AE 308 Project

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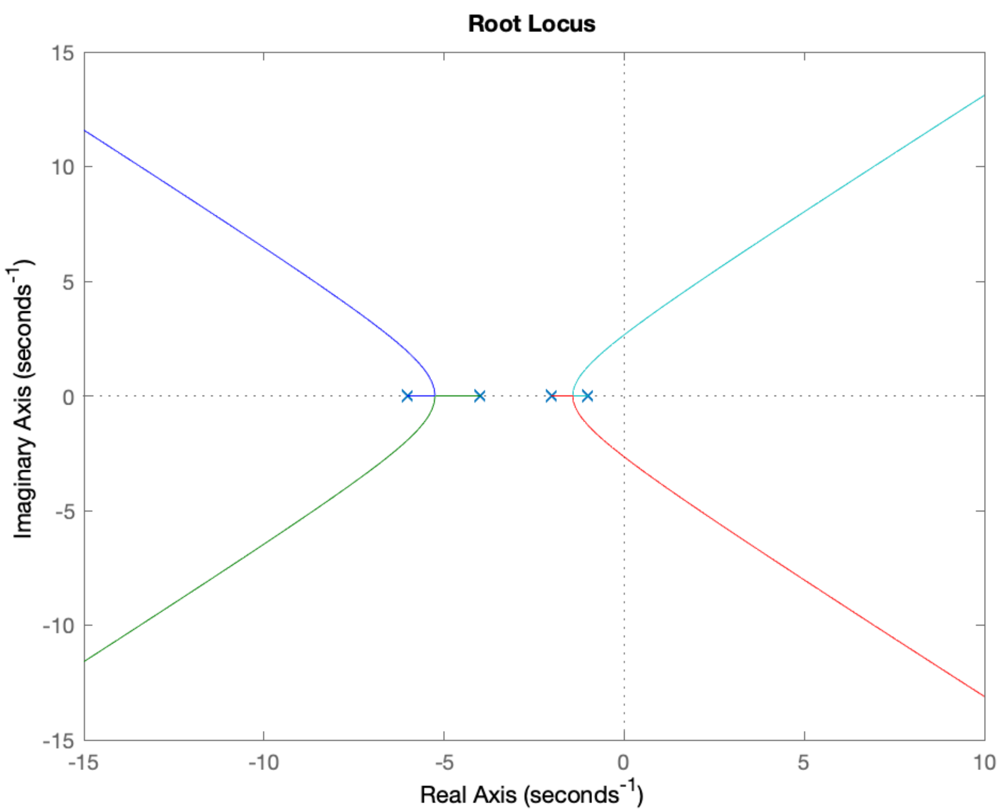
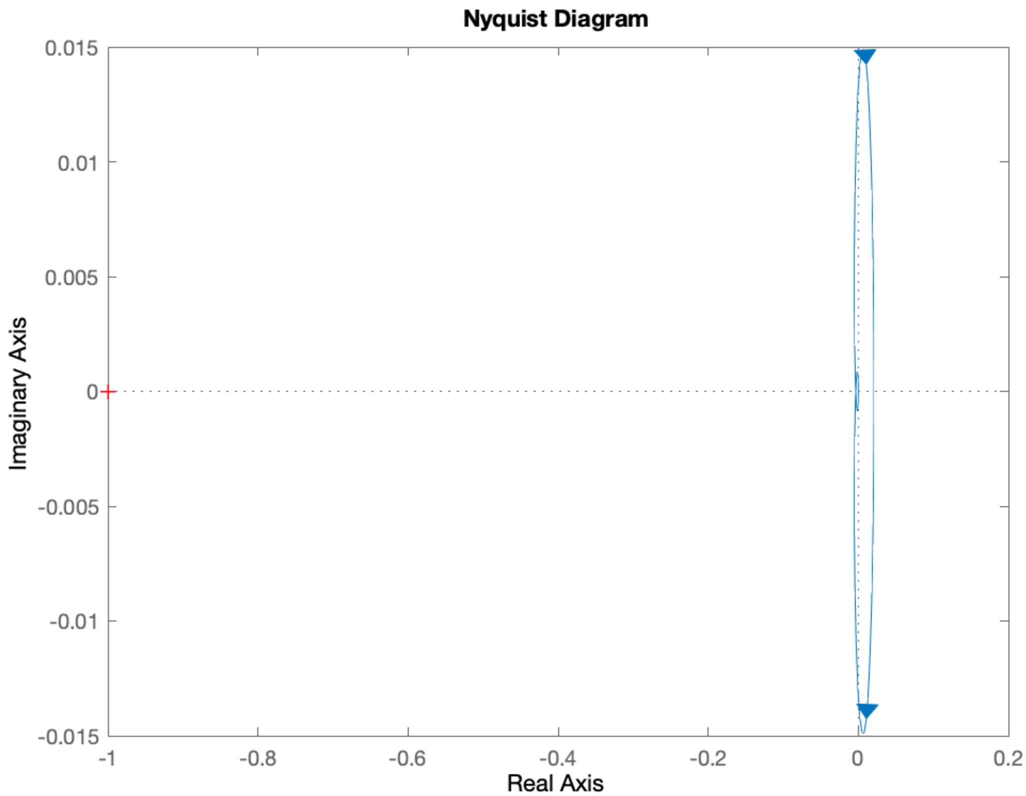
1 Analysis of Uncompensated plant

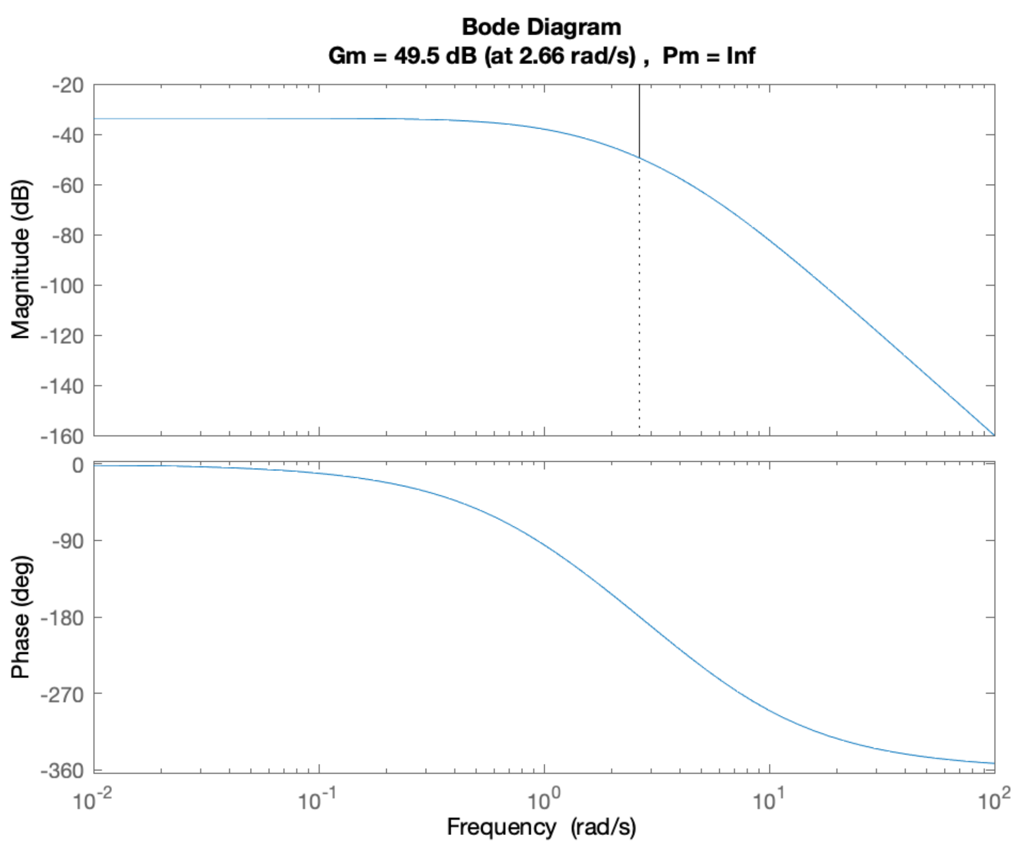
The uncompensated plant is as follows:

$$G(s) = \frac{K}{(s+1)(s+2)(s+4)(s+6)}$$

Root Locus, Nyquist plot, Bode plot of the plant is as follows:



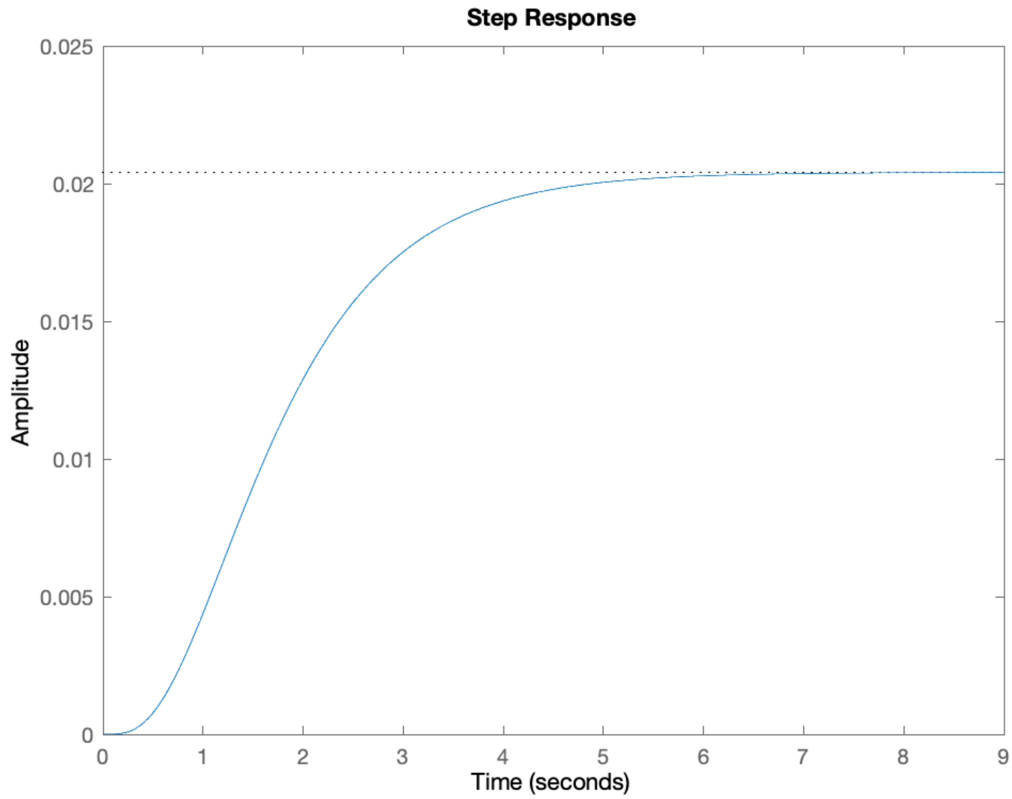




Bandwidth of the plant is 0.7937 rad/s.

From the Nyquist plot, we can see that the closed loop response is stable as there is no encircling of -1 and no poles are in right half plane (N=Z=P=0).

Closed loop step response of the plant is as follows:



The above Step Response characteristics are as follows:

RiseTime: 2.6291
SettlingTime: 4.8860
SettlingMin: 0.0184
SettlingMax: 0.0204
Overshoot: 0
Undershoot: 0
Peak: 0.0204
PeakTime: 9.9071

2 Desired Dominant Pole from Transient Response Specifications

The requirements specified by the problem statement are as follows:

- $\zeta = 0.5$
- $T_s(2\%) < 4s$
- Real part of non-dominant poles to be at least 10 times the real part of the dominant poles

The two conditions on the settling time and damping can be converted to a pole. Since the settling time is required to be less than 4 seconds, I will be extra cautious and solve for settling time of 3 seconds. So, solving for the pole that captures these two requirements:

$$T_s = \frac{4}{\sigma} = 3$$
$$\implies \sigma = 1.33$$
$$\implies \sigma = \zeta\omega_n = 1.33$$
$$\implies \omega_n = 2.667$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$\omega_d = 2.3094$$

Hence, the desired pole location for the given transient response requirement is as follows:

$$s = -\sigma \pm j\omega_d$$
$$\implies s = -1.33 \pm j2.3094$$

3 Design Procedures

I employed the following strategies to achieve these specifications:

- Back propagation from estimated closed loop response
- Approximating using 2 Lag or Lead compensators

4 Back propagation from estimated closed loop response

We know the closed loop dominant poles from our settling time and damping requirement. We know that the real part of non-dominating pole needs to be at least 10 times that of the dominant poles. So, let us the non-dominant poles to be equal and be equal to 15. Preserving the (n-m) of the original uncompensated plant, our assumed form of the final closed loop response of the compensated system will be:

$$\frac{C}{R} = \frac{K'}{(s^2 + 2.66s + 7.102)(s+15)^2}$$

Let the controller transfer function be G_c. So,

$$\frac{GGc}{1+GGc} = \frac{K'}{(s^2 + 2.66s + 7.102)(s+15)^2}$$
$$\implies G_cG = \frac{K'}{(s+15)^2(s^2+2.667s+7.102) - K'}$$
$$\implies G_c = \frac{1}{G}\frac{K'}{(s+15)^2(s^2+2.667s+7.102) - K'}$$
$$\implies G_c = \frac{(s+1)(s+2)(s+4)(s+6)}{K}\frac{K'}{(s+15)^2(s^2+2.667s+7.102) - K'}$$

We would want G_c to represent a known type of controller that we have studied. The way the response has been assumed also ensures that the dominant pole is on the root locus so essentially K' is a free parameter. By hit and trial, I chose $K' = 1000$ as this value allows me to represent G' as a cascade of 4 lag compensators. So our final controller looks as follows:

$$G_c = \frac{1000}{K}\frac{(s+1)(s+2)(s+4)(s+6)}{(s+15)^2(s^2+2.667s+7.102) - 1000}$$

$$\implies G_c = \frac{1000}{K}\frac{(s+1)(s+2)(s+4)(s+6)}{(s+1.2964)(s+2.2357)(s+12.1376)(s+16.9973)}$$

$$\implies G_c = \frac{1000}{K}\Big(\frac{s+1}{s+1.2964}\Big)\Big(\frac{s+2}{2.2357}\Big)\Big(\frac{s+4}{s+12.1376}\Big)\Big(\frac{s+6}{s+16.9973}\Big)$$

This controller transfer function can be understood as a cascade of 4 Lead Compensators. We can view this controller transfer function as:

$$G_c = K_1\Big(\frac{s+1/T_1}{s+1/\alpha_1 T_1}\Big)K_2\Big(\frac{s+1/T_2}{s+1/\alpha_2 T_2}\Big)K_3\Big(\frac{s+1/T_3}{s+1/\alpha_3 T_3}\Big)K_4\Big(\frac{s+1/T_4}{s+1/\alpha_4 T_4}\Big)$$

where the parameters are as follows:
$K_1K_2K_3K_4 = \frac{1000}{K}$
$T_1 = 1, \alpha_1 = 0.7714$
$T_2 = 0.5, \alpha_2 = 0.8946$
$T_3 = 0.25, \alpha_3 = 0.3296$
$T_4 = 0.1667, \alpha_4 = 0.3530$

So using this methodology our final controller transfer function, compensated plant and closed loop response to unit feedback is as follows:

$$G_c = \frac{1000}{K}\Big(\frac{s+1}{s+1.2964}\Big)\Big(\frac{s+2}{2.2357}\Big)\Big(\frac{s+4}{s+12.1376}\Big)\Big(\frac{s+6}{s+16.9973}\Big)$$

$$GG_c = 1000\frac{1}{(s+1.2964)(s+2.2357)(s+12.1376)(s+16.9973)}$$

$$\frac{GG_c}{1+GGc} = \frac{1000}{(s^2+2.66s+7.102)(s+15)^2}$$

Note that the controller transfer function could also have been manipulated to have a combination of 2 lead and 2 lag compensators in cascade. In that case, we might have expressed our controller transfer function as follows: (pole is closer to the origin in case of lag compensator)

$$G_c = \frac{1000}{K}\Big(\frac{s+2}{s+1.2964}\Big)\Big(\frac{s+4}{2.2357}\Big)\Big(\frac{s+1}{s+12.1376}\Big)\Big(\frac{s+6}{s+16.9973}\Big)$$

5 Approximation using 2 compensators

In the previous method, I used 4 lag compensators to satisfy the given requirements. In this method, I am exploring the possibility of using a simpler controller to solve the given problem. I will use 2 compensators to model a similar effect as that provided by the previous controller transfer function that used 4 lag compensators in cascade.

The idea in this procedure is that the new transfer function should add similar amount of phase as that added by the transfer function we obtained from the method used in the precious section. So first let us calculate the angle contribution of the controller transfer function at the dominant pole ($s = -1.33 \pm j2.3094$):

$$\angle G_c = \angle\Big(\frac{s+1}{s+1.2964}\Big) + \angle\Big(\frac{s+2}{2.2357}\Big) + \angle\Big(\frac{s+4}{s+12.1376}\Big) + \angle\Big(\frac{s+6}{s+16.9973}\Big)$$

Substituting $s = -1.33 + j2.3094$, we get:

$$\angle G_c = tan^{-1}\frac{2.3094}{-0.33} + tan^{-1}\frac{2.3094}{0.67} + tan^{-1}\frac{2.3094}{2.67} + tan^{-1}\frac{2.3094}{4.67}$$

$$-tan^{-1}\frac{2.3094}{-0.0336} - tan^{-1}\frac{2.3094}{0.9057} - tan^{-1}\frac{2.3094}{10.8076} - tan^{-1}\frac{2.3094}{15.6673}$$

$$\angle G_c = 59.259°$$

Let us assume our new controller transfer function as follows:

$$G'_c = K_c\Big(\frac{s+z_1}{s+p_1}\Big)\Big(\frac{s+z_2}{s+p_2}\Big)$$

So there are 5 parameters involved in the above transfer function.

The value of K will be determined from the root locus condition, i.e., $|GG'_c| = 1$ at the dominant pole.

So we just have 1 equation for the phase added by the new controller at the dominant pole. Drawing analogy from the alternative root locus technique discussed in the class and the controller transfer function obtained in the previous section, I am assuming the values of z_1 and z_2 such that it cancels the nearest roots of the uncompensated plant. Therefore, assuming $z_1 = 1$ and $z_2 = 2$. Now, we have our transfer function as follows:

$$G'_c = K_c\Big(\frac{s+1}{s+p_1}\Big)\Big(\frac{s+2}{s+p_2}\Big)$$

So we expect this transfer function to add a phase of around 59° at the dominant poles. However we cannot say for sure that the angle added by the simplified transfer function should be exactly 59.259° at the dominant poles. So essentially, we have another unknown in phase angle although we have a better idea about the range of phase angle, which is around 59°. It is interesting to stress the point that we are dealing with an undetermined system as we have 2 equations (magnitude and phase equation) and 4 unknowns (p_1, p_2, K and phase angle).

Strategy employed to deal with the undetermined system is described below:

I wrote a MATLAB script leadlagDesign.m to solve for the parameters of this transfer function. In this script, I iterate over a range of values for p_1 and phase angle and based on these values, I determine the corresponding K and p_2. With these four values, I construct the corresponding compensated plant and get the poles and step response characteristics of the closed loop response of the compensated plant. Then I have defined 2 errors - ratio error and damping error. Ratio error is the amount by which ratio of real part of dominant and non-dominant poles differ from the required value of 10 and damping error is the amount by which the closed damping coefficient ζ deviates from the required value of 0.5. Mathematically the two errors can be expressed as:

$$ratio\ error = \frac{(10 - ratio)}{10} if\ ratio < 10, else = 0$$

$$damping\ error = \frac{abs(\zeta - 0.5)}{0.5}$$

After calculating the above defined errors, I set a threshold of storing all such parameters where the damping error and ratio error is less than 0.2. On doing this procedure, I got around 800 sets of parameters.

To select the optimum set of parameter from these 800 sets involved some tradeoff. For one set, the damping error was 8 percent but the corresponding ratio error was 19.99 percent whereas for another set, the ratio error was 1 percent but the damping error was 19.12 percent. To reduce the tradeoff, I further reduced the threshold to 0.15 instead of 0.2 for both the errors and from the around 90 values obtained, I chose the set of parameters with the lowest sum of both errors.

With the above described procedure, I got the controller parameter values as follows:

$$phase = 57°$$
$$p_1 = 12$$
$$p_2 = 0.4533$$
$$K_c = 495.98$$
$$\zeta = 0.43128$$
$$ratio = 9.0219$$
$$damp\ error = 0.13744$$
$$ratio\ error = 0.097805$$

So my final simplified controller obtained is:

$$G_c' = 495.98\Big(\frac{s+1}{s+0.4533}\Big)\Big(\frac{s+2}{s+12}\Big)$$

So we can see that we have obtained a lag compensator in cascade with a lead compensator.

The compensated plant in this case will be as follows:

$$GG_c' = \frac{495.98}{(s+4)(s+6)(s+0.4533)(s+12)}$$

Derivation of p_2 and K in terms of p_1 and phase:

Before moving onto the next section, I will derive the formulas used for calculation of p_2 and K for the range of values of p_1 and phase angle (This has been implemented in the script get_leadlagParameters.m) and also the expression to obtain ζ from the peak overshoot of the closed loop response of the compensated plant (implemented in get_zeta.m).

Since the angle contribution by the numerator of the controller transfer funnction is constant (close to 8°), I can safely ignore that into my derivation for p_2 by chosing a larger range of values of the phase angle. In my script, I have varied the phase angle from 52° to 82° taking a step of 0.25°. Range of values of p_1 was taken from 5 to 30 (again with a step of 0.25). This range was determined through extensive hit and trial.

Let the phase angle of the denominator be ϕ (p_1 is known).

$$\phi = \tan^{-1}\frac{2.3094}{p_1-1.33} + \tan^{-1}\frac{2.3094}{p_2-1.33}$$
$$\implies tan\phi = \frac{\frac{2.3094}{p_1-1.33}+\frac{2.3094}{p_2-1.33}}{1-\frac{2.3094}{p_1-1.33}\frac{2.3094}{p_2-1.33}}$$
$$\implies tan\phi = 2.3094\frac{p_1+p_2-2.66}{(p_1-1.33)(p_2-1.33)-5.333}$$
$$\implies tan\phi[(p_1-1.33)(p_2-1.33)-5.333] = 2.3094(p_1+p_2-2.66)$$
$$\implies tan\phi(p_1p_2-1.33(p_1+p_2)-3.5641) = 2.3094(p_1+p_2)-6.143004$$

$$\implies p_2(p_1 tan\phi - 1.33 tan\phi - 2.3094) = -6.143004 + 3.5641 tan\phi + 1.33 p_1 tan\phi + 2.3094 p_1$$

$$\implies p_2 = \frac{-6.143004 + 3.5641 tan\phi + 1.33 p_1 tan\phi + 2.3094 p_1}{p_1 tan\phi - 1.33 tan\phi - 2.3094}$$

For K (at dominant pole) (combining the plant K and controller K_c as K):

$$|GG_c'| = 1$$
$$\frac{K}{|s+4||s+6||s+p_1||s+p_2|} = 1$$

Substituting $s = -1.33 + j2.3094$:

$$K = 3.5302 \times 5.2098 \times [(p_1 - 1.33)^2 + 2.3094^2]^{0.5} \times [(p_2 - 1.33)^2 + 2.3094^2]^{0.5}$$

To get ζ, We can obtain the peak overshoot from the stepinfo(closedLoopResonse) routine of MATLAB. Let the peak overshoot be M.

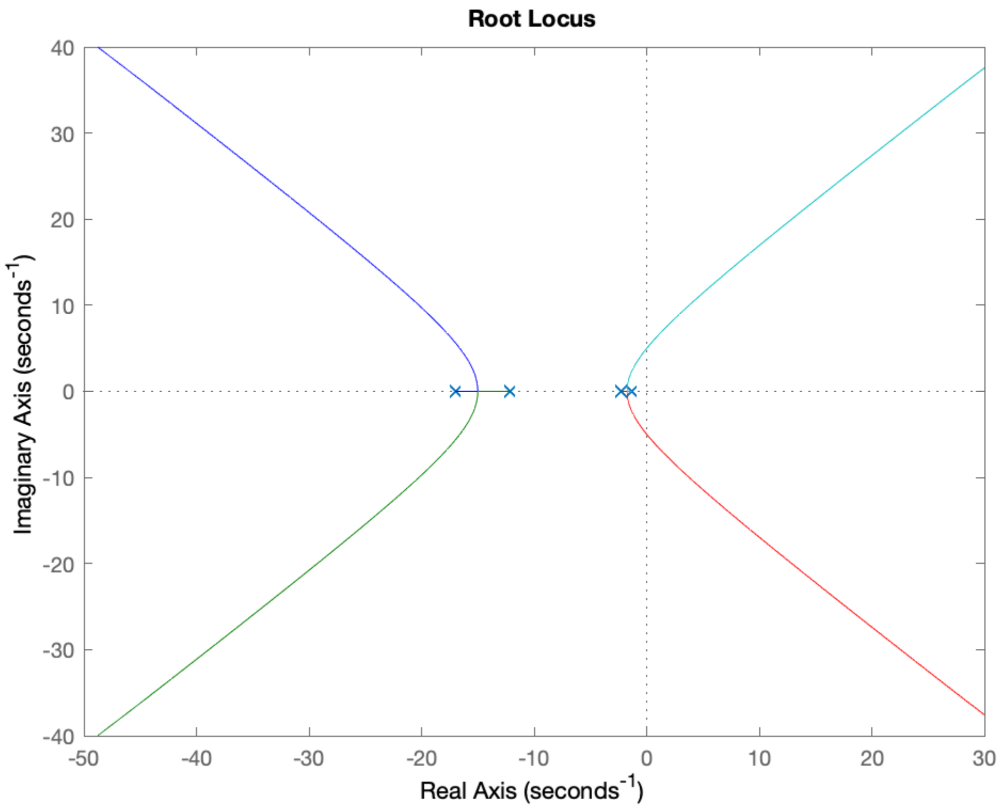
$$M = exp\Big(-\frac{\zeta\pi}{1-\zeta^2}\Big)$$
$$\implies \frac{\zeta\pi}{1-\zeta^2} = lnM$$
$$\implies \zeta^2 + b\zeta - 1 = 0 \qquad [b = \pi/lnM]$$
$$\implies \zeta = \frac{-b + \sqrt{b^2+4}}{2}$$

6 Error analysis and Compensated plant analysis

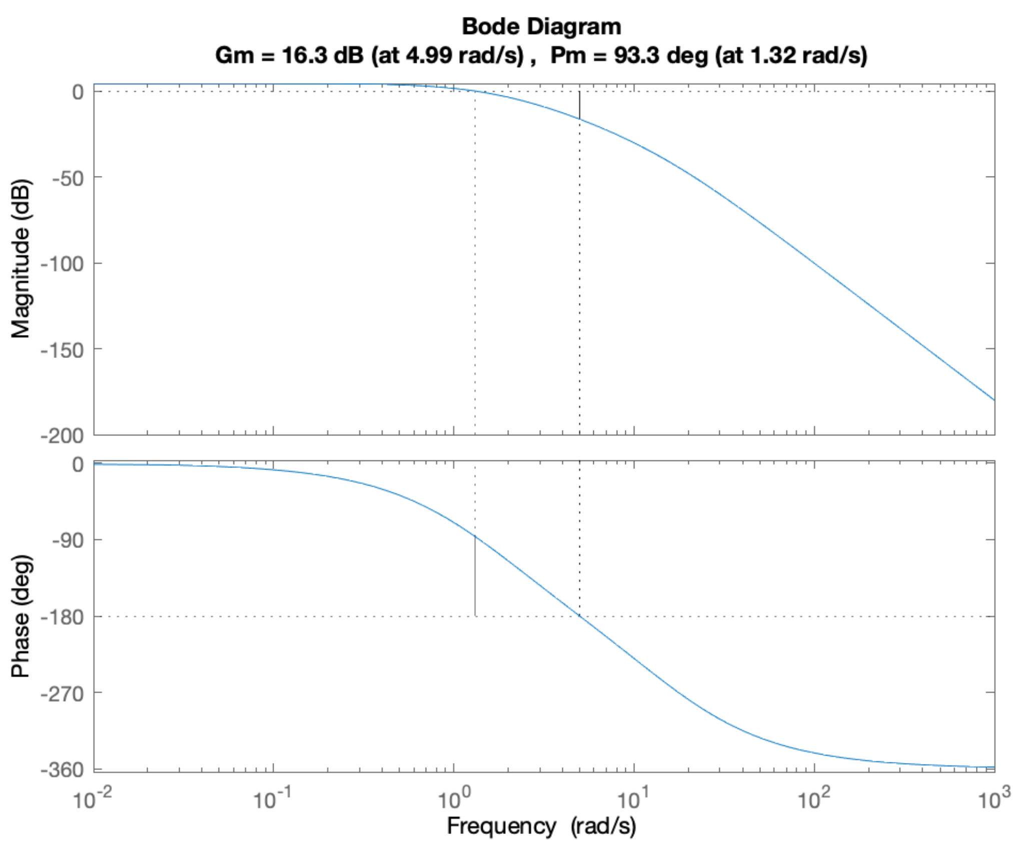
6.1 Four Lag compensator in cascade solution

$$G_c = \frac{1000}{K}\Big(\frac{s+1}{s+1.2964}\Big)\Big(\frac{s+2}{2.2357}\Big)\Big(\frac{s+4}{s+12.1376}\Big)\Big(\frac{s+6}{s+16.9973}\Big)$$
$$GG_c = 1000\frac{1}{(s+1.2964)(s+2.2357)(s+12.1376)(s+16.9973)}$$
$$\frac{GG_c}{1+GGc} = \frac{1000}{(s^2+2.66s+7.102)(s+15)^2}$$

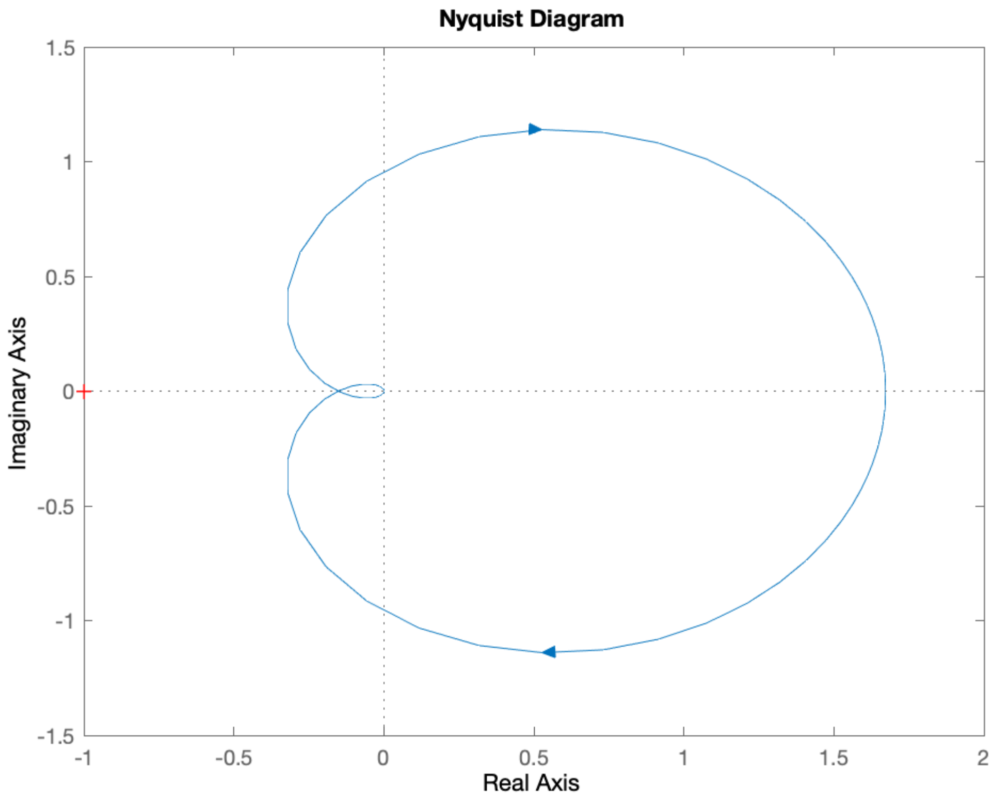
Root Locus, Nyquist plot, Bode plot of the compensated plant is as follows:



Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
40
30
20
10
0
-10
-20
-30
-40
-50
-40
-30
-20
-10
0
10
20
30



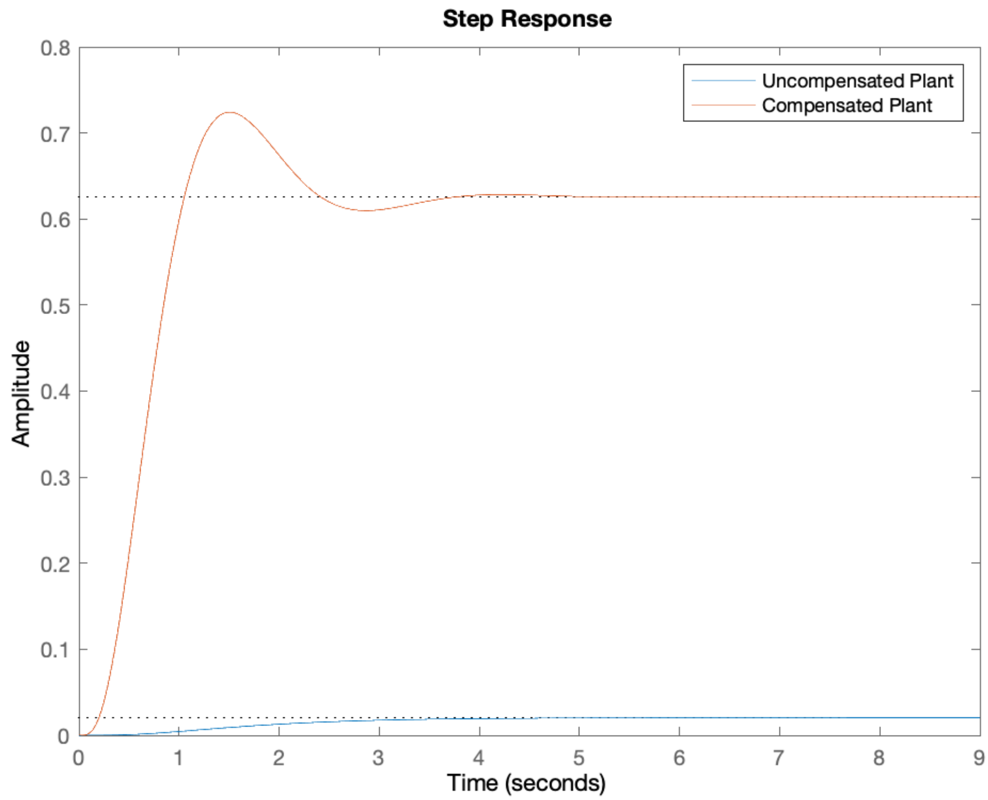
Bode Diagram
Gm = 16.3 dB (at 4.99 rad/s) , Pm = 93.3 deg (at 1.32 rad/s)
Magnitude (dB)
0
-50
-100
-150
-200
Phase (deg)
0
-90
-180
-270
-360
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)



From the nyquist, we can see that our compensated plant is indeed stable (N=Z=P=0).

An interesting thing to note here is the margin of the compensated plant. The Gain Margin has reduced from 49.5dB to 16.3dB whereas ω_{PCO} has increased from 2.66 to almost 5 rad/s. While the Phase margin of the uncompensated plant was infinite, we have introduced a **phase margin in our compensated plant of 93.3dB with ω_{PCO} at 1.32 rad/s**. Also note that the **bandwidth of the compensated plant has increased from 0.7937 rad/s to 1.0283 rad/s**.

Let us look at the closed loop step response of the compensated plant:



The characteristics of the compensated plant unity feedback step response are as follows:

RiseTime: 0.6424
SettlingTime: 3.1535
SettlingMin: 0.5635
SettlingMax: 0.7241
Overshoot: 15.7096
Undershoot: 0
Peak: 0.7241
PeakTime: 1.5052

Closed Loop poles:
-15.0080 + 0.0000i
-14.9920 + 0.0000i
-1.3335 + 2.3073i
-1.3335 - 2.3073i

Error Analysis: The requirement of settling time less than 4s and non-dominating pole greater having a real part more than 10 times of the dominant pole are exactly satisfied. The closed damping that we get from an overshoot of 15.7% is $\zeta = 0.463$ which means there is an error of around 7% in the required damping which is 0.5. This error is due to the final form of transfer function assumed by us through which we back propagated to get the controller transfer function. Again there was a trade-off involved here, we could have assumed a final closed loop response at the expense of either of the two conditions.

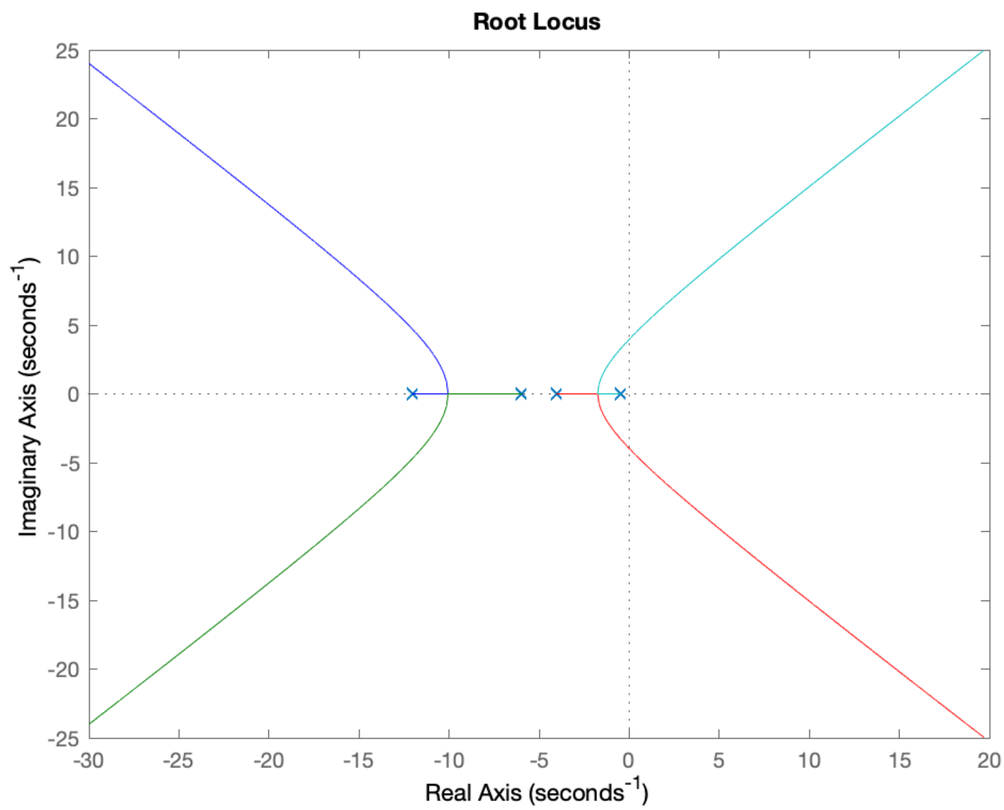
6.2 Lag and Lead compensator in cascade solution

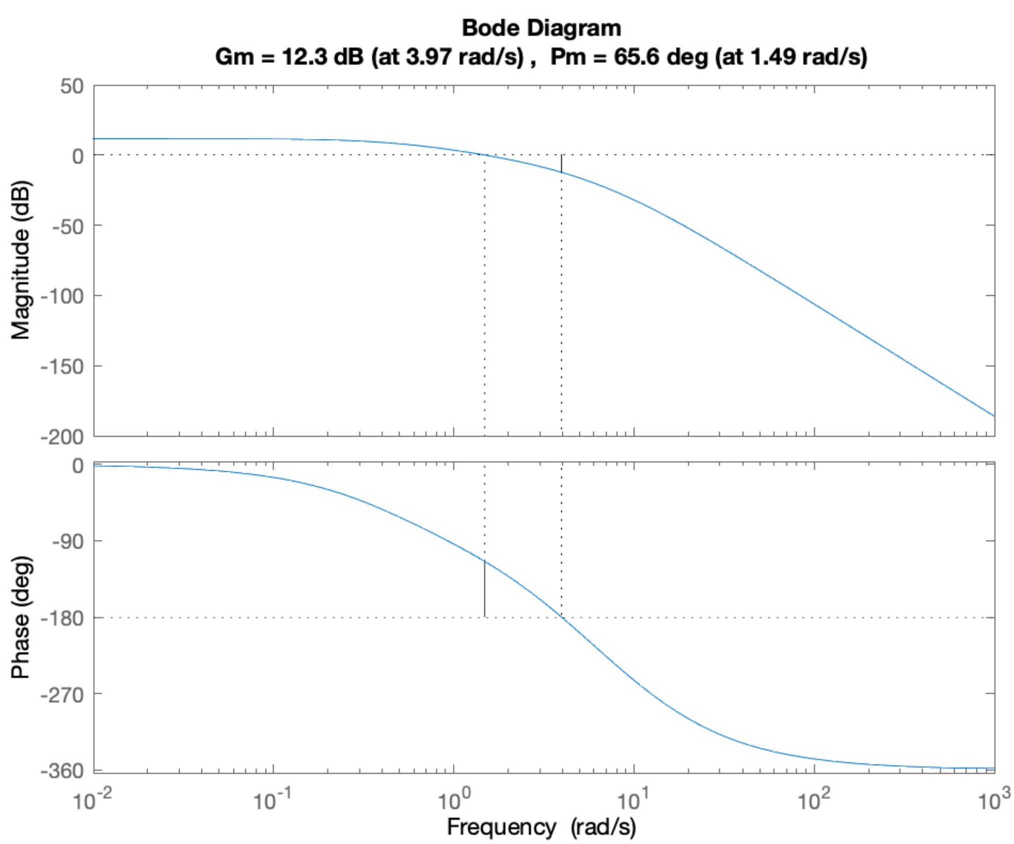
$$G_c' = 495.98\Big(\frac{s+1}{s+0.4533}\Big)\Big(\frac{s+2}{s+12}\Big)$$

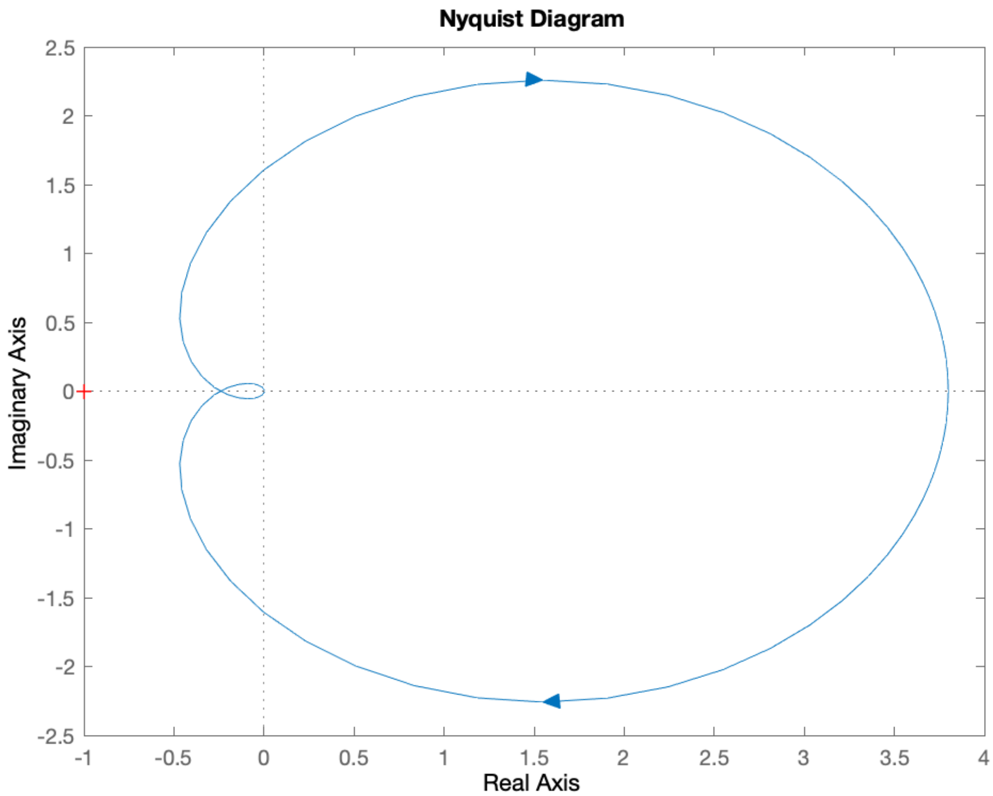
$$GG_c' = \frac{495.98}{(s+4)(s+6)(s+0.4533)(s+12)}$$

$$\frac{GG_c'}{1+GG_c'} = \frac{495.98}{s^4 + 22.45s^3 + 154s^2 + 353.3s + 626.5}$$

Root Locus, Nyquist plot, Bode plot of the compensated plant in this case is as follows:



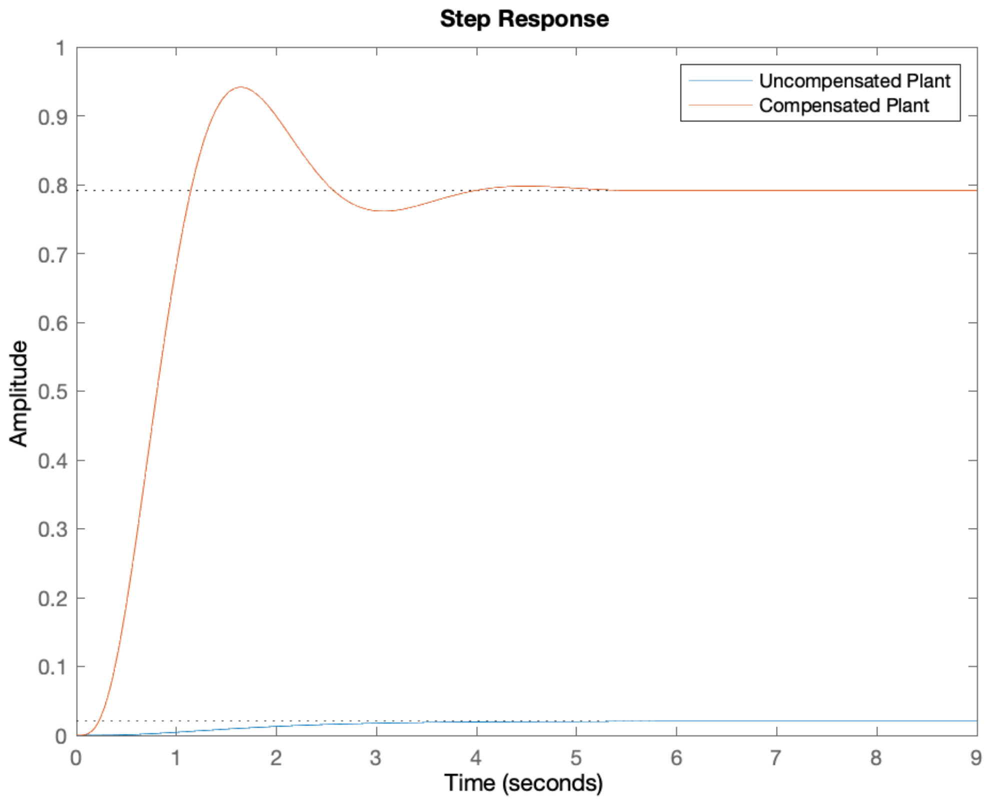




The nyquist plot assures us that the compensated plant is stable.

The Gain margin in this case is 12.3 dB which even less than the Gain Margin obtained from previous controller design. ω_{PCO} also reduces to almost 4 rad/s. **Phase Margin also reduces considerably from the previous design to 65.6 $^\circ$ at $\omega_{GCO} = 1.49$ rad/s.** An interesting comparison between the two compensated plants is in relation to their bandwidths. **The bandwidth of the previous compensated plant was more than the uncompensated plant whereas the bandwidth of ths compensated plant with the simpler controller is 0.4436 rad/s, which is even less than that of the uncompensated plant.** This implies that as a low pass filter, the simpler compensated plant allows less frequencied to pass through whereas the compensated plant with more complicated transfer function allowed more frequencies to pass through.

Let us look at the closed loop step response of the compensated plant:



The characteristics of the compensated plant unity feedback step response are as follows:

RiseTime: 0.6804
SettlingTime: 3.5661
SettlingMin: 0.7159
SettlingMax: 0.9415
Overshoot: 18.9287
Undershoot: 0
Peak: 0.9415
PeakTime: 1.6404

Closed Loop poles:
-10.1064 + 0.6630i
-10.1064 - 0.6630i
-1.1202 + 2.2029i
-1.1202 - 2.2029i

Error Analysis: We managed to exactly satisfy the settling time requirement in this case as well (less than 4s). Ratio of real part of dominant and non-dominant poles is 9.024 (9.7% error from the ratio of 10). The closed damping we get from an overshoot of 18.9287% is $\zeta = 0.432$, which accounts for an error of 13.6%. I had discussed about the tradeoff for this case in section 5. We could have chosen the set of parameters to reduce the error on damping to 8 % and that on ratio error to around 1% but that would have come at the cost of increasing the error on the other quantity to almost 20%.

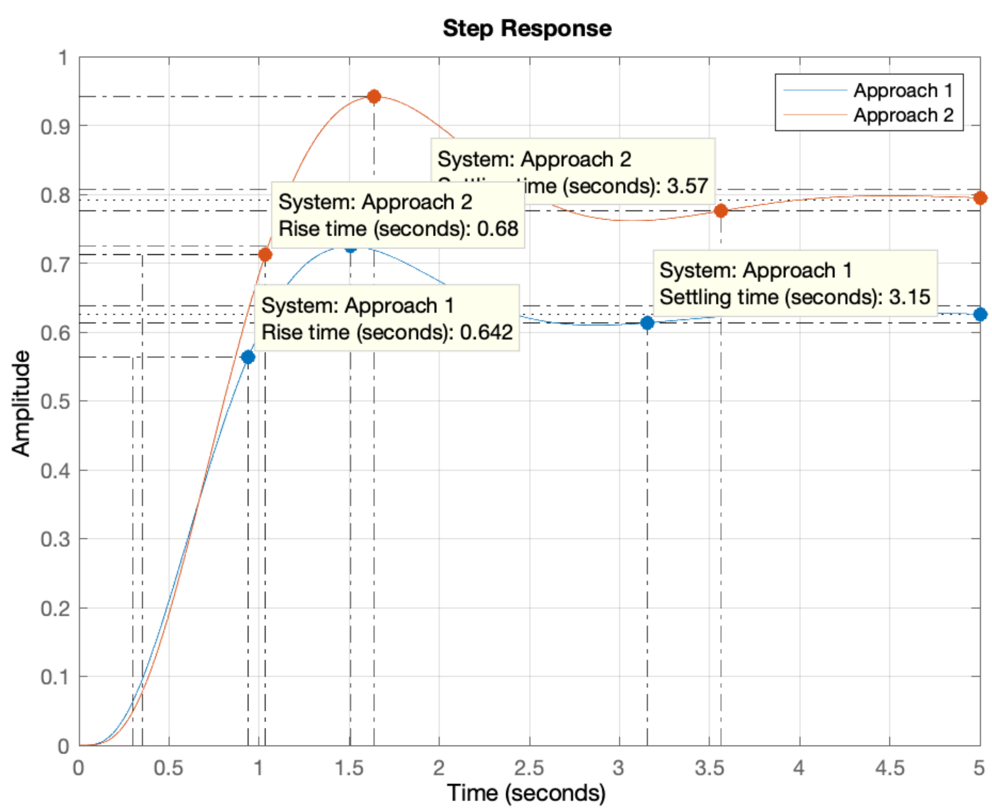
7 Conclusion

I selected Lag and Lead compensators as my controllers because of the various advantages they offer over PID controllers such as retaining the system type, not changing the (n-m) and having greater degree of freedoms. I have been very cautious of the settling time requirement and hence, have managed to satisfy the settling time requirement of being less than 4 seconds with every design.

My first approach was based on a more basic approach on how we go about designing a controller from the fundamental equation of $\frac{C}{R} = \frac{GGc}{1+GGc}$. From there, I manipulated the constants by a very extensive hit and trial method to get a form of 4 lag compensators in cascade. The compensated plant transfer function obtained from this method is better than the second method as the phase margin (93.3dB) is higher than that from the second method (65.6dB) which implies that more phase has to be added to the system to make it unstable. The bandwidth obtained from the first approach is higher while the rise time, settling time and overshoot are lower, so we essentially get a better transient response for this controller. Higher bandwidth also ensures that it allows more frequencies to pass through when acted as a low pass filter. This might be good or not depending on the application for which the filter will need to be employed. this approach satisfies the settling time and ratio of dominant and non-dominant pole exactly but gives a 7% error on damping requirement.

My second approach has a more simpler design and does a reasonably good job of satisfying the given requirements. It gives a higher peak for the step response. It combines a lead and lag compensator in cascade. The ideas on assuming the zeros of the controller were taken from the alternate root locus technique and the controller transfer function we obtained from the previous approach as they involved cancelling of the roots. This approach satisfied settling requirement exactly but gives approaximately 13.7% and 9.7% requirement on damping and ratio respectively.

Here is a comparison of the step response of the two compensated transfer functions obtained:



Overall this assignment is a result of extensive calculations and manipulations, comprehensive understanding and abbundance of problem solving skills. It was a great learning experience and I would appreciate any feedback on what I could have done better on this assignment.

8 MATLAB Code Information

The following MATLAB files have been included along with this report.

- **comparison.m** - plots the step response of the closed loop responses of the the 2 compensated plants
- **fourLag.m** - plots the graphs and displays characteresitics associated with the compensated plant made using 4 lag compensators in cascade
- **get_leadlagParameters.m** - returns K and p_2 for given p_1 and phase; used in the 2nd approach
- **get_zeta.m** - returns damping coefficient ζ for a given peak overshoot M
- **LagLeadPlot.m** - plots the graphs and displays characteresitics associated with the compensated plant made using a lag and lead compensators in cascade
- **leadDesign.m** - the main script described in section 5 of this report that is used to design the controller using 2nd approach.
- **uncompensatedPlantAnalysis.m** - plots the graphs and displays characteresitics associated with the uncompensated plant
- **LeadLagParameters.csv** - Parameters of the cascading lead and lag compensators that is analysed to select the final parameters. The 9 columns of the csv file are p_1, phase angle, p_2, K, ζ, ratio, damping error, ratio error and (damping error+ratio error) respectively
- **pidDesign.slx** - Simulink Model used to tune a PID Controller for the given requirement. I was unable to satisfy all the three requirement using this approach within acceptable error rates (was ale to satisfy two of the requirements of settling time and damping coefficient). In any case, lead and lag compensators can perform the same job and have more advantages compared to a PID design. Therefore this approach has not been included in the report.