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RELATIONS

Introduction :-

Consider the following example:

$A = \{$ Mohan, Sohan, David, Karim $\}$

$B = \{$ Rita, Marry, Fatima $\}$

Suppose Rita has two brothers Mohan & Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R from A to B is a subset of the cartesian product $A \times B$. Suppose R is a relation from A to B, then it is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

Every ordered pair is written as aRb and read as 'a is related to b'.

R is binary relation from A to B since the elements of set R are ordered pairs

Ex, If $A = \{1, 2, 5\}$ and $B = \{2, 4\}$

then $A \times B = \{(1,2), (1,4), (2,2), (2,4), (5,2), (5,4)\}$

If R is defined as $x<y$, then $R = \{(1,2), (1,4), (2,4)\}$

Domain and Range:-

The set $\{a \in A : (a,b) \in R$ for some $b \in B\}$ is called the domain of R and denoted by Dom(R).

The set $\{b \in B : (a,b) \in R$ for some $a \in A\}$ is called the range of R and denoted by Ran(R).

Thus, the domain of a relation R is the set of all the first element of the ordered pairs which belong to R and range of R is set of second element.

Total no. of distinct relation from set A to set B:-

Let no. of elements of A & B be m and n respectively. Then the no. of elements of $A \times B$ is mn. Therefore, the no. of elements of the power set $A \times B$ is 2^{mn}. Thus $A \times B$ has 2^{mn} different subsets. Now every subset of $A \times B$ is a relation from $A \times B$. Hence, the no. of different relations A to B is 2^{mn}.

Some operations on sets:-

Some binary relations are set of ordered pairs, all set operations can be done on relations. The resulting set contains ordered pairs and are therefore relations. If R and S denote two relations, then $R \cap S$ known as intersection of R & S.

$$x(R \cap S)y = xRy \wedge xSy$$

Similarly $R \cup S$.

$$x(R \cup S)y = xRy \cup xSy$$
$$x(R-S)y = xRy - xSy$$
$$x(R')y = x \not R y.$$

Ex If $A = \{x, y, z\}$ $B = \{X, Y, Z\}$ $C = \{x, y\}$ $D = \{Y, Z\}$

$R : A \to B = \{(x, X), (x, Y), (y, Z)\}$
$S : C \to D = \{(x, Y), (y, Z)\}$

Find R', $R \cup S$, $R \cap S$ & $R - S$

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$A \times B = \{(x,X), (x,Y), (x,Z), (y,X), (y,Y), (y,Z), (z,X), (z,Y), (z,Z)\}$

$R' = \{(x,Z), (y,X), (y,Y), (z,X), (z,Y), (z,Z)\}$

$R \cup S = \{(x,X), (x,Y), (y,Z)\}$

$R \cap S = \{(x,Y), (y,Z)\}$

$R - S = \{(x,X)\}$

Types of Relation in a set :-

① **Inverse Relation :-**

Let R be any relation from a set A to set B. The inverse of R, denoted by R^{-1} is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is

$R^{-1} = \{(b,a) : (a,b) \in R\}$ $\qquad x R y$

$(R^{-1})^{-1} = R$ $\qquad y R^{-1} x$

② **Identity relation :-**

A relation R in a set is said to be identity relation, generally denoted by I_A, if $I_A = \{(x,x), x \in A\}$

Ex, $A = \{1,2,3\}$

then $I_A = \{(1,1), (2,2), (3,3)\}$

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3. ## n-ary relation :-

Let $\{A_1, A_2, \dots A_n\}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \dots \times A_n$ is c/d n-ary relation on $A_1, A_2, \dots, A_n$

(i) If $R = \emptyset$ then R is called void or empty relation

(ii) If $R = A_1 \times A_2 \times \dots \times A_n$ then R is c/d universal relation

(iii) If $A_i = A$ for i, then R is called an n-ary relation on A.

(iv) For n = 1, 2 or 3, R is c/d unary, binary or ternary relation respectively.

Properties of Relation :-

A relation R on set A satisfies certain properties. These properties are defined as follows:-

1) **Reflexive Property**

A relation R on a set A is reflexive if aRa for every $a \in A$, that is, if $(a,a) \in R$ for every $a \in A$.

Ex. a) $R = \{(1,1), (2,2), (1,2), (3,2), (3,3)\}$ be a relation on $A = \{1,2,3\}$ then R_1 is c/d reflexive.

b) If $R_2 = \{(1,1), (2,2), (2,3), (3,3)\}$ on $A = \{1,2,3\}$ then R_2 is not reflexive since $2 \in A, (2,2) \notin R_2$

c) $R_3 = \{(x,y) \in R^2 : x \leq y\}$ is reflexive relation

2. Irreflexive Relation :- A relation R on a set A is irreflexive if, for every $a \in A$, $(a,a) \notin R$. For ex,

a) $R_1 = \{(1,2), (1,3), (2,1), (2,3)\}$ on $A = \{1,2,3\}$ is irreflexive since $(x,x) \notin R_1$

3. Non Reflexive Relation :- A relation R on a set A is non reflexive if R is neither reflexive nor irreflexive i.e. if aRa is true for some a and false for others

Ex $R = \{(1,2), (2,3), (2,2), (3,1)\}$ on $A = \{1,2,3\}$ is non-reflexive.

4. Symmetric Relation :- A relation R on a set A is symmetric if whenever $(a,b) \in R$ then $(b,a) \in R$ i.e. $\Rightarrow aRb \Rightarrow bRa$.

Eg. $A = \{a,b,c\}$

$R = \{(a,b), (b,a), (a,c), (c,a), (b,c), (c,b)\}$

or

$R = \{(a,b), (b,a)\}$

$R = \{(a,b), (b,a), (a,c), (c,a)\}$ are all ~~reflexive~~ symmetric

→ It is possible for a relation having only 1 or odd no. of ordered pair to be symmetric.

E.g. $\{(a,a)\}$, $\{(a,a), (b,b), (c,c)\}$

→ $R = \{(a,a), (b,b)\}$ is symmetric as well as reflexive if $A = \{a,b\}$

→ All reflexive relations are not symmetric.

→ Empty relation can be symmetric but not reflexive.

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5. **Antisymmetric relation**: A binary relation R defined on a set A is called anti symmetric if $(a,b) \in R \Rightarrow (b,a) \notin R$ unless $a=b$.

In other words, R is antisymmetric under a condition if (a,b) and (b,a) are present in R then $a=b$.

e.g.:- $A = \{a,b,c\}$

$R_1 = \{(a,a)\}$ anti symmetric

$R_2 = \{(a,a),(b,b),(a,b)\}$

$R_3 = \{(a,a),(b,b),(c,c),(a;c)\}$

Note:- we can also define a relation which can be both symmetric and anti-symmetric.

6. **Transitive**:- A binary relation R defined as transitive if $(a,c) \in R$ whenever (a,b) and $(b,c) \in R$.

$A = \{a,b,c\}$

$R_1 = \{(a,a)\}$ $R_2 = \{(a,a),(b,b)\}$

$R_3 = \{(a,b),(a,c)\}$

$R_4 = \{(a,c),(c,a),(a,a);(c,c)\}$

$R_5 = \{(a,b)\}$

Note:- Union of two transitive relation is not transitive

$R_1 = \{(a,b)\}$

$R_2 = \{(b,c)\}$

$R_1 \cup R_2 = \{(a,b)(b,c)\}$ not transitive

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Q1. Let $A = \{2, 4, 6\}$ and $B = \{1, 4, 5, 6\}$ then find out the relation from A to B defined by " is less than or equal to". Find out the domain & range of the relation

Sol^n.

$$A \times B = \{(2,1), (2,4), (2,5), (2,6), (4,1), (4,4), (4,5), (4,6), (6,1), (6,4), (6,5), (6,6)\}$$

$$R = \{(x,y) : x \in A \text{ and } y \in B,\ x \leq y\}$$

$$R = \{(2,4), (2,5), (2,6), (4,4), (4,5), (4,6), (6,6)\}$$

$$d(R) = \{x : x \in A \text{ and } (x,y) \in R\}$$
$$= \{2, 4, 6\}$$

$$R(R) = \{y : y \in B \text{ . } (x,y) \in R\} = \{4, 5, 6\}.$$

Q2. Let $A = \{a, b\}$ and A^2 is the set of all words of length 2.

a) Find the elements of A^2

$$A^2 = \{(a,a), (a,b), (b,a), (b,b)\}$$

b) The relation R on A^2 is defined $xRy \Rightarrow$ first letter in x is same as first letter in y when $x, y \in A^2$, write R as set of ordered pairs.

$$R = \{(aa, ab), (ab, ab), (ba, bb), (bb, ba)\} \{ab, aa), (aa, aa), (bb, bb), (ba, ba)\}$$

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Q3. Let R be relation from X to Y, where $X = \{1,2,3\}$ and $Y = \{8,9\}$.

$$R = \{(1,8), (2,8), (1,9), (3,9)\}$$

Find complement of relation R.

Sol.ⁿ

$$X \times Y = \{(1,8), (1,9), (2,8), (2,9), (3,8), (3,9)\}$$

$$\bar{R} = \{(2,9), (3,8)\}$$

Q4. Find R^{-1} to relation R on A defined "$x+y$" divisible by 2. For $A = \{1,2,3,4,6\}$

Sol.ⁿ

$$R = \{(1,1), (2,2), (3,3), (4,4), (6,6), (1,3), (2,4), (2,6), (3,1), (4,2), (4,6), (6,2), (6,4)\}$$

$$R^{-1} = \{(1,1), (2,2), (3,3), (4,4), (6,6), (3,1), (4,2), (6,2), (1,3), (2,4), (6,4), (2,6), (4,6)\}$$

Q5. Give an example of a relation which is:

$A = \{1,2,3\}$

(i) reflexive & transitive but not symmetric;

$$R_1 = \{(1,1), (2,2), (3,3), (1,3)\}$$

(ii) Symmetric, transitive but not reflexive

$$R_2 = \{(1,1), (3,3), (1,3), (3,1)\}$$

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(iii) reflexive & symmetric but not transitive;

$$R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$$

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Ques

1. Let $A = \{2,5,7\}$ and R be a relation defined as aRb ($a \neq b$) if and only if a divides b.

$R = \varnothing$. $A \times A$ is a void relation

2. Let Z be set of all integers, then the property 'x is an even integer' can be characterized as a relation which is unary. Thus, the relation $R = \{x \in Z;\ x \text{ is an even}\}$ is unary.

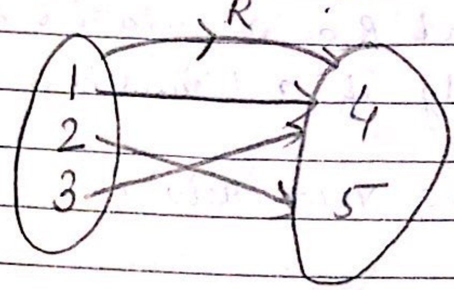
3. Let $A = \{1,2,5,8\}$ and let R be the relation defined by the property 'x' is less than y then
$R = \{(1,2), (1,5), (1,8)\ (2,5), (2,8), (5,8)\}$ is binary.

4. Let $A = \{1,3,5\}$ R defined as '$x+y$ less than z',
$R = \{(1,3,5), (1,1,3), (1,1,5), (3,1,5)\}$
is ternary relation

Binary Relation :-

A relation between two sets (same or different)
A subset of cartesian product of two sets is called a binary relation.

$$R \subseteq (A \times B)$$



Ques: A set A contain 3 elements, how many possible binary relation can be defined on A.

$|A \times A| = 9$ elements

$\therefore$ No. of possible binary relations $= 2^9$ No. of subsets

Note $\emptyset$, empty set is an empty relation

Ques If $|A| = 10$, then how many reflexive relations are possible?

Total no. of binary relation $= 2^{10^2}$

Minimum 10 ordered pairs of (a,a) is required

$\therefore$ Reflexive relation $= 2^{100-10} = 2^{90}$.

$$\boxed{\text{Reflexive relation of a set having cardinality } 2^{n^2-n}.}$$

For a relation to be reflexive all the ordered pairs (i,i) for $i = 1, 2, 3, \ldots, n$ must be present. The remaining $n^2 - n$ can either be present or not.

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Ques If $|A| = 10$, how many symmetric relations are possible?

$A = \{1, 2, \ldots 10\}$

$A \times A = \{ \underbrace{(1,1), (2,2), \ldots (10,10)}_{\text{10 ordered pairs}}, (1,2), (2,1) \ldots \}$

remaining 90 ordered pairs out of which 45 ordered pairs are available

$R = \{(a,b)(b,a)\}$

Now = $A \times A$ contains 55 elements

$\therefore$ Total no. of relations which are symmetric $= 2^{55}$.

$2^{55} = 2^{10 + \left(\frac{100-10}{2}\right)}$

$$\boxed{\begin{array}{l} |A| = n \\ \text{No. of symmetric relation} = 2^{n + \left(\frac{n^2-n}{2}\right)} \\ \qquad = 2^{\frac{n^2+n}{2}} \end{array}}$$

Ques A set contains 10 elements, how many relations are both reflexive and symmetric.

$A = \{1, 2\}$

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

$2^4 = 16$

$A \times A = \{ (1,1), (2,2) \ldots (10,10) \quad (1,2), (2,1) \ldots \}$

10 mandatory — 45 ordered pairs

$\boxed{|R| = 2^{\frac{n^2-n}{2}}}$ No. of relations which are both reflexive and symmetric.

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Equivalence Relation:-

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric and transitive. That is, R is an equivalence relation if it has the following properties:

1. $(a,a) \in R$ for all $a \in R$ (reflexive)
2. $(a,b) \in R \Rightarrow (b,a) \in R$ (symmetric)
3. $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ (transitive)

Ex. If R be a relation in the set of integers Z defined by $R = \{(x,y) : x \in Z, y \in Z, (x-y)$ is divisible by $6\}$. Then prove that R is an equivalence relation.

Solⁿ

Let $x \in Z$, Then $x - x = 0$ and 0 is divisible by 6.

Therefore, xRx for all $x \in Z$ Here we can say R is reflexive.

Again, $xRy \Rightarrow (x-y)$ is divisible by 6.

$\Rightarrow (y-x)$ is also divisible by 6

$\Rightarrow yRx$.

Hence, R is symmetric.

xRy and $yRz \Rightarrow (x-y)$ is divisible by 6 and $(y-z)$ is also divisible by 6.

$\Rightarrow [(x-y)+(y-z)]$ is also divisible by 6.

$x-y+y-z$

$\Rightarrow (x-z)$ divisible by 6.

xRz. Transitive Relation

Therefore, we can say that R is an equivalence relation.

THEOREM Let R and S be relation from A to B, show that

(i) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$

Suppose $R \subseteq S$. If $(a,b) \in R^{-1}$, then $(b,a) \in R$ and also $(b,a) \in S$. Since $R \subseteq S$ Again $(b,a) \in S \Rightarrow (a,b) \in S^{-1}$

$\therefore R^{-1} \subseteq S^{-1}$

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(ii) $(R\cap S)^{-1} = R^{-1}\cap S^{-1}$

Let $(a,b) \in (R\cap S)^{-1}$. Then $(b,a) \in R\cap S$, so that $(b,a) \in R$ and $(b,a) \in S$.

$\Rightarrow (a,b) \in R^{-1}$ and $(a,b) \in S^{-1}$ Hence $(a,b) \in R^{-1}\cap S^{-1}$.

Therefore $(R\cap S)^{-1} \subseteq R^{-1}\cap S^{-1}$

Conversely, let $(a,b) \in R^{-1}\cap S^{-1}$ Then $(a,b) \in R^{-1}$ and $(a,b) \in S^{-1}$

$\Rightarrow (b,a) \in R$ and $(b,a) \in S$

$\Rightarrow (b,a) \in (R\cap S)$

$\Rightarrow (a,b) \in (R\cap S)^{-1}$

$\therefore R^{-1}\cap S^{-1} \subseteq (R\cap S)^{-1}$

Thus $(R\cap S)^{-1} = R^{-1}\cap S^{-1}$.

(iii) $(R\cup S)^{-1} = R^{-1}\cup S^{-1}$

Let $(a,b) \in (R\cup S)^{-1} \Rightarrow (a,b) \in (R\cup S)^{-1}$

$\Rightarrow (b,a) \in R\cup S$

$\Rightarrow (b,a) \in R$ or $(b,a) \in S$

$\Rightarrow (a,b) \in R^{-1}$ or $(a,b) \in S^{-1}$

$\Rightarrow (a,b) \in R^{-1}\cup S^{-1}$

$\therefore (R\cup S)^{-1} \subseteq (R^{-1})\cup(S^{-1})$

Conversely, $(a,b) \in R^{-1}\cup S^{-1} \Rightarrow (a,b) \in R^{-1}$ or $(a,b) \in S^{-1}$

$\Rightarrow (b,a) \in R$ or $(b,a) \in S$

$\Rightarrow (b,a) \in R\cup S$

$\Rightarrow (a,b) \in (R\cup S)^{-1}$

$\therefore R^{-1}\cup S^{-1} \subseteq (R\cup S)^{-1}$

Therefore, $R^{-1}\cup S^{-1} = (R\cup S)^{-1}$

Examples.

① $a \in N$, $b \in N$ $R =$ ' Greater than equal to'

Reflexive as a Greater than equal to a

Not symmetric

Antisymmetric

Transitive $a \geq b, b \geq c \Rightarrow a \geq c$.

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2. R = 'Brother of' from set of people
Not reflexive
Not symmetric
Not antisymmetric
Transitive.

3. R = {(a,b), (b,c)}   a = sandwich, b = nothing
R = 'Better than'   c = happiness
There we get an absurd results.

4. A = set of books
R = {(a,b)} such that 'a' costs more than book 'b' and contains fewer pages than 'b'.
Not reflexive
Not symmetric
Anti symmetric
Transitive
Not an equivalence relation

5. A = set of all binary words
= {1101, 111010, 1111, 1010, ...}
R = {(a,b) | 'a' and 'b' have same no. of zeros}
Reflexive
Symmetric
Not - antisymmetric
Transitive

It is an equivalence relation

Equivalence classes :-

If R is an equivalence relation on set S and xRy, then x and y are called equivalent w.r.t. R. The set of all elements of S that are equivalent to a given element x constitute the equivalence class of x, denoted by $[x]_R$

When only one relation is under consideration, the subscript of R is deleted and equivalence class is denoted by $[x]$.

Thus, $[x] = \{ y \in S : yRx \}$.

The collection of all equivalence classes of elements of S under an equivalence relation R is denoted by S/R that is

$$S/R = \{ [x] : x \in S \}$$

It is called quotient set of S by R.

Important Properties of Equivalence classes:-

Let A be a non-empty and R be an equivalence relation defined in A. Let a and $b \in A$ be arbitrary. Then

(i) $a \in [a]$

R being an equivalence relation, it is reflexive, that is aRa and

$$[a] = [x; x \in A \text{ and } xRa]$$

From this, we have $aRa \Rightarrow a \in [a]$

(ii) $b \in [a] \Rightarrow [b] = [a]$

We have, $b \in [a] \Rightarrow bRa$

Suppose, $x \in [b]$, then $x \in [b] \Rightarrow xRb$

Again R being transitive, xRb, $bRa \Rightarrow xRa \Rightarrow x \in [a]$

Therefore, $x \in [b] \Rightarrow x \in [a]$, that is, $[b] \subseteq [a]$ — (1)

Let if $y \in [a] \Rightarrow yRa$

As R is symmetric, therefore $bRa \Rightarrow aRb$

Hence, yRa and $aRb \Rightarrow yRb$, as R is transitive.

$y \in [b]$

$\therefore y \in [a] \Rightarrow y \in [b]$

$[a] \subseteq [b]$ — (2)

From (1) & (2) $[a] = [b]$

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(iii) $[a]=[b] \Leftrightarrow (a,b)\in R$

We assume that $[a]=[b]$

Since R is reflexive, we have aRa

$aRa \Rightarrow a\in[a] \Rightarrow a\in[b] \quad \because [a]=[b]$

$\Rightarrow aRb$

$\therefore [a]=[b] \Rightarrow aRb$, i.e., $(a,b)\in R$

Conversely, we assume aRb

Let $x\in[a]$ so that xRa, but aRb, Hence being R as transitive $xRa, aRb \Rightarrow xRb$

$x\in[b]$ and $x\in[a] \Rightarrow x\in[b]$

$\Rightarrow [a]\subseteq[b]$ —(1)

Let $y\in[b] \Rightarrow yRb$

But R being symmetric $aRb \Rightarrow bRa$

Now R being transitive $yRb, bRa \Rightarrow yRa \Rightarrow y\in[a]$

$y\in[b] \Rightarrow y\in[a]$ i.e. $[b]\subseteq[a]$ —(2)

From (1) & (2) $[b]=[a] \Leftrightarrow aRb$.

(iv) either $[a]=[b]$ or $[a]\cap[b]=\emptyset$

Here we assume that $[a]\cap[b]\neq\emptyset$

This implies that for some $x\in A$ such that

$x\in[a]\cap[b]$

But $x\in[a]\cap[b] \Rightarrow x\in[a]$ and $x\in[b]$

$\Rightarrow aRx$ and xRb

$\Rightarrow aRb$

$\Rightarrow [a]=[b]$

$\therefore [a]\cap[b]\neq\emptyset \Rightarrow [a]=[b]$

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Partitions of a Set :-

A partition of a set A is non empty subsets of A denoted by $\{A_1, A_2 \ldots A_n\}$ such that union of A_i's is equal to A and intersection of A_i and A_j is $\emptyset$. These subsets are called blocks of the partition.

From an equivalence relation on A one can define partition of A so that every two elements in blocks are related and any two elements of a block are not. This partition is said to be the partitions induced by equivalence relation. From a partition of set one can define an equivalence relation on A so that every two elements in the same block of the partition are related, and any two elements in a block are not related.

Ex $R = \{(1,2), (2,1), (1,1), (2,2), (3,3), (4,4)\}$ on $S = \{1,2,3,4\}$

$[1] = [2] = \{1,2\}$. $[3] = \{3\}$ $[4] = \{4\}$

$[1] \cap [3] = \{1,2\} \cap \{3\} = \emptyset$. $S = [1] \cup [3] \cup [4]$.

Thus equivalence classes form the partition of S.

Partial Order Relation :-

A relation R on set S is called a partial order if it is reflexive, antisymmetric and transitive i.e.

1. Reflexivity : aRa for all $a \in S$
2. Antisymmetric : aRb, $bRa \Rightarrow a = b$
3. Transitive : aRb $bRc \Rightarrow aRc$

A set S together with partial order R is called a partial order set or a poset and is denoted by (S,R)

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Ex, Let / be the divide relation R on a set N of positive integers. That is, for all $a, b \in N$, $a/b \Leftrightarrow b = Ka$ for some integer K. Prove that / is a partial order relation on N.

Reflexive: We have $a \in N$, a is a divisor of a i.e. aRa.

Antisymmetric: If a is a divisor of b then b cannot be divisor of a unless $a=b$. Therefore R is antisymmetric.

Transitive: Finally, a is a divisor of b and b is a divisor of c $\Rightarrow$ a is a divisor of c. Therefore R is transitive

Since R is reflexive, antisymmetric and transitive, therefore R is a partial order relation.

Representation of Relations :-

There are many ways of representing relations on finite sets.

1. Graphs

→ For graphical representation of a relation on a set, each element of the set is represented by a point.

→ If pair $x \in A, y \in B$ is in the relation, the corresponding nodes are connected by arcs called edges. The direction of edges represented by an arrow and is from first element to second element.

→ The graph with directed edges is called directed graph or digraph.

→ An edge of the form (a,a) is represented using an arc from the vertex to itself is called a loop.

NOTE: The digraph of the inverse of a relation has exactly the same edges of digraph of original relation but the direction of edges are reversed.

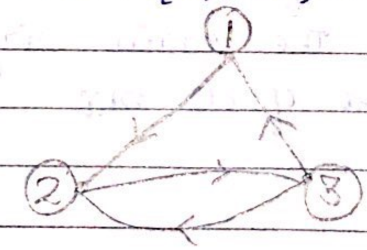
The digraph representing a relation can be used to determine whether the relation has various properties:

(i) A relation is reflexive iff there is a loop at every vertex of the digraph. If no vertex has loop the relation is irreflexive.

(ii) A relation is symmetric iff every edge between distinct vertices in its digraph there is an edge in the opposite direction. A relation is antisymmetric if no two distinct in digraph have an edge going between them in both directions.

(iii) A relation is transitive iff whenever there is a directed edge from a vertex 'a' to 'b' and from vertex 'b' to 'c' then there is also a directed edge from 'a' to 'c'.

Ex. Draw the directed graph that represents the relation.
$R = \{(1,1), (1,2), (2,2), (2,3), (3,2), (3,1), (3,3)\}$ on
$X = \{1,2,3\}$



2. Matrix of a relation :-

Let $A = \{a1, a2, \ldots, a_m\}$ and $B = \{b1, b2, \ldots, b_n\}$ are finite sets containing m and n elements respectively and let R be a relation from A to B.

Then R can be represented by mn matrix $M_R = [m_{ij}]$ where

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$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$ M_R is matrix of relation of R

If a relation is given by matrix, then the domain is given by the rows that contain at least one non zero entry, and the range is similarly, given by the columns with atleast one nonzero entry.

If R and S are relation on set A using operations on Boolean matrix one can show:

$$M_{R \cup S} = M_R \cup M_S$$

$$M_{R \cap S} = M_R \cap M_S$$

$$M_{R^{-1}} = M_R^T$$

The matrix representing a relation can be used to determine whether the relation has various properties. Let $M_R = [m_{ij}]$ be the matrix of R.

(i) Reflexive :-

If all the elements in the main diagonal of the matrix representation of a relation are 1, then the relation is reflexive. The main diagonal elements of matrix M_R is set of all m_{ii}. If all $m_{ii} = 1$ then relation is reflexive.

If all $m_{ii} = 0$, then relation is irreflexive

(ii) Symmetric :-

If the matrix of a relation is symmetric i.e. $m_{ij} = m_{ji}$ for all values of i and j, then relation is symmetric.

A relation is antisymmetric iff $m_{ij} = 1$ necessitates that $m_{ji} = 0$

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$ Symmetric Relation

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ Antisymmetric

(iii) Transitive :-

A relation R is transitive iff its matrix $M_R = [m_{ij}]$ has the property, if $m_{ij} = 1$, $m_{jk} = 1$ then $m_{ik} = 1$. The transitivity of R means that if $M_R^2 = M_R \cdot M_R$ has 1 in any position then M_R must have 1 in same position. Thus R is transitive iff $M_R^2 + M_R = M_R$.

Ex, Let $A = \{1, 2, 3, 4\}$ and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Show that M_R is transitive or relation is transitive relation.

$$M_R^2 = M_R \cdot M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, evaluate $$M_R^2 + M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = M_R$$

Hence the given relation is transitive.

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Composition of Relations :-

Let A, B, C be sets. Let R be a relation from A to B and S be a relation from B to C. That $R \subseteq A \times B$ and $S \subseteq B \times C$. The composite of R and S, denoted by $R \circ S$, is the relation consisting of ordered pairs (a,c) when $a \in A$ and $c \in C$, and for which there exists an element $b \in B$, such that $(a,b) \in R$ and $(b,c) \in S$.

Thus

$$R \circ S = \{ (a,c) \in A \times C : \text{for some } b \in B, (a,b) \in R \text{ \& } (b,c) \in S \}$$

NOTE:- $R \circ S$ is $\emptyset$ if the intersection of range R and domain of S is empty or $\emptyset$.

If R be a relation from A to B, S is a relation from B to C, and P be a relation from C to D, one can find $R \circ (S \circ P)$ and also $(R \circ S) \circ P$ i.e. is associative in nature. Lets see how it is proved:-

let $(a,b) \in R$, $(b,c) \in S$ and $(c,d) \in P$

$(a,b) \in R$, $(b,d) \in (S \circ P)$ $\qquad$ $(a,c) \in (R \circ S)$, $(c,d) \in P$

$(a,d) \in R \circ (S \circ P)$ $\qquad$ $(a,d) \in (R \circ S) \circ P$.

$\therefore \quad R \circ (S \circ P) = (R \circ S) \circ P$.

Ex. Let $R = \{(1,1), (2,1), (3,2)\}$ compute R^2

$$R^2 = R \circ R = \{ (1,1), (2,1), (3,1) \}$$

Theorem:- If R1 and R2 are relations from A to B, R_3 and R_4 are from B to C, then

(i) If $R_1 \subseteq R_2$ and $R_3 \subseteq R_4$ then $R_1 \circ R_3 \subseteq R_2 \circ R_4$

let $x \in A$, $y \in B$ and $z \in C$

$(x,z) \in R_1 \circ R_3$. Then for some $y \in B$

$(x,y) \in R_1 \quad (y,z) \in R_3$

Since, $R_1 \subseteq R_2$ and $R_3 \subseteq R_4$ $(x,y) \in R_2$ $(y,z) \in R_4$

$\therefore (x,z) \in R_2 \circ R_4$

Hence it is proved if $R_1 \subseteq R_2$ & $R_3 \subseteq R_4$ then $R_1 \circ R_3 \subseteq R_2 \circ R_4$

(ii) $(R_1 \cup R_2) \circ R_3 = (R_1 \circ R_3) \cup (R_2 \circ R_3)$

Matrix Representation of Composition of Relation :-

Suppose R is a relation from A to B; S be a relation from B to C. Suppose that A, B and C have elements m, n and p respectively. The matrices represented by R, S and $S \circ R$ are denoted by $M_R = [r_{ij}]_{m \times n}$, $M_S = [s_{ij}]_{n \times p}$ and $M_{R \circ S} = [t_{ij}]_{m \times p}$

$T_{ij} = 1$, $r_{ik} = s_{kj} = 1$ for some k.

Theorem : Let A, B and C be finite sets : Let R be a relation from A to B and S be a relation from B to C. Show that

$M_{R \circ S} = M_R \cdot M_S$ where M_R and M_S represents relation matrices of R and S respectively.

Let $A = \{a_1, a_2, \ldots a_n\}$

$B = \{b_1, b_2, \ldots b_n\}$

$C = \{c_1, c_2, \ldots c_p\}$.

Suppose $M_R = [a_{ij}]$ $M_S = [b_{ij}]$

$M_{R \circ S} = [d_{ij}]$ -

$d_{ij} = 1$ iff $(a_i, c_j) \in R \circ S \Rightarrow (a, b_K) \in R)\ (b_K, c_j) \in S$

$d_{ij} = 0$ if either $(a_i, a_K) \notin R$ or $(a_K, a_j) \notin S$

This condition is identical to the condition needed for $M_R \cdot M_S$ to have 1 or 0 in position i,j and Thus

$M_{R \circ S} = M_R \cdot M_S$

1. $R \circ S \neq S \circ R$ (Not commutative)
2. $R \circ (S \circ T) = (R \circ S) \circ T$ (Associative)
3. $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$ (Distributive)

Closure of Relations :-

Consider a given set A and the collection of all relations on A. Let P be a property of such relations, such as being symmetric or being transitive. A relation with property P will be called a P-relation. The P- closure of an arbitrary relation R on A, written P(R), is a P- relation such that.

$R \subseteq P(R) \subseteq S$ for every P- relation S containing R. we will write reflexive (R), Symmetric(S) and transitive (R) for reflexive, Symmetric, & transitive closures of R.

$P(R) = \cap (S \mid S$ is a P-relation and $R \subseteq S)$

1. **Reflexive Closure :-**

The reflexive closure $R^{(1)}$ of a relation R is the smallest reflexive relation that contains R as a subset. Given a relation R on set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a,a) with $A \in R$, not already in R. Thus,

$$R^{(\lambda)} = R \cup I_A \text{ where } I_A = \{(a,a) : a \in A\} \text{ diagonal relation on } A$$

Ex, $S = \{1,2,3,4\}$ $R = \{(1,2),(2,1),(1,1),(2,2)\}$ is not reflexive

$R^{(\lambda)} = R \cup I_A = R \cup \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$

$= R \cup \{(3,3),(4,4)\}$

$= \{(1,1),(2,1),(1,1),(2,2),(3,3),(4,4)\}$

2. Symmetric Closure :-

The symmetric closure $R^{(s)}$ is the smallest symmetric relation that contains R as a subset. Since the inverse relation R^{-1} contains (y,x) if (x,y) is in R, the symmetric closure of a relation can be constructed by taking union of R & R^{-1} i.e. $R \cup R^{-1}$ is symmetric closure of R, where

$$R^{-1} = \{(y,x) : (x,y) \in R\}$$

Ex,

$R = \{(1,2),(4,3),(2,2),(2,1),(3,1)\}$ $S = \{1,2,3,4\}$

$R^{-1} = \{(2,1),(3,4),(2,2),(1,2),(1,3)\}$

$R^{(s)} = R \cup R^{-1} = \{(1,2),(4,3),(2,2),(2,1),(3,1),\cancel{(2,1)},(3,4),(1,3)\}$

3. Transitive Closure :-

The relation obtained by adding the least no. of ordered pairs to ensure transitivity called transitive closure of the relation. The transitive closure of R is denoted by R^+. Let relation R is defined on A and A contains m elements, one never needs more than m steps.

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$R^+ = R \cup R^2 \cup \dots \cup R^m$

If A be a set and R be a binary relation on A. The transitive closure of R is denoted by R^+, satisfies 3 properties

↳ R^+ is transitive

↳ $R \subseteq R^+$

↳ If S in any other transitive relation that contains R then $R^+ \subseteq S$.

In, matrix terminology, the closures can be obtained as:-

$M_R = M \vee I_n$ Reflexive closure

$M_S = M \vee M^T$ Symmetric closure

$M_T = M \vee M^2 \vee M^3 \dots \vee M^n$ transitive closure

We can also express this idea in graphical representation of R as follows:-

- Reflexive closure :- we add all the missing arrows from points to themselves
- Symmetric closure :- we add missing reverses of all the arrows in graph.
- Transitive closure :- we add an arrow connecting a point x to y whenever some sequence of arrows in digraph of R connected x to y & there was not an arrow from x to y already.

NOTE If R is reflexive so, $R^{(s)}$ and R^+
If R is symmetric so, $R^{(s)}$ and R^+
If R is transitive so is $R^{(s)}$

Ques Let $A = R \times R$ (R be set of real numbers) & define the following relation on A.

$$(a,b)\ R\ (c,d) \Rightarrow a^2+b^2 = c^2+d^2.$$

Verify that R is an equivalence relation.

Ques Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$

i) $[\{1,3,5\}, \{2,6\}, \{4,8,9\}]$

ii) $[\{1,3,5\}, \{2,4,6,8\}, \{5,7,9\}]$

iii) $[\{1,3,5\}, \{2,4,6,8\}, \{7,9\}]$

Then (i) is not a partition of S since 7 in S does not belong to any of subsets. Furthermore (ii) is not partition since $\{1,3,5\} \cap \{5,7,9\} \neq \emptyset$.

Ques If X has 10 members, how many do $P(X)$ has? How many members of $P(X)$ are proper subset of X.

If X has 10 members then $P(X)$ has 2^{10} members. out of which $(2^{10}-1)$ are proper subset of X, as one of the member is set X itself.

Ques Salad is made with combination of one or more eatables, how many different salads can be prepared from onion, tomato, carrot, cabbage & cucumber?

Let $A = \{$ onion, tomato, carrot, cabbage, cucumber $\}$

Set of all salad is all possible subsets of A except $\emptyset$ as no salad can be prepared without atleast

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one of the eatables.

Hence, no. of salads is $|P(A)| - 1$ & is equal to $2^5 - 1$
$= 32 - 1 = 31$.

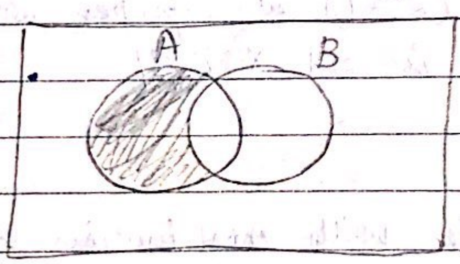
Ques Let $U = \{1, 2, ..., 9\}$ and the sets

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7\}$ $C = \{5,6,7,8,9\}$
$D = \{1,3,5,7,9\}$ $E = \{2,4,6,8\}$ $F = \{1,5,9\}$

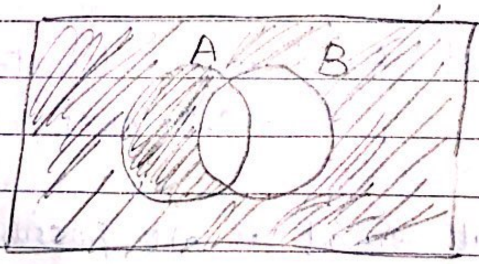
Find
1. A/B, B/A, D/E, F/D
2. $A \oplus B$, $C \oplus D$, $E \oplus F$
3. $(A/E)^c$
4. $(A \cap D)^c / B$

Ques Draw venn diagram for.

a) $A \cap B^c$



b) $(B/A)^c$



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Ques A survey among 1000 people, 595 are democrats, 595 wear glasses and 550 like icecream. 395 of them are democrats who wear glasses, 350 of them are democrats who like ice-cream. 400 of them wear glasses & like icecream and 250 all the three

a) How many of them are not democrats, do not wear glasses and do not like icecream?

$|D| = 595 \quad |G| = 595 \quad |I| = 550 \quad |D \cap G| = 395$
$|D \cap I| = 350, \quad |G \cap I| = 400 \quad |D \cap G \cap I| = 250$

$$|D \cup G \cup I| = 595 + 595 + 550 - 395 - 350 - 400 + 250$$
$$= 845$$

$\therefore$ people are not democrats, do not wear glasses, do not like icecreams $= 1000 - 845 = 155$

b) How many of them are democrats who do not wear glasses and do not like icecreams.

$$= 845 - |G \cup I|$$
$$= 845 - [\,|G| + |I| - |G \cap I|\,]$$
$$= 845 - [595 + 550 - 400]$$
$$= 745 - 745 = 100$$

Ques Consider the following relations on set $A = \{1, 2, 3\}$:

$R = \{(1,1), (1,2), (1,3), (3,3)\}$
$S = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$
$T = \{(1,1), (1,2), (2,2), (2,3)\}$
$\emptyset$ = Empty relation $\quad A \times A$ = universal relation.

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Determine whether each of the relation is reflexive, symmetric, antisymmetric or transitive.

a) R

Not reflexive $(2,2) \notin R$

Not symmetric $(1,2) \in R$ but $(2,1) \notin R$

Antisymmetric

Transitive

b) S

Reflexive

Symmetric

Not antisymmetric

Transitive

c) I

Not reflexive

Not symmetric

Antisymmetric

Not transitive

d) $\emptyset$

Not reflexive

Symmetric

Antisymmetric

Transitive

d) Universal $A \times A$

reflexive

symmetric

Not Antisymmetric

Transitive

Ques

Given $A = \{1,2,3,4\}$

Consider $R = \{(1,1),(2,2),(2,3),(3,2),(4,2),(4,4)\}$

R is Not reflexive

Not symmetric

Not Antisymmetric

Not Transitive

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Ques Give examples of relation $R = \{1,2,3\}$ having stated property

a) R is both symmetric & antisymmetric

$R = \{(1,1), (2,2)\}$

b) R is neither symmetric nor antisymmetric

$R = \{(1,2), (2,3), (3,2)\}$

c) R is transitive but $R \cup R^{-1}$ is not transitive

$R = \{(1,2)\}$

Questions Based on Relations :-

Q1. Consider the following relation on $\{1,2,3,4,5,6\}$
$R = \{(i,j) : |i-j| = 2\}$
Find R is transitive, reflexive or symmetric?

Sol.^{n} $R = \{(1,3),(3,1),(2,4),(4,2),(5,3),(3,5),(4,6),(6,4)\}$

R is not reflexive, symmetric but not transitive.

Q2. Let $A = \{1,2,3,4,5\}$ and $R = \{(1,2),(1,1),(2,1),(2,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$ be an equivalence relation on A. Determine the partitions corresponding to R^{-1}, if it is an equivalence relation.

$[1] = [2] = \{1,2\}$
$[3] = \{3\}$
$[4] = [5] = \{4,5\}$

$P = \{\{1,2\},\{3\},\{4,5\}\}$

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Q3. Let R be an equivalence relation on set $A=\{a,b,c,d\}$ defined by partitions $P=\{\{a,d\},\{b,c\}\}$. Determine the elements of equivalence relation & also find the equivalence classes of R.

$$R=\{(a,a),(a,d),(d,a),(d,d),(b,b),(b,c),(c,b),(c,c)\}$$

Equivalence classes

$[a]=[d]=\{a,d\}$

$[b]=[c]=\{b,c\}$

Q4. Let A be set of first 7 natural no. & R is a relation on A defined by $(x,y)\in R \iff x+2y=10$ i.e. $R=\{(x,y): x\in A,\ y\in A \ \&\ x+2y=10\}$ Express R and R^{-1} as sets of ordered pairs.

$x+2y=10$

$y=\frac{10-x}{2}$ $x,y\in A$ $\qquad x=\{1,3,5,7\}\ y\notin A$

$\qquad x=\{2,4,6\}\ y\in A$

$\therefore R=\{(2,4),(4,3),(6,2)\}$

$R^{-1}=\{(4,2),(3,4),(2,6)\}$

Q5. Let A denote set of real numbers and $B=A\times A$. A relation R is defined on A such that

$(a,b)R(c,d)$ $\quad a^2+b^2=c^2+d^2$

Show that R is an equivalence relation.

Q6. Let A be set $\{1,2,3\}$ define the following types of binary relation on A.

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i) A relation that is both symmetric & antisymmetric.

$R = \{(1,1), (2,2), (3,3)\}$

ii) A relation that is neither symmetric nor antisymmetric

$R = \{(1,2), (2,1), (1,3)\}$

Q7. Let $S = \{1, 2, 3, \ldots, 19, 20\}$. Let R be the equivalence relation defined by $x \equiv y \pmod 5$ i.e. $x-y$ is divisible by 5. Find the partition of S induced by R.

$[\{1, 6, 11, 16\}, \{2, 7, 12, 17\}, \{3, 8, 13, 18\}, \{4, 9, 14, 19\}, \{5, 10, 15, 20\}]$

Q8. Consider the following relations on set $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,3)\}$

$S = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$T = \{(1,1), (1,2), (2,2), (2,3)\}$

Determine whether or not each of the above relations on A is

a) reflexive b) symmetric c) transitive.

Q9. Let $P = \{(1,2), (2,4), (3,3)\}$ and $Q = \{(1,3), (2,4), (4,2)\}$. Find $P \cup Q$, $P \cap Q$. dom(P), dom(Q), ran(P). (ran(Q), and ran$(P \cap Q)$.

Show that dom.$(P \cup Q) =$ dom.$(P) \cup$ dom.(Q)

ran$(P \cap Q) \subseteq$ ran$(P) \cap$ ran(Q)

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Q10. Determine whether relation R on the set of all integers is reflexive, symmetric, antisymmetric and/or transitive where $(x,y) \in R$ iff

i) $x \neq y$ ii) $xy \geq 1$ iii) $x = y^2$ iv) $|x+y| = 2$

v) $y - x + 2$ is a prime no.

Q11. Below is the list of relations among people. For each of the following relations, state whether the relation is reflexive, symmetric, antisymmetric or transitive.

a) xRy stands for x is a child of y – Antisymmetric.

b) xRy stands for x is a spouse of y – Symmetric.

c) xRy stands for x & y have the same parents – R, S, T

Q12. Let $A = \{1,2,3\}$ and R & R be relations on A such that

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $M_{R'}$, $M_{R^{-1}}$, $M_{R \cup S}$ and $M_{R \cap S}$

Q13. Consider the relation R on $A = \{1,2,3\}$ whose matrix representation is given below. Determine its R^{-1} & R'

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

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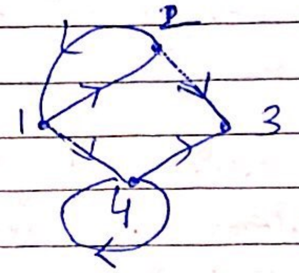
Q14. Consider a relation R whose directed graph is shown below. Determine R^{-1} & R'



Q15. Suppose C is a collection of relations S on a set A, and let T be the intersection of the relations S in C, that is, $T = \cap(S \mid S \in C)$ Prove:

a) If every S is symmetric, then T is symmetric
Suppose $(a,b) \in T$ $(a,b) \in S$ for every S. Since each S is symmetric, $(b,a) \in S$ for every S. Hence $(b,a) \in T$

b) If every S is transitive, then T is transitive.
Suppose $(a,b), (b,c) \in T$ then $(a,b)(b,c) \in S$ for every S. Since S is transitive, $(a,c) \in S$ for every S, $\therefore (a,c) \in T$

Q16. Give an example of relation R on $A = \{1,2,3\}$ such that
a) R is both symmetric and antisymmetric $\{(1,1),(2,2)\}$
b) R is neither symmetric nor antisymmetric $\{(1,2),(2,3)\}$
c) R is transitive but $R \cup R^{-1}$ is not transitive $\{(1,2)\}$.

Q17. Let R be the following equivalence relation on set $A = \{1,2,\ldots 6\}$
$R = \{(1,1),(1,5),(2,3),(2,2),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1)$
$(5,5),(6,2),(6,3)$ & $(6,6)\}$

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Find the partition of A induced by R i.e. find the equivalence classes of R.

$$\{\{1,5\}, \{2,3,6\}, \{4\}\}$$

Q18. Consider the set of integers. Define aRb by $b=a^r$ for some positive integer r. Show that R is a partial order on Z, that is show that R is reflexive, antisymmetric or transitive.

a) R is reflexive $a=a^1$ for 1

b) Suppose aRb, bRa $b=a^r$ and $a=b^s$
then $a=a^{rs}$. Then there are 3 possibilities
i) $rs=1$ ii) $a=1$ iii) $a=-1$

For $rs=1$ then $r=1$ $s=1$ $\therefore a=b$
If $a=1$ then $b=1^r=1=a$
If $a=-1$ then $b=-1$ $(b\neq 1)$ $a=b$

$\therefore$ antisymmetric

c) aRb, bRc $b=a^r$ $c=b^s$ $c=a^{rs}$ $\therefore$ aRc
Hence R is transitive.

Q19. Let $R=\{(1,2),(2,3),(3,1)\}$ and $A=\{1,2,3\}$ find the reflexive, symmetric and transitive closure of R, using
i) composition of relation R
ii) Matrix representation of R
iii) Graphical representation of R

i) $R^{(r)} = R \cup I_A = \{(1,2),(2,3),(3,1)\} \cup \{(1,1),(2,2),(3,3)\}$
$= \{(1,2),(2,3),(3,1),(1,1),(2,2),(3,3)\}$

$R^{(S)} = R \cup R^{-1} = \{(1,2),(2,3),(3,1)\} \cup \{(2,1),(3,2),(1,3)\}$
$= \{(1,2),(2,3),(3,1),(2,1),(3,2),(3,1)\}$

To find R^T $R \circ R = \{(1,2),(2,3),(3,1)\} \circ \{(1,2),(2,3),(3,1)\}$
$= \{(1,3),(2,1),(3,2)\}$
$R^3 = R^2 \circ R = \{(1,3),(2,1),(3,2)\} \circ \{(1,2),(2,3),(3,1)\}$
$= \{(1,1),(2,2),(3,3)\}$

$R^4 = R^3 \circ R = \{(1,1),(2,2),(3,3)\} \circ \{(1,2),(2,3),(3,1)\}$
$= \{(1,2),(2,3),(3,1)\} = R$

$R^T = R \cup R^2 \cup R^3$
$= \{(1,2),(2,3),(3,1),(1,3),(2,1),(3,2),(1,1),(2,2),(3,3)\}$

(ii) <u>Using Matrix</u> $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$M_R = M \cup I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ $R = \{(1,1),(2,2),(1,2),(2,3),(3,1),(3,3)\}$

$M_S = M \cup M^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $R = \{(1,2),(1,3),(2,1),(3,1),(3,2)\}$

M_R^T

$M_R^2 = M \cdot M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$M_R^3 = M^2 \cdot M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M^4 = M^3 \cdot M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = M$$

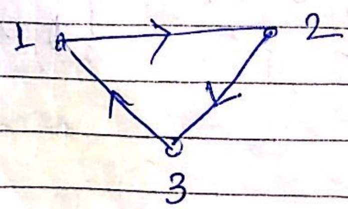
$$M_T = M + M^2 + M^3$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

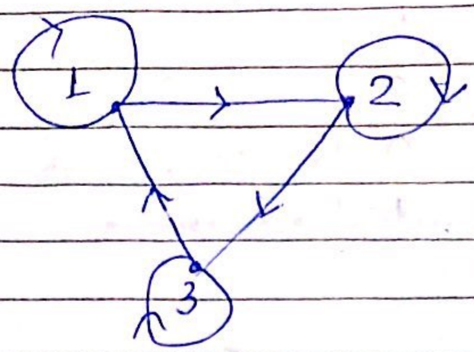
$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,1)$
$(2,3), (3,1), (3,2), (3,3)\}$

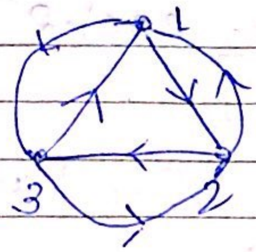
iii) Using Graphical Representation



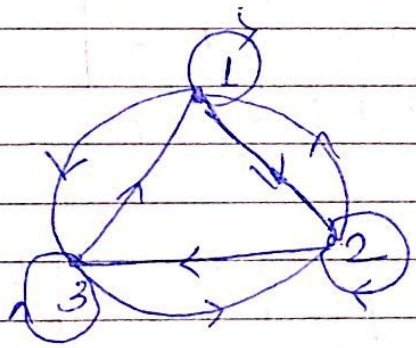
Reflexive closure :-



Symmetric closure :-



Transitive closure :-



Q20. let $A = \{1,2,3\}$ $B = \{a,b,c,d\}$ R & S be relations from A to B with Boolean matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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a) Find Boolean matrices for R^{-1} & S^{-1}

$$M_{R^{-1}} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ a & 1 & 0 & 1 \\ b & 0 & 1 & 0 \\ c & 1 & 0 & 0 \\ d & 0 & 0 & 1 \end{array} = M_R^T$$

$$M_{S^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = M_S^T$$

b) Find Boolean matrices for $(R \cap S) \circ R^{-1}$ & $R \circ R^{-1} \cap S \circ R^{-1}$

$M_R \cap M_S = M_R \cdot M_S$

$$M_{R \cap S} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \phi. \text{ (Null matrix)}$$

$$M_{(R \cap S) \circ R^{-1}} = \underset{3 \times 4}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \circ \underset{4 \times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R \circ R^{-1}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{S \circ R^{-1}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{(R \circ R^{-1}) \cap (S \circ R^{-1})} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cap \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Questions Based on Functions

Ques Let R be the set of real numbers, let $f: R \to R$ such that $f(x) = 2x+3 \; \forall x \in R$. Show that f is one-one and onto

Sol. To show that f is one-one

Let x_1 & x_2 be two elements in domain R. Their images in co-domain R shall be $2x_1+3$ and $2x_2+3$.

If these images are different

Q2. Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$. In each case, state whether the given function is injective, surjective & bijective

i) $f = \{(1,a), (2,d), (3,b)\}$

ii) $g = \{(1,a), (2,a), (3,d)\}$

iii) $h = \{(1,a), (1,b), (2,d), (3,c)\}$

iv) $j = \{(1,a)\ (2,b)\}$.

Q3. A function h is defined on the set of integers as follows:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ x+2 & \text{if } 1 \le x < 3 \\ 4x-5 & \text{if } 3 \le x < 5 \end{cases}$$

i) Find domain of f.

ii) Find range of f

iii) Find co-domain of f.

(iv) State whether f is one-one or many-one function

Q4. Let the function f and g be defined by $f(x) = 2x+1$ and $g(x) = x^2 - 2$. Find the formula defining the composition function gof.

$$4x^2 + 4x - 1$$

Q5. Determine if each function is one-one, many-one, onto

a) To each person on earth assign the no. which corresponds to his age.

b) To each country in the world assign the latitude & longitude of its capital.

c) To each book written by only one author assign the author.

d) To each country in the world assign its prime minister.

Q6. Consider $A = B = C = R$ and let $f: A \to B$ and $g: B \to C$ be defined by $f(x) = x + 9$ and $g(y) = y^2 + 3$. Find the following composition function

i) $f \circ f(a)$ ii) $g \circ g(a)$ iii) $f \circ g(a)$

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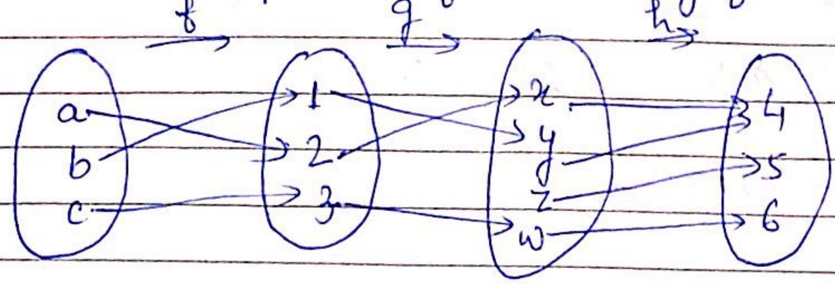
iv) $g \circ f(a)$ v) $g \circ f(3)$ vi) $f \circ g(-3)$

Q7. let the function $f: A \to B$, $g: B \to C$, $h: C \to D$

a) Determine if each function is onto

b) Find the composition function $h \circ g \circ f$.



Q8. Let $f: R \to R$ be defined by $f(x) = 2x-3$. Now f is one-to-one and onto, hence f has an inverse function f^{-1}. Find formula for f^{-1}

$$y = f(x) = 2x-3$$

$$f^{-1}(y) = x$$

$$y = 2x-3$$

$$x = \frac{y+3}{2}$$

$$f^{-1}(x) = \frac{x+3}{2}$$

Q9. Let f, g, h be function from $N \times N$, where N is the set of natural numbers so that

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$f(n) = n+1$ $\quad g(n) = 2n$

$h(n) = 0$, n is even

$h(n) = 1$, n is odd.

Determine

a) $f \circ g$ $\quad$ c) $g \circ f$

b) $f \circ f$ $\quad$ d) $g \circ g$

e) $g \circ h$ $\quad$ f) $f \circ h$

g) $(f \circ g) \cdot h$.

Q9. Show that the function $f(x) = x^4$ and $g(x) = x^{1/4}$ for $x \in R$ are inverses of one another. Here R is set of real no.

Q10. Consider the function $f: N \times N$ so that $f(x,y) = (2x+1) 2^y - 1$, where N is set of natural no. including 0, show that is bijective

$$N = \{0, 1, 2, 3 \ldots\}$$

$$f: N \times N$$

let $a, b \in N$

$x = a$ — (1), $y = b$ — (2)

on multiplying both sides by 2

Therefore $2a = 2x$ — (3)

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On adding ① to ③ eq. ①

$2a+1 = 2x+1$ — ④

Taking powers of 2 on both sides of eq. ②

$2^y = 2^b$ — ⑤

Now Multiply ④ & ⑤

$(2a+1)2^y = (2x+1)2^b$

$(2a+1)2^y - 1 = (2x+1)2^b - 1$

$f(a,y) = f(x,b)$

$\therefore$ f is one-to one

Now to prove f is onto let $f(x,y)=0$

$f(a,b)=0$

$(2a+1)2^b - 1 = 0$

$(2a+1)2^b = 1$

$(2a+1)2^b = 1 \cdot 2^0$

$2a+1 = 1 \qquad 2^b = 2^0$

$2a = 1-1 \qquad b=0$

$a=0$

$\therefore$ for $f(a,b)$ - $a=0, b=0$ exists

Hence f is onto.

Therefore $f(x,y)$ is bijective

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Q11. Let f be the set of one-one functions from A to B,

where $A = \{1, 2, 3, \ldots, n\}$

$B = \{1, 2, 3, \ldots, m\}$ and $m \geq n \geq 1$

a) How many functions are members of f

As $|A| = n$ & $|B| = m$

$\therefore$ Total $n \times m$ total functions are members of f.

b) How many functions f in F satisfy the property $f(i) = 1$ for some i.

In all n functions of F satisfy $f(i) = 1$

c) How many functions in f satisfy property $f(i) < f(j)$ for all $1 \leq j \leq n$?

$f(i) < f(j) = 1 + 2 + 3 + \ldots + n$

$= \frac{n(n-1)}{2}$