine function in each of these intervals. We denote the inverse of the cosine function by  $\cos^{-1}$  (arc cosine function). Thus,  $\cos^{-1}$  is a function whose domain is  $[-1, 1]$  and range could be any of the intervals  $[-\pi, 0]$ ,  $[0, \pi]$ ,  $[\pi, 2\pi]$  etc. Corresponding to each such interval, we get a branch of the function  $\cos^{-1}$ . The branch with range  $[0, \pi]$  is called the *principal value branch* of the function  $\cos^{-1}$ . We write

$$\cos^{-1} : [-1, 1] \rightarrow [0, \pi].$$

The graph of the function given by  $y = \cos^{-1} x$  can be drawn in the same way as discussed about the graph of  $y = \sin^{-1} x$ . The graphs of  $y = \cos x$  and  $y = \cos^{-1} x$  are given in Fig 2.2 (i) and (ii).



Let us now discuss  $\operatorname{cosec}^{-1} x$  and  $\sec^{-1} x$  as follows:

Since,  $\operatorname{cosec} x = \frac{1}{\sin x}$ , the domain of the cosec function is the set  $\{x : x \in \mathbf{R} \text{ and } x \neq n\pi, n \in \mathbf{Z}\}$  and the range is the set  $\{y : y \in \mathbf{R}, y \geq 1 \text{ or } y \leq -1\}$  i.e., the set  $\mathbf{R} - (-1, 1)$ . It means that  $y = \operatorname{cosec} x$  assumes all real values except  $-1 < y < 1$  and is not defined for integral multiple of  $\pi$ . If we restrict the domain of cosec function to  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ , then it is one to one and onto with its range as the set  $\mathbf{R} - (-1, 1)$ . Actually,

cosec function restricted to any of the intervals  $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right] - \{-\pi\}$ ,  $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ ,  $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$  etc., is bijective and its range is the set of all real numbers  $\mathbf{R} - (-1, 1)$ .

Thus  $\text{cosec}^{-1}$  can be defined as a function whose domain is  $\mathbf{R} - (-1, 1)$  and range could be any of the intervals  $\left[ \frac{-3\pi}{2}, \frac{-\pi}{2} \right] - \{-\pi\}$ ,  $\left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] - \{0\}$ ,  $\left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] - \{\pi\}$  etc. The function corresponding to the range  $\left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] - \{0\}$  is called the *principal value branch* of  $\text{cosec}^{-1}$ . We thus have principal branch as

$$\text{cosec}^{-1} : \mathbf{R} - (-1, 1) \rightarrow \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] - \{0\}$$

The graphs of  $y = \text{cosec } x$  and  $y = \text{cosec}^{-1} x$  are given in Fig 2.3 (i), (ii).

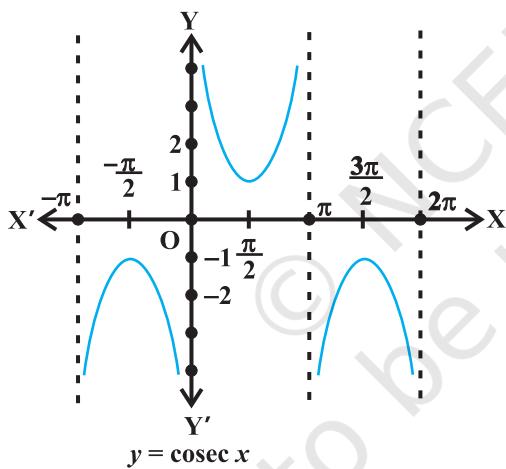


Fig 2.3 (i)

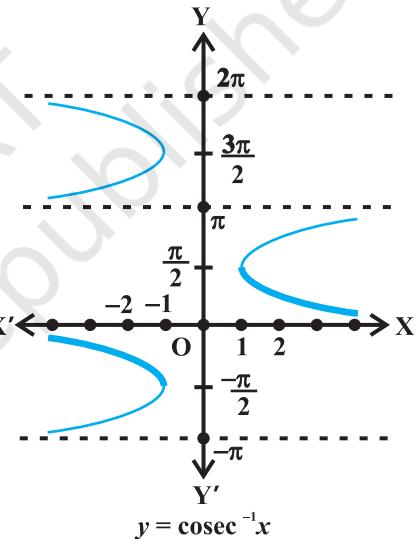


Fig 2.3 (ii)

Also, since  $\sec x = \frac{1}{\cos x}$ , the domain of  $y = \sec x$  is the set  $\mathbf{R} - \{x : x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}\}$  and range is the set  $\mathbf{R} - (-1, 1)$ . It means that  $\sec$  (secant function) assumes all real values except  $-1 < y < 1$  and is not defined for odd multiples of  $\frac{\pi}{2}$ . If we restrict the domain of secant function to  $[0, \pi] - \{ \frac{\pi}{2} \}$ , then it is one-one and onto with

its range as the set  $\mathbf{R} - (-1, 1)$ . Actually, secant function restricted to any of the intervals  $[-\pi, 0] - \left\{-\frac{\pi}{2}\right\}$ ,  $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ ,  $[\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}$  etc., is bijective and its range is  $\mathbf{R} - \{-1, 1\}$ . Thus  $\sec^{-1}$  can be defined as a function whose domain is  $\mathbf{R} - (-1, 1)$  and range could be any of the intervals  $[-\pi, 0] - \left\{-\frac{\pi}{2}\right\}$ ,  $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ ,  $[\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}$  etc. Corresponding to each of these intervals, we get different branches of the function  $\sec^{-1}$ . The branch with range  $[0, \pi] - \left\{\frac{\pi}{2}\right\}$  is called the *principal value branch* of the function  $\sec^{-1}$ . We thus have

$$\sec^{-1} : \mathbf{R} - (-1, 1) \rightarrow [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

The graphs of the functions  $y = \sec x$  and  $y = \sec^{-1} x$  are given in Fig 2.4 (i), (ii).

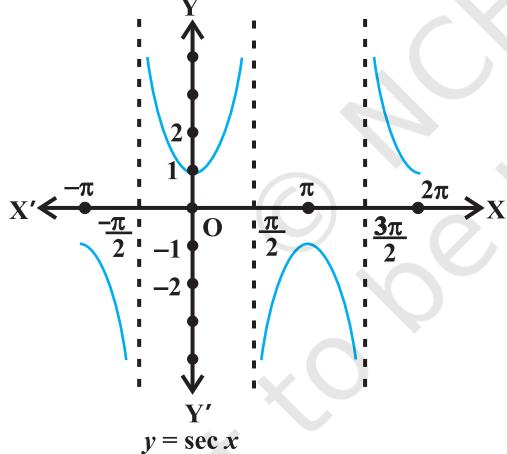


Fig 2.4 (i)

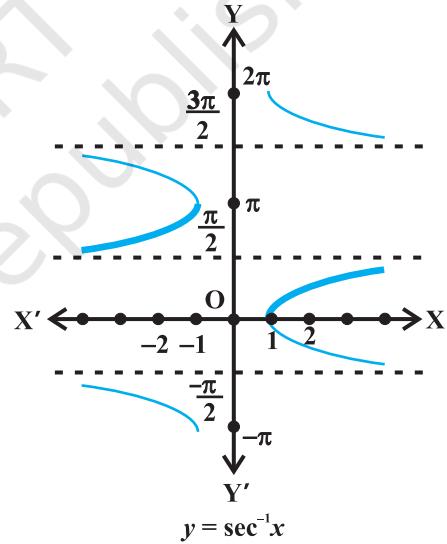


Fig 2.4 (ii)

Finally, we now discuss  $\tan^{-1}$  and  $\cot^{-1}$

We know that the domain of the tan function (tangent function) is the set

$\{x : x \in \mathbf{R} \text{ and } x \neq (2n+1) \frac{\pi}{2}, n \in \mathbf{Z}\}$  and the range is  $\mathbf{R}$ . It means that tan function

is not defined for odd multiples of  $\frac{\pi}{2}$ . If we restrict the domain of tangent function to

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , then it is one-one and onto with its range as  $\mathbf{R}$ . Actually, tangent function restricted to any of the intervals  $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ ,  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$  etc., is bijective and its range is  $\mathbf{R}$ . Thus  $\tan^{-1}$  can be defined as a function whose domain is  $\mathbf{R}$  and range could be any of the intervals  $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ ,  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$  and so on. These intervals give different branches of the function  $\tan^{-1}$ . The branch with range  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  is called the *principal value branch* of the function  $\tan^{-1}$ .

We thus have

$$\tan^{-1} : \mathbf{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the function  $y = \tan x$  and  $y = \tan^{-1} x$  are given in Fig 2.5 (i), (ii).

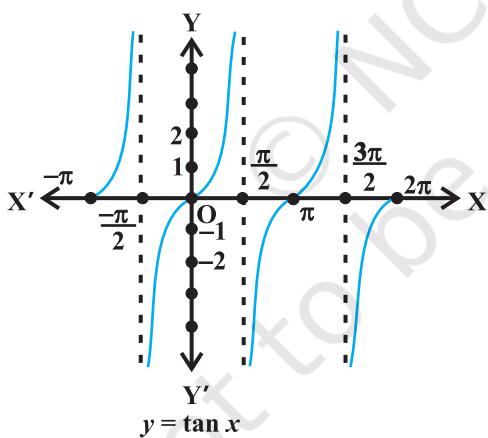


Fig 2.5 (i)

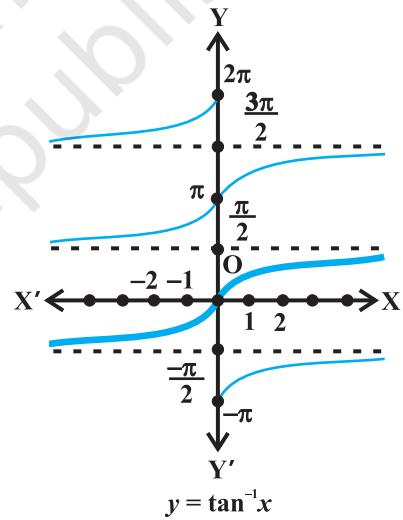


Fig 2.5 (ii)

We know that domain of the cot function (cotangent function) is the set  $\{x : x \in \mathbf{R} \text{ and } x \neq n\pi, n \in \mathbf{Z}\}$  and range is  $\mathbf{R}$ . It means that cotangent function is not defined for integral multiples of  $\pi$ . If we restrict the domain of cotangent function to  $(0, \pi)$ , then it is bijective with and its range as  $\mathbf{R}$ . In fact, cotangent function restricted to any of the intervals  $(-\pi, 0)$ ,  $(0, \pi)$ ,  $(\pi, 2\pi)$  etc., is bijective and its range is  $\mathbf{R}$ . Thus  $\cot^{-1}$  can be defined as a function whose domain is the  $\mathbf{R}$  and range as any of the

intervals  $(-\pi, 0)$ ,  $(0, \pi)$ ,  $(\pi, 2\pi)$  etc. These intervals give different branches of the function  $\cot^{-1}$ . The function with range  $(0, \pi)$  is called the *principal value branch* of the function  $\cot^{-1}$ . We thus have

$$\cot^{-1} : \mathbf{R} \rightarrow (0, \pi)$$

The graphs of  $y = \cot x$  and  $y = \cot^{-1} x$  are given in Fig 2.6 (i), (ii).

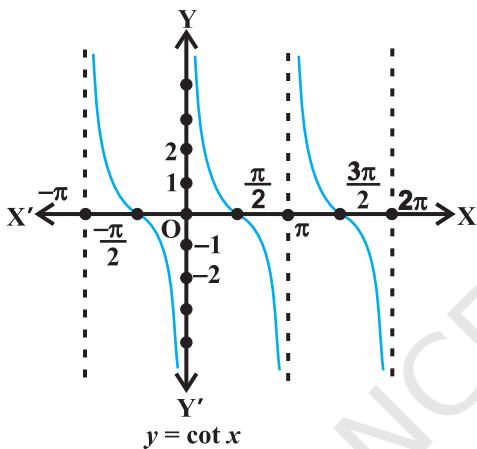


Fig 2.6 (i)

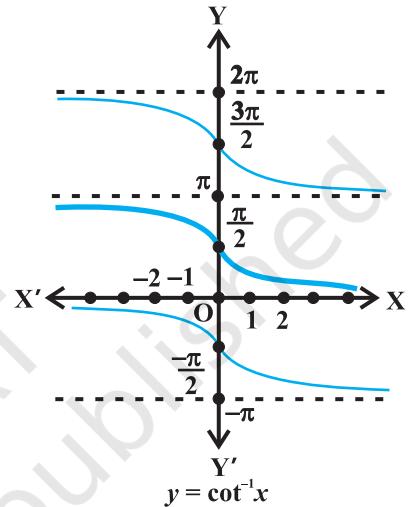


Fig 2.6 (ii)

The following table gives the inverse trigonometric function (principal value branches) along with their domains and ranges.

|             |                        |               |  |
|-------------|------------------------|---------------|--|
| $\sin^{-1}$ | $[-1, 1]$              | $\rightarrow$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$         |
| $\cos^{-1}$ | $[-1, 1]$              | $\rightarrow$ | $[0, \pi]$   |
| $\csc^{-1}$ | $\mathbf{R} - (-1, 1)$ | $\rightarrow$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ |
| $\sec^{-1}$ | $\mathbf{R} - (-1, 1)$ | $\rightarrow$ | $[0, \pi] - \{\frac{\pi}{2}\}$                       |
| $\tan^{-1}$ | $\mathbf{R}$           | $\rightarrow$ | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$         |
| $\cot^{-1}$ | $\mathbf{R}$           | $\rightarrow$ | $(0, \pi)$   |

 **Note**

1.  $\sin^{-1}x$  should not be confused with  $(\sin x)^{-1}$ . In fact  $(\sin x)^{-1} = \frac{1}{\sin x}$  and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

We now consider some examples:

**Example 1** Find the principal value of  $\sin^{-1} \left( \frac{1}{\sqrt{2}} \right)$ .

**Solution** Let  $\sin^{-1} \left( \frac{1}{\sqrt{2}} \right) = y$ . Then,  $\sin y = \frac{1}{\sqrt{2}}$ .

We know that the range of the principal value branch of  $\sin^{-1}$  is  $\left( \frac{-\pi}{2}, \frac{\pi}{2} \right)$  and  $\sin \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}$ . Therefore, principal value of  $\sin^{-1} \left( \frac{1}{\sqrt{2}} \right)$  is  $\frac{\pi}{4}$ .

**Example 2** Find the principal value of  $\cot^{-1} \left( \frac{-1}{\sqrt{3}} \right)$

**Solution** Let  $\cot^{-1} \left( \frac{-1}{\sqrt{3}} \right) = y$ . Then,

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot \left( \frac{\pi}{3} \right) = \cot \left( \pi - \frac{\pi}{3} \right) = \cot \left( \frac{2\pi}{3} \right)$$

We know that the range of principal value branch of  $\cot^{-1}$  is  $(0, \pi)$  and  $\cot \left( \frac{2\pi}{3} \right) = \frac{-1}{\sqrt{3}}$ . Hence, principal value of  $\cot^{-1} \left( \frac{-1}{\sqrt{3}} \right)$  is  $\frac{2\pi}{3}$ .

### EXERCISE 2.1

Find the principal values of the following:

1.  $\sin^{-1} \left( -\frac{1}{2} \right)$       2.  $\cos^{-1} \left( \frac{\sqrt{3}}{2} \right)$       3.  $\operatorname{cosec}^{-1} (2)$

4.  $\tan^{-1} (-\sqrt{3})$       5.  $\cos^{-1} \left( -\frac{1}{2} \right)$       6.  $\tan^{-1} (-1)$

7.  $\sec^{-1} \left( \frac{2}{\sqrt{3}} \right)$

8.  $\cot^{-1} (\sqrt{3})$

9.  $\cos^{-1} \left( -\frac{1}{\sqrt{2}} \right)$

10.  $\operatorname{cosec}^{-1} (-\sqrt{2})$

Find the values of the following:

11.  $\tan^{-1}(1) + \cos^{-1} -\frac{1}{2} + \sin^{-1} -\frac{1}{2}$

12.  $\cos^{-1} \frac{1}{2} + 2 \sin^{-1} \frac{1}{2}$

13. If  $\sin^{-1} x = y$ , then

(A)  $0 \leq y \leq \pi$

(B)  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(C)  $0 < y < \pi$

(D)  $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14.  $\tan^{-1} \sqrt{3} - \sec^{-1} (-2)$  is equal to

(A)  $\pi$

(B)  $-\frac{\pi}{3}$

(C)  $\frac{\pi}{3}$

(D)  $\frac{2\pi}{3}$

### 2.3 Properties of Inverse Trigonometric Functions

In this section, we shall prove some important properties of inverse trigonometric functions. It may be mentioned here that these results are valid within the principal value branches of the corresponding inverse trigonometric functions and wherever they are defined. Some results may not be valid for all values of the domains of inverse trigonometric functions. In fact, they will be valid only for some values of  $x$  for which inverse trigonometric functions are defined. We will not go into the details of these values of  $x$  in the domain as this discussion goes beyond the scope of this textbook.

Let us recall that if  $y = \sin^{-1} x$ , then  $x = \sin y$  and if  $x = \sin y$ , then  $y = \sin^{-1} x$ . This is equivalent to

$$\sin(\sin^{-1} x) = x, x \in [-1, 1] \text{ and } \sin^{-1}(\sin x) = x, x \in \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right]$$

For suitable values of domain similar results follow for remaining trigonometric functions.

We now consider some examples.

**Example 3** Show that

$$(i) \sin^{-1} (2x\sqrt{1-x^2}) = 2 \sin^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$$

$$(ii) \sin^{-1} (2x\sqrt{1-x^2}) = 2 \cos^{-1} x, \frac{1}{\sqrt{2}} \leq x \leq 1$$

**Solution**

(i) Let  $x = \sin \theta$ . Then  $\sin^{-1} x = \theta$ . We have

$$\begin{aligned} \sin^{-1} (2x\sqrt{1-x^2}) &= \sin^{-1} (2\sin \theta \sqrt{1-\sin^2 \theta}) \\ &= \sin^{-1} (2\sin \theta \cos \theta) = \sin^{-1} (\sin 2\theta) = 2\theta \\ &= 2 \sin^{-1} x \end{aligned}$$

(ii) Take  $x = \cos \theta$ , then proceeding as above, we get,  $\sin^{-1} (2x\sqrt{1-x^2}) = 2 \cos^{-1} x$

**Example 4** Express  $\tan^{-1} \frac{\cos x}{1-\sin x}$ ,  $-\frac{3\pi}{2} < x < \frac{\pi}{2}$  in the simplest form.

**Solution** We write

$$\tan^{-1} \left( \frac{\cos x}{1-\sin x} \right) = \tan^{-1} \left[ \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}} \right]$$

$$= \tan^{-1} \left[ \frac{\left( \cos \frac{x}{2} + \sin \frac{x}{2} \right) \left( \cos \frac{x}{2} - \sin \frac{x}{2} \right)}{\left( \cos \frac{x}{2} - \sin \frac{x}{2} \right)^2} \right]$$

$$= \tan^{-1} \left[ \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} \right] = \tan^{-1} \left[ \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right]$$

$$= \tan^{-1} \left[ \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \right] = \frac{\pi}{4} + \frac{x}{2}$$

**Example 5** Write  $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$ ,  $x > 1$  in the simplest form.

**Solution** Let  $x = \sec \theta$ , then  $\sqrt{x^2-1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$

Therefore,  $\cot^{-1}\frac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot \theta) = \theta = \sec^{-1} x$ , which is the simplest form.

### EXERCISE 2.2

Prove the following:

1.  $3\sin^{-1} x = \sin^{-1} (3x - 4x^3)$ ,  $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

2.  $3\cos^{-1} x = \cos^{-1} (4x^3 - 3x)$ ,  $x \in \left[\frac{1}{2}, 1\right]$

Write the following functions in the simplest form:

3.  $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ ,  $x \neq 0$       4.  $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$ ,  $0 < x < \pi$

5.  $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ ,  $\frac{-\pi}{4} < x < \frac{3\pi}{4}$

6.  $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$ ,  $|x| < a$

7.  $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$ ,  $a > 0$ ;  $\frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Find the values of each of the following:

8.  $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

9.  $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$ ,  $|x| < 1$ ,  $y > 0$  and  $xy < 1$

Find the values of each of the expressions in Exercises 16 to 18.

10.  $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

11.  $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

12.  $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

13.  $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$  is equal to

(A)  $\frac{7\pi}{6}$       (B)  $\frac{5\pi}{6}$       (C)  $\frac{\pi}{3}$       (D)  $\frac{\pi}{6}$

14.  $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$  is equal to

(A)  $\frac{1}{2}$       (B)  $\frac{1}{3}$       (C)  $\frac{1}{4}$       (D) 1

15.  $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$  is equal to

(A)  $\pi$       (B)  $-\frac{\pi}{2}$       (C) 0      (D)  $2\sqrt{3}$

### Miscellaneous Examples

**Example 6** Find the value of  $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

**Solution** We know that  $\sin^{-1}(\sin x) = x$ . Therefore,  $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{3\pi}{5}$

But  $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , which is the principal branch of  $\sin^{-1} x$

However  $\sin\left(\frac{3\pi}{5}\right) = \sin\left(\pi - \frac{3\pi}{5}\right) = \sin\frac{2\pi}{5}$  and  $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore  $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left(\sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$

### Miscellaneous Exercise on Chapter 2

Find the value of the following:

1.  $\cos^{-1} \left( \cos \frac{13\pi}{6} \right)$

2.  $\tan^{-1} \left( \tan \frac{7\pi}{6} \right)$

Prove that

3.  $2\sin^{-1} \frac{3}{5} = \tan^{-1} \frac{24}{7}$

4.  $\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$

5.  $\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$

6.  $\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$

7.  $\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$

Prove that

8.  $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \frac{1-x}{1+x}, x \in [0, 1]$

9.  $\cot^{-1} \left( \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, x \in \left( 0, \frac{\pi}{4} \right)$

10.  $\tan^{-1} \left( \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$  [Hint: Put  $x = \cos 2\theta$ ]

Solve the following equations:

11.  $2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$  12.  $\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x, (x > 0)$

13.  $\sin(\tan^{-1} x), |x| < 1$  is equal to

(A)  $\frac{x}{\sqrt{1-x^2}}$  (B)  $\frac{1}{\sqrt{1-x^2}}$  (C)  $\frac{1}{\sqrt{1+x^2}}$  (D)  $\frac{x}{\sqrt{1+x^2}}$

14.  $\sin^{-1}(1-x) - 2 \sin^{-1} x = \frac{\pi}{2}$ , then  $x$  is equal to

(A) 0,  $\frac{1}{2}$  (B) 1,  $\frac{1}{2}$  (C) 0 (D)  $\frac{1}{2}$

### Summary

- ◆ The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

| Functions                         | Domain                 | Range<br>(Principal Value Branches)                    |
|-----------------------------------|------------------------|--|
| $y = \sin^{-1} x$                 | $[-1, 1]$              | $\left[ \frac{-\pi}{2}, \frac{\pi}{2} \right]$         |
| $y = \cos^{-1} x$                 | $[-1, 1]$              | $[0, \pi]$   |
| $y = \operatorname{cosec}^{-1} x$ | $\mathbf{R} - (-1, 1)$ | $\left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] - \{0\}$ |
| $y = \sec^{-1} x$                 | $\mathbf{R} - (-1, 1)$ | $[0, \pi] - \left\{ \frac{\pi}{2} \right\}$            |
| $y = \tan^{-1} x$                 | $\mathbf{R}$           | $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$         |
| $y = \cot^{-1} x$                 | $\mathbf{R}$           | $(0, \pi)$   |

- ◆  $\sin^{-1} x$  should not be confused with  $(\sin x)^{-1}$ . In fact  $(\sin x)^{-1} = \frac{1}{\sin x}$  and similarly for other trigonometric functions.
- ◆ The value of an inverse trigonometric functions which lies in its principal value branch is called the *principal value* of that inverse trigonometric functions.

For suitable values of domain, we have

- |   |   |
|---|---|
| <ul style="list-style-type: none"> <li>◆ <math>y = \sin^{-1} x \Rightarrow x = \sin y</math></li> <li>◆ <math>\sin(\sin^{-1} x) = x</math></li> </ul> | <ul style="list-style-type: none"> <li>◆ <math>x = \sin y \Rightarrow y = \sin^{-1} x</math></li> <li>◆ <math>\sin^{-1}(\sin x) = x</math></li> </ul> |
|---|---|

### Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhata (476 A.D.), Brahmagupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) got important results of trigonometry. All this knowledge went from India to Arabia and then from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents one of the main contribution of the *siddhantas* (Sanskrit astronomical works) to mathematics.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine functions for angles more than  $90^\circ$ . A sixteenth century Malayalam work *Yuktibhasa* contains a proof for the expansion of  $\sin(A + B)$ . Exact expression for sines or cosines of  $18^\circ, 36^\circ, 54^\circ, 72^\circ$ , etc., were given by Bhaskara II.

The symbols  $\sin^{-1} x, \cos^{-1} x$ , etc., for  $\arcsin x, \arccos x$ , etc., were suggested by the astronomer Sir John F.W. Hersehel (1813). The name of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{H}{S} = \frac{h}{s} = \tan(\text{sun's altitude})$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.





# MATRICES

❖ *The essence of Mathematics lies in its freedom. — CANTOR* ❖

## 3.1 Introduction

The knowledge of matrices is necessary in various branches of mathematics. Matrices are one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evolution of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Matrices are also used in cryptography. This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

In this chapter, we shall find it interesting to become acquainted with the fundamentals of matrix and matrix algebra.

## 3.2 Matrix

Suppose we wish to express the information that Radha has 15 notebooks. We may express it as [15] with the understanding that the number inside [ ] is the number of notebooks that Radha has. Now, if we have to express that Radha has 15 notebooks and 6 pens. We may express it as [15 6] with the understanding that first number inside [ ] is the number of notebooks while the other one is the number of pens possessed by Radha. Let us now suppose that we wish to express the information of possession

of notebooks and pens by Radha and her two friends Fauzia and Simran which is as follows:

Radha has 15 notebooks and 6 pens,  
Fauzia has 10 notebooks and 2 pens,  
Simran has 13 notebooks and 5 pens.

Now this could be arranged in the tabular form as follows:

|        | Notebooks | Pens |
|--------|-----------|------|
| Radha  | 15        | 6    |
| Fauzia | 10        | 2    |
| Simran | 13        | 5    |

and this can be expressed as

|    |   |              |
|----|---|--------------|
| 15 | 6 | ← First row  |
| 10 | 2 | ← Second row |
| 13 | 5 | ← Third row  |

or

|           | <b>Radha</b> | <b>Fauzia</b> | <b>Simran</b> |
|-----------|--------------|---------------|---------------|
| Notebooks | 15           | 10            | 13            |
| Pens      | 6            | 2             | 5             |

which can be expressed as:

$$\begin{bmatrix}
 15 & 10 & 13 \\
 6 & 2 & 5
 \end{bmatrix}
 \begin{array}{l}
 \leftarrow \text{First row} \\
 \leftarrow \text{Second row}
 \end{array}
 \begin{array}{l}
 \uparrow \\
 \uparrow \\
 \uparrow
 \end{array}
 \begin{array}{l}
 \text{First} \\
 \text{Column}
 \end{array}
 \begin{array}{l}
 \text{Second} \\
 \text{Column}
 \end{array}
 \begin{array}{l}
 \text{Third} \\
 \text{Column}
 \end{array}$$

In the first arrangement the entries in the first column represent the number of note books possessed by Radha, Fauzia and Simran, respectively and the entries in the second column represent the number of pens possessed by Radha, Fauzia and Simran,

respectively. Similarly, in the second arrangement, the entries in the first row represent the number of notebooks possessed by Radha, Fauzia and Simran, respectively. The entries in the second row represent the number of pens possessed by Radha, Fauzia and Simran, respectively. An arrangement or display of the above kind is called a *matrix*. Formally, we define matrix as:

**Definition 1** A *matrix* is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

We denote matrices by capital letters. The following are some examples of matrices:

$$A = \begin{bmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & 6 \end{bmatrix}, B = \begin{bmatrix} 2+i & 3 & -\frac{1}{2} \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & \frac{5}{7} \end{bmatrix}, C = \begin{bmatrix} 1+x & x^3 & 3 \\ \cos x & \sin x + 2 & \tan x \end{bmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, **rows** of the matrix and the vertical lines of elements are said to constitute, **columns** of the matrix. Thus A has 3 rows and 2 columns, B has 3 rows and 3 columns while C has 2 rows and 3 columns.

### 3.2.1 Order of a matrix

A matrix having  $m$  rows and  $n$  columns is called a matrix of *order*  $m \times n$  or simply  $m \times n$  matrix (read as an  $m$  by  $n$  matrix). So referring to the above examples of matrices, we have A as  $3 \times 2$  matrix, B as  $3 \times 3$  matrix and C as  $2 \times 3$  matrix. We observe that A has  $3 \times 2 = 6$  elements, B and C have 9 and 6 elements, respectively.

In general, an  $m \times n$  matrix has the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$\text{or } A = [a_{ij}]_{m \times n}, 1 \leq i \leq m, 1 \leq j \leq n \quad i, j \in \mathbb{N}$$

Thus the  $i^{\text{th}}$  row consists of the elements  $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$ , while the  $j^{\text{th}}$  column consists of the elements  $a_{1j}, a_{2j}, a_{3j}, \dots, a_{mj}$ .

In general  $a_{ij}$  is an element lying in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. We can also call it as the  $(i, j)^{\text{th}}$  element of A. The number of elements in an  $m \times n$  matrix will be equal to  $mn$ .



**Note** In this chapter

1. We shall follow the notation, namely  $A = [a_{ij}]_{m \times n}$  to indicate that  $A$  is a matrix of order  $m \times n$ .
2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

We can also represent any point  $(x, y)$  in a plane by a matrix (column or row) as

$\begin{bmatrix} x \\ y \end{bmatrix}$  (or  $[x, y]$ ). For example point  $P(0, 1)$  as a matrix representation may be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0 \ 1].$$

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (-1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{bmatrix} A & B & C & D \\ 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} \text{ or } Y = \begin{bmatrix} A & 1 & 0 \\ B & 3 & 2 \\ C & 1 & 3 \\ D & -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used as representation of vertices of geometrical figures in a plane.

Now, let us consider some examples.

**Example 1** Consider the following information regarding the number of men and women workers in three factories I, II and III

|     | Men workers | Women workers |
|-----|-------------|---------------|
| I   | 30          | 25            |
| II  | 25          | 31            |
| III | 27          | 26            |

Represent the above information in the form of a  $3 \times 2$  matrix. What does the entry in the third row and second column represent?

**Solution** The information is represented in the form of a  $3 \times 2$  matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

**Example 2** If a matrix has 8 elements, what are the possible orders it can have?

**Solution** We know that if a matrix is of order  $m \times n$ , it has  $mn$  elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are  $(1, 8), (8, 1), (4, 2), (2, 4)$

Hence, possible orders are  $1 \times 8, 8 \times 1, 4 \times 2, 2 \times 4$

**Example 3** Construct a  $3 \times 2$  matrix whose elements are given by  $a_{ij} = \frac{1}{2}|i - 3j|$ .

**Solution** In general a  $3 \times 2$  matrix is given by  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ .

Now  $a_{ij} = \frac{1}{2}|i - 3j|$ ,  $i = 1, 2, 3$  and  $j = 1, 2$ .

$$\text{Therefore } a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1 \quad a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$$

$$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2} \quad a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$$

$$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0 \quad a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$$

Hence the required matrix is given by  $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$ .

### 3.3 Types of Matrices

In this section, we shall discuss different types of matrices.

#### (i) Column matrix

A matrix is said to be a *column matrix* if it has only one column.

For example,  $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$  is a column matrix of order  $4 \times 1$ .

In general,  $A = [a_{ij}]_{m \times 1}$  is a column matrix of order  $m \times 1$ .

#### (ii) Row matrix

A matrix is said to be a *row matrix* if it has only one row.

For example,  $B = \begin{bmatrix} -\frac{1}{2} & \sqrt{5} & 2 & 3 \end{bmatrix}_{1 \times 4}$  is a row matrix.

In general,  $B = [b_{ij}]_{1 \times n}$  is a row matrix of order  $1 \times n$ .

#### (iii) Square matrix

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus an  $m \times n$  matrix is said to be a square matrix if  $m = n$  and is known as a square matrix of order ‘ $n$ ’.

For example  $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$  is a square matrix of order 3.

In general,  $A = [a_{ij}]_{m \times m}$  is a square matrix of order  $m$ .

 **Note** If  $A = [a_{ij}]$  is a square matrix of order  $n$ , then elements (entries)  $a_{11}, a_{22}, \dots, a_{nn}$

are said to constitute the *diagonal*, of the matrix  $A$ . Thus, if  $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$ .

Then the elements of the diagonal of  $A$  are 1, 4, 6.

(iv) **Diagonal matrix**

A square matrix  $B = [b_{ij}]_{m \times m}$  is said to be a *diagonal matrix* if all its non-diagonal elements are zero, that is a matrix  $B = [b_{ij}]_{m \times m}$  is said to be a diagonal matrix if  $b_{ij} = 0$ , when  $i \neq j$ .

For example,  $A = [4]$ ,  $B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ ,  $C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ , are diagonal matrices of order 1, 2, 3, respectively.

(v) **Scalar matrix**

A diagonal matrix is said to be a *scalar matrix* if its diagonal elements are equal, that is, a square matrix  $B = [b_{ij}]_{n \times n}$  is said to be a scalar matrix if

$$b_{ij} = 0, \quad \text{when } i \neq j$$

$$b_{ij} = k, \quad \text{when } i = j, \text{ for some constant } k.$$

For example

$$A = [3], \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

are scalar matrices of order 1, 2 and 3, respectively.

(vi) **Identity matrix**

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an *identity matrix*. In other words, the square matrix  $A = [a_{ij}]_{n \times n}$  is an

$$\text{identity matrix, if } a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We denote the identity matrix of order  $n$  by  $I_n$ . When order is clear from the context, we simply write it as  $I$ .

For example  $[1]$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  are identity matrices of order 1, 2 and 3,

respectively.

Observe that a scalar matrix is an identity matrix when  $k = 1$ . But every identity matrix is clearly a scalar matrix.

### (vii) Zero matrix

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero.

For example,  $[0]$ ,  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $[0, 0]$  are all zero matrices. We denote zero matrix by  $O$ . Its order will be clear from the context.

#### 3.3.1 Equality of matrices

**Definition 2** Two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are said to be equal if

- (i) they are of the same order
- (ii) each element of  $A$  is equal to the corresponding element of  $B$ , that is  $a_{ij} = b_{ij}$  for all  $i$  and  $j$ .

For example,  $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$  are equal matrices but  $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$  are not equal matrices. Symbolically, if two matrices  $A$  and  $B$  are equal, we write  $A = B$ .

If  $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$ , then  $x = -1.5$ ,  $y = 0$ ,  $z = 2$ ,  $a = \sqrt{6}$ ,  $b = 3$ ,  $c = 2$

**Example 4** If  $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Find the values of  $a$ ,  $b$ ,  $c$ ,  $x$ ,  $y$  and  $z$ .

**Solution** As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$$\begin{aligned} x+3 &= 0, & z+4 &= 6, & 2y-7 &= 3y-2 \\ a-1 &= -3, & 0 &= 2c+2, & b-3 &= 2b+4, \end{aligned}$$

Simplifying, we get

$$a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$$

**Example 5** Find the values of  $a$ ,  $b$ ,  $c$ , and  $d$  from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

**Solution** By equality of two matrices, equating the corresponding elements, we get

$$2a + b = 4 \quad 5c - d = 11$$

$$a - 2b = -3 \quad 4c + 3d = 24$$

Solving these equations, we get

$$a = 1, b = 2, c = 3 \text{ and } d = 4$$

### EXERCISE 3.1

1. In the matrix  $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$ , write:
  - The order of the matrix,
  - The number of elements,
  - Write the elements  $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$ .
2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?
3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?
4. Construct a  $2 \times 2$  matrix,  $A = [a_{ij}]$ , whose elements are given by:
  - $a_{ij} = \frac{(i+j)^2}{2}$
  - $a_{ij} = \frac{i}{j}$
  - $a_{ij} = \frac{(i+2j)^2}{2}$
5. Construct a  $3 \times 4$  matrix, whose elements are given by:
  - $a_{ij} = \frac{1}{2} |-3i + j|$
  - $a_{ij} = 2i - j$
6. Find the values of  $x, y$  and  $z$  from the following equations:
 
$$(i) \begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix} \quad (ii) \begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix} \quad (iii) \begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$
7. Find the value of  $a, b, c$  and  $d$  from the equation:
 
$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

8.  $A = [a_{ij}]_{m \times n}$  is a square matrix, if  
 (A)  $m < n$       (B)  $m > n$       (C)  $m = n$       (D) None of these
9. Which of the given values of  $x$  and  $y$  make the following pair of matrices equal  

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$
  
 (A)  $x = \frac{-1}{3}, y = 7$       (B) Not possible to find  
 (C)  $y = 7, x = \frac{-2}{3}$       (D)  $x = \frac{-1}{3}, y = \frac{-2}{3}$
10. The number of all possible matrices of order  $3 \times 3$  with each entry 0 or 1 is:  
 (A) 27      (B) 18      (C) 81      (D) 512

### 3.4 Operations on Matrices

In this section, we shall introduce certain operations on matrices, namely, addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

#### 3.4.1 Addition of matrices

Suppose Fatima has two factories at places A and B. Each factory produces sport shoes for boys and girls in three different price categories labelled 1, 2 and 3. The quantities produced by each factory are represented as matrices given below:

| Factory at A |   | Factory at B |   |
|--------------|---|--------------|---|
| Boys         | Girls                                   | Boys         | Girls                                   |
| 1            | $\begin{bmatrix} 80 & 60 \end{bmatrix}$ | 1            | $\begin{bmatrix} 90 & 50 \end{bmatrix}$ |
| 2            | $\begin{bmatrix} 75 & 65 \end{bmatrix}$ | 2            | $\begin{bmatrix} 70 & 55 \end{bmatrix}$ |
| 3            | $\begin{bmatrix} 90 & 85 \end{bmatrix}$ | 3            | $\begin{bmatrix} 75 & 75 \end{bmatrix}$ |

Suppose Fatima wants to know the total production of sport shoes in each price category. Then the total production

In category 1 : for boys  $(80 + 90)$ , for girls  $(60 + 50)$

In category 2 : for boys  $(75 + 70)$ , for girls  $(65 + 55)$

In category 3 : for boys  $(90 + 75)$ , for girls  $(85 + 75)$

This can be represented in the matrix form as  $\begin{bmatrix} 80+90 & 60+50 \\ 75+70 & 65+55 \\ 90+75 & 85+75 \end{bmatrix}$ .

This new matrix is the **sum** of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

Thus, if  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$  is a  $2 \times 3$  matrix and  $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$  is another

$2 \times 3$  matrix. Then, we define  $A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$ .

In general, if  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are two matrices of the same order, say  $m \times n$ . Then, the sum of the two matrices A and B is *defined* as a matrix  $C = [c_{ij}]_{m \times n}$ , where  $c_{ij} = a_{ij} + b_{ij}$ , for all possible values of  $i$  and  $j$ .

**Example 6** Given  $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$ , find  $A + B$

Since A, B are of the same order  $2 \times 3$ . Therefore, addition of A and B is defined and is given by

$$A + B = \begin{bmatrix} 2 + \sqrt{3} & 1 + \sqrt{5} & 1 - 1 \\ 2 - 2 & 3 + 3 & 0 + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 + \sqrt{3} & 1 + \sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$$

### Note

1. We emphasise that if A and B are not of the same order, then  $A + B$  is not defined. For example if  $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ , then  $A + B$  is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

### 3.4.2 Multiplication of a matrix by a scalar

Now suppose that Fatima has doubled the production at a factory A in all categories (refer to 3.4.1).

Previously quantities (in standard units) produced by factory A were

$$\begin{array}{cc} \text{Boys} & \text{Girls} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array}$$

Revised quantities produced by factory A are as given below:

$$\begin{array}{cc} \text{Boys} & \text{Girls} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 2 \times 80 & 2 \times 60 \\ 2 \times 75 & 2 \times 65 \\ 2 \times 90 & 2 \times 85 \end{bmatrix} \end{array}$$

This can be represented in the matrix form as  $\begin{bmatrix} 160 & 120 \\ 150 & 130 \\ 180 & 170 \end{bmatrix}$ . We observe that

the new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, we may define *multiplication of a matrix* by a scalar as follows: if  $A = [a_{ij}]_{m \times n}$  is a matrix and  $k$  is a scalar, then  $kA$  is another matrix which is obtained by multiplying each element of  $A$  by the scalar  $k$ .

In other words,  $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$ , that is,  $(i, j)^{\text{th}}$  element of  $kA$  is  $ka_{ij}$  for all possible values of  $i$  and  $j$ .

For example, if  $A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$ , then

$$3A = 3 \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

**Negative of a matrix** The negative of a matrix is denoted by  $-A$ . We define  $-A = (-1)A$ .

For example, let

$$A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}, \text{ then } -A \text{ is given by}$$

$$-A = (-1)A = (-1) \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

**Difference of matrices** If  $A = [a_{ij}]$ ,  $B = [b_{ij}]$  are two matrices of the same order, say  $m \times n$ , then difference  $A - B$  is defined as a matrix  $D = [d_{ij}]$ , where  $d_{ij} = a_{ij} - b_{ij}$ , for all value of  $i$  and  $j$ . In other words,  $D = A - B = A + (-1)B$ , that is sum of the matrix  $A$  and the matrix  $-B$ .

**Example 7** If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ , then find  $2A - B$ .

**Solution** We have

$$\begin{aligned} 2A - B &= 2 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix} \end{aligned}$$

### 3.4.3 Properties of matrix addition

The addition of matrices satisfy the following properties:

- (i) **Commutative Law** If  $A = [a_{ij}]$ ,  $B = [b_{ij}]$  are matrices of the same order, say  $m \times n$ , then  $A + B = B + A$ .

$$\begin{aligned} \text{Now } A + B &= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \text{ (addition of numbers is commutative)} \\ &= ([b_{ij}] + [a_{ij}]) = B + A \end{aligned}$$

- (ii) **Associative Law** For any three matrices  $A = [a_{ij}]$ ,  $B = [b_{ij}]$ ,  $C = [c_{ij}]$  of the same order, say  $m \times n$ ,  $(A + B) + C = A + (B + C)$ .

$$\begin{aligned} \text{Now } (A + B) + C &= ([a_{ij}] + [b_{ij}]) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \quad (\text{Why?}) \\ &= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C) \end{aligned}$$

- (iii) **Existence of additive identity** Let  $A = [a_{ij}]$  be an  $m \times n$  matrix and  $O$  be an  $m \times n$  zero matrix, then  $A + O = O + A = A$ . In other words,  $O$  is the additive identity for matrix addition.
- (iv) **The existence of additive inverse** Let  $A = [a_{ij}]_{m \times n}$  be any matrix, then we have another matrix as  $-A = [-a_{ij}]_{m \times n}$  such that  $A + (-A) = (-A) + A = O$ . So  $-A$  is the additive inverse of  $A$  or negative of  $A$ .

### 3.4.4 Properties of scalar multiplication of a matrix

If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be two matrices of the same order, say  $m \times n$ , and  $k$  and  $l$  are scalars, then

- (i)  $k(A + B) = kA + kB$ , (ii)  $(k + l)A = kA + lA$
- (ii)  $k(A + B) = k([a_{ij}] + [b_{ij}])$   
 $= k[a_{ij} + b_{ij}] = [k(a_{ij} + b_{ij})] = [(ka_{ij}) + (kb_{ij})]$   
 $= [ka_{ij}] + [kb_{ij}] = k[a_{ij}] + k[b_{ij}] = kA + kB$
- (iii)  $(k + l)A = (k + l)[a_{ij}]$   
 $= [(k + l)a_{ij}] + [ka_{ij}] + [la_{ij}] = k[a_{ij}] + l[a_{ij}] = kA + lA$

**Example 8** If  $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$ , then find the matrix  $X$ , such that

$$2A + 3X = 5B.$$

**Solution** We have  $2A + 3X = 5B$

- or  $2A + 3X - 2A = 5B - 2A$   
or  $2A - 2A + 3X = 5B - 2A$  (Matrix addition is commutative)  
or  $O + 3X = 5B - 2A$  ( $-2A$  is the additive inverse of  $2A$ )  
or  $3X = 5B - 2A$  ( $O$  is the additive identity)

or  $X = \frac{1}{3}(5B - 2A)$

or  $X = \frac{1}{3} \left( 5 \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2 \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix} \right) = \frac{1}{3} \left( \begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix} \right)$

$$= \frac{1}{3} \begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

**Example 9** Find  $X$  and  $Y$ , if  $X+Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$  and  $X-Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ .

**Solution** We have  $(X+Y)+(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ .

or  $(X+X)+(Y-Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$

or  $X = \frac{1}{2} \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also  $(X+Y)-(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or  $(X-X)+(Y+Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$

or  $Y = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

**Example 10** Find the values of  $x$  and  $y$  from the following equation:

$$2 \begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

**Solution** We have

$$2 \begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{aligned}
 \text{or } & \begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \\
 \text{or } & 2x+3 = 7 \quad \text{and} \quad 2y-4 = 14 \quad (\text{Why?}) \\
 \text{or } & 2x = 7-3 \quad \text{and} \quad 2y = 18 \\
 \text{or } & x = \frac{4}{2} \quad \text{and} \quad y = \frac{18}{2} \\
 \text{i.e. } & x = 2 \quad \text{and} \quad y = 9.
 \end{aligned}$$

**Example 11** Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

$$\begin{array}{c}
 \text{September Sales (in Rupees)} \\
 \text{Basmati} \quad \text{Permal} \quad \text{Naura} \\
 \mathbf{A} = \begin{bmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{October Sales (in Rupees)} \\
 \text{Basmati} \quad \text{Permal} \quad \text{Naura} \\
 \mathbf{B} = \begin{bmatrix} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}
 \end{array}$$

- Find the combined sales in September and October for each farmer in each variety.
- Find the decrease in sales from September to October.
- If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

### Solution

- Combined sales in September and October for each farmer in each variety is given by

$$\mathbf{A} + \mathbf{B} = \begin{array}{c}
 \text{Basmati} \quad \text{Permal} \quad \text{Naura} \\
 \begin{bmatrix} 15,000 & 30,000 & 36,000 \\ 70,000 & 40,000 & 20,000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}
 \end{array}$$

(ii) Change in sales from September to October is given by

$$A - B = \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 5000 & 10,000 & 24,000 \\ 30,000 & 20,000 & 0 \end{bmatrix} \begin{matrix} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{matrix}$$

$$(iii) 2\% \text{ of } B = \frac{2}{100} \times B = 0.02 \times B$$

$$= 0.02 \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \begin{matrix} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{matrix}$$

$$= \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 100 & 200 & 120 \\ 400 & 200 & 200 \end{bmatrix} \begin{matrix} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{matrix}$$

Thus, in October Ramkishan receives ₹ 100, ₹ 200 and ₹ 120 as profit in the sale of each variety of rice, respectively, and Gurcharan Singh receives profit of ₹ 400, ₹ 200 and ₹ 200 in the sale of each variety of rice, respectively.

### 3.4.5 Multiplication of matrices

Suppose Meera and Nadeem are two friends. Meera wants to buy 2 pens and 5 story books, while Nadeem needs 8 pens and 10 story books. They both go to a shop to enquire about the rates which are quoted as follows:

Pen – ₹ 5 each, story book – ₹ 50 each.

How much money does each need to spend? Clearly, Meera needs ₹  $(5 \times 2 + 50 \times 5)$  that is ₹ 260, while Nadeem needs  $(8 \times 5 + 50 \times 10)$  ₹, that is ₹ 540. In terms of matrix representation, we can write the above information as follows:

| Requirements | Prices per piece (in Rupees) | Money needed (in Rupees) |
|--------------|------------------------------|--------------------------|
|--------------|------------------------------|--------------------------|

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} 5 \\ 50 \end{bmatrix} \begin{bmatrix} 5 \times 2 + 5 \times 50 \\ 8 \times 5 + 10 \times 50 \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$$

Suppose that they enquire about the rates from another shop, quoted as follows:

pen – ₹ 4 each, story book – ₹ 40 each.

Now, the money required by Meera and Nadeem to make purchases will be respectively ₹  $(4 \times 2 + 40 \times 5)$  = ₹ 208 and ₹  $(8 \times 4 + 10 \times 40)$  = ₹ 432

Again, the above information can be represented as follows:

**Requirements    Prices per piece (in Rupees)    Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 40 \end{bmatrix} \quad \begin{bmatrix} 4 \times 2 + 40 \times 5 \\ 8 \times 4 + 10 \times 40 \end{bmatrix} = \begin{bmatrix} 208 \\ 432 \end{bmatrix}$$

Now, the information in both the cases can be combined and expressed in terms of matrices as follows:

**Requirements    Prices per piece (in Rupees)    Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \quad \begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix} \quad \begin{bmatrix} 5 \times 2 + 5 \times 50 & 4 \times 2 + 40 \times 5 \\ 8 \times 5 + 10 \times 50 & 8 \times 4 + 10 \times 40 \end{bmatrix} \\ = \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$$

The above is an example of multiplication of matrices. We observe that, for multiplication of two matrices A and B, the number of columns in A should be equal to the number of rows in B. Furthermore for getting the elements of the product matrix, we take rows of A and columns of B, multiply them element-wise and take the sum. Formally, we define multiplication of matrices as follows:

The *product* of two matrices A and B is *defined* if the number of columns of A is equal to the number of rows of B. Let  $A = [a_{ij}]$  be an  $m \times n$  matrix and  $B = [b_{jk}]$  be an  $n \times p$  matrix. Then the product of the matrices A and B is the matrix C of order  $m \times p$ . To get the  $(i, k)^{\text{th}}$  element  $c_{ik}$  of the matrix C, we take the  $i^{\text{th}}$  row of A and  $k^{\text{th}}$  column of B, multiply them elementwise and take the sum of all these products. In other words, if  $A = [a_{ij}]_{m \times n}$ ,  $B = [b_{jk}]_{n \times p}$ , then the  $i^{\text{th}}$  row of A is  $[a_{i1} \ a_{i2} \ \dots \ a_{in}]$  and the  $k^{\text{th}}$  column of

B is  $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$ , then  $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \dots + a_{in} b_{nk} = \sum_{j=1}^n a_{ij} b_{jk}$ .

The matrix  $C = [c_{ik}]_{m \times p}$  is the product of A and B.

For example, if  $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$  and  $D = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$ , then the product CD is defined

and is given by  $CD = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$ . This is a  $2 \times 2$  matrix in which each

entry is the sum of the products across some row of C with the corresponding entries down some column of D. These four computations are

Entry in first row first column  $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} (1)(2) + (-1)(-1) + (2)(5) & ? \\ ? & ? \end{bmatrix}$

Entry in first row second column  $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & (1)(7) + (-1)(1) + 2(-4) \\ ? & ? \end{bmatrix}$

Entry in second row first column  $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 0(2) + 3(-1) + 4(5) & ? \end{bmatrix}$

Entry in second row second column  $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 17 & 0(7) + 3(1) + 4(-4) \end{bmatrix}$

Thus  $CD = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

**Example 12** Find  $AB$ , if  $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$ .

**Solution** The matrix A has 2 columns which is equal to the number of rows of B. Hence  $AB$  is defined. Now

$$AB = \begin{bmatrix} 6(2) + 9(7) & 6(6) + 9(9) & 6(0) + 9(8) \\ 2(2) + 3(7) & 2(6) + 3(9) & 2(0) + 3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 63 & 36 + 81 & 0 + 72 \\ 4 + 21 & 12 + 27 & 0 + 24 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

**Remark** If  $AB$  is defined, then  $BA$  need not be defined. In the above example,  $AB$  is defined but  $BA$  is not defined because  $B$  has 3 column while  $A$  has only 2 (and not 3) rows. If  $A, B$  are, respectively  $m \times n, k \times l$  matrices, then both  $AB$  and  $BA$  are defined if and only if  $n = k$  and  $l = m$ . In particular, if both  $A$  and  $B$  are square matrices of the same order, then both  $AB$  and  $BA$  are defined.

### Non-commutativity of multiplication of matrices

Now, we shall see by an example that even if  $AB$  and  $BA$  are both defined, it is not necessary that  $AB = BA$ .

**Example 13** If  $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ , then find  $AB$ ,  $BA$ . Show that

$$AB \neq BA.$$

**Solution** Since  $A$  is a  $2 \times 3$  matrix and  $B$  is  $3 \times 2$  matrix. Hence  $AB$  and  $BA$  are both defined and are matrices of order  $2 \times 2$  and  $3 \times 3$ , respectively. Note that

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

$$\text{Clearly } AB \neq BA$$

In the above example both  $AB$  and  $BA$  are of different order and so  $AB \neq BA$ . But one may think that perhaps  $AB$  and  $BA$  could be the same if they were of the same order. But it is not so, here we give an example to show that even if  $AB$  and  $BA$  are of same order they may not be same.

**Example 14** If  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , then  $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

$$\text{and } BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \text{ Clearly } AB \neq BA.$$

Thus matrix multiplication is not commutative.

 **Note** This does not mean that  $AB \neq BA$  for every pair of matrices  $A, B$  for which  $AB$  and  $BA$  are defined. For instance,

$$\text{If } A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \text{ then } AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$$

Observe that multiplication of diagonal matrices of same order will be commutative.

### Zero matrix as the product of two non zero matrices

We know that, for real numbers  $a, b$  if  $ab = 0$ , then either  $a = 0$  or  $b = 0$ . This need not be true for matrices, we will observe this through an example.

**Example 15** Find  $AB$ , if  $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$ .

**Solution** We have  $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

#### 3.4.6 Properties of multiplication of matrices

The multiplication of matrices possesses the following properties, which we state without proof.

1. **The associative law** For any three matrices  $A, B$  and  $C$ . We have  $(AB)C = A(BC)$ , whenever both sides of the equality are defined.
2. **The distributive law** For three matrices  $A, B$  and  $C$ .
  - (i)  $A(B+C) = AB + AC$
  - (ii)  $(A+B)C = AC + BC$ , whenever both sides of equality are defined.
3. **The existence of multiplicative identity** For every square matrix  $A$ , there exist an identity matrix of same order such that  $IA = AI = A$ .

Now, we shall verify these properties by examples.

**Example 16** If  $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$  and  $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$ , find  $A(BC)$ ,  $(AB)C$  and show that  $(AB)C = A(BC)$ .

**Solution** We have  $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3+0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$

$$(AB) (C) = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

Now  $BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

Therefore  $A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-4 & -9+4-22 & -3-2+16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

**Example 17** If  $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ ,  $C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Calculate  $AC$ ,  $BC$  and  $(A + B)C$ . Also, verify that  $(A + B)C = AC + BC$

**Solution** Now,  $A + B = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}$

So  $(A + B)C = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-14+24 \\ -10+0+30 \\ 16+12+0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$

Further  $AC = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-12+21 \\ -12+0+24 \\ 14+16+0 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$

and  $BC = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-2+3 \\ 2+0+6 \\ 2-4+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}$

So  $AC + BC = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$

Clearly,  $(A + B)C = AC + BC$

**Example 18** If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ , then show that  $A^3 - 23A - 40I = O$

**Solution** We have  $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

$$\text{So } A^3 = A A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now

$$\begin{aligned} A^3 - 23A - 40I &= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23 \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix} \\ &= \begin{bmatrix} 63 - 23 - 40 & 46 - 46 + 0 & 69 - 69 + 0 \\ 69 - 69 + 0 & -6 + 46 - 40 & 23 - 23 + 0 \\ 92 - 92 + 0 & 46 - 46 + 0 & 63 - 23 - 40 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \end{aligned}$$

**Example 19** In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{bmatrix} \text{Cost per contact} \\ 40 \\ 100 \\ 50 \end{bmatrix} \begin{array}{l} \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{bmatrix} \text{Telephone} & \text{Housecall} & \text{Letter} \\ 1000 & 500 & 5000 \\ 3000 & 1000 & 10,000 \end{bmatrix} \begin{array}{l} \rightarrow X \\ \rightarrow Y \end{array} \text{. Find the total amount spent by the group in the two cities X and Y.}$$

**Solution** We have

$$\begin{aligned} BA &= \begin{bmatrix} 40,000 + 50,000 + 250,000 \\ 120,000 + 100,000 + 500,000 \end{bmatrix} \rightarrow X \\ &= \begin{bmatrix} 340,000 \\ 720,000 \end{bmatrix} \rightarrow Y \end{aligned}$$

So the total amount spent by the group in the two cities is 340,000 paise and 720,000 paise, i.e., ₹3400 and ₹7200, respectively.

### EXERCISE 3.2

1. Let  $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ ,  $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

$$\begin{array}{lll} \text{(i)} \ A + B & \text{(ii)} \ A - B & \text{(iii)} \ 3A - C \\ \text{(iv)} \ AB & \text{(v)} \ BA & \end{array}$$

2. Compute the following:

$$\text{(i)} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} a^2 + b^2 & b^2 + c^2 \\ a^2 + c^2 & a^2 + b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$$

$$\text{(iii)} \begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

3. Compute the indicated products.

$$\text{(i)} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \ 3 \ 4] \quad \text{(iii)} \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\text{(iv)} \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix} \quad \text{(v)} \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\text{(vi)} \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

4. If  $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$  and  $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ , then compute  $(A+B)$  and  $(B-C)$ . Also, verify that  $A + (B - C) = (A + B) - C$ .

5. If  $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$  and  $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$ , then compute  $3A - 5B$ .

6. Simplify  $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

7. Find X and Y, if

(i)  $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$  and  $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii)  $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$  and  $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$

8. Find X, if  $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$  and  $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

9. Find x and y, if  $2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

10. Solve the equation for x, y, z and t, if  $2 \begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3 \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

11. If  $x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$ , find the values of x and y.

12. Given  $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$ , find the values of x, y, z and w.

13. If  $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , show that  $F(x) F(y) = F(x + y)$ .

14. Show that

$$(i) \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

15. Find  $A^2 - 5A + 6I$ , if  $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

16. If  $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ , prove that  $A^3 - 6A^2 + 7A + 2I = 0$

17. If  $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$  and  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , find  $k$  so that  $A^2 = kA - 2I$

18. If  $A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$  and  $I$  is the identity matrix of order 2, show that

$$I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

19. A trust fund has ₹30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

- (a) ₹1800 (b) ₹2000

- 20.** The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹80, ₹60 and ₹40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume  $X, Y, Z, W$  and  $P$  are matrices of order  $2 \times n, 3 \times k, 2 \times p, n \times 3$  and  $p \times k$ , respectively. Choose the correct answer in Exercises 21 and 22.

- 21.** The restriction on  $n, k$  and  $p$  so that  $PY + WY$  will be defined are:
- (A)  $k = 3, p = n$  (B)  $k$  is arbitrary,  $p = 2$   
 (C)  $p$  is arbitrary,  $k = 3$  (D)  $k = 2, p = 3$
- 22.** If  $n = p$ , then the order of the matrix  $7X - 5Z$  is:
- (A)  $p \times 2$  (B)  $2 \times n$  (C)  $n \times 3$  (D)  $p \times n$

### 3.5. Transpose of a Matrix

In this section, we shall learn about transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

**Definition 3** If  $A = [a_{ij}]$  be an  $m \times n$  matrix, then the matrix obtained by interchanging the rows and columns of  $A$  is called the *transpose* of  $A$ . Transpose of the matrix  $A$  is denoted by  $A'$  or  $(A^T)$ . In other words, if  $A = [a_{ij}]_{m \times n}$ , then  $A' = [a_{ji}]_{n \times m}$ . For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & -1 \\ 5 \end{bmatrix}_{3 \times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2 \times 3}$$

#### 3.5.1 Properties of transpose of the matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any matrices  $A$  and  $B$  of suitable orders, we have

- (i)  $(A')' = A$ , (ii)  $(kA)' = kA'$  (where  $k$  is any constant)  
 (iii)  $(A + B)' = A' + B'$  (iv)  $(AB)' = B' A'$

**Example 20** If  $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ , verify that

- (i)  $(A')' = A$ , (ii)  $(A + B)' = A' + B'$ ,  
 (iii)  $(kB)' = kB'$ , where  $k$  is any constant.

**Solution**

(i) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$$

Thus  $(A')' = A$

(ii) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 5 & \sqrt{3} - 1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

Therefore

$$(A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3} - 1 & 4 \\ 4 & 4 \end{bmatrix}$$

Now

$$A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}, B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix},$$

So

$$A' + B' = \begin{bmatrix} 5 & 5 \\ \sqrt{3} - 1 & 4 \\ 4 & 4 \end{bmatrix}$$

Thus

$$(A + B)' = A' + B'$$

(iii) We have

$$kB = k \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$$

Then

$$(kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$$

Thus

$$(kB)' = kB'$$

**Example 21** If  $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$ , verify that  $(AB)' = B'A'$ .

**Solution** We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$$

then  $AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

Now  $A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$ ,  $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly  $(AB)' = B'A'$

### 3.6 Symmetric and Skew Symmetric Matrices

**Definition 4** A square matrix  $A = [a_{ij}]$  is said to be *symmetric* if  $A' = A$ , that is,  $[a_{ij}] = [a_{ji}]$  for all possible values of  $i$  and  $j$ .

For example  $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$  is a symmetric matrix as  $A' = A$

**Definition 5** A square matrix  $A = [a_{ij}]$  is said to be *skew symmetric* matrix if  $A' = -A$ , that is  $a_{ji} = -a_{ij}$  for all possible values of  $i$  and  $j$ . Now, if we put  $i = j$ , we have  $a_{ii} = -a_{ii}$ . Therefore  $2a_{ii} = 0$  or  $a_{ii} = 0$  for all  $i$ 's.

This means that all the diagonal elements of a skew symmetric matrix are zero.

For example, the matrix  $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$  is a skew symmetric matrix as  $B' = -B$

Now, we are going to prove some results of symmetric and skew-symmetric matrices.

**Theorem 1** For any square matrix  $A$  with real number entries,  $A + A'$  is a symmetric matrix and  $A - A'$  is a skew symmetric matrix.

**Proof** Let  $B = A + A'$ , then

$$\begin{aligned} B' &= (A + A')' \\ &= A' + (A')' \text{ (as } (A + B)' = A' + B') \\ &= A' + A \text{ (as } (A')' = A) \\ &= A + A' \text{ (as } A + B = B + A) \\ &= B \end{aligned}$$

Therefore

$B = A + A'$  is a symmetric matrix

Now let

$$\begin{aligned} C &= A - A' \\ C' &= (A - A')' = A' - (A')' \quad (\text{Why?}) \\ &= A' - A \quad (\text{Why?}) \\ &= - (A - A') = -C \end{aligned}$$

Therefore

$C = A - A'$  is a skew symmetric matrix.

**Theorem 2** Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

**Proof** Let  $A$  be a square matrix, then we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

From the Theorem 1, we know that  $(A + A')$  is a symmetric matrix and  $(A - A')$  is a skew symmetric matrix. Since for any matrix  $A$ ,  $(kA)' = kA'$ , it follows that  $\frac{1}{2}(A + A')$

is symmetric matrix and  $\frac{1}{2}(A - A')$  is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

**Example 22** Express the matrix  $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$  as the sum of a symmetric and a

skew symmetric matrix.

**Solution** Here

$$B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(B + B') = \frac{1}{2} \begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

$$\text{Now } P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Thus  $P = \frac{1}{2}(B + B')$  is a symmetric matrix.

$$\text{Also, let } Q = \frac{1}{2}(B - B') = \frac{1}{2} \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then  $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q$

Thus  $Q = \frac{1}{2}(B - B')$  is a skew symmetric matrix.

Now  $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

### EXERCISE 3.3

1. Find the transpose of each of the following matrices:

(i)  $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$  (ii)  $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$  (iii)  $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

2. If  $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ , then verify that

(i)  $(A + B)' = A' + B'$ , (ii)  $(A - B)' = A' - B'$

3. If  $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ , then verify that

(i)  $(A + B)' = A' + B'$ , (ii)  $(A - B)' = A' - B'$

4. If  $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ , then find  $(A + 2B)'$
5. For the matrices A and B, verify that  $(AB)' = B'A'$ , where
- (i)  $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$  (ii)  $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$
6. If (i)  $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ , then verify that  $A' A = I$
- (ii) If  $A = \begin{bmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{bmatrix}$ , then verify that  $A' A = I$
7. (i) Show that the matrix  $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$  is a symmetric matrix.
- (ii) Show that the matrix  $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$  is a skew symmetric matrix.
8. For the matrix  $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ , verify that
- (i)  $(A + A')$  is a symmetric matrix
- (ii)  $(A - A')$  is a skew symmetric matrix
9. Find  $\frac{1}{2}(A + A')$  and  $\frac{1}{2}(A - A')$ , when  $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$
10. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) 
$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$$

(ii) 
$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

(iii) 
$$\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

(iv) 
$$\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$$

Choose the correct answer in the Exercises 11 and 12.

11. If A, B are symmetric matrices of same order, then  $AB - BA$  is a

- (A) Skew symmetric matrix      (B) Symmetric matrix  
 (C) Zero matrix      (D) Identity matrix

12. If  $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ , and  $A + A' = I$ , then the value of  $\alpha$  is

- (A)  $\frac{\pi}{6}$       (B)  $\frac{\pi}{3}$   
 (C)  $\pi$       (D)  $\frac{3\pi}{2}$

### 3.7 Invertible Matrices

**Definition 6** If A is a square matrix of order  $m$ , and if there exists another square matrix B of the same order  $m$ , such that  $AB = BA = I$ , then B is called the *inverse* matrix of A and it is denoted by  $A^{-1}$ . In that case A is said to be invertible.

For example, let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \text{ be two matrices.}$$

Now

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Also

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I. \text{ Thus } B \text{ is the inverse of } A, \text{ in other}$$

words  $B = A^{-1}$  and A is inverse of B, i.e.,  $A = B^{-1}$

 Note

1. A rectangular matrix does not possess inverse matrix, since for products  $BA$  and  $AB$  to be defined and to be equal, it is necessary that matrices  $A$  and  $B$  should be square matrices of the same order.
2. If  $B$  is the inverse of  $A$ , then  $A$  is also the inverse of  $B$ .

**Theorem 3** (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

**Proof** Let  $A = [a_{ij}]$  be a square matrix of order  $m$ . If possible, let  $B$  and  $C$  be two inverses of  $A$ . We shall show that  $B = C$ .

Since  $B$  is the inverse of  $A$

$$AB = BA = I \quad \dots (1)$$

Since  $C$  is also the inverse of  $A$

$$AC = CA = I \quad \dots (2)$$

Thus

$$B = BI = B(AC) = (BA)C = IC = C$$

**Theorem 4** If  $A$  and  $B$  are invertible matrices of the same order, then  $(AB)^{-1} = B^{-1}A^{-1}$ .

**Proof** From the definition of inverse of a matrix, we have

$$(AB)(AB)^{-1} = 1$$

or  $A^{-1}(AB)(AB)^{-1} = A^{-1}I$  (Pre multiplying both sides by  $A^{-1}$ )

or  $(A^{-1}A)B(AB)^{-1} = A^{-1}$  (Since  $A^{-1}I = A^{-1}$ )

or  $IB(AB)^{-1} = A^{-1}$

or  $B(AB)^{-1} = A^{-1}$

or  $B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

or  $I(AB)^{-1} = B^{-1}A^{-1}$

Hence  $(AB)^{-1} = B^{-1}A^{-1}$

### EXERCISE 3.4

1. Matrices  $A$  and  $B$  will be inverse of each other only if
  - (A)  $AB = BA$  (B)  $AB = BA = 0$
  - (C)  $AB = 0$ ,  $BA = I$  (D)  $AB = BA = I$

### Miscellaneous Examples

**Example 23** If  $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ , then prove that  $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ ,  $n \in \mathbb{N}$ .

**Solution** We shall prove the result by using principle of mathematical induction.

We have  $P(n) : \text{If } A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in \mathbb{N}$

$$P(1) : A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \text{ so } A^1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Therefore, the result is true for  $n = 1$ .

Let the result be true for  $n = k$ . So

$$P(k) : A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for  $n = k + 1$

$$\begin{aligned} \text{Now } A^{k+1} &= A \cdot A^k = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \cos k\theta - \sin \theta \sin k\theta & \cos \theta \sin k\theta + \sin \theta \cos k\theta \\ -\sin \theta \cos k\theta + \cos \theta \sin k\theta & -\sin \theta \sin k\theta + \cos \theta \cos k\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix} \end{aligned}$$

Therefore, the result is true for  $n = k + 1$ . Thus by principle of mathematical induction,

we have  $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ , holds for all natural numbers.

**Example 24** If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is  $AB = BA$ .

**Solution** Since A and B are both symmetric matrices, therefore  $A' = A$  and  $B' = B$ .

Let  $AB$  be symmetric, then  $(AB)' = AB$

But  $(AB)' = B'A' = BA$  (Why?)

Therefore  $BA = AB$

Conversely, if  $AB = BA$ , then we shall show that  $AB$  is symmetric.

Now 
$$\begin{aligned} (AB)' &= B'A' \\ &= B A \text{ (as } A \text{ and } B \text{ are symmetric)} \\ &= AB \end{aligned}$$

Hence  $AB$  is symmetric.

**Example 25** Let  $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$ ,  $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ . Find a matrix  $D$  such that

$$CD - AB = O.$$

**Solution** Since  $A$ ,  $B$ ,  $C$  are all square matrices of order 2, and  $CD - AB$  is well defined,  $D$  must be a square matrix of order 2.

Let  $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $CD - AB = 0$  gives

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or 
$$\begin{bmatrix} 2a + 5c & 2b + 5d \\ 3a + 8c & 3b + 8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or 
$$\begin{bmatrix} 2a + 5c - 3 & 2b + 5d \\ 3a + 8c - 43 & 3b + 8d - 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \quad \dots (1)$$

$$3a + 8c - 43 = 0 \quad \dots (2)$$

$$2b + 5d = 0 \quad \dots (3)$$

and  $3b + 8d - 22 = 0 \quad \dots (4)$

Solving (1) and (2), we get  $a = -191$ ,  $c = 77$ . Solving (3) and (4), we get  $b = -110$ ,  $d = 44$ .

Therefore

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

### Miscellaneous Exercise on Chapter 3

1. If A and B are symmetric matrices, prove that  $AB - BA$  is a skew symmetric matrix.
2. Show that the matrix  $B'AB$  is symmetric or skew symmetric according as A is symmetric or skew symmetric.

3. Find the values of  $x, y, z$  if the matrix  $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$  satisfy the equation  $A'A = I$ .

4. For what values of  $x$  :  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$ ?

5. If  $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ , show that  $A^2 - 5A + 7I = 0$ .

6. Find  $x$ , if  $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

7. A manufacturer produces three products  $x, y, z$  which he sells in two markets. Annual sales are indicated below:

| Market | Products |        |        |
|--------|----------|--------|--------|
| I      | 10,000   | 2,000  | 18,000 |
| II     | 6,000    | 20,000 | 8,000  |

- (a) If unit sale prices of  $x$ ,  $y$  and  $z$  are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively, find the total revenue in each market with the help of matrix algebra.
- (b) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively. Find the gross profit.
8. Find the matrix  $X$  so that  $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

Choose the correct answer in the following questions:

9. If  $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$  is such that  $A^2 = I$ , then
- (A)  $1 + \alpha^2 + \beta\gamma = 0$  (B)  $1 - \alpha^2 + \beta\gamma = 0$   
 (C)  $1 - \alpha^2 - \beta\gamma = 0$  (D)  $1 + \alpha^2 - \beta\gamma = 0$
10. If the matrix  $A$  is both symmetric and skew symmetric, then
- (A)  $A$  is a diagonal matrix (B)  $A$  is a zero matrix  
 (C)  $A$  is a square matrix (D) None of these
11. If  $A$  is square matrix such that  $A^2 = A$ , then  $(I + A)^3 - 7A$  is equal to
- (A)  $A$  (B)  $I - A$  (C)  $I$  (D)  $3A$

### Summary

- ◆ A matrix is an ordered rectangular array of numbers or functions.
- ◆ A matrix having  $m$  rows and  $n$  columns is called a matrix of order  $m \times n$ .
- ◆  $[a_{ij}]_{m \times 1}$  is a column matrix.
- ◆  $[a_{ij}]_{1 \times n}$  is a row matrix.
- ◆ An  $m \times n$  matrix is a square matrix if  $m = n$ .
- ◆  $A = [a_{ij}]_{m \times m}$  is a diagonal matrix if  $a_{ij} = 0$ , when  $i \neq j$ .
- ◆  $A = [a_{ij}]_{n \times n}$  is a scalar matrix if  $a_{ij} = 0$ , when  $i \neq j$ ,  $a_{ij} = k$ , ( $k$  is some constant), when  $i = j$ .
- ◆  $A = [a_{ij}]_{n \times n}$  is an identity matrix, if  $a_{ij} = 1$ , when  $i = j$ ,  $a_{ij} = 0$ , when  $i \neq j$ .
- ◆ A zero matrix has all its elements as zero.
- ◆  $A = [a_{ij}] = [b_{ij}] = B$  if (i)  $A$  and  $B$  are of same order, (ii)  $a_{ij} = b_{ij}$  for all possible values of  $i$  and  $j$ .

- ◆  $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$
- ◆  $-A = (-1)A$
- ◆  $A - B = A + (-1)B$
- ◆  $A + B = B + A$
- ◆  $(A + B) + C = A + (B + C)$ , where A, B and C are of same order.
- ◆  $k(A + B) = kA + kB$ , where A and B are of same order,  $k$  is constant.
- ◆  $(k + l)A = kA + lA$ , where  $k$  and  $l$  are constant.
- ◆ If  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{jk}]_{n \times p}$ , then  $AB = C = [c_{ik}]_{m \times p}$ , where  $c_{ik} = \sum_{j=1}^n a_{ij} b_{jk}$
- ◆ (i)  $A(BC) = (AB)C$ , (ii)  $A(B + C) = AB + AC$ , (iii)  $(A + B)C = AC + BC$
- ◆ If  $A = [a_{ij}]_{m \times n}$ , then  $A'$  or  $A^T = [a_{ji}]_{n \times m}$
- ◆ (i)  $(A')' = A$ , (ii)  $(kA)' = kA'$ , (iii)  $(A + B)' = A' + B'$ , (iv)  $(AB)' = B'A'$
- ◆ A is a symmetric matrix if  $A' = A$ .
- ◆ A is a skew symmetric matrix if  $A' = -A$ .
- ◆ Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- ◆ If A and B are two square matrices such that  $AB = BA = I$ , then B is the inverse matrix of A and is denoted by  $A^{-1}$  and A is the inverse of B.
- ◆ Inverse of a square matrix, if it exists, is unique.



## NOTES

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# DETERMINANTS

❖ *All Mathematical truths are relative and conditional. — C.P. STEINMETZ* ❖

## 4.1 Introduction

In the previous chapter, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

can be represented as  $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ . Now, this

system of equations has a unique solution or not, is determined by the number  $a_1 b_2 - a_2 b_1$ . (Recall that if

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$  or,  $a_1 b_2 - a_2 b_1 \neq 0$ , then the system of linear equations has a unique solution). The number  $a_1 b_2 - a_2 b_1$

which determines uniqueness of solution is associated with the matrix  $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

and is called the determinant of A or  $\det A$ . Determinants have wide applications in Engineering, Science, Economics, Social Science, etc.

In this chapter, we shall study determinants up to order three only with real entries. Also, we will study various properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equations in two or three variables using inverse of a matrix.

## 4.2 Determinant

To every square matrix  $A = [a_{ij}]$  of order  $n$ , we can associate a number (real or complex) called determinant of the square matrix A, where  $a_{ij} = (i, j)^{\text{th}}$  element of A.



P.S. Laplace  
(1749-1827)

This may be thought of as a function which associates each square matrix with a unique number (real or complex). If  $M$  is the set of square matrices,  $K$  is the set of numbers (real or complex) and  $f: M \rightarrow K$  is defined by  $f(A) = k$ , where  $A \in M$  and  $k \in K$ , then  $f(A)$  is called the determinant of  $A$ . It is also denoted by  $|A|$  or  $\det A$  or  $\Delta$ .

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then determinant of  $A$  is written as  $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

### Remarks

- (i) For matrix  $A$ ,  $|A|$  is read as determinant of  $A$  and not modulus of  $A$ .
- (ii) Only square matrices have determinants.

#### 4.2.1 Determinant of a matrix of order one

Let  $A = [a]$  be the matrix of order 1, then determinant of  $A$  is defined to be equal to  $a$

#### 4.2.2 Determinant of a matrix of order two

Let  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  be a matrix of order  $2 \times 2$ ,

then the determinant of  $A$  is defined as:

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

**Example 1** Evaluate  $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$ .

**Solution** We have  $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1) = 4 + 4 = 8$ .

**Example 2** Evaluate  $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

**Solution** We have

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$$

#### 4.2.3 Determinant of a matrix of order $3 \times 3$

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order

3 corresponding to each of three rows ( $R_1$ ,  $R_2$  and  $R_3$ ) and three columns ( $C_1$ ,  $C_2$  and  $C_3$ ) giving the same value as shown below.

Consider the determinant of square matrix  $A = [a_{ij}]_{3 \times 3}$

$$\text{i.e., } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

### Expansion along first Row ( $R_1$ )

**Step 1** Multiply first element  $a_{11}$  of  $R_1$  by  $(-1)^{1+1}$  [ $(-1)^{\text{sum of suffixes in } a_{11}}$ ] and with the second order determinant obtained by deleting the elements of first row ( $R_1$ ) and first column ( $C_1$ ) of  $|A|$  as  $a_{11}$  lies in  $R_1$  and  $C_1$ ,

$$\text{i.e., } (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

**Step 2** Multiply 2nd element  $a_{12}$  of  $R_1$  by  $(-1)^{1+2}$  [ $(-1)^{\text{sum of suffixes in } a_{12}}$ ] and the second order determinant obtained by deleting elements of first row ( $R_1$ ) and 2nd column ( $C_2$ ) of  $|A|$  as  $a_{12}$  lies in  $R_1$  and  $C_2$ ,

$$\text{i.e., } (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

**Step 3** Multiply third element  $a_{13}$  of  $R_1$  by  $(-1)^{1+3}$  [ $(-1)^{\text{sum of suffixes in } a_{13}}$ ] and the second order determinant obtained by deleting elements of first row ( $R_1$ ) and third column ( $C_3$ ) of  $|A|$  as  $a_{13}$  lies in  $R_1$  and  $C_3$ ,

$$\text{i.e., } (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

**Step 4** Now the expansion of determinant of  $A$ , that is,  $|A|$  written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

$$\begin{aligned} \det A = |A| &= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

$$\text{or } |A| = a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23}) + a_{13} (a_{21} a_{32} - a_{31} a_{22})$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{32} a_{23} - a_{12} a_{21} a_{33} + a_{12} a_{31} a_{23} + a_{13} a_{21} a_{32} \\ - a_{13} a_{31} a_{22} \quad \dots (1)$$

 **Note** We shall apply all four steps together.

### Expansion along second row ( $R_2$ )

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along  $R_2$ , we get

$$|A| = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ = -a_{21} (a_{12} a_{33} - a_{32} a_{13}) + a_{22} (a_{11} a_{33} - a_{31} a_{13}) \\ - a_{23} (a_{11} a_{32} - a_{31} a_{12}) \\ |A| = -a_{21} a_{12} a_{33} + a_{21} a_{32} a_{13} + a_{22} a_{11} a_{33} - a_{22} a_{31} a_{13} - a_{23} a_{11} a_{32} \\ + a_{23} a_{31} a_{12} \\ = a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} \\ - a_{13} a_{23} a_{22} \quad \dots (2)$$

### Expansion along first Column ( $C_1$ )

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

By expanding along  $C_1$ , we get

$$|A| = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21} (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ + a_{31} (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ = a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{21} (a_{12} a_{33} - a_{13} a_{32}) + a_{31} (a_{12} a_{23} - a_{13} a_{22})$$

$$\begin{aligned}
 |A| &= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{21} a_{12} a_{33} + a_{21} a_{13} a_{32} + a_{31} a_{12} a_{23} \\
 &\quad - a_{31} a_{13} a_{22} \\
 &= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} \\
 &\quad - a_{13} a_{31} a_{22} \quad \dots (3)
 \end{aligned}$$

Clearly, values of  $|A|$  in (1), (2) and (3) are equal. It is left as an exercise to the reader to verify that the values of  $|A|$  by expanding along  $R_3$ ,  $C_2$  and  $C_3$  are equal to the value of  $|A|$  obtained in (1), (2) or (3).

Hence, expanding a determinant along any row or column gives same value.

### Remarks

- (i) For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.
- (ii) While expanding, instead of multiplying by  $(-1)^{i+j}$ , we can multiply by +1 or -1 according as  $(i+j)$  is even or odd.
- (iii) Let  $A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ . Then, it is easy to verify that  $A = 2B$ . Also  $|A| = 0 - 8 = -8$  and  $|B| = 0 - 2 = -2$ .

Observe that,  $|A| = 4(-2) = 2^2|B|$  or  $|A| = 2^n|B|$ , where  $n = 2$  is the order of square matrices A and B.

In general, if  $A = kB$  where A and B are square matrices of order  $n$ , then  $|A| = k^n |B|$ , where  $n = 1, 2, 3$ .

**Example 3** Evaluate the determinant  $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$ .

**Solution** Note that in the third column, two entries are zero. So expanding along third column ( $C_3$ ), we get

$$\begin{aligned}
 \Delta &= 4 \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \\
 &= 4(-1 - 12) - 0 + 0 = -52
 \end{aligned}$$

**Example 4** Evaluate  $\Delta = \begin{vmatrix} 0 & \sin \alpha & -\cos \alpha \\ -\sin \alpha & 0 & \sin \beta \\ \cos \alpha & -\sin \beta & 0 \end{vmatrix}$ .

**Solution** Expanding along  $R_1$ , we get

$$\begin{aligned}\Delta &= 0 \begin{vmatrix} 0 & \sin \beta \\ -\sin \beta & 0 \end{vmatrix} - \sin \alpha \begin{vmatrix} -\sin \alpha & \sin \beta \\ \cos \alpha & 0 \end{vmatrix} - \cos \alpha \begin{vmatrix} -\sin \alpha & 0 \\ \cos \alpha & -\sin \beta \end{vmatrix} \\ &= 0 - \sin \alpha (0 - \sin \beta \cos \alpha) - \cos \alpha (\sin \alpha \sin \beta - 0) \\ &= \sin \alpha \sin \beta \cos \alpha - \cos \alpha \sin \alpha \sin \beta = 0\end{aligned}$$

**Example 5** Find values of  $x$  for which  $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ .

**Solution** We have  $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

$$\text{i.e.} \quad 3 - x^2 = 3 - 8$$

$$\text{i.e.} \quad x^2 = 8$$

$$\text{Hence} \quad x = \pm 2\sqrt{2}$$

### EXERCISE 4.1

Evaluate the determinants in Exercises 1 and 2.

$$1. \quad \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$$

$$2. \quad \text{(i)} \quad \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} \quad \text{(ii)} \quad \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

$$3. \quad \text{If } A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \text{ then show that } |2A| = 4|A|$$

$$4. \quad \text{If } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \text{ then show that } |3A| = 27|A|$$

5. Evaluate the determinants

$$\text{(i)} \quad \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$$\text{(ii)} \quad \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$(iii) \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$$

$$(iv) \begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$$6. \text{ If } A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}, \text{ find } |A|$$

7. Find values of  $x$ , if

$$(i) \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$$

$$(ii) \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$

$$8. \text{ If } \begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}, \text{ then } x \text{ is equal to}$$

(A) 6

(B)  $\pm 6$

(C) -6

(D) 0

### 4.3 Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ , is given by the expression  $\frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ . Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \dots (1)$$

#### Remarks

- (i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).
- (ii) If area is given, use both positive and negative values of the determinant for calculation.
- (iii) The area of the triangle formed by three collinear points is zero.

**Example 6** Find the area of the triangle whose vertices are  $(3, 8)$ ,  $(-4, 2)$  and  $(5, 1)$ .

**Solution** The area of triangle is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$\begin{aligned}
 &= \frac{1}{2} [3(2-1) - 8(-4-5) + 1(-4-10)] \\
 &= \frac{1}{2} (3 + 72 - 14) = \frac{61}{2}
 \end{aligned}$$

**Example 7** Find the equation of the line joining A(1, 3) and B (0, 0) using determinants and find  $k$  if D( $k$ , 0) is a point such that area of triangle ABD is 3 sq units.

**Solution** Let P ( $x, y$ ) be any point on AB. Then, area of triangle ABP is zero (Why?). So

$$\frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

This gives

$$\frac{1}{2}(y - 3x) = 0 \text{ or } y = 3x,$$

which is the equation of required line AB.

Also, since the area of the triangle ABD is 3 sq. units, we have

$$\frac{1}{2} \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

This gives,  $\frac{-3k}{2} = \pm 3$ , i.e.,  $k = \mp 2$ .

### EXERCISE 4.2

- Find area of the triangle with vertices at the point given in each of the following :
  - (1, 0), (6, 0), (4, 3)
  - (2, 7), (1, 1), (10, 8)
  - (-2, -3), (3, 2), (-1, -8)
- Show that points A ( $a, b + c$ ), B ( $b, c + a$ ), C ( $c, a + b$ ) are collinear.
- Find values of  $k$  if area of triangle is 4 sq. units and vertices are
  - ( $k$ , 0), (4, 0), (0, 2)
  - (-2, 0), (0, 4), (0,  $k$ )
- (i) Find equation of line joining (1, 2) and (3, 6) using determinants.  
(ii) Find equation of line joining (3, 1) and (9, 3) using determinants.
- If area of triangle is 35 sq units with vertices (2, -6), (5, 4) and ( $k$ , 4). Then  $k$  is
  - 12
  - 2
  - 12, -2
  - 12, -2

#### 4.4 Minors and Cofactors

In this section, we will learn to write the expansion of a determinant in compact form using minors and cofactors.

**Definition 1** Minor of an element  $a_{ij}$  of a determinant is the determinant obtained by deleting its  $i$ th row and  $j$ th column in which element  $a_{ij}$  lies. Minor of an element  $a_{ij}$  is denoted by  $M_{ij}$ .

**Remark** Minor of an element of a determinant of order  $n$  ( $n \geq 2$ ) is a determinant of order  $n - 1$ .

**Example 8** Find the minor of element 6 in the determinant  $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

**Solution** Since 6 lies in the second row and third column, its minor  $M_{23}$  is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6 \text{ (obtained by deleting } R_2 \text{ and } C_3 \text{ in } \Delta\text{).}$$

**Definition 2** Cofactor of an element  $a_{ij}$ , denoted by  $A_{ij}$  is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of } a_{ij}.$$

**Example 9** Find minors and cofactors of all the elements of the determinant  $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

**Solution** Minor of the element  $a_{ij}$  is  $M_{ij}$

Here  $a_{11} = 1$ . So  $M_{11} = \text{Minor of } a_{11} = 3$

$M_{12} = \text{Minor of the element } a_{12} = 4$

$M_{21} = \text{Minor of the element } a_{21} = -2$

$M_{22} = \text{Minor of the element } a_{22} = 1$

Now, cofactor of  $a_{ij}$  is  $A_{ij}$ . So

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (4) = -4$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-2) = 2$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1$$

**Example 10** Find minors and cofactors of the elements  $a_{11}, a_{21}$  in the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

**Solution** By definition of minors and cofactors, we have

$$\text{Minor of } a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} a_{33} - a_{23} a_{32}$$

$$\text{Cofactor of } a_{11} = A_{11} = (-1)^{1+1} M_{11} = a_{22} a_{33} - a_{23} a_{32}$$

$$\text{Minor of } a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12} a_{33} - a_{13} a_{32}$$

$$\text{Cofactor of } a_{21} = A_{21} = (-1)^{2+1} M_{21} = (-1) (a_{12} a_{33} - a_{13} a_{32}) = -a_{12} a_{33} + a_{13} a_{32}$$

**Remark** Expanding the determinant  $\Delta$ , in Example 21, along  $R_1$ , we have

$$\Delta = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}, \text{ where } A_{ij} \text{ is cofactor of } a_{ij}$$

= sum of product of elements of  $R_1$  with their corresponding cofactors

Similarly,  $\Delta$  can be calculated by other five ways of expansion that is along  $R_2, R_3, C_1, C_2$  and  $C_3$ .

Hence  $\Delta = \text{sum of the product of elements of any row (or column) with their corresponding cofactors.}$

**Note** If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example,

$$\begin{aligned} \Delta &= a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} \\ &= a_{11} (-1)^{1+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} (-1)^{1+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} (-1)^{1+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)} \end{aligned}$$

Similarly, we can try for other rows and columns.

**Example 11** Find minors and cofactors of the elements of the determinant

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} \text{ and verify that } a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$$

**Solution** We have  $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$ ;  $A_{11} = (-1)^{1+1} (-20) = -20$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46; \quad A_{12} = (-1)^{1+2} (-46) = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30; \quad A_{13} = (-1)^{1+3} (30) = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4; \quad A_{21} = (-1)^{2+1} (-4) = 4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19; \quad A_{22} = (-1)^{2+2} (-19) = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13; \quad A_{23} = (-1)^{2+3} (13) = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12; \quad A_{31} = (-1)^{3+1} (-12) = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22; \quad A_{32} = (-1)^{3+2} (-22) = 22$$

and  $M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18; \quad A_{33} = (-1)^{3+3} (18) = 18$

Now  $a_{11} = 2, a_{12} = -3, a_{13} = 5; A_{31} = -12, A_{32} = 22, A_{33} = 18$

So  $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$   
 $= 2 (-12) + (-3) (22) + 5 (18) = -24 - 66 + 90 = 0$

### EXERCISE 4.3

Write Minors and Cofactors of the elements of following determinants:

1. (i)  $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$  (ii)  $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2. (i)  $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$  (ii)  $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

3. Using Cofactors of elements of second row, evaluate  $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ .

4. Using Cofactors of elements of third column, evaluate  $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$ .

5. If  $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$  and  $A_{ij}$  is Cofactors of  $a_{ij}$ , then value of  $\Delta$  is given by

(A)  $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$  (B)  $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$   
 (C)  $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$  (D)  $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

## 4.5 Adjoint and Inverse of a Matrix

In the previous chapter, we have studied inverse of a matrix. In this section, we shall discuss the condition for existence of inverse of a matrix.

To find inverse of a matrix  $A$ , i.e.,  $A^{-1}$  we shall first define adjoint of a matrix.

### 4.5.1 Adjoint of a matrix

**Definition 3** The adjoint of a square matrix  $A = [a_{ij}]_{n \times n}$  is defined as the transpose of the matrix  $[A_{ij}]_{n \times n}$ , where  $A_{ij}$  is the cofactor of the element  $a_{ij}$ . Adjoint of the matrix  $A$  is denoted by  $\text{adj } A$ .

Let  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then  $adj A = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

**Example 12** Find  $adj A$  for  $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

**Solution** We have  $A_{11} = 4, A_{12} = -1, A_{21} = -3, A_{22} = 2$

Hence  $adj A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

**Remark** For a square matrix of order 2, given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The  $adj A$  can also be obtained by interchanging  $a_{11}$  and  $a_{22}$  and by changing signs of  $a_{12}$  and  $a_{21}$ , i.e.,

$$adj A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Change sign      Interchange

We state the following theorem without proof.

**Theorem 1** If  $A$  be any given square matrix of order  $n$ , then

$$A(adj A) = (adj A) A = |A|I,$$

where  $I$  is the identity matrix of order  $n$

**Verification**

Let  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ , then  $adj A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to  $|A|$  and otherwise zero, we have

$$A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I$$

Similarly, we can show  $(\text{adj } A) A = |A| I$

Hence  $A(\text{adj } A) = (\text{adj } A) A = |A| I$

**Definition 4** A square matrix A is said to be singular if  $|A| = 0$ .

For example, the determinant of matrix  $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$  is zero

Hence A is a singular matrix.

**Definition 5** A square matrix A is said to be non-singular if  $|A| \neq 0$

Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ . Then  $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$ .

Hence A is a nonsingular matrix

We state the following theorems without proof.

**Theorem 2** If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

**Theorem 3** The determinant of the product of matrices is equal to product of their respective determinants, that is,  $|AB| = |A| |B|$ , where A and B are square matrices of the same order

**Remark** We know that  $(\text{adj } A) A = |A| I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}, |A| \neq 0$

Writing determinants of matrices on both sides, we have

$$|(\text{adj } A) A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

i.e.  $|(adj A)| |A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$  (Why?)

i.e.  $|(adj A)| |A| = |A|^3 \quad (1)$

i.e.  $|(adj A)| = |A|^2$

In general, if  $A$  is a square matrix of order  $n$ , then  $|adj(A)| = |A|^{n-1}$ .

**Theorem 4** A square matrix  $A$  is invertible if and only if  $A$  is nonsingular matrix.

**Proof** Let  $A$  be invertible matrix of order  $n$  and  $I$  be the identity matrix of order  $n$ .

Then, there exists a square matrix  $B$  of order  $n$  such that  $AB = BA = I$

Now  $AB = I$ . So  $|AB| = |I|$  or  $|A| |B| = 1$  (since  $|I|=1, |AB|=|A||B|$ )

This gives  $|A| \neq 0$ . Hence  $A$  is nonsingular.

Conversely, let  $A$  be nonsingular. Then  $|A| \neq 0$

Now  $A (adj A) = (adj A) A = |A| I$  (Theorem 1)

or  $A \left( \frac{1}{|A|} adj A \right) = \left( \frac{1}{|A|} adj A \right) A = I$

or  $AB = BA = I$ , where  $B = \frac{1}{|A|} adj A$

Thus  $A$  is invertible and  $A^{-1} = \frac{1}{|A|} adj A$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

**Example 13** If  $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ , then verify that  $A adj A = |A| I$ . Also find  $A^{-1}$ .

**Solution** We have  $|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1 \neq 0$

Now  $A_{11} = 7, A_{12} = -1, A_{13} = -1, A_{21} = -3, A_{22} = 1, A_{23} = 0, A_{31} = -3, A_{32} = 0, A_{33} = 1$

Therefore  $adj A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now 
$$A \text{ (adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I$$

Also 
$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

**Example 14** If  $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ , then verify that  $(AB)^{-1} = B^{-1}A^{-1}$ .

**Solution** We have  $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Since,  $|AB| = -11 \neq 0$ ,  $(AB)^{-1}$  exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB) = -\frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further,  $|A| = -11 \neq 0$  and  $|B| = 1 \neq 0$ . Therefore,  $A^{-1}$  and  $B^{-1}$  both exist and are given by

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore  $B^{-1}A^{-1} = -\frac{1}{11} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Hence  $(AB)^{-1} = B^{-1}A^{-1}$

**Example 15** Show that the matrix  $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$  satisfies the equation  $A^2 - 4A + I = O$ , where  $I$  is  $2 \times 2$  identity matrix and  $O$  is  $2 \times 2$  zero matrix. Using this equation, find  $A^{-1}$ .

**Solution** We have  $A^2 = A \cdot A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence  $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now  $A^2 - 4A + I = O$

Therefore  $A \cdot A - 4A = -I$

or  $A \cdot A (A^{-1}) - 4A \cdot A A^{-1} = -I A^{-1}$  (Post multiplying by  $A^{-1}$  because  $|A| \neq 0$ )

or  $A (A A^{-1}) - 4I = -A^{-1}$

or  $A I - 4I = -A^{-1}$

or  $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence  $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

#### EXERCISE 4.4

Find adjoint of each of the matrices in Exercises 1 and 2.

1.  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2.  $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Verify  $A (adj A) = (adj A) A = |A| I$  in Exercises 3 and 4

3.  $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

4.  $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5.  $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

6.  $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

7.  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

8. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

9. 
$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

10. 
$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

11. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

 12. Let  $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ . Verify that  $(AB)^{-1} = B^{-1} A^{-1}$ .

 13. If  $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ , show that  $A^2 - 5A + 7I = O$ . Hence find  $A^{-1}$ .

 14. For the matrix  $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ , find the numbers  $a$  and  $b$  such that  $A^2 + aA + bI = O$ .

 15. For the matrix  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ 

 Show that  $A^3 - 6A^2 + 5A + 11I = O$ . Hence, find  $A^{-1}$ .

 16. If  $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ 

 Verify that  $A^3 - 6A^2 + 9A - 4I = O$  and hence find  $A^{-1}$ 

 17. Let  $A$  be a nonsingular square matrix of order  $3 \times 3$ . Then  $|adj A|$  is equal to

- (A)  $|A|$  (B)  $|A|^2$  (C)  $|A|^3$  (D)  $3|A|$

 18. If  $A$  is an invertible matrix of order 2, then  $\det(A^{-1})$  is equal to

- (A)  $\det(A)$  (B)  $\frac{1}{\det(A)}$  (C) 1 (D) 0

#### 4.6 Applications of Determinants and Matrices

In this section, we shall discuss application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

**Consistent system** A system of equations is said to be *consistent* if its solution (one or more) exists.

**Inconsistent system** A system of equations is said to be *inconsistent* if its solution does not exist.

 **Note** In this chapter, we restrict ourselves to the system of linear equations having unique solutions only.

#### 4.6.1 Solution of system of linear equations using inverse of a matrix

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then, the system of equations can be written as,  $AX = B$ , i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

**Case I** If  $A$  is a nonsingular matrix, then its inverse exists. Now

$$AX = B$$

$$\text{or } A^{-1}(AX) = A^{-1}B \quad (\text{premultiplying by } A^{-1})$$

$$\text{or } (A^{-1}A)X = A^{-1}B \quad (\text{by associative property})$$

$$\text{or } IX = A^{-1}B$$

$$\text{or } X = A^{-1}B$$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

**Case II** If  $A$  is a singular matrix, then  $|A| = 0$ .

In this case, we calculate  $(adj A)B$ .

If  $(adj A)B \neq O$ , ( $O$  being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If  $(adj A) B = O$ , then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

**Example 16** Solve the system of equations

$$\begin{aligned} 2x + 5y &= 1 \\ 3x + 2y &= 7 \end{aligned}$$

**Solution** The system of equations can be written in the form  $AX = B$ , where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now,  $|A| = -11 \neq 0$ , Hence, A is nonsingular matrix and so has a unique solution.

Note that

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore

$$X = A^{-1}B = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence

$$x = 3, y = -1$$

**Example 17** Solve the following system of equations by matrix method.

$$\begin{aligned} 3x - 2y + 3z &= 8 \\ 2x + y - z &= 1 \\ 4x - 3y + 2z &= 4 \end{aligned}$$

**Solution** The system of equations can be written in the form  $AX = B$ , where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3(2 - 3) + 2(4 + 4) + 3(-6 - 4) = -17 \neq 0$$

Hence, A is nonsingular and so its inverse exists. Now

$$\begin{array}{lll} A_{11} = -1, & A_{12} = -8, & A_{13} = -10 \\ A_{21} = -5, & A_{22} = -6, & A_{23} = 1 \\ A_{31} = -1, & A_{32} = 9, & A_{33} = 7 \end{array}$$

Therefore

$$A^{-1} = -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

So

$$X = A^{-1} B = -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence

$$x = 1, y = 2 \text{ and } z = 3.$$

**Example 18** The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

**Solution** Let first, second and third numbers be denoted by  $x, y$  and  $z$ , respectively. Then, according to given conditions, we have

$$\begin{aligned} x + y + z &= 6 \\ y + 3z &= 11 \\ x + z &= 2y \text{ or } x - 2y + z = 0 \end{aligned}$$

This system can be written as  $A X = B$ , where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Here  $|A| = 1(1+6) - (0-3) + (0-1) = 9 \neq 0$ . Now we find  $\text{adj } A$

$$\begin{aligned} A_{11} &= 1(1+6) = 7, & A_{12} &= -(0-3) = 3, & A_{13} &= -1 \\ A_{21} &= -(1+2) = -3, & A_{22} &= 0, & A_{23} &= -(-2-1) = 3 \\ A_{31} &= (3-1) = 2, & A_{32} &= -(3-0) = -3, & A_{33} &= (1-0) = 1 \end{aligned}$$

Hence

$$\text{adj } A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Thus

$$A^{-1} = \frac{1}{|A|} \text{adj} (A) = \frac{1}{9} \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Since

$$X = A^{-1} B$$

$$X = \frac{1}{9} \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus

$$x = 1, y = 2, z = 3$$

### EXERCISE 4.5

Examine the consistency of the system of equations in Exercises 1 to 6.

- |                     |                      |                      |
|---------------------|----------------------|----------------------|
| 1. $x + 2y = 2$     | 2. $2x - y = 5$      | 3. $x + 3y = 5$      |
| $2x + 3y = 3$       | $x + y = 4$          | $2x + 6y = 8$        |
| 4. $x + y + z = 1$  | 5. $3x - y - 2z = 2$ | 6. $5x - y + 4z = 5$ |
| $2x + 3y + 2z = 2$  | $2y - z = -1$        | $2x + 3y + 5z = 2$   |
| $ax + ay + 2az = 4$ | $3x - 5y = 3$        | $5x - 2y + 6z = -1$  |

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

- |                        |                            |                     |
|------------------------|----------------------------|---------------------|
| 7. $5x + 2y = 4$       | 8. $2x - y = -2$           | 9. $4x - 3y = 3$    |
| $7x + 3y = 5$          | $3x + 4y = 3$              | $3x - 5y = 7$       |
| 10. $5x + 2y = 3$      | 11. $2x + y + z = 1$       | 12. $x - y + z = 4$ |
| $3x + 2y = 5$          | $x - 2y - z = \frac{3}{2}$ | $2x + y - 3z = 0$   |
|                        | $3y - 5z = 9$              | $x + y + z = 2$     |
| 13. $2x + 3y + 3z = 5$ | 14. $x - y + 2z = 7$       |                     |
| $x - 2y + z = -4$      | $3x + 4y - 5z = -5$        |                     |
| $3x - y - 2z = 3$      | $2x - y + 3z = 12$         |                     |

15. If  $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ , find  $A^{-1}$ . Using  $A^{-1}$  solve the system of equations

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.

### Miscellaneous Examples

**Example 19** Use product  $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix}$  to solve the system of equations

$$x - y + 2z = 1$$

$$2y - 3z = 1$$

$$3x - 2y + 4z = 2$$

**Solution** Consider the product  $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2 - 9 + 12 & 0 - 2 + 2 & 1 + 3 - 4 \\ 0 + 18 - 18 & 0 + 4 - 3 & 0 - 6 + 6 \\ -6 - 18 + 24 & 0 - 4 + 4 & 3 + 6 - 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence  $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

or

$$\begin{aligned}
 x &= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}
 \end{aligned}$$

Hence

$$x = 0, y = 5 \text{ and } z = 3$$

### Miscellaneous Exercises on Chapter 4

- Prove that the determinant  $\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$  is independent of  $\theta$ .
- Evaluate  $\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$ .
- If  $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ , find  $(AB)^{-1}$
- Let  $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ . Verify that
  - $[\text{adj } A]^{-1} = \text{adj } (A^{-1})$
  - $(A^{-1})^{-1} = A$
- Evaluate  $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$
- Evaluate  $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Using properties of determinants in Exercises 11 to 15, prove that:

7. Solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Choose the correct answer in Exercise 17 to 19.

8. If  $x, y, z$  are nonzero real numbers, then the inverse of matrix  $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$  is

$$(A) \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$(B) xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$(C) \frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$(D) \frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Let  $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$ , where  $0 \leq \theta \leq 2\pi$ . Then

- (A)  $\text{Det}(A) = 0$   
 (C)  $\text{Det}(A) \in (2, 4)$

- (B)  $\text{Det}(A) \in (2, \infty)$   
 (D)  $\text{Det}(A) \in [2, 4]$

### Summary

- ◆ Determinant of a matrix  $A = [a_{11}]_{1 \times 1}$  is given by  $|a_{11}| = a_{11}$

- ◆ Determinant of a matrix  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

- ◆ Determinant of a matrix  $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$  is given by (expanding along  $R_1$ )

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

**For any square matrix A, the  $|A|$  satisfy following properties.**

- ◆ Area of a triangle with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- ◆ Minor of an element  $a_{ij}$  of the determinant of matrix A is the determinant obtained by deleting  $i^{\text{th}}$  row and  $j^{\text{th}}$  column and denoted by  $M_{ij}$ .
- ◆ Cofactor of  $a_{ij}$  of given by  $A_{ij} = (-1)^{i+j} M_{ij}$
- ◆ Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example,

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

- ◆ If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example,  $a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0$

- ◆ If  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ , then  $\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$ , where  $A_{ij}$  is cofactor of  $a_{ij}$
  - ◆  $A(\text{adj } A) = (\text{adj } A)A = |A|I$ , where A is square matrix of order  $n$ .
  - ◆ A square matrix A is said to be singular or non-singular according as  $|A| = 0$  or  $|A| \neq 0$ .
  - ◆ If  $AB = BA = I$ , where B is square matrix, then B is called inverse of A. Also  $A^{-1} = B$  or  $B^{-1} = A$  and hence  $(A^{-1})^{-1} = A$ .
  - ◆ A square matrix A has inverse if and only if A is non-singular.
  - ◆  $A^{-1} = \frac{1}{|A|}(\text{adj } A)$
  - ◆ If  $a_1x + b_1y + c_1z = d_1$   
 $a_2x + b_2y + c_2z = d_2$   
 $a_3x + b_3y + c_3z = d_3$ ,
- then these equations can be written as  $A X = B$ , where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- ◆ Unique solution of equation  $AX = B$  is given by  $X = A^{-1}B$ , where  $|A| \neq 0$ .
- ◆ A system of equation is consistent or inconsistent according as its solution exists or not.
- ◆ For a square matrix A in matrix equation  $AX = B$ 
  - (i)  $|A| \neq 0$ , there exists unique solution
  - (ii)  $|A| = 0$  and  $(\text{adj } A)B \neq 0$ , then there exists no solution
  - (iii)  $|A| = 0$  and  $(\text{adj } A)B = 0$ , then system may or may not be consistent.

### Historical Note

The Chinese method of representing the coefficients of the unknowns of several linear equations by using rods on a calculating board naturally led to the discovery of simple method of elimination. The arrangement of rods was precisely that of the numbers in a determinant. The Chinese, therefore, early developed the idea of subtracting columns and rows as in simplification of a determinant *Mikami, China, pp 30, 93.*

Seki Kowa, the greatest of the Japanese Mathematicians of seventeenth century in his work ‘*Kai Fukudai no Ho*’ in 1683 showed that he had the idea of determinants and of their expansion. But he used this device only in eliminating a quantity from two equations and not directly in the solution of a set of simultaneous linear equations. T. Hayashi, “*The Fakudoi and Determinants in Japanese Mathematics*,” in the proc. of the Tokyo Math. Soc., V.

Vendermonde was the first to recognise determinants as independent functions. He may be called the formal founder. Laplace (1772), gave general method of expanding a determinant in terms of its complementary minors. In 1773 Lagrange treated determinants of the second and third orders and used them for purpose other than the solution of equations. In 1801, Gauss used determinants in his theory of numbers.

The next great contributor was Jacques - Philippe - Marie Binet, (1812) who stated the theorem relating to the product of two matrices of  $m$ -columns and  $n$ -rows, which for the special case of  $m = n$  reduces to the multiplication theorem.

Also on the same day, Cauchy (1812) presented one on the same subject. He used the word ‘determinant’ in its present sense. He gave the proof of multiplication theorem more satisfactory than Binet’s.

The greatest contributor to the theory was Carl Gustav Jacob Jacobi, after this the word determinant received its final acceptance.



# CONTINUITY AND DIFFERENTIABILITY

❖ *The whole of science is nothing more than a refinement of everyday thinking.* — ALBERT EINSTEIN ❖

## 5.1 Introduction

This chapter is essentially a continuation of our study of differentiation of functions in Class XI. We had learnt to differentiate certain functions like polynomial functions and trigonometric functions. In this chapter, we introduce the very important concepts of continuity, differentiability and relations between them. We will also learn differentiation of inverse trigonometric functions. Further, we introduce a new class of functions called exponential and logarithmic functions. These functions lead to powerful techniques of differentiation. We illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.



Sir Issac Newton  
(1642-1727)

## 5.2 Continuity

We start the section with two informal examples to get a feel of continuity. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 2, & \text{if } x > 0 \end{cases}$$

This function is of course defined at every point of the real line. Graph of this function is given in the Fig 5.1. One can deduce from the graph that the value of the function at *nearby* points on  $x$ -axis remain *close* to each other except at  $x = 0$ . At the points near and to the left of 0, i.e., at points like  $-0.1, -0.01, -0.001$ , the value of the function is 1. At the points near and to the right of 0, i.e., at points like  $0.1, 0.01,$

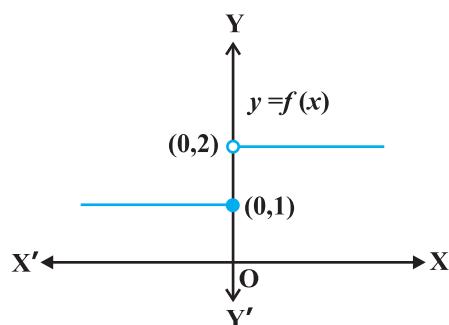


Fig 5.1

0.001, the value of the function is 2. Using the language of left and right hand limits, we may say that the left (respectively right) hand limit of  $f$  at 0 is 1 (respectively 2). In particular the left and right hand limits do not coincide. We also observe that the value of the function at  $x = 0$  concides with the left hand limit. Note that when we try to draw the graph, we cannot draw it in one stroke, i.e., without lifting pen from the plane of the paper, we can not draw the graph of this function. In fact, we need to lift the pen when we come to 0 from left. This is one instance of function being not continuous at  $x = 0$ .

Now, consider the function defined as

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$

This function is also defined at every point. Left and the right hand limits at  $x = 0$  are both equal to 1. But the value of the function at  $x = 0$  equals 2 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is yet another instance of a function being not continuous at  $x = 0$ .

Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function *around* that point without lifting the pen from the plane of the paper.

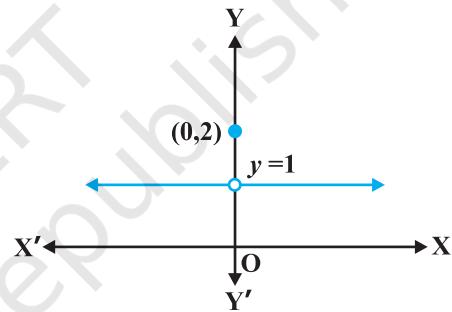


Fig 5.2

Mathematically, it may be phrased precisely as follows:

**Definition 1** Suppose  $f$  is a real function on a subset of the real numbers and let  $c$  be a point in the domain of  $f$ . Then  $f$  is continuous at  $c$  if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at  $x = c$  exist and equal to each other, then  $f$  is said to be continuous at  $x = c$ . Recall that if the right hand and left hand limits at  $x = c$  coincide, then we say that the common value is the limit of the function at  $x = c$ . Hence we may also rephrase the definition of continuity as follows: *a function is continuous at  $x = c$  if the function is defined at  $x = c$  and if the value of the function at  $x = c$  equals the limit of the function at  $x = c$ .* If  $f$  is not continuous at  $c$ , we say  $f$  is *discontinuous* at  $c$  and  $c$  is called a *point of discontinuity* of  $f$ .

**Example 1** Check the continuity of the function  $f$  given by  $f(x) = 2x + 3$  at  $x = 1$ .

**Solution** First note that the function is defined at the given point  $x = 1$  and its value is 5. Then find the limit of the function at  $x = 1$ . Clearly

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (2x + 3) = 2(1) + 3 = 5$$

Thus  $\lim_{x \rightarrow 1} f(x) = 5 = f(1)$

Hence,  $f$  is continuous at  $x = 1$ .

**Example 2** Examine whether the function  $f$  given by  $f(x) = x^2$  is continuous at  $x = 0$ .

**Solution** First note that the function is defined at the given point  $x = 0$  and its value is 0. Then find the limit of the function at  $x = 0$ . Clearly

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 = 0^2 = 0$$

Thus  $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$

Hence,  $f$  is continuous at  $x = 0$ .

**Example 3** Discuss the continuity of the function  $f$  given by  $f(x) = |x|$  at  $x = 0$ .

**Solution** By definition

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly the function is defined at 0 and  $f(0) = 0$ . Left hand limit of  $f$  at 0 is

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

Similarly, the right hand limit of  $f$  at 0 is

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

Thus, the left hand limit, right hand limit and the value of the function coincide at  $x = 0$ . Hence,  $f$  is continuous at  $x = 0$ .

**Example 4** Show that the function  $f$  given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is not continuous at  $x = 0$ .

**Solution** The function is defined at  $x = 0$  and its value at  $x = 0$  is 1. When  $x \neq 0$ , the function is given by a polynomial. Hence,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x^3 + 3) = 0^3 + 3 = 3$$

Since the limit of  $f$  at  $x = 0$  does not coincide with  $f(0)$ , the function is not continuous at  $x = 0$ . It may be noted that  $x = 0$  is the only point of discontinuity for this function.

**Example 5** Check the points where the constant function  $f(x) = k$  is continuous.

**Solution** The function is defined at all real numbers and by definition, its value at any real number equals  $k$ . Let  $c$  be any real number. Then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} k = k$$

Since  $f(c) = k = \lim_{x \rightarrow c} f(x)$  for any real number  $c$ , the function  $f$  is continuous at every real number.

**Example 6** Prove that the identity function on real numbers given by  $f(x) = x$  is continuous at every real number.

**Solution** The function is clearly defined at every point and  $f(c) = c$  for every real number  $c$ . Also,

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c$$

Thus,  $\lim_{x \rightarrow c} f(x) = c = f(c)$  and hence the function is continuous at every real number.

Having defined continuity of a function at a given point, now we make a natural extension of this definition to discuss continuity of a function.

**Definition 2** A real function  $f$  is said to be continuous if it is continuous at every point in the domain of  $f$ .

This definition requires a bit of elaboration. Suppose  $f$  is a function defined on a closed interval  $[a, b]$ , then for  $f$  to be continuous, it needs to be continuous at every point in  $[a, b]$  including the end points  $a$  and  $b$ . Continuity of  $f$  at  $a$  means

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and continuity of  $f$  at  $b$  means

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$

Observe that  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow b^+} f(x)$  do not make sense. As a consequence of this definition, if  $f$  is defined only at one point, it is continuous there, i.e., if the domain of  $f$  is a singleton,  $f$  is a continuous function.

**Example 7** Is the function defined by  $f(x) = |x|$ , a continuous function?

**Solution** We may rewrite  $f$  as

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

By Example 3, we know that  $f$  is continuous at  $x = 0$ .

Let  $c$  be a real number such that  $c < 0$ . Then  $f(c) = -c$ . Also

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-x) = -c \quad (\text{Why?})$$

Since  $\lim_{x \rightarrow c} f(x) = f(c)$ ,  $f$  is continuous at all negative real numbers.

Now, let  $c$  be a real number such that  $c > 0$ . Then  $f(c) = c$ . Also

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c \quad (\text{Why?})$$

Since  $\lim_{x \rightarrow c} f(x) = f(c)$ ,  $f$  is continuous at all positive real numbers. Hence,  $f$  is continuous at all points.

**Example 8** Discuss the continuity of the function  $f$  given by  $f(x) = x^3 + x^2 - 1$ .

**Solution** Clearly  $f$  is defined at every real number  $c$  and its value at  $c$  is  $c^3 + c^2 - 1$ . We also know that

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus  $\lim_{x \rightarrow c} f(x) = f(c)$ , and hence  $f$  is continuous at every real number. This means  $f$  is a continuous function.

**Example 9** Discuss the continuity of the function  $f$  defined by  $f(x) = \frac{1}{x}$ ,  $x \neq 0$ .

**Solution** Fix any non zero real number  $c$ , we have

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$$

Also, since for  $c \neq 0$ ,  $f(c) = \frac{1}{c}$ , we have  $\lim_{x \rightarrow c} f(x) = f(c)$  and hence,  $f$  is continuous at every point in the domain of  $f$ . Thus  $f$  is a continuous function.

We take this opportunity to explain the concept of *infinity*. This we do by analysing the function  $f(x) = \frac{1}{x}$  near  $x = 0$ . To carry out this analysis we follow the usual trick of finding the value of the function at real numbers *close* to 0. Essentially we are trying to find the right hand limit of  $f$  at 0. We tabulate this in the following (Table 5.1).

Table 5.1

|        |   |          |     |                 |                  |                   |           |
|--------|---|----------|-----|-----------------|------------------|-------------------|-----------|
| $x$    | 1 | 0.3      | 0.2 | $0.1 = 10^{-1}$ | $0.01 = 10^{-2}$ | $0.001 = 10^{-3}$ | $10^{-n}$ |
| $f(x)$ | 1 | 3.333... | 5   | 10              | $100 = 10^2$     | $1000 = 10^3$     | $10^n$    |

We observe that as  $x$  gets closer to 0 from the right, the value of  $f(x)$  shoots up higher. This may be rephrased as: the value of  $f(x)$  may be made larger than any given number by choosing a positive real number *very close* to 0. In symbols, we write

$$\lim_{x \rightarrow 0^+} f(x) = +\infty$$

(to be read as: the right hand limit of  $f(x)$  at 0 is plus infinity). We wish to emphasise that  $+\infty$  is NOT a real number and hence the right hand limit of  $f$  at 0 does not exist (as a real number).

Similarly, the left hand limit of  $f$  at 0 may be found. The following table is self explanatory.

Table 5.2

|        |    |           |      |            |            |            |            |
|--------|----|-----------|------|------------|------------|------------|------------|
| $x$    | -1 | -0.3      | -0.2 | $-10^{-1}$ | $-10^{-2}$ | $-10^{-3}$ | $-10^{-n}$ |
| $f(x)$ | -1 | -3.333... | -5   | -10        | $-10^2$    | $-10^3$    | $-10^n$    |

From the Table 5.2, we deduce that the value of  $f(x)$  may be made smaller than any given number by choosing a negative real number *very close* to 0. In symbols, we write

$$\lim_{x \rightarrow 0^-} f(x) = -\infty$$

(to be read as: the left hand limit of  $f(x)$  at 0 is minus infinity). Again, we wish to emphasise that  $-\infty$  is NOT a real number and hence the left hand limit of  $f$  at 0 does not exist (as a real number). The graph of the reciprocal function given in Fig 5.3 is a geometric representation of the above mentioned facts.

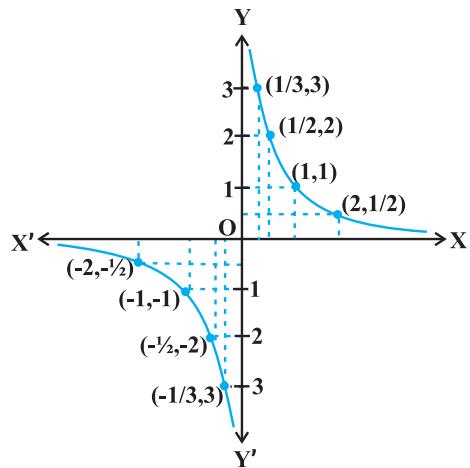


Fig 5.3

**Example 10** Discuss the continuity of the function  $f$  defined by

$$f(x) = \begin{cases} x+2, & \text{if } x \leq 1 \\ x-2, & \text{if } x > 1 \end{cases}$$

**Solution** The function  $f$  is defined at all points of the real line.

**Case 1** If  $c < 1$ , then  $f(c) = c + 2$ . Therefore,  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 2) = c + 2$

Thus,  $f$  is continuous at all real numbers less than 1.

**Case 2** If  $c > 1$ , then  $f(c) = c - 2$ . Therefore,

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - 2) = c - 2 = f(c)$$

Thus,  $f$  is continuous at all points  $x > 1$ .

**Case 3** If  $c = 1$ , then the left hand limit of  $f$  at  $x = 1$  is

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 2) = 1 + 2 = 3$$

The right hand limit of  $f$  at  $x = 1$  is

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x - 2) = 1 - 2 = -1$$

Since the left and right hand limits of  $f$  at  $x = 1$  do not coincide,  $f$  is not continuous at  $x = 1$ . Hence  $x = 1$  is the only point of discontinuity of  $f$ . The graph of the function is given in Fig 5.4.

**Example 11** Find all the points of discontinuity of the function  $f$  defined by

$$f(x) = \begin{cases} x+2, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \\ x-2, & \text{if } x > 1 \end{cases}$$

**Solution** As in the previous example we find that  $f$  is continuous at all real numbers  $x \neq 1$ . The left hand limit of  $f$  at  $x = 1$  is

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 2) = 1 + 2 = 3$$

The right hand limit of  $f$  at  $x = 1$  is

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x - 2) = 1 - 2 = -1$$

Since, the left and right hand limits of  $f$  at  $x = 1$  do not coincide,  $f$  is not continuous at  $x = 1$ . Hence  $x = 1$  is the only point of discontinuity of  $f$ . The graph of the function is given in the Fig 5.5.

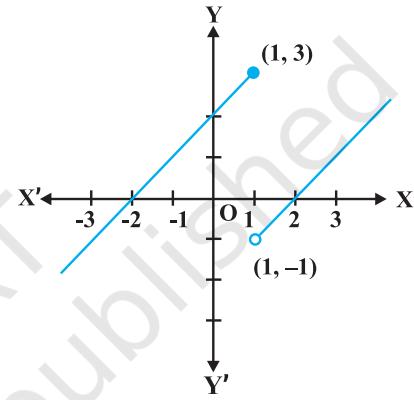


Fig 5.4

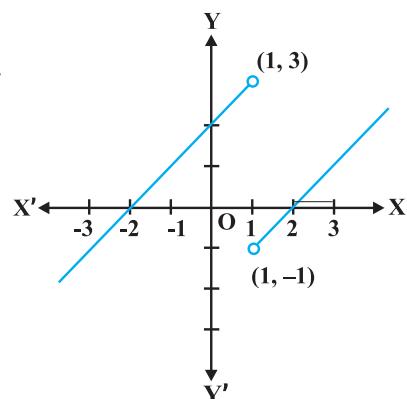


Fig 5.5

**Example 12** Discuss the continuity of the function defined by

$$f(x) = \begin{cases} x + 2, & \text{if } x < 0 \\ -x + 2, & \text{if } x > 0 \end{cases}$$

**Solution** Observe that the function is defined at all real numbers except at 0. Domain of definition of this function is

$$D_1 \cup D_2 \text{ where } D_1 = \{x \in \mathbf{R} : x < 0\} \text{ and } D_2 = \{x \in \mathbf{R} : x > 0\}$$

**Case 1** If  $c \in D_1$ , then  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 2) = c + 2 = f(c)$  and hence  $f$  is continuous in  $D_1$ .

**Case 2** If  $c \in D_2$ , then  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-x + 2) = -c + 2 = f(c)$  and hence  $f$  is continuous in  $D_2$ .

Since  $f$  is continuous at all points in the domain of  $f$ , we deduce that  $f$  is continuous. Graph of this function is given in the Fig 5.6. Note that to graph this function we need to lift the pen from the plane of the paper, but we need to do that only for those points where the function is not defined.

**Example 13** Discuss the continuity of the function  $f$  given by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$$

**Solution** Clearly the function is defined at every real number. Graph of the function is given in Fig 5.7. By inspection, it seems prudent to partition the domain of definition of  $f$  into three disjoint subsets of the real line.

$$\text{Let } D_1 = \{x \in \mathbf{R} : x < 0\}, D_2 = \{0\} \text{ and } D_3 = \{x \in \mathbf{R} : x > 0\}$$

**Case 1** At any point in  $D_1$ , we have  $f(x) = x^2$  and it is easy to see that it is continuous there (see Example 2).

**Case 2** At any point in  $D_3$ , we have  $f(x) = x$  and it is easy to see that it is continuous there (see Example 6).

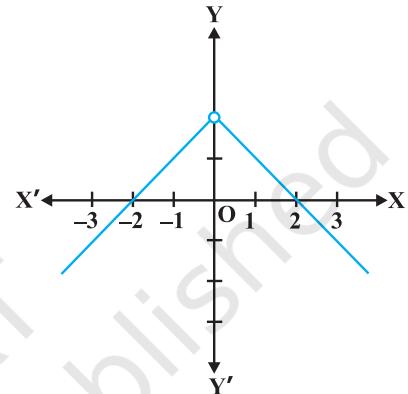


Fig 5.6

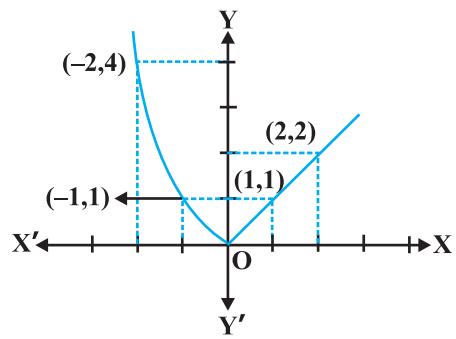


Fig 5.7

**Case 3** Now we analyse the function at  $x = 0$ . The value of the function at 0 is  $f(0) = 0$ . The left hand limit of  $f$  at 0 is

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0^2 = 0$$

The right hand limit of  $f$  at 0 is

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

Thus  $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$  and hence  $f$  is continuous at 0. This means that  $f$  is continuous at every point in its domain and hence,  $f$  is a continuous function.

**Example 14** Show that every polynomial function is continuous.

**Solution** Recall that a function  $p$  is a polynomial function if it is defined by  $p(x) = a_0 + a_1 x + \dots + a_n x^n$  for some natural number  $n$ ,  $a_n \neq 0$  and  $a_i \in \mathbf{R}$ . Clearly this function is defined for every real number. For a fixed real number  $c$ , we have

$$\lim_{x \rightarrow c} p(x) = p(c)$$

By definition,  $p$  is continuous at  $c$ . Since  $c$  is any real number,  $p$  is continuous at every real number and hence  $p$  is a continuous function.

**Example 15** Find all the points of discontinuity of the greatest integer function defined by  $f(x) = [x]$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ .

**Solution** First observe that  $f$  is defined for all real numbers. Graph of the function is given in Fig 5.8. From the graph it looks like that  $f$  is discontinuous at every integral point. Below we explore, if this is true.

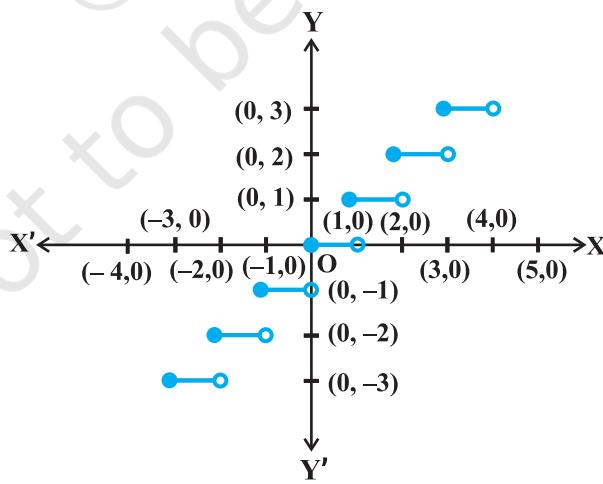


Fig 5.8

**Case 1** Let  $c$  be a real number which is not equal to any integer. It is evident from the graph that for all real numbers *close* to  $c$  the value of the function is equal to  $[c]$ ; i.e.,  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} [x] = [c]$ . Also  $f(c) = [c]$  and hence the function is continuous at all real numbers not equal to integers.

**Case 2** Let  $c$  be an integer. Then we can find a sufficiently small real number  $r > 0$  such that  $[c - r] = c - 1$  whereas  $[c + r] = c$ .

This, in terms of limits mean that

$$\lim_{x \rightarrow c} f(x) = c - 1, \quad \lim_{x \rightarrow c^+} f(x) = c$$

Since these limits cannot be equal to each other for any  $c$ , the function is discontinuous at every integral point.

### 5.2.1 Algebra of continuous functions

In the previous class, after having understood the concept of limits, we learnt some algebra of limits. Analogously, now we will study some algebra of continuous functions. Since continuity of a function at a point is entirely dictated by the limit of the function at that point, it is reasonable to expect results analogous to the case of limits.

**Theorem 1** Suppose  $f$  and  $g$  be two real functions continuous at a real number  $c$ . Then

- (1)  $f + g$  is continuous at  $x = c$ .
- (2)  $f - g$  is continuous at  $x = c$ .
- (3)  $f \cdot g$  is continuous at  $x = c$ .
- (4)  $\left(\frac{f}{g}\right)$  is continuous at  $x = c$ , (provided  $g(c) \neq 0$ ).

**Proof** We are investigating continuity of  $(f + g)$  at  $x = c$ . Clearly it is defined at  $x = c$ . We have

$$\begin{aligned} \lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} [f(x) + g(x)] && \text{(by definition of } f + g) \\ &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) && \text{(by the theorem on limits)} \\ &= f(c) + g(c) && \text{(as } f \text{ and } g \text{ are continuous)} \\ &= (f + g)(c) && \text{(by definition of } f + g) \end{aligned}$$

Hence,  $f + g$  is continuous at  $x = c$ .

Proofs for the remaining parts are similar and left as an exercise to the reader.

**Remarks**

- (i) As a special case of (3) above, if  $f$  is a constant function, i.e.,  $f(x) = \lambda$  for some real number  $\lambda$ , then the function  $(\lambda \cdot g)$  defined by  $(\lambda \cdot g)(x) = \lambda \cdot g(x)$  is also continuous. In particular if  $\lambda = -1$ , the continuity of  $f$  implies continuity of  $-f$ .
- (ii) As a special case of (4) above, if  $f$  is the constant function  $f(x) = \lambda$ , then the function  $\frac{\lambda}{g}$  defined by  $\frac{\lambda}{g}(x) = \frac{\lambda}{g(x)}$  is also continuous wherever  $g(x) \neq 0$ . In particular, the continuity of  $g$  implies continuity of  $\frac{1}{g}$ .

The above theorem can be exploited to generate many continuous functions. They also aid in deciding if certain functions are continuous or not. The following examples illustrate this:

**Example 16** Prove that every rational function is continuous.

**Solution** Recall that every rational function  $f$  is given by

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where  $p$  and  $q$  are polynomial functions. The domain of  $f$  is all real numbers except points at which  $q$  is zero. Since polynomial functions are continuous (Example 14),  $f$  is continuous by (4) of Theorem 1.

**Example 17** Discuss the continuity of sine function.

**Solution** To see this we use the following facts

$$\lim_{x \rightarrow 0} \sin x = 0$$

We have not proved it, but is intuitively clear from the graph of  $\sin x$  near 0.

Now, observe that  $f(x) = \sin x$  is defined for every real number. Let  $c$  be a real number. Put  $x = c + h$ . If  $x \rightarrow c$  we know that  $h \rightarrow 0$ . Therefore

$$\begin{aligned} \lim_{x \rightarrow c} f(x) &= \lim_{x \rightarrow c} \sin x \\ &= \lim_{h \rightarrow 0} \sin(c + h) \\ &= \lim_{h \rightarrow 0} [\sin c \cos h + \cos c \sin h] \\ &= \lim_{h \rightarrow 0} [\sin c \cos h] + \lim_{h \rightarrow 0} [\cos c \sin h] \\ &= \sin c + 0 = \sin c = f(c) \end{aligned}$$

Thus  $\lim_{x \rightarrow c} f(x) = f(c)$  and hence  $f$  is a continuous function.

**Remark** A similar proof may be given for the continuity of cosine function.

**Example 18** Prove that the function defined by  $f(x) = \tan x$  is a continuous function.

**Solution** The function  $f(x) = \tan x = \frac{\sin x}{\cos x}$ . This is defined for all real numbers such

that  $\cos x \neq 0$ , i.e.,  $x \neq (2n+1)\frac{\pi}{2}$ . We have just proved that both sine and cosine functions are continuous. Thus  $\tan x$  being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if  $f$  and  $g$  are two real functions, then

$$(f \circ g)(x) = f(g(x))$$

is defined whenever the range of  $g$  is a subset of domain of  $f$ . The following theorem (stated without proof) captures the continuity of composite functions.

**Theorem 2** Suppose  $f$  and  $g$  are real valued functions such that  $(f \circ g)$  is defined at  $c$ . If  $g$  is continuous at  $c$  and if  $f$  is continuous at  $g(c)$ , then  $(f \circ g)$  is continuous at  $c$ .

The following examples illustrate this theorem.

**Example 19** Show that the function defined by  $f(x) = \sin(x^2)$  is a continuous function.

**Solution** Observe that the function is defined for every real number. The function  $f$  may be thought of as a composition  $g \circ h$  of the two functions  $g$  and  $h$ , where  $g(x) = \sin x$  and  $h(x) = x^2$ . Since both  $g$  and  $h$  are continuous functions, by Theorem 2, it can be deduced that  $f$  is a continuous function.

**Example 20** Show that the function  $f$  defined by

$$f(x) = |1 - x + |x||,$$

where  $x$  is any real number, is a continuous function.

**Solution** Define  $g$  by  $g(x) = 1 - x + |x|$  and  $h$  by  $h(x) = |x|$  for all real  $x$ . Then

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) \\ &= h(1 - x + |x|) \\ &= |1 - x + |x|| = f(x) \end{aligned}$$

In Example 7, we have seen that  $h$  is a continuous function. Hence  $g$  being a sum of a polynomial function and the modulus function is continuous. But then  $f$  being a composite of two continuous functions is continuous.

## EXERCISE 5.1

1. Prove that the function  $f(x) = 5x - 3$  is continuous at  $x = 0$ , at  $x = -3$  and at  $x = 5$ .
2. Examine the continuity of the function  $f(x) = 2x^2 - 1$  at  $x = 3$ .
3. Examine the following functions for continuity.

$$(a) \quad f(x) = x - 5$$

$$(b) \quad f(x) = \frac{1}{x-5}, \quad x \neq 5$$

$$(c) \quad f(x) = \frac{x^2 - 25}{x+5}, \quad x \neq -5 \quad (d) \quad f(x) = |x - 5|$$

4. Prove that the function  $f(x) = x^n$  is continuous at  $x = n$ , where  $n$  is a positive integer.

5. Is the function  $f$  defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at  $x = 0$ ? At  $x = 1$ ? At  $x = 2$ ?

Find all points of discontinuity of  $f$ , where  $f$  is defined by

$$6. \quad f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2 \\ 2x - 3, & \text{if } x > 2 \end{cases}$$

$$7. \quad f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

$$8. \quad f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$9. \quad f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$$10. \quad f(x) = \begin{cases} x + 1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases}$$

$$11. \quad f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

$$12. \quad f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

13. Is the function defined by

$$f(x) = \begin{cases} x + 5, & \text{if } x \leq 1 \\ x - 5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Discuss the continuity of the function  $f$ , where  $f$  is defined by

$$14. \quad f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$$

$$15. \quad f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

$$16. \quad f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

17. Find the relationship between  $a$  and  $b$  so that the function  $f$  defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at  $x = 3$ .

18. For what value of  $\lambda$  is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at  $x = 0$ ? What about continuity at  $x = 1$ ?

19. Show that the function defined by  $g(x) = x - [x]$  is discontinuous at all integral points. Here  $[x]$  denotes the greatest integer less than or equal to  $x$ .

20. Is the function defined by  $f(x) = x^2 - \sin x + 5$  continuous at  $x = \pi$ ?

21. Discuss the continuity of the following functions:

- (a)  $f(x) = \sin x + \cos x$       (b)  $f(x) = \sin x - \cos x$   
 (c)  $f(x) = \sin x \cdot \cos x$

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

23. Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}$$

24. Determine if  $f$  defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

25. Examine the continuity of  $f$ , where  $f$  is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

Find the values of  $k$  so that the function  $f$  is continuous at the indicated point in Exercises 26 to 29.

26.  $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$  at  $x = \frac{\pi}{2}$

27.  $f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$  at  $x = 2$

28.  $f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$  at  $x = \pi$

29.  $f(x) = \begin{cases} kx + 1, & \text{if } x \leq 5 \\ 3x - 5, & \text{if } x > 5 \end{cases}$  at  $x = 5$

30. Find the values of  $a$  and  $b$  such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

is a continuous function.

31. Show that the function defined by  $f(x) = \cos(x^2)$  is a continuous function.  
 32. Show that the function defined by  $f(x) = |\cos x|$  is a continuous function.  
 33. Examine that  $\sin|x|$  is a continuous function.  
 34. Find all the points of discontinuity of  $f$  defined by  $f(x) = |x| - |x + 1|$ .

### 5.3. Differentiability

Recall the following facts from previous class. We had defined the derivative of a real function as follows:

Suppose  $f$  is a real function and  $c$  is a point in its domain. The derivative of  $f$  at  $c$  is defined by

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit exists. Derivative of  $f$  at  $c$  is denoted by  $f'(c)$  or  $\frac{d}{dx}(f(x))|_c$ . The function defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of  $f$ . The derivative of  $f$  is denoted by  $f'(x)$  or  $\frac{d}{dx}(f(x))$  or if  $y = f(x)$  by  $\frac{dy}{dx}$  or  $y'$ . The process of finding derivative of a function is called differentiation. We also use the phrase *differentiate  $f(x)$  with respect to  $x$*  to mean *find  $f'(x)$* .

The following rules were established as a part of algebra of derivatives:

- (1)  $(u \pm v)' = u' \pm v'$
- (2)  $(uv)' = u'v + uv'$  (Leibnitz or product rule)
- (3)  $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ , wherever  $v \neq 0$  (Quotient rule).

The following table gives a list of derivatives of certain standard functions:

Table 5.3

|         |            |          |           |            |
|---------|------------|----------|-----------|------------|
| $f(x)$  | $x^n$      | $\sin x$ | $\cos x$  | $\tan x$   |
| $f'(x)$ | $nx^{n-1}$ | $\cos x$ | $-\sin x$ | $\sec^2 x$ |

Whenever we defined derivative, we had put a caution *provided the limit exists*. Now the natural question is; what if it doesn't? The question is quite pertinent and so is

its answer. If  $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$  does not exist, we say that  $f$  is not differentiable at  $c$ .

In other words, we say that a function  $f$  is differentiable at a point  $c$  in its domain if both  $\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h}$  and  $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$  are finite and equal. A function is said to be differentiable in an interval  $[a, b]$  if it is differentiable at every point of  $[a, b]$ . As in case of continuity, at the end points  $a$  and  $b$ , we take the right hand limit and left hand limit, which are nothing but left hand derivative and right hand derivative of the function at  $a$  and  $b$  respectively. Similarly, a function is said to be differentiable in an interval  $(a, b)$  if it is differentiable at every point of  $(a, b)$ .

**Theorem 3** If a function  $f$  is differentiable at a point  $c$ , then it is also continuous at that point.

**Proof** Since  $f$  is differentiable at  $c$ , we have

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

But for  $x \neq c$ , we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$

Therefore  $\lim_{x \rightarrow c} [f(x) - f(c)] = \lim_{x \rightarrow c} \left[ \frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$

or  $\lim_{x \rightarrow c} [f(x)] - \lim_{x \rightarrow c} [f(c)] = \lim_{x \rightarrow c} \left[ \frac{f(x) - f(c)}{x - c} \right] \cdot \lim_{x \rightarrow c} [(x - c)]$   
 $= f'(c) \cdot 0 = 0$

or  $\lim_{x \rightarrow c} f(x) = f(c)$

Hence  $f$  is continuous at  $x = c$ .

**Corollary 1** Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by  $f(x) = |x|$  is a continuous function. Consider the left hand limit

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$$

The right hand limit

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1$$

Since the above left and right hand limits at 0 are not equal,  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$

does not exist and hence  $f$  is not differentiable at 0. Thus  $f$  is not a differentiable function.

### 5.3.1 Derivatives of composite functions

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of  $f$ , where

$$f(x) = (2x + 1)^3$$

One way is to expand  $(2x + 1)^3$  using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\begin{aligned}\frac{d}{dx} f(x) &= \frac{d}{dx} [(2x+1)^3] \\ &= \frac{d}{dx} (8x^3 + 12x^2 + 6x + 1) \\ &= 24x^2 + 24x + 6 \\ &= 6 (2x + 1)^2\end{aligned}$$

Now, observe that

$$f(x) = (h \circ g)(x)$$

where  $g(x) = 2x + 1$  and  $h(x) = x^3$ . Put  $t = g(x) = 2x + 1$ . Then  $f(x) = h(t) = t^3$ . Thus

$$\frac{df}{dx} = 6(2x+1)^2 = 3(2x+1)^2 \cdot 2 = 3t^2 \cdot 2 = \frac{dh}{dt} \cdot \frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say,  $(2x + 1)^{100}$ . We may formalise this observation in the following theorem called the chain rule.

**Theorem 4 (Chain Rule)** Let  $f$  be a real valued function which is a composite of two functions  $u$  and  $v$ ; i.e.,  $f = v \circ u$ . Suppose  $t = u(x)$  and if both  $\frac{dt}{dx}$  and  $\frac{dv}{dt}$  exist, we have

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose  $f$  is a real valued function which is a composite of three functions  $u$ ,  $v$  and  $w$ ; i.e.,

$f = (w \circ u) \circ v$ . If  $t = v(x)$  and  $s = u(t)$ , then

$$\frac{df}{dx} = \frac{d(w \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

**Example 21** Find the derivative of the function given by  $f(x) = \sin(x^2)$ .

**Solution** Observe that the given function is a composite of two functions. Indeed, if  $t = u(x) = x^2$  and  $v(t) = \sin t$ , then

$$f(x) = (v \circ u)(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put  $t = u(x) = x^2$ . Observe that  $\frac{dv}{dt} = \cos t$  and  $\frac{dt}{dx} = 2x$  exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of  $x$ . Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

### EXERCISE 5.2

Differentiate the functions with respect to  $x$  in Exercises 1 to 8.

1.  $\sin(x^2 + 5)$
2.  $\cos(\sin x)$
3.  $\sin(ax + b)$
4.  $\sec(\tan(\sqrt{x}))$
5.  $\frac{\sin(ax + b)}{\cos(cx + d)}$
6.  $\cos x^3 \cdot \sin^2(x^5)$
7.  $2\sqrt{\cot(x^2)}$
8.  $\cos(\sqrt{x})$

9. Prove that the function  $f$  given by

$$f(x) = |x - 1|, x \in \mathbf{R}$$

is not differentiable at  $x = 1$ .

10. Prove that the greatest integer function defined by

$$f(x) = [x], 0 < x < 3$$

is not differentiable at  $x = 1$  and  $x = 2$ .

#### 5.3.2 Derivatives of implicit functions

Until now we have been differentiating various functions given in the form  $y = f(x)$ . But it is not necessary that functions are always expressed in this form. For example, consider one of the following relationships between  $x$  and  $y$ :

$$x - y - \pi = 0$$

$$x + \sin xy - y = 0$$

In the first case, we can *solve for  $y$*  and rewrite the relationship as  $y = x - \pi$ . In the second case, it does not seem that there is an easy way to *solve for  $y$* . Nevertheless, there is no doubt about the dependence of  $y$  on  $x$  in either of the cases. When a relationship between  $x$  and  $y$  is expressed in a way that it is easy to *solve for  $y$*  and write  $y = f(x)$ , we say that  $y$  is given as an *explicit function* of  $x$ . In the latter case it

is implicit that  $y$  is a function of  $x$  and we say that the relationship of the second type, above, gives function *implicitly*. In this subsection, we learn to differentiate implicit functions.

**Example 22** Find  $\frac{dy}{dx}$  if  $x - y = \pi$ .

**Solution** One way is to solve for  $y$  and rewrite the above as

$$y = x - \pi$$

But then

$$\frac{dy}{dx} = 1$$

**Alternatively**, directly differentiating the relationship w.r.t.,  $x$ , we have

$$\frac{d}{dx}(x - y) = \frac{d\pi}{dx}$$

Recall that  $\frac{d\pi}{dx}$  means to differentiate the constant function taking value  $\pi$  everywhere w.r.t.,  $x$ . Thus

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

which implies that

$$\frac{dy}{dx} = \frac{dx}{dx} = 1$$

**Example 23** Find  $\frac{dy}{dx}$ , if  $y + \sin y = \cos x$ .

**Solution** We differentiate the relationship directly with respect to  $x$ , i.e.,

$$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$

which implies using chain rule

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = -\sin x$$

This gives

$$\frac{dy}{dx} = -\frac{\sin x}{1 + \cos y}$$

where

$$y \neq (2n + 1)\pi$$

### 5.3.3 Derivatives of inverse trigonometric functions

We remark that inverse trigonometric functions are continuous functions, but we will not prove this. Now we use chain rule to find derivatives of these functions.

**Example 24** Find the derivative of  $f$  given by  $f(x) = \sin^{-1} x$  assuming it exists.

**Solution** Let  $y = \sin^{-1} x$ . Then,  $x = \sin y$ .

Differentiating both sides w.r.t.  $x$ , we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for  $\cos y \neq 0$ , i.e.,  $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$ , i.e.,  $x \neq -1, 1$ , i.e.,  $x \in (-1, 1)$ .

To make this result a bit more attractive, we carry out the following manipulation. Recall that for  $x \in (-1, 1)$ ,  $\sin(\sin^{-1} x) = x$  and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since  $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $\cos y$  is positive and hence  $\cos y = \sqrt{1-x^2}$

Thus, for  $x \in (-1, 1)$ ,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

| $f(x)$     | $\sin^{-1} x$            | $\cos^{-1} x$             | $\tan^{-1} x$     |
|------------|--------------------------|---------------------------|-------------------|
| $f'(x)$    | $\frac{1}{\sqrt{1-x^2}}$ | $-\frac{1}{\sqrt{1-x^2}}$ | $\frac{1}{1+x^2}$ |
| Domain off | $(-1, 1)$                | $(-1, 1)$                 | $\mathbb{R}$      |

### EXERCISE 5.3

Find  $\frac{dy}{dx}$  in the following:

1.  $2x + 3y = \sin x$
2.  $2x + 3y = \sin y$
3.  $ax + by^2 = \cos y$
4.  $xy + y^2 = \tan x + y$
5.  $x^2 + xy + y^2 = 100$
6.  $x^3 + x^2y + xy^2 + y^3 = 81$
7.  $\sin^2 y + \cos xy = \kappa$
8.  $\sin^2 x + \cos^2 y = 1$
9.  $y = \sin^{-1} \left( \frac{2x}{1+x^2} \right)$
10.  $y = \tan^{-1} \left( \frac{3x - x^3}{1 - 3x^2} \right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$
11.  $y = \cos^{-1} \left( \frac{1 - x^2}{1 + x^2} \right), 0 < x < 1$
12.  $y = \sin^{-1} \left( \frac{1 - x^2}{1 + x^2} \right), 0 < x < 1$
13.  $y = \cos^{-1} \left( \frac{2x}{1 + x^2} \right), -1 < x < 1$
14.  $y = \sin^{-1} \left( 2x \sqrt{1 - x^2} \right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
15.  $y = \sec^{-1} \left( \frac{1}{2x^2 - 1} \right), 0 < x < \frac{1}{\sqrt{2}}$

### 5.4 Exponential and Logarithmic Functions

Till now we have learnt some aspects of different classes of functions like polynomial functions, rational functions and trigonometric functions. In this section, we shall learn about a new class of (related) functions called exponential functions and logarithmic functions. It needs to be emphasized that many statements made in this section are motivational and precise proofs of these are well beyond the scope of this text.

The Fig 5.9 gives a sketch of  $y = f_1(x) = x$ ,  $y = f_2(x) = x^2$ ,  $y = f_3(x) = x^3$  and  $y = f_4(x) = x^4$ . Observe that the curves get steeper as the power of  $x$  increases. Steeper the curve, faster is the rate of growth. What this means is that for a fixed increment in the value of  $x (> 1)$ , the increment in the value of  $y = f_n(x)$  increases as  $n$  increases for  $n = 1, 2, 3, 4$ . It is conceivable that such a statement is true for all positive values of  $n$ ,

where  $f_n(x) = x^n$ . Essentially, this means that the graph of  $y = f_n(x)$  leans more towards the  $y$ -axis as  $n$  increases. For example, consider  $f_{10}(x) = x^{10}$  and  $f_{15}(x) = x^{15}$ . If  $x$  increases from 1 to 2,  $f_{10}$  increases from 1 to  $2^{10}$  whereas  $f_{15}$  increases from 1 to  $2^{15}$ . Thus, for the same increment in  $x$ ,  $f_{15}$  grows faster than  $f_{10}$ .

Upshot of the above discussion is that the growth of polynomial functions is dependent on the degree of the polynomial function – higher the degree, greater is the growth. The next natural question is:

Is there a function which grows faster than any polynomial function. The answer is in affirmative and an example of such a function is

$$y = f(x) = 10^x.$$

Our claim is that this function  $f$  grows faster than  $f_n(x) = x^n$  for any positive integer  $n$ . For example, we can prove that  $10^x$  grows faster than  $f_{100}(x) = x^{100}$ . For large values of  $x$  like  $x = 10^3$ , note that  $f_{100}(x) = (10^3)^{100} = 10^{300}$  whereas  $f(10^3) = 10^{10^3} = 10^{1000}$ . Clearly  $f(x)$  is much greater than  $f_{100}(x)$ . It is not difficult to prove that for all  $x > 10^3$ ,  $f(x) > f_{100}(x)$ . But we will not attempt to give a proof of this here. Similarly, by choosing large values of  $x$ , one can verify that  $f(x)$  grows faster than  $f_n(x)$  for any positive integer  $n$ .

**Definition 3** The exponential function with positive base  $b > 1$  is the function

$$y = f(x) = b^x$$

The graph of  $y = 10^x$  is given in the Fig 5.9.

It is advised that the reader plots this graph for particular values of  $b$  like 2, 3 and 4. Following are some of the salient features of the exponential functions:

- (1) Domain of the exponential function is  $\mathbf{R}$ , the set of all real numbers.
- (2) Range of the exponential function is the set of all positive real numbers.
- (3) The point  $(0, 1)$  is always on the graph of the exponential function (this is a restatement of the fact that  $b^0 = 1$  for any real  $b > 1$ ).
- (4) Exponential function is ever increasing; i.e., as we move from left to right, the graph rises above.

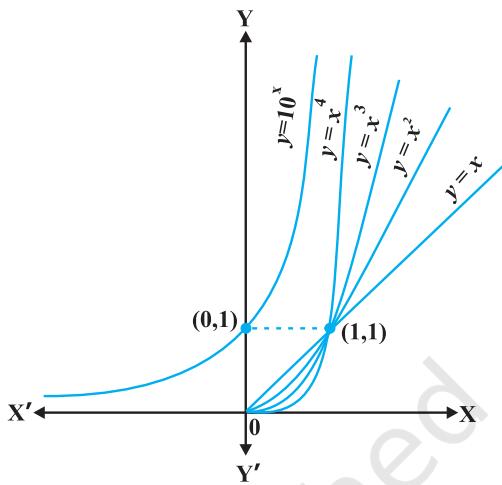


Fig 5.9

- (5) For very large negative values of  $x$ , the exponential function is very close to 0. In other words, in the second quadrant, the graph approaches  $x$ -axis (but never meets it).

Exponential function with base 10 is called the *common exponential function*. In the Appendix A.1.4 of Class XI, it was observed that the sum of the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \dots$$

is a number between 2 and 3 and is denoted by  $e$ . Using this  $e$  as the base we obtain an extremely important exponential function  $y = e^x$ .

This is called *natural exponential function*.

It would be interesting to know if the inverse of the exponential function exists and has *nice* interpretation. This search motivates the following definition.

**Definition 4** Let  $b > 1$  be a real number. Then we say logarithm of  $a$  to base  $b$  is  $x$  if  $b^x = a$ .

Logarithm of  $a$  to base  $b$  is denoted by  $\log_b a$ . Thus  $\log_b a = x$  if  $b^x = a$ . Let us work with a few explicit examples to get a feel for this. We know  $2^3 = 8$ . In terms of logarithms, we may rewrite this as  $\log_2 8 = 3$ . Similarly,  $10^4 = 10000$  is equivalent to saying  $\log_{10} 10000 = 4$ . Also,  $625 = 5^4 = 25^2$  is equivalent to saying  $\log_5 625 = 4$  or  $\log_{25} 625 = 2$ .

On a slightly more mature note, fixing a base  $b > 1$ , we may look at logarithm as a function from positive real numbers to all real numbers. This function, called the *logarithmic function*, is defined by

$$\begin{aligned} \log_b : \mathbf{R}^+ &\rightarrow \mathbf{R} \\ x \rightarrow \log_b x &= y \quad \text{if } b^y = x \end{aligned}$$

As before if the base  $b = 10$ , we say it is *common logarithms* and if  $b = e$ , then we say it is *natural logarithms*. Often natural logarithm is denoted by  $\ln$ . In this chapter,  $\log x$  denotes the logarithm function to base  $e$ , i.e.,  $\ln x$  will be written as simply  $\log x$ . The Fig 5.10 gives the plots of logarithm function to base 2,  $e$  and 10.

Some of the important observations about the logarithm function to any base  $b > 1$  are listed below:

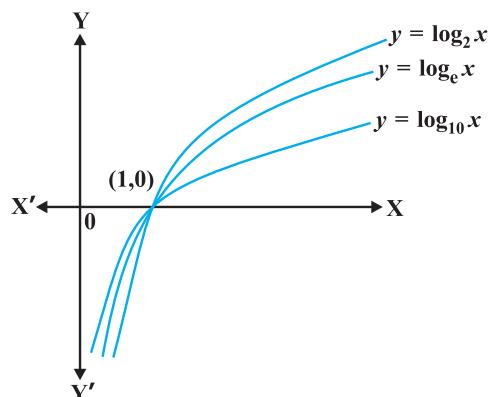


Fig 5.10

- (1) We cannot make a meaningful definition of logarithm of non-positive numbers and hence the domain of log function is  $\mathbf{R}^+$ .
- (2) The range of log function is the set of all real numbers.
- (3) The point  $(1, 0)$  is always on the graph of the log function.
- (4) The log function is ever increasing, i.e., as we move from left to right the graph rises above.
- (5) For  $x$  very near to zero, the value of  $\log x$  can be made lesser than any given real number. In other words in the fourth quadrant the graph approaches  $y$ -axis (but never meets it).
- (6) Fig 5.11 gives the plot of  $y = e^x$  and  $y = \ln x$ . It is of interest to observe that the two curves are the mirror images of each other reflected in the line  $y = x$ .

Two properties of 'log' functions are proved below:

- (1) There is a standard change of base rule to obtain  $\log_a p$  in terms of  $\log_b p$ . Let  $\log_a p = \alpha$ ,  $\log_b p = \beta$  and  $\log_b a = \gamma$ . This means  $a^\alpha = p$ ,  $b^\beta = p$  and  $b^\gamma = a$ . Substituting the third equation in the first one, we have

$$(b^\gamma)^\alpha = b^{\gamma\alpha} = p$$

Using this in the second equation, we get

$$b^\beta = p = b^{\gamma\alpha}$$

which implies  $\beta = \alpha\gamma$  or  $\alpha = \frac{\beta}{\gamma}$ . But then

$$\log_a p = \frac{\log_b p}{\log_b a}$$

- (2) Another interesting property of the log function is its effect on products. Let  $\log_b pq = \alpha$ . Then  $b^\alpha = pq$ . If  $\log_b p = \beta$  and  $\log_b q = \gamma$ , then  $b^\beta = p$  and  $b^\gamma = q$ . But then  $b^\alpha = pq = b^\beta b^\gamma = b^{\beta + \gamma}$

which implies  $\alpha = \beta + \gamma$ , i.e.,

$$\log_b pq = \log_b p + \log_b q$$

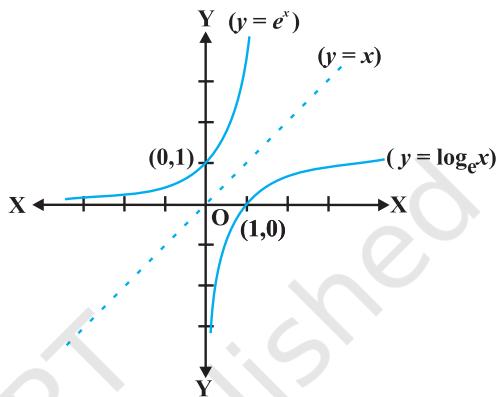


Fig 5.11

A particularly interesting and important consequence of this is when  $p = q$ . In this case the above may be rewritten as

$$\log_b p^2 = \log_b p + \log_b p = 2 \log_b p$$

An easy generalisation of this (left as an exercise!) is

$$\log_b p^n = n \log_b p$$

for any positive integer  $n$ . In fact this is true for any real number  $n$ , but we will not attempt to prove this. On the similar lines the reader is invited to verify

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

**Example 25** Is it true that  $x = e^{\log x}$  for all real  $x$ ?

**Solution** First, observe that the domain of log function is set of all positive real numbers. So the above equation is not true for non-positive real numbers. Now, let  $y = e^{\log x}$ . If  $y > 0$ , we may take logarithm which gives us  $\log y = \log(e^{\log x}) = \log x \cdot \log e = \log x$ . Thus  $y = x$ . Hence  $x = e^{\log x}$  is true only for positive values of  $x$ .

One of the striking properties of the natural exponential function in differential calculus is that it doesn't change during the process of differentiation. This is captured in the following theorem whose proof we skip.

**Theorem 5\***

- (1) The derivative of  $e^x$  w.r.t.,  $x$  is  $e^x$ ; i.e.,  $\frac{d}{dx}(e^x) = e^x$ .
- (2) The derivative of  $\log x$  w.r.t.,  $x$  is  $\frac{1}{x}$ ; i.e.,  $\frac{d}{dx}(\log x) = \frac{1}{x}$ .

**Example 26** Differentiate the following w.r.t.  $x$ :

- (i)  $e^{-x}$
- (ii)  $\sin(\log x)$ ,  $x > 0$
- (iii)  $\cos^{-1}(e^x)$
- (iv)  $e^{\cos x}$

**Solution**

- (i) Let  $y = e^{-x}$ . Using chain rule, we have

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

- (ii) Let  $y = \sin(\log x)$ . Using chain rule, we have

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

---

\* Please see supplementary material on Page 222.

(iii) Let  $y = \cos^{-1}(e^x)$ . Using chain rule, we have

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x) = \frac{-e^x}{\sqrt{1-e^{2x}}}$$

(iv) Let  $y = e^{\cos x}$ . Using chain rule, we have

$$\frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -(\sin x) e^{\cos x}$$

### EXERCISE 5.4

Differentiate the following w.r.t.  $x$ :

1.  $\frac{e^x}{\sin x}$

2.  $e^{\sin^{-1} x}$

3.  $e^{x^3}$

4.  $\sin(\tan^{-1} e^{-x})$

5.  $\log(\cos e^x)$

6.  $e^x + e^{x^2} + \dots + e^{x^5}$

7.  $\sqrt{e^{\sqrt{x}}}, x > 0$

8.  $\log(\log x), x > 1$

9.  $\frac{\cos x}{\log x}, x > 0$

10.  $\cos(\log x + e^x), x > 0$

## 5.5. Logarithmic Differentiation

In this section, we will learn to differentiate certain special class of functions given in the form

$$y = f(x) = [u(x)]^{v(x)}$$

By taking logarithm (to base  $e$ ) the above may be rewritten as

$$\log y = v(x) \log [u(x)]$$

Using chain rule we may differentiate this to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = v(x) \cdot \frac{1}{u(x)} \cdot u'(x) + v'(x) \cdot \log [u(x)]$$

which implies that

$$\frac{dy}{dx} = y \left[ \frac{v(x)}{u(x)} \cdot u'(x) + v'(x) \cdot \log [u(x)] \right]$$

The main point to be noted in this method is that  $f(x)$  and  $u(x)$  must always be positive as otherwise their logarithms are not defined. This process of differentiation is known as *logarithms differentiation* and is illustrated by the following examples:

**Example 27** Differentiate  $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$  w.r.t.  $x$ .

**Solution** Let  $y = \sqrt{\frac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$

Taking logarithm on both sides, we have

$$\log y = \frac{1}{2} [\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)]$$

Now, differentiating both sides w.r.t.  $x$ , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \left[ \frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5} \right]$$

$$\text{or } \frac{dy}{dx} = \frac{y}{2} \left[ \frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5} \right]$$

$$= \frac{1}{2} \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}} \left[ \frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5} \right]$$

**Example 28** Differentiate  $a^x$  w.r.t.  $x$ , where  $a$  is a positive constant.

**Solution** Let  $y = a^x$ . Then

$$\log y = x \log a$$

Differentiating both sides w.r.t.  $x$ , we have

$$\frac{1}{y} \frac{dy}{dx} = \log a$$

$$\text{or } \frac{dy}{dx} = y \log a$$

$$\text{Thus } \frac{d}{dx}(a^x) = a^x \log a$$

$$\begin{aligned} \text{Alternatively } \frac{d}{dx}(a^x) &= \frac{d}{dx}(e^{x \log a}) = e^{x \log a} \frac{d}{dx}(x \log a) \\ &= e^{x \log a} \cdot \log a = a^x \log a. \end{aligned}$$

**Example 29** Differentiate  $x^{\sin x}$ ,  $x > 0$  w.r.t.  $x$ .

**Solution** Let  $y = x^{\sin x}$ . Taking logarithm on both sides, we have

$$\log y = \sin x \log x$$

Therefore

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

or

$$\frac{1}{y} \frac{dy}{dx} = (\sin x) \frac{1}{x} + \log x \cos x$$

or

$$\frac{dy}{dx} = y \left[ \frac{\sin x}{x} + \cos x \log x \right]$$

$$= x^{\sin x} \left[ \frac{\sin x}{x} + \cos x \log x \right]$$

$$= x^{\sin x - 1} \cdot \sin x + x^{\sin x} \cdot \cos x \log x$$

**Example 30** Find  $\frac{dy}{dx}$ , if  $y^x + x^y + x^x = a^b$ .

**Solution** Given that  $y^x + x^y + x^x = a^b$ .

Putting  $u = y^x$ ,  $v = x^y$  and  $w = x^x$ , we get  $u + v + w = a^b$

Therefore

$$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \quad \dots (1)$$

Now,  $u = y^x$ . Taking logarithm on both sides, we have

$$\log u = x \log y$$

Differentiating both sides w.r.t.  $x$ , we have

$$\frac{1}{u} \cdot \frac{du}{dx} = x \frac{d}{dx}(\log y) + \log y \frac{d}{dx}(x)$$

$$= x \frac{1}{y} \cdot \frac{dy}{dx} + \log y \cdot 1$$

So

$$\frac{du}{dx} = u \left( \frac{x}{y} \frac{dy}{dx} + \log y \right) = y^x \left[ \frac{x}{y} \frac{dy}{dx} + \log y \right] \quad \dots (2)$$

Also  $v = x^y$

Taking logarithm on both sides, we have

$$\log v = y \log x$$

Differentiating both sides w.r.t.  $x$ , we have

$$\frac{1}{v} \cdot \frac{dv}{dx} = y \frac{d}{dx}(\log x) + \log x \frac{dy}{dx}$$

$$= y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

So

$$\frac{dv}{dx} = v \left[ \frac{y}{x} + \log x \frac{dy}{dx} \right]$$

$$= x^y \left[ \frac{y}{x} + \log x \frac{dy}{dx} \right]$$

... (3)

Again

$$w = x^x$$

Taking logarithm on both sides, we have

$$\log w = x \log x.$$

Differentiating both sides w.r.t.  $x$ , we have

$$\frac{1}{w} \cdot \frac{dw}{dx} = x \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{x} + \log x \cdot 1$$

i.e.

$$\frac{dw}{dx} = w (1 + \log x)$$

$$= x^x (1 + \log x)$$

... (4)

From (1), (2), (3), (4), we have

$$y^x \left( \frac{x}{y} \frac{dy}{dx} + \log y \right) + x^y \left( \frac{y}{x} + \log x \frac{dy}{dx} \right) + x^x (1 + \log x) = 0$$

or  $(x \cdot y^{x-1} + x^y \cdot \log x) \frac{dy}{dx} = -x^x (1 + \log x) - y \cdot x^{y-1} - y^x \log y$

Therefore  $\frac{dy}{dx} = \frac{-[y^x \log y + y \cdot x^{y-1} + x^x (1 + \log x)]}{x \cdot y^{x-1} + x^y \log x}$

## EXERCISE 5.5

Differentiate the functions given in Exercises 1 to 11 w.r.t.  $x$ .

1.  $\cos x \cdot \cos 2x \cdot \cos 3x$

2.  $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

3.  $(\log x)^{\cos x}$

4.  $x^x - 2^{\sin x}$

5.  $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

6.  $\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$

7.  $(\log x)^x + x^{\log x}$

8.  $(\sin x)^x + \sin^{-1} \sqrt{x}$

9.  $x^{\sin x} + (\sin x)^{\cos x}$

10.  $x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

11.  $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Find  $\frac{dy}{dx}$  of the functions given in Exercises 12 to 15.

12.  $x^y + y^x = 1$

13.  $y^x = x^y$

14.  $(\cos x)^y = (\cos y)^x$

15.  $xy = e^{(x-y)}$

16. Find the derivative of the function given by  $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$  and hence find  $f'(1)$ .

17. Differentiate  $(x^2 - 5x + 8)(x^3 + 7x + 9)$  in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

18. If  $u, v$  and  $w$  are functions of  $x$ , then show that

$$\frac{d}{dx} (u \cdot v \cdot w) = \frac{du}{dx} v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

## 5.6 Derivatives of Functions in Parametric Forms

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between

them is expressed via a third variable. The third variable is called the parameter. More precisely, a relation expressed between two variables  $x$  and  $y$  in the form  $x = f(t)$ ,  $y = g(t)$  is said to be parametric form with  $t$  as a parameter.

In order to find derivative of function in such form, we have by chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left( \text{whenever } \frac{dx}{dt} \neq 0 \right)$$

Thus

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \left( \text{as } \frac{dy}{dt} = g'(t) \text{ and } \frac{dx}{dt} = f'(t) \right) \left[ \text{provided } f'(t) \neq 0 \right]$$

**Example 31** Find  $\frac{dy}{dx}$ , if  $x = a \cos \theta$ ,  $y = a \sin \theta$ .

**Solution** Given that

$$x = a \cos \theta, y = a \sin \theta$$

Therefore

$$\frac{dx}{d\theta} = -a \sin \theta, \frac{dy}{d\theta} = a \cos \theta$$

Hence

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta$$

**Example 32** Find  $\frac{dy}{dx}$ , if  $x = at^2$ ,  $y = 2at$ .

**Solution** Given that  $x = at^2$ ,  $y = 2at$

So

$$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

**Example 33** Find  $\frac{dy}{dx}$ , if  $x = a(\theta + \sin \theta)$ ,  $y = a(1 - \cos \theta)$ .

**Solution** We have  $\frac{dx}{d\theta} = a(1 + \cos \theta)$ ,  $\frac{dy}{d\theta} = a(\sin \theta)$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \sin \theta}{a(1 + \cos \theta)} = \tan \frac{\theta}{2}$$

**Note** It may be noted here that  $\frac{dy}{dx}$  is expressed in terms of parameter only without directly involving the main variables  $x$  and  $y$ .

**Example 34** Find  $\frac{dy}{dx}$ , if  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ .

**Solution** Let  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$ . Then

$$\begin{aligned} x^{\frac{2}{3}} + y^{\frac{2}{3}} &= (a \cos^3 \theta)^{\frac{2}{3}} + (a \sin^3 \theta)^{\frac{2}{3}} \\ &= a^{\frac{2}{3}} (\cos^2 \theta + \sin^2 \theta) = a^{\frac{2}{3}} \end{aligned}$$

Hence,  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$  is parametric equation of  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Now  $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$  and  $\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$

Therefore  $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\tan \theta = -\sqrt[3]{\frac{y}{x}}$

## EXERCISE 5.6

If  $x$  and  $y$  are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find  $\frac{dy}{dx}$ .

1.  $x = 2at^2, y = at^4$
2.  $x = a \cos \theta, y = b \cos \theta$
3.  $x = \sin t, y = \cos 2t$
4.  $x = 4t, y = \frac{4}{t}$
5.  $x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$
6.  $x = a(\theta - \sin \theta), y = a(1 + \cos \theta)$
7.  $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
8.  $x = a \left( \cos t + \log \tan \frac{t}{2} \right), y = a \sin t$
9.  $x = a \sec \theta, y = b \tan \theta$
10.  $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$
11. If  $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$ , show that  $\frac{dy}{dx} = -\frac{y}{x}$

## 5.7 Second Order Derivative

Let  $y = f(x)$ . Then

$$\frac{dy}{dx} = f'(x) \quad \dots (1)$$

If  $f'(x)$  is differentiable, we may differentiate (1) again w.r.t.  $x$ . Then, the left hand side becomes  $\frac{d}{dx} \left( \frac{dy}{dx} \right)$  which is called the *second order derivative* of  $y$  w.r.t.  $x$  and is denoted by  $\frac{d^2 y}{dx^2}$ . The second order derivative of  $f(x)$  is denoted by  $f''(x)$ . It is also denoted by  $D^2 y$  or  $y''$  or  $y_2$  if  $y = f(x)$ . We remark that higher order derivatives may be defined similarly.

**Example 35** Find  $\frac{d^2y}{dx^2}$ , if  $y = x^3 + \tan x$ .

**Solution** Given that  $y = x^3 + \tan x$ . Then

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

Therefore

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} (3x^2 + \sec^2 x) \\ &= 6x + 2 \sec x \cdot \sec x \tan x = 6x + 2 \sec^2 x \tan x\end{aligned}$$

**Example 36** If  $y = A \sin x + B \cos x$ , then prove that  $\frac{d^2y}{dx^2} + y = 0$ .

**Solution** We have

$$\frac{dy}{dx} = A \cos x - B \sin x$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} (A \cos x - B \sin x) \\ &= -A \sin x - B \cos x = -y\end{aligned}$$

Hence

$$\frac{d^2y}{dx^2} + y = 0$$

**Example 37** If  $y = 3e^{2x} + 2e^{3x}$ , prove that  $\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$ .

**Solution** Given that  $y = 3e^{2x} + 2e^{3x}$ . Then

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6(e^{2x} + e^{3x})$$

Therefore

$$\frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x} = 6(2e^{2x} + 3e^{3x})$$

Hence

$$\begin{aligned}\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y &= 6(2e^{2x} + 3e^{3x}) \\ &\quad - 30(e^{2x} + e^{3x}) + 6(3e^{2x} + 2e^{3x}) = 0\end{aligned}$$

**Example 38** If  $y = \sin^{-1} x$ , show that  $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 0$ .

**Solution** We have  $y = \sin^{-1} x$ . Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

or

$$\sqrt{(1-x^2)} \frac{dy}{dx} = 1$$

So

$$\frac{d}{dx} \left( \sqrt{(1-x^2)} \cdot \frac{dy}{dx} \right) = 0$$

or

$$\sqrt{(1-x^2)} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx} \left( \sqrt{(1-x^2)} \right) = 0$$

or

$$\sqrt{(1-x^2)} \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} \cdot \frac{2x}{2\sqrt{1-x^2}} = 0$$

Hence  $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 0$

**Alternatively**, Given that  $y = \sin^{-1} x$ , we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}, \text{ i.e., } (1-x^2) y_1^2 = 1$$

So  $(1-x^2) \cdot 2y_1 y_2 + y_1^2 (0-2x) = 0$

Hence  $(1-x^2) y_2 - xy_1 = 0$

### EXERCISE 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

- |   |                  |                     |
|---|------------------|---------------------|
| 1. $x^2 + 3x + 2$   | 2. $x^{20}$      | 3. $x \cdot \cos x$ |
| 4. $\log x$   | 5. $x^3 \log x$  | 6. $e^x \sin 5x$    |
| 7. $e^{6x} \cos 3x$   | 8. $\tan^{-1} x$ | 9. $\log(\log x)$   |
| 10. $\sin(\log x)$  |                  |                     |
| 11. If $y = 5 \cos x - 3 \sin x$ , prove that $\frac{d^2y}{dx^2} + y = 0$ |                  |                     |

12. If  $y = \cos^{-1} x$ , Find  $\frac{d^2y}{dx^2}$  in terms of  $y$  alone.
13. If  $y = 3 \cos(\log x) + 4 \sin(\log x)$ , show that  $x^2 y_2 + xy_1 + y = 0$
14. If  $y = Ae^{mx} + Be^{nx}$ , show that  $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$
15. If  $y = 500e^{7x} + 600e^{-7x}$ , show that  $\frac{d^2y}{dx^2} = 49y$
16. If  $e^y(x+1) = 1$ , show that  $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$
17. If  $y = (\tan^{-1} x)^2$ , show that  $(x^2 + 1)^2 y_2 + 2x(x^2 + 1) y_1 = 2$

### Miscellaneous Examples

**Example 39** Differentiate w.r.t.  $x$ , the following function:

(i)  $\sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}}$       (ii)  $\log_7(\log x)$

#### Solution

(i) Let  $y = \sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}} = (3x+2)^{\frac{1}{2}} + (2x^2+4)^{-\frac{1}{2}}$

Note that this function is defined at all real numbers  $x > -\frac{2}{3}$ . Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(3x+2)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(3x+2) + \left(-\frac{1}{2}\right)(2x^2+4)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(2x^2+4) \\ &= \frac{1}{2}(3x+2)^{-\frac{1}{2}} \cdot (3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}} \cdot 4x \\ &= \frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{\frac{3}{2}}} \end{aligned}$$

This is defined for all real numbers  $x > -\frac{2}{3}$ .

- (ii) Let  $y = \log_7(\log x) = \frac{\log(\log x)}{\log 7}$  (by change of base formula).

The function is defined for all real numbers  $x > 1$ . Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\log 7} \frac{d}{dx}(\log(\log x)) \\ &= \frac{1}{\log 7} \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \\ &= \frac{1}{x \log 7 \log x}\end{aligned}$$

**Example 40** Differentiate the following w.r.t.  $x$ .

- (i)  $\cos^{-1}(\sin x)$       (ii)  $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$       (iii)  $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

**Solution**

- (i) Let  $f(x) = \cos^{-1}(\sin x)$ . Observe that this function is defined for all real numbers. We may rewrite this function as

$$\begin{aligned}f(x) &= \cos^{-1}(\sin x) \\ &= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right] \\ &= \frac{\pi}{2} - x\end{aligned}$$

Thus  $f'(x) = -1$ .

- (ii) Let  $f(x) = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ . Observe that this function is defined for all real

numbers, where  $\cos x \neq -1$ ; i.e., at all odd multiplies of  $\pi$ . We may rewrite this function as

$$\begin{aligned}f(x) &= \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) \\ &= \tan^{-1}\left[\frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{2 \cos^2 \frac{x}{2}}\right]\end{aligned}$$

$$= \tan^{-1} \left[ \tan \left( \frac{x}{2} \right) \right] = \frac{x}{2}$$

Observe that we could cancel  $\cos \left( \frac{x}{2} \right)$  in both numerator and denominator as it is not equal to zero. Thus  $f'(x) = \frac{1}{2}$ .

(iii) Let  $f(x) = \sin^{-1} \left( \frac{2^{x+1}}{1+4^x} \right)$ . To find the domain of this function we need to find all

$x$  such that  $-1 \leq \frac{2^{x+1}}{1+4^x} \leq 1$ . Since the quantity in the middle is always positive,

we need to find all  $x$  such that  $\frac{2^{x+1}}{1+4^x} \leq 1$ , i.e., all  $x$  such that  $2^{x+1} \leq 1+4^x$ . We

may rewrite this as  $2 \leq \frac{1}{2^x} + 2^x$  which is true for all  $x$ . Hence the function is defined at every real number. By putting  $2^x = \tan \theta$ , this function may be rewritten as

$$\begin{aligned} f(x) &= \sin^{-1} \left[ \frac{2^{x+1}}{1+4^x} \right] \\ &= \sin^{-1} \left[ \frac{2^x \cdot 2}{1+(2^x)^2} \right] \\ &= \sin^{-1} \left[ \frac{2 \tan \theta}{1+\tan^2 \theta} \right] \\ &= \sin^{-1} [\sin 2\theta] \\ &= 2\theta = 2 \tan^{-1} (2^x) \end{aligned}$$

Thus

$$\begin{aligned} f'(x) &= 2 \cdot \frac{1}{1+(2^x)^2} \cdot \frac{d}{dx} (2^x) \\ &= \frac{2}{1+4^x} \cdot (2^x) \log 2 \\ &= \frac{2^{x+1} \log 2}{1+4^x} \end{aligned}$$

**Example 41** Find  $f'(x)$  if  $f(x) = (\sin x)^{\sin x}$  for all  $0 < x < \pi$ .

**Solution** The function  $y = (\sin x)^{\sin x}$  is defined for all positive real numbers. Taking logarithms, we have

$$\log y = \log (\sin x)^{\sin x} = \sin x \log (\sin x)$$

Then

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} (\sin x \log (\sin x)) \\ &= \cos x \log (\sin x) + \sin x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \\ &= \cos x \log (\sin x) + \cos x \\ &= (1 + \log (\sin x)) \cos x \end{aligned}$$

Thus

$$\frac{dy}{dx} = y((1 + \log (\sin x)) \cos x) = (1 + \log (\sin x)) (\sin x)^{\sin x} \cos x$$

**Example 42** For a positive constant  $a$  find  $\frac{dy}{dx}$ , where

$$y = a^{\frac{t+1}{t}}, \text{ and } x = \left(t + \frac{1}{t}\right)^a$$

**Solution** Observe that both  $y$  and  $x$  are defined for all real  $t \neq 0$ . Clearly

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt} \left(a^{\frac{t+1}{t}}\right) = a^{\frac{t+1}{t}} \frac{d}{dt} \left(t + \frac{1}{t}\right) \cdot \log a \\ &= a^{\frac{t+1}{t}} \left(1 - \frac{1}{t^2}\right) \log a \end{aligned}$$

Similarly

$$\begin{aligned} \frac{dx}{dt} &= a \left[t + \frac{1}{t}\right]^{a-1} \cdot \frac{d}{dt} \left(t + \frac{1}{t}\right) \\ &= a \left[t + \frac{1}{t}\right]^{a-1} \cdot \left(1 - \frac{1}{t^2}\right) \end{aligned}$$

$\frac{dx}{dt} \neq 0$  only if  $t \neq \pm 1$ . Thus for  $t \neq \pm 1$ ,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a^{\frac{t+1}{t}} \left(1 - \frac{1}{t^2}\right) \log a}{a \left[t + \frac{1}{t}\right]^{a-1} \cdot \left(1 - \frac{1}{t^2}\right)} \\ &= \frac{a^{\frac{t+1}{t}} \log a}{a \left(t + \frac{1}{t}\right)^{a-1}}\end{aligned}$$

**Example 43** Differentiate  $\sin^2 x$  w.r.t.  $e^{\cos x}$ .

**Solution** Let  $u(x) = \sin^2 x$  and  $v(x) = e^{\cos x}$ . We want to find  $\frac{du}{dv} = \frac{du/dx}{dv/dx}$ . Clearly

$$\frac{du}{dx} = 2 \sin x \cos x \text{ and } \frac{dv}{dx} = e^{\cos x} (-\sin x) = -(\sin x) e^{\cos x}$$

Thus

$$\frac{du}{dv} = \frac{2 \sin x \cos x}{-\sin x e^{\cos x}} = -\frac{2 \cos x}{e^{\cos x}}$$

### Miscellaneous Exercise on Chapter 5

Differentiate w.r.t.  $x$  the function in Exercises 1 to 11.

1.  $(3x^2 - 9x + 5)^9$
2.  $\sin^3 x + \cos^6 x$
3.  $(5x)^3 \cos 2x$
4.  $\sin^{-1}(x \sqrt{x})$ ,  $0 \leq x \leq 1$
5.  $\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}$ ,  $-2 < x < 2$
6.  $\cot^{-1} \left[ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right]$ ,  $0 < x < \frac{\pi}{2}$
7.  $(\log x)^{\log x}$ ,  $x > 1$
8.  $\cos(a \cos x + b \sin x)$ , for some constant  $a$  and  $b$ .
9.  $(\sin x - \cos x)^{(\sin x - \cos x)}$ ,  $\frac{\pi}{4} < x < \frac{3\pi}{4}$
10.  $x^x + x^a + a^x + a^a$ , for some fixed  $a > 0$  and  $x > 0$

11.  $x^{x^2-3} + (x-3)^{x^2}$ , for  $x > 3$
12. Find  $\frac{dy}{dx}$ , if  $y = 12(1 - \cos t)$ ,  $x = 10(t - \sin t)$ ,  $-\frac{\pi}{2} < t < \frac{\pi}{2}$
13. Find  $\frac{dy}{dx}$ , if  $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$ ,  $0 < x < 1$
14. If  $x\sqrt{1+y} + y\sqrt{1+x} = 0$ , for  $-1 < x < 1$ , prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

15. If  $(x-a)^2 + (y-b)^2 = c^2$ , for some  $c > 0$ , prove that

$$\frac{d^2y}{dx^2} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of  $a$  and  $b$ .

16. If  $\cos y = x \cos(a+y)$ , with  $\cos a \neq \pm 1$ , prove that  $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$ .
17. If  $x = a(\cos t + t \sin t)$  and  $y = a(\sin t - t \cos t)$ , find  $\frac{d^2y}{dx^2}$ .
18. If  $f(x) = |x|^3$ , show that  $f''(x)$  exists for all real  $x$  and find it.
19. Using the fact that  $\sin(A+B) = \sin A \cos B + \cos A \sin B$  and the differentiation, obtain the sum formula for cosines.
20. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

21. If  $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ , prove that  $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$
22. If  $y = e^{a \cos^{-1} x}$ ,  $-1 \leq x \leq 1$ , show that  $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$ .

### Summary

◆ A real valued function is **continuous** at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.

◆ Sum, difference, product and quotient of continuous functions are continuous. i.e., if  $f$  and  $g$  are continuous functions, then

$(f \pm g)(x) = f(x) \pm g(x)$  is continuous.

$(f \cdot g)(x) = f(x) \cdot g(x)$  is continuous.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$  (wherever  $g(x) \neq 0$ ) is continuous.

◆ Every differentiable function is continuous, but the converse is not true.

◆ Chain rule is rule to differentiate composites of functions. If  $f = v \circ u$ ,  $t = u(x)$

and if both  $\frac{dt}{dx}$  and  $\frac{dv}{dt}$  exist then

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

◆ Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(\log x) = \frac{1}{x}$$

◆ Logarithmic differentiation is a powerful technique to differentiate functions of the form  $f(x) = [u(x)]^{v(x)}$ . Here both  $f(x)$  and  $u(x)$  need to be positive for this technique to make sense.





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# APPLICATION OF DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature.” — WHITEHEAD* ❖

## 6.1 Introduction

In Chapter 5, we have learnt how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions. In this chapter, we will study applications of the derivative in various disciplines, e.g., in engineering, science, social science, and many other fields. For instance, we will learn how the derivative can be used (i) to determine rate of change of quantities, (ii) to find the equations of tangent and normal to a curve at a point, (iii) to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs. We will also use derivative to find intervals on which a function is increasing or decreasing. Finally, we use the derivative to find approximate value of certain quantities.

## 6.2 Rate of Change of Quantities

Recall that by the derivative  $\frac{ds}{dt}$ , we mean the rate of change of distance  $s$  with respect to the time  $t$ . In a similar fashion, whenever one quantity  $y$  varies with another quantity  $x$ , satisfying some rule  $y = f(x)$ , then  $\frac{dy}{dx}$  (or  $f'(x)$ ) represents the rate of

change of  $y$  with respect to  $x$  and  $\left. \frac{dy}{dx} \right|_{x=x_0}$  (or  $f'(x_0)$ ) represents the rate of change of  $y$  with respect to  $x$  at  $x = x_0$ .

Further, if two variables  $x$  and  $y$  are varying with respect to another variable  $t$ , i.e., if  $x = f(t)$  and  $y = g(t)$ , then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of  $y$  with respect to  $x$  can be calculated using the rate of change of  $y$  and that of  $x$  both with respect to  $t$ .

Let us consider some examples.

**Example 1** Find the rate of change of the area of a circle per second with respect to its radius  $r$  when  $r = 5$  cm.

**Solution** The area  $A$  of a circle with radius  $r$  is given by  $A = \pi r^2$ . Therefore, the rate of change of the area  $A$  with respect to its radius  $r$  is given by  $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$ .

When  $r = 5$  cm,  $\frac{dA}{dr} = 10\pi$ . Thus, the area of the circle is changing at the rate of  $10\pi$   $\text{cm}^2/\text{s}$ .

**Example 2** The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

**Solution** Let  $x$  be the length of a side,  $V$  be the volume and  $S$  be the surface area of the cube. Then,  $V = x^3$  and  $S = 6x^2$ , where  $x$  is a function of time  $t$ .

Now 
$$\frac{dV}{dt} = 9 \text{ cm}^3/\text{s} \text{ (Given)}$$

Therefore 
$$\begin{aligned} 9 &= \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3) \cdot \frac{dx}{dt} \text{ (By Chain Rule)} \\ &= 3x^2 \cdot \frac{dx}{dt} \end{aligned}$$

or 
$$\frac{dx}{dt} = \frac{3}{x^2} \quad \dots (1)$$

Now 
$$\begin{aligned} \frac{dS}{dt} &= \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2) \cdot \frac{dx}{dt} \text{ (By Chain Rule)} \\ &= 12x \cdot \left( \frac{3}{x^2} \right) = \frac{36}{x} \quad \text{(Using (1))} \end{aligned}$$

Hence, when  $x = 10$  cm,  $\frac{dS}{dt} = 3.6 \text{ cm}^2/\text{s}$

**Example 3** A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

**Solution** The area A of a circle with radius  $r$  is given by  $A = \pi r^2$ . Therefore, the rate of change of area A with respect to time  $t$  is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \quad (\text{By Chain Rule})$$

It is given that

$$\frac{dr}{dt} = 4 \text{ cm/s}$$

Therefore, when  $r = 10$  cm,  $\frac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Thus, the enclosed area is increasing at the rate of  $80\pi$  cm<sup>2</sup>/s, when  $r = 10$  cm.

**Note**  $\frac{dy}{dx}$  is positive if  $y$  increases as  $x$  increases and is negative if  $y$  decreases as  $x$  increases.

**Example 4** The length  $x$  of a rectangle is decreasing at the rate of 3 cm/minute and the width  $y$  is increasing at the rate of 2cm/minute. When  $x = 10$ cm and  $y = 6$ cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

**Solution** Since the length  $x$  is decreasing and the width  $y$  is increasing with respect to time, we have

$$\frac{dx}{dt} = -3 \text{ cm/min} \quad \text{and} \quad \frac{dy}{dt} = 2 \text{ cm/min}$$

(a) The perimeter P of a rectangle is given by

$$P = 2(x + y)$$

$$\text{Therefore } \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-3 + 2) = -2 \text{ cm/min}$$

(b) The area A of the rectangle is given by

$$A = x \cdot y$$

$$\begin{aligned} \text{Therefore } \frac{dA}{dt} &= \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} \\ &= -3(6) + 10(2) \quad (\text{as } x = 10 \text{ cm and } y = 6 \text{ cm}) \\ &= 2 \text{ cm}^2/\text{min} \end{aligned}$$

**Example 5** The total cost  $C(x)$  in Rupees, associated with the production of  $x$  units of an item is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

**Solution** Since marginal cost is the rate of change of total cost with respect to the output, we have

Marginal cost (MC) =  $\frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$

When  $x = 3$ ,  $MC = 0.015(3^2) - 0.04(3) + 30$   
 $= 0.135 - 0.12 + 30 = 30.015$

Hence, the required marginal cost is ₹ 30.02 (nearly).

**Example 6** The total revenue in Rupees received from the sale of  $x$  units of a product is given by  $R(x) = 3x^2 + 36x + 5$ . Find the marginal revenue, when  $x = 5$ , where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

**Solution** Since marginal revenue is the rate of change of total revenue with respect to the number of units sold, we have

Marginal Revenue (MR) =  $\frac{dR}{dx} = 6x + 36$

When  $x = 5$ ,  $MR = 6(5) + 36 = 66$

Hence, the required marginal revenue is ₹ 66.

### EXERCISE 6.1

- Find the rate of change of the area of a circle with respect to its radius  $r$  when
  - $r = 3$  cm
  - $r = 4$  cm
- The volume of a cube is increasing at the rate of  $8 \text{ cm}^3/\text{s}$ . How fast is the surface area increasing when the length of an edge is 12 cm?
- The radius of a circle is increasing uniformly at the rate of  $3 \text{ cm/s}$ . Find the rate at which the area of the circle is increasing when the radius is 10 cm.
- An edge of a variable cube is increasing at the rate of  $3 \text{ cm/s}$ . How fast is the volume of the cube increasing when the edge is 10 cm long?
- A stone is dropped into a quiet lake and waves move in circles at the speed of  $5 \text{ cm/s}$ . At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
7. The length  $x$  of a rectangle is decreasing at the rate of 5 cm/minute and the width  $y$  is increasing at the rate of 4 cm/minute. When  $x = 8$  cm and  $y = 6$  cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.
8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.
10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?
11. A particle moves along the curve  $6y = x^3 + 2$ . Find the points on the curve at which the  $y$ -coordinate is changing 8 times as fast as the  $x$ -coordinate.
12. The radius of an air bubble is increasing at the rate of  $\frac{1}{2}$  cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
13. A balloon, which always remains spherical, has a variable diameter  $\frac{3}{2}(2x + 1)$ . Find the rate of change of its volume with respect to  $x$ .
14. Sand is pouring from a pipe at the rate of 12 cm<sup>3</sup>/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
15. The total cost  $C(x)$  in Rupees associated with the production of  $x$  units of an item is given by  

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$
Find the marginal cost when 17 units are produced.
16. The total revenue in Rupees received from the sale of  $x$  units of a product is given by  

$$R(x) = 13x^2 + 26x + 15.$$
Find the marginal revenue when  $x = 7$ .

Choose the correct answer for questions 17 and 18.

17. The rate of change of the area of a circle with respect to its radius  $r$  at  $r = 6$  cm is  
 (A)  $10\pi$       (B)  $12\pi$       (C)  $8\pi$       (D)  $11\pi$

18. The total revenue in Rupees received from the sale of  $x$  units of a product is given by

$R(x) = 3x^2 + 36x + 5$ . The marginal revenue, when  $x = 15$  is

- (A) 116 (B) 96 (C) 90 (D) 126

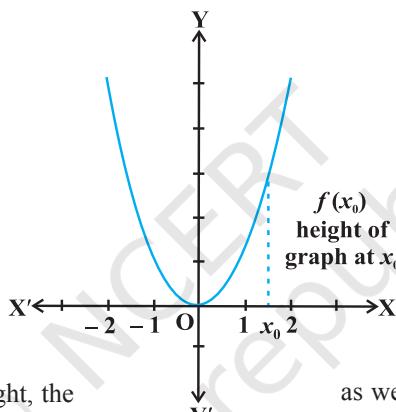
### 6.3 Increasing and Decreasing Functions

In this section, we will use differentiation to find out whether a function is increasing or decreasing or none.

Consider the function  $f$  given by  $f(x) = x^2$ ,  $x \in \mathbf{R}$ . The graph of this function is a parabola as given in Fig 6.1.

Values left to origin

| $x$            | $f(x) = x^2$  |
|----------------|---------------|
| -2             | 4             |
| $-\frac{3}{2}$ | $\frac{9}{4}$ |
| -1             | 1             |
| $-\frac{1}{2}$ | $\frac{1}{4}$ |
| 0              | 0             |



Values right to origin

| $x$           | $f(x) = x^2$  |
|---------------|---------------|
| 0             | 0             |
| $\frac{1}{2}$ | $\frac{1}{4}$ |
| 1             | 1             |
| $\frac{3}{2}$ | $\frac{9}{4}$ |
| 2             | 4             |

as we move from left to right, the height of the graph decreases

as we move from left to right, the height of the graph increases

Fig 6.1

First consider the graph (Fig 6.1) to the right of the origin. Observe that as we move from left to right along the graph, the height of the graph continuously increases. For this reason, the function is said to be increasing for the real numbers  $x > 0$ .

Now consider the graph to the left of the origin and observe here that as we move from left to right along the graph, the height of the graph continuously decreases. Consequently, the function is said to be decreasing for the real numbers  $x < 0$ .

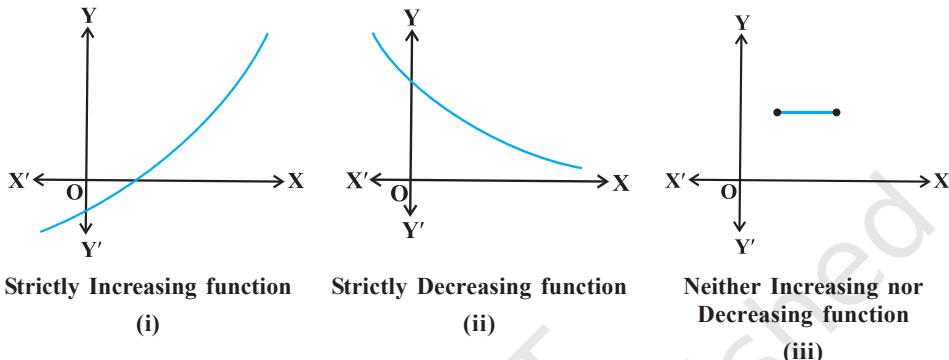
We shall now give the following analytical definitions for a function which is increasing or decreasing on an interval.

**Definition 1** Let  $I$  be an interval contained in the domain of a real valued function  $f$ . Then  $f$  is said to be

- increasing on  $I$  if  $x_1 < x_2$  in  $I \Rightarrow f(x_1) \leq f(x_2)$  for all  $x_1, x_2 \in I$ .
- decreasing on  $I$ , if  $x_1 < x_2$  in  $I \Rightarrow f(x_1) \geq f(x_2)$  for all  $x_1, x_2 \in I$ .
- constant on  $I$ , if  $f(x) = c$  for all  $x \in I$ , where  $c$  is a constant.

- (iv) strictly increasing on  $I$  if  $x_1 < x_2$  in  $I \Rightarrow f(x_1) < f(x_2)$  for all  $x_1, x_2 \in I$ .  
 (v) strictly decreasing on  $I$  if  $x_1 < x_2$  in  $I \Rightarrow f(x_1) > f(x_2)$  for all  $x_1, x_2 \in I$ .

For graphical representation of such functions see Fig 6.2.



**Fig 6.2**

We shall now define when a function is increasing or decreasing at a point.

**Definition 2** Let  $x_0$  be a point in the domain of definition of a real valued function  $f$ . Then  $f$  is said to be increasing, decreasing at  $x_0$  if there exists an open interval  $I$  containing  $x_0$  such that  $f$  is increasing, decreasing, respectively, in  $I$ .

Let us clarify this definition for the case of increasing function.

**Example 7** Show that the function given by  $f(x) = 7x - 3$  is increasing on  $\mathbf{R}$ .

**Solution** Let  $x_1$  and  $x_2$  be any two numbers in  $\mathbf{R}$ . Then

$$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, by Definition 1, it follows that  $f$  is strictly increasing on  $\mathbf{R}$ .

We shall now give the first derivative test for increasing and decreasing functions. The proof of this test requires the Mean Value Theorem studied in Chapter 5.

**Theorem 1** Let  $f$  be continuous on  $[a, b]$  and differentiable on the open interval  $(a, b)$ . Then

- $f$  is increasing in  $[a, b]$  if  $f'(x) \geq 0$  for each  $x \in (a, b)$
- $f$  is decreasing in  $[a, b]$  if  $f'(x) \leq 0$  for each  $x \in (a, b)$
- $f$  is a constant function in  $[a, b]$  if  $f'(x) = 0$  for each  $x \in (a, b)$

**Proof** (a) Let  $x_1, x_2 \in [a, b]$  be such that  $x_1 < x_2$ .

Then, by Mean Value Theorem (Theorem 8 in Chapter 5), there exists a point  $c$  between  $x_1$  and  $x_2$  such that

$$f(x_2) - f(x_1) = f'(c) (x_2 - x_1)$$

i.e.  $f(x_2) - f(x_1) > 0$  (as  $f'(c) > 0$  (given))

i.e.  $f(x_2) > f(x_1)$

Thus, we have

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \text{ for all } x_1, x_2 \in [a, b]$$

Hence,  $f$  is an increasing function in  $[a, b]$ .

The proofs of part (b) and (c) are similar. It is left as an exercise to the reader.

### Remarks

There is a more generalised theorem, which states that if  $f'(x) > 0$  for  $x$  in an interval excluding the end points and  $f$  is continuous in the interval, then  $f$  is increasing. Similarly, if  $f'(x) < 0$  for  $x$  in an interval excluding the end points and  $f$  is continuous in the interval, then  $f$  is decreasing.

**Example 8** Show that the function  $f$  given by

$$f(x) = x^3 - 3x^2 + 4x, x \in \mathbf{R}$$

is increasing on  $\mathbf{R}$ .

**Solution** Note that

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 4 \\ &= 3(x^2 - 2x + 1) + 1 \\ &= 3(x - 1)^2 + 1 > 0, \text{ in every interval of } \mathbf{R} \end{aligned}$$

Therefore, the function  $f$  is increasing on  $\mathbf{R}$ .

**Example 9** Prove that the function given by  $f(x) = \cos x$  is

- (a) decreasing in  $(0, \pi)$
- (b) increasing in  $(\pi, 2\pi)$ , and
- (c) neither increasing nor decreasing in  $(0, 2\pi)$ .

**Solution** Note that  $f'(x) = -\sin x$

- Since for each  $x \in (0, \pi)$ ,  $\sin x > 0$ , we have  $f'(x) < 0$  and so  $f$  is decreasing in  $(0, \pi)$ .
- Since for each  $x \in (\pi, 2\pi)$ ,  $\sin x < 0$ , we have  $f'(x) > 0$  and so  $f$  is increasing in  $(\pi, 2\pi)$ .
- Clearly by (a) and (b) above,  $f$  is neither increasing nor decreasing in  $(0, 2\pi)$ .

**Example 10** Find the intervals in which the function  $f$  given by  $f(x) = x^2 - 4x + 6$  is

- increasing
- decreasing

**Solution** We have

$$\begin{aligned} f(x) &= x^2 - 4x + 6 \\ \text{or} \quad f'(x) &= 2x - 4 \end{aligned}$$



Fig 6.3

Therefore,  $f'(x) = 0$  gives  $x = 2$ . Now the point  $x = 2$  divides the real line into two disjoint intervals namely,  $(-\infty, 2)$  and  $(2, \infty)$  (Fig 6.3). In the interval  $(-\infty, 2)$ ,  $f'(x) = 2x - 4 < 0$ .

Therefore,  $f$  is decreasing in this interval. Also, in the interval  $(2, \infty)$ ,  $f'(x) > 0$  and so the function  $f$  is increasing in this interval.

**Example 11** Find the intervals in which the function  $f$  given by  $f(x) = 4x^3 - 6x^2 - 72x + 30$  is (a) increasing (b) decreasing.

**Solution** We have

$$\begin{aligned} f(x) &= 4x^3 - 6x^2 - 72x + 30 \\ \text{or} \quad f'(x) &= 12x^2 - 12x - 72 \\ &= 12(x^2 - x - 6) \\ &= 12(x - 3)(x + 2) \end{aligned}$$

Therefore,  $f'(x) = 0$  gives  $x = -2, 3$ . The points  $x = -2$  and  $x = 3$  divides the real line into three disjoint intervals, namely,  $(-\infty, -2)$ ,  $(-2, 3)$  and  $(3, \infty)$ .



Fig 6.4

In the intervals  $(-\infty, -2)$  and  $(3, \infty)$ ,  $f'(x)$  is positive while in the interval  $(-2, 3)$ ,  $f'(x)$  is negative. Consequently, the function  $f$  is increasing in the intervals  $(-\infty, -2)$  and  $(3, \infty)$  while the function is decreasing in the interval  $(-2, 3)$ . However,  $f$  is neither increasing nor decreasing in  $\mathbf{R}$ .

| Interval        | Sign of $f'(x)$ | Nature of function $f$ |
|-----------------|-----------------|------------------------|
| $(-\infty, -2)$ | $(-) (-) > 0$   | $f$ is increasing      |
| $(-2, 3)$       | $(-) (+) < 0$   | $f$ is decreasing      |
| $(3, \infty)$   | $(+) (+) > 0$   | $f$ is increasing      |

**Example 12** Find intervals in which the function given by  $f(x) = \sin 3x$ ,  $x \in \left[0, \frac{\pi}{2}\right]$  is  
 (a) increasing (b) decreasing.

**Solution** We have

$$f(x) = \sin 3x$$

or

$$f'(x) = 3\cos 3x$$

Therefore,  $f'(x) = 0$  gives  $\cos 3x = 0$  which in turn gives  $3x = \frac{\pi}{2}, \frac{3\pi}{2}$  (as  $x \in \left[0, \frac{\pi}{2}\right]$ )

implies  $3x \in \left[0, \frac{3\pi}{2}\right]$ . So  $x = \frac{\pi}{6}$  and  $\frac{\pi}{2}$ . The point  $x = \frac{\pi}{6}$  divides the interval  $\left[0, \frac{\pi}{2}\right]$  into two disjoint intervals  $\left[0, \frac{\pi}{6}\right]$  and  $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ .

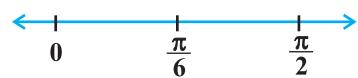


Fig 6.5

Now,  $f'(x) > 0$  for all  $x \in \left[0, \frac{\pi}{6}\right]$  as  $0 \leq x < \frac{\pi}{6} \Rightarrow 0 \leq 3x < \frac{\pi}{2}$  and  $f'(x) < 0$  for all  $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$  as  $\frac{\pi}{6} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3x < \frac{3\pi}{2}$ .

Therefore,  $f$  is increasing in  $\left[0, \frac{\pi}{6}\right]$  and decreasing in  $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ .

Also, the given function is continuous at  $x=0$  and  $x=\frac{\pi}{6}$ . Therefore, by Theorem 1,

$f$  is increasing on  $\left[0, \frac{\pi}{6}\right]$  and decreasing on  $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ .

**Example 13** Find the intervals in which the function  $f$  given by

$$f(x) = \sin x + \cos x, 0 \leq x \leq 2\pi$$

is increasing or decreasing.

**Solution** We have

$$f(x) = \sin x + \cos x,$$

or

$$f'(x) = \cos x - \sin x$$

Now  $f'(x) = 0$  gives  $\sin x = \cos x$  which gives that  $x = \frac{\pi}{4}, \frac{5\pi}{4}$  as  $0 \leq x \leq 2\pi$

The points  $x = \frac{\pi}{4}$  and  $x = \frac{5\pi}{4}$  divide the interval  $[0, 2\pi]$  into three disjoint intervals,

namely,  $\left[0, \frac{\pi}{4}\right)$ ,  $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$  and  $\left(\frac{5\pi}{4}, 2\pi\right]$ .

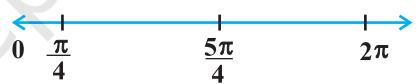


Fig 6.6

Note that  $f'(x) > 0$  if  $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

or  $f$  is increasing in the intervals  $\left[0, \frac{\pi}{4}\right)$  and  $\left(\frac{5\pi}{4}, 2\pi\right]$

Also  $f'(x) < 0$  if  $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

or  $f$  is decreasing in  $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

| Interval                                     | Sign of $f'(x)$ | Nature of function |
|--|-----------------|--------------------|
| $\left[0, \frac{\pi}{4}\right)$              | $> 0$           | $f$ is increasing  |
| $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ | $< 0$           | $f$ is decreasing  |
| $\left(\frac{5\pi}{4}, 2\pi\right]$          | $> 0$           | $f$ is increasing  |

### EXERCISE 6.2

1. Show that the function given by  $f(x) = 3x + 17$  is increasing on  $\mathbf{R}$ .
2. Show that the function given by  $f(x) = e^{2x}$  is increasing on  $\mathbf{R}$ .
3. Show that the function given by  $f(x) = \sin x$  is
  - increasing in  $\left(0, \frac{\pi}{2}\right)$
  - decreasing in  $\left(\frac{\pi}{2}, \pi\right)$
  - neither increasing nor decreasing in  $(0, \pi)$
4. Find the intervals in which the function  $f$  given by  $f(x) = 2x^2 - 3x$  is
  - increasing
  - decreasing
5. Find the intervals in which the function  $f$  given by  $f(x) = 2x^3 - 3x^2 - 36x + 7$  is
  - increasing
  - decreasing
6. Find the intervals in which the following functions are strictly increasing or decreasing:
  - $x^2 + 2x - 5$
  - $10 - 6x - 2x^2$
  - $-2x^3 - 9x^2 - 12x + 1$
  - $6 - 9x - x^2$
  - $(x + 1)^3 (x - 3)^3$
7. Show that  $y = \log(1+x) - \frac{2x}{2+x}$ ,  $x > -1$ , is an increasing function of  $x$  throughout its domain.
8. Find the values of  $x$  for which  $y = [x(x-2)]^2$  is an increasing function.
9. Prove that  $y = \frac{4 \sin \theta}{(2 + \cos \theta)} - \theta$  is an increasing function of  $\theta$  in  $\left[0, \frac{\pi}{2}\right]$ .

10. Prove that the logarithmic function is increasing on  $(0, \infty)$ .
11. Prove that the function  $f$  given by  $f(x) = x^2 - x + 1$  is neither strictly increasing nor decreasing on  $(-1, 1)$ .
12. Which of the following functions are decreasing on  $0, \frac{\pi}{2}$ ?
  - (A)  $\cos x$
  - (B)  $\cos 2x$
  - (C)  $\cos 3x$
  - (D)  $\tan x$
13. On which of the following intervals is the function  $f$  given by  $f(x) = x^{100} + \sin x - 1$  decreasing?
  - (A)  $(0, 1)$
  - (B)  $\frac{\pi}{2}, \pi$
  - (C)  $0, \frac{\pi}{2}$
  - (D) None of these
14. For what values of  $a$  the function  $f$  given by  $f(x) = x^2 + ax + 1$  is increasing on  $[1, 2]$ ?
15. Let  $I$  be any interval disjoint from  $[-1, 1]$ . Prove that the function  $f$  given by  $f(x) = x + \frac{1}{x}$  is increasing on  $I$ .
16. Prove that the function  $f$  given by  $f(x) = \log \sin x$  is increasing on  $\left(0, \frac{\pi}{2}\right)$  and decreasing on  $\left(\frac{\pi}{2}, \pi\right)$ .
17. Prove that the function  $f$  given by  $f(x) = \log |\cos x|$  is decreasing on  $\left(0, \frac{\pi}{2}\right)$  and increasing on  $\left(\frac{3\pi}{2}, 2\pi\right)$ .
18. Prove that the function given by  $f(x) = x^3 - 3x^2 + 3x - 100$  is increasing in  $\mathbf{R}$ .
19. The interval in which  $y = x^2 e^{-x}$  is increasing is
  - (A)  $(-\infty, \infty)$
  - (B)  $(-2, 0)$
  - (C)  $(2, \infty)$
  - (D)  $(0, 2)$

#### 6.4 Maxima and Minima

In this section, we will use the concept of derivatives to calculate the maximum or minimum values of various functions. In fact, we will find the ‘turning points’ of the graph of a function and thus find points at which the graph reaches its highest (or

lowest) *locally*. The knowledge of such points is very useful in sketching the graph of a given function. Further, we will also find the absolute maximum and absolute minimum of a function that are necessary for the solution of many applied problems.

Let us consider the following problems that arise in day to day life.

- (i) The profit from a grove of orange trees is given by  $P(x) = ax + bx^2$ , where  $a, b$  are constants and  $x$  is the number of orange trees per acre. How many trees per acre will maximise the profit?
- (ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by  $h(x) = 60 + x - \frac{x^2}{60}$ , where  $x$  is the horizontal distance from the building and  $h(x)$  is the height of the ball. What is the maximum height the ball will reach?
- (iii) An Apache helicopter of enemy is flying along the path given by the curve  $f(x) = x^2 + 7$ . A soldier, placed at the point  $(1, 2)$ , wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

**Definition 3** Let  $f$  be a function defined on an interval  $I$ . Then

- (a)  $f$  is said to have a *maximum value* in  $I$ , if there exists a point  $c$  in  $I$  such that  $f(c) > f(x)$ , for all  $x \in I$ .  
The number  $f(c)$  is called the maximum value of  $f$  in  $I$  and the point  $c$  is called a *point of maximum value* of  $f$  in  $I$ .
- (b)  $f$  is said to have a minimum value in  $I$ , if there exists a point  $c$  in  $I$  such that  $f(c) < f(x)$ , for all  $x \in I$ .  
The number  $f(c)$ , in this case, is called the minimum value of  $f$  in  $I$  and the point  $c$ , in this case, is called a *point of minimum value* of  $f$  in  $I$ .
- (c)  $f$  is said to have an extreme value in  $I$  if there exists a point  $c$  in  $I$  such that  $f(c)$  is either a maximum value or a minimum value of  $f$  in  $I$ .

The number  $f(c)$ , in this case, is called an *extreme value* of  $f$  in  $I$  and the point  $c$  is called an *extreme point*.

**Remark** In Fig 6.7(a), (b) and (c), we have exhibited that graphs of certain particular functions help us to find maximum value and minimum value at a point. Infact, through graphs, we can even find maximum/minimum value of a function at a point at which it is not even differentiable (Example 15).

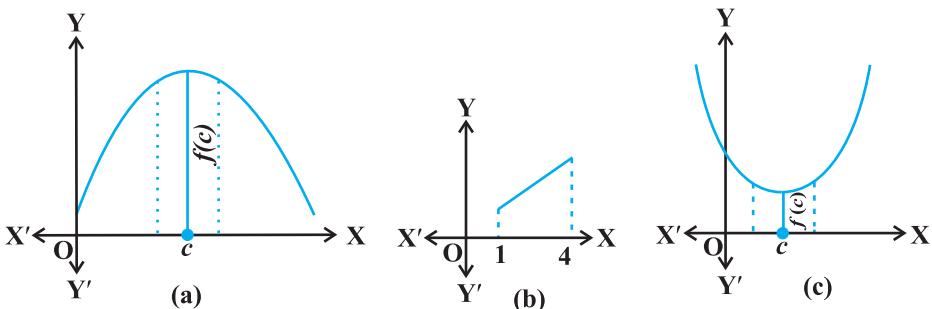


Fig 6.7

**Example 14** Find the maximum and the minimum values, if any, of the function  $f$  given by

$$f(x) = x^2, x \in \mathbf{R}.$$

**Solution** From the graph of the given function (Fig 6.8), we have  $f(x) = 0$  if  $x = 0$ . Also

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R}.$$

Therefore, the minimum value of  $f$  is 0 and the point of minimum value of  $f$  is  $x = 0$ . Further, it may be observed from the graph of the function that  $f$  has no maximum value and hence no point of maximum value of  $f$  in  $\mathbf{R}$ .

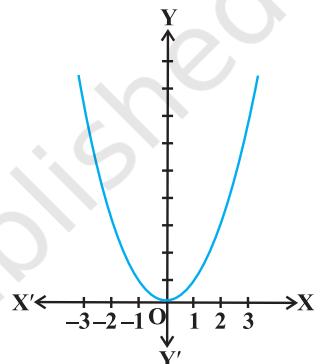


Fig 6.8

**Note** If we restrict the domain of  $f$  to  $[-2, 1]$  only, then  $f$  will have maximum value  $(-2)^2 = 4$  at  $x = -2$ .

**Example 15** Find the maximum and minimum values of  $f$ , if any, of the function given by  $f(x) = |x|, x \in \mathbf{R}$ .

**Solution** From the graph of the given function (Fig 6.9), note that

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R} \text{ and } f(x) = 0 \text{ if } x = 0.$$

Therefore, the function  $f$  has a minimum value 0 and the point of minimum value of  $f$  is  $x = 0$ . Also, the graph clearly shows that  $f$  has no maximum value in  $\mathbf{R}$  and hence no point of maximum value in  $\mathbf{R}$ .

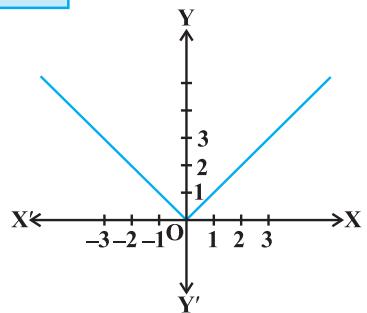


Fig 6.9

**Note**

- If we restrict the domain of  $f$  to  $[-2, 1]$  only, then  $f$  will have maximum value  $|-2| = 2$ .

- (ii) One may note that the function  $f$  in Example 27 is not differentiable at  $x = 0$ .

**Example 16** Find the maximum and the minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1).$$

**Solution** The given function is an increasing (strictly) function in the given interval  $(0, 1)$ . From the graph (Fig 6.10) of the function  $f$ , it seems that, it should have the minimum value at a point closest to 0 on its right and the maximum value at a point closest to 1 on its left. Are such points available? Of course, not. It is not possible to locate such points. Infact, if a point  $x_0$  is closest to 0, then

we find  $\frac{x_0}{2} < x_0$  for all  $x_0 \in (0, 1)$ . Also, if  $x_1$  is closest to 1, then  $\frac{x_1+1}{2} > x_1$  for all  $x_1 \in (0, 1)$ .

Therefore, the given function has neither the maximum value nor the minimum value in the interval  $(0, 1)$ .

**Remark** The reader may observe that in Example 16, if we include the points 0 and 1 in the domain of  $f$ , i.e., if we extend the domain of  $f$  to  $[0, 1]$ , then the function  $f$  has minimum value 0 at  $x = 0$  and maximum value 1 at  $x = 1$ . Infact, we have the following results (The proof of these results are beyond the scope of the present text)

*Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.*

A more general result is

*Every continuous function on a closed interval has a maximum and a minimum value.*

**Note** By a monotonic function  $f$  in an interval  $I$ , we mean that  $f$  is either increasing in  $I$  or decreasing in  $I$ .

Maximum and minimum values of a function defined on a closed interval will be discussed later in this section.

Let us now examine the graph of a function as shown in Fig 6.11. Observe that at points A, B, C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called *turning points* of the given function. Further, observe that at turning points, the graph has either a little hill or a little valley. Roughly speaking, the function has minimum value in some neighbourhood (interval) of each of the points A and C which are at the bottom of their respective

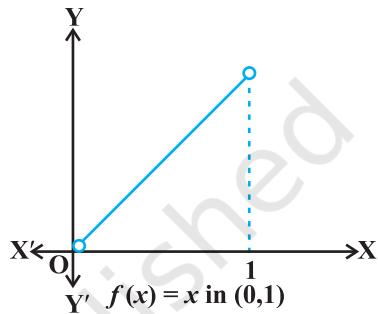


Fig 6.10

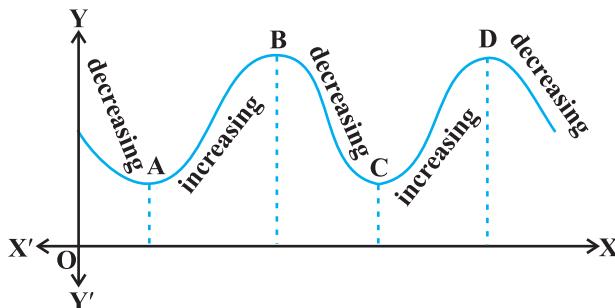


Fig 6.11

valleys. Similarly, the function has maximum value in some neighbourhood of points B and D which are at the top of their respective hills. For this reason, the points A and C may be regarded as points of *local minimum value* (or *relative minimum value*) and points B and D may be regarded as points of *local maximum value* (or *relative maximum value*) for the function. The *local maximum value* and *local minimum value* of the function are referred to as *local maxima* and *local minima*, respectively, of the function.

We now formally give the following definition

**Definition 4** Let  $f$  be a real valued function and let  $c$  be an interior point in the domain of  $f$ . Then

(a)  $c$  is called a point of *local maxima* if there is an  $h > 0$  such that

$$f(c) \geq f(x), \text{ for all } x \text{ in } (c - h, c + h), x \neq c$$

The value  $f(c)$  is called the *local maximum value* of  $f$ .

(b)  $c$  is called a point of *local minima* if there is an  $h > 0$  such that

$$f(c) \leq f(x), \text{ for all } x \text{ in } (c - h, c + h)$$

The value  $f(c)$  is called the *local minimum value* of  $f$ .

Geometrically, the above definition states that if  $x = c$  is a point of local maxima of  $f$ , then the graph of  $f$  around  $c$  will be as shown in Fig 6.12(a). Note that the function  $f$  is increasing (i.e.,  $f'(x) > 0$ ) in the interval  $(c - h, c)$  and decreasing (i.e.,  $f'(x) < 0$ ) in the interval  $(c, c + h)$ .

This suggests that  $f'(c)$  must be zero.

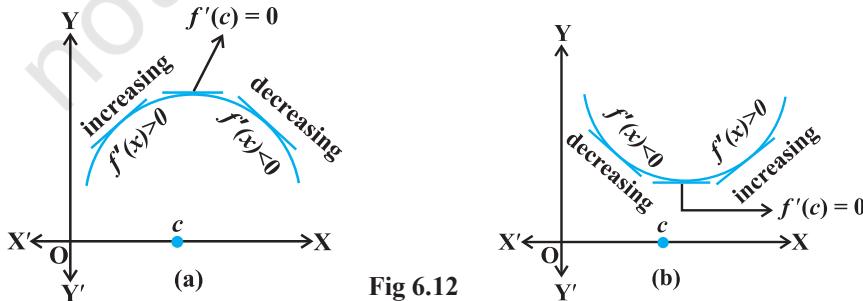


Fig 6.12

Similarly, if  $c$  is a point of local minima of  $f$ , then the graph of  $f$  around  $c$  will be as shown in Fig 6.14(b). Here  $f$  is decreasing (i.e.,  $f'(x) < 0$ ) in the interval  $(c - h, c)$  and increasing (i.e.,  $f'(x) > 0$ ) in the interval  $(c, c + h)$ . This again suggest that  $f'(c)$  must be zero.

The above discussion lead us to the following theorem (without proof).

**Theorem 2** Let  $f$  be a function defined on an open interval  $I$ . Suppose  $c \in I$  be any point. If  $f$  has a local maxima or a local minima at  $x = c$ , then either  $f'(c) = 0$  or  $f$  is not differentiable at  $c$ .

**Remark** The converse of above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima or local minima. For example, if  $f(x) = x^3$ , then  $f'(x) = 3x^2$  and so  $f'(0) = 0$ . But 0 is neither a point of local maxima nor a point of local minima (Fig 6.13).



**Note** A point  $c$  in the domain of a function  $f$  at which either  $f'(c) = 0$  or  $f$  is not differentiable is called a *critical point* of  $f$ . Note that if  $f$  is continuous at  $c$  and  $f'(c) = 0$ , then there exists an  $h > 0$  such that  $f$  is differentiable in the interval  $(c - h, c + h)$ .

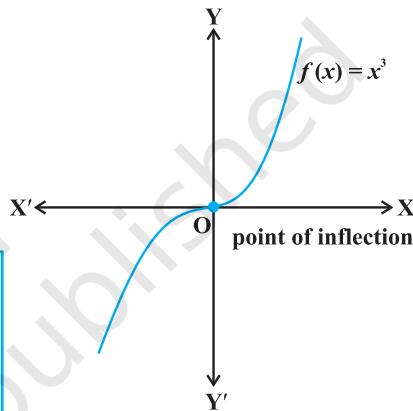


Fig 6.13

We shall now give a working rule for finding points of local maxima or points of local minima using only the first order derivatives.

**Theorem 3 (First Derivative Test)** Let  $f$  be a function defined on an open interval  $I$ . Let  $f$  be continuous at a critical point  $c$  in  $I$ . Then

- If  $f'(x)$  changes sign from positive to negative as  $x$  increases through  $c$ , i.e., if  $f'(x) > 0$  at every point sufficiently close to and to the left of  $c$ , and  $f'(x) < 0$  at every point sufficiently close to and to the right of  $c$ , then  $c$  is a point of *local maxima*.
- If  $f'(x)$  changes sign from negative to positive as  $x$  increases through  $c$ , i.e., if  $f'(x) < 0$  at every point sufficiently close to and to the left of  $c$ , and  $f'(x) > 0$  at every point sufficiently close to and to the right of  $c$ , then  $c$  is a point of *local minima*.
- If  $f'(x)$  does not change sign as  $x$  increases through  $c$ , then  $c$  is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.13).

**Note** If  $c$  is a point of local maxima of  $f$ , then  $f(c)$  is a local maximum value of  $f$ . Similarly, if  $c$  is a point of local minima of  $f$ , then  $f(c)$  is a local minimum value of  $f$ .

Figures 6.13 and 6.14, geometrically explain Theorem 3.

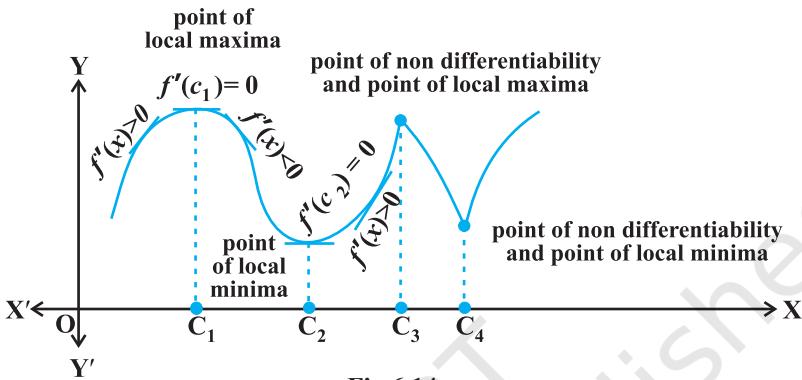


Fig 6.14

**Example 17** Find all points of local maxima and local minima of the function  $f$  given by

$$f(x) = x^3 - 3x + 3.$$

**Solution** We have

$$f(x) = x^3 - 3x + 3$$

or

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$$

or

$$f'(x) = 0 \text{ at } x = 1 \text{ and } x = -1$$

Thus,  $x = \pm 1$  are the only critical points which could possibly be the points of local maxima and/or local minima of  $f$ . Let us first examine the point  $x = 1$ .

Note that for values close to 1 and to the right of 1,  $f'(x) > 0$  and for values close to 1 and to the left of 1,  $f'(x) < 0$ . Therefore, by first derivative test,  $x = 1$  is a point of local minima and local minimum value is  $f(1) = 1$ . In the case of  $x = -1$ , note that  $f'(x) > 0$ , for values close to and to the left of  $-1$  and  $f'(x) < 0$ , for values close to and to the right of  $-1$ . Therefore, by first derivative test,  $x = -1$  is a point of local maxima and local maximum value is  $f(-1) = 5$ .

| Values of $x$  | Sign of $f'(x) = 3(x - 1)(x + 1)$ |
|--|-----------------------------------|
| Close to 1<br>to the right (say 1.1 etc.)<br>to the left (say 0.9 etc.)          | $>0$<br>$<0$                      |
| Close to $-1$<br>to the right (say $-0.9$ etc.)<br>to the left (say $-1.1$ etc.) | $<0$<br>$>0$                      |

**Example 18** Find all the points of local maxima and local minima of the function  $f$  given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

**Solution** We have

$$\begin{aligned} f(x) &= 2x^3 - 6x^2 + 6x + 5 \\ \text{or} \quad f'(x) &= 6x^2 - 12x + 6 = 6(x - 1)^2 \\ \text{or} \quad f'(x) &= 0 \quad \text{at } x = 1 \end{aligned}$$

Thus,  $x = 1$  is the only critical point of  $f$ . We shall now examine this point for local maxima and/or local minima of  $f$ . Observe that  $f'(x) \geq 0$ , for all  $x \in \mathbf{R}$  and in particular  $f'(x) > 0$ , for values close to 1 and to the left and to the right of 1. Therefore, by first derivative test, the point  $x = 1$  is neither a point of local maxima nor a point of local minima. Hence  $x = 1$  is a point of inflexion.

**Remark** One may note that since  $f'(x)$ , in Example 30, never changes its sign on  $\mathbf{R}$ , graph of  $f$  has no turning points and hence no point of local maxima or local minima.

We shall now give another test to examine local maxima and local minima of a given function. This test is often easier to apply than the first derivative test.

**Theorem 4 (Second Derivative Test)** Let  $f$  be a function defined on an interval  $I$  and  $c \in I$ . Let  $f$  be twice differentiable at  $c$ . Then

(i)  $x = c$  is a point of local maxima if  $f'(c) = 0$  and  $f''(c) < 0$

The value  $f(c)$  is local maximum value of  $f$ .

(ii)  $x = c$  is a point of local minima if  $f'(c) = 0$  and  $f''(c) > 0$

In this case,  $f(c)$  is local minimum value of  $f$ .

(iii) The test fails if  $f'(c) = 0$  and  $f''(c) = 0$ .

In this case, we go back to the first derivative test and find whether  $c$  is a point of local maxima, local minima or a point of inflexion.

**Note** As  $f$  is twice differentiable at  $c$ , we mean second order derivative of  $f$  exists at  $c$ .

**Example 19** Find local minimum value of the function  $f$  given by  $f(x) = 3 + |x|, x \in \mathbf{R}$ .

**Solution** Note that the given function is not differentiable at  $x = 0$ . So, second derivative test fails. Let us try first derivative test. Note that 0 is a critical point of  $f$ . Now to the left of 0,  $f(x) = 3 - x$  and so  $f'(x) = -1 < 0$ . Also to

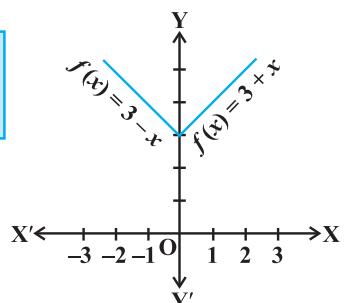


Fig 6.15

the right of 0,  $f(x) = 3 + x$  and so  $f'(x) = 1 > 0$ . Therefore, by first derivative test,  $x = 0$  is a point of local minima of  $f$  and local minimum value of  $f$  is  $f(0) = 3$ .

**Example 20** Find local maximum and local minimum values of the function  $f$  given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

**Solution** We have

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

or

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$$

or

$$f'(x) = 0 \text{ at } x = 0, x = 1 \text{ and } x = -2.$$

Now

$$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

or

$$\begin{cases} f''(0) = -24 < 0 \\ f''(1) = 36 > 0 \\ f''(-2) = 72 > 0 \end{cases}$$

Therefore, by second derivative test,  $x = 0$  is a point of local maxima and local maximum value of  $f$  at  $x = 0$  is  $f(0) = 12$  while  $x = 1$  and  $x = -2$  are the points of local minima and local minimum values of  $f$  at  $x = -1$  and  $-2$  are  $f(1) = 7$  and  $f(-2) = -20$ , respectively.

**Example 21** Find all the points of local maxima and local minima of the function  $f$  given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

**Solution** We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or

$$\begin{cases} f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2 \\ f''(x) = 12(x-1) \end{cases}$$

Now  $f'(x) = 0$  gives  $x = 1$ . Also  $f''(1) = 0$ . Therefore, the second derivative test fails in this case. So, we shall go back to the first derivative test.

We have already seen (Example 18) that, using first derivative test,  $x = 1$  is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

**Example 22** Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

**Solution** Let one of the numbers be  $x$ . Then the other number is  $(15 - x)$ . Let  $S(x)$  denote the sum of the squares of these numbers. Then

$$S(x) = x^2 + (15 - x)^2 = 2x^2 - 30x + 225$$

or

$$\begin{cases} S'(x) = 4x - 30 \\ S''(x) = 4 \end{cases}$$

Now  $S'(x) = 0$  gives  $x = \frac{15}{2}$ . Also  $S''\left(\frac{15}{2}\right) = 4 > 0$ . Therefore, by second derivative

test,  $x = \frac{15}{2}$  is the point of local minima of  $S$ . Hence the sum of squares of numbers is

minimum when the numbers are  $\frac{15}{2}$  and  $15 - \frac{15}{2} = \frac{15}{2}$ .

**Remark** Proceeding as in Example 34 one may prove that the two positive numbers, whose sum is  $k$  and the sum of whose squares is minimum, are  $\frac{k}{2}$  and  $\frac{k}{2}$ .

**Example 23** Find the shortest distance of the point  $(0, c)$  from the parabola  $y = x^2$ , where  $\frac{1}{2} \leq c \leq 5$ .

**Solution** Let  $(h, k)$  be any point on the parabola  $y = x^2$ . Let  $D$  be the required distance between  $(h, k)$  and  $(0, c)$ . Then

$$D = \sqrt{(h-0)^2 + (k-c)^2} = \sqrt{h^2 + (k-c)^2} \quad \dots (1)$$

Since  $(h, k)$  lies on the parabola  $y = x^2$ , we have  $k = h^2$ . So (1) gives

$$D \equiv D(k) = \sqrt{k + (k-c)^2}$$

or

$$D'(k) = \frac{1 + 2(k-c)}{2\sqrt{k + (k-c)^2}}$$

Now

$$D'(k) = 0 \text{ gives } k = \frac{2c-1}{2}$$

Observe that when  $k < \frac{2c-1}{2}$ , then  $2(k-c) + 1 < 0$ , i.e.,  $D'(k) < 0$ . Also when

$k > \frac{2c-1}{2}$ , then  $D'(k) > 0$ . So, by first derivative test,  $D(k)$  is minimum at  $k = \frac{2c-1}{2}$ .

Hence, the required shortest distance is given by

$$D\left(\frac{2c-1}{2}\right) = \sqrt{\frac{2c-1}{2} + \left(\frac{2c-1}{2} - c\right)^2} = \frac{\sqrt{4c-1}}{2}$$

**Note** The reader may note that in Example 35, we have used first derivative test instead of the second derivative test as the former is easy and short.

**Example 24** Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that  $RP^2 + RQ^2$  is minimum.

**Solution** Let R be a point on AB such that  $AR = x$  m. Then  $RB = (20 - x)$  m (as  $AB = 20$  m). From Fig 6.16, we have

$$RP^2 = AR^2 + AP^2$$

and

$$RQ^2 = RB^2 + BQ^2$$

Therefore

$$\begin{aligned} RP^2 + RQ^2 &= AR^2 + AP^2 + RB^2 + BQ^2 \\ &= x^2 + (16)^2 + (20 - x)^2 + (22)^2 \\ &= 2x^2 - 40x + 1140 \end{aligned}$$

Let

$$S \equiv S(x) = RP^2 + RQ^2 = 2x^2 - 40x + 1140.$$

Therefore

$$S'(x) = 4x - 40.$$

Now  $S'(x) = 0$  gives  $x = 10$ . Also  $S''(x) = 4 > 0$ , for all  $x$  and so  $S''(10) > 0$ . Therefore, by second derivative test,  $x = 10$  is the point of local minima of  $S$ . Thus, the distance of R from A on AB is  $AR = x = 10$  m.

**Example 25** If length of three sides of a trapezium other than base are equal to 10 cm, then find the area of the trapezium when it is maximum.

**Solution** The required trapezium is as given in Fig 6.17. Draw perpendiculars DP and

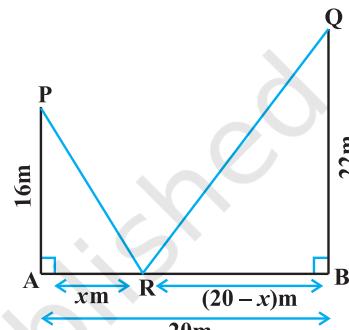


Fig 6.16

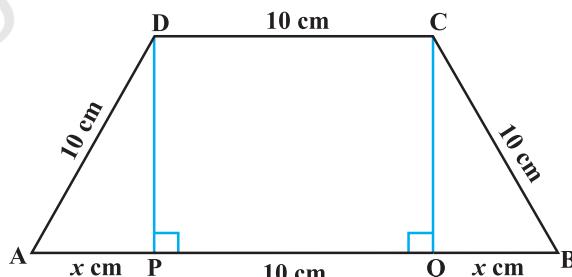


Fig 6.17

CQ on AB. Let AP =  $x$  cm. Note that  $\Delta APD \sim \Delta BQC$ . Therefore, QB =  $x$  cm. Also, by Pythagoras theorem, DP = QC =  $\sqrt{100 - x^2}$ . Let A be the area of the trapezium. Then

$$\begin{aligned} A &\equiv A(x) = \frac{1}{2}(\text{sum of parallel sides})(\text{height}) \\ &= \frac{1}{2}(2x + 10 + 10)(\sqrt{100 - x^2}) \\ &= (x + 10)(\sqrt{100 - x^2}) \end{aligned}$$

or

$$\begin{aligned} A'(x) &= (x + 10) \frac{(-2x)}{2\sqrt{100 - x^2}} + (\sqrt{100 - x^2}) \\ &= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} \end{aligned}$$

Now  $A'(x) = 0$  gives  $2x^2 + 10x - 100 = 0$ , i.e.,  $x = 5$  and  $x = -10$ .

Since  $x$  represents distance, it can not be negative.

So,  $x = 5$ . Now

$$\begin{aligned} A''(x) &= \frac{\sqrt{100 - x^2}(-4x - 10) - (-2x^2 - 10x + 100) \frac{(-2x)}{2\sqrt{100 - x^2}}}{100 - x^2} \\ &= \frac{2x^3 - 300x - 1000}{(100 - x^2)^{\frac{3}{2}}} \text{ (on simplification)} \end{aligned}$$

or  $A''(5) = \frac{2(5)^3 - 300(5) - 1000}{(100 - (5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$

Thus, area of trapezium is maximum at  $x = 5$  and the area is given by

$$A(5) = (5 + 10)\sqrt{100 - (5)^2} = 15\sqrt{75} = 75\sqrt{3} \text{ cm}^2$$

**Example 26** Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

**Solution** Let OC =  $r$  be the radius of the cone and OA =  $h$  be its height. Let a cylinder with radius OE =  $x$  inscribed in the given cone (Fig 6.18). The height QE of the cylinder is given by

$$\frac{QE}{OA} = \frac{EC}{OC} \quad (\text{since } \Delta QEC \sim \Delta AOC)$$

$$\text{or} \quad \frac{QE}{h} = \frac{r-x}{r}$$

$$\text{or} \quad \text{QE} = \frac{h(r-x)}{r}$$

Let  $S$  be the curved surface area of the given cylinder. Then

$$S \equiv S(x) = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r} (rx - x^2)$$

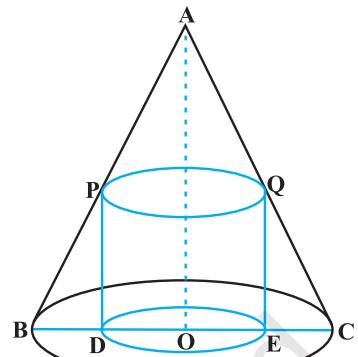


Fig 6.18

or

$$\begin{cases} S'(x) = \frac{2\pi h}{r}(r - 2x) \\ S''(x) = \frac{-4\pi h}{r} \end{cases}$$

Now  $S'(x) = 0$  gives  $x = \frac{r}{2}$ . Since  $S''(x) < 0$  for all  $x$ ,  $S''\left(\frac{r}{2}\right) < 0$ . So  $x = \frac{r}{2}$  is a point of maxima of  $S$ . Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

#### 6.4.1 Maximum and Minimum Values of a Function in a Closed Interval

Let us consider a function  $f$  given by

$$f(x) = x + 2, x \in (0, 1)$$

Observe that the function is continuous on  $(0, 1)$  and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of  $f$  to the closed interval  $[0, 1]$ , then  $f$  still may not have a local maximum (minimum) values but it certainly does have maximum value  $3 = f(1)$  and minimum value  $2 = f(0)$ . The maximum value  $3$  of  $f$  at  $x = 1$  is called *absolute maximum value (global maximum or greatest value)* of  $f$  on the interval  $[0, 1]$ . Similarly, the minimum value  $2$  of  $f$  at  $x = 0$  is called the *absolute minimum value (global minimum or least value)* of  $f$  on  $[0, 1]$ .

Consider the graph given in Fig 6.19 of a continuous function defined on a closed interval  $[a, d]$ . Observe that the function  $f$  has a local minima at  $x = b$  and local

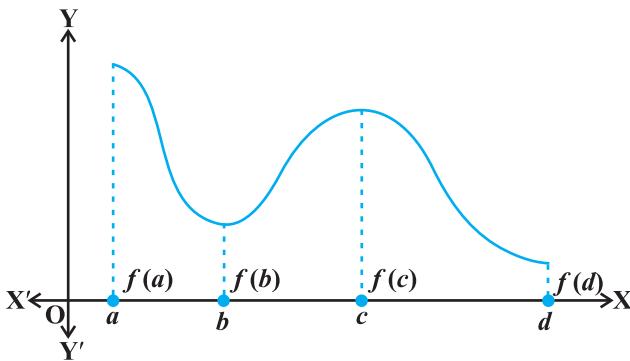


Fig 6.19

minimum value is  $f(b)$ . The function also has a local maxima at  $x = c$  and local maximum value is  $f(c)$ .

Also from the graph, it is evident that  $f$  has absolute maximum value  $f(a)$  and absolute minimum value  $f(d)$ . Further note that the absolute maximum (minimum) value of  $f$  is different from local maximum (minimum) value of  $f$ .

We will now state two results (without proof) regarding absolute maximum and absolute minimum values of a function on a closed interval  $I$ .

**Theorem 5** Let  $f$  be a continuous function on an interval  $I = [a, b]$ . Then  $f$  has the absolute maximum value and  $f$  attains it at least once in  $I$ . Also,  $f$  has the absolute minimum value and attains it at least once in  $I$ .

**Theorem 6** Let  $f$  be a differentiable function on a closed interval  $I$  and let  $c$  be any interior point of  $I$ . Then

- (i)  $f'(c) = 0$  if  $f$  attains its absolute maximum value at  $c$ .
- (ii)  $f'(c) = 0$  if  $f$  attains its absolute minimum value at  $c$ .

In view of the above results, we have the following working rule for finding absolute maximum and/or absolute minimum values of a function in a given closed interval  $[a, b]$ .

#### Working Rule

**Step 1:** Find all critical points of  $f$  in the interval, i.e., find points  $x$  where either  $f'(x) = 0$  or  $f$  is not differentiable.

**Step 2:** Take the end points of the interval.

**Step 3:** At all these points (listed in Step 1 and 2), calculate the values of  $f$ .

**Step 4:** Identify the maximum and minimum values of  $f$  out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of  $f$  and the minimum value will be the absolute minimum (least) value of  $f$ .

**Example 27** Find the absolute maximum and minimum values of a function  $f$  given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5].$$

**Solution** We have

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

or

$$f'(x) = 6x^2 - 30x + 36 = 6(x - 3)(x - 2)$$

Note that  $f'(x) = 0$  gives  $x = 2$  and  $x = 3$ .

We shall now evaluate the value of  $f$  at these points and at the end points of the interval  $[1, 5]$ , i.e., at  $x = 1, x = 2, x = 3$  and at  $x = 5$ . So

$$f(1) = 2(1^3) - 15(1^2) + 36(1) + 1 = 24$$

$$f(2) = 2(2^3) - 15(2^2) + 36(2) + 1 = 29$$

$$f(3) = 2(3^3) - 15(3^2) + 36(3) + 1 = 28$$

$$f(5) = 2(5^3) - 15(5^2) + 36(5) + 1 = 56$$

Thus, we conclude that absolute maximum value of  $f$  on  $[1, 5]$  is 56, occurring at  $x = 5$ , and absolute minimum value of  $f$  on  $[1, 5]$  is 24 which occurs at  $x = 1$ .

**Example 28** Find absolute maximum and minimum values of a function  $f$  given by

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}, x \in [-1, 1]$$

**Solution** We have

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$$

or

$$f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}} = \frac{2(8x - 1)}{x^{\frac{2}{3}}}$$

Thus,  $f'(x) = 0$  gives  $x = \frac{1}{8}$ . Further note that  $f'(x)$  is not defined at  $x = 0$ . So the

critical points are  $x = 0$  and  $x = \frac{1}{8}$ . Now evaluating the value of  $f$  at critical points

$x = 0, \frac{1}{8}$  and at end points of the interval  $x = -1$  and  $x = 1$ , we have

$$f(-1) = 12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}} = 18$$

$$f(0) = 12(0) - 6(0) = 0$$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^3 - 6\left(\frac{1}{8}\right)^3 = \frac{-9}{4}$$

$$f(1) = 12(1)^3 - 6(1)^3 = 6$$

Hence, we conclude that absolute maximum value of  $f$  is 18 that occurs at  $x = -1$

and absolute minimum value of  $f$  is  $\frac{-9}{4}$  that occurs at  $x = \frac{1}{8}$ .

**Example 29** An Apache helicopter of enemy is flying along the curve given by  $y = x^2 + 7$ . A soldier, placed at  $(3, 7)$ , wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

**Solution** For each value of  $x$ , the helicopter's position is at point  $(x, x^2 + 7)$ . Therefore, the distance between the helicopter and the soldier placed at  $(3, 7)$  is

$$\sqrt{(x-3)^2 + (x^2 + 7 - 7)^2}, \text{ i.e., } \sqrt{(x-3)^2 + x^4}.$$

Let

$$f(x) = (x-3)^2 + x^4$$

or

$$f'(x) = 2(x-3) + 4x^3 = 2(x-1)(2x^2 + 2x + 3)$$

Thus,  $f'(x) = 0$  gives  $x = 1$  or  $2x^2 + 2x + 3 = 0$  for which there are no real roots. Also, there are no end points of the interval to be added to the set for which  $f'$  is zero, i.e., there is only one point, namely,  $x = 1$ . The value of  $f$  at this point is given by  $f(1) = (1-3)^2 + (1)^4 = 5$ . Thus, the distance between the soldier and the helicopter is  $\sqrt{f(1)} = \sqrt{5}$ .

Note that  $\sqrt{5}$  is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5},$$

it follows that  $\sqrt{5}$  is the minimum value of  $\sqrt{f(x)}$ . Hence,  $\sqrt{5}$  is the minimum distance between the soldier and the helicopter.

### EXERCISE 6.3

1. Find the maximum and minimum values, if any, of the following functions given by

- |                              |                              |
|------------------------------|------------------------------|
| (i) $f(x) = (2x-1)^2 + 3$    | (ii) $f(x) = 9x^2 + 12x + 2$ |
| (iii) $f(x) = -(x-1)^2 + 10$ | (iv) $g(x) = x^3 + 1$        |

2. Find the maximum and minimum values, if any, of the following functions given by
- (i)  $f(x) = |x + 2| - 1$       (ii)  $g(x) = -|x + 1| + 3$   
 (iii)  $h(x) = \sin(2x) + 5$       (iv)  $f(x) = |\sin 4x + 3|$   
 (v)  $h(x) = x + 1, x \in (-1, 1)$
3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:
- (i)  $f(x) = x^2$       (ii)  $g(x) = x^3 - 3x$   
 (iii)  $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$   
 (iv)  $f(x) = \sin x - \cos x, 0 < x < 2\pi$   
 (v)  $f(x) = x^3 - 6x^2 + 9x + 15$       (vi)  $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$   
 (vii)  $g(x) = \frac{1}{x^2 + 2}$       (viii)  $f(x) = x\sqrt{1-x}, 0 < x < 1$
4. Prove that the following functions do not have maxima or minima:
- (i)  $f(x) = e^x$       (ii)  $g(x) = \log x$   
 (iii)  $h(x) = x^3 + x^2 + x + 1$
5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:
- (i)  $f(x) = x^3, x \in [-2, 2]$       (ii)  $f(x) = \sin x + \cos x, x \in [0, \pi]$   
 (iii)  $f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right]$  (iv)  $f(x) = (x-1)^2 + 3, x \in [-3, 1]$
6. Find the maximum profit that a company can make, if the profit function is given by
- $$p(x) = 41 - 72x - 18x^2$$
7. Find both the maximum value and the minimum value of  $3x^4 - 8x^3 + 12x^2 - 48x + 25$  on the interval  $[0, 3]$ .
8. At what points in the interval  $[0, 2\pi]$ , does the function  $\sin 2x$  attain its maximum value?
9. What is the maximum value of the function  $\sin x + \cos x$ ?
10. Find the maximum value of  $2x^3 - 24x + 107$  in the interval  $[1, 3]$ . Find the maximum value of the same function in  $[-3, -1]$ .

11. It is given that at  $x = 1$ , the function  $x^4 - 62x^2 + ax + 9$  attains its maximum value, on the interval  $[0, 2]$ . Find the value of  $a$ .
12. Find the maximum and minimum values of  $x + \sin 2x$  on  $[0, 2\pi]$ .
13. Find two numbers whose sum is 24 and whose product is as large as possible.
14. Find two positive numbers  $x$  and  $y$  such that  $x + y = 60$  and  $xy^3$  is maximum.
15. Find two positive numbers  $x$  and  $y$  such that their sum is 35 and the product  $x^2y^5$  is a maximum.
16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.
17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.
18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?
19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.
20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.
21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?
22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?
23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius  $R$  is  $\frac{8}{27}$  of the volume of the sphere.
24. Show that the right circular cone of least curved surface and given volume has an altitude equal to  $\sqrt{2}$  time the radius of the base.
25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is  $\tan^{-1}\sqrt{2}$ .
26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is  $\sin^{-1}\left(\frac{1}{3}\right)$ .

Choose the correct answer in Questions 27 and 29.

**27.** The point on the curve  $x^2 = 2y$  which is nearest to the point  $(0, 5)$  is

- (A)  $(2\sqrt{2}, 4)$       (B)  $(2\sqrt{2}, 0)$       (C)  $(0, 0)$       (D)  $(2, 2)$

**28.** For all real values of  $x$ , the minimum value of  $\frac{1-x+x^2}{1+x+x^2}$  is

- (A) 0      (B) 1      (C) 3      (D)  $\frac{1}{3}$

**29.** The maximum value of  $[x(x-1)+1]^{\frac{1}{3}}$ ,  $0 \leq x \leq 1$  is

- (A)  $\left(\frac{1}{3}\right)^{\frac{1}{3}}$       (B)  $\frac{1}{2}$       (C) 1      (D) 0

### Miscellaneous Examples

**Example 30** A car starts from a point P at time  $t = 0$  seconds and stops at point Q. The distance  $x$ , in metres, covered by it, in  $t$  seconds is given by

$$x = t^2 \left( 2 - \frac{t}{3} \right)$$

Find the time taken by it to reach Q and also find distance between P and Q.

**Solution** Let  $v$  be the velocity of the car at  $t$  seconds.

Now 
$$x = t^2 \left( 2 - \frac{t}{3} \right)$$

Therefore 
$$v = \frac{dx}{dt} = 4t - t^2 = t(4 - t)$$

Thus,  $v = 0$  gives  $t = 0$  and/or  $t = 4$ .

Now  $v = 0$  at P as well as at Q and at P,  $t = 0$ . So, at Q,  $t = 4$ . Thus, the car will reach the point Q after 4 seconds. Also the distance travelled in 4 seconds is given by

$$x|_{t=4} = 4^2 \left( 2 - \frac{4}{3} \right) = 16 \left( \frac{2}{3} \right) = \frac{32}{3} \text{ m}$$

**Example 31** A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is  $\tan^{-1}(0.5)$ . Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

**Solution** Let  $r$ ,  $h$  and  $\alpha$  be as in Fig 6.20. Then  $\tan \alpha = \frac{r}{h}$ .

So

$$\alpha = \tan^{-1}\left(\frac{r}{h}\right).$$

But

$$\alpha = \tan^{-1}(0.5) \quad (\text{given})$$

or

$$\frac{r}{h} = 0.5$$

or

$$r = \frac{h}{2}$$

Let  $V$  be the volume of the cone. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Therefore

$$\frac{dV}{dt} = \frac{d}{dh}\left(\frac{\pi h^3}{12}\right) \cdot \frac{dh}{dt}$$

Fig 6.20

(by Chain Rule)

$$= \frac{\pi}{4} h^2 \frac{dh}{dt}$$

Now rate of change of volume, i.e.,  $\frac{dV}{dt} = 5 \text{ m}^3/\text{h}$  and  $h = 4 \text{ m}$ .

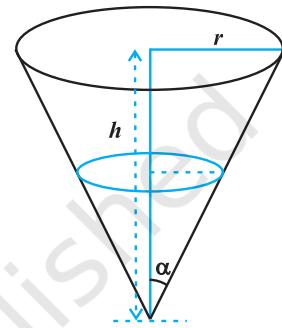
Therefore

$$5 = \frac{\pi}{4} (4)^2 \cdot \frac{dh}{dt}$$

or

$$\frac{dh}{dt} = \frac{5}{4\pi} = \frac{35}{88} \text{ m/h} \quad \left(\pi = \frac{22}{7}\right)$$

Thus, the rate of change of water level is  $\frac{35}{88} \text{ m/h}$ .



**Example 32** A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.

**Solution** In Fig 6.21, Let AB be the lamp-post, the lamp being at the position B and let MN be the man at a particular time  $t$  and let  $AM = l$  metres. Then, MS is the shadow of the man. Let  $MS = s$  metres.

Note that

$$\Delta MSN \sim \Delta ASB$$

or

$$\frac{MS}{AS} = \frac{MN}{AB}$$

or

$$AS = 3s \text{ (as } MN =$$

2 \text{ and } AB = 6 \text{ (given))}

Thus

$$AM = 3s - s = 2s. \text{ But } AM = l$$

So

$$l = 2s$$

Therefore

$$\frac{dl}{dt} = 2 \frac{ds}{dt}$$

Since  $\frac{dl}{dt} = 5$  km/h. Hence, the length of the shadow increases at the rate  $\frac{5}{2}$  km/h.

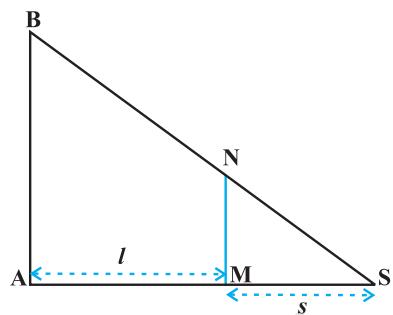


Fig 6.21

**Example 33** Find intervals in which the function given by

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

is (a) increasing (b) decreasing.

**Solution** We have

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

Therefore

$$\begin{aligned} f'(x) &= \frac{3}{10}(4x^3) - \frac{4}{5}(3x^2) - 3(2x) + \frac{36}{5} \\ &= \frac{6}{5}(x-1)(x+2)(x-3) \quad (\text{on simplification}) \end{aligned}$$

Now  $f'(x) = 0$  gives  $x = 1$ ,  $x = -2$ , or  $x = 3$ . The points  $x = 1$ ,  $-2$ , and  $3$  divide the real line into four disjoint intervals namely,  $(-\infty, -2)$ ,  $(-2, 1)$ ,  $(1, 3)$  and  $(3, \infty)$  (Fig 6.22).



Fig 6.22

Consider the interval  $(-\infty, -2)$ , i.e., when  $-\infty < x < -2$ .

In this case, we have  $x - 1 < 0$ ,  $x + 2 < 0$  and  $x - 3 < 0$ .

(In particular, observe that for  $x = -3$ ,  $f'(x) = (x - 1)(x + 2)(x - 3) = (-4)(-1)(-6) < 0$ )

Therefore,  $f'(x) < 0$  when  $-\infty < x < -2$ .

Thus, the function  $f$  is decreasing in  $(-\infty, -2)$ .

Consider the interval  $(-2, 1)$ , i.e., when  $-2 < x < 1$ .

In this case, we have  $x - 1 < 0$ ,  $x + 2 > 0$  and  $x - 3 < 0$

(In particular, observe that for  $x = 0$ ,  $f'(x) = (x - 1)(x + 2)(x - 3) = (-1)(2)(-3) = 6 > 0$ )

So  $f'(x) > 0$  when  $-2 < x < 1$ .

Thus,  $f$  is increasing in  $(-2, 1)$ .

Now consider the interval  $(1, 3)$ , i.e., when  $1 < x < 3$ . In this case, we have  $x - 1 > 0$ ,  $x + 2 > 0$  and  $x - 3 < 0$ .

So,  $f'(x) < 0$  when  $1 < x < 3$ .

Thus,  $f$  is decreasing in  $(1, 3)$ .

Finally, consider the interval  $(3, \infty)$ , i.e., when  $x > 3$ . In this case, we have  $x - 1 > 0$ ,  $x + 2 > 0$  and  $x - 3 > 0$ . So  $f'(x) > 0$  when  $x > 3$ .

Thus,  $f$  is increasing in the interval  $(3, \infty)$ .

**Example 34** Show that the function  $f$  given by

$$f(x) = \tan^{-1}(\sin x + \cos x), x > 0$$

is always an increasing function in  $\left(0, \frac{\pi}{4}\right)$ .

**Solution** We have

$$f(x) = \tan^{-1}(\sin x + \cos x), x > 0$$

Therefore

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} (\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{2 + \sin 2x} \quad (\text{on simplification})$$

Note that  $2 + \sin 2x > 0$  for all  $x$  in  $0, \frac{\pi}{4}$ .

Therefore  $f'(x) > 0$  if  $\cos x - \sin x > 0$

or  $f'(x) > 0$  if  $\cos x > \sin x$  or  $\cot x > 1$

Now  $\cot x > 1$  if  $\tan x < 1$ , i.e., if  $0 < x < \frac{\pi}{4}$

Thus  $f'(x) > 0$  in  $\left(0, \frac{\pi}{4}\right)$

Hence  $f$  is increasing function in  $\left(0, \frac{\pi}{4}\right)$ .

**Example 35** A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.

**Solution** Let  $r$  be the radius of the given disc and  $A$  be its area. Then

$$A = \pi r^2$$

or  $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$  (by Chain Rule)

Now approximate rate of increase of radius  $= dr = \frac{dr}{dt} \Delta t = 0.05$  cm/s.

Therefore, the approximate rate of increase in area is given by

$$\begin{aligned} dA &= \frac{dA}{dt}(\Delta t) = 2\pi r \left( \frac{dr}{dt} \Delta t \right) \\ &= 2\pi (3.2) (0.05) = 0.320\pi \text{ cm}^2/\text{s} \quad (r = 3.2 \text{ cm}) \end{aligned}$$

**Example 36** An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

**Solution** Let  $x$  metre be the length of a side of the removed squares. Then, the height of the box is  $x$ , length is  $8 - 2x$  and breadth is  $3 - 2x$  (Fig 6.23). If  $V(x)$  is the volume of the box, then

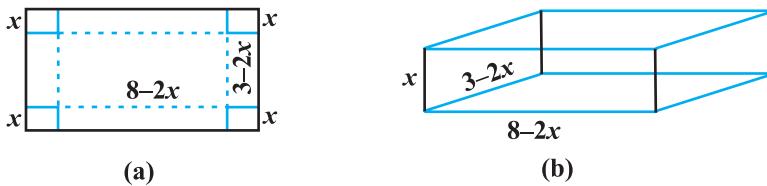


Fig 6.23

$$\begin{aligned} V(x) &= x(3 - 2x)(8 - 2x) \\ &= 4x^3 - 22x^2 + 24x \end{aligned}$$

Therefore

$$\begin{cases} V'(x) = 12x^2 - 44x + 24 = 4(x - 3)(3x - 2) \\ V''(x) = 24x - 44 \end{cases}$$

Now  $V'(x) = 0$  gives  $x = 3, \frac{2}{3}$ . But  $x \neq 3$  (Why?)

Thus, we have  $x = \frac{2}{3}$ . Now  $V''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 44 = -28 < 0$ .

Therefore,  $x = \frac{2}{3}$  is the point of maxima, i.e., if we remove a square of side  $\frac{2}{3}$

metre from each corner of the sheet and make a box from the remaining sheet, then the volume of the box such obtained will be the largest and it is given by

$$\begin{aligned} V\left(\frac{2}{3}\right) &= 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right) \\ &= \frac{200}{27} \text{ m}^3 \end{aligned}$$

**Example 37** Manufacturer can sell  $x$  items at a price of rupees  $\left(5 - \frac{x}{100}\right)$  each. The

cost price of  $x$  items is Rs  $\left(\frac{x}{5} + 500\right)$ . Find the number of items he should sell to earn maximum profit.

**Solution** Let  $S(x)$  be the selling price of  $x$  items and let  $C(x)$  be the cost price of  $x$  items. Then, we have

$$S(x) = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and

$$C(x) = \frac{x}{5} + 500$$

Thus, the profit function  $P(x)$  is given by

$$P(x) = S(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

i.e.

$$P(x) = \frac{24}{5}x - \frac{x^2}{100} - 500$$

or

$$P'(x) = \frac{24}{5} - \frac{x}{50}$$

Now  $P'(x) = 0$  gives  $x = 240$ . Also  $P''(x) = \frac{-1}{50}$ . So  $P''(240) = \frac{-1}{50} < 0$

Thus,  $x = 240$  is a point of maxima. Hence, the manufacturer can earn maximum profit, if he sells 240 items.

### Miscellaneous Exercise on Chapter 6

1. Show that the function given by  $f(x) = \frac{\log x}{x}$  has maximum at  $x = e$ .
2. The two equal sides of an isosceles triangle with fixed base  $b$  are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?
3. Find the intervals in which the function  $f$  given by

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing.

4. Find the intervals in which the function  $f$  given by  $f(x) = x^3 + \frac{1}{x^3}$ ,  $x \neq 0$  is
  - (i) increasing
  - (ii) decreasing.

5. Find the maximum area of an isosceles triangle inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with its vertex at one end of the major axis.

6. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m<sup>3</sup>. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

7. The sum of the perimeter of a circle and square is  $k$ , where  $k$  is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

8. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

9. A point on the hypotenuse of a triangle is at distance  $a$  and  $b$  from the sides of the triangle.

Show that the minimum length of the hypotenuse is  $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$ .



a sphere of radius R is  $\frac{2R}{\sqrt{3}}$ . Also find the maximum volume.

15. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height  $h$  and semi vertical angle  $\alpha$  is one-third that of the cone and the greatest volume of cylinder is  $\frac{4}{27}\pi h^3 \tan^2 \alpha$ .

16. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of
- (A) 1 m/h (B) 0.1 m/h  
 (C) 1.1 m/h (D) 0.5 m/h

### Summary

- ◆ If a quantity  $y$  varies with another quantity  $x$ , satisfying some rule  $y = f(x)$ , then  $\frac{dy}{dx}$  (or  $f'(x)$ ) represents the rate of change of  $y$  with respect to  $x$  and  $\frac{dy}{dx} \Big|_{x=x_0}$  (or  $f'(x_0)$ ) represents the rate of change of  $y$  with respect to  $x$  at  $x = x_0$ .
- ◆ If two variables  $x$  and  $y$  are varying with respect to another variable  $t$ , i.e., if  $x = f(t)$  and  $y = g(t)$ , then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0.$$

- ◆ A function  $f$  is said to be
  - (a) increasing on an interval  $(a, b)$  if  $x_1 < x_2$  in  $(a, b) \Rightarrow f(x_1) < f(x_2)$  for all  $x_1, x_2 \in (a, b)$ . Alternatively, if  $f'(x) \geq 0$  for each  $x$  in  $(a, b)$
  - (b) decreasing on  $(a, b)$  if  $x_1 < x_2$  in  $(a, b) \Rightarrow f(x_1) > f(x_2)$  for all  $x_1, x_2 \in (a, b)$ .
  - (c) constant in  $(a, b)$ , if  $f(x) = c$  for all  $x \in (a, b)$ , where  $c$  is a constant.
- ◆ A point  $c$  in the domain of a function  $f$  at which either  $f'(c) = 0$  or  $f$  is not differentiable is called a *critical point* of  $f$ .
- ◆ **First Derivative Test** Let  $f$  be a function defined on an open interval  $I$ . Let  $f$  be continuous at a critical point  $c$  in  $I$ . Then
  - (i) If  $f'(x)$  changes sign from positive to negative as  $x$  increases through  $c$ , i.e., if  $f'(x) > 0$  at every point sufficiently close to and to the left of  $c$ , and  $f'(x) < 0$  at every point sufficiently close to and to the right of  $c$ , then  $c$  is a point of *local maxima*.

(ii) If  $f'(x)$  changes sign from negative to positive as  $x$  increases through  $c$ , i.e., if  $f'(x) < 0$  at every point sufficiently close to and to the left of  $c$ , and  $f'(x) > 0$  at every point sufficiently close to and to the right of  $c$ , then  $c$  is a point of *local minima*.

(iii) If  $f'(x)$  does not change sign as  $x$  increases through  $c$ , then  $c$  is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflexion*.

◆ **Second Derivative Test** Let  $f$  be a function defined on an interval  $I$  and  $c \in I$ . Let  $f$  be twice differentiable at  $c$ . Then

(i)  $x = c$  is a point of local maxima if  $f'(c) = 0$  and  $f''(c) < 0$

The values  $f(c)$  is local maximum value of  $f$ .

(ii)  $x = c$  is a point of local minima if  $f'(c) = 0$  and  $f''(c) > 0$

In this case,  $f(c)$  is local minimum value of  $f$ .

(iii) The test fails if  $f'(c) = 0$  and  $f''(c) = 0$ .

In this case, we go back to the first derivative test and find whether  $c$  is a point of maxima, minima or a point of inflexion.

◆ Working rule for finding absolute maxima and/or absolute minima

**Step 1:** Find all critical points of  $f$  in the interval, i.e., find points  $x$  where either  $f'(x) = 0$  or  $f$  is not differentiable.

**Step 2:** Take the end points of the interval.

**Step 3:** At all these points (listed in Step 1 and 2), calculate the values of  $f$ .

**Step 4:** Identify the maximum and minimum values of  $f$  out of the values calculated in Step 3. This maximum value will be the absolute maximum value of  $f$  and the minimum value will be the absolute minimum value of  $f$ .



# ANSWERS

## EXERCISE 7.1

1.  $-\frac{1}{2} \cos 2x$       2.  $\frac{1}{3} \sin 3x$       3.  $\frac{1}{2} e^{2x}$   
4.  $\frac{1}{3a} (ax+b)^3$       5.  $-\frac{1}{2} \cos 2x - \frac{4}{3} e^{3x}$       6.  $\frac{4}{3} e^{3x} + x + C$   
7.  $\frac{x^3}{3} - x + C$       8.  $\frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$       9.  $\frac{2}{3} x^3 + e^x + C$   
10.  $\frac{x^2}{2} + \log|x| - 2x + C$       11.  $\frac{x^2}{2} + 5x + \frac{4}{x} + C$   
12.  $\frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$       13.  $\frac{x^3}{3} + x + C$   
14.  $\frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$       15.  $\frac{6}{7} x^{\frac{7}{2}} + \frac{4}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$   
16.  $x^2 - 3 \sin x + e^x + C$       17.  $\frac{2}{3} x^3 + 3 \cos x + \frac{10}{3} x^{\frac{3}{2}} + C$   
18.  $\tan x + \sec x + C$       19.  $\tan x - x + C$   
20.  $2 \tan x - 3 \sec x + C$       21. C  
22. A

## EXERCISE 7.2

1.  $\log(1+x^2) + C$       2.  $\frac{1}{3}(\log|x|)^3 + C$       3.  $\log|1+\log x| + C$   
4.  $\cos(\cos x) + C$       5.  $-\frac{1}{4a} \cos 2(ax+b) + C$   
6.  $\frac{2}{3a} (ax+b)^{\frac{3}{2}} + C$       7.  $\frac{2}{5} (x+2)^{\frac{5}{2}} - \frac{4}{3} (x+2)^{\frac{3}{2}} + C$

8.  $\frac{1}{6}(1+2x^2)^{\frac{3}{2}}+C$     9.  $\frac{4}{3}(x^2+x+1)^{\frac{3}{2}}+C$     10.  $2\log|\sqrt{x}-1|+C$
11.  $\frac{2}{3}\sqrt{x+4}(x-8)+C$
12.  $\frac{1}{7}(x^3-1)^{\frac{7}{3}}+\frac{1}{4}(x^3-1)^{\frac{4}{3}}+C$     13.  $-\frac{1}{18(2+3x^3)^2}+C$
14.  $\frac{(\log x)^{1-m}}{1-m}+C$     15.  $-\frac{1}{8}\log|9-4x^2|+C$     16.  $\frac{1}{2}e^{2x+3}+C$
17.  $-\frac{1}{2e^{x^2}}+C$     18.  $e^{\tan^{-1}x}+C$     19.  $\log(e^x+e^{-x})+C$
20.  $\frac{1}{2}\log(e^{2x}+e^{-2x})+C$     21.  $\frac{1}{2}\tan(2x-3)-x+C$
22.  $-\frac{1}{4}\tan(7-4x)+C$     23.  $\frac{1}{2}(\sin^{-1}x)^2+C$
24.  $\frac{1}{2}\log|2\sin x+3\cos x|+C$     25.  $\frac{1}{(1-\tan x)}+C$
26.  $2\sin\sqrt{x}+C$     27.  $\frac{1}{3}(\sin 2x)^{\frac{3}{2}}+C$     28.  $2\sqrt{1+\sin x}+C$
29.  $\frac{1}{2}(\log \sin x)^2+C$     30.  $-\log|1+\cos x|+C$     31.  $\frac{1}{1+\cos x}+C$
32.  $\frac{x}{2}-\frac{1}{2}\log|\cos x+\sin x|+C$     33.  $\frac{x}{2}-\frac{1}{2}\log|\cos x-\sin x|+C$
34.  $2\sqrt{\tan x}+C$     35.  $\frac{1}{3}(1+\log x)^3+C$     36.  $\frac{1}{3}(x+\log x)^3+C$
37.  $-\frac{1}{4}\cos(\tan^{-1}x^4)+C$     38. D
39. B

## EXERCISE 7.3

1.  $\frac{x}{2} - \frac{1}{8} \sin(4x+10) + C$

2.  $-\frac{1}{14} \cos 7x + \frac{1}{2} \cos x + C$

3.  $\frac{1}{4} \left[ \frac{1}{12} \sin 12x + x + \frac{1}{8} \sin 8x + \frac{1}{4} \sin 4x \right] + C$

4.  $-\frac{1}{2} \cos(2x+1) + \frac{1}{6} \cos^3(2x+1) + C$

5.  $\frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + C$

6.  $\frac{1}{4} \left[ \frac{1}{6} \cos 6x - \frac{1}{4} \cos 4x - \frac{1}{2} \cos 2x \right] + C$

7.  $\frac{1}{2} \left[ \frac{1}{4} \sin 4x - \frac{1}{12} \sin 12x \right] + C$

8.  $2 \tan \frac{x}{2} - x + C$

9.  $x - \tan \frac{x}{2} + C$

10.  $\frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$

11.  $\frac{3x}{8} + \frac{1}{8} \sin 4x + \frac{1}{64} \sin 8x + C$

12.  $x - \sin x + C$

13.  $2(\sin x + x \cos \alpha) + C$

14.  $-\frac{1}{\cos x + \sin x} + C$

15.  $\frac{1}{6} \sec^3 2x - \frac{1}{2} \sec 2x + C$

16.  $\frac{1}{3} \tan^3 x - \tan x + x + C$

17.  $\sec x - \operatorname{cosec} x + C$

18.  $\tan x + C$

19.  $\log |\tan x| + \frac{1}{2} \tan^2 x + C$

20.  $\log |\cos x + \sin x| + C$

21.  $\frac{\pi x}{2} - \frac{x^2}{2} + C$

22.  $\frac{1}{\sin(a-b)} \log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| + C$

23. A

24. B

## EXERCISE 7.4

1.  $\tan^{-1} x^3 + C$

2.  $\frac{1}{2} \log \left| 2x + \sqrt{1+4x^2} \right| + C$

3.  $\log \left| \frac{1}{2-x+\sqrt{x^2-4x+5}} \right| + C$       4.  $\frac{1}{5} \sin^{-1} \frac{5x}{3} + C$
5.  $\frac{3}{2\sqrt{2}} \tan^{-1} \sqrt{2} x^2 + C$       6.  $\frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C$
7.  $\sqrt{x^2-1} - \log \left| x + \sqrt{x^2-1} \right| + C$       8.  $\frac{1}{3} \log \left| x^3 + \sqrt{x^6+a^6} \right| + C$
9.  $\log \left| \tan x + \sqrt{\tan^2 x + 4} \right| + C$       10.  $\log \left| x+1+\sqrt{x^2+2x+2} \right| + C$
11.  $\frac{1}{6} \tan^{-1} \left( \frac{3x+1}{2} \right) + C$       12.  $\sin^{-1} \left( \frac{x+3}{4} \right) + C$
13.  $\log \left| x - \frac{3}{2} + \sqrt{x^2-3x+2} \right| + C$       14.  $\sin^{-1} \left( \frac{2x-3}{\sqrt{41}} \right) + C$
15.  $\log \left| x - \frac{a+b}{2} + \sqrt{(x-a)(x-b)} \right| + C$
16.  $2\sqrt{2x^2+x-3} + C$       17.  $\sqrt{x^2-1} + 2 \log \left| x + \sqrt{x^2-1} \right| + C$
18.  $\frac{5}{6} \log |3x^2+2x+1| - \frac{11}{3\sqrt{2}} \tan^{-1} \left( \frac{3x+1}{\sqrt{2}} \right) + C$
19.  $6\sqrt{x^2-9x+20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2-9x+20} \right| + C$
20.  $-\sqrt{4x-x^2} + 4 \sin^{-1} \left( \frac{x-2}{2} \right) + C$
21.  $\sqrt{x^2+2x+3} + \log \left| x+1+\sqrt{x^2+2x+3} \right| + C$
22.  $\frac{1}{2} \log |x^2-2x-5| + \frac{2}{\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C$

23.  $5\sqrt{x^2 + 4x + 10} - 7 \log|x + 2 + \sqrt{x^2 + 4x + 10}| + C$

24. B

25. B

**EXERCISE 7.5**

1.  $\log \frac{(x+2)^2}{|x+1|} + C$

2.  $\frac{1}{6} \log \left| \frac{x-3}{x+3} \right| + C$

3.  $\log|x-1| - 5 \log|x-2| + 4 \log|x-3| + C$

4.  $\frac{1}{2} \log|x-1| - 2 \log|x-2| + \frac{3}{2} \log|x-3| + C$

5.  $4 \log|x+2| - 2 \log|x+1| + C$

6.  $\frac{x}{2} + \log|x| - \frac{3}{4} \log|1-2x| + C$

7.  $\frac{1}{2} \log|x-1| - \frac{1}{4} \log(x^2 + 1) + \frac{1}{2} \tan^{-1}x + C$

8.  $\frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C$

9.  $\frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{x-1} + C$

10.  $\frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{12}{5} \log|2x+3| + C$

11.  $\frac{5}{3} \log|x+1| - \frac{5}{2} \log|x+2| + \frac{5}{6} \log|x-2| + C$

12.  $\frac{x^2}{2} + \frac{1}{2} \log|x+1| + \frac{3}{2} \log|x-1| + C$

13.  $-\log|x-1| + \frac{1}{2} \log(1+x^2) + \tan^{-1}x + C$

14.  $3 \log|x+2| + \frac{7}{x+2} + C$

15.  $\frac{1}{4} \log \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1}x + C$

16.  $\frac{1}{n} \log \left| \frac{x^n}{x^n + 1} \right| + C$

17.  $\log \left| \frac{2-\sin x}{1-\sin x} \right| + C$

18.  $x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C$

19.  $\frac{1}{2} \log \left( \frac{x^2 + 1}{x^2 + 3} \right) + C$

20.  $\frac{1}{4} \log \left| \frac{x^4 - 1}{x^4} \right| + C$

22. B

21.  $\log \left( \frac{e^x - 1}{e^x} \right) + C$

23. A

**EXERCISE 7.6**

1.  $-x \cos x + \sin x + C$

2.  $-\frac{x}{3} \cos 3x + \frac{1}{9} \sin 3x + C$

3.  $e^x (x^2 - 2x + 2) + C$

4.  $\frac{x^2}{2} \log x - \frac{x^2}{4} + C$

5.  $\frac{x^2}{2} \log 2x - \frac{x^2}{4} + C$

6.  $\frac{x^3}{3} \log x - \frac{x^3}{9} + C$

7.  $\frac{1}{4} (2x^2 - 1) \sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} + C$

8.  $\frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$

9.  $(2x^2 - 1) \frac{\cos^{-1} x}{4} - \frac{x}{4} \sqrt{1-x^2} + C$

10.  $(\sin^{-1} x)^2 x + 2\sqrt{1-x^2} \sin^{-1} x - 2x + C$

11.  $-\sqrt{1-x^2} \cos^{-1} x + x + C$

12.  $x \tan x + \log |\cos x| + C$

13.  $x \tan^{-1} x - \frac{1}{2} \log(1+x^2) + C$

14.  $\frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + C$

15.  $\left( \frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C$

16.  $e^x \sin x + C$

17.  $\frac{e^x}{1+x} + C$

18.  $e^x \tan \frac{x}{2} + C$

19.  $\frac{e^x}{x} + C$

20.  $\frac{e^x}{(x-1)^2} + C$

21.  $\frac{e^{2x}}{5} (2 \sin x - \cos x) + C$

22.  $2x \tan^{-1} x - \log(1+x^2) + C$

23. A

24. B

**EXERCISE 7.7**

1.  $\frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$
2.  $\frac{1}{4}\sin^{-1}2x + \frac{1}{2}x\sqrt{1-4x^2} + C$
3.  $\frac{(x+2)}{2}\sqrt{x^2+4x+6} + \log\left|x+2+\sqrt{x^2+4x+6}\right| + C$
4.  $\frac{(x+2)}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|x+2+\sqrt{x^2+4x+1}\right| + C$
5.  $\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + \frac{x+2}{2}\sqrt{1-4x-x^2} + C$
6.  $\frac{(x+2)}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|x+2+\sqrt{x^2+4x-5}\right| + C$
7.  $\frac{(2x-3)}{4}\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$
8.  $\frac{2x+3}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right| + C$
9.  $\frac{x}{6}\sqrt{x^2+9} + \frac{3}{2}\log\left|x+\sqrt{x^2+9}\right| + C$
10. A
11. D

**EXERCISE 7.8**

1. 2
2.  $\log\frac{3}{2}$
3.  $\frac{64}{3}$
4.  $\frac{1}{2}$
5. 0
6.  $e^4(e-1)$
7.  $\frac{1}{2}\log 2$
8.  $\log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$
9.  $\frac{\pi}{2}$
10.  $\frac{\pi}{4}$
11.  $\frac{1}{2}\log\frac{3}{2}$
12.  $\frac{\pi}{4}$

13.  $\frac{1}{2} \log 2$

14.  $\frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}$

15.  $\frac{1}{2}(e - 1)$

16.  $5 - \frac{5}{2} \left( 9 \log \frac{5}{4} - \log \frac{3}{2} \right)$

17.  $\frac{\pi^4}{1024} + \frac{\pi}{2} + 2$

18. 0

19.  $3 \log 2 + \frac{3\pi}{8}$

20.  $1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$

21. D

22. C

**EXERCISE 7.9**

1.  $\frac{1}{2} \log 2$

2.  $\frac{64}{231}$

3.  $\frac{\pi}{2} - \log 2$

4.  $\frac{16\sqrt{2}}{15}(\sqrt{2} + 1)$

5.  $\frac{\pi}{4}$

6.  $\frac{1}{\sqrt{17}} \log \frac{21+5\sqrt{17}}{4}$

7.  $\frac{\pi}{8}$

8.  $\frac{e^2(e^2-2)}{4}$

9. D

10. B

**EXERCISE 7.10**

1.  $\frac{\pi}{4}$

2.  $\frac{\pi}{4}$

3.  $\frac{\pi}{4}$

4.  $\frac{\pi}{4}$

5. 29

6. 9

7.  $\frac{1}{(n+1)(n+2)}$

8.  $\frac{\pi}{8} \log 2$

9.  $\frac{16\sqrt{2}}{15}$

10.  $\frac{\pi}{2} \log \frac{1}{2}$

11.  $\frac{\pi}{2}$

12.  $\pi$

13. 0

14. 0

15. 0

16.  $-\pi \log 2$

17.  $\frac{a}{2}$

18. 5

20. C

21. C

**MISCELLANEOUS EXERCISE ON CHAPTER 7**

1.  $\frac{1}{2} \log \left| \frac{x^2}{1-x^2} \right| + C$

2.  $\frac{2}{3(a-b)} \left[ (x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + C$

3.  $-\frac{2}{a} \sqrt{\frac{(a-x)}{x}} + C$

4.  $-\left( 1 + \frac{1}{x^4} \right)^{\frac{1}{4}} + C$

5.  $2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^6 - 6\log(1+x^6) + C$

6.  $-\frac{1}{2} \log|x+1| + \frac{1}{4} \log(x^2+9) + \frac{3}{2} \tan^{-1} \frac{x}{3} + C$

7.  $\sin a \log|\sin(x-a)| + x \cos a + C$  8.  $\frac{x^3}{3} + C$

9.  $\sin^{-1} \left( \frac{\sin x}{2} \right) + C$

10.  $-\frac{1}{2} \sin 2x + C$

11.  $\frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C$  12.  $\frac{1}{4} \sin^{-1}(x^4) + C$

13.  $\log \left( \frac{1+e^x}{2+e^x} \right) + C$

14.  $\frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C$

15.  $-\frac{1}{4} \cos^4 x + C$

16.  $\frac{1}{4} \log(x^4+1) + C$

17.  $\frac{[f(ax+b)]^{n+1}}{a(n+1)} + C$

18.  $\frac{-2}{\sin \alpha} \sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$

19.  $-2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x-x^2} + C$

20.  $e^x \tan x + C$

21.  $-2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + C$

22.  $\frac{1}{2} \left[ x \cos^{-1} x - \sqrt{1-x^2} \right] + C$

23.  $-\frac{1}{3} \left( 1 + \frac{1}{x^2} \right)^{\frac{3}{2}} \left[ \log \left( 1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + C$

24.  $e^{\frac{\pi}{2}}$

25.  $\frac{\pi}{8}$

26.  $\frac{\pi}{6}$

27.  $2\sin^{-1}\frac{(\sqrt{3}-1)}{2}$

28.  $\frac{4\sqrt{2}}{3}$

29.  $\frac{1}{40}\log 9$

30.  $\frac{\pi}{2}-1$

31.  $\frac{19}{2}$

38. A

39. B

40. D

**EXERCISE 8.1**

1.  $12\pi$

2.  $6\pi$

3. A

4. B

*Miscellaneous Exercise on Chapter 8*

1. (i)  $\frac{7}{3}$

(ii) 624.8

2. 9

3. 4

4. D

5. C

**EXERCISE 9.1**

1. Order 4; Degree not defined

2. Order 1; Degree 1

3. Order 2; Degree 1

4. Order 2; Degree not defined

5. Order 2; Degree 1

6. Order 3; Degree 2

7. Order 3; Degree 1

8. Order 1; Degree 1

9. Order 2; Degree 1

10. Order 2; Degree 1

11. D

12. A

**EXERCISE 9.2**

11. D

12. D

## EXERCISE 9.3

1.  $y = 2 \tan \frac{x}{2} - x + C$
2.  $y = 2 \sin(x + C)$
3.  $y = 1 + Ae^{-x}$
4.  $\tan x \tan y = C$
5.  $y = \log(e^x + e^{-x}) + C$
6.  $\tan^{-1} y = x + \frac{x^3}{3} + C$
7.  $y = e^{cx}$
8.  $x^{-4} + y^{-4} = C$
9.  $y = x \sin^{-1} x + \sqrt{1-x^2} + C$
10.  $\tan y = C(1 - e^x)$
11.  $y = \frac{1}{4} \log[(x+1)^2(x^2+1)^3] - \frac{1}{2} \tan^{-1} x + 1$
12.  $y = \frac{1}{2} \log\left(\frac{x^2-1}{x^2}\right) - \frac{1}{2} \log \frac{3}{4}$
13.  $\cos\left(\frac{y-2}{x}\right) = a$
14.  $y = \sec x$
15.  $2y - 1 = e^x(\sin x - \cos x)$
16.  $y - x + 2 = \log(x^2(y+2)^2)$
17.  $y^2 - x^2 = 4$
18.  $(x+4)^2 = y+3$
19.  $(63t+27)^{\frac{1}{3}}$
20. 6.93%
21. Rs 1648
22.  $\frac{2 \log 2}{\log\left(\frac{11}{10}\right)}$
23. A

## EXERCISE 9.4

1.  $(x-y)^2 = Cx e^{\frac{-y}{x}}$
2.  $y = x \log|x| + Cx$
3.  $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2} \log(x^2 + y^2) + C$
4.  $x^2 + y^2 = Cx$
5.  $\frac{1}{2\sqrt{2}} \log \left| \frac{x+\sqrt{2}y}{x-\sqrt{2}y} \right| = \log|x| + C$
6.  $y + \sqrt{x^2 + y^2} = Cx^2$
7.  $xy \cos \left| \frac{y}{x} \right| = C$
8.  $x \left[ 1 - \cos\left(\frac{y}{x}\right) \right] = C \sin\left(\frac{y}{x}\right)$

9.  $cy = \log \left| \frac{y}{x} \right| - 1$

10.  $ye^{\frac{x}{y}} + x = C$

11.  $\log(x^2 + y^2) + 2 \tan^{-1} \frac{y}{x} = \frac{\pi}{2} + \log 2$

12.  $y + 2x = 3x^2 y$

13.  $\cot\left(\frac{y}{x}\right) = \log|ex|$

14.  $\cos\left(\frac{y}{x}\right) = \log|ex|$

15.  $y = \frac{2x}{1 - \log|x|} \quad (x \neq 0, x \neq e)$

16. C

17. D

## EXERCISE 9.5

1.  $y = \frac{1}{5}(2\sin x - \cos x) + C e^{-2x}$

2.  $y = e^{-2x} + C e^{-3x}$

3.  $xy = \frac{x^4}{4} + C$

4.  $y(\sec x + \tan x) = \sec x + \tan x - x + C$

5.  $y = (\tan x - 1) + C e^{-\tan x}$

6.  $y = \frac{x^2}{16}(4 \log|x| - 1) + C x^{-2}$

7.  $y \log x = \frac{-2}{x}(1 + \log|x|) + C$

8.  $y = (1+x)^{-1} \log|\sin x| + C(1+x^2)^{-1}$

9.  $y = \frac{1}{x} - \cot x + \frac{C}{x \sin x}$

10.  $(x + y + 1) = C e^y$

11.  $x = \frac{y^2}{3} + \frac{C}{y}$

12.  $x = 3y^2 + Cy$

13.  $y = \cos x - 2 \cos^2 x$

14.  $y(1 + x^2) = \tan^{-1} x - \frac{\pi}{4}$

15.  $y = 4 \sin^3 x - 2 \sin^2 x$

16.  $x + y + 1 = e^x$

17.  $y = 4 - x - 2 e^x$

18. C

19. D

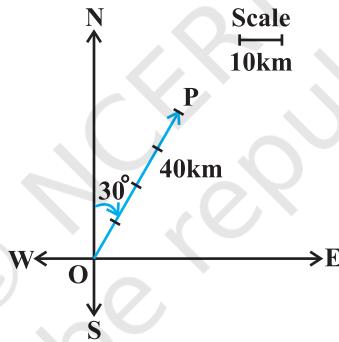
**Miscellaneous Exercise on Chapter 9**

1. (i) Order 2; Degree 1      (ii) Order 1; Degree 3  
 (iii) Order 4; Degree not defined

4.  $\sin^{-1}y + \sin^{-1}x = C$       6.  $\cos y = \frac{\sec x}{\sqrt{2}}$
7.  $\tan^{-1} y + \tan^{-1}(e^x) = \frac{\pi}{2}$       8.  $e^{\frac{x}{y}} = y + C$
9.  $\log |x - y| = x + y + 1$       10.  $ye^{2\sqrt{x}} = (2\sqrt{x} + C)$
11.  $y \sin x = 2x^2 - \frac{\pi^2}{2}$  ( $\sin x \neq 0$ )      12.  $y = \log \left| \frac{2x+1}{x+1} \right|, x \neq -1$
13. C      14. C
15. C

### EXERCISE 10.1

1. In the adjoining figure, the vector  $\overrightarrow{OP}$  represents the required displacement.



2. (i) scalar (ii) vector (iii) scalar (iv) scalar (v) scalar  
 (vi) vector
3. (i) scalar (ii) scalar (iii) vector (iv) vector (v) scalar
4. (i) Vectors  $\vec{a}$  and  $\vec{b}$  are coinitial  
 (ii) Vectors  $\vec{b}$  and  $\vec{d}$  are equal  
 (iii) Vectors  $\vec{a}$  and  $\vec{c}$  are collinear but not equal
5. (i) True (ii) False (iii) False (iv) False

### EXERCISE 10.2

1.  $|\vec{a}| = \sqrt{3}, |\vec{b}| = \sqrt{62}, |\vec{c}| = 1$
2. An infinite number of possible answers.

3. An infinite number of possible answers.
4.  $x = 2, y = 3$
5.  $-7$  and  $6$ ;  $-7\hat{i}$  and  $6\hat{j}$
6.  $-4\hat{j} - \hat{k}$
7.  $\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$
8.  $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$
9.  $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$
10.  $\frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$
12.  $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$
13.  $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$
15. (i)  $-\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$  (ii)  $-3\hat{i} + 3\hat{k}$
16.  $3\hat{i} + 2\hat{j} + \hat{k}$
18. (C)
19. (B), (C), (D)

### EXERCISE 10.3

1.  $\frac{\pi}{4}$
2.  $\cos^{-1}\left(\frac{5}{7}\right)$
3. 0
4.  $\frac{60}{\sqrt{114}}$
6.  $\frac{16\sqrt{2}}{3\sqrt{7}}, \frac{2\sqrt{2}}{3\sqrt{7}}$
7.  $6|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$
8.  $|\vec{a}|=1, |\vec{b}|=1$
9.  $\sqrt{13}$
10. 8
12. Vector  $\vec{b}$  can be any vector
13.  $\frac{-3}{2}$
14. Take any two non-zero perpendicular vectors  $\vec{a}$  and  $\vec{b}$
15.  $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$
18. (D)

### EXERCISE 10.4

1.  $19\sqrt{2}$
2.  $\pm\frac{2}{3}\hat{i}, \frac{2}{3}\hat{j}, \frac{1}{3}\hat{k}$
3.  $\frac{\pi}{3}; \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$
5.  $3, \frac{27}{2}$
6. Either  $|\vec{a}|=0$  or  $|\vec{b}|=0$

8. No; take any two nonzero collinear vectors

9.  $\frac{\sqrt{61}}{2}$

10.  $15\sqrt{2}$

11. (B)

12. (C)

### Miscellaneous Exercise on Chapter 10

1.  $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

2.  $x_2 - x_1, y_2 - y_1, z_2 - z_1; \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

3.  $\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

4. No; take  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  to represent the sides of a triangle.

5.  $\pm \frac{1}{\sqrt{3}}$

6.  $\frac{3}{2}\sqrt{10}\hat{i} + \frac{\sqrt{10}}{2}\hat{j}$

7.  $\frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}$

8.  $2:3$

9.  $3\vec{a} + 5\vec{b}$

10.  $\frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}); 11\sqrt{5}$

12.  $\frac{1}{3}(160\hat{i} - 5\hat{j} + 70\hat{k})$

13.  $\lambda = 1$

16. (B)

17. (D)

18. (C)

19. (B)

### EXERCISE 11.1

1.  $0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

2.  $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}$

3.  $\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

5.  $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{17}; \frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}; \frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$

### EXERCISE 11.2

4.  $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$ , where  $\lambda$  is a real number

5.  $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$  and cartesian form is

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

6.  $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$

7.  $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

8. (i)  $\theta = \cos^{-1}\left(\frac{19}{21}\right)$  (ii)  $\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$

9. (i)  $\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$  (ii)  $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

10.  $p = \frac{70}{11}$

12.  $\frac{3\sqrt{2}}{2}$

13.  $2\sqrt{29}$

14.  $\frac{3}{\sqrt{19}}$

15.  $\frac{8}{\sqrt{29}}$

### Miscellaneous Exercise on Chapter 11

1.  $90^\circ$

2.  $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

3.  $k = \frac{-10}{7}$

4. 9

5.  $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

#### EXERCISE 12.1

1. Maximum  $Z = 16$  at  $(0, 4)$

2. Minimum  $Z = -12$  at  $(4, 0)$

3. Maximum  $Z = \frac{235}{19}$  at  $\left(\frac{20}{19}, \frac{45}{19}\right)$

4. Minimum  $Z = 7$  at  $\left(\frac{3}{2}, \frac{1}{2}\right)$

5. Maximum  $Z = 18$  at  $(4, 3)$

6. Minimum  $Z = 6$  at all the points on the line segment joining the points  $(6, 0)$  and  $(0, 3)$ .

7. Minimum  $Z = 300$  at  $(60, 0)$ ;

Maximum  $Z = 600$  at all the points on the line segment joining the points  $(120, 0)$  and  $(60, 30)$ .

8. Minimum  $Z = 100$  at all the points on the line segment joining the points  $(0, 50)$  and  $(20, 40)$ ;

Maximum  $Z = 400$  at  $(0, 200)$

9.  $Z$  has no maximum value

10. No feasible region, hence no maximum value of  $Z$ .

### EXERCISE 13.1

1.  $P(E|F) = \frac{2}{3}$ ,  $P(F|E) = \frac{1}{3}$       2.  $P(A|B) = \frac{16}{25}$
3. (i) 0.32      (ii) 0.64      (iii) 0.98
4.  $\frac{11}{26}$
5. (i)  $\frac{4}{11}$       (ii)  $\frac{4}{5}$       (iii)  $\frac{2}{3}$
6. (i)  $\frac{1}{2}$       (ii)  $\frac{3}{7}$       (iii)  $\frac{6}{7}$
7. (i) 1      (ii) 0
8.  $\frac{1}{6}$       9. 1      10. (a)  $\frac{1}{3}$ , (b)  $\frac{1}{9}$
11. (i)  $\frac{1}{2}, \frac{1}{3}$       (ii)  $\frac{1}{2}, \frac{2}{3}$       (iii)  $\frac{3}{4}, \frac{1}{4}$
12. (i)  $\frac{1}{2}$       (ii)  $\frac{1}{3}$       13.  $\frac{5}{9}$
14.  $\frac{1}{15}$       15. 0      16. C      17. D

**EXERCISE 13.2**

1.  $\frac{3}{25}$

2.  $\frac{25}{102}$

3.  $\frac{44}{91}$

4. A and B are independent

5. A and B are not independent

6. E and F are not independent

7. (i)  $p = \frac{1}{10}$

(ii)  $p = \frac{1}{5}$

8. (i) 0.12 (ii) 0.58 (iii) 0.3 (iv) 0.4

9.  $\frac{3}{8}$

10. A and B are not independent

11. (i) 0.18 (ii) 0.12 (iii) 0.72 (iv) 0.28

12.  $\frac{7}{8}$  13. (i)  $\frac{16}{81}$ , (ii)  $\frac{20}{81}$ , (iii)  $\frac{40}{81}$

14. (i)  $\frac{2}{3}$ , (ii)  $\frac{1}{2}$  15. (i), (ii) 16. (a)  $\frac{1}{5}$ , (b)  $\frac{1}{3}$ , (c)  $\frac{1}{2}$

17. D 18. B

**EXERCISE 13.3**

1.  $\frac{1}{2}$

2.  $\frac{2}{3}$

3.  $\frac{9}{13}$

4.  $\frac{12}{13}$

5.  $\frac{22}{133}$

6.  $\frac{4}{9}$

7.  $\frac{1}{52}$

8.  $\frac{1}{4}$

9.  $\frac{2}{9}$

10.  $\frac{8}{11}$

11.  $\frac{5}{34}$

12.  $\frac{11}{50}$

13. A

14. C

**Miscellaneous Exercise on Chapter 13**

1. (i) 1 (ii) 0

2. (i)  $\frac{1}{3}$  (ii)  $\frac{1}{2}$

3.  $\frac{20}{21}$

4.  $1 - \sum_{r=7}^{10} {}^{10}C_r (0.9)^r (0.1)^{10-r}$

5.  $\frac{2}{7}$

6.  $\frac{1}{15}, \frac{2}{5}, \frac{8}{15}$

7.  $\frac{14}{29}$

8.  $\frac{3}{16}$

9. (i) 0.5      (ii) 0.05

10.  $\frac{16}{31}$

11. A

12. C

13. B



## Notes

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# MATHEMATICS

## PART II

*Textbook for Class XII*



12080



राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्  
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## Foreword

The National Curriculum Framework, 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

NCERT appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisor for this book, Professor P.K. Jain for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

New Delhi  
20 November 2006

*Director*  
National Council of Educational  
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## **Rationalisation of Content in the Textbooks**

In view of the COVID-19 pandemic, it is imperative to reduce content load on students. The National Education Policy 2020, also emphasises reducing the content load and providing opportunities for experiential learning with creative mindset. In this background, the NCERT has undertaken the exercise to rationalise the textbooks across all classes. Learning Outcomes already developed by the NCERT across classes have been taken into consideration in this exercise.

**Contents of the textbooks have been rationalised in view of the following:**

- Overlapping with similar content included in other subject areas in the same class
- Similar content included in the lower or higher class in the same subject
- Difficulty level
- Content, which is easily accessible to students without much interventions from teachers and can be learned by children through self-learning or peer-learning
- Content, which is irrelevant in the present context

This present edition, is a reformatted version after carrying out the changes given above.

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## Preface

The National Council of Educational Research and Training (NCERT) had constituted 21 Focus Groups on Teaching of various subjects related to School Education, to review the National Curriculum Framework for School Education - 2000 (NCFSE - 2000) in face of new emerging challenges and transformations occurring in the fields of content and pedagogy under the contexts of National and International spectrum of school education. These Focus Groups made general and specific comments in their respective areas. Consequently, based on these reports of Focus Groups, National Curriculum Framework (NCF)-2005 was developed.

NCERT designed the new syllabi and constituted Textbook Development Teams for Classes XI and XII to prepare textbooks in Mathematics under the new guidelines and new syllabi. The textbook for Class XI is already in use, which was brought in 2005.

The first draft of the present book (Class XII) was prepared by the team consisting of NCERT faculty, experts and practicing teachers. The draft was refined by the development team in different meetings. This draft of the book was exposed to a group of practicing teachers teaching Mathematics at higher secondary stage in different parts of the country, in a review workshop organised by the NCERT at Delhi. The teachers made useful comments and suggestions which were incorporated in the draft textbook. The draft textbook was finalised by an editorial board constituted out of the development team. Finally, the Advisory Group in Science and Mathematics and the Monitoring Committee constituted by the HRD Ministry, Government of India have approved the draft of the textbook.

In the fitness of things, let us cite some of the essential features dominating the textbook. These characteristics have reflections in almost all the chapters. The existing textbook contains thirteen main chapters and two appendices. Each chapter contains the followings :

- Introduction: Highlighting the importance of the topic; connection with earlier studied topics; brief mention about the new concepts to be discussed in the chapter.
- Organisation of chapter into sections comprising one or more concepts/ subconcepts.
- Motivating and introducing the concepts/subconcepts. Illustrations have been provided wherever possible.

- Proofs/problem solving involving deductive or inductive reasoning, multiplicity of approaches wherever possible have been inducted.
- Geometric viewing / visualisation of concepts have been emphasized whenever needed.
- Applications of mathematical concepts have also been integrated with allied subjects like Science and Social Sciences.
- Adequate and variety of examples/exercises have been given in each section.
- For refocusing and strengthening the understanding and skill of problem solving and applicabilities, miscellaneous types of examples/exercises have been provided involving two or more subconcepts at a time at the end of the chapter. The scope of challenging problems to talented minority have been reflected conducive to the recommendation as reflected in NCF-2005.
- For more motivational purpose, brief historical background of topics have been provided at the end of the chapter and at the beginning of each chapter, relevant quotation and photograph of eminent mathematician who have contributed significantly in the development of the topic undertaken, are also provided.
- Lastly, for direct recapitulation of main concepts, formulas and results, brief summary of the chapter has also been provided.

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PAWAN K. JAIN  
*Chief Advisor*

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# **THE CONSTITUTION OF INDIA**

## **PREAMBLE**

**WE, THE PEOPLE OF INDIA**, having solemnly resolved to constitute India into a <sup>1</sup>**[SOVEREIGN SOCIALIST SECULAR  
DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

**JUSTICE**, social, economic and political;

**LIBERTY** of thought, expression, belief, faith and worship;

**EQUALITY** of status and of opportunity; and to promote among them all

**FRATERNITY** assuring the dignity of the individual and the <sup>2</sup>[unity and integrity of the Nation];

**IN OUR CONSTITUENT ASSEMBLY** this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

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# INTEGRALS

❖ *Just as a mountaineer climbs a mountain – because it is there, so a good mathematics student studies new material because it is there. — JAMES B. BRISTOL* ❖

## 7.1 Introduction

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.

If a function  $f$  is differentiable in an interval  $I$ , i.e., its derivative  $f'$  exists at each point of  $I$ , then a natural question arises that given  $f'$  at each point of  $I$ , can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the *indefinite integral* of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types:

- the problem of finding a function whenever its derivative is given,
- the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the *Integral Calculus*.



G .W. Leibnitz  
(1646 -1716)

There is a connection, known as the ***Fundamental Theorem of Calculus***, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering. The definite integral is also used to solve many interesting problems from various disciplines like economics, finance and probability.

In this Chapter, we shall confine ourselves to the study of indefinite and definite integrals and their elementary properties including some techniques of integration.

## 7.2 Integration as an Inverse Process of Differentiation

Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*. Let us consider the following examples:

We know that 
$$\frac{d}{dx}(\sin x) = \cos x \quad \dots (1)$$

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \quad \dots (2)$$

and 
$$\frac{d}{dx}(e^x) = e^x \quad \dots (3)$$

We observe that in (1), the function  $\cos x$  is the derived function of  $\sin x$ . We say that  $\sin x$  is an anti derivative (or an integral) of  $\cos x$ . Similarly, in (2) and (3),  $\frac{x^3}{3}$  and  $e^x$  are the anti derivatives (or integrals) of  $x^2$  and  $e^x$ , respectively. Again, we note that for any real number  $C$ , treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows :

$$\frac{d}{dx}(\sin x + C) = \cos x, \quad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2 \text{ and } \frac{d}{dx}(e^x + C) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing  $C$  arbitrarily from the set of real numbers. For this reason  $C$  is customarily referred to as ***arbitrary constant***. In fact,  $C$  is the ***parameter*** by varying which one gets different anti derivatives (or integrals) of the given function.

More generally, if there is a function  $F$  such that  $\frac{d}{dx}F(x) = f(x), \forall x \in I$  (interval), then for any arbitrary real number  $C$ , (also called *constant of integration*)

$$\frac{d}{dx}[F(x) + C] = f(x), \quad x \in I$$

Thus,  $\{F + C, C \in \mathbf{R}\}$  denotes a family of anti derivatives of  $f$ .

**Remark** Functions with same derivatives differ by a constant. To show this, let  $g$  and  $h$  be two functions having the same derivatives on an interval  $I$ .

Consider the function  $f = g - h$  defined by  $f(x) = g(x) - h(x), \forall x \in I$

Then  $\frac{df}{dx} = f' = g' - h'$  giving  $f'(x) = g'(x) - h'(x) \forall x \in I$

or  $f'(x) = 0, \forall x \in I$  by hypothesis,

i.e., the rate of change of  $f$  with respect to  $x$  is zero on  $I$  and hence  $f$  is constant.

In view of the above remark, it is justified to infer that the family  $\{F + C, C \in \mathbf{R}\}$  provides all possible anti derivatives of  $f$ .

We introduce a new symbol, namely,  $\int f(x) dx$  which will represent the entire class of anti derivatives read as the indefinite integral of  $f$  with respect to  $x$ .

Symbolically, we write  $\int f(x) dx = F(x) + C$ .

**Notation** Given that  $\frac{dy}{dx} = f(x)$ , we write  $y = \int f(x) dx$ .

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the Table (7.1).

Table 7.1

| Symbols/Terms/Phrases    | Meaning   |
|--------------------------|---|
| $\int f(x) dx$           | Integral of $f$ with respect to $x$                   |
| $f(x)$ in $\int f(x) dx$ | Integrand   |
| $x$ in $\int f(x) dx$    | Variable of integration                               |
| Integrate                | Find the integral                                     |
| An integral of $f$       | A function $F$ such that<br>$F'(x) = f(x)$            |
| Integration              | The process of finding the integral                   |
| Constant of Integration  | Any real number $C$ , considered as constant function |

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down immediately the corresponding formulae (referred to as standard formulae) for the integrals of these functions, as listed below which will be used to find integrals of other functions.

### Derivatives

$$(i) \frac{d}{dx} \left( \frac{x^{n+1}}{n+1} \right) = x^n ;$$

Particularly, we note that

$$\frac{d}{dx} (x) = 1 ;$$

$$(ii) \frac{d}{dx} (\sin x) = \cos x ;$$

$$(iii) \frac{d}{dx} (-\cos x) = \sin x ;$$

$$(iv) \frac{d}{dx} (\tan x) = \sec^2 x ;$$

$$(v) \frac{d}{dx} (-\cot x) = \operatorname{cosec}^2 x ;$$

$$(vi) \frac{d}{dx} (\sec x) = \sec x \tan x ;$$

$$(vii) \frac{d}{dx} (-\operatorname{cosec} x) = \operatorname{cosec} x \cot x ;$$

$$(viii) \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} ;$$

$$(ix) \frac{d}{dx} (-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} ;$$

$$(x) \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} ;$$

$$(xi) \frac{d}{dx} (e^x) = e^x ;$$

### Integrals (Anti derivatives)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int dx = x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int e^x dx = e^x + C$$

$$(xii) \frac{d}{dx}(\log|x|) = \frac{1}{x}; \quad \int \frac{1}{x} dx = \log|x| + C$$

$$(xiii) \frac{d}{dx} \left( \frac{a^x}{\log a} \right) = a^x; \quad \int a^x dx = \frac{a^x}{\log a} + C$$

 **Note** In practice, we normally do not mention the interval over which the various functions are defined. However, in any specific problem one has to keep it in mind.

### 7.2.1 Some properties of indefinite integral

In this sub section, we shall derive some properties of indefinite integrals.

- (I) The process of differentiation and integration are inverses of each other in the sense of the following results :

$$\frac{d}{dx} \int f(x) dx = f(x)$$

and  $\int f'(x) dx = f(x) + C$ , where C is any arbitrary constant.

**Proof** Let F be any anti derivative of f, i.e.,

$$\frac{d}{dx} F(x) = f(x)$$

Then  $\int f(x) dx = F(x) + C$

$$\begin{aligned} \text{Therefore } \frac{d}{dx} \int f(x) dx &= \frac{d}{dx} (F(x) + C) \\ &= \frac{d}{dx} F(x) = f(x) \end{aligned}$$

Similarly, we note that

$$f'(x) = \frac{d}{dx} f(x)$$

and hence  $\int f'(x) dx = f(x) + C$

where C is arbitrary constant called constant of integration.

- (II) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

**Proof** Let  $f$  and  $g$  be two functions such that

$$\frac{d}{dx} \int f(x) dx = \frac{d}{dx} \int g(x) dx$$

or  $\frac{d}{dx} \left[ \int f(x) dx - \int g(x) dx \right] = 0$

Hence  $\int f(x) dx - \int g(x) dx = C$ , where  $C$  is any real number (Why?)

or  $\int f(x) dx = \int g(x) dx + C$

So the families of curves  $\left\{ \int f(x) dx + C_1, C_1 \in \mathbf{R} \right\}$

and  $\left\{ \int g(x) dx + C_2, C_2 \in \mathbf{R} \right\}$  are identical.

Hence, in this sense,  $\int f(x) dx$  and  $\int g(x) dx$  are equivalent.

 **Note** The equivalence of the families  $\left\{ \int f(x) dx + C_1, C_1 \in \mathbf{R} \right\}$  and  $\left\{ \int g(x) dx + C_2, C_2 \in \mathbf{R} \right\}$  is customarily expressed by writing  $\int f(x) dx = \int g(x) dx$ , without mentioning the parameter.

$$(III) \quad \int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

**Proof** By Property (I), we have

$$\frac{d}{dx} \left[ \int [f(x) + g(x)] dx \right] = f(x) + g(x) \quad \dots (1)$$

On the otherhand, we find that

$$\begin{aligned} \frac{d}{dx} \left[ \int f(x) dx + \int g(x) dx \right] &= \frac{d}{dx} \int f(x) dx + \frac{d}{dx} \int g(x) dx \\ &= f(x) + g(x) \end{aligned} \quad \dots (2)$$

Thus, in view of Property (II), it follows by (1) and (2) that

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx.$$

$$(IV) \quad \text{For any real number } k, \int k f(x) dx = k \int f(x) dx$$

**Proof** By the Property (I),  $\frac{d}{dx} \int k f(x) dx = k f(x)$ .

$$\text{Also } \frac{d}{dx} \left[ k \int f(x) dx \right] = k \frac{d}{dx} \int f(x) dx = k f(x)$$

Therefore, using the Property (II), we have  $\int k f(x) dx = k \int f(x) dx$ .

- (V) Properties (III) and (IV) can be generalised to a finite number of functions  $f_1, f_2, \dots, f_n$  and the real numbers,  $k_1, k_2, \dots, k_n$  giving

$$\begin{aligned} & \int [k_1 f_1(x) + k_2 f_2(x) + \dots + k_n f_n(x)] dx \\ &= k_1 \int f_1(x) dx + k_2 \int f_2(x) dx + \dots + k_n \int f_n(x) dx. \end{aligned}$$

To find an anti derivative of a given function, we search intuitively for a function whose derivative is the given function. The search for the requisite function for finding an anti derivative is known as integration by the method of inspection. We illustrate it through some examples.

**Example 1** Write an anti derivative for each of the following functions using the method of inspection:

$$(i) \cos 2x \quad (ii) 3x^2 + 4x^3 \quad (iii) \frac{1}{x}, x \neq 0$$

### Solution

- (i) We look for a function whose derivative is  $\cos 2x$ . Recall that

$$\frac{d}{dx} \sin 2x = 2 \cos 2x$$

$$\text{or } \cos 2x = \frac{1}{2} \frac{d}{dx} (\sin 2x) = \frac{d}{dx} \left( \frac{1}{2} \sin 2x \right)$$

Therefore, an anti derivative of  $\cos 2x$  is  $\frac{1}{2} \sin 2x$ .

- (ii) We look for a function whose derivative is  $3x^2 + 4x^3$ . Note that

$$\frac{d}{dx} (x^3 + x^4) = 3x^2 + 4x^3.$$

Therefore, an anti derivative of  $3x^2 + 4x^3$  is  $x^3 + x^4$ .

(iii) We know that

$$\frac{d}{dx}(\log x) = \frac{1}{x}, x > 0 \text{ and } \frac{d}{dx}[\log(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}, x < 0$$

Combining above, we get  $\frac{d}{dx}(\log|x|) = \frac{1}{x}, x \neq 0$

Therefore,  $\int \frac{1}{x} dx = \log|x|$  is one of the anti derivatives of  $\frac{1}{x}$ .

**Example 2** Find the following integrals:

$$(i) \int \frac{x^3 - 1}{x^2} dx \quad (ii) \int (x^{\frac{2}{3}} + 1) dx \quad (iii) \int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x}) dx$$

### Solution

(i) We have

$$\begin{aligned} \int \frac{x^3 - 1}{x^2} dx &= \int x dx - \int x^{-2} dx \quad (\text{by Property V}) \\ &= \left( \frac{x^{1+1}}{1+1} + C_1 \right) - \left( \frac{x^{-2+1}}{-2+1} + C_2 \right); \quad C_1, C_2 \text{ are constants of integration} \\ &= \frac{x^2}{2} + C_1 - \frac{x^{-1}}{-1} - C_2 = \frac{x^2}{2} + \frac{1}{x} + C_1 - C_2 \\ &= \frac{x^2}{2} + \frac{1}{x} + C, \text{ where } C = C_1 - C_2 \text{ is another constant of integration.} \end{aligned}$$

 **Note** From now onwards, we shall write only one constant of integration in the final answer.

(ii) We have

$$\begin{aligned} \int (x^{\frac{2}{3}} + 1) dx &= \int x^{\frac{2}{3}} dx + \int dx \\ &= \frac{\frac{2}{3}+1}{\frac{2}{3}+1} x^{\frac{5}{3}} + x + C = \frac{3}{5} x^{\frac{5}{3}} + x + C \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \text{We have } \int \left( x^{\frac{3}{2}} + 2e^x - \frac{1}{x} \right) dx &= \int x^{\frac{3}{2}} dx + \int 2e^x dx - \int \frac{1}{x} dx \\
 &= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2e^x - \log|x| + C \\
 &= \frac{2}{5}x^{\frac{5}{2}} + 2e^x - \log|x| + C
 \end{aligned}$$

**Example 3** Find the following integrals:

$$\begin{aligned}
 \text{(i)} \quad \int (\sin x + \cos x) dx &\quad \text{(ii)} \int \operatorname{cosec} x (\operatorname{cosec} x + \cot x) dx \\
 \text{(iii)} \quad \int \frac{1 - \sin x}{\cos^2 x} dx
 \end{aligned}$$

**Solution**

(i) We have

$$\begin{aligned}
 \int (\sin x + \cos x) dx &= \int \sin x dx + \int \cos x dx \\
 &= -\cos x + \sin x + C
 \end{aligned}$$

(ii) We have

$$\begin{aligned}
 \int (\operatorname{cosec} x (\operatorname{cosec} x + \cot x) dx &= \int \operatorname{cosec}^2 x dx + \int \operatorname{cosec} x \cot x dx \\
 &= -\cot x - \operatorname{cosec} x + C
 \end{aligned}$$

(iii) We have

$$\begin{aligned}
 \int \frac{1 - \sin x}{\cos^2 x} dx &= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx \\
 &= \int \sec^2 x dx - \int \tan x \sec x dx \\
 &= \tan x - \sec x + C
 \end{aligned}$$

**Example 4** Find the anti derivative F of f defined by  $f(x) = 4x^3 - 6$ , where  $F(0) = 3$

**Solution** One anti derivative of  $f(x)$  is  $x^4 - 6x$  since

$$\frac{d}{dx}(x^4 - 6x) = 4x^3 - 6$$

Therefore, the anti derivative F is given by

$$F(x) = x^4 - 6x + C, \text{ where } C \text{ is constant.}$$

Given that

 $F(0) = 3$ , which gives,

$$3 = 0 - 6 \times 0 + C \quad \text{or} \quad C = 3$$

Hence, the required anti derivative is the unique function  $F$  defined by

$$F(x) = x^4 - 6x + 3.$$

### Remarks

- We see that if  $F$  is an anti derivative of  $f$ , then so is  $F + C$ , where  $C$  is any constant. Thus, if we know one anti derivative  $F$  of a function  $f$ , we can write down an infinite number of anti derivatives of  $f$  by adding any constant to  $F$  expressed by  $F(x) + C$ ,  $C \in \mathbf{R}$ . In applications, it is often necessary to satisfy an additional condition which then determines a specific value of  $C$  giving unique anti derivative of the given function.
- Sometimes,  $F$  is not expressible in terms of elementary functions viz., polynomial, logarithmic, exponential, trigonometric functions and their inverses etc. We are therefore blocked for finding  $\int f(x) dx$ . For example, it is not possible to find  $\int e^{-x^2} dx$  by inspection since we can not find a function whose derivative is  $e^{-x^2}$ .
- When the variable of integration is denoted by a variable other than  $x$ , the integral formulae are modified accordingly. For instance

$$\int y^4 dy = \frac{y^{4+1}}{4+1} + C = \frac{1}{5} y^5 + C$$

### EXERCISE 7.1

Find an anti derivative (or integral) of the following functions by the method of inspection.

1.  $\sin 2x$
2.  $\cos 3x$
3.  $e^{2x}$
4.  $(ax + b)^2$
5.  $\sin 2x - 4 e^{3x}$

Find the following integrals in Exercises 6 to 20:

6.  $\int (4 e^{3x} + 1) dx$
7.  $\int x^2 (1 - \frac{1}{x^2}) dx$
8.  $\int (ax^2 + bx + c) dx$
9.  $\int (2x^2 + e^x) dx$
10.  $\int \left( \sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$
11.  $\int \frac{x^3 + 5x^2 - 4}{x^2} dx$
12.  $\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$
13.  $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$
14.  $\int (1 - x) \sqrt{x} dx$

15.  $\int \sqrt{x}(3x^2 + 2x + 3) dx$

16.  $\int (2x - 3\cos x + e^x) dx$

17.  $\int (2x^2 - 3\sin x + 5\sqrt{x}) dx$

18.  $\int \sec x (\sec x + \tan x) dx$

19.  $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$

20.  $\int \frac{2 - 3\sin x}{\cos^2 x} dx.$

Choose the correct answer in Exercises 21 and 22.

21. The anti derivative of  $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$  equals

(A)  $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$

(B)  $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C)  $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(D)  $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

22. If  $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$  such that  $f(2) = 0$ . Then  $f(x)$  is

(A)  $x^4 + \frac{1}{x^3} - \frac{129}{8}$

(B)  $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C)  $x^4 + \frac{1}{x^3} + \frac{129}{8}$

(D)  $x^3 + \frac{1}{x^4} - \frac{129}{8}$

### 7.3 Methods of Integration

In previous section, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function  $F$  whose derivative is  $f$  which led us to the integral of  $f$ . However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on:

1. Integration by Substitution
2. Integration using Partial Fractions
3. Integration by Parts

#### 7.3.1 Integration by substitution

In this section, we consider the method of integration by substitution.

The given integral  $\int f(x) dx$  can be transformed into another form by changing the independent variable  $x$  to  $t$  by substituting  $x = g(t)$ .

Consider

$$I = \int f(x) dx$$

Put  $x = g(t)$  so that  $\frac{dx}{dt} = g'(t)$ .

We write

$$dx = g'(t) dt$$

Thus

$$I = \int f(x) dx = \int f(g(t)) g'(t) dt$$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

**Example 5** Integrate the following functions w.r.t.  $x$ :

$$(i) \sin mx \quad (ii) 2x \sin (x^2 + 1)$$

$$(iii) \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} \quad (iv) \frac{\sin(\tan^{-1} x)}{1+x^2}$$

### Solution

(i) We know that derivative of  $mx$  is  $m$ . Thus, we make the substitution  $mx = t$  so that  $mdx = dt$ .

$$\text{Therefore, } \int \sin mx dx = \frac{1}{m} \int \sin t dt = -\frac{1}{m} \cos t + C = -\frac{1}{m} \cos mx + C$$

(ii) Derivative of  $x^2 + 1$  is  $2x$ . Thus, we use the substitution  $x^2 + 1 = t$  so that  $2x dx = dt$ .

$$\text{Therefore, } \int 2x \sin(x^2 + 1) dx = \int \sin t dt = -\cos t + C = -\cos(x^2 + 1) + C$$

(iii) Derivative of  $\sqrt{x}$  is  $\frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ . Thus, we use the substitution

$$\sqrt{x} = t \text{ so that } \frac{1}{2\sqrt{x}} dx = dt \text{ giving } dx = 2t dt.$$

$$\text{Thus, } \int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \int \frac{2t \tan^4 t \sec^2 t dt}{t} = 2 \int \tan^4 t \sec^2 t dt$$

Again, we make another substitution  $\tan t = u$  so that  $\sec^2 t dt = du$

Therefore,  $2 \int \tan^4 t \sec^2 t \, dt = 2 \int u^4 \, du = 2 \frac{u^5}{5} + C$

$$= \frac{2}{5} \tan^5 t + C \quad (\text{since } u = \tan t)$$

$$= \frac{2}{5} \tan^5 \sqrt{x} + C \quad (\text{since } t = \sqrt{x})$$

Hence,  $\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} \, dx = \frac{2}{5} \tan^5 \sqrt{x} + C$

**Alternatively**, make the substitution  $\tan \sqrt{x} = t$

- (iv) Derivative of  $\tan^{-1} x = \frac{1}{1+x^2}$ . Thus, we use the substitution

$$\tan^{-1} x = t \text{ so that } \frac{dx}{1+x^2} = dt.$$

Therefore,  $\int \frac{\sin(\tan^{-1} x)}{1+x^2} \, dx = \int \sin t \, dt = -\cos t + C = -\cos(\tan^{-1} x) + C$

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i)  $\int \tan x \, dx = \log |\sec x| + C$

We have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Put  $\cos x = t$  so that  $\sin x \, dx = -dt$

Then  $\int \tan x \, dx = - \int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$

or  $\int \tan x \, dx = \log |\sec x| + C$

(ii)  $\int \cot x \, dx = \log |\sin x| + C$

We have  $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

Put  $\sin x = t$  so that  $\cos x dx = dt$

Then  $\int \cot x dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$

(iii)  $\int \sec x dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx$$

Put  $\sec x + \tan x = t$  so that  $\sec x (\tan x + \sec x) dx = dt$

Therefore,  $\int \sec x dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv)  $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

We have

$$\int \operatorname{cosec} x dx = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} dx$$

Put  $\operatorname{cosec} x + \cot x = t$  so that  $-\operatorname{cosec} x (\operatorname{cosec} x + \cot x) dx = dt$

$$\begin{aligned} \text{So } \int \operatorname{cosec} x dx &= -\int \frac{dt}{t} = -\log|t| = -\log|\operatorname{cosec} x + \cot x| + C \\ &= -\log \left| \frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x - \cot x} \right| + C \\ &= \log|\operatorname{cosec} x - \cot x| + C \end{aligned}$$

**Example 6** Find the following integrals:

(i)  $\int \sin^3 x \cos^2 x dx$       (ii)  $\int \frac{\sin x}{\sin(x+a)} dx$       (iii)  $\int \frac{1}{1+\tan x} dx$

**Solution**

(i) We have

$$\begin{aligned} \int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x (\sin x) dx \\ &= \int (1 - \cos^2 x) \cos^2 x (\sin x) dx \end{aligned}$$

Put  $t = \cos x$  so that  $dt = -\sin x dx$

$$\begin{aligned}
 \text{Therefore, } \int \sin^2 x \cos^2 x (\sin x) dx &= - \int (1-t^2) t^2 dt \\
 &= - \int (t^2 - t^4) dt = - \left( \frac{t^3}{3} - \frac{t^5}{5} \right) + C \\
 &= - \frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C
 \end{aligned}$$

(ii) Put  $x + a = t$ . Then  $dx = dt$ . Therefore

$$\begin{aligned}
 \int \frac{\sin x}{\sin(x+a)} dx &= \int \frac{\sin(t-a)}{\sin t} dt \\
 &= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} dt \\
 &= \cos a \int dt - \sin a \int \cot t dt \\
 &= (\cos a) t - (\sin a) [\log |\sin t| + C_1] \\
 &= (\cos a) (x+a) - (\sin a) [\log |\sin(x+a)| + C_1] \\
 &= x \cos a + a \cos a - (\sin a) \log |\sin(x+a)| - C_1 \sin a
 \end{aligned}$$

$$\text{Hence, } \int \frac{\sin x}{\sin(x+a)} dx = x \cos a - \sin a \log |\sin(x+a)| + C,$$

where,  $C = -C_1 \sin a + a \cos a$ , is another arbitrary constant.

$$\begin{aligned}
 \text{(iii)} \quad \int \frac{dx}{1+\tan x} &= \int \frac{\cos x dx}{\cos x + \sin x} \\
 &= \frac{1}{2} \int \frac{(\cos x + \sin x + \cos x - \sin x) dx}{\cos x + \sin x} \\
 &= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{\cos x - \sin x}{\cos x + \sin x} dx \\
 &= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2} \int \frac{\cos x - \sin x}{\cos x + \sin x} dx \quad \dots (1)
 \end{aligned}$$

Now, consider  $I = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Put  $\cos x + \sin x = t$  so that  $(\cos x - \sin x) dx = dt$

Therefore  $I = \int \frac{dt}{t} = \log |t| + C_2 = \log |\cos x + \sin x| + C_2$

Putting it in (1), we get

$$\begin{aligned} \int \frac{dx}{1 + \tan x} &= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2} \log |\cos x + \sin x| + \frac{C_2}{2} \\ &= \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x| + \frac{C_1}{2} + \frac{C_2}{2} \\ &= \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x| + C, \left( C = \frac{C_1}{2} + \frac{C_2}{2} \right) \end{aligned}$$

### EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

1.  $\frac{2x}{1+x^2}$

2.  $\frac{(\log x)^2}{x}$

3.  $\frac{1}{x+x \log x}$

4.  $\sin x \sin (\cos x)$

5.  $\sin (ax+b) \cos (ax+b)$

6.  $\sqrt{ax+b}$

7.  $x \sqrt{x+2}$

8.  $x \sqrt{1+2x^2}$

9.  $(4x+2) \sqrt{x^2+x+1}$

10.  $\frac{1}{x-\sqrt{x}}$

11.  $\frac{x}{\sqrt{x+4}}, x > 0$

12.  $(x^3-1)^{\frac{1}{3}} x^5$

13.  $\frac{x^2}{(2+3x^3)^3}$

14.  $\frac{1}{x(\log x)^m}, x > 0, m \neq 1$

15.  $\frac{x}{9-4x^2}$

16.  $e^{2x+3}$

17.  $\frac{x}{e^{x^2}}$

18.  $\frac{e^{\tan^{-1} x}}{1+x^2}$

19.  $\frac{e^{2x}-1}{e^{2x}+1}$

20.  $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21.  $\tan^2(2x - 3)$

22.  $\sec^2(7 - 4x)$

23.  $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

24.  $\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

25.  $\frac{1}{\cos^2 x (1 - \tan x)^2}$

26.  $\frac{\cos \sqrt{x}}{\sqrt{x}}$

27.  $\sqrt{\sin 2x} \cos 2x$

28.  $\frac{\cos x}{\sqrt{1 + \sin x}}$

29.  $\cot x \log \sin x$

30.  $\frac{\sin x}{1 + \cos x}$

31.  $\frac{\sin x}{(1 + \cos x)^2}$

32.  $\frac{1}{1 + \cot x}$

33.  $\frac{1}{1 - \tan x}$

34.  $\frac{\sqrt{\tan x}}{\sin x \cos x}$

35.  $\frac{(1 + \log x)^2}{x}$

36.  $\frac{(x+1)(x+\log x)^2}{x}$

37.  $\frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38.  $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$  equals

- (A)  $10^x - x^{10} + C$   
 (C)  $(10^x - x^{10})^{-1} + C$

- (B)  $10^x + x^{10} + C$   
 (D)  $\log(10^x + x^{10}) + C$

39.  $\int \frac{dx}{\sin^2 x \cos^2 x}$  equals

- (A)  $\tan x + \cot x + C$   
 (C)  $\tan x \cot x + C$

- (B)  $\tan x - \cot x + C$   
 (D)  $\tan x - \cot 2x + C$

### 7.3.2 Integration using trigonometric identities

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

**Example 7** Find (i)  $\int \cos^2 x \, dx$  (ii)  $\int \sin 2x \cos 3x \, dx$  (iii)  $\int \sin^3 x \, dx$

**Solution**

(i) Recall the identity  $\cos 2x = 2 \cos^2 x - 1$ , which gives

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\begin{aligned} \text{Therefore, } \int \cos^2 x \, dx &= \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx \\ &= \frac{x}{2} + \frac{1}{4} \sin 2x + C \end{aligned}$$

(ii) Recall the identity  $\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$  (Why?)

$$\begin{aligned} \text{Then } \int \sin 2x \cos 3x \, dx &= \frac{1}{2} \left[ \int \sin 5x \, dx - \int \sin x \, dx \right] \\ &= \frac{1}{2} \left[ -\frac{1}{5} \cos 5x + \cos x \right] + C \\ &= -\frac{1}{10} \cos 5x + \frac{1}{2} \cos x + C \end{aligned}$$

(iii) From the identity  $\sin 3x = 3 \sin x - 4 \sin^3 x$ , we find that

$$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$\begin{aligned} \text{Therefore, } \int \sin^3 x \, dx &= \frac{3}{4} \int \sin x \, dx - \frac{1}{4} \int \sin 3x \, dx \\ &= -\frac{3}{4} \cos x + \frac{1}{12} \cos 3x + C \end{aligned}$$

$$\text{Alternatively, } \int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

Put  $\cos x = t$  so that  $-\sin x \, dx = dt$

$$\begin{aligned} \text{Therefore, } \int \sin^3 x \, dx &= - \int (1 - t^2) \, dt = - \int dt + \int t^2 \, dt = -t + \frac{t^3}{3} + C \\ &= -\cos x + \frac{1}{3} \cos^3 x + C \end{aligned}$$

**Remark** It can be shown using trigonometric identities that both answers are equivalent.

### EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

1.  $\sin^2(2x + 5)$
2.  $\sin 3x \cos 4x$
3.  $\cos 2x \cos 4x \cos 6x$
4.  $\sin^3(2x + 1)$
5.  $\sin^3 x \cos^3 x$
6.  $\sin x \sin 2x \sin 3x$
7.  $\sin 4x \sin 8x$
8.  $\frac{1 - \cos x}{1 + \cos x}$
9.  $\frac{\cos x}{1 + \cos x}$
10.  $\sin^4 x$
11.  $\cos^4 2x$
12.  $\frac{\sin^2 x}{1 + \cos x}$
13.  $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$
14.  $\frac{\cos x - \sin x}{1 + \sin 2x}$
15.  $\tan^3 2x \sec 2x$
16.  $\tan^4 x$
17.  $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
18.  $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
19.  $\frac{1}{\sin x \cos^3 x}$
20.  $\frac{\cos 2x}{(\cos x + \sin x)^2}$
21.  $\sin^{-1}(\cos x)$
22.  $\frac{1}{\cos(x - a) \cos(x - b)}$

Choose the correct answer in Exercises 23 and 24.

23.  $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$  is equal to
  - (A)  $\tan x + \cot x + C$
  - (B)  $\tan x + \operatorname{cosec} x + C$
  - (C)  $-\tan x + \cot x + C$
  - (D)  $\tan x + \sec x + C$
24.  $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$  equals
  - (A)  $-\cot(ex^x) + C$
  - (B)  $\tan(xe^x) + C$
  - (C)  $\tan(e^x) + C$
  - (D)  $\cot(e^x) + C$

## 7.4 Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

$$(1) \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$(2) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

$$(3) \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$(4) \int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$(5) \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

$$(6) \int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

We now prove the above results:

$$(1) \text{ We have } \frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)}$$

$$= \frac{1}{2a} \left[ \frac{(x+a) - (x-a)}{(x-a)(x+a)} \right] = \frac{1}{2a} \left[ \frac{1}{x-a} - \frac{1}{x+a} \right]$$

$$\text{Therefore, } \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \left[ \int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right]$$

$$= \frac{1}{2a} [\log |(x-a)| - \log |(x+a)|] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a} \left[ \frac{(a+x) + (a-x)}{(a+x)(a-x)} \right] = \frac{1}{2a} \left[ \frac{1}{a-x} + \frac{1}{a+x} \right]$$

$$\begin{aligned}
 \text{Therefore, } \int \frac{dx}{a^2 - x^2} &= \frac{1}{2a} \left[ \int \frac{dx}{a-x} + \int \frac{dx}{a+x} \right] \\
 &= \frac{1}{2a} [-\log|a-x| + \log|a+x|] + C \\
 &= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C
 \end{aligned}$$

 **Note** The technique used in (1) will be explained in Section 7.5.

- (3) Put  $x = a \tan \theta$ . Then  $dx = a \sec^2 \theta d\theta$ .

$$\begin{aligned}
 \text{Therefore, } \int \frac{dx}{x^2 + a^2} &= \int \frac{a \sec^2 \theta d\theta}{a^2 \tan^2 \theta + a^2} \\
 &= \frac{1}{a} \int d\theta = \frac{1}{a} \theta + C = \frac{1}{a} \tan^{-1} \frac{x}{a} + C
 \end{aligned}$$

- (4) Let  $x = a \sec \theta$ . Then  $dx = a \sec \theta \tan \theta d\theta$ .

$$\begin{aligned}
 \text{Therefore, } \int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a \sec \theta \tan \theta d\theta}{\sqrt{a^2 \sec^2 \theta - a^2}} \\
 &= \int \sec \theta d\theta = \log |\sec \theta + \tan \theta| + C_1 \\
 &= \log \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + C_1 \\
 &= \log \left| x + \sqrt{x^2 - a^2} \right| - \log |a| + C_1 \\
 &= \log \left| x + \sqrt{x^2 - a^2} \right| + C, \text{ where } C = C_1 - \log |a|
 \end{aligned}$$

- (5) Let  $x = a \sin \theta$ . Then  $dx = a \cos \theta d\theta$ .

$$\begin{aligned}
 \text{Therefore, } \int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{a \cos \theta d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \\
 &= \int d\theta = \theta + C = \sin^{-1} \frac{x}{a} + C
 \end{aligned}$$

- (6) Let  $x = a \tan \theta$ . Then  $dx = a \sec^2 \theta d\theta$ .

$$\begin{aligned}
 \text{Therefore, } \int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{a \sec^2 \theta d\theta}{\sqrt{a^2 \tan^2 \theta + a^2}} \\
 &= \int \sec \theta d\theta = \log |(\sec \theta + \tan \theta)| + C_1
 \end{aligned}$$

$$\begin{aligned}
 &= \log \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1} \right| + C_1 \\
 &= \log \left| x + \sqrt{x^2 + a^2} \right| - \log |a| + C_1 \\
 &= \log \left| x + \sqrt{x^2 + a^2} \right| + C, \text{ where } C = C_1 - \log |a|
 \end{aligned}$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

- (7) **To find the integral**  $\int \frac{dx}{ax^2 + bx + c}$ , we write

$$ax^2 + bx + c = a \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right] = a \left[ \left( x + \frac{b}{2a} \right)^2 + \left( \frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$$

Now, put  $x + \frac{b}{2a} = t$  so that  $dx = dt$  and writing  $\frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$ . We find the

integral reduced to the form  $\frac{1}{a} \int \frac{dt}{t^2 \pm k^2}$  depending upon the sign of  $\left( \frac{c}{a} - \frac{b^2}{4a^2} \right)$  and hence can be evaluated.

- (8) **To find the integral of the type**  $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ , proceeding as in (7), we obtain the integral using the standard formulae.

- (9) **To find the integral of the type**  $\int \frac{px + q}{ax^2 + bx + c} dx$ , where  $p, q, a, b, c$  are constants, we are to find real numbers  $A, B$  such that

$$px + q = A \frac{d}{dx} (ax^2 + bx + c) + B = A(2ax + b) + B$$

To determine  $A$  and  $B$ , we equate from both sides the coefficients of  $x$  and the constant terms.  $A$  and  $B$  are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type  $\int \frac{(px+q) dx}{\sqrt{ax^2+bx+c}}$ , we proceed

as in (9) and transform the integral into known standard forms.

Let us illustrate the above methods by some examples.

**Example 8** Find the following integrals:

$$(i) \int \frac{dx}{x^2 - 16}$$

$$(ii) \int \frac{dx}{\sqrt{2x - x^2}}$$

**Solution**

$$(i) \text{ We have } \int \frac{dx}{x^2 - 16} = \int \frac{dx}{x^2 - 4^2} = \frac{1}{8} \log \left| \frac{x-4}{x+4} \right| + C \text{ [by 7.4 (1)]}$$

$$(ii) \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1 - (x-1)^2}}$$

Put  $x - 1 = t$ . Then  $dx = dt$ .

$$\text{Therefore, } \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dt}{\sqrt{1 - t^2}} = \sin^{-1}(t) + C \text{ [by 7.4 (5)]}$$

$$= \sin^{-1}(x-1) + C$$

**Example 9** Find the following integrals :

$$(i) \int \frac{dx}{x^2 - 6x + 13}$$

$$(ii) \int \frac{dx}{3x^2 + 13x - 10}$$

$$(iii) \int \frac{dx}{\sqrt{5x^2 - 2x}}$$

**Solution**

$$(i) \text{ We have } x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$$

$$\text{So, } \int \frac{dx}{x^2 - 6x + 13} = \int \frac{1}{(x-3)^2 + 2^2} dx$$

Let

$x - 3 = t$ . Then  $dx = dt$

$$\text{Therefore, } \int \frac{dx}{x^2 - 6x + 13} = \int \frac{dt}{t^2 + 2^2} = \frac{1}{2} \tan^{-1} \frac{t}{2} + C \text{ [by 7.4 (3)]}$$

$$= \frac{1}{2} \tan^{-1} \frac{x-3}{2} + C$$

(ii) The given integral is of the form 7.4 (7). We write the denominator of the integrand,

$$\begin{aligned} 3x^2 + 13x - 10 &= 3\left(x^2 + \frac{13x}{3} - \frac{10}{3}\right) \\ &= 3\left[\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right] \text{ (completing the square)} \end{aligned}$$

$$\text{Thus } \int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3} \int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}$$

Put  $x + \frac{13}{6} = t$ . Then  $dx = dt$ .

$$\begin{aligned} \text{Therefore, } \int \frac{dx}{3x^2 + 13x - 10} &= \frac{1}{3} \int \frac{dt}{t^2 - \left(\frac{17}{6}\right)^2} \\ &= \frac{1}{3 \times 2 \times \frac{17}{6}} \log \left| \frac{t - \frac{17}{6}}{t + \frac{17}{6}} \right| + C_1 \quad [\text{by 7.4 (i)}] \\ &= \frac{1}{17} \log \left| \frac{x + \frac{13}{6} - \frac{17}{6}}{x + \frac{13}{6} + \frac{17}{6}} \right| + C_1 \\ &= \frac{1}{17} \log \left| \frac{6x - 4}{6x + 30} \right| + C_1 \\ &= \frac{1}{17} \log \left| \frac{3x - 2}{x + 5} \right| + C_1 + \frac{1}{17} \log \frac{1}{3} \\ &= \frac{1}{17} \log \left| \frac{3x - 2}{x + 5} \right| + C, \text{ where } C = C_1 + \frac{1}{17} \log \frac{1}{3} \end{aligned}$$

$$\begin{aligned}
 \text{(iii) We have } \int \frac{dx}{\sqrt{5x^2 - 2x}} &= \int \frac{dx}{\sqrt{5\left(x^2 - \frac{2x}{5}\right)}} \\
 &= \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} \text{ (completing the square)}
 \end{aligned}$$

Put  $x - \frac{1}{5} = t$ . Then  $dx = dt$ .

$$\begin{aligned}
 \text{Therefore, } \int \frac{dx}{\sqrt{5x^2 - 2x}} &= \frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}} \\
 &= \frac{1}{\sqrt{5}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C \quad [\text{by 7.4 (4)}] \\
 &= \frac{1}{\sqrt{5}} \log \left| x - \frac{1}{5} + \sqrt{x^2 - \frac{2x}{5}} \right| + C
 \end{aligned}$$

**Example 10** Find the following integrals:

$$\text{(i) } \int \frac{x+2}{2x^2+6x+5} dx \quad \text{(ii) } \int \frac{x+3}{\sqrt{5-4x-x^2}} dx$$

**Solution**

(i) Using the formula 7.4 (9), we express

$$x+2 = A \frac{d}{dx}(2x^2+6x+5) + B = A(4x+6) + B$$

Equating the coefficients of  $x$  and the constant terms from both sides, we get

$$4A = 1 \text{ and } 6A + B = 2 \quad \text{or} \quad A = \frac{1}{4} \text{ and } B = \frac{1}{2}.$$

$$\begin{aligned}
 \text{Therefore, } \int \frac{x+2}{2x^2+6x+5} dx &= \frac{1}{4} \int \frac{4x+6}{2x^2+6x+5} dx + \frac{1}{2} \int \frac{dx}{2x^2+6x+5} \\
 &= \frac{1}{4} I_1 + \frac{1}{2} I_2 \quad (\text{say}) \quad \dots (1)
 \end{aligned}$$

In  $I_1$ , put  $2x^2 + 6x + 5 = t$ , so that  $(4x + 6) dx = dt$

Therefore,

$$I_1 = \int \frac{dt}{t} = \log |t| + C_1$$

$$= \log |2x^2 + 6x + 5| + C_1 \quad \dots (2)$$

and

$$I_2 = \int \frac{dx}{2x^2 + 6x + 5} = \frac{1}{2} \int \frac{dx}{x^2 + 3x + \frac{5}{2}}$$

$$= \frac{1}{2} \int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

Put  $x + \frac{3}{2} = t$ , so that  $dx = dt$ , we get

$$I_2 = \frac{1}{2} \int \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2 \times \frac{1}{2}} \tan^{-1} 2t + C_2 \quad [\text{by 7.4 (3)}]$$

$$= \tan^{-1} 2 \left(x + \frac{3}{2}\right) + C_2 = \tan^{-1} (2x + 3) + C_2 \quad \dots (3)$$

Using (2) and (3) in (1), we get

$$\int \frac{x+2}{2x^2+6x+5} dx = \frac{1}{4} \log |2x^2 + 6x + 5| + \frac{1}{2} \tan^{-1} (2x + 3) + C$$

where,

$$C = \frac{C_1}{4} + \frac{C_2}{2}$$

(ii) This integral is of the form given in 7.4 (10). Let us express

$$x + 3 = A \frac{d}{dx} (5 - 4x - x^2) + B = A (-4 - 2x) + B$$

Equating the coefficients of  $x$  and the constant terms from both sides, we get

$$-2A = 1 \text{ and } -4A + B = 3, \text{ i.e., } A = -\frac{1}{2} \text{ and } B = 1$$

$$\text{Therefore, } \int \frac{x+3}{\sqrt{5-4x-x^2}} dx = -\frac{1}{2} \int \frac{(-4-2x) dx}{\sqrt{5-4x-x^2}} + \int \frac{dx}{\sqrt{5-4x-x^2}} \\ = -\frac{1}{2} I_1 + I_2 \quad \dots (1)$$

In  $I_1$ , put  $5-4x-x^2 = t$ , so that  $(-4-2x) dx = dt$ .

$$\text{Therefore, } I_1 = \int \frac{(-4-2x) dx}{\sqrt{5-4x-x^2}} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C_1 \\ = 2\sqrt{5-4x-x^2} + C_1 \quad \dots (2)$$

Now consider

$$I_2 = \int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}}$$

Put  $x+2 = t$ , so that  $dx = dt$ .

$$\text{Therefore, } I_2 = \int \frac{dt}{\sqrt{3^2-t^2}} = \sin^{-1} \frac{t}{3} + C_2 \quad [\text{by 7.4 (5)}] \\ = \sin^{-1} \frac{x+2}{3} + C_2 \quad \dots (3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{\sqrt{5-4x-x^2}} = -\sqrt{5-4x-x^2} + \sin^{-1} \frac{x+2}{3} + C, \text{ where } C = C_2 - \frac{C_1}{2}$$

### EXERCISE 7.4

Integrate the functions in Exercises 1 to 23.

- |                               |                                 |   |
|-------------------------------|---------------------------------|---|
| 1. $\frac{3x^2}{x^6+1}$       | 2. $\frac{1}{\sqrt{1+4x^2}}$    | 3. $\frac{1}{\sqrt{(2-x)^2+1}}$         |
| 4. $\frac{1}{\sqrt{9-25x^2}}$ | 5. $\frac{3x}{1+2x^4}$          | 6. $\frac{x^2}{1-x^6}$                  |
| 7. $\frac{x-1}{\sqrt{x^2-1}}$ | 8. $\frac{x^2}{\sqrt{x^6+a^6}}$ | 9. $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$ |

10. 
$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

11. 
$$\frac{1}{9x^2 + 6x + 5}$$

12. 
$$\frac{1}{\sqrt{7 - 6x - x^2}}$$

13. 
$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

14. 
$$\frac{1}{\sqrt{8 + 3x - x^2}}$$

15. 
$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

16. 
$$\frac{4x+1}{\sqrt{2x^2 + x - 3}}$$

17. 
$$\frac{x+2}{\sqrt{x^2 - 1}}$$

18. 
$$\frac{5x-2}{1 + 2x + 3x^2}$$

19. 
$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

20. 
$$\frac{x+2}{\sqrt{4x - x^2}}$$

21. 
$$\frac{x+2}{\sqrt{x^2 + 2x + 3}}$$

22. 
$$\frac{x+3}{x^2 - 2x - 5}$$

23. 
$$\frac{5x+3}{\sqrt{x^2 + 4x + 10}}$$

Choose the correct answer in Exercises 24 and 25.

24. 
$$\int \frac{dx}{x^2 + 2x + 2}$$
 equals

- (A)  $x \tan^{-1} (x + 1) + C$       (B)  $\tan^{-1} (x + 1) + C$   
 (C)  $(x + 1) \tan^{-1} x + C$       (D)  $\tan^{-1} x + C$

25. 
$$\int \frac{dx}{\sqrt{9x - 4x^2}}$$
 equals

- (A)  $\frac{1}{9} \sin^{-1} \left( \frac{9x-8}{8} \right) + C$       (B)  $\frac{1}{2} \sin^{-1} \left( \frac{8x-9}{9} \right) + C$   
 (C)  $\frac{1}{3} \sin^{-1} \left( \frac{9x-8}{8} \right) + C$       (D)  $\frac{1}{2} \sin^{-1} \left( \frac{9x-8}{9} \right) + C$

## 7.5 Integration by Partial Fractions

Recall that a rational function is defined as the ratio of two polynomials in the form

$\frac{P(x)}{Q(x)}$ , where  $P(x)$  and  $Q(x)$  are polynomials in  $x$  and  $Q(x) \neq 0$ . If the degree of  $P(x)$

is less than the degree of  $Q(x)$ , then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be reduced to the proper rational

functions by long division process. Thus, if  $\frac{P(x)}{Q(x)}$  is improper, then  $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ ,

where  $T(x)$  is a polynomial in  $x$  and  $\frac{P_1(x)}{Q(x)}$  is a proper rational function. As we know

how to integrate polynomials, the integration of any rational function is reduced to the integration of a proper rational function. The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into

linear and quadratic factors. Assume that we want to evaluate  $\int \frac{P(x)}{Q(x)} dx$ , where  $\frac{P(x)}{Q(x)}$

is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods. The following Table 7.2 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.2

| S.No.   | Form of the rational function          | Form of the partial fraction                        |
|---|--|---|
| 1.  | $\frac{px+q}{(x-a)(x-b)}$ , $a \neq b$ | $\frac{A}{x-a} + \frac{B}{x-b}$                     |
| 2.  | $\frac{px+q}{(x-a)^2}$                 | $\frac{A}{x-a} + \frac{B}{(x-a)^2}$                 |
| 3.  | $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$    | $\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$     |
| 4.  | $\frac{px^2+qx+r}{(x-a)^2(x-b)}$       | $\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$ |
| 5.  | $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$    | $\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$ ,           |
| where $x^2 + bx + c$ cannot be factorised further |  |   |

In the above table, A, B and C are real numbers to be determined suitably.

**Example 11** Find  $\int \frac{dx}{(x+1)(x+2)}$

**Solution** The integrand is a proper rational function. Therefore, by using the form of partial fraction [Table 7.2 (i)], we write

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \quad \dots (1)$$

where, real numbers A and B are to be determined suitably. This gives

$$1 = A(x+2) + B(x+1).$$

Equating the coefficients of  $x$  and the constant term, we get

$$A + B = 0$$

and

$$2A + B = 1$$

Solving these equations, we get  $A = 1$  and  $B = -1$ .

Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

Therefore,

$$\begin{aligned} \int \frac{dx}{(x+1)(x+2)} &= \int \frac{dx}{x+1} - \int \frac{dx}{x+2} \\ &= \log|x+1| - \log|x+2| + C \\ &= \log \left| \frac{x+1}{x+2} \right| + C \end{aligned}$$

**Remark** The equation (1) above is an identity, i.e. a statement true for all (permissible) values of  $x$ . Some authors use the symbol ' $\equiv$ ' to indicate that the statement is an identity and use the symbol '=' to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of  $x$ .

**Example 12** Find  $\int \frac{x^2+1}{x^2-5x+6} dx$

**Solution** Here the integrand  $\frac{x^2+1}{x^2-5x+6}$  is not proper rational function, so we divide  $x^2+1$  by  $x^2-5x+6$  and find that

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6} = 1 + \frac{5x-5}{(x-2)(x-3)}$$

Let

$$\frac{5x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

So that

$$5x-5 = A(x-3) + B(x-2)$$

Equating the coefficients of  $x$  and constant terms on both sides, we get  $A + B = 5$  and  $3A + 2B = 5$ . Solving these equations, we get  $A = -5$  and  $B = 10$

Thus,

$$\frac{x^2+1}{x^2-5x+6} = 1 - \frac{5}{x-2} + \frac{10}{x-3}$$

Therefore,

$$\begin{aligned} \int \frac{x^2+1}{x^2-5x+6} dx &= \int dx - 5 \int \frac{1}{x-2} dx + 10 \int \frac{dx}{x-3} \\ &= x - 5 \log|x-2| + 10 \log|x-3| + C. \end{aligned}$$

**Example 13** Find  $\int \frac{3x-2}{(x+1)^2(x+3)} dx$

**Solution** The integrand is of the type as given in Table 7.2 (4). We write

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$$

So that

$$\begin{aligned} 3x-2 &= A(x+1)(x+3) + B(x+3) + C(x+1)^2 \\ &= A(x^2+4x+3) + B(x+3) + C(x^2+2x+1) \end{aligned}$$

Comparing coefficient of  $x^2$ ,  $x$  and constant term on both sides, we get  $A + C = 0$ ,  $4A + B + 2C = 3$  and  $3A + 3B + C = -2$ . Solving these equations, we get

$A = \frac{11}{4}$ ,  $B = -\frac{5}{2}$  and  $C = -\frac{11}{4}$ . Thus the integrand is given by

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)}$$

Therefore,

$$\begin{aligned} \int \frac{3x-2}{(x+1)^2(x+3)} dx &= \frac{11}{4} \int \frac{dx}{x+1} - \frac{5}{2} \int \frac{dx}{(x+1)^2} - \frac{11}{4} \int \frac{dx}{x+3} \\ &= \frac{11}{4} \log|x+1| + \frac{5}{2(x+1)} - \frac{11}{4} \log|x+3| + C \\ &= \frac{11}{4} \log \left| \frac{x+1}{x+3} \right| + \frac{5}{2(x+1)} + C \end{aligned}$$

**Example 14** Find  $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

**Solution** Consider  $\frac{x^2}{(x^2+1)(x^2+4)}$  and put  $x^2 = y$ .

Then

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$

Write

$$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$$

So that

$$y = A(y+4) + B(y+1)$$

Comparing coefficients of  $y$  and constant terms on both sides, we get  $A + B = 1$  and  $4A + B = 0$ , which give

$$A = -\frac{1}{3} \quad \text{and} \quad B = \frac{4}{3}$$

Thus,

$$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

Therefore,

$$\begin{aligned} \int \frac{x^2 dx}{(x^2+1)(x^2+4)} &= -\frac{1}{3} \int \frac{dx}{x^2+1} + \frac{4}{3} \int \frac{dx}{x^2+4} \\ &= -\frac{1}{3} \tan^{-1} x + \frac{4}{3} \times \frac{1}{2} \tan^{-1} \frac{x}{2} + C \\ &= -\frac{1}{3} \tan^{-1} x + \frac{2}{3} \tan^{-1} \frac{x}{2} + C \end{aligned}$$

In the above example, the substitution was made only for the partial fraction part and not for the integration part. Now, we consider an example, where the integration involves a combination of the substitution method and the partial fraction method.

**Example 15** Find  $\int \frac{(3 \sin \phi - 2) \cos \phi}{5 - \cos^2 \phi - 4 \sin \phi} d\phi$

**Solution** Let  $y = \sin \phi$

Then

$$dy = \cos \phi \, d\phi$$

$$\begin{aligned}
 \text{Therefore, } \int \frac{(3 \sin \phi - 2) \cos \phi}{5 - \cos^2 \phi - 4 \sin \phi} d\phi &= \int \frac{(3y - 2) dy}{5 - (1 - y^2) - 4y} \\
 &= \int \frac{3y - 2}{y^2 - 4y + 4} dy \\
 &= \int \frac{3y - 2}{(y - 2)^2} = I \text{ (say)}
 \end{aligned}$$

Now, we write

$$\frac{3y - 2}{(y - 2)^2} = \frac{A}{y - 2} + \frac{B}{(y - 2)^2} \quad [\text{by Table 7.2 (2)}]$$

$$\text{Therefore, } 3y - 2 = A(y - 2) + B$$

Comparing the coefficients of  $y$  and constant term, we get  $A = 3$  and  $B - 2A = -2$ , which gives  $A = 3$  and  $B = 4$ .

Therefore, the required integral is given by

$$\begin{aligned}
 I &= \int \left[ \frac{3}{y - 2} + \frac{4}{(y - 2)^2} \right] dy = 3 \int \frac{dy}{y - 2} + 4 \int \frac{dy}{(y - 2)^2} \\
 &= 3 \log |y - 2| + 4 \left( -\frac{1}{y - 2} \right) + C \\
 &= 3 \log | \sin \phi - 2 | + \frac{4}{2 - \sin \phi} + C \\
 &= 3 \log (2 - \sin \phi) + \frac{4}{2 - \sin \phi} + C \quad (\text{since, } 2 - \sin \phi \text{ is always positive})
 \end{aligned}$$

**Example 16** Find  $\int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx$

**Solution** The integrand is a proper rational function. Decompose the rational function into partial fraction [Table 2.2(5)]. Write

$$\frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} = \frac{A}{x + 2} + \frac{Bx + C}{(x^2 + 1)}$$

Therefore,

$$x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2)$$

Equating the coefficients of  $x^2$ ,  $x$  and of constant term of both sides, we get  $A + B = 1$ ,  $2B + C = 1$  and  $A + 2C = 1$ . Solving these equations, we get

$$A = \frac{3}{5}, B = \frac{2}{5} \text{ and } C = \frac{1}{5}$$

Thus, the integrand is given by

$$\frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} = \frac{3}{5(x + 2)} + \frac{\frac{2}{5}x + \frac{1}{5}}{x^2 + 1} = \frac{3}{5(x + 2)} + \frac{1}{5} \left( \frac{2x + 1}{x^2 + 1} \right)$$

$$\begin{aligned} \text{Therefore, } \int \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} dx &= \frac{3}{5} \int \frac{dx}{x + 2} + \frac{1}{5} \int \frac{2x}{x^2 + 1} dx + \frac{1}{5} \int \frac{1}{x^2 + 1} dx \\ &= \frac{3}{5} \log|x + 2| + \frac{1}{5} \log|x^2 + 1| + \frac{1}{5} \tan^{-1}x + C \end{aligned}$$

### EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

$$1. \frac{x}{(x+1)(x+2)}$$

$$2. \frac{1}{x^2 - 9}$$

$$3. \frac{3x-1}{(x-1)(x-2)(x-3)}$$

$$4. \frac{x}{(x-1)(x-2)(x-3)}$$

$$5. \frac{2x}{x^2 + 3x + 2}$$

$$6. \frac{1-x^2}{x(1-2x)}$$

$$7. \frac{x}{(x^2 + 1)(x - 1)}$$

$$8. \frac{x}{(x-1)^2(x+2)}$$

$$9. \frac{3x+5}{x^3 - x^2 - x + 1}$$

$$10. \frac{2x-3}{(x^2-1)(2x+3)}$$

$$11. \frac{5x}{(x+1)(x^2-4)}$$

$$12. \frac{x^3+x+1}{x^2-1}$$

$$13. \frac{2}{(1-x)(1+x^2)}$$

$$14. \frac{3x-1}{(x+2)^2}$$

$$15. \frac{1}{x^4-1}$$

$$16. \frac{1}{x(x^n+1)} \text{ [Hint: multiply numerator and denominator by } x^{n-1} \text{ and put } x^n = t]$$

$$17. \frac{\cos x}{(1-\sin x)(2-\sin x)} \text{ [Hint : Put } \sin x = t]$$

18.  $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$     19.  $\frac{2x}{(x^2+1)(x^2+3)}$     20.  $\frac{1}{x(x^4-1)}$
21.  $\frac{1}{(e^x-1)}$  [Hint : Put  $e^x = t$ ]

Choose the correct answer in each of the Exercises 22 and 23.

22.  $\int \frac{x \, dx}{(x-1)(x-2)}$  equals

(A)  $\log \left| \frac{(x-1)^2}{x-2} \right| + C$     (B)  $\log \left| \frac{(x-2)^2}{x-1} \right| + C$

(C)  $\log \left| \left( \frac{x-1}{x-2} \right)^2 \right| + C$     (D)  $\log |(x-1)(x-2)| + C$

23.  $\int \frac{dx}{x(x^2+1)}$  equals

(A)  $\log|x| - \frac{1}{2} \log(x^2+1) + C$     (B)  $\log|x| + \frac{1}{2} \log(x^2+1) + C$

(C)  $-\log|x| + \frac{1}{2} \log(x^2+1) + C$     (D)  $\frac{1}{2} \log|x| + \log(x^2+1) + C$

## 7.6 Integration by Parts

In this section, we describe one more method of integration, that is found quite useful in integrating products of functions.

If  $u$  and  $v$  are any two differentiable functions of a single variable  $x$  (say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Integrating both sides, we get

$$uv = \int u \frac{dv}{dx} \, dx + \int v \frac{du}{dx} \, dx$$

or  $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx \quad \dots (1)$

Let  $u = f(x)$  and  $\frac{dv}{dx} = g(x)$ . Then

$$\frac{du}{dx} = f'(x) \text{ and } v = \int g(x) \, dx$$

Therefore, expression (1) can be rewritten as

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int [\int g(x) dx] f'(x) dx$$

i.e.,

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int [f'(x) \int g(x) dx] dx$$

If we take  $f$  as the first function and  $g$  as the second function, then this formula may be stated as follows:

**“The integral of the product of two functions = (first function)  $\times$  (integral of the second function) – Integral of [(differential coefficient of the first function)  $\times$  (integral of the second function)]”**

**Example 17** Find  $\int x \cos x dx$

**Solution** Put  $f(x) = x$  (first function) and  $g(x) = \cos x$  (second function).

Then, integration by parts gives

$$\begin{aligned} \int x \cos x dx &= x \int \cos x dx - \int \left[ \frac{d}{dx}(x) \int \cos x dx \right] dx \\ &= x \sin x - \int \sin x dx = x \sin x + \cos x + C \end{aligned}$$

Suppose, we take

$f(x) = \cos x$  and  $g(x) = x$ . Then

$$\begin{aligned} \int x \cos x dx &= \cos x \int x dx - \int \left[ \frac{d}{dx}(\cos x) \int x dx \right] dx \\ &= (\cos x) \frac{x^2}{2} + \int \sin x \frac{x^2}{2} dx \end{aligned}$$

Thus, it shows that the integral  $\int x \cos x dx$  is reduced to the comparatively more complicated integral having more power of  $x$ . Therefore, the proper choice of the first function and the second function is significant.

### Remarks

- It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for  $\int \sqrt{x} \sin x dx$ . The reason is that there does not exist any function whose derivative is  $\sqrt{x} \sin x$ .
- Observe that while finding the integral of the second function, we did not add any constant of integration. If we write the integral of the second function  $\cos x$

as  $\sin x + k$ , where  $k$  is any constant, then

$$\begin{aligned}\int x \cos x \, dx &= x(\sin x + k) - \int (\sin x + k) \, dx \\ &= x(\sin x + k) - \int (\sin x \, dx - \int k \, dx) \\ &= x(\sin x + k) - \cos x - kx + C = x \sin x + \cos x + C\end{aligned}$$

This shows that adding a constant to the integral of the second function is superfluous so far as the final result is concerned while applying the method of integration by parts.

- (iii) Usually, if any function is a power of  $x$  or a polynomial in  $x$ , then we take it as the first function. However, in cases where other function is inverse trigonometric function or logarithmic function, then we take them as first function.

**Example 18** Find  $\int \log x \, dx$

**Solution** To start with, we are unable to guess a function whose derivative is  $\log x$ . We take  $\log x$  as the first function and the constant function 1 as the second function. Then, the integral of the second function is  $x$ .

Hence,

$$\begin{aligned}\int (\log x \cdot 1) \, dx &= \log x \int 1 \, dx - \int \left[ \frac{d}{dx}(\log x) \int 1 \, dx \right] dx \\ &= (\log x) \cdot x - \int \frac{1}{x} x \, dx = x \log x - x + C.\end{aligned}$$

**Example 19** Find  $\int x e^x \, dx$

**Solution** Take first function as  $x$  and second function as  $e^x$ . The integral of the second function is  $e^x$ .

Therefore,

$$\int x e^x \, dx = x e^x - \int 1 \cdot e^x \, dx = x e^x - e^x + C.$$

**Example 20** Find  $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$

**Solution** Let first function be  $\sin^{-1} x$  and second function be  $\frac{x}{\sqrt{1-x^2}}$ .

First we find the integral of the second function, i.e.,  $\int \frac{x \, dx}{\sqrt{1-x^2}}$ .

Put  $t = 1 - x^2$ . Then  $dt = -2x \, dx$

Therefore,  $\int \frac{x \, dx}{\sqrt{1-x^2}} = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$

Hence,  $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx = (\sin^{-1} x) \left( -\sqrt{1-x^2} \right) - \int \frac{1}{\sqrt{1-x^2}} (-\sqrt{1-x^2}) \, dx$   
 $= -\sqrt{1-x^2} \sin^{-1} x + x + C = x - \sqrt{1-x^2} \sin^{-1} x + C$

**Alternatively**, this integral can also be worked out by making substitution  $\sin^{-1} x = \theta$  and then integrating by parts.

**Example 21** Find  $\int e^x \sin x \, dx$

**Solution** Take  $e^x$  as the first function and  $\sin x$  as second function. Then, integrating by parts, we have

$$\begin{aligned} I &= \int e^x \sin x \, dx = e^x (-\cos x) + \int e^x \cos x \, dx \\ &= -e^x \cos x + I_1 \text{ (say)} \end{aligned} \quad \dots (1)$$

Taking  $e^x$  and  $\cos x$  as the first and second functions, respectively, in  $I_1$ , we get

$$I_1 = e^x \sin x - \int e^x \sin x \, dx$$

Substituting the value of  $I_1$  in (1), we get

$$I = -e^x \cos x + e^x \sin x - I_1 \text{ or } 2I = e^x (\sin x - \cos x)$$

Hence,  $I = \int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C$

**Alternatively**, above integral can also be determined by taking  $\sin x$  as the first function and  $e^x$  the second function.

**7.6.1 Integral of the type**  $\int e^x [f(x) + f'(x)] \, dx$

We have  $I = \int e^x [f(x) + f'(x)] \, dx = \int e^x f(x) \, dx + \int e^x f'(x) \, dx$   
 $= I_1 + \int e^x f'(x) \, dx$ , where  $I_1 = \int e^x f(x) \, dx \quad \dots (1)$

Taking  $f(x)$  and  $e^x$  as the first function and second function, respectively, in  $I_1$  and integrating it by parts, we have  $I_1 = f(x) e^x - \int f'(x) e^x \, dx + C$

Substituting  $I_1$  in (1), we get

$$I = e^x f(x) - \int f'(x) e^x \, dx + \int e^x f'(x) \, dx + C = e^x f(x) + C$$

Thus,

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

**Example 22** Find (i)  $\int e^x (\tan^{-1} x + \frac{1}{1+x^2}) dx$  (ii)  $\int \frac{(x^2+1)e^x}{(x+1)^2} dx$

### Solution

(i) We have  $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2}) dx$

Consider  $f(x) = \tan^{-1} x$ , then  $f'(x) = \frac{1}{1+x^2}$

Thus, the given integrand is of the form  $e^x [f(x) + f'(x)]$ .

Therefore,  $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2}) dx = e^x \tan^{-1} x + C$

(ii) We have  $I = \int \frac{(x^2+1)e^x}{(x+1)^2} dx = \int e^x \left[ \frac{x^2-1+1+1}{(x+1)^2} \right] dx$

$$= \int e^x \left[ \frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2} \right] dx = \int e^x \left[ \frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx$$

Consider  $f(x) = \frac{x-1}{x+1}$ , then  $f'(x) = \frac{2}{(x+1)^2}$

Thus, the given integrand is of the form  $e^x [f(x) + f'(x)]$ .

Therefore,  $\int \frac{x^2+1}{(x+1)^2} e^x dx = \frac{x-1}{x+1} e^x + C$

### EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

- |                    |                       |  |                    |
|--------------------|-----------------------|--|--------------------|
| 1. $x \sin x$      | 2. $x \sin 3x$        | 3. $x^2 e^x$                             | 4. $x \log x$      |
| 5. $x \log 2x$     | 6. $x^2 \log x$       | 7. $x \sin^{-1} x$                       | 8. $x \tan^{-1} x$ |
| 9. $x \cos^{-1} x$ | 10. $(\sin^{-1} x)^2$ | 11. $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$ | 12. $x \sec^2 x$   |
| 13. $\tan^{-1} x$  | 14. $x (\log x)^2$    | 15. $(x^2+1) \log x$                     |                    |

16.  $e^x (\sin x + \cos x)$

17.  $\frac{x e^x}{(1+x)^2}$

18.  $e^x \left( \frac{1+\sin x}{1+\cos x} \right)$

19.  $e^x \left( \frac{1}{x} - \frac{1}{x^2} \right)$

20.  $\frac{(x-3) e^x}{(x-1)^3}$

21.  $e^{2x} \sin x$

22.  $\sin^{-1} \left( \frac{2x}{1+x^2} \right)$

Choose the correct answer in Exercises 23 and 24.

23.  $\int x^2 e^{x^3} dx$  equals

(A)  $\frac{1}{3} e^{x^3} + C$

(B)  $\frac{1}{3} e^{x^2} + C$

(C)  $\frac{1}{2} e^{x^3} + C$

(D)  $\frac{1}{2} e^{x^2} + C$

24.  $\int e^x \sec x (1 + \tan x) dx$  equals

(A)  $e^x \cos x + C$

(B)  $e^x \sec x + C$

(C)  $e^x \sin x + C$

(D)  $e^x \tan x + C$

### 7.6.2 Integrals of some more types

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i)  $\int \sqrt{x^2 - a^2} dx$       (ii)  $\int \sqrt{x^2 + a^2} dx$       (iii)  $\int \sqrt{a^2 - x^2} dx$

(i) Let  $I = \int \sqrt{x^2 - a^2} dx$

Taking constant function 1 as the second function and integrating by parts, we have

$$I = x \sqrt{x^2 - a^2} - \int \frac{1}{2} \frac{2x}{\sqrt{x^2 - a^2}} x dx$$

$$= x \sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx = x \sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

$$= x \sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} \, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= x \sqrt{x^2 - a^2} - I - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

or  $2I = x \sqrt{x^2 - a^2} - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$

or  $I = \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$

Similarly, integrating other two integrals by parts, taking constant function 1 as the second function, we get

$$(ii) \quad \int \sqrt{x^2 + a^2} \, dx = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$(iii) \quad \int \sqrt{a^2 - x^2} \, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

**Alternatively**, integrals (i), (ii) and (iii) can also be found by making trigonometric substitution  $x = a \sec \theta$  in (i),  $x = a \tan \theta$  in (ii) and  $x = a \sin \theta$  in (iii) respectively.

**Example 23** Find  $\int \sqrt{x^2 + 2x + 5} \, dx$

**Solution** Note that

$$\int \sqrt{x^2 + 2x + 5} \, dx = \int \sqrt{(x+1)^2 + 4} \, dx$$

Put  $x+1 = y$ , so that  $dx = dy$ . Then

$$\begin{aligned} \int \sqrt{x^2 + 2x + 5} \, dx &= \int \sqrt{y^2 + 2^2} \, dy \\ &= \frac{1}{2} y \sqrt{y^2 + 4} + \frac{4}{2} \log \left| y + \sqrt{y^2 + 4} \right| + C \quad [\text{using 7.6.2 (ii)}] \\ &= \frac{1}{2} (x+1) \sqrt{x^2 + 2x + 5} + 2 \log \left| x+1 + \sqrt{x^2 + 2x + 5} \right| + C \end{aligned}$$

**Example 24** Find  $\int \sqrt{3 - 2x - x^2} \, dx$

**Solution** Note that  $\int \sqrt{3 - 2x - x^2} \, dx = \int \sqrt{4 - (x+1)^2} \, dx$

Put  $x + 1 = y$  so that  $dx = dy$ .

Thus 
$$\begin{aligned} \int \sqrt{3-2x-x^2} \, dx &= \int \sqrt{4-y^2} \, dy \\ &= \frac{1}{2} y \sqrt{4-y^2} + \frac{4}{2} \sin^{-1} \frac{y}{2} + C \quad [\text{using 7.6.2 (iii)}] \\ &= \frac{1}{2} (x+1) \sqrt{3-2x-x^2} + 2 \sin^{-1} \left( \frac{x+1}{2} \right) + C \end{aligned}$$

### EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

1.  $\sqrt{4-x^2}$

2.  $\sqrt{1-4x^2}$

3.  $\sqrt{x^2+4x+6}$

4.  $\sqrt{x^2+4x+1}$

5.  $\sqrt{1-4x-x^2}$

6.  $\sqrt{x^2+4x-5}$

7.  $\sqrt{1+3x-x^2}$

8.  $\sqrt{x^2+3x}$

9.  $\sqrt{1+\frac{x^2}{9}}$

Choose the correct answer in Exercises 10 to 11.

10.  $\int \sqrt{1+x^2} \, dx$  is equal to

(A)  $\frac{x}{2} \sqrt{1+x^2} + \frac{1}{2} \log \left| \left( x + \sqrt{1+x^2} \right) \right| + C$

(B)  $\frac{2}{3} (1+x^2)^{\frac{3}{2}} + C$  (C)  $\frac{2}{3} x (1+x^2)^{\frac{3}{2}} + C$

(D)  $\frac{x^2}{2} \sqrt{1+x^2} + \frac{1}{2} x^2 \log \left| x + \sqrt{1+x^2} \right| + C$

11.  $\int \sqrt{x^2-8x+7} \, dx$  is equal to

(A)  $\frac{1}{2} (x-4) \sqrt{x^2-8x+7} + 9 \log \left| x-4+\sqrt{x^2-8x+7} \right| + C$

(B)  $\frac{1}{2} (x+4) \sqrt{x^2-8x+7} + 9 \log \left| x+4+\sqrt{x^2-8x+7} \right| + C$

(C)  $\frac{1}{2} (x-4) \sqrt{x^2-8x+7} - 3\sqrt{2} \log \left| x-4+\sqrt{x^2-8x+7} \right| + C$

(D)  $\frac{1}{2} (x-4) \sqrt{x^2-8x+7} - \frac{9}{2} \log \left| x-4+\sqrt{x^2-8x+7} \right| + C$

## 7.7 Definite Integral

In the previous sections, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by  $\int_a^b f(x) dx$ , where  $a$  is called the lower limit of the integral and  $b$  is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative  $F$  in the interval  $[a, b]$ , then its value is the difference between the values of  $F$  at the end points, i.e.,  $F(b) - F(a)$ .

## 7.8 Fundamental Theorem of Calculus

### 7.8.1 Area function

We have defined  $\int_a^b f(x) dx$  as the area of the region bounded by the curve  $y = f(x)$ , the ordinates  $x = a$  and  $x = b$  and  $x$ -axis. Let  $x$  be a given point in  $[a, b]$ . Then  $\int_a^x f(x) dx$

represents the area of the light shaded region in Fig 7.1 [Here it is assumed that  $f(x) > 0$  for  $x \in [a, b]$ , the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of  $x$ .

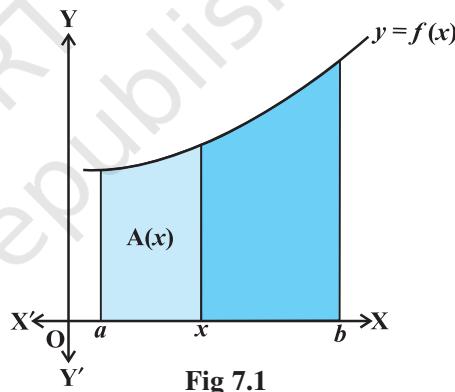


Fig 7.1

In other words, the area of this shaded region is a function of  $x$ . We denote this function of  $x$  by  $A(x)$ . We call the function  $A(x)$  as *Area function* and is given by

$$A(x) = \int_a^x f(x) dx \quad \dots (1)$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

### 7.8.2 First fundamental theorem of integral calculus

**Theorem 1** Let  $f$  be a continuous function on the closed interval  $[a, b]$  and let  $A(x)$  be the area function. Then  $A'(x) = f(x)$ , for all  $x \in [a, b]$ .

### 7.8.3 Second fundamental theorem of integral calculus

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

**Theorem 2** Let  $f$  be continuous function defined on the closed interval  $[a, b]$  and  $F$  be an anti derivative of  $f$ . Then  $\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$ .

#### Remarks

- (i) In words, the Theorem 2 tells us that  $\int_a^b f(x) dx = (\text{value of the anti derivative } F \text{ of } f \text{ at the upper limit } b) - (\text{value of the same anti derivative at the lower limit } a)$ .
- (ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.
- (iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.
- (iv) In  $\int_a^b f(x) dx$ , the function  $f$  needs to be well defined and continuous in  $[a, b]$ .

For instance, the consideration of definite integral  $\int_{-2}^3 x(x^2 - 1)^{\frac{1}{2}} dx$  is erroneous

since the function  $f$  expressed by  $f(x) = x(x^2 - 1)^{\frac{1}{2}}$  is not defined in a portion  $-1 < x < 1$  of the closed interval  $[-2, 3]$ .

**Steps for calculating  $\int_a^b f(x) dx$ .**

- (i) Find the indefinite integral  $\int f(x) dx$ . Let this be  $F(x)$ . There is no need to keep integration constant  $C$  because if we consider  $F(x) + C$  instead of  $F(x)$ , we get  $\int_a^b f(x) dx = [F(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$ .

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

- (ii) Evaluate  $F(b) - F(a) = [F(x)]_a^b$ , which is the value of  $\int_a^b f(x) dx$ .

We now consider some examples

**Example 25** Evaluate the following integrals:

$$(i) \int_2^3 x^2 \, dx$$

$$(ii) \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2} \, dx$$

$$(iii) \int_1^2 \frac{x \, dx}{(x+1)(x+2)}$$

$$(iv) \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t \, dt$$

**Solution**

$$(i) \text{ Let } I = \int_2^3 x^2 \, dx. \text{ Since } \int x^2 \, dx = \frac{x^3}{3} = F(x),$$

Therefore, by the second fundamental theorem, we get

$$I = F(3) - F(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

$$(ii) \text{ Let } I = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2} \, dx. \text{ We first find the anti derivative of the integrand.}$$

$$\text{Put } 30 - x^{\frac{3}{2}} = t. \text{ Then } -\frac{3}{2} \sqrt{x} \, dx = dt \text{ or } \sqrt{x} \, dx = -\frac{2}{3} dt$$

$$\text{Thus, } \int \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2} \, dx = -\frac{2}{3} \int \frac{dt}{t^2} = \frac{2}{3} \left[ \frac{1}{t} \right] = \frac{2}{3} \left[ \frac{1}{(30 - x^{\frac{3}{2}})} \right] = F(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$\begin{aligned} I &= F(9) - F(4) = \frac{2}{3} \left[ \frac{1}{(30 - x^{\frac{3}{2}})} \right]_4^9 \\ &= \frac{2}{3} \left[ \frac{1}{(30 - 27)} - \frac{1}{30 - 8} \right] = \frac{2}{3} \left[ \frac{1}{3} - \frac{1}{22} \right] = \frac{19}{99} \end{aligned}$$

$$(iii) \text{ Let } I = \int_1^2 \frac{x \, dx}{(x+1)(x+2)}$$

Using partial fraction, we get  $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So  $\int \frac{x \, dx}{(x+1)(x+2)} = -\log|x+1| + 2\log|x+2| = F(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$\begin{aligned} I &= F(2) - F(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3] \\ &= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right) \end{aligned}$$

(iv) Let  $I = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t \, dt$ . Consider  $\int \sin^3 2t \cos 2t \, dt$

Put  $\sin 2t = u$  so that  $2 \cos 2t \, dt = du$  or  $\cos 2t \, dt = \frac{1}{2} \, du$

$$\begin{aligned} \text{So } \int \sin^3 2t \cos 2t \, dt &= \frac{1}{2} \int u^3 \, du \\ &= \frac{1}{8} [u^4] = \frac{1}{8} \sin^4 2t = F(t) \text{ say} \end{aligned}$$

Therefore, by the second fundamental theorem of integral calculus

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{8} \left[ \sin^4 \frac{\pi}{2} - \sin^4 0 \right] = \frac{1}{8}$$

### EXERCISE 7.8

Evaluate the definite integrals in Exercises 1 to 20.

1.  $\int_{-1}^1 (x+1) \, dx$     2.  $\int_2^3 \frac{1}{x} \, dx$     3.  $\int_1^2 (4x^3 - 5x^2 + 6x + 9) \, dx$

4.  $\int_0^{\frac{\pi}{4}} \sin 2x \, dx$     5.  $\int_0^{\frac{\pi}{2}} \cos 2x \, dx$     6.  $\int_4^5 e^x \, dx$     7.  $\int_0^{\frac{\pi}{4}} \tan x \, dx$

8.  $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$     9.  $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$     10.  $\int_0^1 \frac{dx}{1+x^2}$     11.  $\int_2^3 \frac{dx}{x^2-1}$

12.  $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

13.  $\int_2^3 \frac{x \, dx}{x^2 + 1}$

14.  $\int_0^1 \frac{2x+3}{5x^2+1} \, dx$

15.  $\int_0^1 x e^{x^2} \, dx$

16.  $\int_1^2 \frac{5x^2}{x^2 + 4x + 3} \, dx$

17.  $\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) \, dx$

18.  $\int_0^{\pi} (\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}) \, dx$

19.  $\int_0^2 \frac{6x+3}{x^2+4} \, dx$

20.  $\int_0^1 (x e^x + \sin \frac{\pi x}{4}) \, dx$

Choose the correct answer in Exercises 21 and 22.

21.  $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$  equals

(A)  $\frac{\pi}{3}$

(B)  $\frac{2\pi}{3}$

(C)  $\frac{\pi}{6}$

(D)  $\frac{\pi}{12}$

22.  $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$  equals

(A)  $\frac{\pi}{6}$

(B)  $\frac{\pi}{12}$

(C)  $\frac{\pi}{24}$

(D)  $\frac{\pi}{4}$

## 7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate  $\int_a^b f(x) \, dx$ , by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute,  $y = f(x)$  or  $x = g(y)$  to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

**Note** In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

**Example 26** Evaluate  $\int_{-1}^1 5x^4 \sqrt{x^5 + 1} dx$ .

**Solution** Put  $t = x^5 + 1$ , then  $dt = 5x^4 dx$ .

$$\text{Therefore, } \int 5x^4 \sqrt{x^5 + 1} dx = \int \sqrt{t} dt = \frac{2}{3} t^{\frac{3}{2}} = \frac{2}{3} (x^5 + 1)^{\frac{3}{2}}$$

$$\begin{aligned} \text{Hence, } \int_{-1}^1 5x^4 \sqrt{x^5 + 1} dx &= \frac{2}{3} \left[ (x^5 + 1)^{\frac{3}{2}} \right]_{-1}^1 \\ &= \frac{2}{3} \left[ (1^5 + 1)^{\frac{3}{2}} - ((-1)^5 + 1)^{\frac{3}{2}} \right] \\ &= \frac{2}{3} \left[ 2^{\frac{3}{2}} - 0^{\frac{3}{2}} \right] = \frac{2}{3} (2\sqrt{2}) = \frac{4\sqrt{2}}{3} \end{aligned}$$

**Alternatively**, first we transform the integral and then evaluate the transformed integral with new limits.

Let  $t = x^5 + 1$ . Then  $dt = 5x^4 dx$ .

Note that, when  $x = -1$ ,  $t = 0$  and when  $x = 1$ ,  $t = 2$

Thus, as  $x$  varies from  $-1$  to  $1$ ,  $t$  varies from  $0$  to  $2$

$$\begin{aligned} \text{Therefore } \int_{-1}^1 5x^4 \sqrt{x^5 + 1} dx &= \int_0^2 \sqrt{t} dt \\ &= \frac{2}{3} \left[ t^{\frac{3}{2}} \right]_0^2 = \frac{2}{3} \left[ 2^{\frac{3}{2}} - 0^{\frac{3}{2}} \right] = \frac{2}{3} (2\sqrt{2}) = \frac{4\sqrt{2}}{3} \end{aligned}$$

**Example 27** Evaluate  $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

**Solution** Let  $t = \tan^{-1} x$ , then  $dt = \frac{1}{1+x^2} dx$ . The new limits are, when  $x=0$ ,  $t=0$  and when  $x=1$ ,  $t=\frac{\pi}{4}$ . Thus, as  $x$  varies from 0 to 1,  $t$  varies from 0 to  $\frac{\pi}{4}$ .

Therefore

$$\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \int_0^{\frac{\pi}{4}} t dt \left[ \frac{t^2}{2} \right]_0^{\frac{\pi}{4}} = \frac{1}{2} \left[ \frac{\pi^2}{16} - 0 \right] = \frac{\pi^2}{32}$$

### EXERCISE 7.9

Evaluate the integrals in Exercises 1 to 8 using substitution.

1.  $\int_0^1 \frac{x}{x^2+1} dx$
2.  $\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi$
3.  $\int_0^1 \sin^{-1} \left( \frac{2x}{1+x^2} \right) dx$
4.  $\int_0^2 x \sqrt{x+2} dx$  (Put  $x+2=t^2$ )
5.  $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$
6.  $\int_0^2 \frac{dx}{x+4-x^2}$
7.  $\int_{-1}^1 \frac{dx}{x^2+2x+5}$
8.  $\int_1^2 \left( \frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral  $\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$  is
 

(A) 6      (B) 0      (C) 3      (D) 4
10. If  $f(x) = \int_0^x t \sin t dt$ , then  $f'(x)$  is
 

(A)  $\cos x + x \sin x$       (B)  $x \sin x$   
     (C)  $x \cos x$       (D)  $\sin x + x \cos x$

### 7.10 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

$$P_0: \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$P_1: \int_a^b f(x) dx = - \int_b^a f(x) dx. \text{ In particular, } \int_a^a f(x) dx = 0$$

$$P_2: \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$P_3 : \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$P_4 : \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

(Note that  $P_4$  is a particular case of  $P_3$ )

$$P_5 : \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$$

$$P_6 : \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x) \text{ and} \\ 0 \text{ if } f(2a-x) = -f(x)$$

$$P_7 : \begin{aligned} \text{(i)} \quad & \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f \text{ is an even function, i.e., if } f(-x) = f(x). \\ \text{(ii)} \quad & \int_{-a}^a f(x) dx = 0, \text{ if } f \text{ is an odd function, i.e., if } f(-x) = -f(x). \end{aligned}$$

We give the proofs of these properties one by one.

**Proof of  $P_0$**  It follows directly by making the substitution  $x = t$ .

**Proof of  $P_1$**  Let  $F$  be anti derivative of  $f$ . Then, by the second fundamental theorem of calculus, we have  $\int_a^b f(x) dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x) dx$

Here, we observe that, if  $a = b$ , then  $\int_a^a f(x) dx = 0$ .

**Proof of  $P_2$**  Let  $F$  be anti derivative of  $f$ . Then

$$\int_a^b f(x) dx = F(b) - F(a) \quad \dots (1)$$

$$\int_a^c f(x) dx = F(c) - F(a) \quad \dots (2)$$

$$\text{and} \quad \int_c^b f(x) dx = F(b) - F(c) \quad \dots (3)$$

Adding (2) and (3), we get  $\int_a^c f(x) dx + \int_c^b f(x) dx = F(b) - F(a) = \int_a^b f(x) dx$

This proves the property  $P_2$ .

**Proof of  $P_3$**  Let  $t = a + b - x$ . Then  $dt = -dx$ . When  $x = a$ ,  $t = b$  and when  $x = b$ ,  $t = a$ . Therefore

$$\int_a^b f(x) dx = - \int_b^a f(a+b-t) dt$$

$$\begin{aligned}
 &= \int_a^b f(a+b-t) dt \text{ (by P}_1\text{)} \\
 &= \int_a^b f(a+b-x) dx \text{ by P}_0
 \end{aligned}$$

**Proof of P<sub>4</sub>** Put  $t = a - x$ . Then  $dt = -dx$ . When  $x = 0$ ,  $t = a$  and when  $x = a$ ,  $t = 0$ . Now proceed as in P<sub>3</sub>.

**Proof of P<sub>5</sub>** Using P<sub>2</sub>, we have  $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_a^{2a} f(x) dx$ .

Let  $t = 2a - x$  in the second integral on the right hand side. Then  $dt = -dx$ . When  $x = a$ ,  $t = a$  and when  $x = 2a$ ,  $t = 0$ . Also  $x = 2a - t$ .

Therefore, the second integral becomes

$$\int_a^{2a} f(x) dx = - \int_a^0 f(2a-t) dt = \int_0^a f(2a-t) dt = \int_0^a f(2a-x) dx$$

Hence  $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$

**Proof of P<sub>6</sub>** Using P<sub>5</sub>, we have  $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx \dots (1)$

Now, if  $f(2a-x) = f(x)$ , then (1) becomes

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(x) dx = 2 \int_0^a f(x) dx,$$

and if  $f(2a-x) = -f(x)$ , then (1) becomes

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx - \int_0^a f(x) dx = 0$$

**Proof of P<sub>7</sub>** Using P<sub>2</sub>, we have

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx. \text{ Then}$$

Let  $t = -x$  in the first integral on the right hand side.  $dt = -dx$ . When  $x = -a$ ,  $t = a$  and when  $x = 0$ ,  $t = 0$ . Also  $x = -t$ .

Therefore  $\int_{-a}^a f(x) dx = - \int_a^0 f(-t) dt + \int_0^a f(x) dx$

$$\begin{aligned}
 &= \int_0^a f(-x) dx + \int_0^a f(x) dx \quad (\text{by P}_0) \dots (1)
 \end{aligned}$$

(i) Now, if  $f$  is an even function, then  $f(-x) = f(x)$  and so (1) becomes

$$\int_{-a}^a f(x) dx = \int_0^a f(x) dx + \int_0^a f(x) dx = 2 \int_0^a f(x) dx$$

(ii) If  $f$  is an odd function, then  $f(-x) = -f(x)$  and so (1) becomes

$$\int_{-a}^a f(x) dx = - \int_0^a f(x) dx + \int_0^a f(x) dx = 0$$

**Example 28** Evaluate  $\int_{-1}^2 |x^3 - x| dx$

**Solution** We note that  $x^3 - x \geq 0$  on  $[-1, 0]$  and  $x^3 - x \leq 0$  on  $[0, 1]$  and that  $x^3 - x \geq 0$  on  $[1, 2]$ . So by P<sub>2</sub> we write

$$\begin{aligned} \int_{-1}^2 |x^3 - x| dx &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 -(x^3 - x) dx + \int_1^2 (x^3 - x) dx \\ &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx + \int_1^2 (x^3 - x) dx \\ &= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 + \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 + \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_1^2 \\ &= -\left( \frac{1}{4} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + (4 - 2) - \left( \frac{1}{4} - \frac{1}{2} \right) \\ &= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4} \end{aligned}$$

**Example 29** Evaluate  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

**Solution** We observe that  $\sin^2 x$  is an even function. Therefore, by P<sub>7</sub> (i), we get

$$\begin{aligned} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx &= 2 \int_0^{\frac{\pi}{4}} \sin^2 x dx \\ &= 2 \int_0^{\frac{\pi}{4}} \frac{(1 - \cos 2x)}{2} dx = \int_0^{\frac{\pi}{4}} (1 - \cos 2x) dx \end{aligned}$$

$$= \left[ x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} = \left( \frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} \right) - 0 = \frac{\pi}{4} - \frac{1}{2}$$

**Example 30** Evaluate  $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

**Solution** Let  $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ . Then, by P<sub>4</sub>, we have

$$\begin{aligned} I &= \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x) dx}{1 + \cos^2(\pi - x)} \\ &= \int_0^{\pi} \frac{(\pi - x) \sin x dx}{1 + \cos^2 x} = \pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} - I \end{aligned}$$

or

$$2I = \pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x}$$

or

$$I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x}$$

Put  $\cos x = t$  so that  $-\sin x dx = dt$ . When  $x = 0$ ,  $t = 1$  and when  $x = \pi$ ,  $t = -1$ . Therefore, (by P<sub>1</sub>) we get

$$\begin{aligned} I &= \frac{-\pi}{2} \int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2} \int_{-1}^1 \frac{dt}{1+t^2} \\ &= \pi \int_0^1 \frac{dt}{1+t^2} \text{ (by P}_7\text{, since } \frac{1}{1+t^2} \text{ is even function)} \\ &= \pi \left[ \tan^{-1} t \right]_0^1 = \pi \left[ \tan^{-1} 1 - \tan^{-1} 0 \right] = \pi \left[ \frac{\pi}{4} - 0 \right] = \frac{\pi^2}{4} \end{aligned}$$

**Example 31** Evaluate  $\int_{-1}^1 \sin^5 x \cos^4 x dx$

**Solution** Let  $I = \int_{-1}^1 \sin^5 x \cos^4 x dx$ . Let  $f(x) = \sin^5 x \cos^4 x$ . Then

$f(-x) = \sin^5(-x) \cos^4(-x) = -\sin^5 x \cos^4 x = -f(x)$ , i.e.,  $f$  is an odd function. Therefore, by P<sub>7</sub> (ii),  $I = 0$

**Example 32** Evaluate  $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

**Solution** Let  $I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx \dots (1)$

Then, by  $P_4$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 \left(\frac{\pi}{2} - x\right)}{\sin^4 \left(\frac{\pi}{2} - x\right) + \cos^4 \left(\frac{\pi}{2} - x\right)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} dx \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence  $I = \frac{\pi}{4}$

**Example 33** Evaluate  $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}}$

**Solution** Let  $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x} dx}{\sqrt{\cos x} + \sqrt{\sin x}} \dots (1)$

Then, by  $P_3$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos \left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} dx}{\sqrt{\cos \left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin \left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}} \\ = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \text{ Hence } I = \frac{\pi}{12}$$

**Example 34** Evaluate  $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

**Solution** Let  $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Then, by P<sub>4</sub>

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

Adding the two values of I, we get

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx \\ &= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x + \log 2 - \log 2) dx \quad (\text{by adding and subtracting } \log 2) \\ &= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx \quad (\text{Why?}) \end{aligned}$$

Put  $2x = t$  in the first integral. Then  $2 \, dx = dt$ , when  $x = 0, t = 0$  and when  $x = \frac{\pi}{2}, t = \pi$ ,

$t = \pi$ .

$$\begin{aligned} \text{Therefore } 2I &= \frac{1}{2} \int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2 \\ &= \frac{2}{2} \int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2} \log 2 \quad [\text{by P}_6 \text{ as } \sin(\pi - t) = \sin t] \\ &= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2 \quad (\text{by changing variable } t \text{ to } x) \\ &= I - \frac{\pi}{2} \log 2 \end{aligned}$$

$$\text{Hence } \int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{-\pi}{2} \log 2.$$

## EXERCISE 7.10

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1.  $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

2.  $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$

3.  $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$

4.  $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x}$

5.  $\int_{-5}^5 |x+2| \, dx$

6.  $\int_2^8 |x-5| \, dx$

7.  $\int_0^1 x(1-x)^n \, dx$

8.  $\int_0^{\frac{\pi}{4}} \log(1+\tan x) \, dx$

9.  $\int_0^2 x \sqrt{2-x} \, dx$

10.  $\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) \, dx$

11.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$

12.  $\int_0^{\pi} \frac{x \, dx}{1+\sin x}$

13.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$

14.  $\int_0^{2\pi} \cos^5 x \, dx$

15.  $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \, dx$

16.  $\int_0^{\pi} \log(1+\cos x) \, dx$

17.  $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} \, dx$

18.  $\int_0^4 |x-1| \, dx$

19. Show that  $\int_0^a f(x)g(x) \, dx = 2 \int_0^a f(x) \, dx$ , if  $f$  and  $g$  are defined as  $f(x) = f(a-x)$  and  $g(x) + g(a-x) = 4$

Choose the correct answer in Exercises 20 and 21.

20. The value of  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) \, dx$  is

(A) 0 (B) 2 (C)  $\pi$  (D) 1

21. The value of  $\int_0^{\frac{\pi}{2}} \log \left( \frac{4+3 \sin x}{4+3 \cos x} \right) \, dx$  is

(A) 2 (B)  $\frac{3}{4}$  (C) 0 (D) -2

### Miscellaneous Examples

**Example 35** Find  $\int \cos 6x \sqrt{1 + \sin 6x} dx$

**Solution** Put  $t = 1 + \sin 6x$ , so that  $dt = 6 \cos 6x dx$

$$\begin{aligned} \text{Therefore } \int \cos 6x \sqrt{1 + \sin 6x} dx &= \frac{1}{6} \int t^{\frac{1}{2}} dt \\ &= \frac{1}{6} \times \frac{2}{3} (t)^{\frac{3}{2}} + C = \frac{1}{9} (1 + \sin 6x)^{\frac{3}{2}} + C \end{aligned}$$

**Example 36** Find  $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx$

$$\text{Solution} \text{ We have } \int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \int \frac{(1 - \frac{1}{x^3})^{\frac{1}{4}}}{x^4} dx$$

Put  $1 - \frac{1}{x^3} = 1 - x^{-3} = t$ , so that  $\frac{3}{x^4} dx = dt$

$$\text{Therefore } \int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \frac{1}{3} \int t^{\frac{1}{4}} dt = \frac{1}{3} \times \frac{4}{5} t^{\frac{5}{4}} + C = \frac{4}{15} \left(1 - \frac{1}{x^3}\right)^{\frac{5}{4}} + C$$

**Example 37** Find  $\int \frac{x^4}{(x-1)(x^2+1)} dx$

**Solution** We have

$$\begin{aligned} \frac{x^4}{(x-1)(x^2+1)} &= (x+1) + \frac{1}{x^3 - x^2 + x - 1} \\ &= (x+1) + \frac{1}{(x-1)(x^2+1)} \end{aligned} \quad \dots (1)$$

Now express

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)} \quad \dots (2)$$

So

$$1 = A(x^2 + 1) + (Bx + C)(x - 1)$$

$$= (A + B)x^2 + (C - B)x + A - C$$

Equating coefficients on both sides, we get  $A + B = 0$ ,  $C - B = 0$  and  $A - C = 1$ ,

which give  $A = \frac{1}{2}$ ,  $B = C = -\frac{1}{2}$ . Substituting values of  $A$ ,  $B$  and  $C$  in (2), we get

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2} \frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)} \quad \dots (3)$$

Again, substituting (3) in (1), we have

$$\frac{x^4}{(x-1)(x^2+x+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2} \frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)}$$

Therefore

$$\int \frac{x^4}{(x-1)(x^2+x+1)} dx = \frac{x^2}{2} + x + \frac{1}{2} \log|x-1| - \frac{1}{4} \log(x^2+1) - \frac{1}{2} \tan^{-1}x + C$$

**Example 38** Find  $\int \left[ \log(\log x) + \frac{1}{(\log x)^2} \right] dx$

$$\begin{aligned} \text{Solution} \quad I &= \int \left[ \log(\log x) + \frac{1}{(\log x)^2} \right] dx \\ &= \int \log(\log x) dx + \int \frac{1}{(\log x)^2} dx \end{aligned}$$

In the first integral, let us take 1 as the second function. Then integrating it by parts, we get

$$\begin{aligned} I &= x \log(\log x) - \int \frac{1}{x \log x} x dx + \int \frac{dx}{(\log x)^2} \\ &= x \log(\log x) - \int \frac{dx}{\log x} + \int \frac{dx}{(\log x)^2} \quad \dots (1) \end{aligned}$$

Again, consider  $\int \frac{dx}{\log x}$ , take 1 as the second function and integrate it by parts,

$$\text{we have } \int \frac{dx}{\log x} = \left[ \frac{x}{\log x} - \int x \left\{ -\frac{1}{(\log x)^2} \left( \frac{1}{x} \right) \right\} dx \right] \quad \dots (2)$$

Putting (2) in (1), we get

$$I = x \log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2} + \int \frac{dx}{(\log x)^2} = x \log(\log x) - \frac{x}{\log x} + C$$

**Example 39** Find  $\int [\sqrt{\cot x} + \sqrt{\tan x}] dx$

**Solution** We have

$$I = \int [\sqrt{\cot x} + \sqrt{\tan x}] dx = \int \sqrt{\tan x} (1 + \cot x) dx$$

Put  $\tan x = t^2$ , so that  $\sec^2 x dx = 2t dt$

$$\text{or } dx = \frac{2t dt}{1+t^4}$$

$$\text{Then } I = \int t \left(1 + \frac{1}{t^2}\right) \frac{2t}{(1+t^4)} dt$$

$$= 2 \int \frac{(t^2+1)}{t^4+1} dt = 2 \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t^2 + \frac{1}{t^2}\right)} = 2 \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2}$$

Put  $t - \frac{1}{t} = y$ , so that  $\left(1 + \frac{1}{t^2}\right) dt = dy$ . Then

$$I = 2 \int \frac{dy}{y^2 + (\sqrt{2})^2} = \sqrt{2} \tan^{-1} \frac{y}{\sqrt{2}} + C = \sqrt{2} \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C$$

$$= \sqrt{2} \tan^{-1} \left( \frac{t^2 - 1}{\sqrt{2} t} \right) + C = \sqrt{2} \tan^{-1} \left( \frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + C$$

**Example 40** Find  $\int \frac{\sin 2x \cos 2x dx}{\sqrt{9 - \cos^4(2x)}}$

**Solution** Let  $I = \int \frac{\sin 2x \cos 2x dx}{\sqrt{9 - \cos^4 2x}}$

Put  $\cos^2(2x) = t$  so that  $4 \sin 2x \cos 2x \, dx = -dt$

Therefore  $I = -\frac{1}{4} \int \frac{dt}{\sqrt{9-t^2}} = -\frac{1}{4} \sin^{-1}\left(\frac{t}{3}\right) + C = -\frac{1}{4} \sin^{-1}\left[\frac{1}{3} \cos^2 2x\right] + C$

**Example 41** Evaluate  $\int_{-1}^{\frac{3}{2}} |x \sin(\pi x)| \, dx$

**Solution** Here  $f(x) = |x \sin \pi x| = \begin{cases} x \sin \pi x & \text{for } -1 \leq x \leq 1 \\ -x \sin \pi x & \text{for } 1 \leq x \leq \frac{3}{2} \end{cases}$

Therefore  $\int_{-1}^{\frac{3}{2}} |x \sin \pi x| \, dx = \int_{-1}^1 x \sin \pi x \, dx + \int_1^{\frac{3}{2}} -x \sin \pi x \, dx$   
 $= \int_{-1}^1 x \sin \pi x \, dx - \int_1^{\frac{3}{2}} x \sin \pi x \, dx$

Integrating both integrals on righthand side, we get

$$\begin{aligned} \int_{-1}^{\frac{3}{2}} |x \sin \pi x| \, dx &= \left[ \frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} \right]_1^1 - \left[ \frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} \right]_1^{\frac{3}{2}} \\ &= \frac{2}{\pi} - \left[ -\frac{1}{\pi^2} - \frac{1}{\pi} \right] = \frac{3}{\pi} + \frac{1}{\pi^2} \end{aligned}$$

**Example 42** Evaluate  $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

**Solution** Let  $I = \int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi} \frac{(\pi-x) \, dx}{a^2 \cos^2(\pi-x) + b^2 \sin^2(\pi-x)}$  (using P<sub>4</sub>)  
 $= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - \int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$   
 $= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - I$

Thus  $2I = \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

or 
$$\begin{aligned} I &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} \text{ (using P}_6\text{)} \\ &= \pi \left[ \int_0^{\frac{\pi}{4}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} \right] \\ &= \pi \left[ \int_0^{\frac{\pi}{4}} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\operatorname{cosec}^2 x dx}{a^2 \cot^2 x + b^2} \right] \\ &= \pi \left[ \int_0^1 \frac{dt}{a^2 + b^2 t^2} - \int_1^0 \frac{du}{a^2 u^2 + b^2} \right] \left( \text{put } \tan x = t \text{ and } \cot x = u \right) \\ &= \frac{\pi}{ab} \left[ \tan^{-1} \frac{bt}{a} \right]_0^1 - \frac{\pi}{ab} \left[ \tan^{-1} \frac{au}{b} \right]_1^0 = \frac{\pi}{ab} \left[ \tan^{-1} \frac{b}{a} + \tan^{-1} \frac{a}{b} \right] = \frac{\pi^2}{2ab} \end{aligned}$$

### Miscellaneous Exercise on Chapter 7

Integrate the functions in Exercises 1 to 23.

1.  $\frac{1}{x - x^3}$
2.  $\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$
3.  $\frac{1}{x \sqrt{ax-x^2}}$  [Hint: Put  $x = \frac{a}{t}$ ]
4.  $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$
5.  $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$  [Hint:  $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}} \left( 1 + x^{\frac{1}{6}} \right)}$ , put  $x = t^6$ ]
6.  $\frac{5x}{(x+1)(x^2+9)}$
7.  $\frac{\sin x}{\sin(x-a)}$
8.  $\frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}}$
9.  $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$
10.  $\frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x}$
11.  $\frac{1}{\cos(x+a) \cos(x+b)}$
12.  $\frac{x^3}{\sqrt{1-x^8}}$
13.  $\frac{e^x}{(1+e^x)(2+e^x)}$
14.  $\frac{1}{(x^2+1)(x^2+4)}$
15.  $\cos^3 x \ e^{\log \sin x}$
16.  $e^{3 \log x} (x^4 + 1)^{-1}$
17.  $f'(ax+b) [f(ax+b)]^n$

18. 
$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

19. 
$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

20. 
$$\frac{2+\sin 2x}{1+\cos 2x} e^x$$

21. 
$$\frac{x^2+x+1}{(x+1)^2(x+2)}$$

22. 
$$\tan^{-1} \sqrt{\frac{1-x}{1+x}}$$

23. 
$$\frac{\sqrt{x^2+1} [\log(x^2+1) - 2\log x]}{x^4}$$

Evaluate the definite integrals in Exercises 24 to 31.

24. 
$$\int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{1-\sin x}{1-\cos x} \right) dx$$

25. 
$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

26. 
$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x dx}{\cos^2 x + 4 \sin^2 x}$$

27. 
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

28. 
$$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

29. 
$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9+16 \sin 2x} dx$$

30. 
$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

31. 
$$\int_1^4 [|x-1| + |x-2| + |x-3|] dx$$

Prove the following (Exercises 32 to 37)

32. 
$$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$$

33. 
$$\int_0^1 x e^x dx = 1$$

34. 
$$\int_{-1}^1 x^{17} \cos^4 x dx = 0$$

35. 
$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$$

36. 
$$\int_0^{\frac{\pi}{4}} 2 \tan^3 x dx = 1 - \log 2$$

37. 
$$\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$$

Choose the correct answers in Exercises 38 to 40

38. 
$$\int \frac{dx}{e^x + e^{-x}}$$
 is equal to

- (A)  $\tan^{-1}(e^x) + C$   
 (C)  $\log(e^x - e^{-x}) + C$

- (B)  $\tan^{-1}(e^{-x}) + C$   
 (D)  $\log(e^x + e^{-x}) + C$

39. 
$$\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$$
 is equal to

(A) 
$$\frac{-1}{\sin x + \cos x} + C$$

(B) 
$$\log |\sin x + \cos x| + C$$

(C) 
$$\log |\sin x - \cos x| + C$$

(D) 
$$\frac{1}{(\sin x + \cos x)^2}$$

40. If  $f(a+b-x) = f(x)$ , then  $\int_a^b x f(x) dx$  is equal to

(A) 
$$\frac{a+b}{2} \int_a^b f(b-x) dx$$

(B) 
$$\frac{a+b}{2} \int_a^b f(b+x) dx$$

(C) 
$$\frac{b-a}{2} \int_a^b f(x) dx$$

(D) 
$$\frac{a+b}{2} \int_a^b f(x) dx$$

### Summary

- ◆ Integration is the inverse process of differentiation. In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation.

Let  $\frac{d}{dx} F(x) = f(x)$ . Then we write  $\int f(x) dx = F(x) + C$ . These integrals are called indefinite integrals or general integrals,  $C$  is called constant of integration. All these integrals differ by a constant.

- ◆ Some properties of indefinite integrals are as follows:

1.  $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$

2. For any real number  $k$ ,  $\int k f(x) dx = k \int f(x) dx$

More generally, if  $f_1, f_2, f_3, \dots, f_n$  are functions and  $k_1, k_2, \dots, k_n$  are real numbers. Then

$$\begin{aligned} \int [k_1 f_1(x) + k_2 f_2(x) + \dots + k_n f_n(x)] dx \\ = k_1 \int f_1(x) dx + k_2 \int f_2(x) dx + \dots + k_n \int f_n(x) dx \end{aligned}$$

◆ **Some standard integrals**

- (i)  $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$ . Particularly,  $\int dx = x + C$

(ii)  $\int \cos x \, dx = \sin x + C$   
 (iv)  $\int \sec^2 x \, dx = \tan x + C$   
 (vi)  $\int \sec x \tan x \, dx = \sec x + C$

(vii)  $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$  (viii)  $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(ix)  $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$  (x)  $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(xi)  $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$  (xii)  $\int e^x \, dx = e^x + C$

(xiii)  $\int a^x \, dx = \frac{a^x}{\log a} + C$  (xiv)  $\int \frac{1}{x} \, dx = \log |x| + C$

◆ **Integration by partial fractions**

Recall that a rational function is ratio of two polynomials of the form  $\frac{P(x)}{Q(x)}$ ,

where  $P(x)$  and  $Q(x)$  are polynomials in  $x$  and  $Q(x) \neq 0$ . If degree of the polynomial  $P(x)$  is greater than the degree of the polynomial  $Q(x)$ , then we

may divide  $P(x)$  by  $Q(x)$  so that  $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ , where  $T(x)$  is a

polynomial in  $x$  and degree of  $P_1(x)$  is less than the degree of  $Q(x)$ .  $T(x)$

being polynomial can be easily integrated.  $\frac{P_1(x)}{Q(x)}$  can be integrated by

expressing  $\frac{P_1(x)}{Q(x)}$  as the sum of partial fractions of the following type:

$$1. \quad \frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, \quad a \neq b$$

$$2. \quad \frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$3. \frac{px^2 + qx + r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$4. \frac{px^2 + qx + r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$$

$$5. \frac{px^2 + qx + r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

where  $x^2 + bx + c$  can not be factorised further.

#### ◆ Integration by substitution

A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method in which we change the variable to some other variable is called the method of substitution. When the integrand involves some trigonometric functions, we use some well known identities to find the integrals. Using substitution technique, we obtain the following standard integrals.

$$(i) \int \tan x \, dx = \log |\sec x| + C \quad (ii) \int \cot x \, dx = \log |\sin x| + C$$

$$(iii) \int \sec x \, dx = \log |\sec x + \tan x| + C$$

$$(iv) \int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + C$$

#### ◆ Integrals of some special functions

$$(i) \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$(ii) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C \quad (iii) \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$(iv) \int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C \quad (v) \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

$$(vi) \int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + C$$

#### ◆ Integration by parts

For given functions  $f_1$  and  $f_2$ , we have

$\int f_1(x) \cdot f_2(x) dx = f_1(x) \int f_2(x) dx - \int \left[ \frac{d}{dx} f_1(x) \cdot \int f_2(x) dx \right] dx$ , i.e., the

integral of the product of two functions = first function  $\times$  integral of the second function – integral of {differential coefficient of the first function  $\times$  integral of the second function}. Care must be taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

◆  $\int e^x [f(x) + f'(x)] dx = \int e^x f(x) dx + C$

◆ **Some special types of integrals**

(i)  $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$

(ii)  $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$

(iii)  $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

(iv) Integrals of the types  $\int \frac{dx}{ax^2 + bx + c}$  or  $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$  can be

transformed into standard form by expressing

$$ax^2 + bx + c = a \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right] = a \left[ \left( x + \frac{b}{2a} \right)^2 + \left( \frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$$

(v) Integrals of the types  $\int \frac{px + q}{ax^2 + bx + c} dx$  or  $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$  can be

transformed into standard form by expressing

$px + q = A \frac{d}{dx} (ax^2 + bx + c) + B = A(2ax + b) + B$ , where A and B are determined by comparing coefficients on both sides.

◆ We have defined  $\int_a^b f(x) dx$  as the area of the region bounded by the curve  $y = f(x)$ ,  $a \leq x \leq b$ , the  $x$ -axis and the ordinates  $x = a$  and  $x = b$ . Let  $x$  be a

given point in  $[a, b]$ . Then  $\int_a^x f(x) dx$  represents the **Area function**  $A(x)$ .

This concept of area function leads to the Fundamental Theorems of Integral Calculus.

◆ **First fundamental theorem of integral calculus**

Let the area function be defined by  $A(x) = \int_a^x f(x) dx$  for all  $x \geq a$ , where the function  $f$  is assumed to be continuous on  $[a, b]$ . Then  $A'(x) = f(x)$  for all  $x \in [a, b]$ .

◆ **Second fundamental theorem of integral calculus**

Let  $f$  be a continuous function of  $x$  defined on the closed interval  $[a, b]$  and

let  $F$  be another function such that  $\frac{d}{dx} F(x) = f(x)$  for all  $x$  in the domain of

$f$ , then  $\int_a^b f(x) dx = [F(x) + C]_a^b = F(b) - F(a)$ .

This is called the definite integral of  $f$  over the range  $[a, b]$ , where  $a$  and  $b$  are called the limits of integration,  $a$  being the lower limit and  $b$  the upper limit.





# APPLICATION OF INTEGRALS

❖ *One should study Mathematics because it is only through Mathematics that nature can be conceived in harmonious form. – BIRKHOFF* ❖

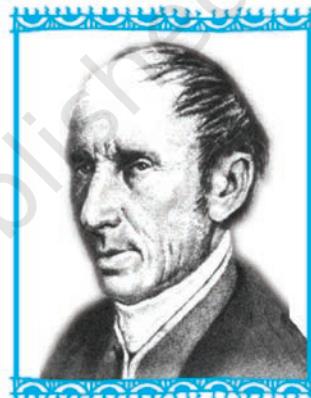
## 8.1 Introduction

In geometry, we have learnt formulae to calculate areas of various geometrical figures including triangles, rectangles, trapezias and circles. Such formulae are fundamental in the applications of mathematics to many real life problems. The formulae of elementary geometry allow us to calculate areas of many simple figures. However, they are inadequate for calculating the areas enclosed by curves. For that we shall need some concepts of Integral Calculus.

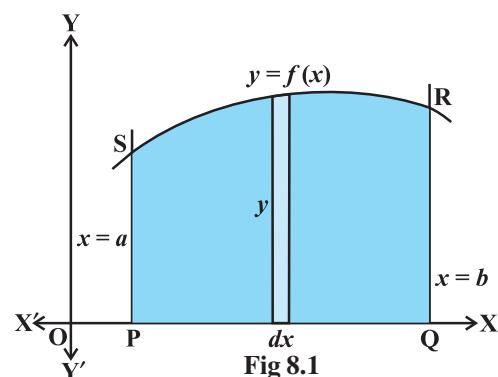
In the previous chapter, we have studied to find the area bounded by the curve  $y = f(x)$ , the ordinates  $x = a$ ,  $x = b$  and  $x$ -axis, while calculating definite integral as the limit of a sum. Here, in this chapter, we shall study a specific application of integrals to find the area under simple curves, area between lines and arcs of circles, parabolas and ellipses (standard forms only). We shall also deal with finding the area bounded by the above said curves.

## 8.2 Area under Simple Curves

In the previous chapter, we have studied definite integral as the limit of a sum and how to evaluate definite integral using Fundamental Theorem of Calculus. Now, we consider the easy and intuitive way of finding the area bounded by the curve  $y = f(x)$ ,  $x$ -axis and the ordinates  $x = a$  and  $x = b$ . From Fig 8.1, we can think of area under the curve as composed of large number of very thin vertical strips. Consider an arbitrary strip of height  $y$  and width  $dx$ , then  $dA$  (area of the elementary strip) =  $ydx$ , where,  $y = f(x)$ .



A.L. Cauchy  
(1789-1857)



This area is called the *elementary area* which is located at an arbitrary position within the region which is specified by some value of  $x$  between  $a$  and  $b$ . We can think of the total area  $A$  of the region between  $x$ -axis, ordinates  $x = a$ ,  $x = b$  and the curve  $y = f(x)$  as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically, we express

$$A = \int_a^b dA = \int_a^b ydx = \int_a^b f(x) dx$$

The area  $A$  of the region bounded by the curve  $x = g(y)$ ,  $y$ -axis and the lines  $y = c$ ,  $y = d$  is given by

$$A = \int_c^d xdy = \int_c^d g(y) dy$$

Here, we consider horizontal strips as shown in the Fig 8.2

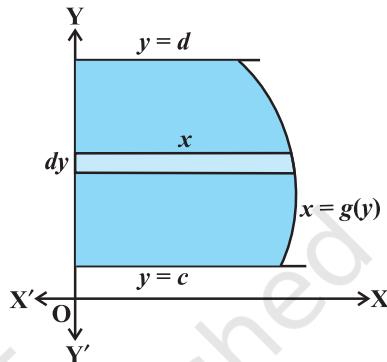


Fig 8.2

**Remark** If the position of the curve under consideration is below the  $x$ -axis, then since  $f(x) < 0$  from  $x = a$  to  $x = b$ , as shown in Fig 8.3, the area bounded by the curve,  $x$ -axis and the ordinates  $x = a$ ,  $x = b$  come out to be negative. But, it is only the numerical value of the area which is taken into consideration. Thus, if the area is negative, we

take its absolute value, i.e.,  $\left| \int_a^b f(x) dx \right|$ .

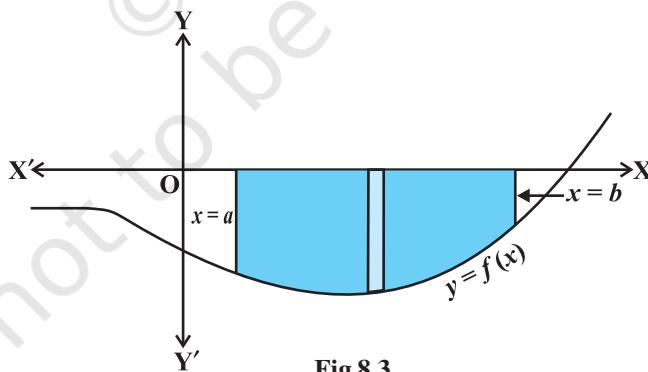


Fig 8.3

Generally, it may happen that some portion of the curve is above  $x$ -axis and some is below the  $x$ -axis as shown in the Fig 8.4. Here,  $A_1 < 0$  and  $A_2 > 0$ . Therefore, the area  $A$  bounded by the curve  $y = f(x)$ ,  $x$ -axis and the ordinates  $x = a$  and  $x = b$  is given by  $A = |A_1| + A_2$ .

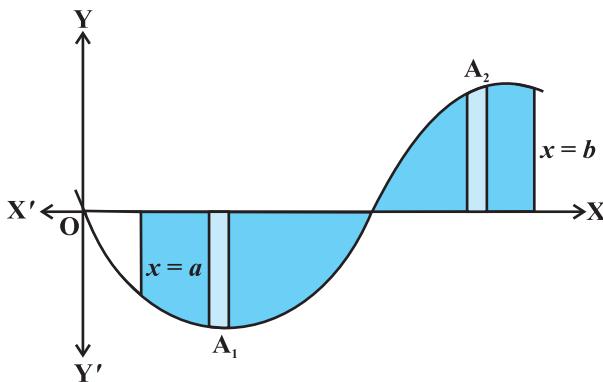


Fig 8.4

**Example 1** Find the area enclosed by the circle  $x^2 + y^2 = a^2$ .

**Solution** From Fig 8.5, the whole area enclosed by the given circle

$= 4$  (area of the region AOBA bounded by the curve,  $x$ -axis and the ordinates  $x = 0$  and  $x = a$ ) [as the circle is symmetrical about both  $x$ -axis and  $y$ -axis]

$$= 4 \int_0^a y dx \text{ (taking vertical strips)}$$

$$= 4 \int_0^a \sqrt{a^2 - x^2} dx$$

Since  $x^2 + y^2 = a^2$  gives  $y = \pm \sqrt{a^2 - x^2}$

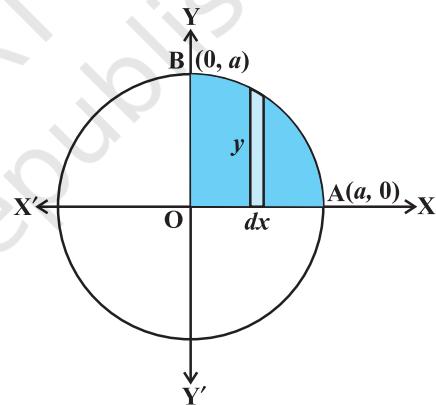


Fig 8.5

As the region AOBA lies in the first quadrant,  $y$  is taken as positive. Integrating, we get the whole area enclosed by the given circle

$$= 4 \left[ \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= 4 \left[ \left( \frac{a}{2} \times 0 + \frac{a^2}{2} \sin^{-1} 1 \right) - 0 \right] = 4 \left( \frac{a^2}{2} \right) \left( \frac{\pi}{2} \right) = \pi a^2$$

**Alternatively**, considering horizontal strips as shown in Fig 8.6, the whole area of the region enclosed by circle

$$\begin{aligned}
 &= 4 \int_0^a x dy = 4 \int_0^a \sqrt{a^2 - y^2} dy \quad (\text{Why?}) \\
 &= 4 \left[ \frac{y}{2} \sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} \right]_0^a \\
 &= 4 \left[ \left( \frac{a}{2} \times 0 + \frac{a^2}{2} \sin^{-1} 1 \right) - 0 \right] \\
 &= 4 \frac{a^2}{2} \frac{\pi}{2} = \pi a^2
 \end{aligned}$$

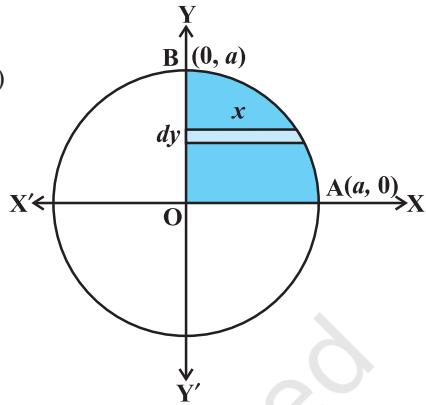


Fig 8.6

**Example 2** Find the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**Solution** From Fig 8.7, the area of the region ABA'B'A bounded by the ellipse

$$\begin{aligned}
 &= 4 \left( \text{area of the region } AOB \text{ in the first quadrant bounded} \right. \\
 &\quad \left. \text{by the curve, } x\text{-axis and the ordinates } x=0, x=a \right) \\
 &\quad (\text{as the ellipse is symmetrical about both } x\text{-axis and } y\text{-axis}) \\
 &= 4 \int_0^a y dx \quad (\text{taking vertical strips})
 \end{aligned}$$

Now  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  gives  $y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$ , but as the region AOB lies in the first quadrant,  $y$  is taken as positive. So, the required area is

$$\begin{aligned}
 &= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx \\
 &= \frac{4b}{a} \left[ \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \quad (\text{Why?}) \\
 &= \frac{4b}{a} \left[ \left( \frac{a}{2} \times 0 + \frac{a^2}{2} \sin^{-1} 1 \right) - 0 \right]
 \end{aligned}$$

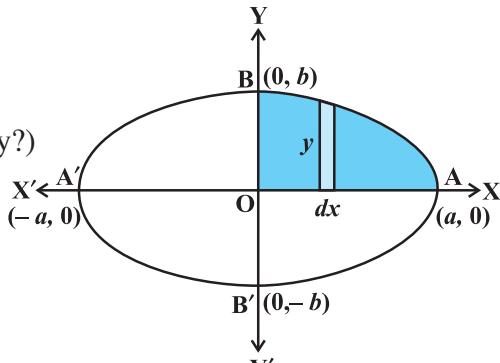


Fig 8.7

**Alternatively**, considering horizontal strips as shown in the Fig 8.8, the area of the ellipse is

$$\begin{aligned}
 &= 4 \int_0^b x dy = 4 \frac{a}{b} \int_0^b \sqrt{b^2 - y^2} dy \quad (\text{Why?}) \\
 &= \frac{4a}{b} \left[ \frac{y}{2} \sqrt{b^2 - y^2} + \frac{b^2}{2} \sin^{-1} \frac{y}{b} \right]_0^b \\
 &= \frac{4a}{b} \left[ \left( \frac{b}{2} \times 0 + \frac{b^2}{2} \sin^{-1} 1 \right) - 0 \right] \\
 &= \frac{4a}{b} \frac{b^2}{2} \frac{\pi}{2} = \pi ab
 \end{aligned}$$

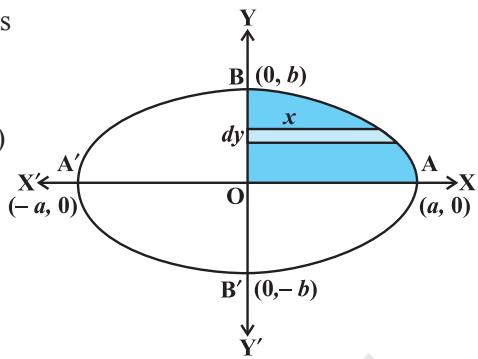


Fig 8.8

### EXERCISE 8.1

1. Find the area of the region bounded by the ellipse  $\frac{x^2}{16} + \frac{y^2}{9} = 1$ .

2. Find the area of the region bounded by the ellipse  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ .

Choose the correct answer in the following Exercises 3 and 4.

3. Area lying in the first quadrant and bounded by the circle  $x^2 + y^2 = 4$  and the lines  $x = 0$  and  $x = 2$  is

- (A)  $\pi$       (B)  $\frac{\pi}{2}$       (C)  $\frac{\pi}{3}$       (D)  $\frac{\pi}{4}$

4. Area of the region bounded by the curve  $y^2 = 4x$ ,  $y$ -axis and the line  $y = 3$  is

- (A) 2      (B)  $\frac{9}{4}$       (C)  $\frac{9}{3}$       (D)  $\frac{9}{2}$

### Miscellaneous Examples

**Example 3** Find the area of the region bounded by the line  $y = 3x + 2$ , the  $x$ -axis and the ordinates  $x = -1$  and  $x = 1$ .

**Solution** As shown in the Fig 8.9, the line

$y = 3x + 2$  meets  $x$ -axis at  $x = \frac{-2}{3}$  and its graph

lies below  $x$ -axis for  $x \in \left(-1, \frac{-2}{3}\right)$  and above

$x$ -axis for  $x \in \left(\frac{-2}{3}, 1\right)$ .

The required area = Area of the region ACBA + Area of the region ADEA

$$\begin{aligned}
 &= \left| \int_{-1}^{\frac{-2}{3}} (3x + 2) dx \right| + \int_{\frac{-2}{3}}^1 (3x + 2) dx \\
 &= \left[ \frac{3x^2}{2} + 2x \right]_{-1}^{\frac{-2}{3}} + \left[ \frac{3x^2}{2} + 2x \right]_{-2}^1 = \frac{1}{6} + \frac{25}{6} = \frac{13}{3}
 \end{aligned}$$

**Example 4** Find the area bounded by the curve  $y = \cos x$  between  $x = 0$  and  $x = 2\pi$ .

**Solution** From the Fig 8.10, the required area = area of the region OABO + area of the region BCDB + area of the region DEFD.

Thus, we have the required area

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{2}} \cos x dx + \left| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx \right| + \int_{\frac{3\pi}{2}}^{2\pi} \cos x dx \\
 &= [\sin x]_0^{\frac{\pi}{2}} + \left| [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right| + [\sin x]_{\frac{3\pi}{2}}^{2\pi} \\
 &= 1 + 2 + 1 = 4
 \end{aligned}$$

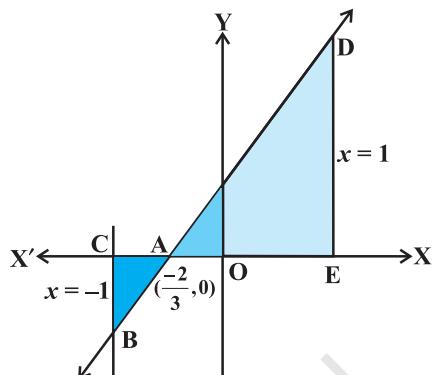


Fig 8.9

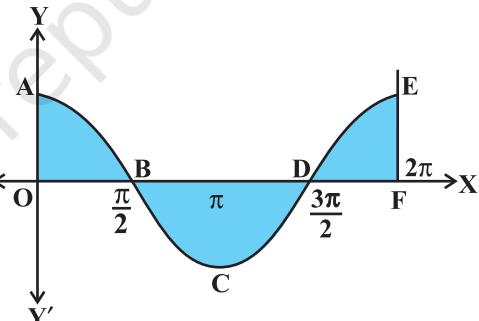


Fig 8.10

### Miscellaneous Exercise on Chapter 8

1. Find the area under the given curves and given lines:
  - (i)  $y = x^2$ ,  $x = 1$ ,  $x = 2$  and  $x$ -axis
  - (ii)  $y = x^4$ ,  $x = 1$ ,  $x = 5$  and  $x$ -axis
2. Sketch the graph of  $y = |x+3|$  and evaluate  $\int_{-6}^0 |x+3| dx$ .
3. Find the area bounded by the curve  $y = \sin x$  between  $x = 0$  and  $x = 2\pi$ .

Choose the correct answer in the following Exercises from 4 to 5.

4. Area bounded by the curve  $y = x^3$ , the  $x$ -axis and the ordinates  $x = -2$  and  $x = 1$  is
 

|          |                     |                    |                    |
|----------|---------------------|--------------------|--------------------|
| (A) $-9$ | (B) $\frac{-15}{4}$ | (C) $\frac{15}{4}$ | (D) $\frac{17}{4}$ |
|----------|---------------------|--------------------|--------------------|
5. The area bounded by the curve  $y = x|x|$ ,  $x$ -axis and the ordinates  $x = -1$  and  $x = 1$  is given by
 

|         |                   |                   |                   |
|---------|-------------------|-------------------|-------------------|
| (A) $0$ | (B) $\frac{1}{3}$ | (C) $\frac{2}{3}$ | (D) $\frac{4}{3}$ |
|---------|-------------------|-------------------|-------------------|

[Hint :  $y = x^2$  if  $x > 0$  and  $y = -x^2$  if  $x < 0$ ].

### Summary

- ◆ The area of the region bounded by the curve  $y = f(x)$ ,  $x$ -axis and the lines  $x = a$  and  $x = b$  ( $b > a$ ) is given by the formula:  $\text{Area} = \int_a^b y dx = \int_a^b f(x) dx$ .
- ◆ The area of the region bounded by the curve  $x = \phi(y)$ ,  $y$ -axis and the lines  $y = c$ ,  $y = d$  is given by the formula:  $\text{Area} = \int_c^d x dy = \int_c^d \phi(y) dy$ .

### Historical Note

The origin of the Integral Calculus goes back to the early period of development of Mathematics and it is related to the method of exhaustion developed by the mathematicians of ancient Greece. This method arose in the solution of problems on calculating areas of plane figures, surface areas and volumes of solid bodies etc. In this sense, the method of exhaustion can be regarded as an early method

of integration. The greatest development of method of exhaustion in the early period was obtained in the works of Eudoxus (440 B.C.) and Archimedes (300 B.C.)

Systematic approach to the theory of Calculus began in the 17th century. In 1665, Newton began his work on the Calculus described by him as the theory of fluxions and used his theory in finding the tangent and radius of curvature at any point on a curve. Newton introduced the basic notion of inverse function called the anti derivative (indefinite integral) or the inverse method of tangents.

During 1684-86, Leibnitz published an article in the *Acta Eruditorum* which he called *Calculus summatorius*, since it was connected with the summation of a number of infinitely small areas, whose sum, he indicated by the symbol ‘ $\int$ ’. In 1696, he followed a suggestion made by J. Bernoulli and changed this article to *Calculus integrali*. This corresponded to Newton’s inverse method of tangents.

Both Newton and Leibnitz adopted quite independent lines of approach which was radically different. However, respective theories accomplished results that were practically identical. Leibnitz used the notion of definite integral and what is quite certain is that he first clearly appreciated tie up between the antiderivative and the definite integral.

Conclusively, the fundamental concepts and theory of Integral Calculus and primarily its relationships with Differential Calculus were developed in the work of P.de Fermat, I. Newton and G. Leibnitz at the end of 17th century. However, this justification by the concept of limit was only developed in the works of A.L. Cauchy in the early 19th century. Lastly, it is worth mentioning the following quotation by Lie Sophie’s:

“It may be said that the conceptions of differential quotient and integral which in their origin certainly go back to Archimedes were introduced in Science by the investigations of Kepler, Descartes, Cavalieri, Fermat and Wallis .... The discovery that differentiation and integration are inverse operations belongs to Newton and Leibnitz”.





# DIFFERENTIAL EQUATIONS

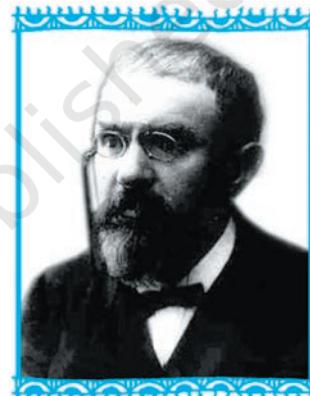
❖ *He who seeks for methods without having a definite problem in mind  
seeks for the most part in vain. – D. HILBERT* ❖

## 9.1 Introduction

In Class XI and in Chapter 5 of the present book, we discussed how to differentiate a given function  $f$  with respect to an independent variable, i.e., how to find  $f'(x)$  for a given function  $f$  at each  $x$  in its domain of definition. Further, in the chapter on Integral Calculus, we discussed how to find a function  $f$  whose derivative is the function  $g$ , which may also be formulated as follows:

For a given function  $g$ , find a function  $f$  such that

$$\frac{dy}{dx} = g(x), \text{ where } y = f(x) \quad \dots (1)$$



Henri Poincaré  
(1854-1912)

An equation of the form (1) is known as a *differential equation*. A formal definition will be given later.

These equations arise in a variety of applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics etc. Hence, an indepth study of differential equations has assumed prime importance in all modern scientific investigations.

In this chapter, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation, formation of differential equations, some methods to solve a first order - first degree differential equation and some applications of differential equations in different areas.

## 9.2 Basic Concepts

We are already familiar with the equations of the type:

$$x^2 - 3x + 3 = 0 \quad \dots (1)$$

$$\sin x + \cos x = 0 \quad \dots (2)$$

$$x + y = 7 \quad \dots (3)$$

Let us consider the equation:

$$x \frac{dy}{dx} + y = 0 \quad \dots (4)$$

We see that equations (1), (2) and (3) involve independent and/or dependent variable (variables) only but equation (4) involves variables as well as derivative of the dependent variable  $y$  with respect to the independent variable  $x$ . Such an equation is called a *differential equation*.

In general, an equation involving derivative (derivatives) of the dependent variable with respect to independent variable (variables) is called a differential equation.

A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation, e.g.,

$$2 \frac{d^2 y}{dx^2} + \left( \frac{dy}{dx} \right)^3 = 0 \text{ is an ordinary differential equation} \quad \dots (5)$$

Of course, there are differential equations involving derivatives with respect to more than one independent variables, called partial differential equations but at this stage we shall confine ourselves to the study of ordinary differential equations only. Now onward, we will use the term ‘differential equation’ for ‘ordinary differential equation’.

### Note

1. We shall prefer to use the following notations for derivatives:

$$\frac{dy}{dx} = y', \frac{d^2 y}{dx^2} = y'', \frac{d^3 y}{dx^3} = y'''$$

2. For derivatives of higher order, it will be inconvenient to use so many dashes

as supersuffix therefore, we use the notation  $y_n$  for  $n$ th order derivative  $\frac{d^n y}{dx^n}$ .

#### 9.2.1. Order of a differential equation

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Consider the following differential equations:

$$\frac{dy}{dx} = e^x \quad \dots (6)$$

$$\frac{d^2y}{dx^2} + y = 0 \quad \dots (7)$$

$$\left( \frac{d^3y}{dx^3} \right) + x^2 \left( \frac{d^2y}{dx^2} \right)^3 = 0 \quad \dots (8)$$

The equations (6), (7) and (8) involve the highest derivative of first, second and third order respectively. Therefore, the order of these equations are 1, 2 and 3 respectively.

### 9.2.2 Degree of a differential equation

To study the degree of a differential equation, the key point is that the differential equation must be a polynomial equation in derivatives, i.e.,  $y'$ ,  $y''$ ,  $y'''$  etc. Consider the following differential equations:

$$\frac{d^3y}{dx^3} + 2 \left( \frac{d^2y}{dx^2} \right)^2 - \frac{dy}{dx} + y = 0 \quad \dots (9)$$

$$\left( \frac{dy}{dx} \right)^2 + \left( \frac{dy}{dx} \right) - \sin^2 y = 0 \quad \dots (10)$$

$$\frac{dy}{dx} + \sin \left( \frac{dy}{dx} \right) = 0 \quad \dots (11)$$

We observe that equation (9) is a polynomial equation in  $y'''$ ,  $y''$  and  $y'$ , equation (10) is a polynomial equation in  $y'$  (not a polynomial in  $y$  though). Degree of such differential equations can be defined. But equation (11) is not a polynomial equation in  $y'$  and degree of such a differential equation can not be defined.

By the degree of a differential equation, when it is a polynomial equation in derivatives, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

In view of the above definition, one may observe that differential equations (6), (7), (8) and (9) each are of degree one, equation (10) is of degree two while the degree of differential equation (11) is not defined.

 **Note** Order and degree (if defined) of a differential equation are always positive integers.

**Example 1** Find the order and degree, if defined, of each of the following differential equations:

$$(i) \frac{dy}{dx} - \cos x = 0 \quad (ii) xy \frac{d^2y}{dx^2} + x \left( \frac{dy}{dx} \right)^2 - y \frac{dy}{dx} = 0$$

$$(iii) y''' + y^2 + e^{y'} = 0$$

### Solution

- (i) The highest order derivative present in the differential equation is  $\frac{dy}{dx}$ , so its order is one. It is a polynomial equation in  $y'$  and the highest power raised to  $\frac{dy}{dx}$  is one, so its degree is one.
- (ii) The highest order derivative present in the given differential equation is  $\frac{d^2y}{dx^2}$ , so its order is two. It is a polynomial equation in  $\frac{d^2y}{dx^2}$  and  $\frac{dy}{dx}$  and the highest power raised to  $\frac{d^2y}{dx^2}$  is one, so its degree is one.
- (iii) The highest order derivative present in the differential equation is  $y'''$ , so its order is three. The given differential equation is not a polynomial equation in its derivatives and so its degree is not defined.

### EXERCISE 9.1

Determine order and degree (if defined) of differential equations given in Exercises 1 to 10.

1.  $\frac{d^4y}{dx^4} + \sin(y''') = 0$
2.  $y' + 5y = 0$
3.  $\left( \frac{ds}{dt} \right)^4 + 3s \frac{d^2s}{dt^2} = 0$
4.  $\left( \frac{d^2y}{dx^2} \right)^2 + \cos \left( \frac{dy}{dx} \right) = 0$
5.  $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$
6.  $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$
7.  $y''' + 2y'' + y' = 0$

8.  $y' + y = e^x$       9.  $y'' + (y')^2 + 2y = 0$       10.  $y'' + 2y' + \sin y = 0$

11. The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is}$$

- (A) 3      (B) 2      (C) 1      (D) not defined

12. The order of the differential equation

$$2x^2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + y = 0 \text{ is}$$

- (A) 2      (B) 1      (C) 0      (D) not defined

### 9.3. General and Particular Solutions of a Differential Equation

In earlier Classes, we have solved the equations of the type:

$$x^2 + 1 = 0 \quad \dots (1)$$

$$\sin^2 x - \cos x = 0 \quad \dots (2)$$

Solution of equations (1) and (2) are numbers, real or complex, that will satisfy the given equation i.e., when that number is substituted for the unknown  $x$  in the given equation, L.H.S. becomes equal to the R.H.S..

Now consider the differential equation  $\frac{d^2y}{dx^2} + y = 0$   $\dots (3)$

In contrast to the first two equations, the solution of this differential equation is a function  $\phi$  that will satisfy it i.e., when the function  $\phi$  is substituted for the unknown  $y$  (dependent variable) in the given differential equation, L.H.S. becomes equal to R.H.S..

The curve  $y = \phi(x)$  is called the solution curve (integral curve) of the given differential equation. Consider the function given by

$$y = \phi(x) = a \sin(x + b), \quad \dots (4)$$

where  $a, b \in \mathbf{R}$ . When this function and its derivative are substituted in equation (3), L.H.S. = R.H.S.. So it is a solution of the differential equation (3).

Let  $a$  and  $b$  be given some particular values say  $a = 2$  and  $b = \frac{\pi}{4}$ , then we get a

function  $y = \phi_1(x) = 2 \sin\left(x + \frac{\pi}{4}\right)$   $\dots (5)$

When this function and its derivative are substituted in equation (3) again L.H.S. = R.H.S.. Therefore  $\phi_1$  is also a solution of equation (3).

Function  $\phi$  consists of two arbitrary constants (parameters)  $a, b$  and it is called *general solution* of the given differential equation. Whereas function  $\phi_1$  contains no arbitrary constants but only the particular values of the parameters  $a$  and  $b$  and hence is called a *particular solution* of the given differential equation.

The solution which contains arbitrary constants is called the *general solution (primitive)* of the differential equation.

The solution free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a *particular solution* of the differential equation.

**Example 2** Verify that the function  $y = e^{-3x}$  is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

**Solution** Given function is  $y = e^{-3x}$ . Differentiating both sides of equation with respect to  $x$ , we get

$$\frac{dy}{dx} = -3e^{-3x} \quad \dots (1)$$

Now, differentiating (1) with respect to  $x$ , we have

$$\frac{d^2y}{dx^2} = 9e^{-3x}$$

Substituting the values of  $\frac{d^2y}{dx^2}, \frac{dy}{dx}$  and  $y$  in the given differential equation, we get

$$\text{L.H.S.} = 9e^{-3x} + (-3e^{-3x}) - 6e^{-3x} = 9e^{-3x} - 9e^{-3x} = 0 = \text{R.H.S.}$$

Therefore, the given function is a solution of the given differential equation.

**Example 3** Verify that the function  $y = a \cos x + b \sin x$ , where,  $a, b \in \mathbf{R}$  is a solution of the differential equation  $\frac{d^2y}{dx^2} + y = 0$

**Solution** The given function is

$$y = a \cos x + b \sin x \quad \dots (1)$$

Differentiating both sides of equation (1) with respect to  $x$ , successively, we get

$$\frac{dy}{dx} = -a \sin x + b \cos x$$

$$\frac{d^2y}{dx^2} = -a \cos x - b \sin x$$

Substituting the values of  $\frac{d^2 y}{dx^2}$  and  $y$  in the given differential equation, we get

$$\text{L.H.S.} = (-a \cos x - b \sin x) + (a \cos x + b \sin x) = 0 = \text{R.H.S.}$$

Therefore, the given function is a solution of the given differential equation.

### EXERCISE 9.2

In each of the Exercises 1 to 10 verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

1.  $y = e^x + 1$  :  $y'' - y' = 0$
2.  $y = x^2 + 2x + C$  :  $y' - 2x - 2 = 0$
3.  $y = \cos x + C$  :  $y' + \sin x = 0$
4.  $y = \sqrt{1+x^2}$  :  $y' = \frac{xy}{1+x^2}$
5.  $y = Ax$  :  $xy' = y (x \neq 0)$
6.  $y = x \sin x$  :  $xy' = y + x \sqrt{x^2 - y^2} \quad (x \neq 0 \text{ and } x > y \text{ or } x < -y)$
7.  $xy = \log y + C$  :  $y' = \frac{y^2}{1-xy} \quad (xy \neq 1)$
8.  $y - \cos y = x$  :  $(y \sin y + \cos y + x) y' = y$
9.  $x + y = \tan^{-1} y$  :  $y^2 y' + y^2 + 1 = 0$
10.  $y = \sqrt{a^2 - x^2} \quad x \in (-a, a)$  :  $x + y \frac{dy}{dx} = 0 \quad (y \neq 0)$
11. The number of arbitrary constants in the general solution of a differential equation of fourth order are:  
 (A) 0      (B) 2      (C) 3      (D) 4
12. The number of arbitrary constants in the particular solution of a differential equation of third order are:  
 (A) 3      (B) 2      (C) 1      (D) 0

## 9.4. Methods of Solving First Order, First Degree Differential Equations

In this section we shall discuss three methods of solving first order first degree differential equations.

### 9.4.1 Differential equations with variables separable

A first order-first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y) \quad \dots (1)$$

If  $F(x, y)$  can be expressed as a product  $g(x) h(y)$ , where,  $g(x)$  is a function of  $x$  and  $h(y)$  is a function of  $y$ , then the differential equation (1) is said to be of variable separable type. The differential equation (1) then has the form

$$\frac{dy}{dx} = h(y) \cdot g(x) \quad \dots (2)$$

If  $h(y) \neq 0$ , separating the variables, (2) can be rewritten as

$$\frac{1}{h(y)} dy = g(x) dx \quad \dots (3)$$

Integrating both sides of (3), we get

$$\int \frac{1}{h(y)} dy = \int g(x) dx \quad \dots (4)$$

Thus, (4) provides the solutions of given differential equation in the form

$$H(y) = G(x) + C$$

Here,  $H(y)$  and  $G(x)$  are the anti derivatives of  $\frac{1}{h(y)}$  and  $g(x)$  respectively and

$C$  is the arbitrary constant.

**Example 4** Find the general solution of the differential equation  $\frac{dy}{dx} = \frac{x+1}{2-y}$ , ( $y \neq 2$ )

**Solution** We have

$$\frac{dy}{dx} = \frac{x+1}{2-y} \quad \dots (1)$$

Separating the variables in equation (1), we get

$$(2-y) dy = (x+1) dx \quad \dots (2)$$

Integrating both sides of equation (2), we get

$$\int (2-y) dy = \int (x+1) dx$$

$$\text{or} \quad 2y - \frac{y^2}{2} = \frac{x^2}{2} + x + C_1$$

$$\text{or} \quad x^2 + y^2 + 2x - 4y + 2C_1 = 0$$

$$\text{or} \quad x^2 + y^2 + 2x - 4y + C = 0, \text{ where } C = 2C_1$$

which is the general solution of equation (1).

**Example 5** Find the general solution of the differential equation  $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ .

**Solution** Since  $1+y^2 \neq 0$ , therefore separating the variables, the given differential equation can be written as

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \dots (1)$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or

$$\tan^{-1} y = \tan^{-1} x + C$$

which is the general solution of equation (1).

**Example 6** Find the particular solution of the differential equation  $\frac{dy}{dx} = -4xy^2$  given

that  $y = 1$ , when  $x = 0$ .

**Solution** If  $y \neq 0$ , the given differential equation can be written as

$$\frac{dy}{y^2} = -4x \, dx \quad \dots (1)$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{y^2} = -4 \int x \, dx$$

or 
$$-\frac{1}{y} = -2x^2 + C$$
 
$$\dots (2)$$

or 
$$y = \frac{1}{2x^2 - C}$$
 
$$\dots (2)$$

Substituting  $y = 1$  and  $x = 0$  in equation (2), we get,  $C = -1$ .

Now substituting the value of  $C$  in equation (2), we get the particular solution of the given differential equation as  $y = \frac{1}{2x^2 + 1}$ .

**Example 7** Find the equation of the curve passing through the point  $(1, 1)$  whose differential equation is  $x \, dy = (2x^2 + 1) \, dx$  ( $x \neq 0$ ).

**Solution** The given differential equation can be expressed as

$$dy^* = \left( \frac{2x^2 + 1}{x} \right) dx^*$$

or  $dy = \left( 2x + \frac{1}{x} \right) dx \quad \dots (1)$

Integrating both sides of equation (1), we get

$$\int dy = \int \left( 2x + \frac{1}{x} \right) dx$$

or  $y = x^2 + \log |x| + C \quad \dots (2)$

Equation (2) represents the family of solution curves of the given differential equation but we are interested in finding the equation of a particular member of the family which passes through the point  $(1, 1)$ . Therefore substituting  $x = 1, y = 1$  in equation (2), we get  $C = 0$ .

Now substituting the value of  $C$  in equation (2) we get the equation of the required curve as  $y = x^2 + \log |x|$ .

**Example 8** Find the equation of a curve passing through the point  $(-2, 3)$ , given that the slope of the tangent to the curve at any point  $(x, y)$  is  $\frac{2x}{y^2}$ .

**Solution** We know that the slope of the tangent to a curve is given by  $\frac{dy}{dx}$ .

so,  $\frac{dy}{dx} = \frac{2x}{y^2} \quad \dots (1)$

Separating the variables, equation (1) can be written as

$$y^2 dy = 2x dx \quad \dots (2)$$

Integrating both sides of equation (2), we get

$$\int y^2 dy = \int 2x dx$$

or  $\frac{y^3}{3} = x^2 + C \quad \dots (3)$

\* The notation  $\frac{dy}{dx}$  due to Leibnitz is extremely flexible and useful in many calculation and formal transformations, where, we can deal with symbols  $dy$  and  $dx$  exactly as if they were ordinary numbers. By treating  $dx$  and  $dy$  like separate entities, we can give neater expressions to many calculations.

Refer: Introduction to Calculus and Analysis, volume-I page 172, By Richard Courant, Fritz John Springer – Verlog New York.

Substituting  $x = -2, y = 3$  in equation (3), we get  $C = 5$ .

Substituting the value of  $C$  in equation (3), we get the equation of the required curve as

$$\frac{y^3}{3} = x^2 + 5 \quad \text{or} \quad y = (3x^2 + 15)^{\frac{1}{3}}$$

**Example 9** In a bank, principal increases continuously at the rate of 5% per year. In how many years Rs 1000 double itself?

**Solution** Let  $P$  be the principal at any time  $t$ . According to the given problem,

$$\frac{dp}{dt} = \left(\frac{5}{100}\right) \times P$$

$$\text{or} \quad \frac{dp}{dt} = \frac{P}{20} \quad \dots (1)$$

separating the variables in equation (1), we get

$$\frac{dp}{P} = \frac{dt}{20} \quad \dots (2)$$

Integrating both sides of equation (2), we get

$$\log P = \frac{t}{20} + C_1$$

$$\text{or} \quad P = e^{\frac{t}{20}} \cdot e^{C_1} \quad \dots (3)$$

$$\text{or} \quad P = C e^{\frac{t}{20}} \quad (\text{where } e^{C_1} = C) \quad \dots (3)$$

$$\text{Now} \quad P = 1000, \quad \text{when } t = 0$$

Substituting the values of  $P$  and  $t$  in (3), we get  $C = 1000$ . Therefore, equation (3), gives

$$P = 1000 e^{\frac{t}{20}}$$

Let  $t$  years be the time required to double the principal. Then

$$2000 = 1000 e^{\frac{t}{20}} \Rightarrow t = 20 \log 2$$

### EXERCISE 9.3

For each of the differential equations in Exercises 1 to 10, find the general solution:

$$1. \quad \frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$$

$$2. \quad \frac{dy}{dx} = \sqrt{4 - y^2} \quad (-2 < y < 2)$$

3.  $\frac{dy}{dx} + y = 1 \ (y \neq 1)$

4.  $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

5.  $(e^x + e^{-x}) \, dy - (e^x - e^{-x}) \, dx = 0$

6.  $\frac{dy}{dx} = (1+x^2)(1+y^2)$

7.  $y \log y \, dx - x \, dy = 0$

8.  $x^5 \frac{dy}{dx} = -y^5$

9.  $\frac{dy}{dx} = \sin^{-1} x$

10.  $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

For each of the differential equations in Exercises 11 to 14, find a particular solution satisfying the given condition:

11.  $(x^3 + x^2 + x + 1) \frac{dy}{dx} = 2x^2 + x; \ y = 1 \text{ when } x = 0$

12.  $x(x^2 - 1) \frac{dy}{dx} = 1; \ y = 0 \text{ when } x = 2$

13.  $\cos\left(\frac{dy}{dx}\right) = a \ (a \in \mathbf{R}); \ y = 1 \text{ when } x = 0$

14.  $\frac{dy}{dx} = y \tan x; \ y = 1 \text{ when } x = 0$

15. Find the equation of a curve passing through the point  $(0, 0)$  and whose differential equation is  $y' = e^x \sin x$ .

16. For the differential equation  $xy \frac{dy}{dx} = (x+2)(y+2)$ , find the solution curve passing through the point  $(1, -1)$ .

17. Find the equation of a curve passing through the point  $(0, -2)$  given that at any point  $(x, y)$  on the curve, the product of the slope of its tangent and  $y$  coordinate of the point is equal to the  $x$  coordinate of the point.

18. At any point  $(x, y)$  of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point  $(-4, -3)$ . Find the equation of the curve given that it passes through  $(-2, 1)$ .

19. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after  $t$  seconds.

20. In a bank, principal increases continuously at the rate of  $r\%$  per year. Find the value of  $r$  if Rs 100 double itself in 10 years ( $\log_2 = 0.6931$ ).

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ( $e^{0.5} = 1.648$ ).

22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

23. The general solution of the differential equation  $\frac{dy}{dx} = e^{x+y}$  is

(A)  $e^x + e^{-y} = C$       (B)  $e^x + e^y = C$   
(C)  $e^{-x} + e^y = C$       (D)  $e^{-x} + e^{-y} = C$

### 9.4.2 Homogeneous differential equations

Consider the following functions in  $x$  and  $y$

$$F_1(x, y) = y^2 + 2xy, \quad F_2(x, y) = 2x - 3y,$$

$$F_3(x, y) = \cos\left(\frac{y}{x}\right), \quad F_4(x, y) = \sin x + \cos y$$

If we replace  $x$  and  $y$  by  $\lambda x$  and  $\lambda y$  respectively in the above functions, for any nonzero constant  $\lambda$ , we get

$$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$$

$$F_2(\lambda x, \lambda y) = \lambda(2x - 3y) = \lambda F_2(x, y)$$

$$F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right) = \lambda^0 \ F_3(x, y)$$

$$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n F_4(x, y), \text{ for any } n \in \mathbb{N}$$

Here, we observe that the functions  $F_1, F_2, F_3$  can be written in the form  $F(\lambda x, \lambda y) = \lambda^n F(x, y)$  but  $F_4$  can not be written in this form. This leads to the following definition:

A function  $F(x, y)$  is said to be *homogeneous function of degree n* if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y) \text{ for any nonzero constant } \lambda.$$

We note that in the above examples,  $F_1$ ,  $F_2$ ,  $F_3$  are homogeneous functions of degree 2, 1, 0 respectively but  $F_4$  is not a homogeneous function.

We also observe that

$$F_1(x, y) = x^2 \left( \frac{y^2}{x^2} + \frac{2y}{x} \right) = x^2 h_1 \left( \frac{y}{x} \right)$$

or

$$F_1(x, y) = y^2 \left( 1 + \frac{2x}{y} \right) = y^2 h_2 \left( \frac{x}{y} \right)$$

$$F_2(x, y) = x^1 \left( 2 - \frac{3y}{x} \right) = x^1 h_3 \left( \frac{y}{x} \right)$$

or

$$F_2(x, y) = y^1 \left( 2 \frac{x}{y} - 3 \right) = y^1 h_4 \left( \frac{x}{y} \right)$$

$$F_3(x, y) = x^0 \cos \left( \frac{y}{x} \right) = x^0 h_5 \left( \frac{y}{x} \right)$$

$$F_4(x, y) \neq x^n h_6 \left( \frac{y}{x} \right), \text{ for any } n \in \mathbf{N}$$

or

$$F_4(x, y) \neq y^n h_7 \left( \frac{x}{y} \right), \text{ for any } n \in \mathbf{N}$$

Therefore, a function  $F(x, y)$  is a homogeneous function of degree  $n$  if

$$F(x, y) = x^n g \left( \frac{y}{x} \right) \quad \text{or} \quad y^n h \left( \frac{x}{y} \right)$$

A differential equation of the form  $\frac{dy}{dx} = F(x, y)$  is said to be *homogenous* if

$F(x, y)$  is a homogenous function of degree zero.

To solve a homogeneous differential equation of the type

$$\frac{dy}{dx} = F(x, y) = g \left( \frac{y}{x} \right) \quad \dots (1)$$

We make the substitution  $y = v \cdot x$   $\dots (2)$

Differentiating equation (2) with respect to  $x$ , we get

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (3)$$

Substituting the value of  $\frac{dy}{dx}$  from equation (3) in equation (1), we get

$$v + x \frac{dv}{dx} = g(v)$$

$$\text{or } x \frac{dv}{dx} = g(v) - v \quad \dots (4)$$

Separating the variables in equation (4), we get

$$\frac{dv}{g(v) - v} = \frac{dx}{x} \quad \dots (5)$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{g(v) - v} = \int \frac{1}{x} dx + C \quad \dots (6)$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace  $v$  by  $\frac{y}{x}$ .

**Note** If the homogeneous differential equation is in the form  $\frac{dx}{dy} = F(x, y)$

where,  $F(x, y)$  is homogenous function of degree zero, then we make substitution

$\frac{x}{y} = v$  i.e.,  $x = vy$  and we proceed further to find the general solution as discussed

above by writing  $\frac{dx}{dy} = F(x, y) = h\left(\frac{x}{y}\right)$ .

**Example 10** Show that the differential equation  $(x - y) \frac{dy}{dx} = x + 2y$  is homogeneous and solve it.

**Solution** The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y} \quad \dots (1)$$

Let

$$F(x, y) = \frac{x + 2y}{x - y}$$

Now

$$F(\lambda x, \lambda y) = \frac{\lambda(x + 2y)}{\lambda(x - y)} = \lambda^0 \cdot f(x, y)$$

Therefore,  $F(x, y)$  is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

**Alternatively,**

$$\frac{dy}{dx} = \left( \frac{1 + \frac{2y}{x}}{1 - \frac{y}{x}} \right) = g\left(\frac{y}{x}\right) \quad \dots (2)$$

R.H.S. of differential equation (2) is of the form  $g\left(\frac{y}{x}\right)$  and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \quad \dots (3)$$

Differentiating equation (3) with respect to,  $x$  we get

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (4)$$

Substituting the value of  $y$  and  $\frac{dy}{dx}$  in equation (1) we get

$$v + x \frac{dv}{dx} = \frac{1 + 2v}{1 - v}$$

or  $x \frac{dv}{dx} = \frac{1 + 2v}{1 - v} - v$

or  $x \frac{dv}{dx} = \frac{v^2 + v + 1}{1 - v}$

or  $\frac{v - 1}{v^2 + v + 1} dv = \frac{-dx}{x}$

Integrating both sides of equation (5), we get

$$\int \frac{v - 1}{v^2 + v + 1} dv = - \int \frac{dx}{x}$$

or  $\frac{1}{2} \int \frac{2v + 1 - 3}{v^2 + v + 1} dv = - \log |x| + C_1$

$$\text{or } \frac{1}{2} \int \frac{2v+1}{v^2+v+1} dv - \frac{3}{2} \int \frac{1}{v^2+v+1} dv = -\log|x| + C_1$$

$$\text{or } \frac{1}{2} \log|v^2+v+1| - \frac{3}{2} \int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dv = -\log|x| + C_1$$

$$\text{or } \frac{1}{2} \log|v^2+v+1| - \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x| + C_1$$

$$\text{or } \frac{1}{2} \log|v^2+v+1| + \frac{1}{2} \log x^2 = \sqrt{3} \tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C_1 \quad (\text{Why?})$$

Replacing  $v$  by  $\frac{y}{x}$ , we get

$$\text{or } \frac{1}{2} \log\left|\frac{y^2}{x^2} + \frac{y}{x} + 1\right| + \frac{1}{2} \log x^2 = \sqrt{3} \tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C_1$$

$$\text{or } \frac{1}{2} \log\left|\left(\frac{y^2}{x^2} + \frac{y}{x} + 1\right)x^2\right| = \sqrt{3} \tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C_1$$

$$\text{or } \log|(y^2 + xy + x^2)| = 2\sqrt{3} \tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + 2C_1$$

$$\text{or } \log|(x^2 + xy + y^2)| = 2\sqrt{3} \tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C$$

which is the general solution of the differential equation (1)

**Example 11** Show that the differential equation  $x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$  is homogeneous and solve it.

**Solution** The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)} \quad \dots (1)$$

It is a differential equation of the form  $\frac{dy}{dx} = F(x, y)$ .

Here

$$F(x, y) = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)}$$

Replacing  $x$  by  $\lambda x$  and  $y$  by  $\lambda y$ , we get

$$F(\lambda x, \lambda y) = \frac{\lambda[y \cos\left(\frac{y}{x}\right) + x]}{\lambda\left(x \cos\frac{y}{x}\right)} = \lambda^0 [F(x, y)]$$

Thus,  $F(x, y)$  is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \quad \dots (2)$$

Differentiating equation (2) with respect to  $x$ , we get

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (3)$$

Substituting the value of  $y$  and  $\frac{dy}{dx}$  in equation (1), we get

$$v + x \frac{dv}{dx} = \frac{v \cos v + 1}{\cos v}$$

or  $x \frac{dv}{dx} = \frac{v \cos v + 1}{\cos v} - v$

or  $x \frac{dv}{dx} = \frac{1}{\cos v}$

or  $\cos v \, dv = \frac{dx}{x}$

Therefore  $\int \cos v \, dv = \int \frac{1}{x} \, dx$

$$\begin{aligned} \text{or} \quad \sin v &= \log |x| + \log |C| \\ \text{or} \quad \sin v &= \log |Cx| \end{aligned}$$

Replacing  $v$  by  $\frac{y}{x}$ , we get

$$\sin\left(\frac{y}{x}\right) = \log |Cx|$$

which is the general solution of the differential equation (1).

**Example 12** Show that the differential equation  $2y e^y dx + \left(y - 2x e^y\right) dy = 0$  is homogeneous and find its particular solution, given that,  $x = 0$  when  $y = 1$ .

**Solution** The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x e^y - y}{2y e^y} \quad \dots (1)$$

Let

$$F(x, y) = \frac{2x e^y - y}{2y e^y}$$

Then

$$F(\lambda x, \lambda y) = \frac{\lambda \left(2x e^{\frac{x}{y}} - y\right)}{\lambda \left(2y e^{\frac{x}{y}}\right)} = \lambda^0 [F(x, y)]$$

Thus,  $F(x, y)$  is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \quad \dots (2)$$

Differentiating equation (2) with respect to  $y$ , we get

$$\frac{dx}{dy} = v + y \frac{dv}{dy}$$

Substituting the value of  $x$  and  $\frac{dx}{dy}$  in equation (1), we get

$$v + y \frac{dv}{dy} = \frac{2v e^v - 1}{2e^v}$$

or  $y \frac{dv}{dy} = \frac{2v e^v - 1}{2e^v} - v$

or  $y \frac{dv}{dy} = -\frac{1}{2e^v}$

or  $2e^v dv = \frac{-dy}{y}$

or  $\int 2e^v \cdot dv = -\int \frac{dy}{y}$

or  $2 e^v = -\log |y| + C$

and replacing  $v$  by  $\frac{x}{y}$ , we get

$$2 e^{\frac{x}{y}} + \log |y| = C \quad \dots (3)$$

Substituting  $x = 0$  and  $y = 1$  in equation (3), we get

$$2 e^0 + \log |1| = C \Rightarrow C = 2$$

Substituting the value of  $C$  in equation (3), we get

$$2 e^{\frac{x}{y}} + \log |y| = 2$$

which is the particular solution of the given differential equation.

**Example 13** Show that the family of curves for which the slope of the tangent at any point  $(x, y)$  on it is  $\frac{x^2 + y^2}{2xy}$ , is given by  $x^2 - y^2 = cx$ .

**Solution** We know that the slope of the tangent at any point on a curve is  $\frac{dy}{dx}$ .

Therefore,

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

or

$$\frac{dy}{dx} = \frac{\frac{1+y^2}{x^2}}{\frac{2y}{x}} \quad \dots (1)$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating  $y = vx$  with respect to  $x$ , we get

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

or

$$v + x \frac{dv}{dx} = \frac{1+v^2}{2v}$$

or

$$x \frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2} dv = \frac{dx}{x}$$

or

$$\frac{2v}{v^2-1} dv = -\frac{dx}{x}$$

Therefore

$$\int \frac{2v}{v^2-1} dv = -\int \frac{1}{x} dx$$

or

$$\log |v^2 - 1| = -\log |x| + \log |C_1|$$

or

$$\log |(v^2 - 1) (x)| = \log |C_1|$$

or

$$(v^2 - 1) x = \pm C_1$$

Replacing  $v$  by  $\frac{y}{x}$ , we get

$$\left( \frac{y^2}{x^2} - 1 \right) x = \pm C_1$$

or

$$(y^2 - x^2) = \pm C_1 x \text{ or } x^2 - y^2 = Cx$$

## EXERCISE 9.4

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1.  $(x^2 + xy) dy = (x^2 + y^2) dx$

2.  $y' = \frac{x+y}{x}$

3.  $(x-y) dy - (x+y) dx = 0$

4.  $(x^2 - y^2) dx + 2xy dy = 0$

5.  $x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$

6.  $x dy - y dx = \sqrt{x^2 + y^2} dx$

7.  $\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y dx = \left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x dy$

8.  $x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$

9.  $y dx + x \log\left(\frac{y}{x}\right) dy - 2x dy = 0$

10.  $\left( 1 + e^{\frac{x}{y}} \right) dx + e^{\frac{x}{y}} \left( 1 - \frac{x}{y} \right) dy = 0$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11.  $(x+y) dy + (x-y) dx = 0; y = 1 \text{ when } x = 1$

12.  $x^2 dy + (xy + y^2) dx = 0; y = 1 \text{ when } x = 1$

13.  $\left[ x \sin^2\left(\frac{y}{x}\right) - y \right] dx + x dy = 0; y = \frac{\pi}{4} \text{ when } x = 1$

14.  $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0; y = 0 \text{ when } x = 1$

15.  $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0; y = 2 \text{ when } x = 1$

16. A homogeneous differential equation of the form  $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$  can be solved by making the substitution.

- (A)  $y = vx$       (B)  $v = yx$       (C)  $x = vy$       (D)  $x = v$

17. Which of the following is a homogeneous differential equation?

- (A)  $(4x + 6y + 5) dy - (3y + 2x + 4) dx = 0$
- (B)  $(xy) dx - (x^3 + y^3) dy = 0$
- (C)  $(x^3 + 2y^2) dx + 2xy dy = 0$
- (D)  $y^2 dx + (x^2 - xy - y^2) dy = 0$

### 9.4.3 Linear differential equations

A differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + P_1 x = Q_1$$

where,  $P_1$  and  $Q_1$  are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} + Py = Q \quad \dots (1)$$

Multiply both sides of the equation by a function of x say  $g(x)$  to get

$$g(x) \frac{dy}{dx} + P \cdot (g(x)) y = Q \cdot g(x) \quad \dots (2)$$

Choose  $g(x)$  in such a way that R.H.S. becomes a derivative of  $y \cdot g(x)$ .

i.e. 
$$g(x) \frac{dy}{dx} + P \cdot g(x) y = \frac{d}{dx} [y \cdot g(x)]$$

or 
$$g(x) \frac{dy}{dx} + P \cdot g(x) y = g(x) \frac{dy}{dx} + y g'(x)$$

$$\Rightarrow P \cdot g(x) = g'(x)$$

or 
$$P = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to  $x$ , we get

$$\int P dx = \int \frac{g'(x)}{g(x)} dx$$

or 
$$\int P \cdot dx = \log(g(x))$$

or 
$$g(x) = e^{\int P dx}$$

On multiplying the equation (1) by  $g(x) = e^{\int P dx}$ , the L.H.S. becomes the derivative of some function of  $x$  and  $y$ . This function  $g(x) = e^{\int P dx}$  is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of  $g(x)$  in equation (2), we get

$$e^{\int P dx} \frac{dy}{dx} + P e^{\int P dx} y = Q \cdot e^{\int P dx}$$

or 
$$\frac{d}{dx} \left( y e^{\int P dx} \right) = Q e^{\int P dx}$$

Integrating both sides with respect to  $x$ , we get

$$y \cdot e^{\int P dx} = \int \left( Q e^{\int P dx} \right) dx$$

or 
$$y = e^{-\int P dx} \cdot \int \left( Q e^{\int P dx} \right) dx + C$$

which is the general solution of the differential equation.

**Steps involved to solve first order linear differential equation:**

- (i) Write the given differential equation in the form  $\frac{dy}{dx} + Py = Q$  where P, Q are constants or functions of x only.
- (ii) Find the Integrating Factor (I.F) =  $e^{\int P dx}$ .
- (iii) Write the solution of the given differential equation as

$$y \cdot (\text{I.F.}) = \int (Q \times \text{I.F.}) dx + C$$

In case, the first order linear differential equation is in the form  $\frac{dx}{dy} + P_1 x = Q_1$ ,

where,  $P_1$  and  $Q_1$  are constants or functions of y only. Then I.F =  $e^{\int P_1 dy}$  and the solution of the differential equation is given by

$$x \cdot (\text{I.F.}) = \int (Q_1 \times \text{I.F.}) dy + C$$

**Example 14** Find the general solution of the differential equation  $\frac{dy}{dx} - y = \cos x$ .

**Solution** Given differential equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -1 \text{ and } Q = \cos x$$

Therefore

$$\text{I.F.} = e^{\int -1 dx} = e^{-x}$$

Multiplying both sides of equation by I.F, we get

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} \cos x$$

or  $\frac{dy}{dx} (y e^{-x}) = e^{-x} \cos x$

On integrating both sides with respect to x, we get

$$y e^{-x} = \int e^{-x} \cos x dx + C \quad \dots (1)$$

Let

$$I = \int e^{-x} \cos x dx$$

$$= \cos x \left( \frac{e^{-x}}{-1} \right) - \int (-\sin x) (-e^{-x}) dx$$

$$\begin{aligned}
 &= -\cos x e^{-x} - \int \sin x e^{-x} dx \\
 &= -\cos x e^{-x} - \left[ \sin x (-e^{-x}) - \int \cos x (-e^{-x}) dx \right] \\
 &= -\cos x e^{-x} + \sin x e^{-x} - \int \cos x e^{-x} dx
 \end{aligned}$$

or

or

or

$$I = -e^{-x} \cos x + \sin x e^{-x} - I$$

$$2I = (\sin x - \cos x) e^{-x}$$

$$I = \frac{(\sin x - \cos x) e^{-x}}{2}$$

Substituting the value of  $I$  in equation (1), we get

$$ye^{-x} = \left( \frac{\sin x - \cos x}{2} \right) e^{-x} + C$$

or

$$y = \left( \frac{\sin x - \cos x}{2} \right) + Ce^x$$

which is the general solution of the given differential equation.

**Example 15** Find the general solution of the differential equation  $x \frac{dy}{dx} + 2y = x^2$  ( $x \neq 0$ ).

**Solution** The given differential equation is

$$x \frac{dy}{dx} + 2y = x^2 \quad \dots (1)$$

Dividing both sides of equation (1) by  $x$ , we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is a linear differential equation of the type  $\frac{dy}{dx} + Py = Q$ , where  $P = \frac{2}{x}$  and  $Q = x$ .

$$\text{So I.F.} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2 \quad [\text{as } e^{\log f(x)} = f(x)]$$

Therefore, solution of the given equation is given by

$$y \cdot x^2 = \int (x) (x^2) dx + C = \int x^3 dx + C$$

$$\text{or} \quad y = \frac{x^2}{4} + Cx^{-2}$$

which is the general solution of the given differential equation.

**Example 16** Find the general solution of the differential equation  $y \, dx - (x + 2y^2) \, dy = 0$ .

**Solution** The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a linear differential equation of the type  $\frac{dx}{dy} + P_1 x = Q_1$ , where  $P_1 = -\frac{1}{y}$  and

$$Q_1 = 2y. \text{ Therefore } I.F = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log(y)^{-1}} = \frac{1}{y}$$

Hence, the solution of the given differential equation is

$$x \frac{1}{y} = \int (2y) \left( \frac{1}{y} \right) dy + C$$

$$\text{or } \frac{x}{y} = \int (2dy) + C$$

$$\text{or } \frac{x}{y} = 2y + C$$

$$\text{or } x = 2y^2 + Cy$$

which is a general solution of the given differential equation.

**Example 17** Find the particular solution of the differential equation

$$\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x \quad (x \neq 0)$$

given that  $y = 0$  when  $x = \frac{\pi}{2}$ .

**Solution** The given equation is a linear differential equation of the type  $\frac{dy}{dx} + Py = Q$ ,

where  $P = \cot x$  and  $Q = 2x + x^2 \cot x$ . Therefore

$$I.F = e^{\int \cot x \, dx} = e^{\log \sin x} = \sin x$$

Hence, the solution of the differential equation is given by

$$y \cdot \sin x = \int (2x + x^2 \cot x) \sin x \, dx + C$$

or  $y \sin x = \int 2x \sin x \, dx + \int x^2 \cos x \, dx + C$

or  $y \sin x = \sin x \left( \frac{2x^2}{2} \right) - \int \cos x \left( \frac{2x^2}{2} \right) dx + \int x^2 \cos x \, dx + C$

or  $y \sin x = x^2 \sin x - \int x^2 \cos x \, dx + \int x^2 \cos x \, dx + C$

or  $y \sin x = x^2 \sin x + C \quad \dots (1)$

Substituting  $y = 0$  and  $x = \frac{\pi}{2}$  in equation (1), we get

$$0 = \left( \frac{\pi}{2} \right)^2 \sin \left( \frac{\pi}{2} \right) + C$$

or  $C = \frac{-\pi^2}{4}$

Substituting the value of  $C$  in equation (1), we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

or  $y = x^2 - \frac{\pi^2}{4 \sin x} \quad (\sin x \neq 0)$

which is the particular solution of the given differential equation.

**Example 18** Find the equation of a curve passing through the point  $(0, 1)$ . If the slope of the tangent to the curve at any point  $(x, y)$  is equal to the sum of the  $x$  coordinate (abscissa) and the product of the  $x$  coordinate and  $y$  coordinate (ordinate) of that point.

**Solution** We know that the slope of the tangent to the curve is  $\frac{dy}{dx}$ .

Therefore,  $\frac{dy}{dx} = x + xy$

or  $\frac{dy}{dx} - xy = x \quad \dots (1)$

This is a linear differential equation of the type  $\frac{dy}{dx} + Py = Q$ , where  $P = -x$  and  $Q = x$ .

Therefore,

$$I.F = e^{\int -x \, dx} = e^{\frac{-x^2}{2}}$$

Hence, the solution of equation is given by

$$y \cdot e^{\frac{-x^2}{2}} = \int (x) \left( e^{\frac{-x^2}{2}} \right) dx + C \quad \dots (2)$$

Let

$$I = \int (x) e^{\frac{-x^2}{2}} dx$$

Let  $\frac{-x^2}{2} = t$ , then  $-x dx = dt$  or  $x dx = -dt$ .

$$\text{Therefore, } I = - \int e^t dt = -e^t = -e^{\frac{-x^2}{2}}$$

Substituting the value of I in equation (2), we get

$$\begin{aligned} y e^{\frac{-x^2}{2}} &= -e^{\frac{-x^2}{2}} + C \\ y &= -1 + C e^{\frac{x^2}{2}} \end{aligned} \quad \dots (3)$$

Now (3) represents the equation of family of curves. But we are interested in finding a particular member of the family passing through  $(0, 1)$ . Substituting  $x = 0$  and  $y = 1$  in equation (3) we get

$$1 = -1 + C \cdot e^0 \quad \text{or} \quad C = 2$$

Substituting the value of C in equation (3), we get

$$y = -1 + 2 e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

### EXERCISE 9.5

For each of the differential equations given in Exercises 1 to 12, find the general solution:

$$1. \frac{dy}{dx} + 2y = \sin x \quad 2. \frac{dy}{dx} + 3y = e^{-2x} \quad 3. \frac{dy}{dx} + \frac{y}{x} = x^2$$

$$4. \frac{dy}{dx} + (\sec x)y = \tan x \left( 0 \leq x < \frac{\pi}{2} \right) \quad 5. \cos^2 x \frac{dy}{dx} + y = \tan x \left( 0 \leq x < \frac{\pi}{2} \right)$$

$$6. x \frac{dy}{dx} + 2y = x^2 \log x \quad 7. x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$$

$$8. (1 + x^2) dy + 2xy dx = \cot x dx (x \neq 0)$$

9.  $x \frac{dy}{dx} + y - x + xy \cot x = 0 \quad (x \neq 0)$  10.  $(x+y) \frac{dy}{dx} = 1$
11.  $y \, dx + (x - y^2) \, dy = 0$  12.  $(x+3y^2) \frac{dy}{dx} = y \quad (y > 0)$ .

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13.  $\frac{dy}{dx} + 2y \tan x = \sin x; y = 0 \text{ when } x = \frac{\pi}{3}$
14.  $(1+x^2) \frac{dy}{dx} + 2xy = \frac{1}{1+x^2}; y = 0 \text{ when } x = 1$
15.  $\frac{dy}{dx} - 3y \cot x = \sin 2x; y = 2 \text{ when } x = \frac{\pi}{2}$
16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point  $(x, y)$  is equal to the sum of the coordinates of the point.
17. Find the equation of a curve passing through the point  $(0, 2)$  given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.
18. The Integrating Factor of the differential equation  $x \frac{dy}{dx} - y = 2x^2$  is
- (A)  $e^{-x}$  (B)  $e^{-y}$  (C)  $\frac{1}{x}$  (D)  $x$
19. The Integrating Factor of the differential equation  $(1-y^2) \frac{dx}{dy} + yx = ay \quad (-1 < y < 1)$  is
- (A)  $\frac{1}{y^2-1}$  (B)  $\frac{1}{\sqrt{y^2-1}}$  (C)  $\frac{1}{1-y^2}$  (D)  $\frac{1}{\sqrt{1-y^2}}$

### Miscellaneous Examples

**Example 19** Verify that the function  $y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx$ , where  $c_1, c_2$  are arbitrary constants is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$$

**Solution** The given function is

$$y = e^{ax} [c_1 \cos bx + c_2 \sin bx] \quad \dots (1)$$

Differentiating both sides of equation (1) with respect to  $x$ , we get

$$\frac{dy}{dx} = e^{ax} [-bc_1 \sin bx + bc_2 \cos bx] + [c_1 \cos bx + c_2 \sin bx] e^{ax} \cdot a$$

or 
$$\frac{dy}{dx} = e^{ax} [(bc_2 + ac_1) \cos bx + (ac_2 - bc_1) \sin bx] \quad \dots (2)$$

Differentiating both sides of equation (2) with respect to  $x$ , we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= e^{ax} [(bc_2 + ac_1)(-b \sin bx) + (ac_2 - bc_1)(b \cos bx)] \\ &\quad + [(bc_2 + ac_1) \cos bx + (ac_2 - bc_1) \sin bx] e^{ax} \cdot a \\ &= e^{ax} [(a^2 c_2 - 2abc_1 - b^2 c_2) \sin bx + (a^2 c_1 + 2abc_2 - b^2 c_1) \cos bx] \end{aligned}$$

Substituting the values of  $\frac{d^2y}{dx^2}$ ,  $\frac{dy}{dx}$  and  $y$  in the given differential equation, we get

$$\begin{aligned} \text{L.H.S.} &= e^{ax} [a^2 c_2 - 2abc_1 - b^2 c_2] \sin bx + (a^2 c_1 + 2abc_2 - b^2 c_1) \cos bx \\ &\quad - 2ae^{ax} [(bc_2 + ac_1) \cos bx + (ac_2 - bc_1) \sin bx] \\ &\quad + (a^2 + b^2) e^{ax} [c_1 \cos bx + c_2 \sin bx] \\ &= e^{ax} \left[ (a^2 c_2 - 2abc_1 - b^2 c_2 - 2a^2 c_2 + 2abc_1 + a^2 c_2 + b^2 c_2) \sin bx \right. \\ &\quad \left. + (a^2 c_1 + 2abc_2 - b^2 c_1 - 2abc_2 - 2a^2 c_1 + a^2 c_1 + b^2 c_1) \cos bx \right] \\ &= e^{ax} [0 \times \sin bx + 0 \cos bx] = e^{ax} \times 0 = 0 = \text{R.H.S.} \end{aligned}$$

Hence, the given function is a solution of the given differential equation.

**Example 20** Find the particular solution of the differential equation  $\log\left(\frac{dy}{dx}\right) = 3x + 4y$  given that  $y = 0$  when  $x = 0$ .

**Solution** The given differential equation can be written as

$$\frac{dy}{dx} = e^{(3x + 4y)}$$

or 
$$\frac{dy}{dx} = e^{3x} \cdot e^{4y} \quad \dots (1)$$

Separating the variables, we get

$$\frac{dy}{e^{4y}} = e^{3x} dx$$

Therefore 
$$\int e^{-4y} dy = \int e^{3x} dx$$

or 
$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

or 
$$4e^{3x} + 3e^{-4y} + 12C = 0 \quad \dots (2)$$

Substituting  $x = 0$  and  $y = 0$  in (2), we get

$$4 + 3 + 12C = 0 \text{ or } C = \frac{-7}{12}$$

Substituting the value of  $C$  in equation (2), we get

$$4e^{3x} + 3e^{-4y} - 7 = 0,$$

which is a particular solution of the given differential equation.

**Example 21** Solve the differential equation

$$(x dy - y dx) y \sin\left(\frac{y}{x}\right) = (y dx + x dy) x \cos\left(\frac{y}{x}\right).$$

**Solution** The given differential equation can be written as

$$\left[ x y \sin\left(\frac{y}{x}\right) - x^2 \cos\left(\frac{y}{x}\right) \right] dy = \left[ x y \cos\left(\frac{y}{x}\right) + y^2 \sin\left(\frac{y}{x}\right) \right] dx$$

or 
$$\frac{dy}{dx} = \frac{x y \cos\left(\frac{y}{x}\right) + y^2 \sin\left(\frac{y}{x}\right)}{x y \sin\left(\frac{y}{x}\right) - x^2 \cos\left(\frac{y}{x}\right)}$$

Dividing numerator and denominator on RHS by  $x^2$ , we get

$$\frac{dy}{dx} = \frac{\frac{y}{x} \cos\left(\frac{y}{x}\right) + \left(\frac{y^2}{x^2}\right) \sin\left(\frac{y}{x}\right)}{\frac{y}{x} \sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)} \quad \dots (1)$$

Clearly, equation (1) is a homogeneous differential equation of the form  $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$ .

To solve it, we make the substitution

$$y = vx \quad \dots (2)$$

or  $\frac{dy}{dx} = v + x \frac{dv}{dx}$

or  $v + x \frac{dv}{dx} = \frac{v \cos v + v^2 \sin v}{v \sin v - \cos v} \quad \text{(using (1) and (2))}$

or  $x \frac{dv}{dx} = \frac{2v \cos v}{v \sin v - \cos v}$

or  $\left(\frac{v \sin v - \cos v}{v \cos v}\right) dv = \frac{2 dx}{x}$

Therefore  $\int \left(\frac{v \sin v - \cos v}{v \cos v}\right) dv = 2 \int \frac{1}{x} dx$

or  $\int \tan v dv - \int \frac{1}{v} dv = 2 \int \frac{1}{x} dx$

or  $\log |\sec v| - \log |v| = 2 \log |x| + \log |C_1|$

or  $\log \left| \frac{\sec v}{v x^2} \right| = \log |C_1|$

or  $\frac{\sec v}{v x^2} = \pm C_1 \quad \dots (3)$

Replacing  $v$  by  $\frac{y}{x}$  in equation (3), we get

$$\frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)(x^2)} = C \text{ where, } C = \pm C_1$$

or  $\sec\left(\frac{y}{x}\right) = C xy$

which is the general solution of the given differential equation.

**Example 22** Solve the differential equation

$$(\tan^{-1}y - x) dy = (1 + y^2) dx.$$

**Solution** The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \quad \dots (1)$$

Now (1) is a linear differential equation of the form  $\frac{dx}{dy} + P_1 x = Q_1$ ,

$$\text{where, } P_1 = \frac{1}{1+y^2} \text{ and } Q_1 = \frac{\tan^{-1}y}{1+y^2}.$$

$$\text{Therefore, } I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

Thus, the solution of the given differential equation is

$$x e^{\tan^{-1}y} = \int \left( \frac{\tan^{-1}y}{1+y^2} \right) e^{\tan^{-1}y} dy + C \quad \dots (2)$$

$$\text{Let } I = \int \left( \frac{\tan^{-1}y}{1+y^2} \right) e^{\tan^{-1}y} dy$$

Substituting  $\tan^{-1}y = t$  so that  $\left( \frac{1}{1+y^2} \right) dy = dt$ , we get

$$I = \int t e^t dt = t e^t - \int 1 \cdot e^t dt = t e^t - e^t = e^t (t - 1)$$

$$\text{or } I = e^{\tan^{-1}y} (\tan^{-1}y - 1)$$

Substituting the value of I in equation (2), we get

$$x \cdot e^{\tan^{-1}y} = e^{\tan^{-1}y} (\tan^{-1}y - 1) + C$$

$$\text{or } x = (\tan^{-1}y - 1) + C e^{-\tan^{-1}y}$$

which is the general solution of the given differential equation.

### Miscellaneous Exercise on Chapter 9

- For each of the differential equations given below, indicate its order and degree (if defined).

$$(i) \frac{d^2y}{dx^2} + 5x \left( \frac{dy}{dx} \right)^2 - 6y = \log x \quad (ii) \left( \frac{dy}{dx} \right)^3 - 4 \left( \frac{dy}{dx} \right)^2 + 7y = \sin x$$

$$(iii) \frac{d^4y}{dx^4} - \sin \left( \frac{d^3y}{dx^3} \right) = 0$$

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

$$(i) xy = a e^x + b e^{-x} + x^2 \quad : \quad x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy + x^2 - 2 = 0$$

$$(ii) y = e^x (a \cos x + b \sin x) \quad : \quad \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

$$(iii) y = x \sin 3x \quad : \quad \frac{d^2y}{dx^2} + 9y - 6 \cos 3x = 0$$

$$(iv) x^2 = 2y^2 \log y \quad : \quad (x^2 + y^2) \frac{dy}{dx} - xy = 0$$

3. Prove that  $x^2 - y^2 = c (x^2 + y^2)^2$  is the general solution of differential equation  $(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy$ , where  $c$  is a parameter.

$$4. \text{ Find the general solution of the differential equation } \frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0.$$

5. Show that the general solution of the differential equation  $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$  is given by  $(x + y + 1) = A (1 - x - y - 2xy)$ , where  $A$  is parameter.

6. Find the equation of the curve passing through the point  $\left( 0, \frac{\pi}{4} \right)$  whose differential equation is  $\sin x \cos y dx + \cos x \sin y dy = 0$ .

7. Find the particular solution of the differential equation  $(1 + e^{2x}) dy + (1 + y^2) e^x dx = 0$ , given that  $y = 1$  when  $x = 0$ .

8. Solve the differential equation  $y e^{\frac{x}{y}} dx = \left( x e^{\frac{x}{y}} + y^2 \right) dy (y \neq 0)$ .

9. Find a particular solution of the differential equation  $(x - y)(dx + dy) = dx - dy$ , given that  $y = -1$ , when  $x = 0$ . (Hint: put  $x - y = t$ )
10. Solve the differential equation  $\left[ \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \right] \frac{dx}{dy} = 1 \ (x \neq 0)$ .
11. Find a particular solution of the differential equation  $\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$  ( $x \neq 0$ ), given that  $y = 0$  when  $x = \frac{\pi}{2}$ .
12. Find a particular solution of the differential equation  $(x + 1) \frac{dy}{dx} = 2e^{-y} - 1$ , given that  $y = 0$  when  $x = 0$ .
13. The general solution of the differential equation  $\frac{y \, dx - x \, dy}{y} = 0$  is  
 (A)  $xy = C$       (B)  $x = Cy^2$       (C)  $y = Cx$       (D)  $y = Cx^2$
14. The general solution of a differential equation of the type  $\frac{dx}{dy} + P_1 x = Q_1$  is  
 (A)  $y e^{\int P_1 \, dy} = \int (Q_1 e^{\int P_1 \, dy}) \, dy + C$   
 (B)  $y \cdot e^{\int P_1 \, dx} = \int (Q_1 e^{\int P_1 \, dx}) \, dx + C$   
 (C)  $x e^{\int P_1 \, dy} = \int (Q_1 e^{\int P_1 \, dy}) \, dy + C$   
 (D)  $x e^{\int P_1 \, dx} = \int (Q_1 e^{\int P_1 \, dx}) \, dx + C$
15. The general solution of the differential equation  $e^x \, dy + (y e^x + 2x) \, dx = 0$  is  
 (A)  $x e^y + x^2 = C$       (B)  $x e^y + y^2 = C$   
 (C)  $y e^x + x^2 = C$       (D)  $y e^y + x^2 = C$

### Summary

- ◆ An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.
- ◆ Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
- ◆ Degree of a differential equation is defined if it is a polynomial equation in its derivatives.
- ◆ Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- ◆ A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.
- ◆ Variable separable method is used to solve such an equation in which variables can be separated completely i.e. terms containing  $y$  should remain with  $dy$  and terms containing  $x$  should remain with  $dx$ .
- ◆ A differential equation which can be expressed in the form  $\frac{dy}{dx} = f(x, y)$  or  $\frac{dx}{dy} = g(x, y)$  where,  $f(x, y)$  and  $g(x, y)$  are homogenous functions of degree zero is called a homogeneous differential equation.
- ◆ A differential equation of the form  $\frac{dy}{dx} + Py = Q$ , where  $P$  and  $Q$  are constants or functions of  $x$  only is called a first order linear differential equation.

### Historical Note

One of the principal languages of Science is that of differential equations. Interestingly, the date of birth of differential equations is taken to be November, 11, 1675, when Gottfried Wilhelm Freiherr Leibnitz (1646 - 1716) first put in black and white the identity  $\int y \, dy = \frac{1}{2} y^2$ , thereby introducing both the symbols  $\int$  and  $dy$ .

Leibnitz was actually interested in the problem of finding a curve whose tangents were prescribed. This led him to discover the '*method of separation of variables*' 1691. A year later he formulated the '*method of solving the homogeneous*

*differential equations of the first order*'. He went further in a very short time to the discovery of the '*method of solving a linear differential equation of the first-order*'. How surprising is it that all these methods came from a single man and that too within 25 years of the birth of differential equations!

In the old days, what we now call the 'solution' of a differential equation, was used to be referred to as 'integral' of the differential equation, the word being coined by James Bernoulli (1654 - 1705) in 1690. The word 'solution' was first used by Joseph Louis Lagrange (1736 - 1813) in 1774, which was almost hundred years since the birth of differential equations. It was Jules Henri Poincare (1854 - 1912) who strongly advocated the use of the word 'solution' and thus the word 'solution' has found its deserved place in modern terminology. The name of the '*method of separation of variables*' is due to John Bernoulli (1667 - 1748), a younger brother of James Bernoulli.

Application to geometric problems were also considered. It was again John Bernoulli who first brought into light the intricate nature of differential equations. In a letter to Leibnitz, dated May 20, 1715, he revealed the solutions of the differential equation

$$x^2 y'' = 2y,$$

which led to three types of curves, viz., parabolas, hyperbolas and a class of cubic curves. This shows how varied the solutions of such innocent looking differential equation can be. From the second half of the twentieth century attention has been drawn to the investigation of this complicated nature of the solutions of differential equations, under the heading '*qualitative analysis of differential equations*'. Now-a-days, this has acquired prime importance being absolutely necessary in almost all investigations.



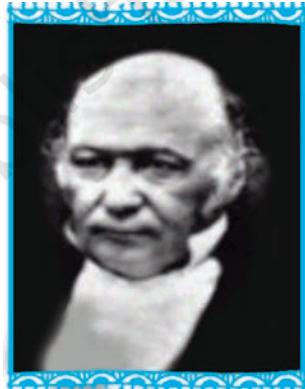


# VECTOR ALGEBRA

❖ *In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure. – HERMAN HANKEL* ❖

## 10.1 Introduction

In our day to day life, we come across many queries such as – What is your height? How should a football player hit the ball to give a pass to another player of his team? Observe that a possible answer to the first query may be 1.6 meters, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors*. In mathematics, physics and engineering, we frequently come across with both types of quantities, namely, scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc.



W.R. Hamilton  
(1805-1865)

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors, and their algebraic and geometric properties. These two type of properties, when considered together give a full realisation to the concept of vectors, and lead to their vital applicability in various areas as mentioned above.

## 10.2 Some Basic Concepts

Let ‘*l*’ be any straight line in plane or three dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a *directed line* (Fig 10.1 (i), (ii)).

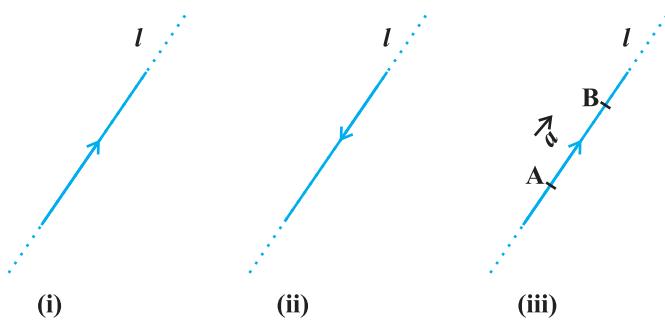


Fig 10.1

Now observe that if we restrict the line  $l$  to the line segment  $AB$ , then a magnitude is prescribed on the line  $l$  with one of the two directions, so that we obtain a *directed line segment* (Fig 10.1(iii)). Thus, a directed line segment has magnitude as well as direction.

**Definition 1** A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig 10.1(iii)), denoted as  $\overrightarrow{AB}$  or simply as  $\vec{a}$ , and read as ‘vector  $\overrightarrow{AB}$ ’ or ‘vector  $\vec{a}$ ’.

The point  $A$  from where the vector  $\overrightarrow{AB}$  starts is called its *initial point*, and the point  $B$  where it ends is called its *terminal point*. The distance between initial and terminal points of a vector is called the *magnitude* (or length) of the vector, denoted as  $|\overrightarrow{AB}|$ , or  $|\vec{a}|$ , or  $a$ . The arrow indicates the direction of the vector.



**Note** Since the length is never negative, the notation  $|\vec{a}| < 0$  has no meaning.

### Position Vector

From Class XI, recall the three dimensional right handed rectangular coordinate system (Fig 10.2(i)). Consider a point  $P$  in space, having coordinates  $(x, y, z)$  with respect to the origin  $O(0, 0, 0)$ . Then, the vector  $\overrightarrow{OP}$  having  $O$  and  $P$  as its initial and terminal points, respectively, is called the *position vector* of the point  $P$  with respect to  $O$ . Using distance formula (from Class XI), the magnitude of  $\overrightarrow{OP}$  (or  $\vec{r}$ ) is given by

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

In practice, the position vectors of points  $A, B, C, \dots$ , with respect to the origin  $O$  are denoted by  $\vec{a}, \vec{b}, \vec{c}, \dots$ , respectively (Fig 10.2 (ii)).

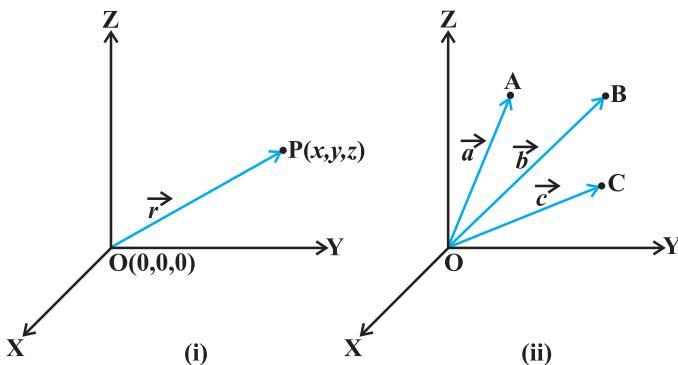


Fig 10.2

### Direction Cosines

Consider the position vector  $\overline{OP}$  (or  $\vec{r}$ ) of a point  $P(x, y, z)$  as in Fig 10.3. The angles  $\alpha, \beta, \gamma$  made by the vector  $\vec{r}$  with the positive directions of  $x$ ,  $y$  and  $z$ -axes respectively, are called its *direction angles*. The cosine values of these angles, i.e.,  $\cos \alpha, \cos \beta$  and  $\cos \gamma$  are called *direction cosines* of the vector  $\vec{r}$ , and usually denoted by  $l, m$  and  $n$ , respectively.

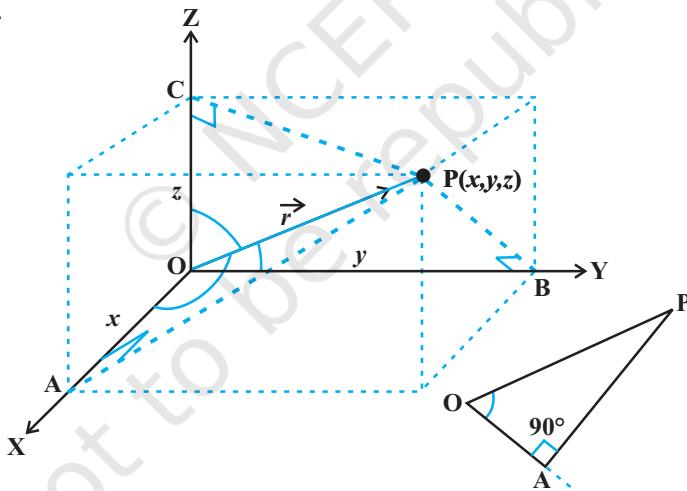


Fig 10.3

From Fig 10.3, one may note that the triangle OAP is right angled, and in it, we have  $\cos \alpha = \frac{x}{r}$  ( $r$  stands for  $|\vec{r}|$ ). Similarly, from the right angled triangles OBP and OCP, we may write  $\cos \beta = \frac{y}{r}$  and  $\cos \gamma = \frac{z}{r}$ . Thus, the coordinates of the point P may also be expressed as  $(lr, mr, nr)$ . The numbers  $lr, mr$  and  $nr$ , proportional to the direction cosines are called as *direction ratios* of vector  $\vec{r}$ , and denoted as  $a, b$  and  $c$ , respectively.



**Note** One may note that  $l^2 + m^2 + n^2 = 1$  but  $a^2 + b^2 + c^2 \neq 1$ , in general.

### 10.3 Types of Vectors

**Zero Vector** A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as  $\vec{0}$ . Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors  $\overrightarrow{AA}$ ,  $\overrightarrow{BB}$  represent the zero vector,

**Unit Vector** A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector  $\vec{a}$  is denoted by  $\hat{a}$ .

**Coinitial Vectors** Two or more vectors having the same initial point are called coinitial vectors.

**Collinear Vectors** Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

**Equal Vectors** Two vectors  $\vec{a}$  and  $\vec{b}$  are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as  $\vec{a} = \vec{b}$ .

**Negative of a Vector** A vector whose magnitude is the same as that of a given vector (say,  $\overrightarrow{AB}$ ), but direction is opposite to that of it, is called *negative* of the given vector. For example, vector  $\overrightarrow{BA}$  is negative of the vector  $\overrightarrow{AB}$ , and written as  $\overrightarrow{BA} = -\overrightarrow{AB}$ .

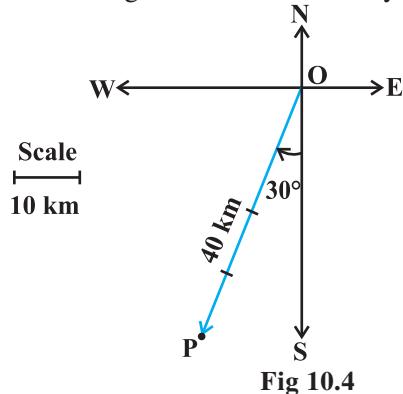
**Remark** The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called *free vectors*. Throughout this chapter, we will be dealing with free vectors only.

**Example 1** Represent graphically a displacement of 40 km,  $30^\circ$  west of south.

**Solution** The vector  $\overrightarrow{OP}$  represents the required displacement (Fig 10.4).

**Example 2** Classify the following measures as scalars and vectors.

- 5 seconds
- 1000 cm<sup>3</sup>



## Solution

- (i) Time-scalar      (ii) Volume-scalar      (iii) Force-vector  
 (iv) Speed-scalar      (v) Density-scalar      (vi) Velocity-vector

**Example 3** In Fig 10.5, which of the vectors are:



## Solution

- (i) Collinear vectors :  $\vec{a}$ ,  $\vec{c}$  and  $\vec{d}$ .
  - (ii) Equal vectors :  $\vec{a}$  and  $\vec{c}$ .
  - (iii) Coinitial vectors :  $\vec{b}$ ,  $\vec{c}$  and  $\vec{d}$ .

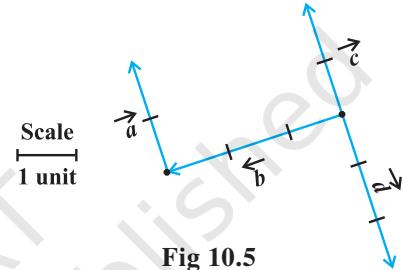


Fig 10.5

## EXERCISE 10.1

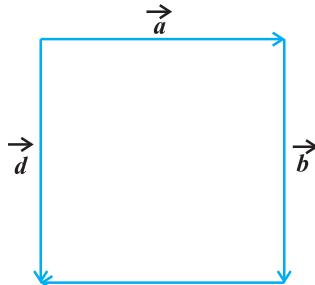


Fig 10.6

## 10.4 Addition of Vectors

A vector  $\overrightarrow{AB}$  simply means the displacement from a point A to the point B. Now consider a situation that a girl moves from A to B and then from B to C (Fig 10.7). The net displacement made by the girl from point A to the point C, is given by the vector  $\overrightarrow{AC}$  and expressed as

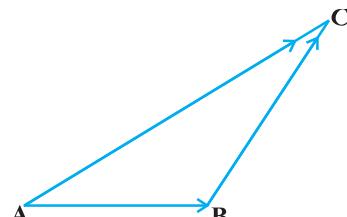


Fig 10.7

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

This is known as the *triangle law of vector addition*.

In general, if we have two vectors  $\vec{a}$  and  $\vec{b}$  (Fig 10.8 (i)), then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other (Fig 10.8(ii)).

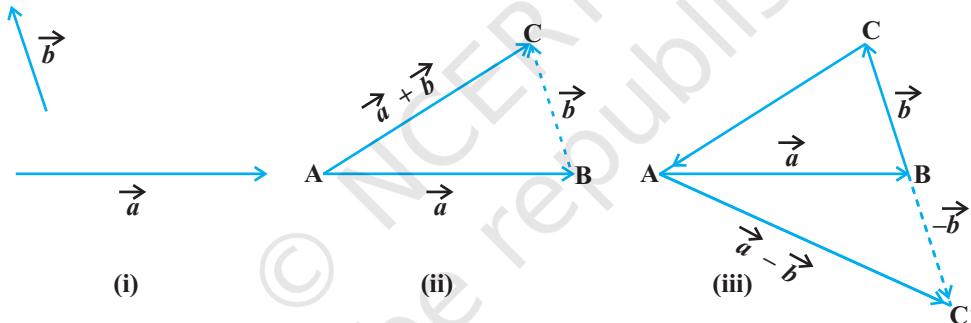


Fig 10.8

For example, in Fig 10.8 (ii), we have shifted vector  $\vec{b}$  without changing its magnitude and direction, so that it's initial point coincides with the terminal point of  $\vec{a}$ . Then, the vector  $\vec{a} + \vec{b}$ , represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors  $\vec{a}$  and  $\vec{b}$  i.e., in triangle ABC (Fig 10.8 (ii)), we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Now again, since  $\overrightarrow{AC} = -\overrightarrow{CA}$ , from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig 10.8(iii)).

Now, construct a vector  $\overrightarrow{BC'}$  so that its magnitude is same as the vector  $\overrightarrow{BC}$ , but the direction opposite to that of it (Fig 10.8 (iii)), i.e.,

$$\overrightarrow{BC'} = -\overrightarrow{BC}$$

Then, on applying triangle law from the Fig 10.8 (iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

The vector  $\overrightarrow{AC'}$  is said to represent the *difference of  $\vec{a}$  and  $\vec{b}$* .

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors—one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat in actual starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

If we have two vectors  $\vec{a}$  and  $\vec{b}$  represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig 10.9), then their sum  $\vec{a} + \vec{b}$  is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the *parallelogram law of vector addition*.

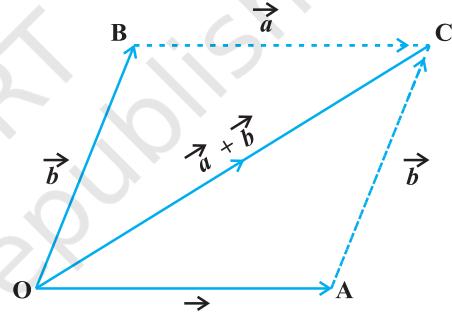


Fig 10.9

**Note** From Fig 10.9, using the triangle law, one may note that

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

or

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

(since  $\overrightarrow{AC} = \overrightarrow{OB}$ )

which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

### Properties of vector addition

**Property 1** For any two vectors  $\vec{a}$  and  $\vec{b}$ ,

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(Commutative property)

**Proof** Consider the parallelogram ABCD (Fig 10.10). Let  $\overrightarrow{AB} = \vec{a}$  and  $\overrightarrow{BC} = \vec{b}$ , then using the triangle law, from triangle ABC, we have

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

Now, since the opposite sides of a parallelogram are equal and parallel, from Fig 10.10, we have,  $\overrightarrow{AD} = \overrightarrow{BC} = \vec{b}$  and  $\overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$ . Again using triangle law, from triangle ADC, we have

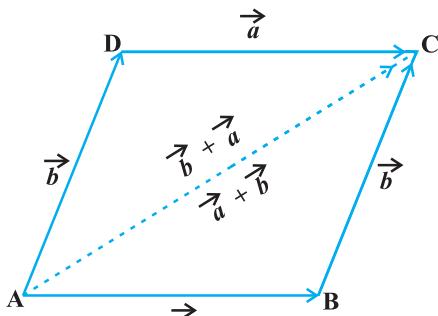


Fig 10.10

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$$

$$\text{Hence } \vec{a} + \vec{b} = \vec{b} + \vec{a}$$

**Property 2** For any three vectors  $a, b$  and  $c$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \quad (\text{Associative property})$$

**Proof** Let the vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$  be represented by  $\overrightarrow{PQ}$ ,  $\overrightarrow{QR}$  and  $\overrightarrow{RS}$ , respectively, as shown in Fig 10.11(i) and (ii).

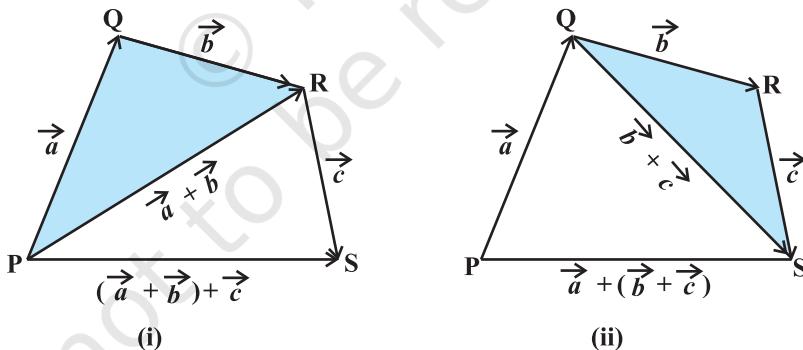


Fig 10.11

Then

$$\vec{a} + \vec{b} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

and

$$\vec{b} + \vec{c} = \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$$

So

$$(\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$$

and

$$\vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$$

Hence

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

**Remark** The associative property of vector addition enables us to write the sum of three vectors  $\vec{a}, \vec{b}, \vec{c}$  as  $\vec{a} + \vec{b} + \vec{c}$  without using brackets.

Note that for any vector  $\vec{a}$ , we have

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

Here, the zero vector  $\vec{0}$  is called the *additive identity* for the vector addition.

## 10.5 Multiplication of a Vector by a Scalar

Let  $\vec{a}$  be a given vector and  $\lambda$  a scalar. Then the product of the vector  $\vec{a}$  by the scalar  $\lambda$ , denoted as  $\lambda\vec{a}$ , is called the multiplication of vector  $\vec{a}$  by the scalar  $\lambda$ . Note that,  $\lambda\vec{a}$  is also a vector, collinear to the vector  $\vec{a}$ . The vector  $\lambda\vec{a}$  has the direction same (or opposite) to that of vector  $\vec{a}$  according as the value of  $\lambda$  is positive (or negative). Also, the magnitude of vector  $\lambda\vec{a}$  is  $|\lambda|$  times the magnitude of the vector  $\vec{a}$ , i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualisation of multiplication of a vector by a scalar is given in Fig 10.12.

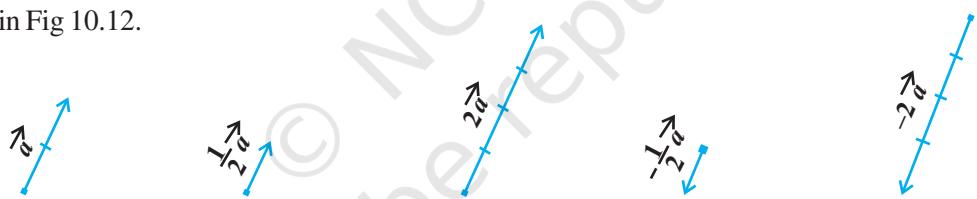


Fig 10.12

When  $\lambda = -1$ , then  $\lambda\vec{a} = -\vec{a}$ , which is a vector having magnitude equal to the magnitude of  $\vec{a}$  and direction opposite to that of the direction of  $\vec{a}$ . The vector  $-\vec{a}$  is called the *negative* (or *additive inverse*) of vector  $\vec{a}$  and we always have

$$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$$

Also, if  $\lambda = \frac{1}{|\vec{a}|}$ , provided  $\vec{a} \neq 0$  i.e.  $\vec{a}$  is not a null vector, then

$$|\lambda\vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

So,  $\lambda \vec{a}$  represents the unit vector in the direction of  $\vec{a}$ . We write it as

$$\hat{a} = \frac{1}{|\vec{a}|} \vec{a}$$

**Note** For any scalar  $k$ ,  $k\vec{0} = \vec{0}$ .

### 10.5.1 Components of a vector

Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the  $x$ -axis,  $y$ -axis and  $z$ -axis, respectively. Then, clearly

$$|\overrightarrow{OA}| = 1, |\overrightarrow{OB}| = 1 \text{ and } |\overrightarrow{OC}| = 1$$

The vectors  $\overrightarrow{OA}$ ,  $\overrightarrow{OB}$  and  $\overrightarrow{OC}$ , each having magnitude 1, are called *unit vectors along the axes OX, OY and OZ*, respectively, and denoted by  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ , respectively (Fig 10.13).

Now, consider the position vector  $\overrightarrow{OP}$  of a point  $P(x, y, z)$  as in Fig 10.14. Let  $P_1$  be the foot of the perpendicular from  $P$  on the plane XOY.

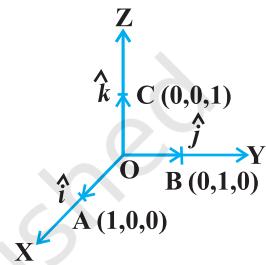


Fig 10.13

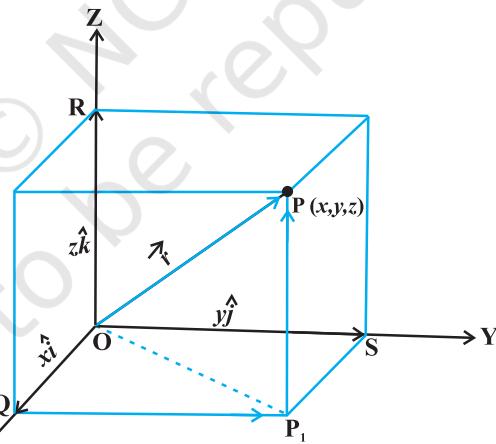


Fig 10.14

We, thus, see that  $P_1 P$  is parallel to  $z$ -axis. As  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$  are the unit vectors along the  $x$ ,  $y$  and  $z$ -axes, respectively, and by the definition of the coordinates of  $P$ , we have  $\overrightarrow{P_1 P} = \overrightarrow{OR} = z\hat{k}$ . Similarly,  $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$  and  $\overrightarrow{OQ} = x\hat{i}$ .

Therefore, it follows that

$$\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$$

and

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$$

Hence, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its *component form*. Here, x, y and z are called as the *scalar components* of  $\vec{r}$ , and  $x\hat{i}$ ,  $y\hat{j}$  and  $z\hat{k}$  are called the *vector components* of  $\vec{r}$  along the respective axes. Sometimes x, y and z are also termed as *rectangular components*.

The length of any vector  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , is readily determined by applying the Pythagoras theorem twice. We note that in the right angle triangle  $OQP_1$  (Fig 10.14)

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2},$$

and in the right angle triangle  $OP_1P$ , we have

$$|\overrightarrow{OP}| = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

If  $\vec{a}$  and  $\vec{b}$  are any two vectors given in the component form  $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ , respectively, then

(i) the sum (or resultant) of the vectors  $\vec{a}$  and  $\vec{b}$  is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) the difference of the vector  $\vec{a}$  and  $\vec{b}$  is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) the vectors  $\vec{a}$  and  $\vec{b}$  are equal if and only if

$$a_1 = b_1, a_2 = b_2 \text{ and } a_3 = b_3$$

(iv) the multiplication of vector  $\vec{a}$  by any scalar  $\lambda$  is given by

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let  $\vec{a}$  and  $\vec{b}$  be any two vectors, and  $k$  and  $m$  be any scalars. Then

- (i)  $k\vec{a} + m\vec{a} = (k + m)\vec{a}$
- (ii)  $k(m\vec{a}) = (km)\vec{a}$
- (iii)  $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

### Remarks

- (i) One may observe that whatever be the value of  $\lambda$ , the vector  $\lambda\vec{a}$  is always collinear to the vector  $\vec{a}$ . In fact, two vectors  $\vec{a}$  and  $\vec{b}$  are collinear if and only if there exists a nonzero scalar  $\lambda$  such that  $\vec{b} = \lambda\vec{a}$ . If the vectors  $\vec{a}$  and  $\vec{b}$  are given in the component form, i.e.  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ , then the two vectors are collinear if and only if

$$\begin{aligned} b_1\hat{i} + b_2\hat{j} + b_3\hat{k} &= \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \\ \Leftrightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} &= (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k} \\ \Leftrightarrow b_1 &= \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3 \\ \Leftrightarrow \frac{b_1}{a_1} &= \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda \end{aligned}$$

- (ii) If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ , then  $a_1, a_2, a_3$  are also called direction ratios of  $\vec{a}$ .
- (iii) In case if it is given that  $l, m, n$  are direction cosines of a vector, then  $l\hat{i} + m\hat{j} + n\hat{k} = (\cos \alpha)\hat{i} + (\cos \beta)\hat{j} + (\cos \gamma)\hat{k}$  is the unit vector in the direction of that vector, where  $\alpha, \beta$  and  $\gamma$  are the angles which the vector makes with  $x, y$  and  $z$  axes respectively.

**Example 4** Find the values of  $x, y$  and  $z$  so that the vectors  $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$  and  $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$  are equal.

**Solution** Note that two vectors are equal if and only if their corresponding components are equal. Thus, the given vectors  $\vec{a}$  and  $\vec{b}$  will be equal if and only if

$$x = 2, y = 2, z = 1$$

**Example 5** Let  $\vec{a} = \hat{i} + 2\hat{j}$  and  $\vec{b} = 2\hat{i} + \hat{j}$ . Is  $|\vec{a}| = |\vec{b}|$ ? Are the vectors  $\vec{a}$  and  $\vec{b}$  equal?

**Solution** We have  $|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$  and  $|\vec{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

So,  $|\vec{a}| = |\vec{b}|$ . But, the two vectors are not equal since their corresponding components are distinct.

**Example 6** Find unit vector in the direction of vector  $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

**Solution** The unit vector in the direction of a vector  $\vec{a}$  is given by  $\hat{a} = \frac{1}{|\vec{a}|} \vec{a}$ .

Now

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

Therefore  $\hat{a} = \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k}) = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$

**Example 7** Find a vector in the direction of vector  $\vec{a} = \hat{i} - 2\hat{j}$  that has magnitude 7 units.

**Solution** The unit vector in the direction of the given vector  $\vec{a}$  is

$$\hat{a} = \frac{1}{|\vec{a}|} \vec{a} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j}) = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of  $\vec{a}$  is

$$7\hat{a} = 7\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

**Example 8** Find the unit vector in the direction of the sum of the vectors,  $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$  and  $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$ .

**Solution** The sum of the given vectors is

$$\vec{a} + \vec{b} (= \vec{c}, \text{ say}) = 4\hat{i} + 3\hat{j} - 2\hat{k}$$

and

$$|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

Thus, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|} \vec{c} = \frac{1}{\sqrt{29}} (4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}} \hat{i} + \frac{3}{\sqrt{29}} \hat{j} - \frac{2}{\sqrt{29}} \hat{k}$$

**Example 9** Write the direction ratio's of the vector  $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$  and hence calculate its direction cosines.

**Solution** Note that the direction ratio's  $a, b, c$  of a vector  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  are just the respective components  $x, y$  and  $z$  of the vector. So, for the given vector, we have  $a = 1, b = 1$  and  $c = -2$ . Further, if  $l, m$  and  $n$  are the direction cosines of the given vector, then

$$l = \frac{a}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad m = \frac{b}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad n = \frac{c}{|\vec{r}|} = \frac{-2}{\sqrt{6}} \quad \text{as } |\vec{r}| = \sqrt{6}$$

Thus, the direction cosines are  $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ .

### 10.5.2 Vector joining two points

If  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  are any two points, then the vector joining  $P_1$  and  $P_2$  is the vector  $\overrightarrow{P_1P_2}$  (Fig 10.15).

Joining the points  $P_1$  and  $P_2$  with the origin  $O$ , and applying triangle law, from the triangle  $OP_1P_2$ , we have

$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$$

Using the properties of vector addition, the above equation becomes

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$\begin{aligned} \text{i.e. } \overrightarrow{P_1P_2} &= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\ &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \end{aligned}$$

The magnitude of vector  $\overrightarrow{P_1P_2}$  is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

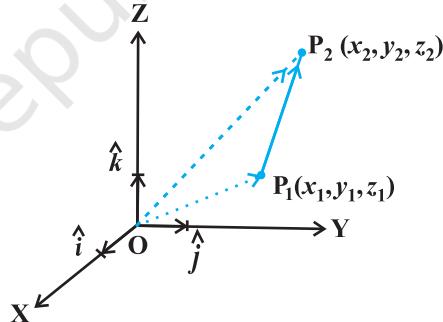


Fig 10.15

**Example 10** Find the vector joining the points  $P(2, 3, 0)$  and  $Q(-1, -2, -4)$  directed from  $P$  to  $Q$ .

**Solution** Since the vector is to be directed from P to Q, clearly P is the initial point and Q is the terminal point. So, the required vector joining P and Q is the vector  $\overrightarrow{PQ}$ , given by

$$\overrightarrow{PQ} = (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{k}$$

i.e.

$$\overrightarrow{PQ} = -3\hat{i} - 5\hat{j} - 4\hat{k}.$$

### 10.5.3 *Section formula*

Let P and Q be two points represented by the position vectors  $\overrightarrow{OP}$  and  $\overrightarrow{OQ}$ , respectively, with respect to the origin O. Then the line segment

joining the points P and Q may be divided by a third point, say R, in two ways – internally (Fig 10.16) and externally (Fig 10.17). Here, we intend to find the position vector  $\overrightarrow{OR}$  for the point R with respect to the origin O. We take the two cases one by one.

### Case I When R divides PQ internally (Fig 10.16).

If R divides  $\overrightarrow{PQ}$  such that  $m \overrightarrow{RQ} = n \overrightarrow{PR}$ ,

where  $m$  and  $n$  are positive scalars, we say that the point  $R$  divides  $\overline{PQ}$  internally in the ratio of  $m : n$ . Now from triangles  $ORQ$  and  $OPR$ , we have

$$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR} = \vec{b} - \vec{r}$$

and

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \vec{r} - \vec{a}$$

Therefore, we have

$$m(\vec{b} - \vec{r}) = n(\vec{r} - \vec{a}) \quad (\text{Why?})$$

$$\vec{r} = \frac{mb + na}{(on \text{ simplification})}$$

$$\vec{r} = \frac{mb + na}{m + n} \quad (\text{on simplification})$$

Hence, the position ratio of  $m : n$  is given by

$$\overrightarrow{\text{OR}} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

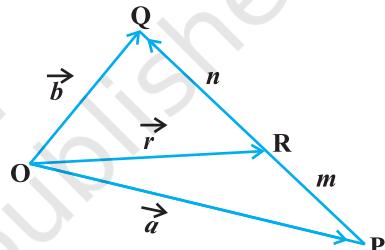


Fig 10.16

**Case II** When R divides PQ externally (Fig 10.17). We leave it to the reader as an exercise to verify that the position vector of the point R which divides the line segment PQ externally in the ratio

$m : n$  i.e.  $\frac{\overline{PR}}{\overline{QR}} = \frac{m}{n}$  is given by

$$\overline{OR} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

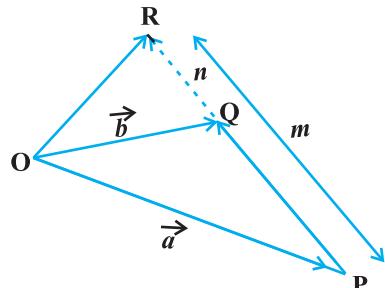


Fig 10.17

**Remark** If R is the midpoint of PQ, then  $m = n$ . And therefore, from Case I, the midpoint R of  $\overline{PQ}$ , will have its position vector as

$$\overline{OR} = \frac{\vec{a} + \vec{b}}{2}$$

**Example 11** Consider two points P and Q with position vectors  $\overline{OP} = 3\vec{a} - 2\vec{b}$  and  $\overline{OQ} = \vec{a} + \vec{b}$ . Find the position vector of a point R which divides the line joining P and Q in the ratio 2:1, (i) internally, and (ii) externally.

### Solution

(i) The position vector of the point R dividing the join of P and Q internally in the ratio 2:1 is

$$\overline{OR} = \frac{2(\vec{a} + \vec{b}) + (3\vec{a} - 2\vec{b})}{2 + 1} = \frac{5\vec{a}}{3}$$

(ii) The position vector of the point R dividing the join of P and Q externally in the ratio 2:1 is

$$\overline{OR} = \frac{2(\vec{a} + \vec{b}) - (3\vec{a} - 2\vec{b})}{2 - 1} = 4\vec{b} - \vec{a}$$

**Example 12** Show that the points  $A(2\hat{i} - \hat{j} + \hat{k})$ ,  $B(\hat{i} - 3\hat{j} - 5\hat{k})$ ,  $C(3\hat{i} - 4\hat{j} - 4\hat{k})$  are the vertices of a right angled triangle.

**Solution** We have

$$\overline{AB} = (1 - 2)\hat{i} + (-3 + 1)\hat{j} + (-5 - 1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overline{BC} = (3 - 1)\hat{i} + (-4 + 3)\hat{j} + (-4 + 5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

and  $\overline{CA} = (2 - 3)\hat{i} + (-1 + 4)\hat{j} + (1 + 4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

Further, note that

$$|\overrightarrow{AB}|^2 = 41 = 6 + 35 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$$

Hence, the triangle is a right angled triangle.

### EXERCISE 10.2

1. Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + k; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \quad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

2. Write two different vectors having same magnitude.
3. Write two different vectors having same direction.
4. Find the values of  $x$  and  $y$  so that the vectors  $2\hat{i} + 3\hat{j}$  and  $x\hat{i} + y\hat{j}$  are equal.
5. Find the scalar and vector components of the vector with initial point  $(2, 1)$  and terminal point  $(-5, 7)$ .
6. Find the sum of the vectors  $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ ,  $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$  and  $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$ .
7. Find the unit vector in the direction of the vector  $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ .
8. Find the unit vector in the direction of vector  $\overrightarrow{PQ}$ , where  $P$  and  $Q$  are the points  $(1, 2, 3)$  and  $(4, 5, 6)$ , respectively.
9. For given vectors,  $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$  and  $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$ , find the unit vector in the direction of the vector  $\vec{a} + \vec{b}$ .
10. Find a vector in the direction of vector  $5\hat{i} - \hat{j} + 2\hat{k}$  which has magnitude 8 units.
11. Show that the vectors  $2\hat{i} - 3\hat{j} + 4\hat{k}$  and  $-4\hat{i} + 6\hat{j} - 8\hat{k}$  are collinear.
12. Find the direction cosines of the vector  $\hat{i} + 2\hat{j} + 3\hat{k}$ .
13. Find the direction cosines of the vector joining the points  $A(1, 2, -3)$  and  $B(-1, -2, 1)$ , directed from  $A$  to  $B$ .
14. Show that the vector  $\hat{i} + \hat{j} + \hat{k}$  is equally inclined to the axes  $OX$ ,  $OY$  and  $OZ$ .
15. Find the position vector of a point  $R$  which divides the line joining two points  $P$  and  $Q$  whose position vectors are  $\hat{i} + 2\hat{j} - \hat{k}$  and  $-\hat{i} + \hat{j} + \hat{k}$  respectively, in the ratio  $2 : 1$ 
  - (i) internally
  - (ii) externally

16. Find the position vector of the mid point of the vector joining the points  $P(2, 3, 4)$  and  $Q(4, 1, -2)$ .
17. Show that the points A, B and C with position vectors,  $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ ,  $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$  and  $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$ , respectively form the vertices of a right angled triangle.
18. In triangle ABC (Fig 10.18), which of the following is not true:
- $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$
  - $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$
  - $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$
  - $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

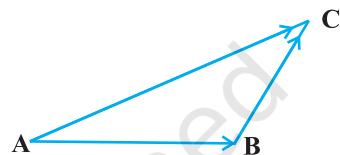


Fig 10.18

19. If  $\vec{a}$  and  $\vec{b}$  are two collinear vectors, then which of the following are incorrect:
- $\vec{b} = \lambda \vec{a}$ , for some scalar  $\lambda$
  - $\vec{a} = \pm \vec{b}$
  - the respective components of  $\vec{a}$  and  $\vec{b}$  are not proportional
  - both the vectors  $\vec{a}$  and  $\vec{b}$  have same direction, but different magnitudes.

## 10.6 Product of Two Vectors

So far we have studied about addition and subtraction of vectors. An other algebraic operation which we intend to discuss regarding vectors is their product. We may recall that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiply them in two ways, namely, multiplication of two functions pointwise and composition of two functions. Similarly, multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector. Based upon these two types of products for vectors, they have found various applications in geometry, mechanics and engineering. In this section, we will discuss these two types of products.

### 10.6.1 *Scalar (or dot) product of two vectors*

**Definition 2** The scalar product of two nonzero vectors  $\vec{a}$  and  $\vec{b}$ , denoted by  $\vec{a} \cdot \vec{b}$ , is

defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta,$$

where,  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ ,  $0 \leq \theta \leq \pi$  (Fig 10.19).

If either  $\vec{a} = 0$  or  $\vec{b} = 0$  then  $\theta$  is not defined, and in this case, we

define  $\vec{a} \cdot \vec{b} = 0$

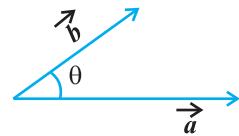


Fig 10.19

### Observations

1.  $\vec{a} \cdot \vec{b}$  is a real number.
2. Let  $\vec{a}$  and  $\vec{b}$  be two nonzero vectors, then  $\vec{a} \cdot \vec{b} = 0$  if and only if  $\vec{a}$  and  $\vec{b}$  are perpendicular to each other. i.e.

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$$

3. If  $\theta = 0$ , then  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$

In particular,  $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ , as  $\theta$  in this case is 0.

4. If  $\theta = \pi$ , then  $\vec{a} \cdot \vec{b} = -|\vec{a}| |\vec{b}|$

In particular,  $\vec{a} \cdot \vec{b} = -|\vec{a}| |\vec{b}|$ , as  $\theta$  in this case is  $\pi$ .

5. In view of the Observations 2 and 3, for mutually perpendicular unit vectors  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ , we have

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1,$$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

6. The angle between two nonzero vectors  $\vec{a}$  and  $\vec{b}$  is given by

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}, \text{ or } \theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

7. The scalar product is commutative. i.e.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \quad (\text{Why?})$$

### Two important properties of scalar product

**Property 1** (Distributivity of scalar product over addition) Let  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  be any three vectors, then

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

**Property 2** Let  $\vec{a}$  and  $\vec{b}$  be any two vectors, and  $l$  be any scalar. Then

$$(\lambda\vec{a}) \cdot \vec{b} = (\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda\vec{b})$$

If two vectors  $\vec{a}$  and  $\vec{b}$  are given in component form as  $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ , then their scalar product is given as

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= a_1\hat{i} \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j} \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k} \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= a_1b_1(\hat{i} \cdot \hat{i}) + a_1b_2(\hat{i} \cdot \hat{j}) + a_1b_3(\hat{i} \cdot \hat{k}) + a_2b_1(\hat{j} \cdot \hat{i}) + a_2b_2(\hat{j} \cdot \hat{j}) + a_2b_3(\hat{j} \cdot \hat{k}) \\ &\quad + a_3b_1(\hat{k} \cdot \hat{i}) + a_3b_2(\hat{k} \cdot \hat{j}) + a_3b_3(\hat{k} \cdot \hat{k}) \text{ (Using the above Properties 1 and 2)} \\ &= a_1b_1 + a_2b_2 + a_3b_3 \text{ (Using Observation 5)}\end{aligned}$$

Thus  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

### 10.6.2 Projection of a vector on a line

Suppose a vector  $\overline{AB}$  makes an angle  $\theta$  with a given directed line  $l$  (say), in the *anticlockwise direction* (Fig 10.20). Then the projection of  $\overline{AB}$  on  $l$  is a vector  $\vec{p}$  (say) with magnitude  $|\overline{AB}| |\cos \theta|$ , and the direction of  $\vec{p}$  being the same (or opposite) to that of the line  $l$ , depending upon whether  $\cos \theta$  is positive or negative. The vector  $\vec{p}$

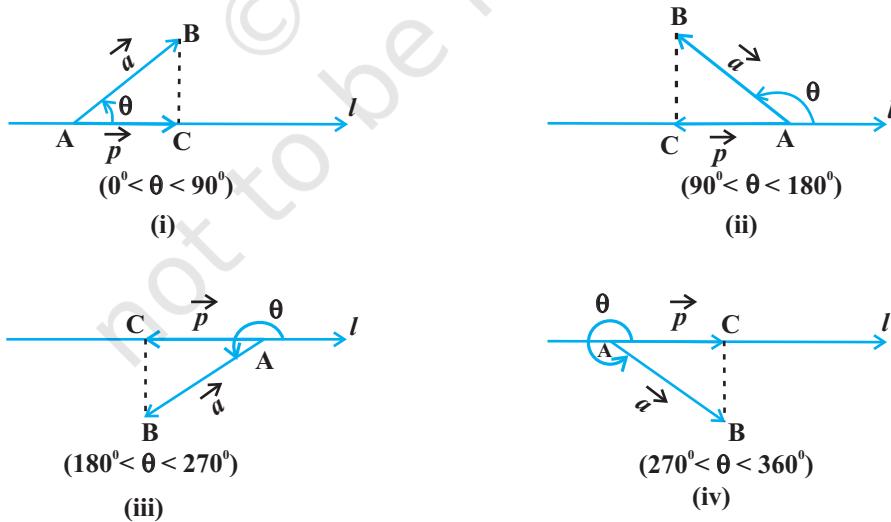


Fig 10.20

is called the *projection vector*, and its magnitude  $|\vec{p}|$  is simply called as the *projection* of the vector  $\overline{AB}$  on the directed line  $l$ .

For example, in each of the following figures (Fig 10.20 (i) to (iv)), projection vector of  $\overline{AB}$  along the line  $l$  is vector  $\overline{AC}$ .

### Observations

1. If  $\hat{p}$  is the unit vector along a line  $l$ , then the projection of a vector  $\vec{a}$  on the line  $l$  is given by  $\vec{a} \cdot \hat{p}$ .
  2. Projection of a vector  $\vec{a}$  on other vector  $\vec{b}$ , is given by
- $$\vec{a} \cdot \hat{b}, \quad \text{or} \quad \vec{a} \cdot \left( \frac{\vec{b}}{|\vec{b}|} \right), \quad \text{or} \quad \frac{1}{|\vec{b}|} (\vec{a} \cdot \vec{b})$$
3. If  $\theta = 0$ , then the projection vector of  $\overline{AB}$  will be  $\overline{AB}$  itself and if  $\theta = \pi$ , then the projection vector of  $\overline{AB}$  will be  $\overline{BA}$ .
  4. If  $\theta = \frac{\pi}{2}$  or  $\theta = \frac{3\pi}{2}$ , then the projection vector of  $\overline{AB}$  will be zero vector.

**Remark** If  $\alpha$ ,  $\beta$  and  $\gamma$  are the direction angles of vector  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ , then its direction cosines may be given as

$$\cos \alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}| \|\hat{i}\|} = \frac{a_1}{|\vec{a}|}, \quad \cos \beta = \frac{a_2}{|\vec{a}|}, \quad \text{and} \quad \cos \gamma = \frac{a_3}{|\vec{a}|}$$

Also, note that  $|\vec{a}| \cos \alpha$ ,  $|\vec{a}| \cos \beta$  and  $|\vec{a}| \cos \gamma$  are respectively the projections of  $\vec{a}$  along OX, OY and OZ. i.e., the scalar components  $a_1$ ,  $a_2$  and  $a_3$  of the vector  $\vec{a}$ , are precisely the projections of  $\vec{a}$  along  $x$ -axis,  $y$ -axis and  $z$ -axis, respectively. Further, if  $\vec{a}$  is a unit vector, then it may be expressed in terms of its direction cosines as

$$\vec{a} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k}$$

**Example 13** Find the angle between two vectors  $\vec{a}$  and  $\vec{b}$  with magnitudes 1 and 2 respectively and when  $\vec{a} \cdot \vec{b} = 1$ .

**Solution** Given  $\vec{a} \cdot \vec{b} = 1$ ,  $|\vec{a}| = 1$  and  $|\vec{b}| = 2$ . We have

$$\theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) = \cos^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{3}$$

**Example 14** Find angle ‘ $\theta$ ’ between the vectors  $\vec{a} = \hat{i} + \hat{j} - \hat{k}$  and  $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ .

**Solution** The angle  $\theta$  between two vectors  $\vec{a}$  and  $\vec{b}$  is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

Now

$$\vec{a} \cdot \vec{b} = (\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{i} - \hat{j} + \hat{k}) = 1 - 1 - 1 = -1.$$

Therefore, we have

$$\cos\theta = \frac{-1}{3}$$

hence the required angle is

$$\theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

**Example 15** If  $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$  and  $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$ , then show that the vectors  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  are perpendicular.

**Solution** We know that two nonzero vectors are perpendicular if their scalar product is zero.

$$\text{Here } \vec{a} + \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k}) = 6\hat{i} + 2\hat{j} - 8\hat{k}$$

$$\text{and } \vec{a} - \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k}) = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

$$\text{So } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k}) \cdot (4\hat{i} - 4\hat{j} + 2\hat{k}) = 24 - 8 - 16 = 0.$$

Hence  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  are perpendicular vectors.

**Example 16** Find the projection of the vector  $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$  on the vector  $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ .

**Solution** The projection of vector  $\vec{a}$  on the vector  $\vec{b}$  is given by

$$\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) = \frac{(2 \times 1 + 3 \times 2 + 2 \times 1)}{\sqrt{(1)^2 + (2)^2 + (1)^2}} = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}$$

**Example 17** Find  $|\vec{a} - \vec{b}|$ , if two vectors  $\vec{a}$  and  $\vec{b}$  are such that  $|\vec{a}| = 2$ ,  $|\vec{b}| = 3$  and  $\vec{a} \cdot \vec{b} = 4$ .

**Solution** We have

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \end{aligned}$$

$$\begin{aligned}
 &= |\vec{a}|^2 - 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2 \\
 &= (2)^2 - 2(4) + (3)^2
 \end{aligned}$$

Therefore

$$|\vec{a} - \vec{b}| = \sqrt{5}$$

**Example 18** If  $\vec{a}$  is a unit vector and  $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$ , then find  $|\vec{x}|$ .

**Solution** Since  $\vec{a}$  is a unit vector,  $|\vec{a}| = 1$ . Also,

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$$

$$\text{or } \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 8$$

$$\text{or } |\vec{x}|^2 - 1 = 8 \text{ i.e. } |\vec{x}|^2 = 9$$

Therefore

$$|\vec{x}| = 3 \text{ (as magnitude of a vector is non negative).}$$

**Example 19** For any two vectors  $\vec{a}$  and  $\vec{b}$ , we always have  $|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$  (Cauchy-Schwartz inequality).

**Solution** The inequality holds trivially when either  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$ . Actually, in such a situation we have  $|\vec{a} \cdot \vec{b}| = 0 = |\vec{a}| |\vec{b}|$ . So, let us assume that  $|\vec{a}| \neq 0 \neq |\vec{b}|$ .

Then, we have

$$\frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}| |\vec{b}|} = |\cos \theta| \leq 1$$

Therefore

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$$

**Example 20** For any two vectors  $\vec{a}$  and  $\vec{b}$ , we always have  $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$  (triangle inequality).

**Solution** The inequality holds trivially in case either  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$  (How?). So, let  $|\vec{a}| \neq 0 \neq |\vec{b}|$ . Then,

$$\begin{aligned}
 |\vec{a} + \vec{b}|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\
 &= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\
 &= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
 &\leq |\vec{a}|^2 + 2|\vec{a} \cdot \vec{b}| + |\vec{b}|^2 \\
 &\leq |\vec{a}|^2 + 2|\vec{a}| |\vec{b}| + |\vec{b}|^2 \\
 &= (|\vec{a}| + |\vec{b}|)^2
 \end{aligned}$$

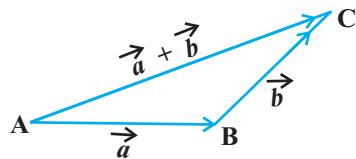


Fig 10.21

(scalar product is commutative)

(since  $x \leq |x| \forall x \in \mathbb{R}$ )

(from Example 19)

Hence

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

**Remark** If the equality holds in triangle inequality (in the above Example 20), i.e.

$$|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|,$$

then

$$|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$$

showing that the points A, B and C are collinear.

**Example 21** Show that the points A( $-2\hat{i} + 3\hat{j} + 5\hat{k}$ ), B( $\hat{i} + 2\hat{j} + 3\hat{k}$ ) and C( $7\hat{i} - \hat{k}$ ) are collinear.

**Solution** We have

$$\begin{aligned}\overrightarrow{AB} &= (1+2)\hat{i} + (2-3)\hat{j} + (3-5)\hat{k} = 3\hat{i} - \hat{j} - 2\hat{k}, \\ \overrightarrow{BC} &= (7-1)\hat{i} + (0-2)\hat{j} + (-1-3)\hat{k} = 6\hat{i} - 2\hat{j} - 4\hat{k}, \\ \overrightarrow{AC} &= (7+2)\hat{i} + (0-3)\hat{j} + (-1-5)\hat{k} = 9\hat{i} - 3\hat{j} - 6\hat{k} \\ |\overrightarrow{AB}| &= \sqrt{14}, \quad |\overrightarrow{BC}| = 2\sqrt{14} \quad \text{and} \quad |\overrightarrow{AC}| = 3\sqrt{14}\end{aligned}$$

Therefore

$$|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$$

Hence the points A, B and C are collinear.

 **Note** In Example 21, one may note that although  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$  but the points A, B and C do not form the vertices of a triangle.

### EXERCISE 10.3

- Find the angle between two vectors  $\vec{a}$  and  $\vec{b}$  with magnitudes  $\sqrt{3}$  and 2, respectively having  $\vec{a} \cdot \vec{b} = \sqrt{6}$ .
- Find the angle between the vectors  $\hat{i} - 2\hat{j} + 3\hat{k}$  and  $3\hat{i} - 2\hat{j} + \hat{k}$ .
- Find the projection of the vector  $\hat{i} - \hat{j}$  on the vector  $\hat{i} + \hat{j}$ .
- Find the projection of the vector  $\hat{i} + 3\hat{j} + 7\hat{k}$  on the vector  $7\hat{i} - \hat{j} + 8\hat{k}$ .
- Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \quad \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

6. Find  $|\vec{a}|$  and  $|\vec{b}|$ , if  $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$  and  $|\vec{a}| = 8|\vec{b}|$ .
7. Evaluate the product  $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$ .
8. Find the magnitude of two vectors  $\vec{a}$  and  $\vec{b}$ , having the same magnitude and such that the angle between them is  $60^\circ$  and their scalar product is  $\frac{1}{2}$ .
9. Find  $|\vec{x}|$ , if for a unit vector  $\vec{a}$ ,  $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$ .
10. If  $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$  and  $\vec{c} = 3\hat{i} + \hat{j}$  are such that  $\vec{a} + \lambda\vec{b}$  is perpendicular to  $\vec{c}$ , then find the value of  $\lambda$ .
11. Show that  $|\vec{a}| \vec{b} + |\vec{b}| \vec{a}$  is perpendicular to  $|\vec{a}| \vec{b} - |\vec{b}| \vec{a}$ , for any two nonzero vectors  $\vec{a}$  and  $\vec{b}$ .
12. If  $\vec{a} \cdot \vec{a} = 0$  and  $\vec{a} \cdot \vec{b} = 0$ , then what can be concluded about the vector  $\vec{b}$ ?
13. If  $\vec{a}, \vec{b}, \vec{c}$  are unit vectors such that  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ , find the value of  $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ .
14. If either vector  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$ , then  $\vec{a} \cdot \vec{b} = 0$ . But the converse need not be true. Justify your answer with an example.
15. If the vertices A, B, C of a triangle ABC are  $(1, 2, 3)$ ,  $(-1, 0, 0)$ ,  $(0, 1, 2)$ , respectively, then find  $\angle ABC$ . [ $\angle ABC$  is the angle between the vectors  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ ].
16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, -1) are collinear.
17. Show that the vectors  $2\hat{i} - \hat{j} + \hat{k}$ ,  $\hat{i} - 3\hat{j} - 5\hat{k}$  and  $3\hat{i} - 4\hat{j} - 4\hat{k}$  form the vertices of a right angled triangle.
18. If  $\vec{a}$  is a nonzero vector of magnitude 'a' and  $\lambda$  a nonzero scalar, then  $\lambda\vec{a}$  is unit vector if
  - (A)  $\lambda = 1$
  - (B)  $\lambda = -1$
  - (C)  $a = |\lambda|$
  - (D)  $a = 1/|\lambda|$

### 10.6.3 Vector (or cross) product of two vectors

In Section 10.2, we have discussed on the three dimensional right handed rectangular coordinate system. In this system, when the positive  $x$ -axis is rotated counterclockwise

into the positive  $y$ -axis, a right handed (standard) screw would advance in the direction of the positive  $z$ -axis (Fig 10.22(i)).

In a right handed coordinate system, the thumb of the right hand points in the direction of the positive  $z$ -axis when the fingers are curled in the direction away from the positive  $x$ -axis toward the positive  $y$ -axis (Fig 10.22(ii)).

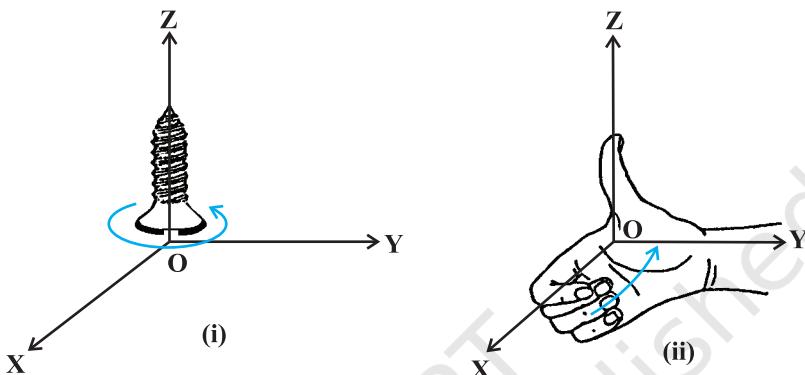


Fig 10.22 (i), (ii)

**Definition 3** The vector product of two nonzero vectors  $\vec{a}$  and  $\vec{b}$ , is denoted by  $\vec{a} \times \vec{b}$  and defined as

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n},$$

where,  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ ,  $0 \leq \theta \leq \pi$  and  $\hat{n}$  is a unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$ , such that  $\vec{a}, \vec{b}$  and  $\hat{n}$  form a right handed system (Fig 10.23). i.e., the right handed system rotated from  $\vec{a}$  to  $\vec{b}$  moves in the direction of  $\hat{n}$ .

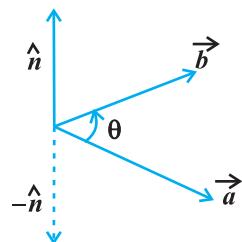


Fig 10.23

If either  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$ , then  $\theta$  is not defined and in this case, we define  $\vec{a} \times \vec{b} = \vec{0}$ .

### Observations

1.  $\vec{a} \times \vec{b}$  is a vector.
2. Let  $\vec{a}$  and  $\vec{b}$  be two nonzero vectors. Then  $\vec{a} \times \vec{b} = \vec{0}$  if and only if  $\vec{a}$  and  $\vec{b}$  are parallel (or collinear) to each other, i.e.,

$$\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$$

In particular,  $\vec{a} \times \vec{a} = \vec{0}$  and  $\vec{a} \times (-\vec{a}) = \vec{0}$ , since in the first situation,  $\theta = 0$  and in the second one,  $\theta = \pi$ , making the value of  $\sin \theta$  to be 0.

3. If  $\theta = \frac{\pi}{2}$  then  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}|$ .
4. In view of the Observations 2 and 3, for mutually perpendicular unit vectors  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$  (Fig 10.24), we have

$$\begin{aligned}\hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0} \\ \hat{i} \times \hat{j} &= \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}\end{aligned}$$

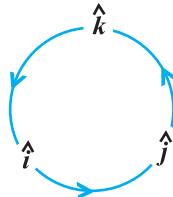


Fig 10.24

5. In terms of vector product, the angle between two vectors  $\vec{a}$  and  $\vec{b}$  may be given as

$$\sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$$

6. It is always true that the vector product is not commutative, as  $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ . Indeed,  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$ , where  $\vec{a}$ ,  $\vec{b}$  and  $\hat{n}$  form a right handed system, i.e.,  $\theta$  is traversed from  $\vec{a}$  to  $\vec{b}$ , Fig 10.25 (i). While,  $\vec{b} \times \vec{a} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}_1$ , where  $\vec{b}$ ,  $\vec{a}$  and  $\hat{n}_1$  form a right handed system i.e.  $\theta$  is traversed from  $\vec{b}$  to  $\vec{a}$ , Fig 10.25(ii).

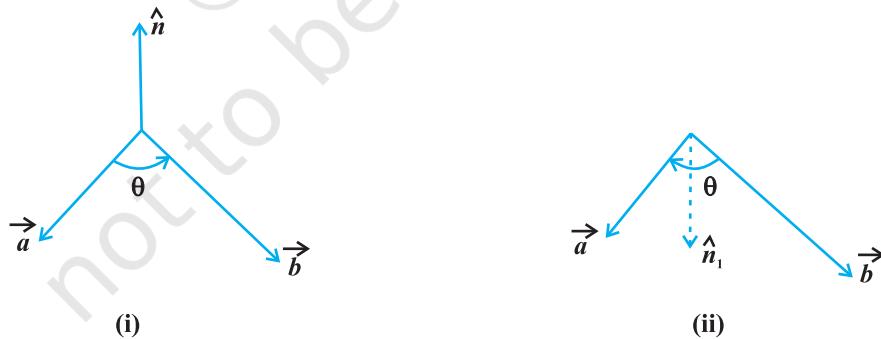


Fig 10.25 (i), (ii)

Thus, if we assume  $\vec{a}$  and  $\vec{b}$  to lie in the plane of the paper, then  $\hat{n}$  and  $\hat{n}_1$  both will be perpendicular to the plane of the paper. But,  $\hat{n}$  being directed above the paper while  $\hat{n}_1$  directed below the paper. i.e.  $\hat{n}_1 = -\hat{n}$ .

Hence  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$

$$= -|\vec{a}| |\vec{b}| \sin \theta \hat{n}_1 = -\vec{b} \times \vec{a}$$

7. In view of the Observations 4 and 6, we have

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i} \quad \text{and} \quad \hat{i} \times \hat{k} = -\hat{j}.$$

8. If  $\vec{a}$  and  $\vec{b}$  represent the adjacent sides of a triangle then its area is given as

$$\frac{1}{2} |\vec{a} \times \vec{b}|.$$

By definition of the area of a triangle, we have from Fig 10.26,

$$\text{Area of triangle } ABC = \frac{1}{2} AB \cdot CD.$$

But  $AB = |\vec{b}|$  (as given), and  $CD = |\vec{a}| \sin \theta$ .

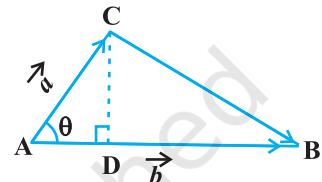


Fig 10.26

$$\text{Thus, Area of triangle } ABC = \frac{1}{2} |\vec{b}| |\vec{a}| \sin \theta = \frac{1}{2} |\vec{a} \times \vec{b}|.$$

9. If  $\vec{a}$  and  $\vec{b}$  represent the adjacent sides of a parallelogram, then its area is given by  $|\vec{a} \times \vec{b}|$ .

From Fig 10.27, we have

$$\text{Area of parallelogram } ABCD = AB \cdot DE.$$

But  $AB = |\vec{b}|$  (as given), and

$$DE = |\vec{a}| \sin \theta.$$

Thus,

$$\text{Area of parallelogram } ABCD = |\vec{b}| |\vec{a}| \sin \theta = |\vec{a} \times \vec{b}|.$$

We now state two important properties of vector product.

**Property 3** (Distributivity of vector product over addition): If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are any three vectors and  $\lambda$  be a scalar, then

$$(i) \quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$(ii) \quad \lambda(\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b})$$

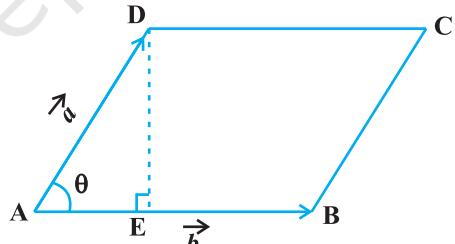


Fig 10.27

Let  $\vec{a}$  and  $\vec{b}$  be two vectors given in component form as  $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ , respectively. Then their cross product may be given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

**Explanation** We have

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= a_1b_1(\hat{i} \times \hat{i}) + a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) \\ &\quad + a_2b_2(\hat{j} \times \hat{j}) + a_2b_3(\hat{j} \times \hat{k}) \\ &\quad + a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k}) \quad (\text{by Property 1}) \\ &= a_1b_2(\hat{i} \times \hat{j}) - a_1b_3(\hat{k} \times \hat{i}) - a_2b_1(\hat{i} \times \hat{j}) \\ &\quad + a_2b_3(\hat{j} \times \hat{k}) + a_3b_1(\hat{k} \times \hat{i}) - a_3b_2(\hat{j} \times \hat{k}) \end{aligned}$$

(as  $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$  and  $\hat{i} \times \hat{k} = -\hat{k} \times \hat{i}$ ,  $\hat{j} \times \hat{i} = -\hat{i} \times \hat{j}$  and  $\hat{k} \times \hat{j} = -\hat{j} \times \hat{k}$ )

$$\begin{aligned} &= a_1b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i} \\ &\quad (\text{as } \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i} \text{ and } \hat{k} \times \hat{i} = \hat{j}) \end{aligned}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

**Example 22** Find  $|\vec{a} \times \vec{b}|$ , if  $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$  and  $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$

**Solution** We have

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix} \\ &= \hat{i}(-2 - 15) - (-4 - 9)\hat{j} + (10 - 3)\hat{k} = -17\hat{i} + 13\hat{j} + 7\hat{k} \end{aligned}$$

$$\text{Hence } |\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2} = \sqrt{507}$$

**Example 23** Find a unit vector perpendicular to each of the vectors  $(\vec{a} + \vec{b})$  and  $(\vec{a} - \vec{b})$ , where  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ .

**Solution** We have  $\vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$  and  $\vec{a} - \vec{b} = -\hat{j} - 2\hat{k}$

A vector which is perpendicular to both  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  is given by

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i} + 4\hat{j} - 2\hat{k} \quad (= \vec{c}, \text{ say})$$

Now

$$|\vec{c}| = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$$

Therefore, the required unit vector is

$$\frac{\vec{c}}{|\vec{c}|} = \frac{-1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$$



**Note** There are two perpendicular directions to any plane. Thus, another unit vector perpendicular to  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  will be  $\frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$ . But that will be a consequence of  $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$ .

**Example 24** Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

**Solution** We have  $\overrightarrow{AB} = \hat{j} + 2\hat{k}$  and  $\overrightarrow{AC} = \hat{i} + 2\hat{j}$ . The area of the given triangle is  $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$ .

Now,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\hat{i} + 2\hat{j} - \hat{k}$$

Therefore

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{16+4+1} = \sqrt{21}$$

Thus, the required area is  $\frac{1}{2}\sqrt{21}$

**Example 25** Find the area of a parallelogram whose adjacent sides are given by the vectors  $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$  and  $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

**Solution** The area of a parallelogram with  $\vec{a}$  and  $\vec{b}$  as its adjacent sides is given by  $|\vec{a} \times \vec{b}|$ .

Now 
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{i} + \hat{j} - 4\hat{k}$$

Therefore  $|\vec{a} \times \vec{b}| = \sqrt{25+1+16} = \sqrt{42}$

and hence, the required area is  $\sqrt{42}$ .

### EXERCISE 10.4

- Find  $|\vec{a} \times \vec{b}|$ , if  $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$  and  $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ .
- Find a unit vector perpendicular to each of the vector  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$ , where  $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$  and  $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ .
- If a unit vector  $\vec{a}$  makes angles  $\frac{\pi}{3}$  with  $\hat{i}$ ,  $\frac{\pi}{4}$  with  $\hat{j}$  and an acute angle  $\theta$  with  $\hat{k}$ , then find  $\theta$  and hence, the components of  $\vec{a}$ .
- Show that  $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$
- Find  $\lambda$  and  $\mu$  if  $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$ .
- Given that  $\vec{a} \cdot \vec{b} = 0$  and  $\vec{a} \times \vec{b} = \vec{0}$ . What can you conclude about the vectors  $\vec{a}$  and  $\vec{b}$ ?
- Let the vectors  $\vec{a}, \vec{b}, \vec{c}$  be given as  $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ ,  $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ ,  $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ . Then show that  $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ .
- If either  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$ , then  $\vec{a} \times \vec{b} = \vec{0}$ . Is the converse true? Justify your answer with an example.
- Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors  $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$  and  $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$ .
11. Let the vectors  $\vec{a}$  and  $\vec{b}$  be such that  $|\vec{a}|=3$  and  $|\vec{b}|=\frac{\sqrt{2}}{3}$ , then  $\vec{a} \times \vec{b}$  is a unit vector, if the angle between  $\vec{a}$  and  $\vec{b}$  is  
 (A)  $\pi/6$       (B)  $\pi/4$       (C)  $\pi/3$       (D)  $\pi/2$
12. Area of a rectangle having vertices A, B, C and D with position vectors  $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$ ,  $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$ ,  $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$  and  $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ , respectively is  
 (A)  $\frac{1}{2}$       (B) 1      (C) 2      (D) 4

### Miscellaneous Examples

**Example 26** Write all the unit vectors in XY-plane.

**Solution** Let  $\vec{r} = x\hat{i} + y\hat{j}$  be a unit vector in XY-plane (Fig 10.28). Then, from the figure, we have  $x = \cos \theta$  and  $y = \sin \theta$  (since  $|\vec{r}| = 1$ ). So, we may write the vector  $\vec{r}$  as

$$\vec{r} (= \overrightarrow{OP}) = \cos \theta \hat{i} + \sin \theta \hat{j} \quad \dots (1)$$

Clearly,

$$|\vec{r}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

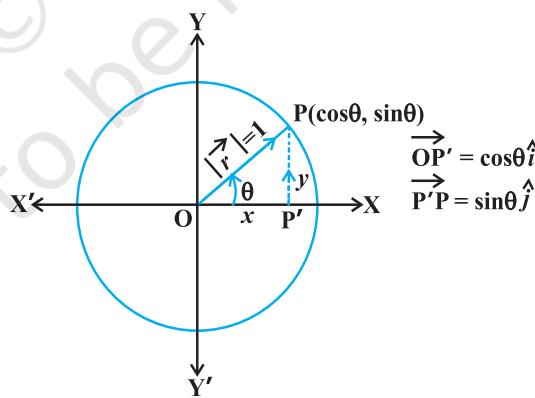


Fig 10.28

Also, as  $\theta$  varies from 0 to  $2\pi$ , the point P (Fig 10.28) traces the circle  $x^2 + y^2 = 1$  counterclockwise, and this covers all possible directions. So, (1) gives every unit vector in the XY-plane.

**Example 27** If  $\hat{i} + \hat{j} + \hat{k}$ ,  $2\hat{i} + 5\hat{j}$ ,  $3\hat{i} + 2\hat{j} - 3\hat{k}$  and  $\hat{i} - 6\hat{j} - \hat{k}$  are the position vectors of points A, B, C and D respectively, then find the angle between  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$ . Deduce that  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  are collinear.

**Solution** Note that if  $\theta$  is the angle between AB and CD, then  $\theta$  is also the angle between  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$ .

Now

$$\begin{aligned}\overrightarrow{AB} &= \text{Position vector of B} - \text{Position vector of A} \\ &= (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}\end{aligned}$$

Therefore

$$|\overrightarrow{AB}| = \sqrt{(1)^2 + (4)^2 + (-1)^2} = 3\sqrt{2}$$

Similarly

$$\overrightarrow{CD} = -2\hat{i} - 8\hat{j} + 2\hat{k} \text{ and } |\overrightarrow{CD}| = 6\sqrt{2}$$

Thus

$$\begin{aligned}\cos \theta &= \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{AB}| |\overrightarrow{CD}|} \\ &= \frac{1(-2) + 4(-8) + (-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1\end{aligned}$$

Since  $0 \leq \theta \leq \pi$ , it follows that  $\theta = \pi$ . This shows that  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  are collinear.

**Alternatively**,  $\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{CD}$  which implies that  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  are collinear vectors.

**Example 28** Let  $\vec{a}, \vec{b}$  and  $\vec{c}$  be three vectors such that  $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$  and each one of them being perpendicular to the sum of the other two, find  $|\vec{a} + \vec{b} + \vec{c}|$ .

**Solution** Given  $\vec{a} \cdot (\vec{b} + \vec{c}) = 0, \vec{b} \cdot (\vec{c} + \vec{a}) = 0, \vec{c} \cdot (\vec{a} + \vec{b}) = 0$ .

Now

$$\begin{aligned}|\vec{a} + \vec{b} + \vec{c}|^2 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ &= \vec{a} \cdot \vec{a} + \vec{a} \cdot (\vec{b} + \vec{c}) + \vec{b} \cdot \vec{b} + \vec{b} \cdot (\vec{a} + \vec{c}) \\ &\quad + \vec{c} \cdot (\vec{a} + \vec{b}) + \vec{c} \cdot \vec{c} \\ &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \\ &= 9 + 16 + 25 = 50\end{aligned}$$

Therefore

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{50} = 5\sqrt{2}$$

**Example 29** Three vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  satisfy the condition  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ . Evaluate the quantity  $\mu = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ , if  $|\vec{a}| = 3$ ,  $|\vec{b}| = 4$  and  $|\vec{c}| = 2$ .

**Solution** Since  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ , we have

$$\vec{a} + \vec{b} + \vec{c} = \vec{0} = 0$$

or

$$\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0$$

Therefore

$$\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = -|\vec{a}|^2 = -9 \quad \dots (1)$$

Again,

$$\vec{b} \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$$

or  $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} = -|\vec{b}|^2 = -16 \quad \dots (2)$

Similarly

$$\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = -4. \quad \dots (3)$$

Adding (1), (2) and (3), we have

$$2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{a} \cdot \vec{c}) = -29$$

or  $2\mu = -29$ , i.e.,  $\mu = \frac{-29}{2}$

**Example 30** If with reference to the right handed system of mutually perpendicular unit vectors  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ ,  $\vec{\alpha} = 3\hat{i} - \hat{j}$ ,  $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$ , then express  $\vec{\beta}$  in the form  $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$ , where  $\vec{\beta}_1$  is parallel to  $\vec{\alpha}$  and  $\vec{\beta}_2$  is perpendicular to  $\vec{\alpha}$ .

**Solution** Let  $\vec{\beta}_1 = \lambda \vec{\alpha}$ ,  $\lambda$  is a scalar, i.e.,  $\vec{\beta}_1 = 3\lambda \hat{i} - \lambda \hat{j}$ .

Now  $\vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1 = (2 - 3\lambda)\hat{i} + (1 + \lambda)\hat{j} - 3\hat{k}$ .

Now, since  $\vec{\beta}_2$  is to be perpendicular to  $\vec{\alpha}$ , we should have  $\vec{\alpha} \cdot \vec{\beta}_2 = 0$ . i.e.,

$$3(2 - 3\lambda) - (1 + \lambda) = 0$$

or  $\lambda = \frac{1}{2}$

Therefore  $\vec{\beta}_1 = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$  and  $\vec{\beta}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

### Miscellaneous Exercise on Chapter 10

1. Write down a unit vector in XY-plane, making an angle of  $30^\circ$  with the positive direction of  $x$ -axis.
2. Find the scalar components and magnitude of the vector joining the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$ .
3. A girl walks 4 km towards west, then she walks 3 km in a direction  $30^\circ$  east of north and stops. Determine the girl's displacement from her initial point of departure.
4. If  $\vec{a} = \vec{b} + \vec{c}$ , then is it true that  $|\vec{a}| = |\vec{b}| + |\vec{c}|$ ? Justify your answer.
5. Find the value of  $x$  for which  $x(\hat{i} + \hat{j} + \hat{k})$  is a unit vector.
6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors  $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$  and  $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ .
7. If  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$  and  $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ , find a unit vector parallel to the vector  $2\vec{a} - \vec{b} + 3\vec{c}$ .
8. Show that the points  $A(1, -2, -8)$ ,  $B(5, 0, -2)$  and  $C(11, 3, 7)$  are collinear, and find the ratio in which  $B$  divides  $AC$ .
9. Find the position vector of a point  $R$  which divides the line joining two points  $P$  and  $Q$  whose position vectors are  $(2\vec{a} + \vec{b})$  and  $(\vec{a} - 3\vec{b})$  externally in the ratio  $1 : 2$ . Also, show that  $P$  is the mid point of the line segment  $RQ$ .
10. The two adjacent sides of a parallelogram are  $2\hat{i} - 4\hat{j} + 5\hat{k}$  and  $\hat{i} - 2\hat{j} - 3\hat{k}$ . Find the unit vector parallel to its diagonal. Also, find its area.
11. Show that the direction cosines of a vector equally inclined to the axes  $OX$ ,  $OY$  and  $OZ$  are  $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ .
12. Let  $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$ ,  $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$  and  $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$ . Find a vector  $\vec{d}$  which is perpendicular to both  $\vec{a}$  and  $\vec{b}$ , and  $\vec{c} \cdot \vec{d} = 15$ .
13. The scalar product of the vector  $\hat{i} + \hat{j} + \hat{k}$  with a unit vector along the sum of vectors  $2\hat{i} + 4\hat{j} - 5\hat{k}$  and  $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$  is equal to one. Find the value of  $\lambda$ .
14. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are mutually perpendicular vectors of equal magnitudes, show that the vector  $\vec{c} \cdot \vec{d} = 15$  is equally inclined to  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ .

15. Prove that  $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ , if and only if  $\vec{a}, \vec{b}$  are perpendicular, given  $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$ .

Choose the correct answer in Exercises 16 to 19.

16. If  $\theta$  is the angle between two vectors  $\vec{a}$  and  $\vec{b}$ , then  $\vec{a} \cdot \vec{b} \geq 0$  only when

- (A)  $0 < \theta < \frac{\pi}{2}$  (B)  $0 \leq \theta \leq \frac{\pi}{2}$   
 (C)  $0 < \theta < \pi$  (D)  $0 \leq \theta \leq \pi$

17. Let  $\vec{a}$  and  $\vec{b}$  be two unit vectors and  $\theta$  is the angle between them. Then  $\vec{a} + \vec{b}$  is a unit vector if

- (A)  $\theta = \frac{\pi}{4}$  (B)  $\theta = \frac{\pi}{3}$  (C)  $\theta = \frac{\pi}{2}$  (D)  $\theta = \frac{2\pi}{3}$

18. The value of  $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$  is

- (A) 0 (B) -1 (C) 1 (D) 3

19. If  $\theta$  is the angle between any two vectors  $\vec{a}$  and  $\vec{b}$ , then  $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$  when  $\theta$  is equal to

- (A) 0 (B)  $\frac{\pi}{4}$  (C)  $\frac{\pi}{2}$  (D)  $\pi$

### Summary

- ◆ Position vector of a point  $P(x, y, z)$  is given as  $\overline{OP} (= \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$ , and its magnitude by  $\sqrt{x^2 + y^2 + z^2}$ .
- ◆ The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- ◆ The magnitude ( $r$ ), direction ratios ( $a, b, c$ ) and direction cosines ( $l, m, n$ ) of any vector are related as:

$$l = \frac{a}{r}, \quad m = \frac{b}{r}, \quad n = \frac{c}{r}$$

- ◆ The vector sum of the three sides of a triangle taken in order is  $\vec{0}$ .
- ◆ The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- ◆ The multiplication of a given vector by a scalar  $\lambda$ , changes the magnitude of the vector by the multiple  $|\lambda|$ , and keeps the direction same (or makes it opposite) according as the value of  $\lambda$  is positive (or negative).
- ◆ For a given vector  $\vec{a}$ , the vector  $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$  gives the unit vector in the direction of  $\vec{a}$ .
- ◆ The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are  $\vec{a}$  and  $\vec{b}$  respectively, in the ratio  $m : n$ 
  - (i) internally, is given by  $\frac{n\vec{a} + m\vec{b}}{m + n}$ .
  - (ii) externally, is given by  $\frac{m\vec{b} - n\vec{a}}{m - n}$ .
- ◆ The scalar product of two given vectors  $\vec{a}$  and  $\vec{b}$  having angle  $\theta$  between them is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta.$$

Also, when  $\vec{a} \cdot \vec{b}$  is given, the angle 'θ' between the vectors  $\vec{a}$  and  $\vec{b}$  may be determined by

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

- ◆ If  $\theta$  is the angle between two vectors  $\vec{a}$  and  $\vec{b}$ , then their cross product is given as

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

where  $\hat{n}$  is a unit vector perpendicular to the plane containing  $\vec{a}$  and  $\vec{b}$ . Such that  $\vec{a}, \vec{b}, \hat{n}$  form right handed system of coordinate axes.

- ◆ If we have two vectors  $\vec{a}$  and  $\vec{b}$ , given in component form as  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$  and  $\lambda$  any scalar,

$$\text{then } \vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k};$$

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k};$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3;$$

$$\text{and } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

### Historical Note

The word *vector* has been derived from a Latin word *vectus*, which means “to carry”. The germinal ideas of modern vector theory date from around 1800 when Caspar Wessel (1745-1818) and Jean Robert Argand (1768-1822) described that how a complex number  $a + ib$  could be given a geometric interpretation with the help of a directed line segment in a coordinate plane. William Rowen Hamilton (1805-1865) an Irish mathematician was the first to use the term vector for a directed line segment in his book *Lectures on Quaternions* (1853). Hamilton’s method of quaternions (an ordered set of four real numbers given as:

$a + b\hat{i} + c\hat{j} + d\hat{k}$ ,  $\hat{i}$ ,  $\hat{j}$ ,  $\hat{k}$  following certain algebraic rules) was a solution to the problem of multiplying vectors in three dimensional space. Though, we must mention here that in practice, the idea of vector concept and their addition was known much earlier ever since the time of Aristotle (384-322 B.C.), a Greek philosopher, and pupil of Plato (427-348 B.C.). That time it was supposed to be known that the combined action of two or more forces could be seen by adding them according to parallelogram law. The correct law for the composition of forces, that forces add vectorially, had been discovered in the case of perpendicular forces by Stevin-Simon (1548-1620). In 1586 A.D., he analysed the principle of geometric addition of forces in his treatise *De Beghinselen der Weeghconst* (“Principles of the Art of Weighing”), which caused a major breakthrough in the development of mechanics. But it took another 200 years for the general concept of vectors to form.

In the 1880, Josiah Willard Gibbs (1839-1903), an American physicist and mathematician, and Oliver Heaviside (1850-1925), an English engineer, created what we now know as *vector analysis*, essentially by separating the real (*scalar*)

part of quaternion from its imaginary (*vector*) part. In 1881 and 1884, Gibbs printed a treatise entitled *Element of Vector Analysis*. This book gave a systematic and concise account of vectors. However, much of the credit for demonstrating the applications of vectors is due to the D. Heaviside and P.G. Tait (1831-1901) who contributed significantly to this subject.



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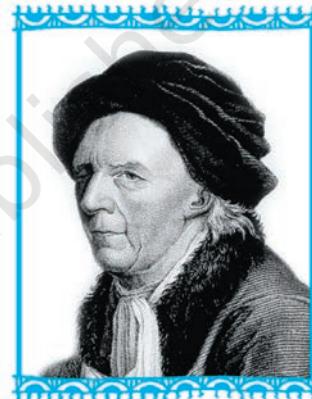
# THREE DIMENSIONAL GEOMETRY

❖ *The moving power of mathematical invention is not reasoning but imagination. – A. DEMORGAN* ❖

## 11.1 Introduction

In Class XI, while studying Analytical Geometry in two dimensions, and the introduction to three dimensional geometry, we confined to the Cartesian methods only. In the previous chapter of this book, we have studied some basic concepts of vectors. We will now use vector algebra to three dimensional geometry. The purpose of this approach to 3-dimensional geometry is that it makes the study simple and elegant\*.

In this chapter, we shall study the direction cosines and direction ratios of a line joining two points and also discuss about the equations of lines and planes in space under different conditions, angle between two lines, two planes, a line and a plane, shortest distance between two skew lines and distance of a point from a plane. Most of the above results are obtained in vector form. Nevertheless, we shall also translate these results in the Cartesian form which, at times, presents a more clear geometric and analytic picture of the situation.



Leonhard Euler  
(1707-1783)

## 11.2 Direction Cosines and Direction Ratios of a Line

From Chapter 10, recall that if a directed line  $L$  passing through the origin makes angles  $\alpha$ ,  $\beta$  and  $\gamma$  with  $x$ ,  $y$  and  $z$ -axes, respectively, called direction angles, then cosine of these angles, namely,  $\cos \alpha$ ,  $\cos \beta$  and  $\cos \gamma$  are called direction cosines of the directed line  $L$ .

If we reverse the direction of  $L$ , then the direction angles are replaced by their supplements, i.e.,  $\pi - \alpha$ ,  $\pi - \beta$  and  $\pi - \gamma$ . Thus, the signs of the direction cosines are reversed.

\* For various activities in three dimensional geometry, one may refer to the Book “*A Hand Book for designing Mathematics Laboratory in Schools*”, NCERT, 2005

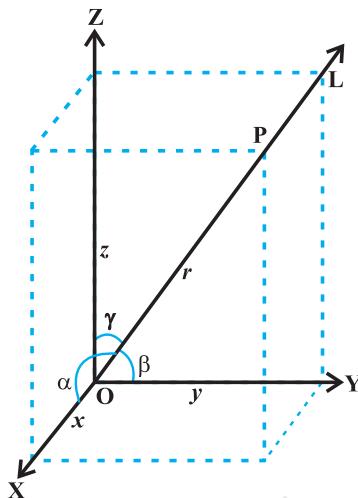


Fig 11.1

Note that a given line in space can be extended in two opposite directions and so it has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by  $l, m$  and  $n$ .

**Remark** If the given line in space does not pass through the origin, then, in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now take one of the directed lines from the origin and find its direction cosines as two parallel line have same set of direction cosines.

Any three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line. If  $l, m, n$  are direction cosines and  $a, b, c$  are direction ratios of a line, then  $a = \lambda l, b = \lambda m$  and  $c = \lambda n$ , for any nonzero  $\lambda \in \mathbf{R}$ .



**Note** Some authors also call direction ratios as direction numbers.

Let  $a, b, c$  be direction ratios of a line and let  $l, m$  and  $n$  be the direction cosines (d.c's) of the line. Then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ (say), } k \text{ being a constant.}$$

Therefore

$$l = ak, m = bk, n = ck \quad \dots (1)$$

But

$$l^2 + m^2 + n^2 = 1$$

Therefore

$$k^2 (a^2 + b^2 + c^2) = 1$$

or

$$k = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

Hence, from (1), the *d.c.*'s of the line are

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where, depending on the desired sign of  $k$ , either a positive or a negative sign is to be taken for  $l$ ,  $m$  and  $n$ .

For any line, if  $a, b, c$  are direction ratios of a line, then  $ka, kb, kc; k \neq 0$  is also a set of direction ratios. So, any two sets of direction ratios of a line are also proportional. Also, for any line there are infinitely many sets of direction ratios.

### 11.2.1 *Direction cosines of a line passing through two points*

Since one and only one line passes through two given points, we can determine the direction cosines of a line passing through the given points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  as follows (Fig 11.2 (a)).

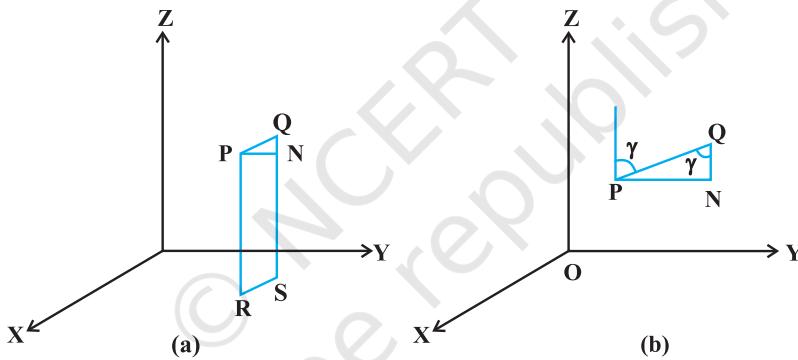


Fig 11.2

Let  $l, m, n$  be the direction cosines of the line  $PQ$  and let it makes angles  $\alpha, \beta$  and  $\gamma$  with the  $x, y$  and  $z$ -axis, respectively.

Draw perpendiculars from  $P$  and  $Q$  to  $XY$ -plane to meet at  $R$  and  $S$ . Draw a perpendicular from  $P$  to  $QS$  to meet at  $N$ . Now, in right angle triangle  $PNQ$ ,  $\angle PQN = \gamma$  (Fig 11.2 (b)).

Therefore, 
$$\cos \gamma = \frac{NQ}{PQ} = \frac{z_2 - z_1}{PQ}$$

Similarly 
$$\cos \alpha = \frac{x_2 - x_1}{PQ} \text{ and } \cos \beta = \frac{y_2 - y_1}{PQ}$$

Hence, the direction cosines of the line segment joining the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Note** The direction ratios of the line segment joining  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  may be taken as

$$x_2 - x_1, y_2 - y_1, z_2 - z_1 \text{ or } x_1 - x_2, y_1 - y_2, z_1 - z_2$$

**Example 1** If a line makes angle  $90^\circ, 60^\circ$  and  $30^\circ$  with the positive direction of  $x, y$  and  $z$ -axis respectively, find its direction cosines.

**Solution** Let the *d.c.*'s of the lines be  $l, m, n$ . Then  $l = \cos 90^\circ = 0, m = \cos 60^\circ = \frac{1}{2}$ ,

$$n = \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

**Example 2** If a line has direction ratios  $2, -1, -2$ , determine its direction cosines.

**Solution** Direction cosines are

$$\frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$\text{or } \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$$

**Example 3** Find the direction cosines of the line passing through the two points  $(-2, 4, -5)$  and  $(1, 2, 3)$ .

**Solution** We know the direction cosines of the line passing through two points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  are given by

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Here  $P$  is  $(-2, 4, -5)$  and  $Q$  is  $(1, 2, 3)$ .

$$\text{So } PQ = \sqrt{(1 - (-2))^2 + (2 - 4)^2 + (3 - (-5))^2} = \sqrt{77}$$

Thus, the direction cosines of the line joining two points is

$$\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

**Example 4** Find the direction cosines of  $x$ ,  $y$  and  $z$ -axis.

**Solution** The  $x$ -axis makes angles  $0^\circ$ ,  $90^\circ$  and  $90^\circ$  respectively with  $x$ ,  $y$  and  $z$ -axis. Therefore, the direction cosines of  $x$ -axis are  $\cos 0^\circ$ ,  $\cos 90^\circ$ ,  $\cos 90^\circ$  i.e.,  $1, 0, 0$ . Similarly, direction cosines of  $y$ -axis and  $z$ -axis are  $0, 1, 0$  and  $0, 0, 1$  respectively.

**Example 5** Show that the points  $A (2, 3, -4)$ ,  $B (1, -2, 3)$  and  $C (3, 8, -11)$  are collinear.

**Solution** Direction ratios of line joining  $A$  and  $B$  are

$$1 - 2, -2 - 3, 3 + 4 \text{ i.e., } -1, -5, 7.$$

The direction ratios of line joining  $B$  and  $C$  are

$$3 - 1, 8 + 2, -11 - 3, \text{ i.e., } 2, 10, -14.$$

It is clear that direction ratios of  $AB$  and  $BC$  are proportional, hence,  $AB$  is parallel to  $BC$ . But point  $B$  is common to both  $AB$  and  $BC$ . Therefore,  $A$ ,  $B$ ,  $C$  are collinear points.

### EXERCISE 11.1

1. If a line makes angles  $90^\circ$ ,  $135^\circ$ ,  $45^\circ$  with the  $x$ ,  $y$  and  $z$ -axes respectively, find its direction cosines.
2. Find the direction cosines of a line which makes equal angles with the coordinate axes.
3. If a line has the direction ratios  $-18, 12, -4$ , then what are its direction cosines ?
4. Show that the points  $(2, 3, 4)$ ,  $(-1, -2, 1)$ ,  $(5, 8, 7)$  are collinear.
5. Find the direction cosines of the sides of the triangle whose vertices are  $(3, 5, -4)$ ,  $(-1, 1, 2)$  and  $(-5, -5, -2)$ .

## 11.3 Equation of a Line in Space

We have studied equation of lines in two dimensions in Class XI, we shall now study the vector and cartesian equations of a line in space.

A line is uniquely determined if

- it passes through a given point and has given direction, or
- it passes through two given points.

### 11.3.1 Equation of a line through a given point and parallel to a given vector $\vec{b}$

Let  $\vec{a}$  be the position vector of the given point  $A$  with respect to the origin  $O$  of the rectangular coordinate system. Let  $l$  be the line which passes through the point  $A$  and is parallel to a given vector  $\vec{b}$ . Let  $\vec{r}$  be the position vector of an arbitrary point  $P$  on the line (Fig 11.3).

Then  $\overrightarrow{AP}$  is parallel to the vector  $\vec{b}$ , i.e.,  
 $\overrightarrow{AP} = \lambda \vec{b}$ , where  $\lambda$  is some real number.

But  $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$   
i.e.  $\lambda \vec{b} = \vec{r} - \vec{a}$

Conversely, for each value of the parameter  $\lambda$ , this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda \vec{b} \quad \dots (1)$$

**Remark** If  $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ , then  $a, b, c$  are direction ratios of the line and conversely, if  $a, b, c$  are direction ratios of a line, then  $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$  will be the parallel to the line. Here,  $b$  should not be confused with  $|\vec{b}|$ .

### Derivation of cartesian form from vector form

Let the coordinates of the given point A be  $(x_1, y_1, z_1)$  and the direction ratios of the line be  $a, b, c$ . Consider the coordinates of any point P be  $(x, y, z)$ . Then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}; \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

and  $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Substituting these values in (1) and equating the coefficients of  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ , we get

$$x = x_1 + \lambda a; y = y_1 + \lambda b; z = z_1 + \lambda c \quad \dots (2)$$

These are parametric equations of the line. Eliminating the parameter  $\lambda$  from (2), we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \dots (3)$$

This is the Cartesian equation of the line.

**Note** If  $l, m, n$  are the direction cosines of the line, the equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

**Example 6** Find the vector and the Cartesian equations of the line through the point  $(5, 2, -4)$  and which is parallel to the vector  $3\hat{i} + 2\hat{j} - 8\hat{k}$ .

**Solution** We have

$$\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$$

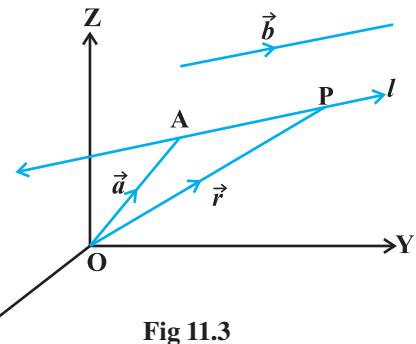


Fig 11.3

Therefore, the vector equation of the line is

$$\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$$

Now,  $\vec{r}$  is the position vector of any point  $P(x, y, z)$  on the line.

$$\begin{aligned} \text{Therefore, } x\hat{i} + y\hat{j} + z\hat{k} &= 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k}) \\ &= (5+3\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4-8\lambda)\hat{k} \end{aligned}$$

Eliminating  $\lambda$ , we get

$$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$$

which is the equation of the line in Cartesian form.

#### 11.4 Angle between Two Lines

Let  $L_1$  and  $L_2$  be two lines passing through the origin and with direction ratios  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$ , respectively. Let  $P$  be a point on  $L_1$  and  $Q$  be a point on  $L_2$ . Consider the directed lines  $OP$  and  $OQ$  as given in Fig 11.6. Let  $\theta$  be the acute angle between  $OP$  and  $OQ$ . Now recall that the directed line segments  $OP$  and  $OQ$  are vectors with components  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$ , respectively. Therefore, the angle  $\theta$  between them is given by

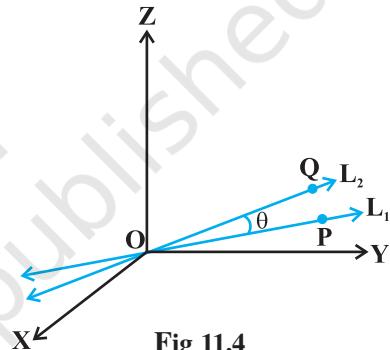


Fig 11.4

$$\cos \theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right| \quad \dots (1)$$

The angle between the lines in terms of  $\sin \theta$  is given by

$$\begin{aligned} \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ &= \sqrt{1 - \frac{(a_1 a_2 + b_1 b_2 + c_1 c_2)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}} \\ &= \frac{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1 a_2 + b_1 b_2 + c_1 c_2)^2}}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \sqrt{(a_2^2 + b_2^2 + c_2^2)}} \\ &= \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \dots (2) \end{aligned}$$

**Note** In case the lines  $L_1$  and  $L_2$  do not pass through the origin, we may take lines  $L'_1$  and  $L'_2$  which are parallel to  $L_1$  and  $L_2$  respectively and pass through the origin.

If instead of direction ratios for the lines  $L_1$  and  $L_2$ , direction cosines, namely,  $l_1, m_1, n_1$  for  $L_1$  and  $l_2, m_2, n_2$  for  $L_2$  are given, then (1) and (2) takes the following form:

$$\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| \quad (\text{as } l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2) \quad \dots (3)$$

$$\text{and} \quad \sin \theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 - (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2} \quad \dots (4)$$

Two lines with direction ratios  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  are

(i) perpendicular i.e. if  $\theta = 90^\circ$  by (1)

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

(ii) parallel i.e. if  $\theta = 0$  by (2)

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Now, we find the angle between two lines when their equations are given. If  $\theta$  is acute the angle between the lines

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \quad \text{and} \quad \vec{r} = \vec{a}_2 + \mu \vec{b}_2$$

$$\text{then} \quad \cos \theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

In Cartesian form, if  $\theta$  is the angle between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \quad \dots (1)$$

$$\text{and} \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \quad \dots (2)$$

where,  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  are the direction ratios of the lines (1) and (2), respectively, then

$$\cos \theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

**Example 7** Find the angle between the pair of lines given by

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

and  $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

**Solution** Here  $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$  and  $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

The angle  $\theta$  between the two lines is given by

$$\begin{aligned}\cos \theta &= \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| |\vec{b}_2|} \right| = \left| \frac{(\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (3\hat{i} + 2\hat{j} + 6\hat{k})}{\sqrt{1+4+4} \sqrt{9+4+36}} \right| \\ &= \left| \frac{3+4+12}{3 \times 7} \right| = \frac{19}{21}\end{aligned}$$

Hence  $\theta = \cos^{-1} \left( \frac{19}{21} \right)$

**Example 8** Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$$

and  $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

**Solution** The direction ratios of the first line are 3, 5, 4 and the direction ratios of the second line are 1, 1, 2. If  $\theta$  is the angle between them, then

$$\cos \theta = \left| \frac{3.1 + 5.1 + 4.2}{\sqrt{3^2 + 5^2 + 4^2} \sqrt{1^2 + 1^2 + 2^2}} \right| = \frac{16}{\sqrt{50} \sqrt{6}} = \frac{16}{5\sqrt{2} \sqrt{6}} = \frac{8\sqrt{3}}{15}$$

Hence, the required angle is  $\cos^{-1} \left( \frac{8\sqrt{3}}{15} \right)$ .

## 11.5 Shortest Distance between Two Lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line.

Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are *non coplanar* and are called *skew lines*. For example, let us consider a room of size 1, 3, 2 units along  $x$ ,  $y$  and  $z$ -axes respectively Fig 11.5.

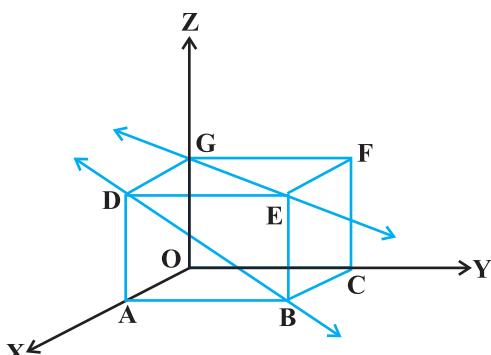


Fig 11.5

The line GE that goes diagonally across the ceiling and the line DB passes through one corner of the ceiling directly above A and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

### 11.5.1 Distance between two skew lines

We now determine the shortest distance between two skew lines in the following way: Let  $l_1$  and  $l_2$  be two skew lines with equations (Fig. 11.6)

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \quad \dots (1)$$

and

$$\vec{r} = \vec{a}_2 + \mu \vec{b}_2 \quad \dots (2)$$

Take any point S on  $l_1$  with position vector  $\vec{a}_1$  and T on  $l_2$  with position vector  $\vec{a}_2$ . Then the magnitude of the shortest distance vector will be equal to that of the projection of ST along the direction of the line of shortest distance (See 10.6.2).

If  $\overline{PQ}$  is the shortest distance vector between  $l_1$  and  $l_2$ , then it being perpendicular to both  $\vec{b}_1$  and  $\vec{b}_2$ , the unit vector  $\hat{n}$  along  $\overline{PQ}$  would therefore be

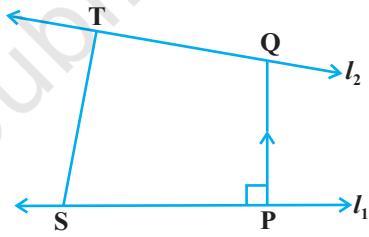


Fig 11.6

$$\hat{n} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|} \quad \dots (3)$$

Then

$$\overline{PQ} = d \hat{n}$$

where,  $d$  is the magnitude of the shortest distance vector. Let  $\theta$  be the angle between  $\overline{ST}$  and  $\overline{PQ}$ . Then

$$PQ = ST |\cos \theta|$$

But

$$\begin{aligned} \cos \theta &= \left| \frac{\overline{PQ} \cdot \overline{ST}}{|\overline{PQ}| |\overline{ST}|} \right| \\ &= \left| \frac{d \hat{n} \cdot (\vec{a}_2 - \vec{a}_1)}{d ST} \right| \quad (\text{since } \overline{ST} = \vec{a}_2 - \vec{a}_1) \\ &= \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{ST |\vec{b}_1 \times \vec{b}_2|} \right| \quad [\text{From (3)}] \end{aligned}$$

Hence, the required shortest distance is

$$d = PQ = ST |\cos \theta|$$

or

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 \times \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

### Cartesian form

The shortest distance between the lines

$$l_1: \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and

$$l_2: \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is

$$\left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}} \right|$$

#### 11.5.2 Distance between parallel lines

If two lines  $l_1$  and  $l_2$  are parallel, then they are coplanar. Let the lines be given by

$$\vec{r} = \vec{a}_1 + \lambda \vec{b} \quad \dots (1)$$

and

$$\vec{r} = \vec{a}_2 + \mu \vec{b} \quad \dots (2)$$

where,  $\vec{a}_1$  is the position vector of a point S on  $l_1$  and  $\vec{a}_2$  is the position vector of a point T on  $l_2$  Fig 11.7.

As  $l_1, l_2$  are coplanar, if the foot of the perpendicular from T on the line  $l_1$  is P, then the distance between the lines  $l_1$  and  $l_2$  =  $|TP|$ .

Let  $\theta$  be the angle between the vectors  $\vec{ST}$  and  $\vec{b}$ .

Then

$$\vec{b} \times \vec{ST} = (|\vec{b}| |\vec{ST}| \sin \theta) \hat{n} \dots \quad (3)$$

where  $\hat{n}$  is the unit vector perpendicular to the plane of the lines  $l_1$  and  $l_2$ .

But

$$\vec{ST} = \vec{a}_2 - \vec{a}_1$$

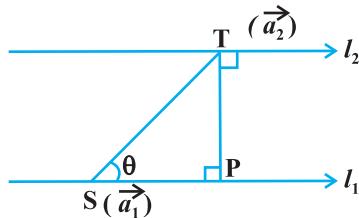


Fig 11.7

Therefore, from (3), we get

$$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = \vec{b} |\text{PT} \hat{n} \quad (\text{since PT} = \text{ST} \sin \theta)$$

i.e.,  $|\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}| \text{PT} \cdot 1 \quad (\text{as } |\hat{n}| = 1)$

Hence, the distance between the given parallel lines is

$$d = |\overline{\text{PT}}| = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

**Example 9** Find the shortest distance between the lines  $l_1$  and  $l_2$  whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \lambda (2\hat{i} - \hat{j} + \hat{k}) \quad \dots (1)$$

and

$$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu (3\hat{i} - 5\hat{j} + 2\hat{k}) \quad \dots (2)$$

**Solution** Comparing (1) and (2) with  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$  respectively,

we get

$$\vec{a}_1 = \hat{i} + \hat{j}, \quad \vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k} \quad \text{and} \quad \vec{b}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}$$

Therefore

$$\vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$$

and

$$\vec{b}_1 \times \vec{b}_2 = (2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} - 5\hat{j} + 2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

So

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9 + 1 + 49} = \sqrt{59}$$

Hence, the shortest distance between the given lines is given by

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| = \frac{|3 - 0 + 7|}{\sqrt{59}} = \frac{10}{\sqrt{59}}$$

**Example 10** Find the distance between the lines  $l_1$  and  $l_2$  given by

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda (2\hat{i} + 3\hat{j} + 6\hat{k})$$

and

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu (2\hat{i} + 3\hat{j} + 6\hat{k})$$

**Solution** The two lines are parallel (Why? ) We have

$$\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}, \vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k} \text{ and } \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Therefore, the distance between the lines is given by

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right| = \left| \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4 + 9 + 36}} \right|$$

$$\text{or } = \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{\sqrt{49}} = \frac{\sqrt{293}}{\sqrt{49}} = \frac{\sqrt{293}}{7}$$

### EXERCISE 11.2

1. Show that the three lines with direction cosines  $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$  are mutually perpendicular.
2. Show that the line through the points  $(1, -1, 2), (3, 4, -2)$  is perpendicular to the line through the points  $(0, 3, 2)$  and  $(3, 5, 6)$ .
3. Show that the line through the points  $(4, 7, 8), (2, 3, 4)$  is parallel to the line through the points  $(-1, -2, 1), (1, 2, 5)$ .
4. Find the equation of the line which passes through the point  $(1, 2, 3)$  and is parallel to the vector  $3\hat{i} + 2\hat{j} - 2\hat{k}$ .
5. Find the equation of the line in vector and in cartesian form that passes through the point with position vector  $2\hat{i} - \hat{j} + 4\hat{k}$  and is in the direction  $\hat{i} + 2\hat{j} - \hat{k}$ .
6. Find the cartesian equation of the line which passes through the point  $(-2, 4, -5)$

and parallel to the line given by  $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ .

7. The cartesian equation of a line is  $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ . Write its vector form.
8. Find the angle between the following pairs of lines:

(i)  $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$  and

$$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$$

(ii)  $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$  and

$$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$$

9. Find the angle between the following pair of lines:

(i)  $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$  and  $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii)  $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$  and  $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

10. Find the values of  $p$  so that the lines  $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$

and  $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$  are at right angles.

11. Show that the lines  $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$  and  $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$  are perpendicular to each other.

12. Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

13. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

14. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$

$$\text{and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

15. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

### Miscellaneous Exercise on Chapter 11

- Find the angle between the lines whose direction ratios are  $a, b, c$  and  $b-c, c-a, a-b$ .
- Find the equation of a line parallel to  $x$ -axis and passing through the origin.

3. If the lines  $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$  and  $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$  are perpendicular, find the value of  $k$ .
4. Find the shortest distance between lines  $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$  and  $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$ .
5. Find the vector equation of the line passing through the point  $(1, 2, -4)$  and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

### Summary

- ◆ **Direction cosines of a line** are the cosines of the angles made by the line with the positive directions of the coordinate axes.
- ◆ If  $l, m, n$  are the direction cosines of a line, then  $l^2 + m^2 + n^2 = 1$ .
- ◆ Direction cosines of a line joining two points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  are  $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$  where  $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- ◆ **Direction ratios of a line** are the numbers which are proportional to the direction cosines of a line.
- ◆ If  $l, m, n$  are the direction cosines and  $a, b, c$  are the direction ratios of a line then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}; m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}; n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- ◆ **Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.
- ◆ **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- ◆ If  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  are the direction cosines of two lines; and  $\theta$  is the acute angle between the two lines; then

$$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$$

- ◆ If  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  are the direction ratios of two lines and  $\theta$  is the acute angle between the two lines; then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- ◆ Vector equation of a line that passes through the given point whose position vector is  $\vec{a}$  and parallel to a given vector  $\vec{b}$  is  $\vec{r} = \vec{a} + \lambda \vec{b}$ .
  - ◆ Equation of a line through a point  $(x_1, y_1, z_1)$  and having direction cosines  $l, m, n$  is
- $$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$
- ◆ The vector equation of a line which passes through two points whose position vectors are  $\vec{a}$  and  $\vec{b}$  is  $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$ .
  - ◆ If  $\theta$  is the acute angle between  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ , then

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| |\vec{b}_2|} \right|$$

- ◆ If  $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$  and  $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are the equations of two lines, then the acute angle between the two lines is given by  $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$ .

- ◆ Shortest distance between two skew lines is the line segment perpendicular to both the lines.
- ◆ Shortest distance between  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$  is

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

- ◆ Shortest distance between the lines:  $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$  and

$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \text{ is}$$

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

- ◆ Distance between parallel lines  $\vec{r} = \vec{a}_1 + \lambda \vec{b}$  and  $\vec{r} = \vec{a}_2 + \mu \vec{b}$  is

$$\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$





# LINEAR PROGRAMMING

❖ *The mathematical experience of the student is incomplete if he never had the opportunity to solve a problem invented by himself. – G. POLYA* ❖

## 12.1 Introduction

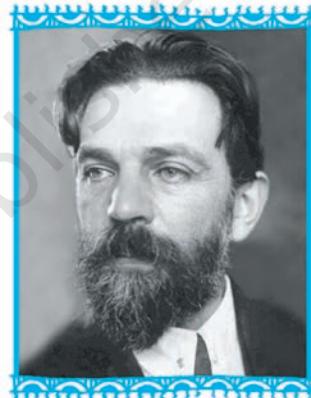
In earlier classes, we have discussed systems of linear equations and their applications in day to day problems. In Class XI, we have studied linear inequalities and systems of linear inequalities in two variables and their solutions by graphical method. Many applications in mathematics involve systems of inequalities/equations. In this chapter, we shall apply the systems of linear inequalities/equations to solve some real life problems of the type as given below:

A furniture dealer deals in only two items—tables and chairs. He has Rs 50,000 to invest and has storage space of at most 60 pieces. A table costs Rs 2500 and a chair Rs 500. He estimates that from the sale of one table, he can make a profit of Rs 250 and that from the sale of one chair a profit of Rs 75. He wants to know how many tables and chairs he should buy from the available money so as to maximise his total profit, assuming that he can sell all the items which he buys.

Such type of problems which seek to maximise (or, minimise) profit (or, cost) form a general class of problems called **optimisation problems**. Thus, an optimisation problem may involve finding maximum profit, minimum cost, or minimum use of resources etc.

A special but a very important class of optimisation problems is **linear programming problem**. The above stated optimisation problem is an example of linear programming problem. Linear programming problems are of much interest because of their wide applicability in industry, commerce, management science etc.

In this chapter, we shall study some linear programming problems and their solutions by graphical method only, though there are many other methods also to solve such problems.



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## 12.2 Linear Programming Problem and its Mathematical Formulation

We begin our discussion with the above example of furniture dealer which will further lead to a mathematical formulation of the problem in two variables. In this example, we observe

- (i) The dealer can invest his money in buying tables or chairs or combination thereof. Further he would earn different profits by following different investment strategies.
- (ii) There are certain **overriding conditions** or **constraints** viz., his investment is limited to a **maximum** of Rs 50,000 and so is his storage space which is for a maximum of 60 pieces.

Suppose he decides to buy tables only and no chairs, so he can buy  $50000 \div 2500$ , i.e., 20 tables. His profit in this case will be  $(250 \times 20)$ , i.e., **Rs 5000**.

Suppose he chooses to buy chairs only and no tables. With his capital of Rs 50,000, he can buy  $50000 \div 500$ , i.e. 100 chairs. But he can store only 60 pieces. Therefore, he is forced to buy only 60 chairs which will give him a total profit of  $(60 \times 75)$ , i.e., **Rs 4500**.

There are many other possibilities, for instance, he may choose to buy 10 tables and 50 chairs, as he can store only 60 pieces. Total profit in this case would be Rs  $(10 \times 250 + 50 \times 75)$ , i.e., **Rs 6250** and so on.

We, thus, find that the dealer can invest his money in different ways and he would earn different profits by following different investment strategies.

Now the problem is : How should he invest his money in order to get maximum profit? To answer this question, let us try to formulate the problem mathematically.

### 12.2.1 Mathematical formulation of the problem

Let  $x$  be the number of tables and  $y$  be the number of chairs that the dealer buys. Obviously,  $x$  and  $y$  must be non-negative, i.e.,

$$x \geq 0 \quad \dots (1)$$

$$y \geq 0 \quad \dots (2)$$

The dealer is constrained by the maximum amount he can invest (Here it is Rs 50,000) and by the maximum number of items he can store (Here it is 60).

Stated mathematically,

$$2500x + 500y \leq 50000 \text{ (investment constraint)}$$

$$\text{or} \quad 5x + y \leq 100 \quad \dots (3)$$

$$\text{and} \quad x + y \leq 60 \text{ (storage constraint)} \quad \dots (4)$$

The dealer wants to invest in such a way so as to maximise his profit, say,  $Z$  which is stated as a function of  $x$  and  $y$  is given by

$$Z = 250x + 75y \text{ (called } \text{objective function)} \quad \dots (5)$$

Mathematically, the given problems now reduces to:

$$\text{Maximise } Z = 250x + 75y$$

subject to the constraints:

$$5x + y \leq 100$$

$$x + y \leq 60$$

$$x \geq 0, y \geq 0$$

So, we have to maximise the linear function  $Z$  subject to certain conditions determined by a set of linear inequalities with variables as non-negative. There are also some other problems where we have to minimise a linear function subject to certain conditions determined by a set of linear inequalities with variables as non-negative. Such problems are called **Linear Programming Problems**.

Thus, a Linear Programming Problem is one that is concerned with finding the **optimal value** (maximum or minimum value) of a linear function (called **objective function**) of several variables (say  $x$  and  $y$ ), subject to the conditions that the variables are **non-negative** and satisfy a set of linear inequalities (called **linear constraints**). The term **linear** implies that all the mathematical relations used in the problem are **linear relations** while the term programming refers to the method of determining a particular **programme** or plan of action.

Before we proceed further, we now formally define some terms (which have been used above) which we shall be using in the linear programming problems:

**Objective function** Linear function  $Z = ax + by$ , where  $a, b$  are constants, which has to be maximised or minimized is called a linear **objective function**.

In the above example,  $Z = 250x + 75y$  is a linear objective function. Variables  $x$  and  $y$  are called **decision variables**.

**Constraints** The linear inequalities or equations or restrictions on the variables of a linear programming problem are called **constraints**. The conditions  $x \geq 0, y \geq 0$  are called non-negative restrictions. In the above example, the set of inequalities (1) to (4) are **constraints**.

**Optimisation problem** A problem which seeks to maximise or minimise a linear function (say of two variables  $x$  and  $y$ ) subject to certain constraints as determined by a set of linear inequalities is called an **optimisation problem**. Linear programming problems are special type of optimisation problems. The above problem of investing a

given sum by the dealer in purchasing chairs and tables is an example of an optimisation problem as well as of a linear programming problem.

We will now discuss how to find solutions to a linear programming problem. In this chapter, we will be concerned only with the graphical method.

### 12.2.2 Graphical method of solving linear programming problems

In Class XI, we have learnt how to graph a system of linear inequalities involving two variables  $x$  and  $y$  and to find its solutions graphically. Let us refer to the problem of investment in tables and chairs discussed in Section 12.2. We will now solve this problem graphically. Let us graph the constraints stated as linear inequalities:

$$5x + y \leq 100 \quad \dots (1)$$

$$x + y \leq 60 \quad \dots (2)$$

$$x \geq 0 \quad \dots (3)$$

$$y \geq 0 \quad \dots (4)$$

The graph of this system (shaded region) consists of the points common to all half planes determined by the inequalities (1) to (4) (Fig 12.1). Each point in this region represents a **feasible choice** open to the dealer for investing in tables and chairs. The region, therefore, is called the **feasible region** for the problem. Every point of this region is called a **feasible solution** to the problem. Thus, we have,

**Feasible region** The common region determined by all the constraints including non-negative constraints  $x, y \geq 0$  of a linear programming problem is called the **feasible region** (or solution region) for the problem. In Fig 12.1, the region OABC (shaded) is the feasible region for the problem. The region other than feasible region is called an **infeasible region**.

**Feasible solutions** Points within and on the boundary of the feasible region represent feasible solutions of the constraints. In Fig 12.1, every point within and on the boundary of the feasible region OABC represents feasible solution to the problem. For example, the point (10, 50) is a feasible solution of the problem and so are the points (0, 60), (20, 0) etc.

Any point outside the feasible region is called an **infeasible solution**. For example, the point (25, 40) is an infeasible solution of the problem.

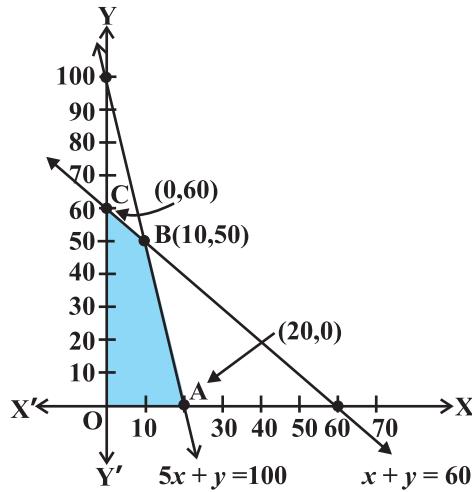


Fig 12.1

**Optimal (feasible) solution:** Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution**.

Now, we see that every point in the feasible region OABC satisfies all the constraints as given in (1) to (4), and since there are **infinitely many points**, it is not evident how we should go about finding a point that gives a maximum value of the objective function  $Z = 250x + 75y$ . To handle this situation, we use the following theorems which are fundamental in solving linear programming problems. The proofs of these theorems are beyond the scope of the book.

**Theorem 1** Let  $R$  be the feasible region (convex polygon) for a linear programming problem and let  $Z = ax + by$  be the objective function. When  $Z$  has an optimal value (maximum or minimum), where the variables  $x$  and  $y$  are subject to constraints described by linear inequalities, this optimal value must occur at a corner point\* (vertex) of the feasible region.

**Theorem 2** Let  $R$  be the feasible region for a linear programming problem, and let  $Z = ax + by$  be the objective function. If  $R$  is **bounded**\*\*, then the objective function  $Z$  has both a **maximum** and a **minimum** value on  $R$  and each of these occurs at a corner point (vertex) of  $R$ .

**Remark** If  $R$  is **unbounded**, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of  $R$ . (By Theorem 1).

In the above example, the corner points (vertices) of the bounded (feasible) region are:  $O$ ,  $A$ ,  $B$  and  $C$  and it is easy to find their coordinates as  $(0, 0)$ ,  $(20, 0)$ ,  $(10, 50)$  and  $(0, 60)$  respectively. Let us now compute the values of  $Z$  at these points.

We have

| Vertex of the Feasible Region | Corresponding value of $Z$ (in Rs) |         |
|-------------------------------|------------------------------------|---------|
| $O (0,0)$                     | 0                                  |         |
| $C (0,60)$                    | 4500                               |         |
| $B (10,50)$                   | 6250 ←                             | Maximum |
| $A (20,0)$                    | 5000                               |         |

\* A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.

\*\* A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a circle. Otherwise, it is called unbounded. Unbounded means that the feasible region does extend indefinitely in any direction.

We observe that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs.

This method of solving linear programming problem is referred as **Corner Point Method**. The method comprises of the following steps:

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Evaluate the objective function  $Z = ax + by$  at each corner point. Let  $M$  and  $m$ , respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is **bounded**,  $M$  and  $m$  are the maximum and minimum values of  $Z$ .  
(ii) In case, the feasible region is **unbounded**, we have:
  - (a)  $M$  is the maximum value of  $Z$ , if the open half plane determined by  $ax + by > M$  has no point in common with the feasible region. Otherwise,  $Z$  has no maximum value.
  - (b) Similarly,  $m$  is the minimum value of  $Z$ , if the open half plane determined by  $ax + by < m$  has no point in common with the feasible region. Otherwise,  $Z$  has no minimum value.

We will now illustrate these steps of Corner Point Method by considering some examples:

**Example 1** Solve the following linear programming problem graphically:

$$\text{Maximise } Z = 4x + y \quad \dots (1)$$

subject to the constraints:

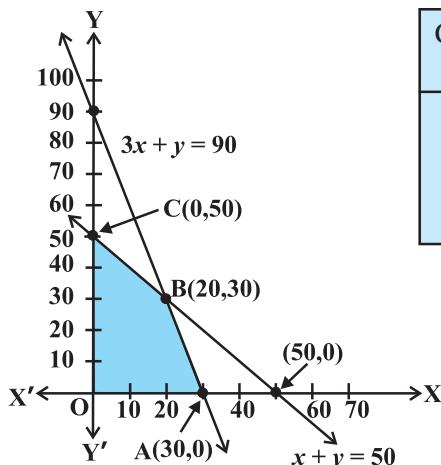
$$x + y \leq 50 \quad \dots (2)$$

$$3x + y \leq 90 \quad \dots (3)$$

$$x \geq 0, y \geq 0 \quad \dots (4)$$

**Solution** The shaded region in Fig 12.2 is the feasible region determined by the system of constraints (2) to (4). We observe that the feasible region OABC is **bounded**. So, we now use Corner Point Method to determine the maximum value of  $Z$ .

The coordinates of the corner points O, A, B and C are (0, 0), (30, 0), (20, 30) and (0, 50) respectively. Now we evaluate  $Z$  at each corner point.



| Corner Point | Corresponding value of Z |
|--------------|--------------------------|
| (0, 0)       | 0                        |
| (30, 0)      | 120 ← Maximum            |
| (20, 30)     | 110                      |
| (0, 50)      | 50                       |

Fig 12.2

Hence, maximum value of  $Z$  is 120 at the point  $(30, 0)$ .

**Example 2** Solve the following linear programming problem graphically:

$$\text{Minimise } Z = 200x + 500y \quad \dots (1)$$

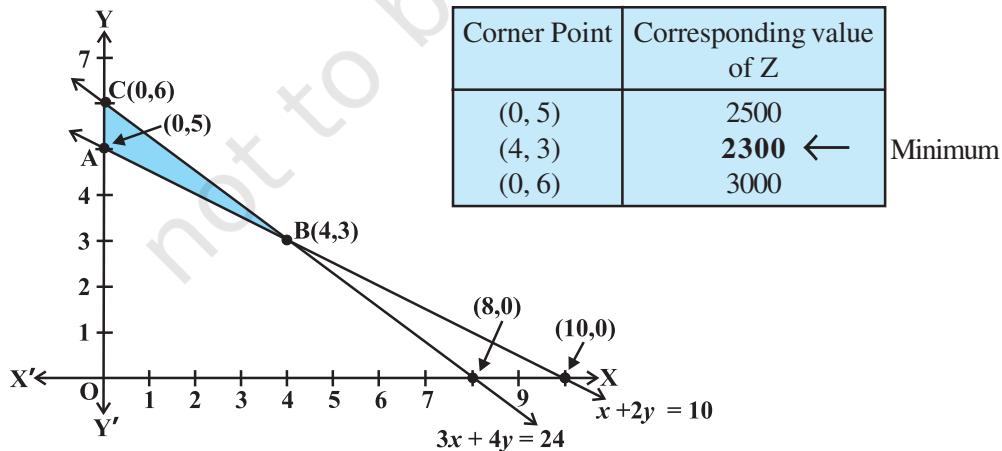
subject to the constraints:

$$x + 2y \geq 10 \quad \dots (2)$$

$$3x + 4y \leq 24 \quad \dots (3)$$

$$x \geq 0, y \geq 0 \quad \dots (4)$$

**Solution** The shaded region in Fig 12.3 is the feasible region ABC determined by the system of constraints (2) to (4), which is **bounded**. The coordinates of corner points



| Corner Point | Corresponding value of Z |
|--------------|--------------------------|
| (0, 5)       | 2500                     |
| (4, 3)       | 2300 ← Minimum           |
| (0, 6)       | 3000                     |

Fig 12.3

A, B and C are  $(0,5)$ ,  $(4,3)$  and  $(0,6)$  respectively. Now we evaluate  $Z = 200x + 500y$  at these points.

Hence, minimum value of  $Z$  is 2300 attained at the point  $(4, 3)$

**Example 3** Solve the following problem graphically:

$$\text{Minimise and Maximise } Z = 3x + 9y \quad \dots (1)$$

$$\text{subject to the constraints: } x + 3y \leq 60 \quad \dots (2)$$

$$x + y \geq 10 \quad \dots (3)$$

$$x \leq y \quad \dots (4)$$

$$x \geq 0, y \geq 0 \quad \dots (5)$$

**Solution** First of all, let us graph the feasible region of the system of linear inequalities (2) to (5). The feasible region ABCD is shown in the Fig 12.4. Note that the region is bounded. The coordinates of the corner points A, B, C and D are  $(0, 10)$ ,  $(5, 5)$ ,  $(15, 15)$  and  $(0, 20)$  respectively.

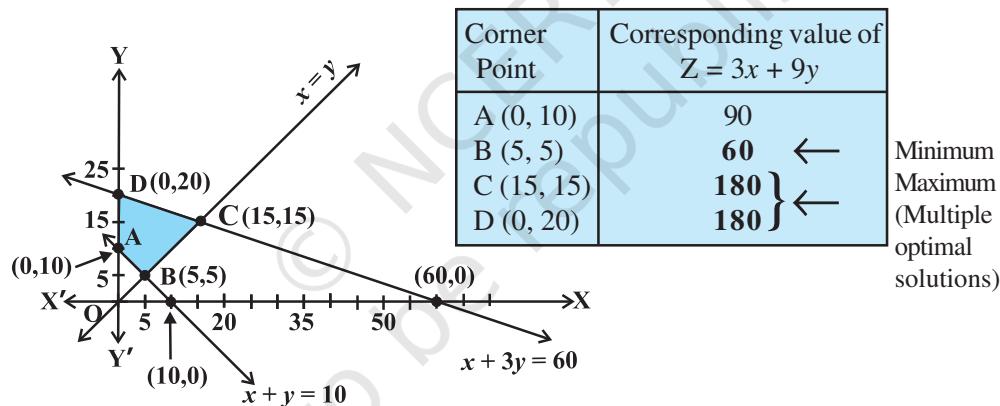


Fig 12.4

We now find the minimum and maximum value of  $Z$ . From the table, we find that the minimum value of  $Z$  is 60 at the point B  $(5, 5)$  of the feasible region.

The maximum value of  $Z$  on the feasible region occurs at the two corner points C  $(15, 15)$  and D  $(0, 20)$  and it is 180 in each case.

**Remark** Observe that in the above example, the problem has multiple optimal solutions at the corner points C and D, i.e. the both points produce same maximum value 180. In such cases, you can see that every point on the line segment CD joining the two corner points C and D also give the same maximum value. Same is also true in the case if the two points produce same minimum value.

**Example 4** Determine graphically the minimum value of the objective function

$$Z = -50x + 20y \quad \dots (1)$$

subject to the constraints:

$$2x - y \geq -5 \quad \dots (2)$$

$$3x + y \geq 3 \quad \dots (3)$$

$$2x - 3y \leq 12 \quad \dots (4)$$

$$x \geq 0, y \geq 0 \quad \dots (5)$$

**Solution** First of all, let us graph the feasible region of the system of inequalities (2) to (5). The feasible region (shaded) is shown in the Fig 12.5. Observe that the feasible region is **unbounded**.

We now evaluate  $Z$  at the corner points.

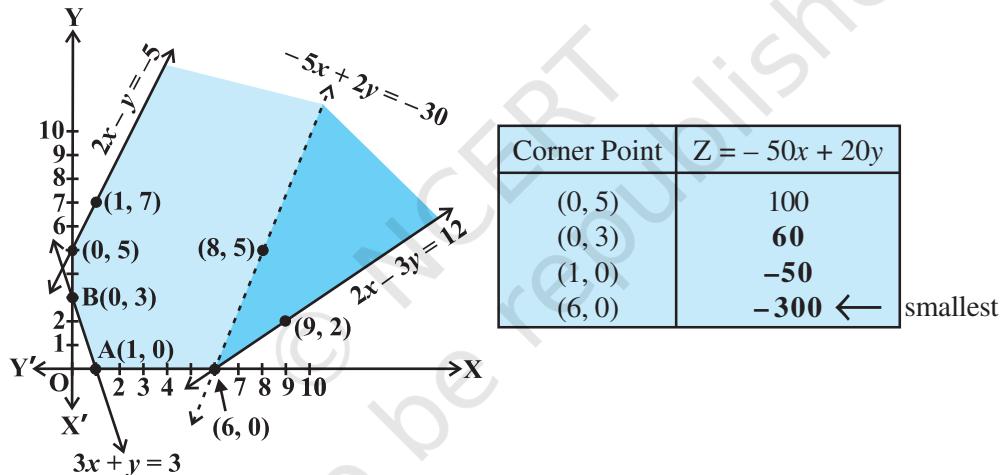


Fig 12.5

From this table, we find that  $-300$  is the smallest value of  $Z$  at the corner point  $(6, 0)$ . Can we say that minimum value of  $Z$  is  $-300$ ? Note that if the region would have been bounded, this smallest value of  $Z$  is the minimum value of  $Z$  (Theorem 2). But here we see that the feasible region is unbounded. Therefore,  $-300$  may or may not be the minimum value of  $Z$ . To decide this issue, we graph the inequality

$$-50x + 20y < -300 \text{ (see Step 3(ii) of corner Point Method.)}$$

i.e.,

$$-5x + 2y < -30$$

and check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then  $-300$  will not be the minimum value of  $Z$ . Otherwise,  $-300$  will be the minimum value of  $Z$ .

As shown in the Fig 12.5, it has common points. Therefore,  $Z = -50x + 20y$  has no minimum value subject to the given constraints.

In the above example, can you say whether  $z = -50x + 20y$  has the maximum value 100 at  $(0,5)$ ? For this, check whether the graph of  $-50x + 20y > 100$  has points in common with the feasible region. (Why?)

**Example 5** Minimise  $Z = 3x + 2y$

subject to the constraints:

$$x + y \geq 8 \quad \dots (1)$$

$$3x + 5y \leq 15 \quad \dots (2)$$

$$x \geq 0, y \geq 0 \quad \dots (3)$$

**Solution** Let us graph the inequalities (1) to (3) (Fig 12.6). Is there any feasible region? Why is so?

From Fig 12.6, you can see that there is no point satisfying all the constraints simultaneously. Thus, the problem is having no feasible region and hence no feasible solution.

**Remarks** From the examples which we have discussed so far, we notice some general features of linear programming problems:

- The feasible region is always a convex region.
- The maximum (or minimum) solution of the objective function occurs at the vertex (corner) of the feasible region. If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

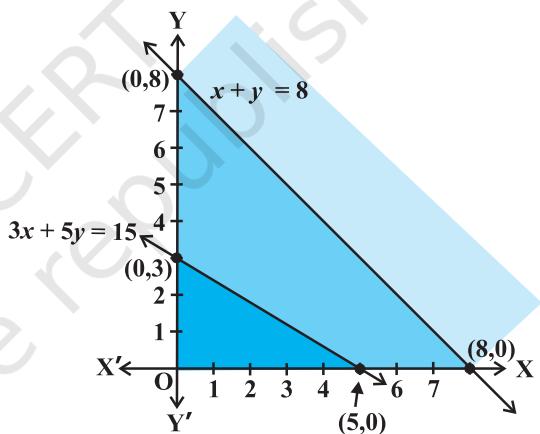


Fig 12.6

### EXERCISE 12.1

Solve the following Linear Programming Problems graphically:

- Maximise  $Z = 3x + 4y$

subject to the constraints :  $x + y \leq 4$ ,  $x \geq 0$ ,  $y \geq 0$ .

2. Minimise  $Z = -3x + 4y$

subject to  $x + 2y \leq 8$ ,  $3x + 2y \leq 12$ ,  $x \geq 0$ ,  $y \geq 0$ .

3. Maximise  $Z = 5x + 3y$

subject to  $3x + 5y \leq 15$ ,  $5x + 2y \leq 10$ ,  $x \geq 0$ ,  $y \geq 0$ .

4. Minimise  $Z = 3x + 5y$

such that  $x + 3y \geq 3$ ,  $x + y \geq 2$ ,  $x, y \geq 0$ .

5. Maximise  $Z = 3x + 2y$

subject to  $x + 2y \leq 10$ ,  $3x + y \leq 15$ ,  $x, y \geq 0$ .

6. Minimise  $Z = x + 2y$

subject to  $2x + y \geq 3$ ,  $x + 2y \geq 6$ ,  $x, y \geq 0$ .

Show that the minimum of  $Z$  occurs at more than two points.

7. Minimise and Maximise  $Z = 5x + 10y$

subject to  $x + 2y \leq 120$ ,  $x + y \geq 60$ ,  $x - 2y \geq 0$ ,  $x, y \geq 0$ .

8. Minimise and Maximise  $Z = x + 2y$

subject to  $x + 2y \geq 100$ ,  $2x - y \leq 0$ ,  $2x + y \leq 200$ ;  $x, y \geq 0$ .

9. Maximise  $Z = -x + 2y$ , subject to the constraints:

$x \geq 3$ ,  $x + y \geq 5$ ,  $x + 2y \geq 6$ ,  $y \geq 0$ .

10. Maximise  $Z = x + y$ , subject to  $x - y \leq -1$ ,  $-x + y \leq 0$ ,  $x, y \geq 0$ .

### Summary

- ◆ A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum) of a linear function of several variables (called **objective function**) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear **constraints**). Variables are sometimes called **decision variables** and are **non-negative**.

### Historical Note

In the World War II, when the war operations had to be planned to economise expenditure, maximise damage to the enemy, linear programming problems came to the forefront.

The first problem in linear programming was formulated in 1941 by the Russian mathematician, L. Kantorovich and the American economist, F. L. Hitchcock,

both of whom worked at it independently of each other. This was the well known *transportation problem*. In 1945, an English economist, G. Stigler, described yet another linear programming problem – that of determining an *optimal diet*.

In 1947, the American economist, G. B. Dantzig suggested an efficient method known as the simplex method which is an iterative procedure to solve any linear programming problem in a finite number of steps.

L. Katorovich and American mathematical economist, T. C. Koopmans were awarded the nobel prize in the year 1975 in economics for their pioneering work in linear programming. With the advent of computers and the necessary softwares, it has become possible to apply linear programming model to increasingly complex problems in many areas.



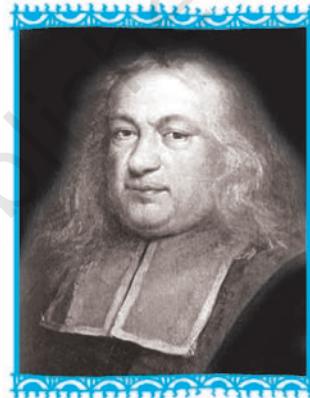


# PROBABILITY

❖ *The theory of probabilities is simply the Science of logic quantitatively treated. – C.S. PEIRCE* ❖

## 13.1 Introduction

In earlier Classes, we have studied the probability as a measure of uncertainty of events in a random experiment. We discussed the axiomatic approach formulated by Russian Mathematician, A.N. Kolmogorov (1903-1987) and treated probability as a function of outcomes of the experiment. We have also established equivalence between the axiomatic theory and the classical theory of probability in case of equally likely outcomes. On the basis of this relationship, we obtained probabilities of events associated with discrete sample spaces. We have also studied the addition rule of probability. In this chapter, we shall discuss the important concept of conditional probability of an event given that another event has occurred, which will be helpful in understanding the Bayes' theorem, multiplication rule of probability and independence of events. We shall also learn an important concept of random variable and its probability distribution and also the mean and variance of a probability distribution. In the last section of the chapter, we shall study an important discrete probability distribution called Binomial distribution. Throughout this chapter, we shall take up the experiments having equally likely outcomes, unless stated otherwise.



Pierre de Fermat  
(1601-1665)

## 13.2 Conditional Probability

Uptill now in probability, we have discussed the methods of finding the probability of events. If we have two events from the same sample space, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur.

Consider the experiment of tossing three fair coins. The sample space of the experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Since the coins are fair, we can assign the probability  $\frac{1}{8}$  to each sample point. Let

$E$  be the event ‘at least two heads appear’ and  $F$  be the event ‘first coin shows tail’. Then

$$E = \{\text{HHH, HHT, HTH, THH}\}$$

and  $F = \{\text{THH, THT, TTH, TTT}\}$

Therefore  $P(E) = P(\{\text{HHH}\}) + P(\{\text{HHT}\}) + P(\{\text{HTH}\}) + P(\{\text{THH}\})$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \text{ (Why ?)}$$

and  $P(F) = P(\{\text{THH}\}) + P(\{\text{THT}\}) + P(\{\text{TTH}\}) + P(\{\text{TTT}\})$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Also  $E \cap F = \{\text{THH}\}$

with  $P(E \cap F) = P(\{\text{THH}\}) = \frac{1}{8}$

Now, suppose we are given that the first coin shows tail, i.e.  $F$  occurs, then what is the probability of occurrence of  $E$ ? With the information of occurrence of  $F$ , we are sure that the cases in which first coin does not result into a tail should not be considered while finding the probability of  $E$ . This information reduces our sample space from the set  $S$  to its subset  $F$  for the event  $E$ . In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favourable to the occurrence of the event  $F$ .

Now, the sample point of  $F$  which is favourable to event  $E$  is  $\text{THH}$ .

Thus, Probability of  $E$  considering  $F$  as the sample space  $= \frac{1}{4}$ ,

or Probability of  $E$  given that the event  $F$  has occurred  $= \frac{1}{4}$

This probability of the event  $E$  is called the *conditional probability of  $E$  given that  $F$  has already occurred*, and is denoted by  $P(E|F)$ .

Thus  $P(E|F) = \frac{1}{4}$

Note that the elements of  $F$  which favour the event  $E$  are the common elements of  $E$  and  $F$ , i.e. the sample points of  $E \cap F$ .

Thus, we can also write the conditional probability of E given that F has occurred as

$$\begin{aligned} P(E|F) &= \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F} \\ &= \frac{n(E \cap F)}{n(F)} \end{aligned}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that  $P(E|F)$  can also be written as

$$P(E|F) = \frac{\frac{n(E \cap F)}{n(S)}}{\frac{n(F)}{n(S)}} = \frac{P(E \cap F)}{P(F)} \dots (1)$$

Note that (1) is valid only when  $P(F) \neq 0$  i.e.,  $F \neq \emptyset$  (Why?)

Thus, we can define the conditional probability as follows :

**Definition 1** If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e.  $P(E|F)$  is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \text{ provided } P(F) \neq 0$$

### 13.2.1 Properties of conditional probability

Let E and F be events of a sample space S of an experiment, then we have

**Property 1**  $P(S|F) = P(F|F) = 1$

We know that

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Also  $P(F|F) = \frac{P(F \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$

Thus  $P(S|F) = P(F|F) = 1$

**Property 2** If A and B are any two events of a sample space S and F is an event of S such that  $P(F) \neq 0$ , then

$$P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$$

In particular, if A and B are disjoint events, then

$$P((A \cup B)|F) = P(A|F) + P(B|F)$$

We have

$$\begin{aligned} P((A \cup B)|F) &= \frac{P[(A \cup B) \cap F]}{P(F)} \\ &= \frac{P[(A \cap F) \cup (B \cap F)]}{P(F)} \\ &\quad \text{(by distributive law of union of sets over intersection)} \\ &= \frac{P(A \cap F) + P(B \cap F) - P(A \cap B \cap F)}{P(F)} \\ &= \frac{P(A \cap F)}{P(F)} + \frac{P(B \cap F)}{P(F)} - \frac{P[(A \cap B) \cap F]}{P(F)} \\ &= P(A|F) + P(B|F) - P((A \cap B)|F) \end{aligned}$$

When A and B are disjoint events, then

$$P((A \cap B)|F) = 0$$

$$\Rightarrow P((A \cup B)|F) = P(A|F) + P(B|F)$$

**Property 3**  $P(E'|F) = 1 - P(E|F)$

From Property 1, we know that  $P(S|F) = 1$

$$\Rightarrow P(E \cup E'|F) = 1 \quad \text{since } S = E \cup E'$$

$$\Rightarrow P(E|F) + P(E'|F) = 1 \quad \text{since } E \text{ and } E' \text{ are disjoint events}$$

$$\text{Thus, } P(E'|F) = 1 - P(E|F)$$

Let us now take up some examples.

**Example 1** If  $P(A) = \frac{7}{13}$ ,  $P(B) = \frac{9}{13}$  and  $P(A \cap B) = \frac{4}{13}$ , evaluate  $P(A|B)$ .

**Solution** We have  $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{13}}{\frac{9}{13}} = \frac{4}{9}$

**Example 2** A family has two children. What is the probability that both the children are boys given that at least one of them is a boy ?

**Solution** Let  $b$  stand for boy and  $g$  for girl. The sample space of the experiment is

$$S = \{(b, b), (g, b), (b, g), (g, g)\}$$

Let  $E$  and  $F$  denote the following events :

$E$  : 'both the children are boys'

$F$  : 'at least one of the child is a boy'

Then

$$E = \{(b, b)\} \text{ and } F = \{(b, b), (g, b), (b, g)\}$$

Now

$$E \cap F = \{(b, b)\}$$

Thus

$$P(F) = \frac{3}{4} \text{ and } P(E \cap F) = \frac{1}{4}$$

Therefore

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

**Example 3** Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

**Solution** Let  $A$  be the event 'the number on the card drawn is even' and  $B$  be the event 'the number on the card drawn is greater than 3'. We have to find  $P(A|B)$ .

Now, the sample space of the experiment is  $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Then

$$A = \{2, 4, 6, 8, 10\}, B = \{4, 5, 6, 7, 8, 9, 10\}$$

and

$$A \cap B = \{4, 6, 8, 10\}$$

Also

$$P(A) = \frac{5}{10}, P(B) = \frac{7}{10} \text{ and } P(A \cap B) = \frac{4}{10}$$

Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{10}}{\frac{7}{10}} = \frac{4}{7}$$

**Example 4** In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in Class XII given that the chosen student is a girl?

**Solution** Let  $E$  denote the event that a student chosen randomly studies in Class XII and  $F$  be the event that the randomly chosen student is a girl. We have to find  $P(E|F)$ .

Now  $P(F) = \frac{430}{1000} = 0.43$  and  $P(E \cap F) = \frac{43}{1000} = 0.043$  (Why?)

Then  $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.043}{0.43} = 0.1$

**Example 5** A die is thrown three times. Events A and B are defined as below:

A : 4 on the third throw

B : 6 on the first and 5 on the second throw

Find the probability of A given that B has already occurred.

**Solution** The sample space has 216 outcomes.

Now  $A = \{(1,1,4), (1,2,4), \dots, (1,6,4), (2,1,4), (2,2,4), \dots, (2,6,4), (3,1,4), (3,2,4), \dots, (3,6,4), (4,1,4), (4,2,4), \dots, (4,6,4), (5,1,4), (5,2,4), \dots, (5,6,4), (6,1,4), (6,2,4), \dots, (6,6,4)\}$

and  $B = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$   
 $A \cap B = \{(6,5,4)\}$ .

Now  $P(B) = \frac{6}{216}$  and  $P(A \cap B) = \frac{1}{216}$

Then  $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{216}}{\frac{6}{216}} = \frac{1}{6}$

**Example 6** A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

**Solution** Let E be the event that ‘number 4 appears at least once’ and F be the event that ‘the sum of the numbers appearing is 6’.

Then,  $E = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (5,4), (6,4)\}$

and  $F = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

We have  $P(E) = \frac{11}{36}$  and  $P(F) = \frac{5}{36}$

Also  $E \cap F = \{(2,4), (4,2)\}$

Therefore  $P(E \cap F) = \frac{2}{36}$

Hence, the required probability

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

For the conditional probability discussed above, we have considered the elementary events of the experiment to be equally likely and the corresponding definition of the probability of an event was used. However, the same definition can also be used in the general case where the elementary events of the sample space are not equally likely, the probabilities  $P(E \cap F)$  and  $P(F)$  being calculated accordingly. Let us take up the following example.

**Example 7** Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Find the conditional probability of the event that ‘the die shows a number greater than 4’ given that ‘there is at least one tail’.

**Solution** The outcomes of the experiment can be represented in following diagrammatic manner called the ‘tree diagram’.

The sample space of the experiment may be described as

$$S = \{(H,H), (H,T), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

where  $(H, H)$  denotes that both the tosses result into head and  $(T, i)$  denote the first toss result into a tail and the number  $i$  appeared on the die for  $i = 1, 2, 3, 4, 5, 6$ .

Thus, the probabilities assigned to the 8 elementary events

$$(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)$$

are  $\frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$  respectively which is clear from the Fig 13.2.

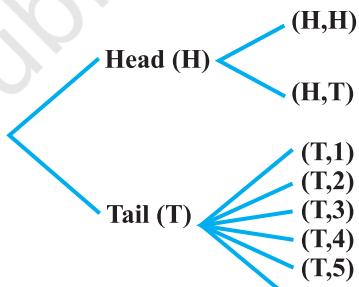


Fig 13.1

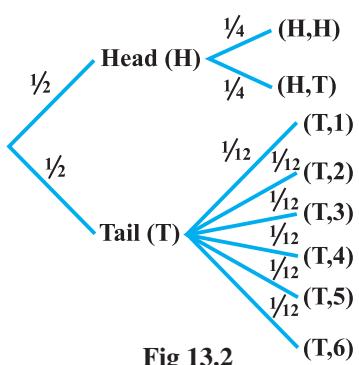


Fig 13.2

Let  $F$  be the event that ‘there is at least one tail’ and  $E$  be the event ‘the die shows a number greater than 4’. Then

$$F = \{(H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$$

$$E = \{(T, 5), (T, 6)\} \text{ and } E \cap F = \{(T, 5), (T, 6)\}$$

$$\begin{aligned} \text{Now } P(F) &= P(\{(H, T)\}) + P(\{(T, 1)\}) + P(\{(T, 2)\}) + P(\{(T, 3)\}) \\ &\quad + P(\{(T, 4)\}) + P(\{(T, 5)\}) + P(\{(T, 6)\}) \\ &= \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4} \end{aligned}$$

$$\text{and } P(E \cap F) = P(\{(T, 5)\}) + P(\{(T, 6)\}) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$\text{Hence } P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{2}{9}$$

### EXERCISE 13.1

- Given that  $E$  and  $F$  are events such that  $P(E) = 0.6$ ,  $P(F) = 0.3$  and  $P(E \cap F) = 0.2$ , find  $P(E|F)$  and  $P(F|E)$
  - Compute  $P(A|B)$ , if  $P(B) = 0.5$  and  $P(A \cap B) = 0.32$
  - If  $P(A) = 0.8$ ,  $P(B) = 0.5$  and  $P(B|A) = 0.4$ , find
    - $P(A \cap B)$
    - $P(A|B)$
    - $P(A \cup B)$
  - Evaluate  $P(A \cup B)$ , if  $2P(A) = P(B) = \frac{5}{13}$  and  $P(A|B) = \frac{2}{5}$
  - If  $P(A) = \frac{6}{11}$ ,  $P(B) = \frac{5}{11}$  and  $P(A \cup B) = \frac{7}{11}$ , find
    - $P(A \cap B)$
    - $P(A|B)$
    - $P(B|A)$
- Determine  $P(E|F)$  in Exercises 6 to 9.
- A coin is tossed three times, where
    - $E$  : head on third toss ,  $F$  : heads on first two tosses
    - $E$  : at least two heads ,  $F$  : at most two heads
    - $E$  : at most two tails ,  $F$  : at least one tail

7. Two coins are tossed once, where  
(i) E : tail appears on one coin, F : one coin shows head  
(ii) E : no tail appears, F : no head appears

8. A die is thrown three times,  
E : 4 appears on the third toss, F : 6 and 5 appears respectively on first two tosses

9. Mother, father and son line up at random for a family picture  
E : son on one end, F : father in middle

10. A black and a red dice are rolled.  
(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.  
(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

11. A fair die is rolled. Consider events  $E = \{1,3,5\}$ ,  $F = \{2,3\}$  and  $G = \{2,3,4,5\}$   
Find  
(i)  $P(E|F)$  and  $P(F|E)$  (ii)  $P(E|G)$  and  $P(G|E)$   
(iii)  $P((E \cup F)|G)$  and  $P((E \cap F)|G)$

12. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that  
(i) the youngest is a girl, (ii) at least one is a girl?

13. An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

14. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

15. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

In each of the Exercises 16 and 17 choose the correct answer:

16. If  $P(A) = \frac{1}{2}$ ,  $P(B) = 0$ , then  $P(A|B)$  is

(A) 0 (B)  $\frac{1}{2}$   
(C) not defined (D) 1

17. If A and B are events such that  $P(A|B) = P(B|A)$ , then

  - (A)  $A \subset B$  but  $A \neq B$
  - (B)  $A = B$
  - (C)  $A \cap B = \emptyset$
  - (D)  $P(A) = P(B)$

### 13.3 Multiplication Theorem on Probability

Let  $E$  and  $F$  be two events associated with a sample space  $S$ . Clearly, the set  $E \cap F$  denotes the event that both  $E$  and  $F$  have occurred. In other words,  $E \cap F$  denotes the simultaneous occurrence of the events  $E$  and  $F$ . The event  $E \cap F$  is also written as  $EF$ .

Very often we need to find the probability of the event EF. For example, in the experiment of drawing two cards one after the other, we may be interested in finding the probability of the event 'a king and a queen'. The probability of event EF is obtained by using the conditional probability as obtained below :

We know that the conditional probability of event E given that F has occurred is denoted by  $P(E|F)$  and is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$$

From this result, we can write

$$P(E \cap F) = P(F) \cdot P(E|F) \quad \dots (1)$$

Also, we know that

$$P(F|E) = \frac{P(F \cap E)}{P(E)}, P(E) \neq 0$$

$$P(F|E) = \frac{P(E \cap F)}{P(E)} \text{ (since } E \cap F = F \cap E\text{)}$$

Thus

$$P(E \cap F) = P(E) \cdot P(F|E) \quad \dots \quad (2)$$

Combining (1) and (2), we find that

$$P(E \cap F) = P(E) \cdot P(F|E)$$

$\equiv P(E) P(E|F)$  provided  $P(E) \neq 0$  and  $P(F) \neq 0$ .

The above result is known as the *multiplication rule of probability*.

Let us now take up an example

**Example 8** An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

**Solution** Let E and F denote respectively the events that first and second ball drawn are black. We have to find  $P(E \cap F)$  or  $P(EF)$ .

Now  $P(E) = P(\text{black ball in first draw}) = \frac{10}{15}$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e.  $P(F|E) = \frac{9}{14}$

By multiplication rule of probability, we have

$$\begin{aligned} P(E \cap F) &= P(E) P(F|E) \\ &= \frac{10}{15} \times \frac{9}{14} = \frac{3}{7} \end{aligned}$$

**Multiplication rule of probability for more than two events** If E, F and G are three events of sample space, we have

$$P(E \cap F \cap G) = P(E) P(F|E) P(G|(E \cap F)) = P(E) P(F|E) P(G|EF)$$

Similarly, the multiplication rule of probability can be extended for four or more events.

The following example illustrates the extension of multiplication rule of probability for three events.

**Example 9** Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

**Solution** Let K denote the event that the card drawn is king and A be the event that the card drawn is an ace. Clearly, we have to find  $P(KKA)$

Now  $P(K) = \frac{4}{52}$

Also,  $P(K|K)$  is the probability of second king with the condition that one king has already been drawn. Now there are three kings in  $(52 - 1) = 51$  cards.

Therefore  $P(K|K) = \frac{3}{51}$

Lastly,  $P(A|KK)$  is the probability of third drawn card to be an ace, with the condition that two kings have already been drawn. Now there are four aces in left 50 cards.

Therefore

$$P(A|KK) = \frac{4}{50}$$

By multiplication law of probability, we have

$$\begin{aligned} P(KKA) &= P(K) \cdot P(K|K) \cdot P(A|KK) \\ &= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525} \end{aligned}$$

### 13.4 Independent Events

Consider the experiment of drawing a card from a deck of 52 playing cards, in which the elementary events are assumed to be equally likely. If E and F denote the events 'the card drawn is a spade' and 'the card drawn is an ace' respectively, then

$$P(E) = \frac{13}{52} = \frac{1}{4} \text{ and } P(F) = \frac{4}{52} = \frac{1}{13}$$

Also E and F is the event 'the card drawn is the ace of spades' so that

$$P(E \cap F) = \frac{1}{52}$$

Hence

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{52}}{\frac{1}{13}} = \frac{1}{4}$$

Since  $P(E) = \frac{1}{4} = P(E|F)$ , we can say that the occurrence of event F has not affected the probability of occurrence of the event E.

We also have

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{1}{52}}{\frac{1}{4}} = \frac{1}{13} = P(F)$$

Again,  $P(F) = \frac{1}{13} = P(F|E)$  shows that occurrence of event E has not affected the probability of occurrence of the event F.

Thus, E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other.

Such events are called *independent events*.

**Definition 2** Two events E and F are said to be independent, if

$$P(F|E) = P(F) \text{ provided } P(E) \neq 0$$

and

$$P(E|F) = P(E) \text{ provided } P(F) \neq 0$$

Thus, in this definition we need to have  $P(E) \neq 0$  and  $P(F) \neq 0$

Now, by the multiplication rule of probability, we have

$$P(E \cap F) = P(E) \cdot P(F|E) \quad \dots (1)$$

If E and F are independent, then (1) becomes

$$P(E \cap F) = P(E) \cdot P(F) \quad \dots (2)$$

Thus, using (2), the independence of two events is also defined as follows:

**Definition 3** Let E and F be two events associated with the same random experiment, then E and F are said to be independent if

$$P(E \cap F) = P(E) \cdot P(F)$$

### Remarks

(i) Two events E and F are said to be dependent if they are not independent, i.e. if

$$P(E \cap F) \neq P(E) \cdot P(F)$$

(ii) Sometimes there is a confusion between independent events and mutually exclusive events. Term ‘independent’ is defined in terms of ‘probability of events’ whereas mutually exclusive is defined in term of events (subset of sample space). Moreover, mutually exclusive events never have an outcome common, but independent events, may have common outcome. Clearly, ‘independent’ and ‘mutually exclusive’ do not have the same meaning.

In other words, two independent events having nonzero probabilities of occurrence can not be mutually exclusive, and conversely, i.e. two mutually exclusive events having nonzero probabilities of occurrence can not be independent.

(iii) Two experiments are said to be independent if for every pair of events E and F, where E is associated with the first experiment and F with the second experiment, the probability of the simultaneous occurrence of the events E and F when the two experiments are performed is the product of  $P(E)$  and  $P(F)$  calculated separately on the basis of two experiments, i.e.,  $P(E \cap F) = P(E) \cdot P(F)$

(iv) Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A) P(B)$$

$$P(A \cap C) = P(A) P(C)$$

$$P(B \cap C) = P(B) P(C)$$

and

$$P(A \cap B \cap C) = P(A) P(B) P(C)$$

If at least one of the above is not true for three given events, we say that the events are not independent.

**Example 10** A die is thrown. If E is the event ‘the number appearing is a multiple of 3’ and F be the event ‘the number appearing is even’ then find whether E and F are independent?

**Solution** We know that the sample space is  $S = \{1, 2, 3, 4, 5, 6\}$

Now  $E = \{3, 6\}$ ,  $F = \{2, 4, 6\}$  and  $E \cap F = \{6\}$

Then  $P(E) = \frac{2}{6} = \frac{1}{3}$ ,  $P(F) = \frac{3}{6} = \frac{1}{2}$  and  $P(E \cap F) = \frac{1}{6}$

Clearly  $P(E \cap F) = P(E) \cdot P(F)$

Hence E and F are independent events.

**Example 11** An unbiased die is thrown twice. Let the event A be ‘odd number on the first throw’ and B the event ‘odd number on the second throw’. Check the independence of the events A and B.

**Solution** If all the 36 elementary events of the experiment are considered to be equally likely, we have

$$P(A) = \frac{18}{36} = \frac{1}{2} \text{ and } P(B) = \frac{18}{36} = \frac{1}{2}$$

Also  $P(A \cap B) = P(\text{odd number on both throws})$

$$= \frac{9}{36} = \frac{1}{4}$$

Now  $P(A) P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Clearly  $P(A \cap B) = P(A) \times P(B)$

Thus, A and B are independent events

**Example 12** Three coins are tossed simultaneously. Consider the event E ‘three heads or three tails’, F ‘at least two heads’ and G ‘at most two heads’. Of the pairs (E,F), (E,G) and (F,G), which are independent? which are dependent?

**Solution** The sample space of the experiment is given by

$$S = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}, \text{HTT}, \text{THT}, \text{TTH}, \text{TTT}\}$$

Clearly  $E = \{\text{HHH}, \text{TTT}\}$ ,  $F = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\}$

and

$$G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Also

$$E \cap F = \{HHH\}, E \cap G = \{TTT\}, F \cap G = \{HHT, HTH, THH\}$$

Therefore

$$P(E) = \frac{2}{8} = \frac{1}{4}, P(F) = \frac{4}{8} = \frac{1}{2}, P(G) = \frac{7}{8}$$

and

$$P(E \cap F) = \frac{1}{8}, P(E \cap G) = \frac{1}{8}, P(F \cap G) = \frac{3}{8}$$

Also

$$P(E) \cdot P(F) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}, P(E) \cdot P(G) = \frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$$

and

$$P(F) \cdot P(G) = \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$$

Thus

$$P(E \cap F) = P(E) \cdot P(F)$$

$$P(E \cap G) \neq P(E) \cdot P(G)$$

and

$$P(F \cap G) \neq P(F) \cdot P(G)$$

Hence, the events (E and F) are independent, and the events (E and G) and (F and G) are dependent.

**Example 13** Prove that if E and F are independent events, then so are the events E and F'.

**Solution** Since E and F are independent, we have

$$P(E \cap F) = P(E) \cdot P(F) \quad \dots(1)$$

From the venn diagram in Fig 13.3, it is clear that  $E \cap F$  and  $E \cap F'$  are mutually exclusive events and also  $E = (E \cap F) \cup (E \cap F')$ .

Therefore  $P(E) = P(E \cap F) + P(E \cap F')$

$$\begin{aligned} \text{or } P(E \cap F') &= P(E) - P(E \cap F) \\ &= P(E) - P(E) \cdot P(F) \\ &\quad (\text{by (1)}) \\ &= P(E) (1 - P(F)) \\ &= P(E) \cdot P(F') \end{aligned}$$

Hence, E and F' are independent

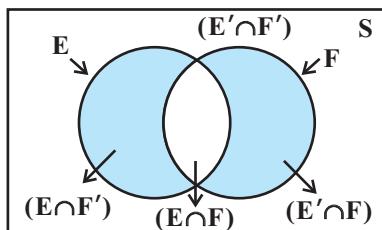


Fig 13.3

 **Note** In a similar manner, it can be shown that if the events E and F are independent, then

- (a)  $E'$  and  $F$  are independent,
- (b)  $E'$  and  $F'$  are independent

**Example 14** If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by  $1 - P(A') P(B')$

**Solution** We have

$$\begin{aligned}
 P(\text{at least one of A and B}) &= P(A \cup B) \\
 &= P(A) + P(B) - P(A \cap B) \\
 &= P(A) + P(B) - P(A) P(B) \\
 &= P(A) + P(B) [1 - P(A)] \\
 &= P(A) + P(B) \cdot P(A') \\
 &= 1 - P(A') + P(B) P(A') \\
 &= 1 - P(A') [1 - P(B)] \\
 &= 1 - P(A') P(B')
 \end{aligned}$$

### EXERCISE 13.2

1. If  $P(A) = \frac{3}{5}$  and  $P(B) = \frac{1}{5}$ , find  $P(A \cap B)$  if A and B are independent events.
2. Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.
3. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.
4. A fair coin and an unbiased die are tossed. Let A be the event ‘head appears on the coin’ and B be the event ‘3 on the die’. Check whether A and B are independent events or not.
5. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, ‘the number is even,’ and B be the event, ‘the number is red’. Are A and B independent?
6. Let E and F be events with  $P(E) = \frac{3}{5}$ ,  $P(F) = \frac{3}{10}$  and  $P(E \cap F) = \frac{1}{5}$ . Are E and F independent?

7. Given that the events A and B are such that  $P(A) = \frac{1}{2}$ ,  $P(A \cup B) = \frac{3}{5}$  and  $P(B) = p$ . Find  $p$  if they are (i) mutually exclusive (ii) independent.
8. Let A and B be independent events with  $P(A) = 0.3$  and  $P(B) = 0.4$ . Find
- (i)  $P(A \cap B)$
  - (ii)  $P(A \cup B)$
  - (iii)  $P(A|B)$
  - (iv)  $P(B|A)$
9. If A and B are two events such that  $P(A) = \frac{1}{4}$ ,  $P(B) = \frac{1}{2}$  and  $P(A \cap B) = \frac{1}{8}$ , find  $P(\text{not } A \text{ and not } B)$ .
10. Events A and B are such that  $P(A) = \frac{1}{2}$ ,  $P(B) = \frac{7}{12}$  and  $P(\text{not } A \text{ or not } B) = \frac{1}{4}$ . State whether A and B are independent ?
11. Given two independent events A and B such that  $P(A) = 0.3$ ,  $P(B) = 0.6$ . Find
- (i)  $P(A \text{ and } B)$
  - (ii)  $P(A \text{ and not } B)$
  - (iii)  $P(A \text{ or } B)$
  - (iv)  $P(\text{neither } A \text{ nor } B)$
12. A die is tossed thrice. Find the probability of getting an odd number at least once.
13. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that
- (i) both balls are red.
  - (ii) first ball is black and second is red.
  - (iii) one of them is black and other is red.
14. Probability of solving specific problem independently by A and B are  $\frac{1}{2}$  and  $\frac{1}{3}$  respectively. If both try to solve the problem independently, find the probability that
- (i) the problem is solved
  - (ii) exactly one of them solves the problem.
15. One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?
- (i) E : 'the card drawn is a spade'  
F : 'the card drawn is an ace'
  - (ii) E : 'the card drawn is black'  
F : 'the card drawn is a king'
  - (iii) E : 'the card drawn is a king or queen'  
F : 'the card drawn is a queen or jack'.

16. In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

  - Find the probability that she reads neither Hindi nor English newspapers.
  - If she reads Hindi newspaper, find the probability that she reads English newspaper.
  - If she reads English newspaper, find the probability that she reads Hindi newspaper.

Choose the correct answer in Exercises 17 and 18.

17. The probability of obtaining an even prime number on each die, when a pair of dice is rolled is



18. Two events A and B will be independent, if

- (A) A and B are mutually exclusive
  - (B)  $P(A'B') = [1 - P(A)] [1 - P(B)]$
  - (C)  $P(A) = P(B)$
  - (D)  $P(A) + P(B) = 1$

## 13.5 Bayes' Theorem

Consider that there are two bags I and II. Bag I contains 2 white and 3 red balls and Bag II contains 4 white and 5 red balls. One ball is drawn at random from one of the

bags. We can find the probability of selecting any of the bags (i.e.  $\frac{1}{2}$ ) or probability of drawing a ball of a particular colour (say white) from a particular bag (say Bag I). In other words, we can find the probability that the ball drawn is of a particular colour, if we are given the bag from which the ball is drawn. But, can we find the probability that the ball drawn is from a particular bag (say Bag II), if the colour of the ball drawn is given? Here, we have to find the reverse probability of Bag II to be selected when an event occurred after it is known. Famous mathematician, John Bayes' solved the problem of finding reverse probability by using conditional probability. The formula developed by him is known as '*Bayes theorem*' which was published posthumously in 1763. Before stating and proving the Bayes' theorem, let us first take up a definition and some preliminary results.

### 13.5.1 Partition of a sample space

A set of events  $E_1, E_2, \dots, E_n$  is said to represent a partition of the sample space  $S$  if

- (a)  $E_i \cap E_j = \emptyset, i \neq j, i, j = 1, 2, 3, \dots, n$

- (b)  $E_1 \cup E_2 \cup \dots \cup E_n = S$  and  
 (c)  $P(E_i) > 0$  for all  $i = 1, 2, \dots, n$ .

In other words, the events  $E_1, E_2, \dots, E_n$  represent a partition of the sample space  $S$  if they are pairwise disjoint, exhaustive and have nonzero probabilities.

As an example, we see that any nonempty event  $E$  and its complement  $E'$  form a partition of the sample space  $S$  since they satisfy  $E \cap E' = \emptyset$  and  $E \cup E' = S$ .

From the Venn diagram in Fig 13.3, one can easily observe that if  $E$  and  $F$  are any two events associated with a sample space  $S$ , then the set  $\{E \cap F', E \cap F, E' \cap F, E' \cap F'\}$  is a partition of the sample space  $S$ . It may be mentioned that the partition of a sample space is not unique. There can be several partitions of the same sample space.

We shall now prove a theorem known as *Theorem of total probability*.

### 13.5.2 Theorem of total probability

Let  $\{E_1, E_2, \dots, E_n\}$  be a partition of the sample space  $S$ , and suppose that each of the events  $E_1, E_2, \dots, E_n$  has nonzero probability of occurrence. Let  $A$  be any event associated with  $S$ , then

$$\begin{aligned} P(A) &= P(E_1) P(A|E_1) + P(E_2) P(A|E_2) + \dots + P(E_n) P(A|E_n) \\ &= \sum_{j=1}^n P(E_j) P(A|E_j) \end{aligned}$$

**Proof** Given that  $E_1, E_2, \dots, E_n$  is a partition of the sample space  $S$  (Fig 13.4). Therefore,

$$S = E_1 \cup E_2 \cup \dots \cup E_n \quad \dots (1)$$

and  $E_i \cap E_j = \emptyset, i \neq j, i, j = 1, 2, \dots, n$

Now, we know that for any event  $A$ ,

$$\begin{aligned} A &= A \cap S \\ &= A \cap (E_1 \cup E_2 \cup \dots \cup E_n) \\ &= (A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n) \end{aligned}$$

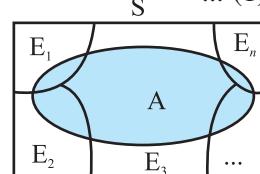


Fig 13.4

Also  $A \cap E_i$  and  $A \cap E_j$  are respectively the subsets of  $E_i$  and  $E_j$ . We know that  $E_i$  and  $E_j$  are disjoint, for  $i \neq j$ , therefore,  $A \cap E_i$  and  $A \cap E_j$  are also disjoint for all  $i \neq j, i, j = 1, 2, \dots, n$ .

$$\begin{aligned} \text{Thus, } P(A) &= P[(A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n)] \\ &= P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n) \end{aligned}$$

Now, by multiplication rule of probability, we have

$$P(A \cap E_i) = P(E_i) P(A|E_i) \text{ as } P(E_i) \neq 0 \forall i = 1, 2, \dots, n$$

Therefore,

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \dots + P(E_n)P(A|E_n)$$

or

$$P(A) = \sum_{j=1}^n P(E_j)P(A|E_j)$$

**Example 15** A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

**Solution** Let A be the event that the construction job will be completed on time, and B be the event that there will be a strike. We have to find P(A).

We have

$$P(B) = 0.65, P(\text{no strike}) = P(B') = 1 - P(B) = 1 - 0.65 = 0.35$$

$$P(A|B) = 0.32, P(A|B') = 0.80$$

Since events B and B' form a partition of the sample space S, therefore, by theorem on total probability, we have

$$\begin{aligned} P(A) &= P(B)P(A|B) + P(B')P(A|B') \\ &= 0.65 \times 0.32 + 0.35 \times 0.8 \\ &= 0.208 + 0.28 = 0.488 \end{aligned}$$

Thus, the probability that the construction job will be completed in time is 0.488.

We shall now state and prove the Bayes' theorem.

**Bayes' Theorem** If  $E_1, E_2, \dots, E_n$  are  $n$  non empty events which constitute a partition of sample space S, i.e.  $E_1, E_2, \dots, E_n$  are pairwise disjoint and  $E_1 \cup E_2 \cup \dots \cup E_n = S$  and A is any event of nonzero probability, then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^n P(E_j)P(A|E_j)} \quad \text{for any } i = 1, 2, 3, \dots, n$$

**Proof** By formula of conditional probability, we know that

$$\begin{aligned} P(E_i|A) &= \frac{P(A \cap E_i)}{P(A)} \\ &= \frac{P(E_i)P(A|E_i)}{P(A)} \quad (\text{by multiplication rule of probability}) \\ &= \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^n P(E_j)P(A|E_j)} \quad (\text{by the result of theorem of total probability}) \end{aligned}$$

**Remark** The following terminology is generally used when Bayes' theorem is applied.

The events  $E_1, E_2, \dots, E_n$  are called *hypotheses*.

The probability  $P(E_i)$  is called the *priori probability* of the hypothesis  $E_i$ .

The conditional probability  $P(E_i|A)$  is called *a posteriori probability* of the hypothesis  $E_i$ .

Bayes' theorem is also called the formula for the probability of "causes". Since the  $E_i$ 's are a partition of the sample space  $S$ , one and only one of the events  $E_i$  occurs (i.e. one of the events  $E_i$  must occur and only one can occur). Hence, the above formula gives us the probability of a particular  $E_i$  (i.e. a "Cause"), given that the event  $A$  has occurred.

The Bayes' theorem has its applications in variety of situations, few of which are illustrated in following examples.

**Example 16** Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

**Solution** Let  $E_1$  be the event of choosing the bag I,  $E_2$  the event of choosing the bag II and  $A$  be the event of drawing a red ball.

Then  $P(E_1) = P(E_2) = \frac{1}{2}$

Also  $P(A|E_1) = P(\text{drawing a red ball from Bag I}) = \frac{3}{7}$

and  $P(A|E_2) = P(\text{drawing a red ball from Bag II}) = \frac{5}{11}$

Now, the probability of drawing a ball from Bag II, being given that it is red, is  $P(E_2|A)$

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)} = \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}} = \frac{35}{68}$$

**Example 17** Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

**Solution** Let  $E_1$ ,  $E_2$  and  $E_3$  be the events that boxes I, II and III are chosen, respectively.

Then  $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then  $P(A|E_1) = P(\text{a gold coin from bag I}) = \frac{2}{2} = 1$

$$P(A|E_2) = P(\text{a gold coin from bag II}) = 0$$

$$P(A|E_3) = P(\text{a gold coin from bag III}) = \frac{1}{2}$$

Now, the probability that the other coin in the box is of gold

$$\begin{aligned} &= \text{the probability that gold coin is drawn from the box I.} \\ &= P(E_1|A) \end{aligned}$$

By Bayes' theorem, we know that

$$\begin{aligned} P(E_1|A) &= \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)} \\ &= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3} \end{aligned}$$

**Example 18** Suppose that the reliability of a HIV test is specified as follows:

Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV-ive but 1% are diagnosed as showing HIV+ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV+ive. What is the probability that the person actually has HIV?

**Solution** Let E denote the event that the person selected is actually having HIV and A the event that the person's HIV test is diagnosed as +ive. We need to find  $P(E|A)$ .

Also  $E'$  denotes the event that the person selected is actually not having HIV.

Clearly,  $\{E, E'\}$  is a partition of the sample space of all people in the population. We are given that

$$P(E) = 0.1\% = \frac{0.1}{100} = 0.001$$

$$P(E') = 1 - P(E) = 0.999$$

$P(A|E) = P(\text{Person tested as HIV+ive given that he/she is actually having HIV})$

$$= 90\% = \frac{90}{100} = 0.9$$

and

$P(A|E') = P(\text{Person tested as HIV+ive given that he/she is actually not having HIV})$

$$= 1\% = \frac{1}{100} = 0.01$$

Now, by Bayes' theorem

$$\begin{aligned} P(E|A) &= \frac{P(E)P(A|E)}{P(E)P(A|E) + P(E')P(A|E')} \\ &= \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089} \\ &= 0.083 \text{ approx.} \end{aligned}$$

Thus, the probability that a person selected at random is actually having HIV given that he/she is tested HIV+ive is 0.083.

**Example 19** In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

**Solution** Let events  $B_1, B_2, B_3$  be the following :

$B_1$  : the bolt is manufactured by machine A

$B_2$  : the bolt is manufactured by machine B

$B_3$  : the bolt is manufactured by machine C

Clearly,  $B_1, B_2, B_3$  are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space.

Let the event E be 'the bolt is defective'.

The event E occurs with  $B_1$  or with  $B_2$  or with  $B_3$ . Given that,

$$P(B_1) = 25\% = 0.25, P(B_2) = 0.35 \text{ and } P(B_3) = 0.40$$

Again  $P(E|B_1) = \text{Probability that the bolt drawn is defective given that it is manufactured by machine A} = 5\% = 0.05$

Similarly,  $P(E|B_2) = 0.04, P(E|B_3) = 0.02$ .

Hence, by Bayes' Theorem, we have

$$\begin{aligned}
 P(B_2|E) &= \frac{P(B_2)P(E|B_2)}{P(B_1)P(E|B_1)+P(B_2)P(E|B_2)+P(B_3)P(E|B_3)} \\
 &= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} \\
 &= \frac{0.0140}{0.0345} = \frac{28}{69}
 \end{aligned}$$

**Example 20** A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport

are respectively  $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$  and  $\frac{2}{5}$ . The probabilities that he will be late are  $\frac{1}{4}, \frac{1}{3}$ , and  $\frac{1}{12}$ , if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

**Solution** Let  $E$  be the event that the doctor visits the patient late and let  $T_1, T_2, T_3, T_4$  be the events that the doctor comes by train, bus, scooter, and other means of transport respectively.

Then  $P(T_1) = \frac{3}{10}, P(T_2) = \frac{1}{5}, P(T_3) = \frac{1}{10}$  and  $P(T_4) = \frac{2}{5}$  (given)

$P(E|T_1) =$  Probability that the doctor arriving late comes by train =  $\frac{1}{4}$

Similarly,  $P(E|T_2) = \frac{1}{3}, P(E|T_3) = \frac{1}{12}$  and  $P(E|T_4) = 0$ , since he is not late if he comes by other means of transport.

Therefore, by Bayes' Theorem, we have

$P(T_1|E) =$  Probability that the doctor arriving late comes by train

$$\begin{aligned}
 &= \frac{P(T_1)P(E|T_1)}{P(T_1)P(E|T_1)+P(T_2)P(E|T_2)+P(T_3)P(E|T_3)+P(T_4)P(E|T_4)} \\
 &= \frac{\frac{3}{10} \times \frac{1}{4}}{\frac{3}{10} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{10} \times \frac{1}{12} + \frac{2}{5} \times 0} = \frac{\frac{3}{40} \times \frac{120}{18}}{\frac{2}{2}} = \frac{1}{2}
 \end{aligned}$$

Hence, the required probability is  $\frac{1}{2}$ .

**Example 21** A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

**Solution** Let  $E$  be the event that the man reports that six occurs in the throwing of the die and let  $S_1$  be the event that six occurs and  $S_2$  be the event that six does not occur.

Then  $P(S_1) = \text{Probability that six occurs} = \frac{1}{6}$

$P(S_2) = \text{Probability that six does not occur} = \frac{5}{6}$

$P(E|S_1) = \text{Probability that the man reports that six occurs when six has actually occurred on the die}$

$= \text{Probability that the man speaks the truth} = \frac{3}{4}$

$P(E|S_2) = \text{Probability that the man reports that six occurs when six has not actually occurred on the die}$

$= \text{Probability that the man does not speak the truth} = 1 - \frac{3}{4} = \frac{1}{4}$

Thus, by Bayes' theorem, we get

$P(S_1|E) = \text{Probability that the report of the man that six has occurred is actually a six}$

$$= \frac{P(S_1)P(E|S_1)}{P(S_1)P(E|S_1) + P(S_2)P(E|S_2)}$$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{1}{8} \times \frac{24}{8} = \frac{3}{8}$$

Hence, the required probability is  $\frac{3}{8}$ .

**Remark** A random variable is a real valued function whose domain is the sample space of a random experiment.

For example, let us consider the experiment of tossing a coin two times in succession.

The sample space of the experiment is  $S = \{HH, HT, TH, TT\}$ .

If  $X$  denotes the number of heads obtained, then  $X$  is a random variable and for each outcome, its value is as given below :

$$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.$$

More than one random variables can be defined on the same sample space. For example, let  $Y$  denote the number of heads minus the number of tails for each outcome of the above sample space  $S$ .

Then  $Y(HH) = 2, Y(HT) = 0, Y(TH) = 0, Y(TT) = -2$ .

Thus,  $X$  and  $Y$  are two different random variables defined on the same sample space  $S$ .

### EXERCISE 13.3

1. An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?
2. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.
3. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostler?
4. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let  $\frac{3}{4}$  be the probability that he knows the answer and  $\frac{1}{4}$  be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability  $\frac{1}{4}$ . What is the probability that the student knows the answer given that he answered it correctly?
5. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population

actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

6. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?
7. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?
8. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?
9. Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.
10. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?
11. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?
12. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.
13. Probability that A speaks truth is  $\frac{4}{5}$ . A coin is tossed. A reports that a head appears. The probability that actually there was head is

- (A)  $\frac{4}{5}$  (B)  $\frac{1}{2}$  (C)  $\frac{1}{5}$  (D)  $\frac{2}{5}$

14. If A and B are two events such that  $A \subset B$  and  $P(B) \neq 0$ , then which of the following is correct?

- (A)  $P(A|B) = \frac{P(B)}{P(A)}$  (B)  $P(A|B) < P(A)$   
 (C)  $P(A|B) \geq P(A)$  (D) None of these

### Miscellaneous Examples

**Example 22** Coloured balls are distributed in four boxes as shown in the following table:

| Box | Colour |       |     |      |
|-----|--------|-------|-----|------|
|     | Black  | White | Red | Blue |
| I   | 3      | 4     | 5   | 6    |
| II  | 2      | 2     | 2   | 2    |
| III | 1      | 2     | 3   | 1    |
| IV  | 4      | 3     | 1   | 5    |

A box is selected at random and then a ball is randomly drawn from the selected box. The colour of the ball is black, what is the probability that ball drawn is from the box III?

**Solution** Let A,  $E_1$ ,  $E_2$ ,  $E_3$  and  $E_4$  be the events as defined below :

A : a black ball is selected  $E_1$  : box I is selected

$E_2$  : box II is selected  $E_3$  : box III is selected

$E_4$  : box IV is selected

Since the boxes are chosen at random,

Therefore  $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

Also  $P(A|E_1) = \frac{3}{18}$ ,  $P(A|E_2) = \frac{2}{8}$ ,  $P(A|E_3) = \frac{1}{7}$  and  $P(A|E_4) = \frac{4}{13}$

$P(\text{box III is selected, given that the drawn ball is black}) = P(E_3|A)$ . By Bayes' theorem,

$$\begin{aligned}
 P(E_3|A) &= \frac{P(E_3) \cdot P(A|E_3)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3) + P(E_4)P(A|E_4)} \\
 &= \frac{\frac{1}{4} \times \frac{1}{7}}{\frac{1}{4} \times \frac{3}{18} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{7} + \frac{1}{4} \times \frac{4}{13}} = 0.165
 \end{aligned}$$

**Example 23** A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

**Solution** Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus,  $P(S) = \frac{1}{6}$ ,  $P(F) = \frac{5}{6}$

$$P(\text{A wins in the first throw}) = P(S) = \frac{1}{6}$$

A gets the third throw, when the first throw by A and second throw by B result into failures.

$$\begin{aligned}
 \text{Therefore, } P(\text{A wins in the 3rd throw}) &= P(FFS) = P(F)P(F)P(S) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \\
 &= \left(\frac{5}{6}\right)^2 \times \frac{1}{6}
 \end{aligned}$$

$$P(\text{A wins in the 5th throw}) = P(\text{FFFFS}) = \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) \text{ and so on.}$$

$$\begin{aligned}
 \text{Hence, } P(\text{A wins}) &= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) + \dots \\
 &= \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}
 \end{aligned}$$

$$P(\text{B wins}) = 1 - P(\text{A wins}) = 1 - \frac{6}{11} = \frac{5}{11}$$

**Remark** If  $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ , where  $|r| < 1$ , then sum of this infinite G.P.

is given by  $\frac{a}{1-r}$ . (Refer A.1.3 of Class XI Text book).

**Example 24** If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly setup.

**Solution** Let A be the event that the machine produces 2 acceptable items.

Also let  $B_1$  represent the event of correct set up and  $B_2$  represent the event of incorrect setup.

Now  $P(B_1) = 0.8, P(B_2) = 0.2$

$$P(A|B_1) = 0.9 \times 0.9 \text{ and } P(A|B_2) = 0.4 \times 0.4$$

Therefore 
$$P(B_1|A) = \frac{P(B_1) P(A|B_1)}{P(B_1) P(A|B_1) + P(B_2) P(A|B_2)}$$

$$= \frac{0.8 \times 0.9 \times 0.9}{0.8 \times 0.9 \times 0.9 + 0.2 \times 0.4 \times 0.4} = \frac{648}{680} = 0.95$$

### Miscellaneous Exercise on Chapter 13

1. A and B are two events such that  $P(A) \neq 0$ . Find  $P(B|A)$ , if
  - (i) A is a subset of B
  - (ii)  $A \cap B = \emptyset$
2. A couple has two children,
  - (i) Find the probability that both children are males, if it is known that at least one of the children is male.
  - (ii) Find the probability that both children are females, if it is known that the elder child is a female.
3. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.
4. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?
5. If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?
6. Suppose we have four boxes A, B, C and D containing coloured marbles as given below:

| Box | Marble colour |       |       |
|-----|---------------|-------|-------|
|     | Red           | White | Black |
| A   | 1             | 6     | 3     |
| B   | 6             | 2     | 2     |
| C   | 8             | 1     | 1     |
| D   | 0             | 6     | 4     |

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A?, box B?, box C?

- Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?
- If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assume that the individual entries of the determinant are chosen independently, each value being

assumed with probability  $\frac{1}{2}$ ).

- An electronic assembly consists of two subsystems, say, A and B. From previous testing procedures, the following probabilities are assumed to be known:

$$P(A \text{ fails}) = 0.2$$

$$P(B \text{ fails alone}) = 0.15$$

$$P(A \text{ and } B \text{ fail}) = 0.15$$

Evaluate the following probabilities

$$(i) P(A \text{ fails} | B \text{ has failed}) \quad (ii) P(A \text{ fails alone})$$

- Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Choose the correct answer in each of the following:

- If A and B are two events such that  $P(A) \neq 0$  and  $P(B | A) = 1$ , then  
 (A)  $A \subset B$       (B)  $B \subset A$       (C)  $B = \emptyset$       (D)  $A = \emptyset$

12. If  $P(A|B) > P(A)$ , then which of the following is correct :
- (A)  $P(B|A) < P(B)$  (B)  $P(A \cap B) < P(A) \cdot P(B)$   
 (C)  $P(B|A) > P(B)$  (D)  $P(B|A) = P(B)$
13. If A and B are any two events such that  $P(A) + P(B) - P(A \text{ and } B) = P(A)$ , then
- (A)  $P(B|A) = 1$  (B)  $P(A|B) = 1$   
 (C)  $P(B|A) = 0$  (D)  $P(A|B) = 0$

### Summary

The salient features of the chapter are –

- ◆ The conditional probability of an event E, given the occurrence of the event F is given by  $P(E|F) = \frac{P(E \cap F)}{P(F)}$ ,  $P(F) \neq 0$
- ◆  $0 \leq P(E|F) \leq 1$ ,  $P(E'|F) = 1 - P(E|F)$
- ◆  $P((E \cup F)|G) = P(E|G) + P(F|G) - P((E \cap F)|G)$
- ◆  $P(E \cap F) = P(E) P(F|E)$ ,  $P(E) \neq 0$   
 $P(E \cap F) = P(F) P(E|F)$ ,  $P(F) \neq 0$
- ◆ If E and F are independent, then  
 $P(E \cap F) = P(E) P(F)$   
 $P(E|F) = P(E)$ ,  $P(F) \neq 0$   
 $P(F|E) = P(F)$ ,  $P(E) \neq 0$
- ◆ **Theorem of total probability**

Let  $\{E_1, E_2, \dots, E_n\}$  be a partition of a sample space and suppose that each of  $E_1, E_2, \dots, E_n$  has nonzero probability. Let A be any event associated with S, then

$$P(A) = P(E_1) P(A|E_1) + P(E_2) P(A|E_2) + \dots + P(E_n) P(A|E_n)$$

- ◆ **Bayes' theorem** If  $E_1, E_2, \dots, E_n$  are events which constitute a partition of sample space S, i.e.  $E_1, E_2, \dots, E_n$  are pairwise disjoint and  $E_1 \cup E_2 \cup \dots \cup E_n = S$  and A be any event with nonzero probability, then

$$P(E_i | A) = \frac{P(E_i) P(A|E_i)}{\sum_{j=1}^n P(E_j) P(A|E_j)}$$

### Historical Note

The earliest indication on measurement of chances in game of dice appeared in 1477 in a commentary on Dante's Divine Comedy. A treatise on gambling named *liber de Ludo Alcae*, by Geronimo Carden (1501-1576) was published posthumously in 1663. In this treatise, he gives the number of favourable cases for each event when two dice are thrown.

Galileo (1564-1642) gave casual remarks concerning the correct evaluation of chance in a game of three dice. Galileo analysed that when three dice are thrown, the sum of the number that appear is more likely to be 10 than the sum 9, because the number of cases favourable to 10 are more than the number of cases for the appearance of number 9.

Apart from these early contributions, it is generally acknowledged that the true origin of the science of probability lies in the correspondence between two great men of the seventeenth century, Pascal (1623-1662) and Pierre de Fermat (1601-1665). A French gambler, Chevalier de Metre asked Pascal to explain some seeming contradiction between his theoretical reasoning and the observation gathered from gambling. In a series of letters written around 1654, Pascal and Fermat laid the first foundation of science of probability. Pascal solved the problem in algebraic manner while Fermat used the method of combinations.

Great Dutch Scientist, Huygens (1629-1695), became acquainted with the content of the correspondence between Pascal and Fermat and published a first book on probability, "*De Ratiociniis in Ludo Aleae*" containing solution of many interesting rather than difficult problems on probability in games of chances.

The next great work on probability theory is by Jacob Bernoulli (1654-1705), in the form of a great book, "*Ars Conjectandi*" published posthumously in 1713 by his nephew, Nicholes Bernoulli. To him is due the discovery of one of the most important probability distribution known as Binomial distribution. The next remarkable work on probability lies in 1993. A. N. Kolmogorov (1903-1987) is credited with the axiomatic theory of probability. His book, 'Foundations of probability' published in 1933, introduces probability as a set function and is considered a 'classic!'.

