Unit-5

Chapter 12

Towards the Self-Organizing Feature Map

Neural Networks: A Classroom Approach
Satish Kumar

Properties of Stochastic Data

- Impinging inputs comprise a stream of stochastic vectors that are drawn from a stationary or non-stationary probability distribution
- Characterization of the properties of the input stream is of paramount importance
  - simple average of the input data
  - correlation matrix of the input vector stream

Properties of Stochastic Data

- Stream of stochastic data vectors:
  - Need to have complete information about the population in order to calculate statistical quantities of interest
  - Difficult since the vectors stream is usually drawn from a real-time sampling process in some environment
- Solution:
  - Make do with estimates which should be computed quickly and be accurate such that they converge to the correct values in the long run

Self-Organization

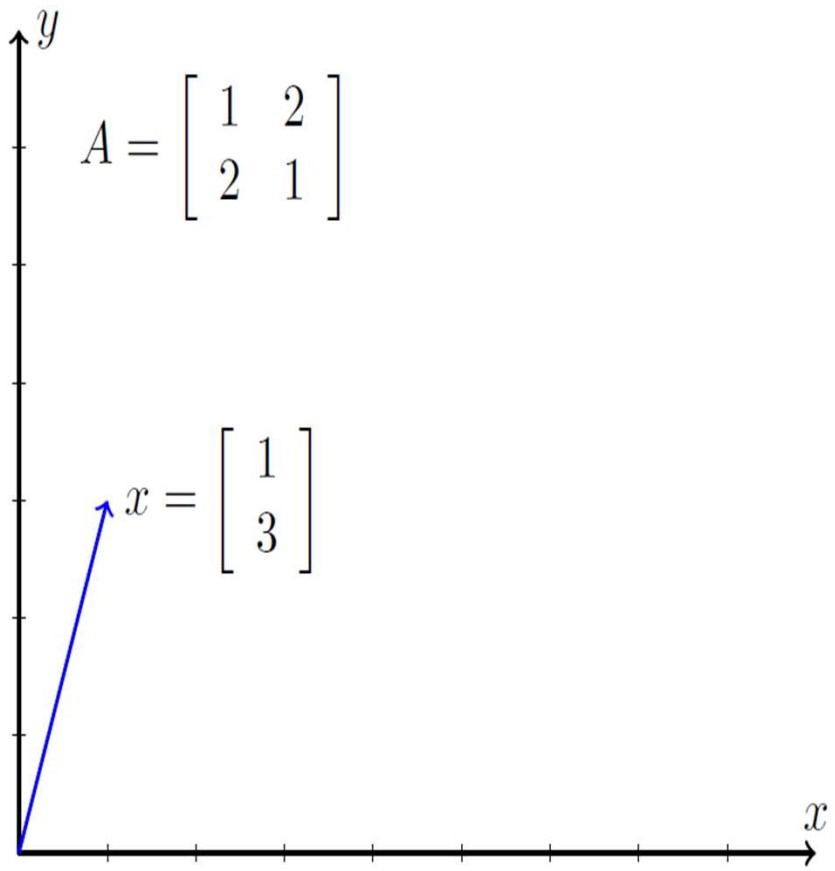
- Focus on the design of *self-organizing systems* that are capable of extracting useful information from the environment
- Primary purpose of self-organization:
  - the discovery of significant patterns or ***invariants*** of the environment without the intervention of a teaching input
- Implementation: Adaptation must be based on information that is available locally to the synapse—from the pre- and postsynaptic neuron signals and activations

Principles of Self-Organization

- Self-organizing systems are based on three principles:
  - Adaptation in synapses is self-reinforcing
  - LTM dynamics are based on competition
  - LTM dynamics involve cooperation as well

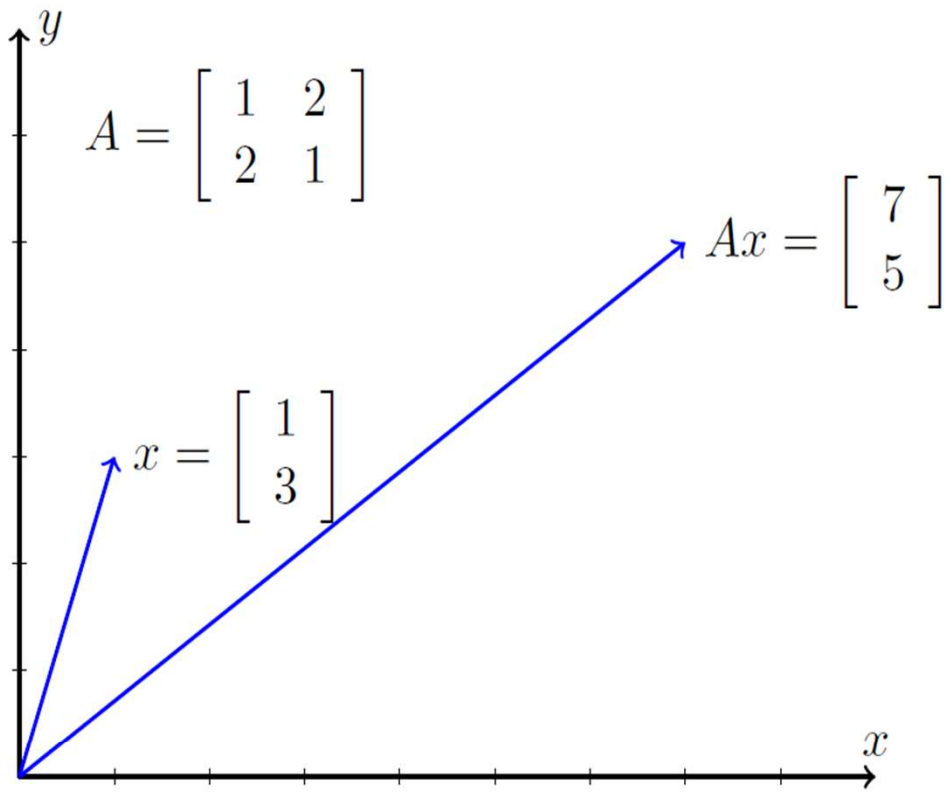
LTM-Long Term Memory

Eigen Values and Eigen Vectors



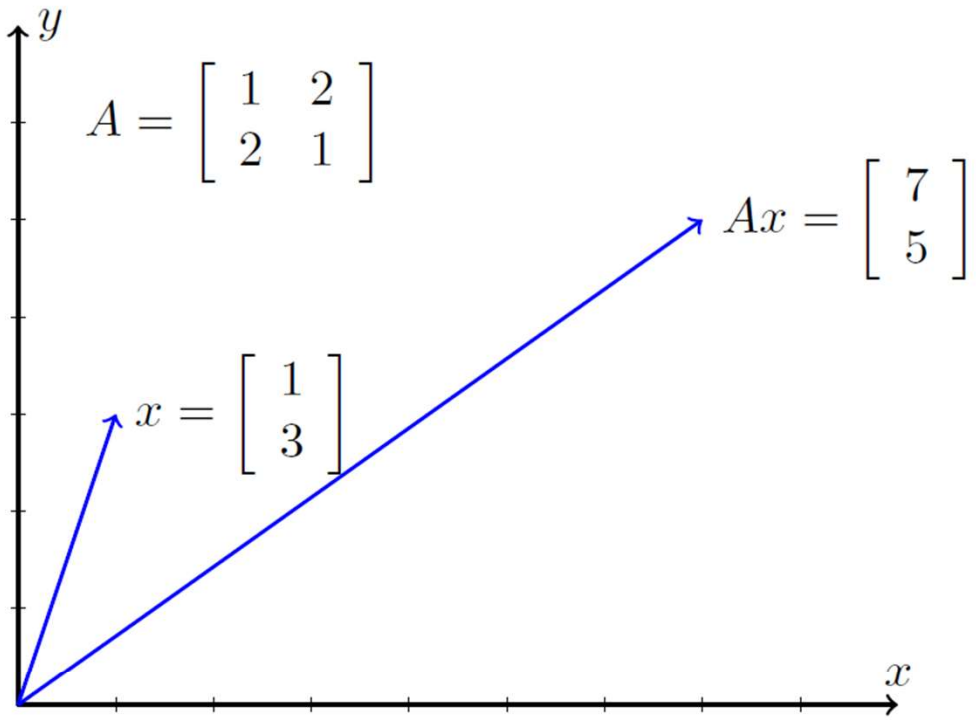
- What happens when a matrix hits a vector?

Eigen Vectors



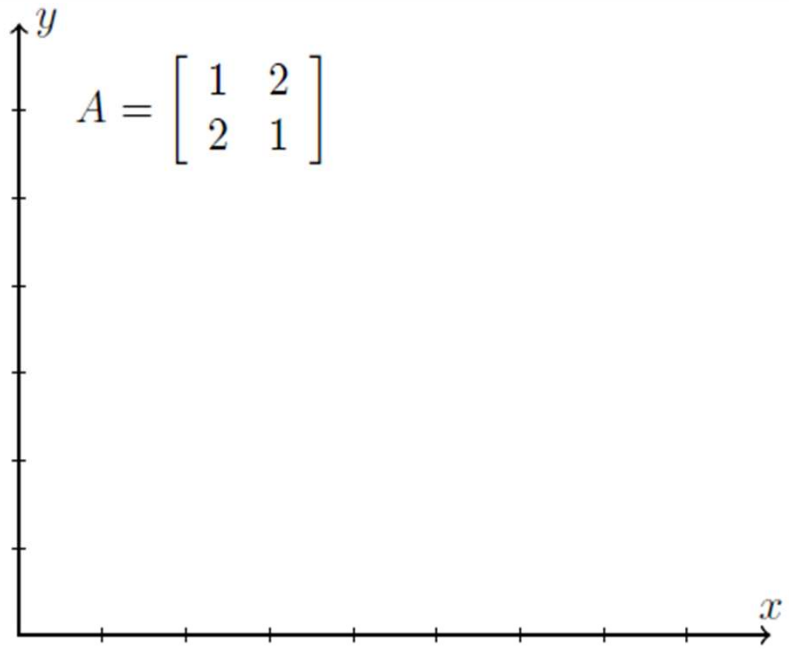
- What happens when a matrix hits a vector?
- The vector gets transformed into a new vector (it strays from its path)

Eigen Vectors



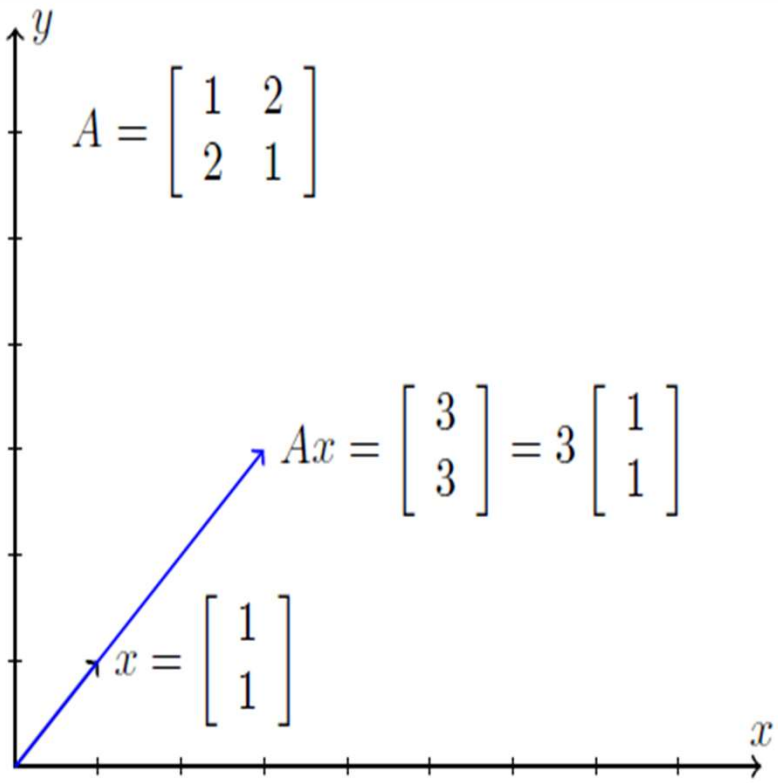
- What happens when a matrix hits a vector?
- The vector gets transformed into a new vector (it strays from its path)
- The vector may also get scaled (elongated or shortened) in the process.

Eigen Vectors



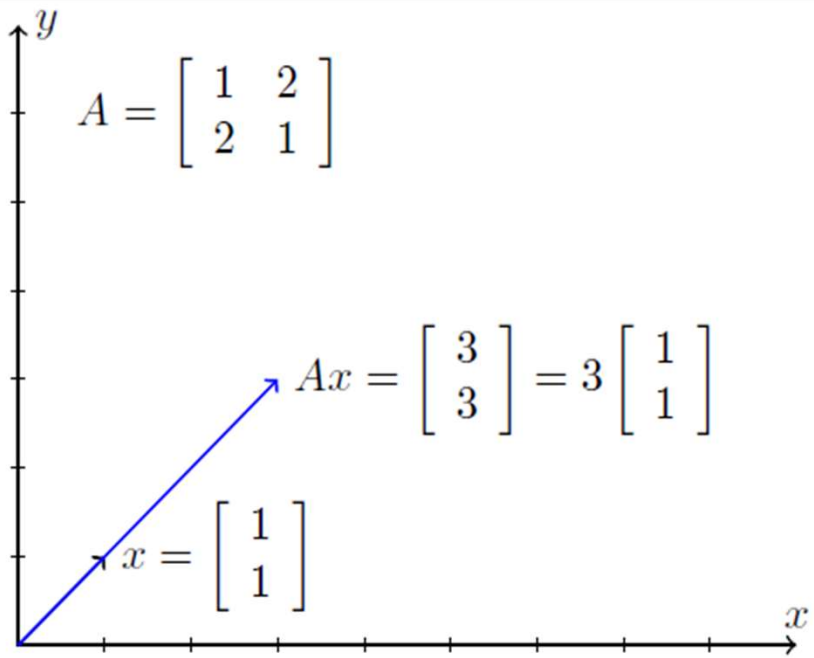
- For a given square matrix A, there exist special vectors which refuse to stray from their path.

Eigen Vectors



- For a given square matrix A, there exist special vectors which refuse to stray from their path.
- These vectors are called eigenvectors.

Eigen Vectors

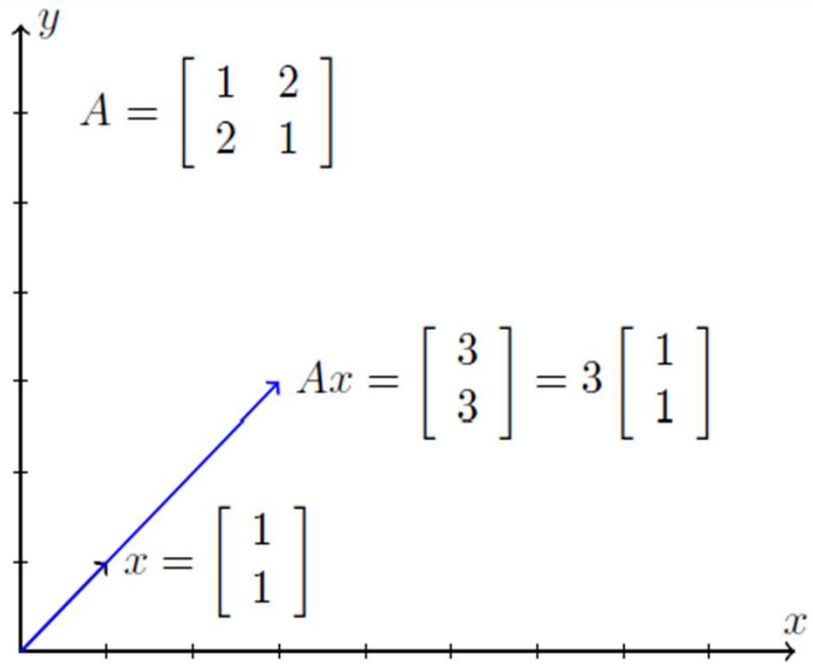


- For a given square matrix A, there exist special vectors which refuse to stray from their path.
- These vectors are called eigenvectors.
- More formally,

  $Ax = \lambda x$ [direction remains the same]

- The vector will only get scaled but will not change its direction.

Eigen Vectors



- So what is so special about eigenvectors?
- Why are they always in the limelight?
- It turns out that several properties of matrices can be analyzed based on their eigenvalues

Eigen Value Decomposition

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU = A\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \ldots & Au_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 u_1 & \lambda_2 u_2 & \ldots & \lambda_n u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} = U\Lambda$$

- where Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A.

Eigen Value Decomposition

$$AU = U\Lambda$$

- If U^{-1} exists, then we can write,

$$A = U\Lambda U^{-1} \quad \text{[eigenvalue decomposition]}$$

$$U^{-1}AU = \Lambda \quad \text{[diagonalization of A]}$$

- Under what conditions would U^{-1} exist?
  - If the columns of U are linearly independent
  - *i.e.* if A has n linearly independent eigenvectors.
  - *i.e.* if A has n distinct eigenvalues

Hebbian Learning

- Incorporates both exponential forgetting of past information and asymptotic encoding of the product of the signals

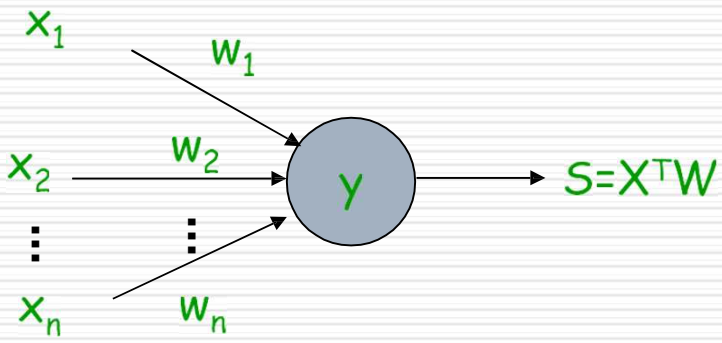
$$\dot{w}_{ij} = -w_{ij} + \mathcal{S}_i(x_i)\mathcal{S}_j(x_j)$$

- The change in the weight is dictated by the product of signals of the pre- and postsynaptic neurons

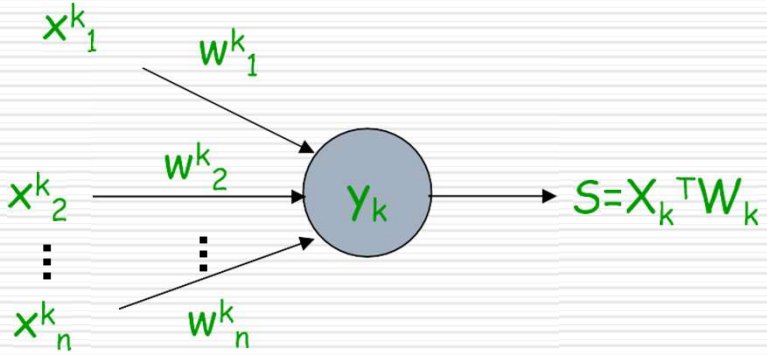
Why is Hebbian Learning Useful?

- A single linear unit using Hebbian learning can extract the dominant eigen-direction of the correlation matrix of an input vector stream that comprises patterns drawn from some unknown probability distribution

Linear Neuron and Discrete Time Formalism



(a) A linear neuron



(b) Discrete time formalism

Activation and Signal Computation

- Input vector X is assumed to be drawn from a stationary stochastic distribution
- $X = (x_1^k, \ldots, x_n^k)^T$, $W = (w_1^k, \ldots, w_n^k)^T$

$$s_k = y_k = \sum_{i=1}^{n} w_i^k x_i^k = X_k^T W_k$$

Discrete ☞

$$w_i^{k+1} = w_i^k + \alpha x_i^k s_k$$
$$= w_i^k + \alpha x_i^k y_k$$

α is the learning rate

Vector Form of Simple Hebbian Learning

- The learning law perturbs the weight vector in the direction of X_k by an amount proportional to
  - the signal, s_k, or
  - the activation y_k (since the signal of the linear neuron is simply its activation)

$$W_{k+1} = W_k + \alpha s_k X_k$$
$$= W_k + \alpha y_k X_k$$
$$= W_k + \alpha_k X_k$$

One can interpret the Hebb learning scheme of as adding the impinging input vector to the weight vector in direct proportion to the similarity between the two

Points worth noting…

- A major problem arises with the magnitude of the weight vector—it grows without bound!
- Patterns continuously perturb the system
- Equilibrium condition of learning is identified by the weight vector remaining within a neighbourhood of an equilibrium weight vector

Some Algebra

- Re-arrangement of the learning law:

$$W_{k+1} = W_k + \alpha(X_k^T W_k)X_k$$
$$= W_k + \alpha X_k(X_k^T W_k)$$

$$W_{k+1} - W_k = \alpha X_k X_k^T W_k$$

- Taking expectations of both sides

$$E[W_{k+1} - W_k] = E[\Delta W_k] = \alpha E[X_k X_k^T]W_k$$
$$= \alpha \mathbf{R} W_k$$

R is correlation matrix

Equilibrium Condition

- $\hat{W}$ denotes the equilibrium weight vector: the vector towards the neighbourhood of which weight vectors converge after sufficient iterations elapse
- Define the equilibrium condition as one such condition that weight changes must average to zero:

$$E[\Delta W_k] = 0 = \alpha \mathbf{R} \hat{W}$$

$$\mathbf{R}\hat{W} = 0 = \lambda_{\text{null}} \hat{W}$$

- Shows that $\hat{W}$ is an eigenvector of **R** corresponding to the degenerate eigenvalue $\lambda_{null} = 0$

Eigen-decomposition of the Weight Vector

- In general, any weight vector can be expressed in terms of the eigenvectors:

$$W = \sum_{i=1}^{m} \beta_i \eta_i + W_{\text{null}}$$

$$= \sum_{i=1}^{m} \beta_i \eta_i + \sum_{j=1}^{p} \gamma_j \eta'_j$$

- W_{null} is the component of W in the null subspace, η_i, η_j' are eigenvectors corresponding to non-zero and zero eigenvalues respectively

Average Weight Perturbation

- Consider a small perturbation about the equilibrium:

$$E[\Delta W_k] = \alpha \mathbf{R}(\hat{W} + \epsilon)$$
$$= \alpha \mathbf{R}\hat{W} + \alpha \mathbf{R}\epsilon$$
$$= \alpha \mathbf{R}\epsilon$$

- Expressing the perturbation using the eigen-decomposition:

$$\epsilon = \sum_{i=1}^{m} \beta_i \eta_i + \sum_{j=1}^{p} \gamma_j \eta'_j$$

- Substituting back yields:

$$E[\Delta W_k] = \alpha \mathbf{R} \left(\sum_{i=1}^{m} \beta_i \eta_i + \sum_{j=1}^{p} \gamma_j \eta'_j \right)$$
$$= \alpha \left(\sum_{i=1}^{m} \beta_i \mathbf{R} \eta_i + \sum_{j=1}^{p} \gamma_j \mathbf{R} \eta'_j \right)$$
$$= \alpha \sum_{i=1}^{m} \beta_i \lambda_i \eta_i$$

Kernel term goes to zero

i^{th} eigenvalue

Searching the Maximal Eigendirection

- $\hat{W}$ represents an unstable equilibrium
- Dominant direction of movement is the one corresponding to the largest eigenvalue, and these components must therefore grow in time
- Weight vector magnitude w grows indefinitely
- Direction approaches the eigenvector corresponding to the largest eigenvalue

$$E[\Delta W_k] = \alpha \sum_{i=1}^{m} \beta_i \lambda_i \eta_i$$

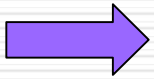
Small perturbations cause weight changes to occur in directions away from that of $\hat{W}$ towards eigenvectors corresponding to non-zero eigenvalues

Oja's Rule

- Modification to the simple Hebbian weight change procedure

$$W_{k+1} = W_k + \alpha s_k (X_k - s_k W_k)$$
$$= W_k + \alpha s_k X'_k$$

Can be re-cast into a different form to clearly see the normalization



$$w_i^{k+1} = \frac{w_i^k + \alpha s_k x_i^k}{\sqrt{\sum_{j=1}^{n} \left(w_j^k + \alpha s_k x_j^k\right)^2}}$$

Re-compute the Average Weight Change

- Compute the expected weight change conditional on W_k

$$E[\Delta W_k] = E[s_k X_k - s_k^2 W_k]$$
$$= \mathbf{R} W_k - (W_k^T \mathbf{R} W_k) W_k$$

- Setting $E[W_k]$ to zero yields the equilibrium weight vector $\hat{W}$

$$E[\Delta W_k] = 0 = \mathbf{R}\hat{W} - (\hat{W}^T \mathbf{R}\hat{W})\hat{W}$$

$$\mathbf{R}\hat{W} = (\hat{W}^T \mathbf{R}\hat{W})\hat{W} = \lambda \hat{W}$$

- Define $\lambda = \hat{W}^T \mathbf{R}\hat{W} = \hat{W}^T \lambda \hat{W} = \lambda \hat{W}^T \hat{W} = \lambda \| \hat{W} \|^2$

- Shows that $\| \hat{W} \|^2 = 1$ → Self-normalizing!

Maximal Eigendirection is the only stable direction…

- Conducting a small neighbourhood analysis as before:

$$W = \eta_i + \epsilon$$

- Then the average weight change is:

$$E[\Delta W] = \mathbf{R}\epsilon - 2\lambda_i(\epsilon^T \eta_i)\eta_i - \lambda_i \epsilon + \mathcal{O}(\epsilon)$$

- Compute the component of the average weight change E[ΔW] along any *other* eigenvector, η_j for $j \neq i$

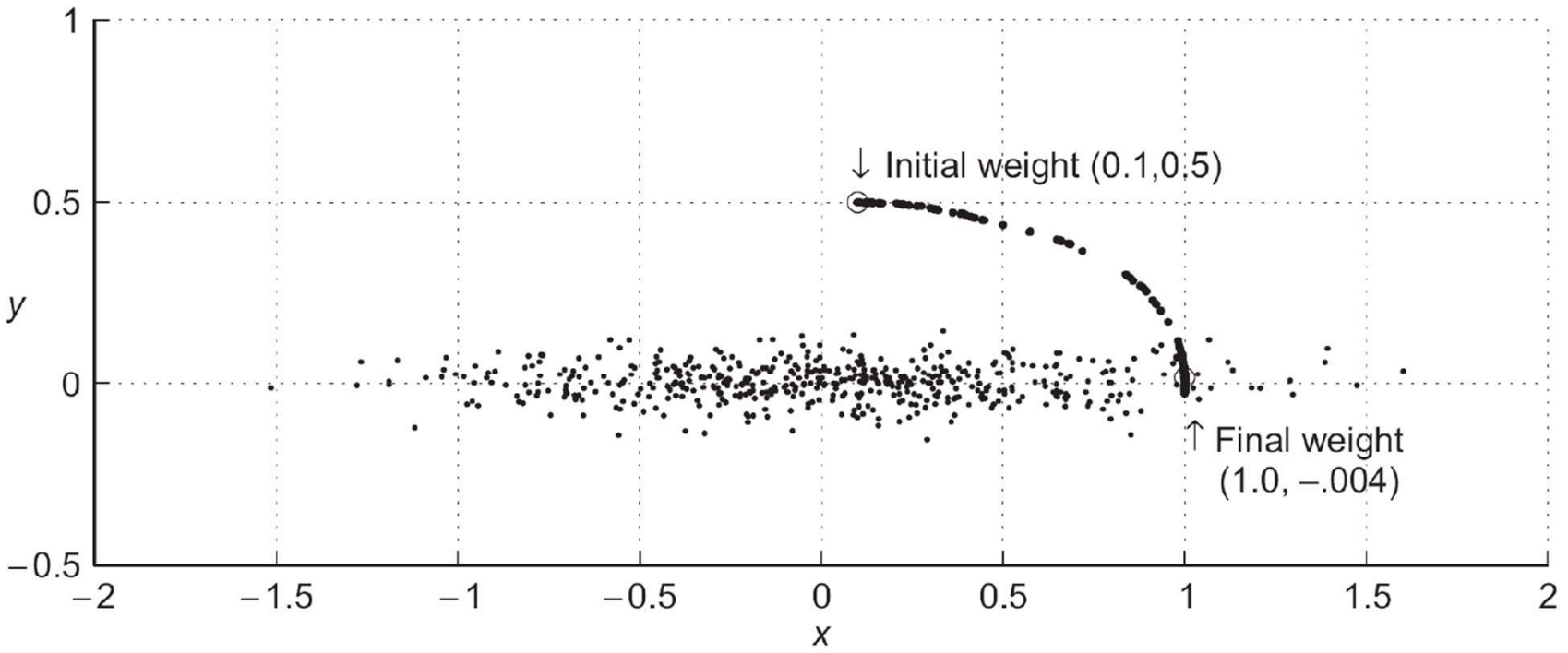
$$\eta_j^T E[\Delta W] = (\lambda_j - \lambda_i)\eta_j^T \epsilon$$

clearly shows that the perturbation component along η_j must grow if $\lambda_j > \lambda_i$

Operational Summary for Simulation of Oja's rule

Given	A set of feature vectors $\mathcal{X} = \{X_i\}$ drawn from a stationary stochastic distribution $p(X)$.
Initialize	↬ Weight vector $W \in \mathbb{R}^n$ of a linear neuron to some small random number (no bias required) ↬ Learning rate α to a small value (say $0.05 - 0.1$) ↬ Average weight change tolerance ϵ
Iterate	↻Repeat { ⇝ Pick an X_k with uniform probability over $\mathcal{X}$ ⇝ Compute the neuron signal $s_k = y_k = X_k^T W_k$ ⇝ Update weights: $W_{k+1} = W_k + \alpha s_k (X_k - s W_k)$ } until (the average weight change $E[\Delta W] < \epsilon$)

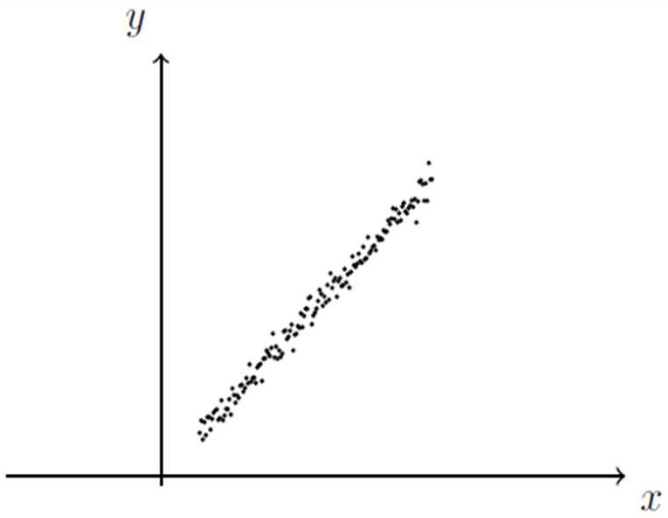
Simulation of Oja's Rule



Note: Maximal Eigen Direction is x-axis

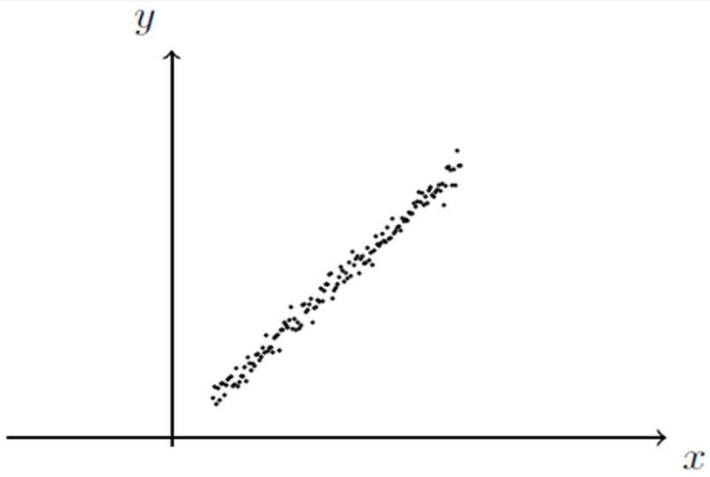
Principal Components Analysis

Principal Components Analysis



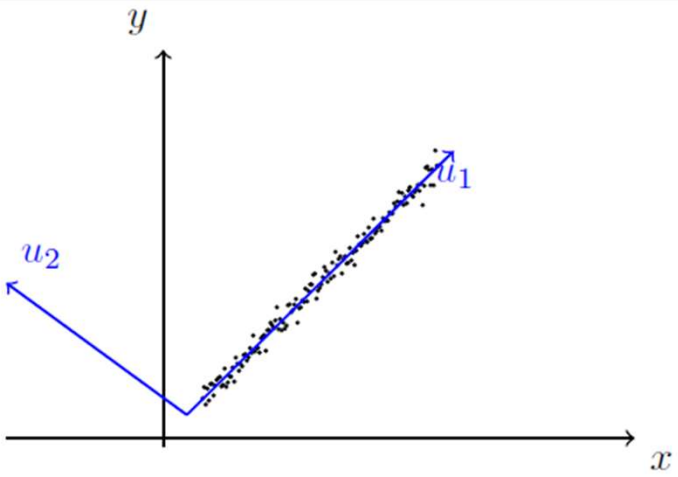
- Consider the following data
- Each point (vector) here is represented using a linear combination of the x and y axes (i.e. using the point's x and y co-ordinates)

Principal Components Analysis



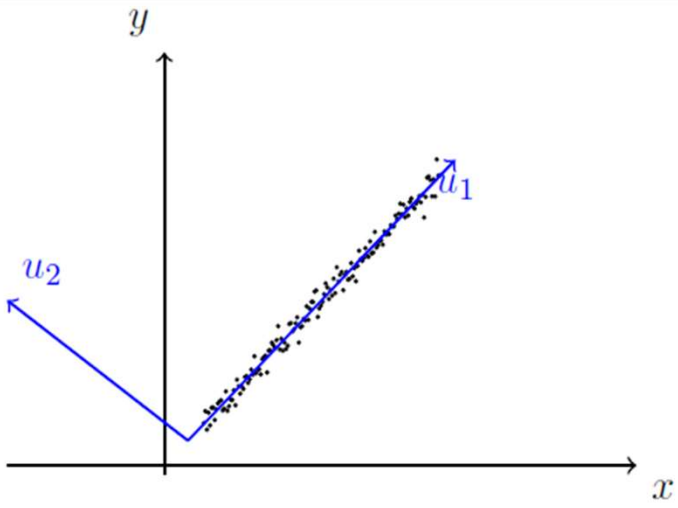
- Consider the following data
- Each point (vector) here is represented using a linear combination of the x and y axes (i.e. using the point's x and y co-ordinates)
- In other words we are using x and y as the basis
- What if we choose a different basis?

Principal Components Analysis



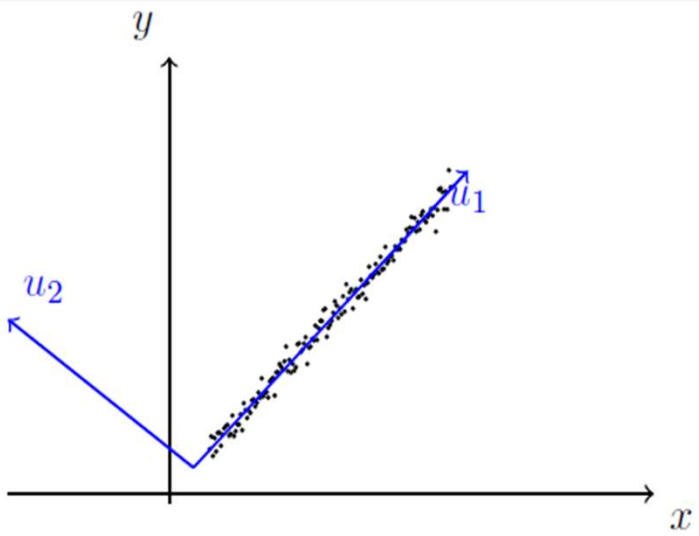
- For example, what if we use u_1 and u_2 as a basis instead of x and y.

Principal Components Analysis



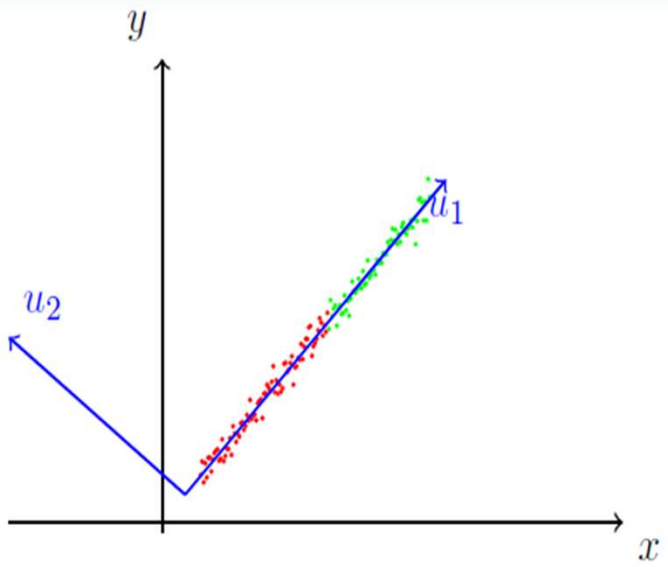
- For example, what if we use u_1 and u_2 as a basis instead of x and y.
- We observe that all the points have a very small component in the direction of u_2 (almost noise)
- It seems that the same data which was originally in $\mathbb{R}^2(x, y)$ can now be represented in $\mathbb{R}^1(u_1)$ by making a smarter choice for the basis

Principal Components Analysis



- Let's try stating this more formally
- Why do we not care about u_2?
- Because the variance in the data in this direction is very small (all data points have almost the same value in the u_2 direction)

Principal Components Analysis



- In general, we are interested in representing the data using fewer dimensions such that the data has high variance along these dimensions
- Is that all?

Principal Components Analysis

x	y	z
1	1	1
0.5	0	0
0.25	1	1
0.35	1.5	1.5
0.45	1	1
0.57	2	2.1
0.62	1.1	1
0.73	0.75	0.76
0.72	0.86	0.87

- Consider the following data
- Is z adding any new information beyond what is already contained in y?

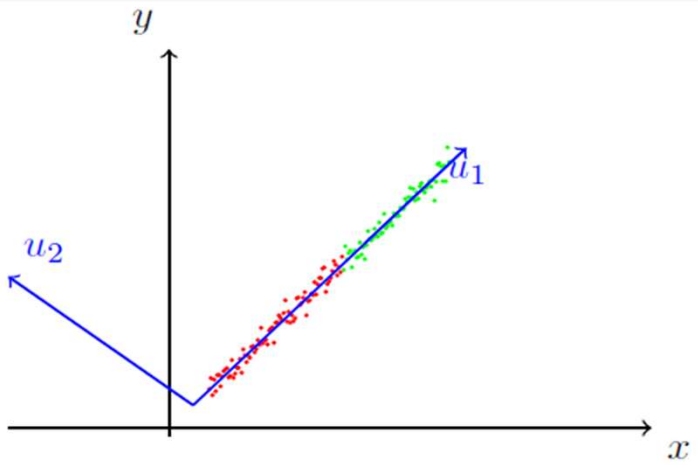
Principal Components Analysis

x	y	z
1	1	1
0.5	0	0
0.25	1	1
0.35	1.5	1.5
0.45	1	1
0.57	2	2.1
0.62	1.1	1
0.73	0.75	0.76
0.72	0.86	0.87

$$\rho_{yz} = \frac{\sum_{i=1}^{n}(y_i - \overline{y})(z_i - \overline{z})}{\sqrt{\sum_{i=1}^{n}(y_i - \overline{y})^2}\sqrt{\sum_{i=1}^{n}(z_i - \overline{z})^2}}$$

- Consider the following data
- Is z adding any new information beyond what is already contained in y?
- The two columns are highly correlated (or they have a high covariance)
- In other words the column z is redundant since it is linearly dependent on y.

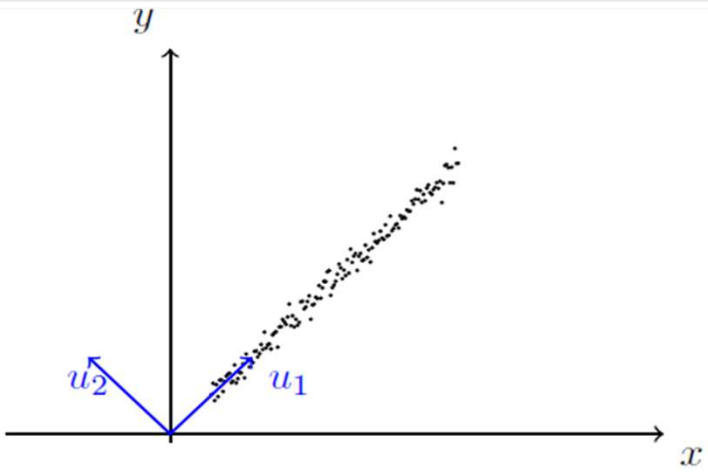
Principal Components Analysis



In general, we are interested in representing the data using fewer dimensions such that

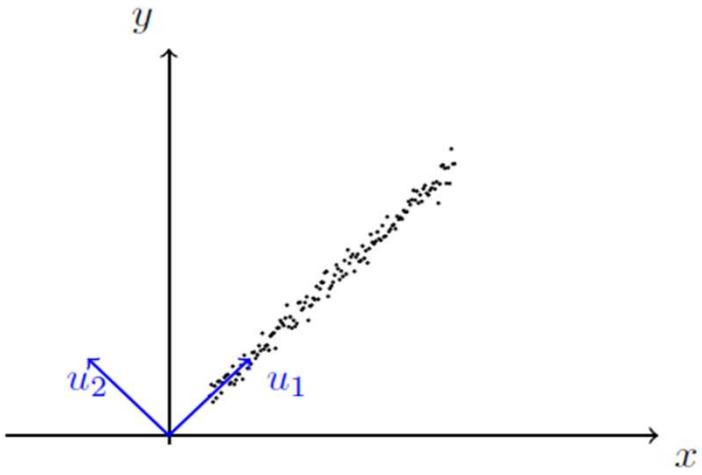
- the data has high variance along these dimensions
- the dimensions are linearly independent (uncorrelated)
- (even better if they are orthogonal because that is a very convenient basis)

Principal Components Analysis



- $u_1 = [1, 1]$ and $u_2 = [-1, 1]$ are the new basis vectors

Principal Components Analysis



- $u_1 = [1, 1]$ and $u_2 = [-1, 1]$ are the new basis vectors
- Let us convert them to unit vectors $u_1 = \left[\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right]$ & $u_2 = \left[\frac{-1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right]$

- Consider the point $x = [3.3, 3]$ in the original data
- $\alpha_1 = x^T u_1 = 6.3/\sqrt{2}$
  $\alpha_2 = x^T u_2 = -0.3/\sqrt{2}$
- the perfect reconstruction of x is given by (using $n = 2$ dimensions)

$$x = \alpha_1 u_1 + \alpha_2 u_2 = [3.3 \quad 3]$$

- But we are going to reconstruct it using fewer (only $k = 1 < n$ dimensions, ignoring the low variance u_2 dimension)

$$\hat{x} = \alpha_1 u_1 = [3.15 \quad 3.15]$$

(reconstruction with minimum error)

Principal Components Analysis

- The eigen vectors of a matrix with distinct eigenvalues are linearly independent
- The eigen vectors of a square symmetric matrix are orthogonal
- PCA exploits this fact by representing the data using a new basis comprising only the top-k eigen vectors
- The $n - k$ dimensions which contribute very little to the reconstruction error are discarded

Principal Components Analysis

- Eigenvectors of the correlation matrix of the input data stream characterize the properties of the data set
- Represent *principal component directions* (orthogonal directions) in the input space that account for the data's variance
- High dimensional applications:
  - possible to neglect information in certain less important directions
  - retaining the information along other more important ones
  - reconstruct the data points to well within an acceptable error tolerance.

Subspace Decomposition

- To reduce dimension
  - Analyze the correlation matrix R of the data stream to find its eigenvectors and eigenvalues
  - Project the data onto the eigendirections.
  - Discard n-m components corresponding to n-m smallest eigenvalues

Sanger's Rule

- m node linear neuron network that accepts n-dimensional inputs can extract the first m principal components

$$\Delta w_{ij} = \alpha s_j (x_i - \sum_{k=1}^{j} s_k w_{ik}) \qquad j = 1, \ldots, m$$

- Sanger's rule reduces to Oja's learning rule for a single neuron
- Searches the first (and maximal) eigenvector or first principal component of the input data stream
- Weight vectors of the m units converge to the first *m* eigenvectors that correspond to eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$

Competitive Neural Networks

- Competitive networks
  - cluster
  - encode
  - classify

  data by identifying
  - vectors which logically belong to the same category
  - vectors that share similar properties
- Competitive learning algorithms use competition between lateral neurons in a layer (via lateral interconnections) to provide selectivity (or localization) of the learning process

Types of Competition

- *Hard competition*
  - exactly one neuron—the one with the largest activation in the layer—is declared the winner
  - ART 1 F_2 layer
- *Soft competition*
  - competition suppresses the activities of all neurons except those that might lie in a neighbourhood of the true winner
  - Mexican Hat Nets

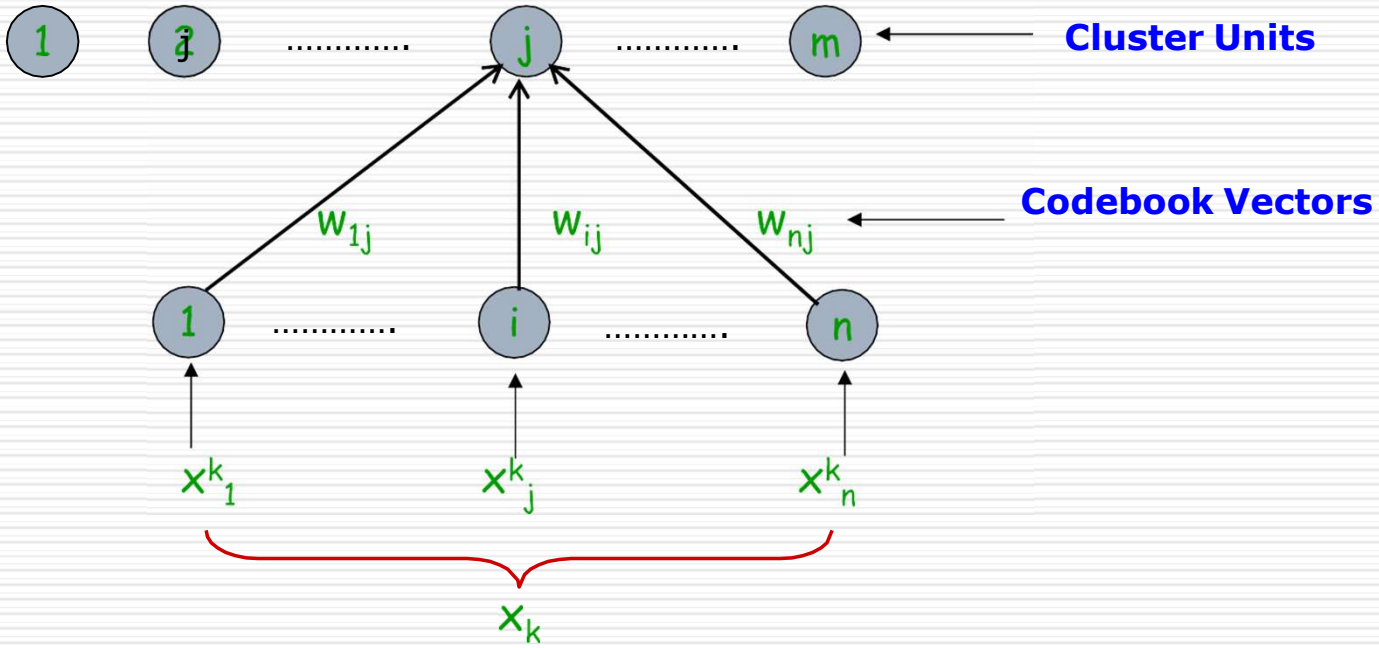
Competitive Learning is Localized

- CL algorithms employ *localized learning*
  - update weights of only the active neuron(s)
- CL algorithms identify *codebook vectors* that represent invariant features of a cluster or class

Vector Quantization

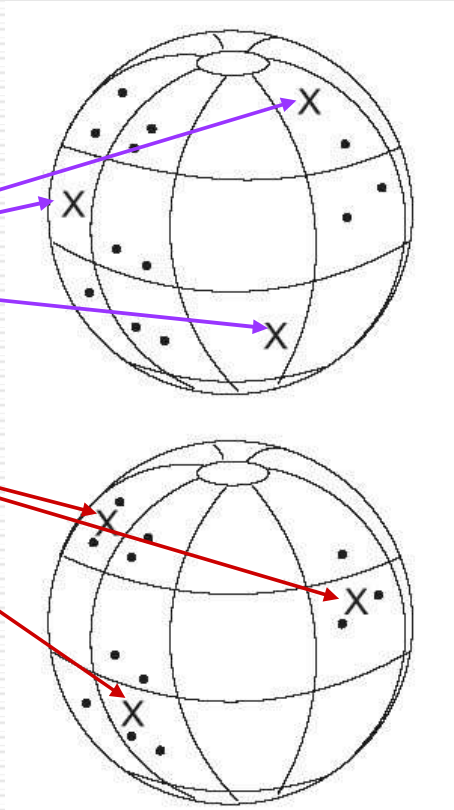
- If many patterns X_k cause cluster neuron j to fire with maximum activation a codebook vector $W_j = (w_{1j}, \ldots, w_{nj})^T$ behaves like a quantizing vector
- Quantizing vector : representative of all members of the cluster or class
- This process of representation is called vector quantization
- Principal Applications
  - signal compression
  - function approximation
  - image processing

Competitive Learning Network



Example of CL

- Three clusters of vectors (denoted by solid dots) distributed on the unit sphere
- Initially randomized codebook vectors (crosses) move under influence of a competitive learning rule to approximate the centroids of the clusters
- Competitive learning schemes use codebook vectors to approximate centroids of data clusters



Principle of Competitive Learning

- Given a sequence of stochastic vectors $X_k \in \Re^n$ drawn from a possibly unknown distribution, each pattern X_k is compared with a set of initially randomized weight vectors $W_j \in \Re^n$ and the vector W_J which best matches X_k is to be updated to match X_k more closely

Inner Product vs Euclidean Distance Based Competition

- Inner Product

$$y_J = \max_j \{ X_k^T W_j \}$$

- Euclidean Distance Based Competition

$$\| X_k - W_J \| = \min_j \{ \| X_k - W_j \| \}$$

Generalized CL Law

- For an n - neuron competitive network

$$w_{ij}^{k+1} = \begin{cases} w_{ij}^{k} + \eta(x_i^k - w_{ij}^k) & i = 1, \ldots, n \quad j = J \\ w_{ij}^{k} & i = 1, \ldots, n \quad j \neq J \end{cases}$$

$$J = \arg\max_j \{y_j^k\}$$

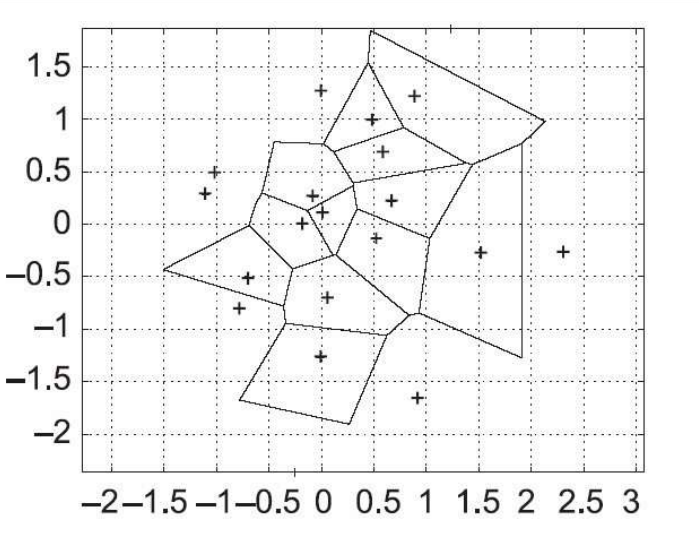
Vector Quantization

- An important application of competitive learning
- Originally developed for information compression applications
- Routinely employed to store and transmit speech or vision data.
- VQ places codebook vectors W_i into the signal space in a way that minimizes the expected quantization error

$$E = \int \|X - W_J\|^2 p(X) dX$$

Example: Voronoi Tesselation

- Depict classification regions that are formed using the 1-nearest neighbour classification rule
- Voronoi bin specified by a codebook vector W_J is simply the set of points in R^n whose nearest neighbour of all W_j is W_J a Euclidean distance measure



20 randomly generated Gaussian distributed points using the MATLAB **voronoi** command

K-nearest neighbour classification with k=1

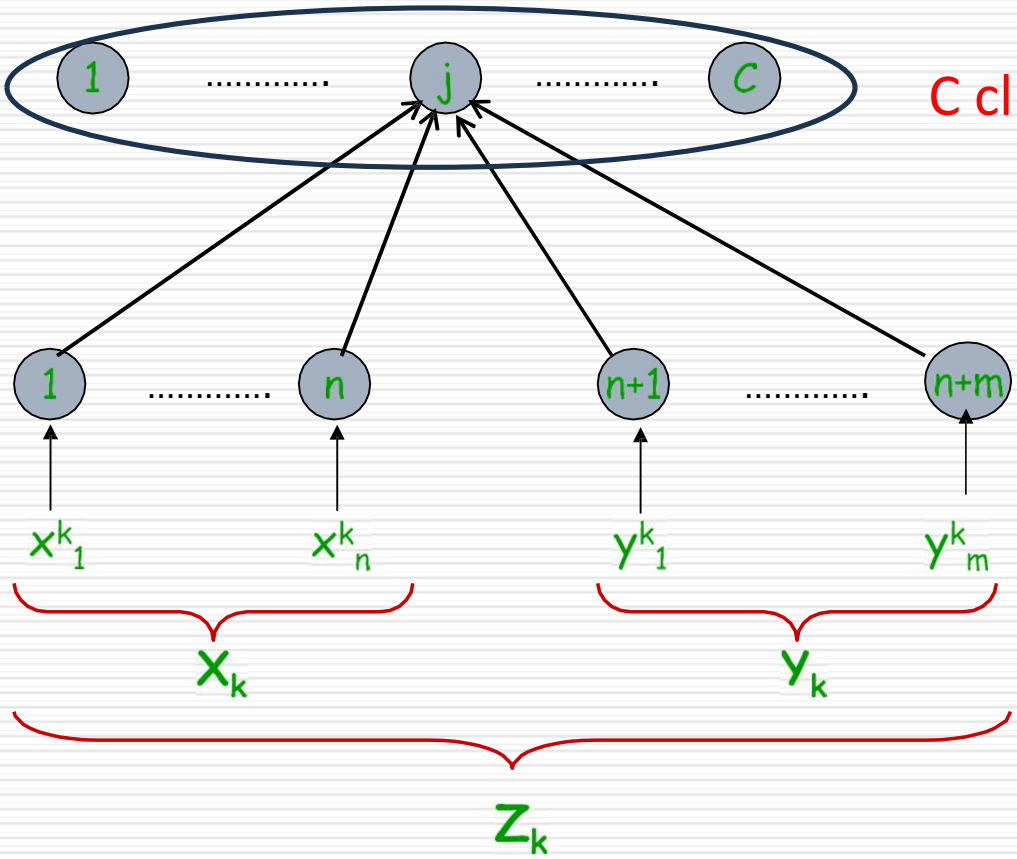
Unsupervised Vector Quantization

- Compares the current random sample vector $Z_k = (X_k \mid Y_k)$ with the C quantizing weight vectors $W_j(k)$ (weight vector W_j at time instant k)
- Neuron J wins based on a standard Euclidean distance competition

$$\| W_{J(k)} - Z_k \| = \min_j \| W_{j(k)} - Z_k \|$$

Winning vector W_j learns ;losing vectors do not learn

Unsupervised Vector Quantization



C cluster neurons

Unsupervised VQ Learning

- Neuron J learns the input pattern in accordance with standard competitive learning in vector form:

$$W_{J(k+1)} = W_{J(k)} + \eta_k (Z_k - W_{J(k)})$$

- Learning coefficient η_k should decrease gradually towards zero
- Example: $\eta_k = \eta_0[1 - k/2Q]$ for an initial learning rate η_0 and Q training samples
- Makes η decrease linearly from η_0 to zero over 2Q iterations

Scaling the Data Components

- Scale data samples $\{Z_k\}$ such that all features have equal weight in the distance measure
- Ensures that no one variable dominates the choice of the winner **-Normalized Euclidean Distance**
- Embedded within the distance computation:

$$(W_{J(k)} - Z_k)^T \Omega (W_{J(k)} - Z_k) = \min_j \left\{ (W_{j(k)} - Z_k)^T \Omega (W_{j(k)} - Z_k) \right\}$$

$$\Omega = \begin{bmatrix} \omega_1^2 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \omega_{m+n}^2 \end{bmatrix}$$

Ω is a data dependent scaling matrix computed from maximum and minimum feature values in each dimension

Operational Summary of AVQ

Given	Concatenated input data, $\{Z_k\}_{k=1}^{Q}$, $Z_k \in \mathbb{R}^{n+m}$, preprocessed for normalization and scaling.
Initialize	$\hookrightarrow$ Number of clusters C to be generated. $\hookrightarrow$ Quantization vectors W_{j0} of all C clusters to random samples of the input data. $\hookrightarrow$ Learning rate schedule: $\eta_k = 0.1\left(1-\left(\frac{k}{2Q}\right)\right)$, $\eta_0 = 0.1$ $\hookrightarrow$ Maximum number of iterations MAXITER = Q or $2Q$ $\hookrightarrow$ Iteration index $k = 0$.

Operational Summary of AVQ

Adaptive Vector Quantization(AVQ)

Iterate ↻Repeat

{

⇝ Pick a data sample Z_k from a random sequence

⇝ Find the winning neuron index J:

$$\|W_{J(k)} - Z_k\| = \min_j\{\|W_{j(k)} - Z_k\|\}$$

⇝ Update the winning neuron synapses:

$$w_{iJ}^{k+1} = w_{iJ}^k + \eta_k(z_i^k - w_{iJ}^k) \quad i = 1, \ldots, n+m$$

⇝ Update learning rate:

$$\eta_k = 0.1\left(1-\left(\frac{k}{2Q}\right)\right).$$

} while ($k <$ MAXITER)

AVQ algorithm extracts the centroids of data clusters

Supervised Vector Quantization

- Suggested by Kohonen
- Uses a supervised version of vector quantization
  - *Learning vector quantization* (LVQ1)
- Data classes defined in advance and each data sample is labelled with its class

$$w_{iJ}^{k+1} = \begin{cases} w_{iJ}^{k} + \eta_k \, (x_i^k - w_{iJ}^{k}) & \text{if} \quad X_k \in \mathcal{C}_J \\ w_{iJ}^{k} - \eta_k \, (x_i^k - w_{iJ}^{k}) & \text{if} \quad X_k \notin \mathcal{C}_J \end{cases}$$

$$w_{ij}^{k+1} = w_{ij}^{k} \qquad \text{for} \quad j \neq J$$

Practical Aspects of LVQ1

- $0 < \eta_k < 1$ decreases monotonically with successive iterations
- Recommended that η_k be kept small: 0.1
- Vectors in a limited training set may be applied cyclically to the system as η_k is made to decrease linearly to zero
- Use an equal number of codebook vectors per class
  - Leads to an optimal approximation of the class borders
- Initialization of codebook vectors may be done to actual samples of each class
- Define the number of iterations in advance:
  - Anything from 50 to 200 times the number of codebook vectors selected for representation

Operational Summary of LVQ1

Learning Vector Quantization(LVQ)

Given	Input stream of labelled vectors $\{X_k\}_{k=1}^{Q}$ $X_k \in \mathbb{R}^n$ that belong to one of C classes $\{\mathcal{C}_j\}_{j=1}^{C}$
Initialize	$\looparrowright$ Number of classes C to be generated. $\looparrowright$ Quantization vectors W_{j0} of all C classes to random samples of the input data. $\looparrowright$ Learning rate schedule: $\eta_k = 0.1\left(1 - \left(\frac{k}{2Q}\right)\right)$, $\eta_0 = 0.1$ $\looparrowright$ Maximum number of iterations MAXITER = Q or $2Q$ $\looparrowright$ Iteration index $k = 0$.
Iterate	⟳Repeat { $\rightsquigarrow$ Pick a data sample X_k from the data stream $\rightsquigarrow$ Find the winning neuron index J: $\|W_{J(k)} - X_k\| = \min_j\{\|W_{j(k)} - X_k\|\}$ $\rightsquigarrow$ Update only the winning neuron synapses: $w_{iJ}^{k+1} = w_{iJ}^{k} + \eta_k(x_i^k - w_{iJ}^k)$ if $X_k \in \mathcal{C}_J$ $w_{iJ}^{k+1} = w_{iJ}^{k} - \eta_k(x_i^k - w_{iJ}^k)$ if $X_k \notin \mathcal{C}_J$ $\rightsquigarrow$ Update learning rate $\eta_k = 0.1\left(1 - \left(\frac{k}{2Q}\right)\right)$. } while ($k <$ MAXITER)

Learning Vector Quantization(LVQ): Example

Consider the following five input vectors and their target class.

VECTOR	CLASS LABEL
[0 0 1 1]	1
[1 0 0 0]	2
[0 0 0 1]	2
[1 1 0 0]	1
[0 1 1 0]	1

https://www.turing.com/kb/application-of-learning-vector-quantization

Learning Vector Quantization(LVQ): Example

- In each input vector, there are four input components (x1, x2, x3, x4) and two target classes (1, 2).
- Weights assigned based on the class:
  - As there are two target classes, the first two vectors can be used as weight vectors as w1 = [0 0 1 1] & w2 = [1 0 0 0].

VECTOR	CLASS LABEL
[0 0 1 1]	1
[1 0 0 0]	2
[0 0 0 1]	2
[1 1 0 0]	1
[0 1 1 0]	1

$$W = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Learning Vector Quantization(LVQ): Example

- Assume learning rate α as 0.1.
- Let's take Input vector: [0 0 0 1]; Target class:2
- Calculate the Euclidean distance between X and W

i.e. calculate D(1) and D(2) which are the distance of the input unit from the first and second weight vectors, respectively.

- D(1) is lesser than D(2) and the winner index is J = 1.

https://www.turing.com/kb/application-of-learning-vector-quantization

Learning Vector Quantization(LVQ): Updating Weights

Since target class 2 is not equal to J, weights are updated as follows:

$$W_{11}(new) = w_{11}(old) - \alpha[x_1 - w_{11}(old)] = 0 - 0.1[0 - 0] = 0$$

$$W_{21}(new) = w_{21}(old) - \alpha[x_2 - w_{21}(old)] = 0 - 0.1[0 - 0] = 0$$

$$W_{31}(new) = w_{31}(old) - \alpha[x_3 - w_{31}(old)] = 1 - 0.1[0 - 1] = 1.1$$

$$W_{41}(new) = w_{41}(old) - \alpha[x_4 - w_{41}(old)] = 1 - 0.1[1 - 1] = 1$$

$$W = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1.1 & 0 \\ 1 & 0 \end{bmatrix}$$

https://www.turing.com/kb/application-of-learning-vector-quantization

Learning Vector Quantization(LVQ): Updating Weights

Input vector: [1 1 0 0]
Target class: 1

$$W = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1.1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$D(1) = (0 - 1)^2 + (0 - 1)^2 + (1.1 - 0)^2 + (1 - 0)^2 = 4.21$$

$$D(2) = (1 - 1)^2 + (0 - 1)^2 + (0 - 0)^2 + (0 - 0)^2 = 1$$

D(2) is less than D(1) and the winner index is J = 2.

https://www.turing.com/kb/application-of-learning-vector-quantization

Learning Vector Quantization(LVQ): Updating Weights

Since the target class 2 is not equal to J, updating the weight can be done by:

$$W_{12}(new) = w_{12}(old) - \alpha[x_1 - w_{12}(old)] = 1 - 0.1[1 - 1] = 1$$

$$W_{22}(new) = w_{22}(old) - \alpha[x_2 - w_{22}(old)] = 0 - 0.1[1 - 0] = -0.1$$

$$W_{32}(new) = w_{32}(old) - \alpha[x_3 - w_{32}(old)] = 0 - 0.1[0 - 0] = 0$$

$$W_{42}(new) = w_{42}(old) - \alpha[x_4 - w_{42}(old)] = 0 - 0.1[0 - 0] = 0$$

$$W = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \\ 1.1 & 0 \\ 1 & 0 \end{bmatrix}$$

https://www.turing.com/kb/application-of-learning-vector-quantization

Learning Vector Quantization(LVQ): Updating Weights

Input vector: [0 1 1 0]
Target class: 1

$$w = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \\ 1.1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$D(1) = (0 - 0)^2 + (0 - 1)^2 + (1.1 - 1)^2 + (1 - 0)^2 = 2.01$$

$$D(2) = (1 - 0)^2 + (-0.1 - 1)^2 + (0 - 1)^2 + (0 - 0)^2 = 3.21$$

https://www.turing.com/kb/application-of-learning-vector-quantization

Learning Vector Quantization(LVQ): Updating Weights

Input vector: [0 1 1 0]
Target class: 1

$$W = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \\ 1.1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$D(1) = (0 - 0)^2 + (0 - 1)^2 + (1.1 - 1)^2 + (1 - 0)^2 = 2.01$$

$$D(2) = (1 - 0)^2 + (-0.1 - 1)^2 + (0 - 1)^2 + (0 - 0)^2 = 3.21$$

D(1) is lesser than D(2) and the winner index is J = 1.

https://www.turing.com/kb/application-of-learning-vector-quantization

Learning Vector Quantization(LVQ): Updating Weights

Since target class 1 is equal to J, weights are updated as follows:

$$W_{11}(new) = w_{13}(old) + \alpha[x_1 - w_{11}(old)] = 0 + 0.1[0 - 0] = 0$$

$$W_{21}(new) = w_{21}(old) + \alpha[x_2 - w_{21}(old)] = 0 + 0.1[1 - 0] = 0.1$$

$$W_{31}(new) = w_{33}(old) + \alpha[x_3 - w_{31}(old)] = 1 + 0.1[1.1 - 1] = 1.09$$

$$W_{41}(new) = w_{41}(old) + \alpha[x_4 - w_{41}(old)] = 1 + 0.1[0 - 1] = 0.9$$

$$W = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.1 \\ 1.09 & 0 \\ 0.9 & 0 \end{bmatrix}$$

https://www.turing.com/kb/application-of-learning-vector-quantization

Self-Organizing Feature Maps

- Dimensionality reduction + preservation of topological information common in normal human subconscious information processing
- Humans
  - routinely compress information by extracting relevant facts
  - develop reduced representations of impinging information while retaining essential knowledge
- Example: Biological vision
  - Three dimensional visual images routinely mapped onto a two dimensional retina
  - Information preserved to permit perfect visualization of a three dimensional world

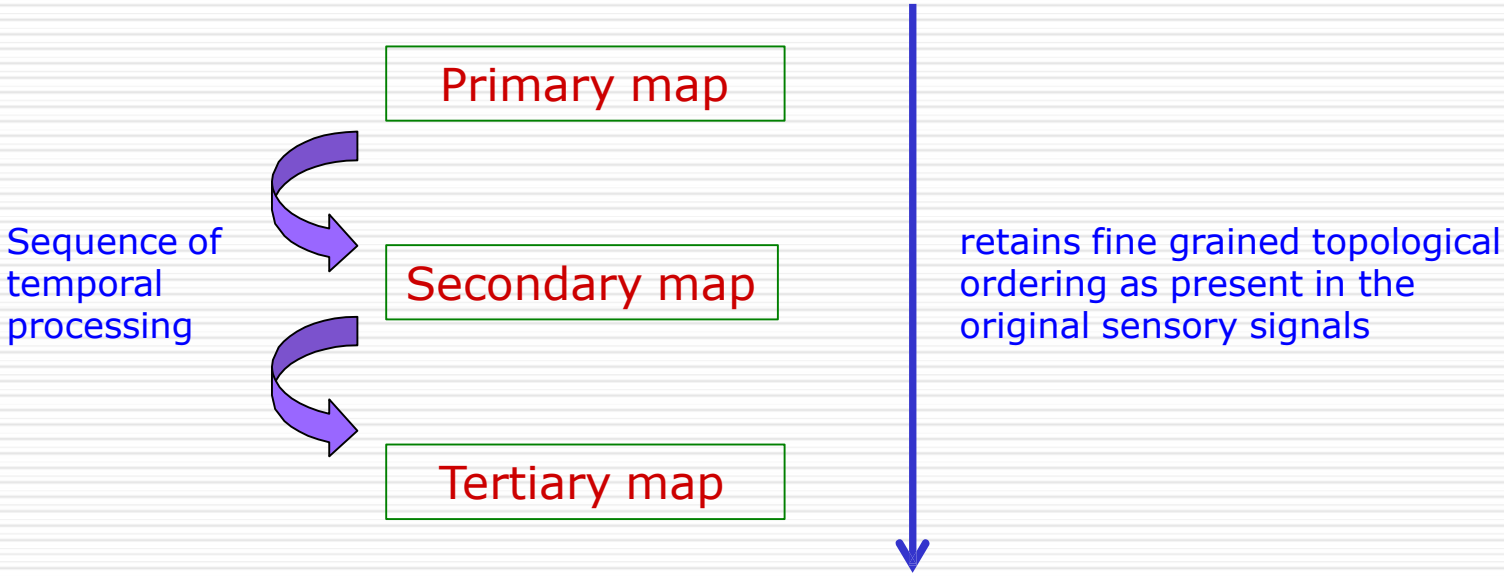
Purpose of Intelligent Information Processing (Kohonen)

- Lies in the creation of simplified internal representations of the external world at different levels of abstraction

Computational Maps

- Early evidence for computational maps comes from the studies of Hubel and Wiesel on the primary visual cortex of cats and monkeys
- Specialized sensory areas of the cortex respond to the available spectrum of real world signals in an ordered fashion

A Hierarchy of Maps in Human brain



Topology Preservation

- Kohonen
  - " . . . it will be intriguing to learn that an almost optimal spatial order, in relation to signal statistics can be completely determined in simple self-organizing processes under control of received information"

Topological Maps

- Topological maps preserve an order or a metric defined on the impinging inputs
- Motivated by the fact that representation of sensory information in the human brain has a geometrical order
- The same functional principle can be responsible for diverse (self-organized) representations of information—possibly even hierarchical

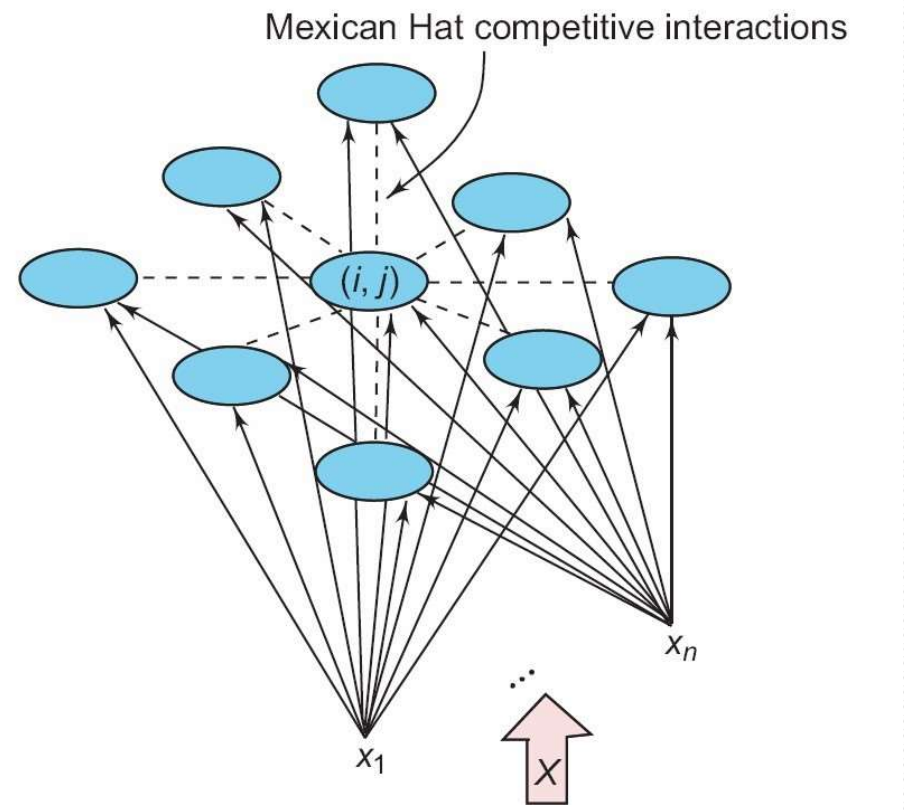
Self-Organizing Feature Map

- Finds its origin in the seminal work of von der Malsburg on self-organization
- Basic idea:
  - In addition to a genetically wired visual cortex there has to be some scope for self-organization of synapses of domain sensitive neurons to allow a local topographic ordering to develop

Self-Organizing Feature Map: Underlying Ideas

- Unsupervised learning process
- Is a competitive vector quantizer
- Real valued patterns are presented sequentially to a linear or planar array of neurons with Mexican hat interactions
- Clusters of neurons win the competition
- Weights of winning neurons are adjusted to bring about a better response to the current input
- Final weights specify clusters of network nodes that are topologically close
  - sensitive to clusters of inputs that are physically close in the input space
- Correspondence between signal features and response locations on the map
  - spatial location of a neuron in the array corresponds to a specific domain of inputs
- *Preserves the topology of the input*

SOFM Network Architecture



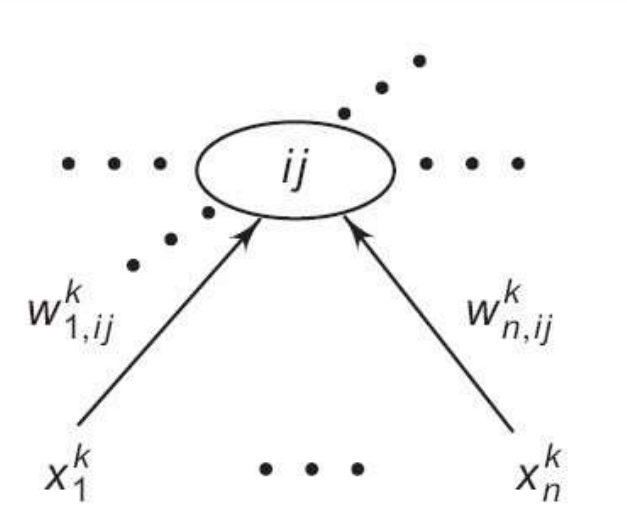
Requirements

- Distance relations in high dimensional spaces should be approximated by the network as the distances in the two dimensional neuronal field:
  - input neurons should be exposed to a sufficient number of different inputs
  - only the winning neuron and its neighbours adapt their connections
  - a similar weight update procedure is employed on neurons which comprise *topologically related subsets*
  - the resulting adjustment enhances the responses to the same or to a similar input that occurs subsequently

Notation

- Each neuron is identified by the double row-column index ij, i, j = 1, . . . ,m
- The ij th neuron has an incoming weight vector

  $W_{ij}(k) = (w^k_{1,ij}, \ldots, w^k_{n,ij})$



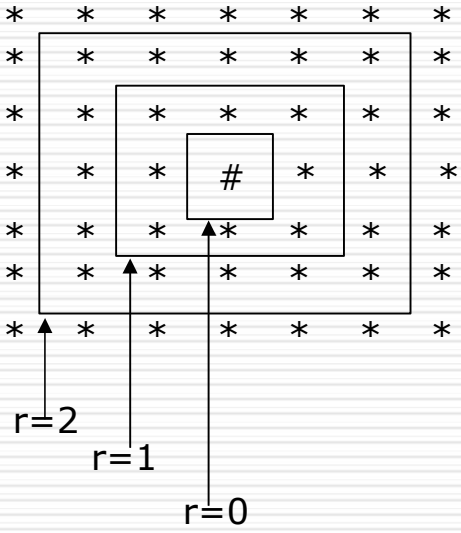
Neighbourhood Computation

- Identify a *neighbourhood* N_{IJ} around the winning neuron
- Winner identified by minimum Euclidean distance to input vector:

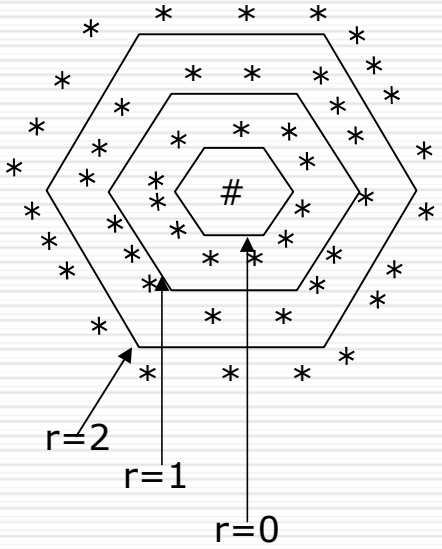
$$\|X_k - W_{IJ(k)}\| = \min_{i,j}\{\|X_k - W_{ij(k)}\|\}$$

- Neighbourhood is a function of time: as epochs of training elapse, the neighbourhood shrinks

Neighbourhood Shapes



Square neighbourhood



Hexagonal neighbourhood

Adaptation in SOFM

- Takes place according to the second generalized law of adaptation

$$\dot{w}_{l,ij} = \eta x_l s_{ij} - \gamma(s_{ij}) w_{l,ij}$$

- $\gamma(s_{ij})$ may be chosen to be linear

$$\dot{w}_{l,ij} = \eta x_l s_{ij} - \beta s_{ij} w_{l,ij}$$

- Choosing $\eta = \beta$

$$\dot{w}_{l,ij} = \eta s_{ij}(x_l - w_{l,ij})$$

γ is scaled function of neuronal signal

SOFM Adaptation

- Continuous time

$$\dot{W}_{ij} = \begin{cases} \eta(X - W_{ij}) & ij \in \mathcal{N}_{IJ} \\ 0 & ij \notin \mathcal{N}_{IJ} \end{cases}$$

- Discrete time

$$W_{ij(k+1)} = \begin{cases} W_{ij(k)} + \eta_k(X_k - W_{ij(k)}) & ij \in \mathcal{N}_{IJ}^k \\ W_{ij(k)} & ij \notin \mathcal{N}_{IJ}^k \end{cases}$$

Some Observations

- *Ordering phase* (initial period of adaptation) : learning rate should be close to unity
- Learning rate should be decreased linearly, exponentially or inversely with iteration over the first 1000 epochs while maintaining its value above 0.1
- *Convergence phase:* learning rate should be maintained at around 0.01 for a large number of epochs
  - may typically run into many tens of thousands of epochs
- During the ordering phase N^k_{IJ} shrinks linearly with k to finally include only a few neurons
- During the convergence phase N^k_{IJ} may comprise only one or no neighbours

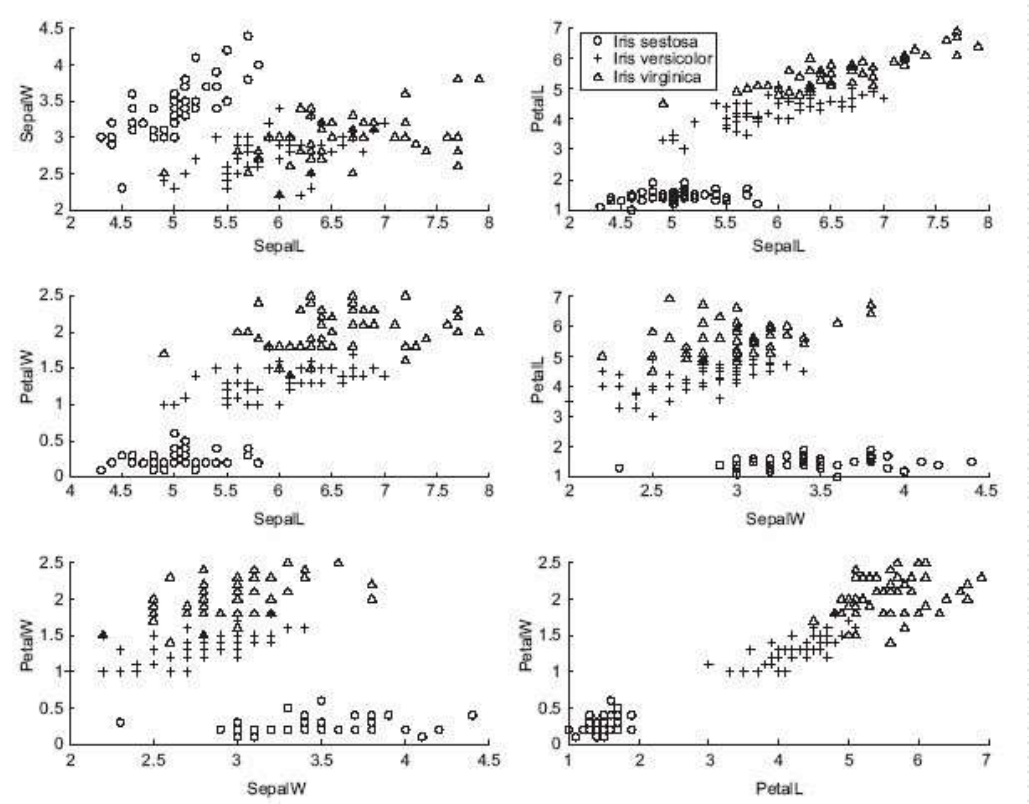
Operational Summary of the SOFM Algorithm

Given	A stream of training vectors $\{X_k\}_{k=1}^{Q}$ drawn uniformly from a possibly unknown probability distribution $p(X)$.
Initialize	↬ Weights $W_{ij(0)}$ to some small random numbers ↬ Value of the neighbourhood $\mathcal{N}_{IJ}^{k}$ ↬ Learning rate η_0
Iterate	↻Repeat { ⇝ *Selection*: Pick a sample X_k ⇝ *Similarity matching*: Find the winning neuron $(I\,J)$ $\|X_k - W_{IJ(k)}\| = \min_{(1\le i\le m)(1\le j\le m)}\{\|X_k - W_{ij(k)}\|\}$ ⇝ *Adaptation*: Update synaptic vectors of ONLY the winning cluster $w_{l,ij}^{k+1} = w_{l,ij}^{k} + \eta_k(x_l^k - w_{l,ij}^{k}) \quad ij \in \mathcal{N}_{IJ}^{k}$ ⇝ *Update*: Update $\eta_k, \mathcal{N}_{IJ}^{k}$ } until (there is no observable change in the map)

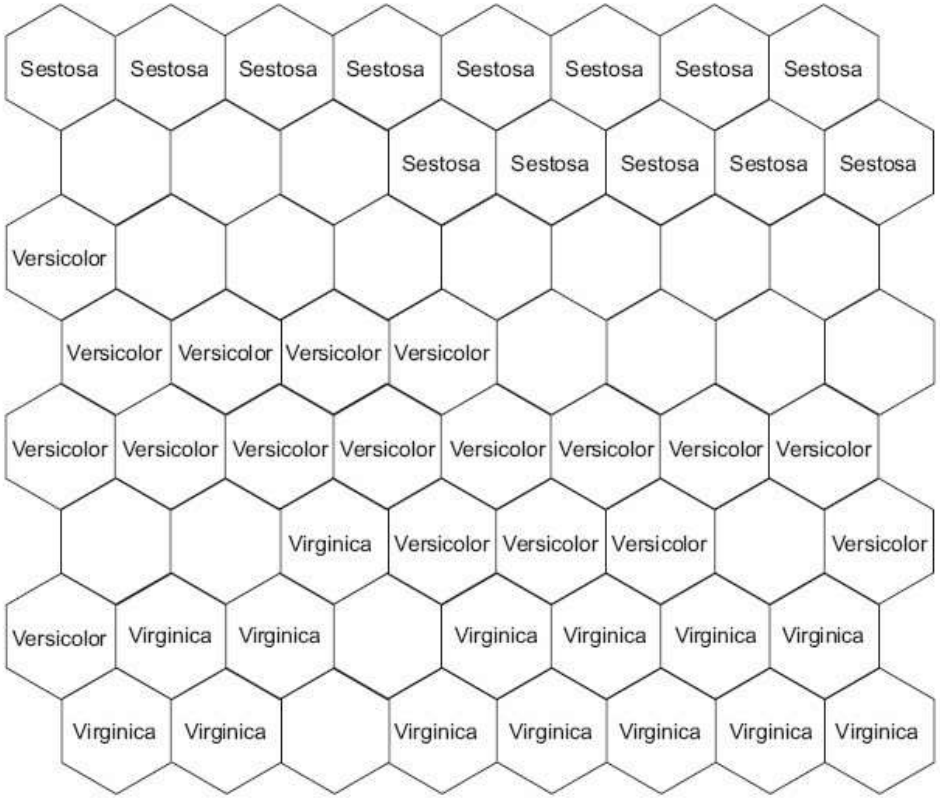
Applications of the Self-organizing Map

- Vector quantization
- Neural phonetic typewriter
- Control of robot arms

Iris Pattern Classification



Iris Pattern Classification



Feature Mapping

Generalized Learning Laws

Generalized Learning Laws

- Generalized *forgetting laws* take the form:

$$\frac{dW}{dt} = \phi(s)X - \gamma(s)W$$

- Assume that the impinging input vector $X \in \Re^n$ is a stochastic variable with stationary stochastic properties; $W \in \Re^n$ is the neuronal weight vector, and $\varphi(\cdot)$ and $\gamma(\cdot)$ are possibly non-linear functions of the neuronal signal
- Assume X is independent of W

Questions to Address

- What kind of information does the weight vector asymptotically encode?
- How does this information depend on the generalized functions $\varphi(\cdot)$ and $\gamma(\cdot)$?

Two Laws to Analyze

Adaptation Law 1

- A simple passive decay of weights proportional to the signal, and a reinforcement proportional to the external input:

$$\dot{W} = -\alpha s W + \beta X$$

Adaptation Law 2

- The standard Hebbian form of adaptation with signal driven passive weight decay:

$$\dot{W} = -\alpha s W + \beta s X$$

Analysis of Adaptation Law 1

$$\dot{W} = -\alpha s W + \beta X$$

- Since X is stochastic (with stationary properties), we are interested in the averaged or expected trajectory of the weight vector W
- Taking the expectation of both sides:

$$E\left[\dot{W}|W\right] = E\left[-\alpha s W + \beta X|W\right]$$

An Intermediate Result

$$\frac{1}{2}\frac{d}{dt}(W^T W) = \frac{1}{2}\frac{d}{dt}(\|W\|^2) = \|W\|\frac{d\|W\|}{dt} = W^T \dot{W}$$

$$= W^T(-\alpha s W + \beta X)$$

$$= -\alpha s\|W\|^2 + \beta X^T W$$

$$= s(\beta - \alpha\|W\|^2)$$

Asymptotic Analysis

- Note that the mean $\bar{X}$ is a constant
- We are interested in the average angle between the weight vector and the mean:

$$E\left[\frac{d\cos\theta}{dt}\right] = E\left[\frac{d}{dt}\left(\frac{\bar{X}^T W}{\|\bar{X}\|\|W\|}\right)\right]$$

$$= E\left[\frac{\bar{X}^T \dot{W}}{\|\bar{X}\|\|W\|} - \frac{\bar{X}^T W}{\|\bar{X}\|\|W\|^2}\frac{d\|W\|}{dt}\right]$$

Asymptotic Analysis

$$E\left[\frac{d\cos\theta}{dt}\right] = E\left[\frac{\bar{X}^T(-\alpha s W + \beta X)}{\|\bar{X}\|\|W\|} - \frac{(\bar{X}^T W)(-\alpha s\|W\|^2 + \beta X^T W)}{\|\bar{X}\|\|W\|^3}\right]$$

$$= \frac{\beta\|\bar{X}\|}{\|W\|} - \frac{\beta(\bar{X}^T W)^2}{\|\bar{X}\|\|W\|^3}$$

$$= \frac{\beta}{\|\bar{X}\|\|W\|^3}\left(\|\bar{X}\|^2\|W\|^2 - (\bar{X}^T W)^2\right)$$

$$\geq 0$$

where in the end we have employed the Cauchy–Schwarz inequality. Since $d\cos\theta/dt$ is non-negative, θ converges uniformly to zero, with $d\cos\theta/dt = 0$ iff $\bar{X}$ and W have the same direction. Therefore, for finite $\bar{X}$ and W, the weight vector direction converges asymptotically to the direction of $\bar{X}$.

Analysis of Adaptation Law 2

$$\begin{aligned}\dot{W} &= -\alpha s W + \beta s X \\ &= -\alpha s W + \beta X s \\ &= -\alpha X^T W W + \beta X X^T W\end{aligned}$$

- Taking the expectation of both sides conditional on W

$$E[\dot{W}] = -\alpha \bar{X}^T W W + \beta \mathbf{R} W$$

Fixed points of W

- To find the fixed points, set the expectation of the expected weight derivative to zero:

$$E[\dot{W}] = 0 = -\alpha \bar{X}^T \hat{W} \hat{W} + \beta \mathbf{R} \hat{W}$$

- From where

$$\mathbf{R}\hat{W} = \left(\frac{\alpha \bar{X}^T \hat{W}}{\beta} \right) \hat{W}$$
$$= \lambda \hat{W}$$

- Clearly, eigenvectors of R are fixed point solutions of W

All Eigensolutions are not Stable

- The i^{th} solution is the eigenvector η_i of **R** with corresponding eigenvalue

$$\lambda_i = \frac{\alpha \bar{X}^T \eta_i}{\beta}$$

- Define θ_i as the angle between W and η_i , and analyze (as before) the average value of rate of change of cos θ_i , conditional on W

Asymptotic Analysis

$$E\left[\frac{d\cos\theta_i}{dt}\right] = E\left[\frac{d}{dt}\left(\frac{\eta_i^T W}{\|\eta_i\|\|W\|}\right)\right]$$

$$= E\left[\frac{\eta_i^T \dot{W}}{\|\eta_i\|\|W\|} - \frac{\eta_i^T W}{\|\eta_i\|\|W\|^2}\frac{d}{dt}\|W\|\right]$$

$$= E\left[\frac{\eta_i^T \dot{W}}{\|\eta_i\|\|W\|} - \frac{\eta_i^T W W^T \dot{W}}{\|\eta_i\|\|W\|^3}\right]$$

$$= \frac{\eta_i^T(-\alpha \bar{X}^T W W + \beta \mathbf{R} W)}{\|\eta_i\|\|W\|} - \frac{\eta_i^T W W^T(-\alpha \bar{X}^T W W + \beta \mathbf{R} W)}{\|\eta_i\|\|W\|^3}$$

Contd.

Asymptotic Analysis

$$
\begin{aligned}
E\left[\frac{d\cos\theta_i}{dt}\right] &= \frac{\beta\eta_i^T\mathbf{R}W}{\|\eta_i\|\|W\|} - \frac{\beta\eta_i^T W(W^T\mathbf{R}W)}{\|\eta_i\|\|W\|^3} \\
&= \frac{\beta\lambda_i\eta_i^T W}{\|\eta_i\|\|W\|} - \frac{\beta\eta_i^T W(W^T\mathbf{R}W)}{\|\eta_i\|\|W\|^3} \\
&= \frac{\beta\eta_i^T W}{\|\eta_i\|\|W\|}\left(\lambda_i - \frac{W^T\mathbf{R}W}{\|W\|^2}\right) \\
&= \beta\cos\theta_i\left(\lambda_i - \frac{W^T\mathbf{R}W}{\|W\|^2}\right)
\end{aligned}
$$

Asymptotic Analysis

- It follows from the Rayleigh quotient that the parenthetic term is guaranteed to be positive only for $\lambda_i = \lambda_{max}$, which means that for the eigenvector η_{max} the angle θ_{max} between W and η_{max} monotonically tends to zero as learning proceeds

$$E\left[\frac{d\cos\theta_i}{dt}\right]$$

$$= \beta\cos\theta_i\left(\lambda_i - \frac{W^T\mathbf{R}W}{\|W\|^2}\right)$$

First Limit Theorem

- Let $\alpha > 0$, and $s = X^T W$. Let $\gamma(s)$ be an arbitrary scalar function of s such that $E[\gamma(s)]$ exists. Let $X(t) \in \Re^n$ be a stochastic vector with stationary stochastic properties, $\overline{X}$ being the mean of $X(t)$ and $X(t)$ being independent of W
- If equations of the form

$$\dot{W} = E[\alpha X - \gamma(s) W]$$

have non-zero bounded asymptotic solutions, then these solutions must have the same direction as that of $\overline{X}$

Second Limit Theorem

- Let α, s and $\gamma(s)$ be the same as in Limit Theorem 1. Let $\mathbf{R} = E[XX^T]$ be the correlation matrix of X. If equations of the form :

$$\dot{W} = E[\alpha s X - \gamma(s) W]$$

have non-zero bounded asymptotic solutions, then these solutions must have the same direction as η_{max} where η_{max}, is the maximal eigenvector of $\mathbf{R}$ with eigenvalue λ_{max}, provided $\eta^T_{max} W(0) = 0$