Navigation and Guidance (AE 410)

Assignment 2

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Contents

1 Question 1

Given Desired Navigation: $\dot{\psi}_M = 3\dot{\psi}_T$. Navigation due to Radome effect: $\dot{\psi}_M = 1.8\dot{\psi}_T$.

As per the given figure, we can derive the formula for navigation due to Radome effect as follows (Let desired be $\dot{\psi}_M = N\dot{\psi}_s$)

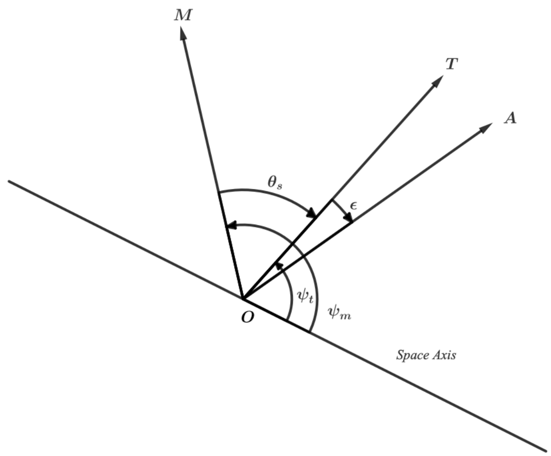

Figure 1: Change in navigation law due to Radome effect

$$\dot{\psi}_M = N(\dot{\psi}_T + \dot{\epsilon})$$

As $\dot{\epsilon} = \alpha\dot{\theta}_s$

$$\implies \dot{\psi}_M = N(\dot{\psi}_s + \alpha\dot{\theta}_s) = N(\dot{\psi}_s + \alpha(\dot{\psi}_s - \dot{\psi}_m))$$

$$\implies \dot{\psi}_M = N\frac{1+\alpha}{1+N\alpha}\dot{\psi}_T$$

For the given problem, $N = 3$. So we can find α as:

$$N\frac{1+\alpha}{1+N\alpha} = 1.8$$

$$\implies \frac{1+\alpha}{1+3\alpha} = 0.6$$

$$\implies \alpha = 0.5$$

As $\dot{\psi}_M = N(\dot{\psi}_T + \dot{\epsilon})$ and given $\dot{\psi}_M = 0.15\ rad/s$:

$$\implies \dot{\psi}_M = 3(\frac{\dot{\psi}_M}{1.8} + \dot{\epsilon})$$

$$\implies \frac{0.15}{3} = \frac{0.15}{1.8} + \dot{\epsilon}$$

$$\implies \dot{\epsilon} = -0.0333\ rad/s$$

2 Question 2

Given:

$$V_M = 400m/s, \theta = 30°, \gamma_T = 60°, r = 7km, V_T = 0.5V_M$$

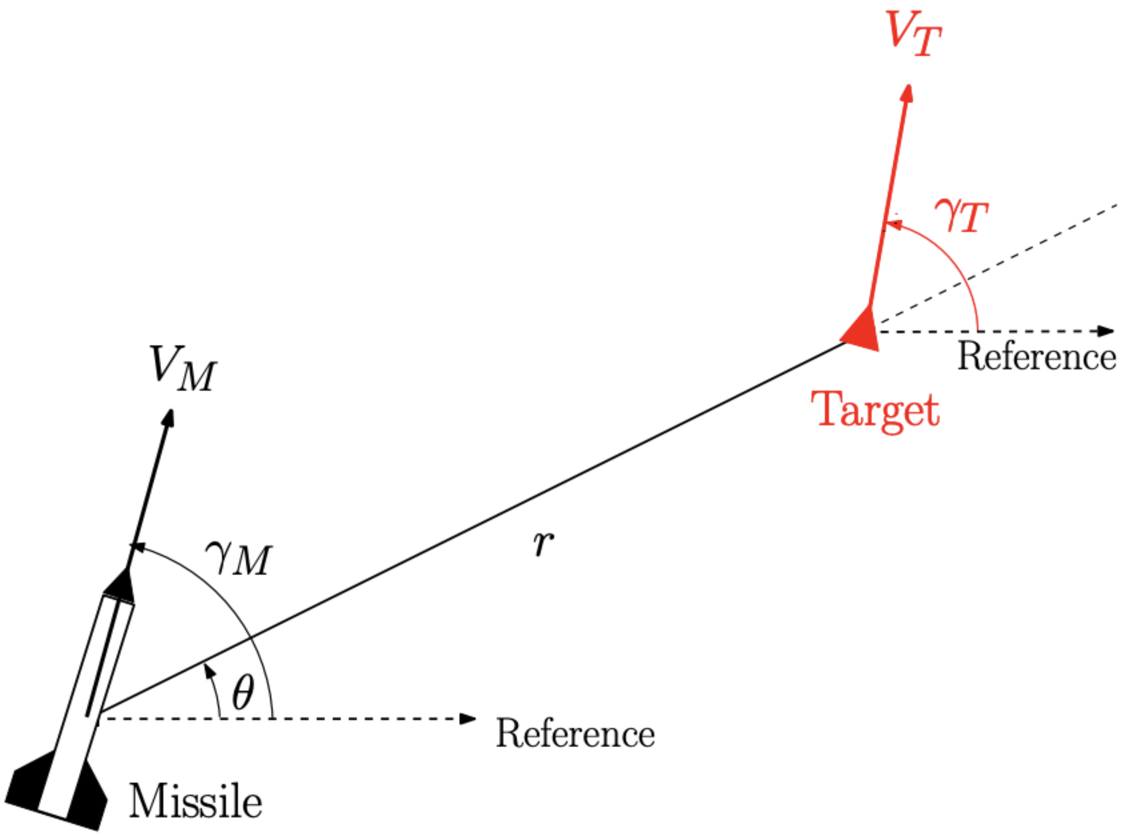

Figure 2: Planar engagement geometry between an interceptor and a target.

We can express V_r and V_θ as follows:

$$V_r = V_T cos(\gamma_T - \theta) - V_M cos(\gamma_M - \theta)$$
$$V_\theta = V_T sin(\gamma_T - \theta) - V_M sin(\gamma_M - \theta)$$

For a realistic true proportional navigation guidance, the condition for capturability for $N > 1$ can be expressed as (2 conditions):

$$V_{\theta_0}^2 + (1 - N)V_{R_0}^2 < 0, V_{R_0} < 0$$

2.1 Initial Heading Angle range

To find the range of heading angle, we will need substitute to substitute the expression for V_{R_0} and V_{θ_0} in the capturability conditions in terms of the heading angles, target and missile velocities.

For the condition $V_{R_0} < 0$:

$$V_T cos(\gamma_T - \theta) - V_M cos(\gamma_{M_0} - \theta) < 0$$

Substituting the given values in the above expression, we get:

$$\implies 200cos(60° - 30°) - 400cos(\gamma_M - 30°) < 0$$

$$\implies 173.205 - 400cos(\gamma_M - 30°) < 0 \quad \ldots eq^n\ 1$$

$$\implies \gamma_M - 30° < 64.3736°$$

$$\implies \gamma_M < 94.3736° \quad \ldots condition\ 1$$

For the condition $V_{\theta_0}^2 + (1-N)V_{R_0}^2 < 0$:

$$\implies V_{\theta_0}^2 - 2V_{R_0}^2 < 0$$

$$\implies (V_T sin(\gamma_T - \theta) - V_M sin(\gamma_{M_0} - \theta))^2 - 2(V_T cos(\gamma_T - \theta) - V_M cos(\gamma_{M_0} - \theta))^2 < 0$$

$$\implies (173.205 - 400cos(30° - 30°))^2 - 2(100 - 400sin(\gamma_{M_0} - 30°))^2 < 0 \quad \ldots eq^n\ 2$$

Expanding equation 2:

$$\implies 1 + 16sin^2(\gamma_{M_0} - 30°) - 8sin(\gamma_{M_0} - 30°) - 6 - 32cos^2(\gamma_{M_0} - 30°) - 27.713cos(\gamma_{M_0} - 30°) < 0$$

Substituting equation 1 in the above expression to obtain a less strict bound on the initial heading angle as follows:

$$\implies 11 - 48(1 - sin^2(\gamma_{M_0} - 30°)) - 8sin(\gamma_{M_0} - 30°) < 173/400$$

$$48sin^2(\gamma_{M_0} - 30°) - 8sin(\gamma_{M_0} - 30°) - 37.4325 < 0$$

Solving the quadratic inequality, we get the range of initial heading angle as follows:

$$\gamma_{M_0} \in (-53.4848°, 76.001°)$$

We cannot have a negative heading angle, so essentially we get a loose upper bound on our initial heading angle as 76.001°.

In order to get strict bounds, I used scipy's fsolve to solve for the roots of LHS of equation 2. With the given inequality, we know that the initial heading angle will lie between the roots of the LHS. Using that, we get:

$$\gamma_{M_0} \in (5.1249°, 72.6592°) \quad \ldots condition\ 2$$

This answers aligns with the less strict upper bound obtained earlier.

Therefore from conditions 1 and 2, we get γ_{M_0} as expressed in condition 2:

$$\gamma_{M_0} \in (5.1249°, 72.6592°)$$

2.2 Capturability Region in $(V_{\theta_0} - V_{R_0})$ space

In $(V_{\theta_0} - V_{R_0})$ space, the conditions for capturability are as follows:

$$V_{\theta_0}^2 + (1-N)V_{R_0}^2 < 0, V_{R_0} < 0$$

Plotting the above equations for 10 values of k:

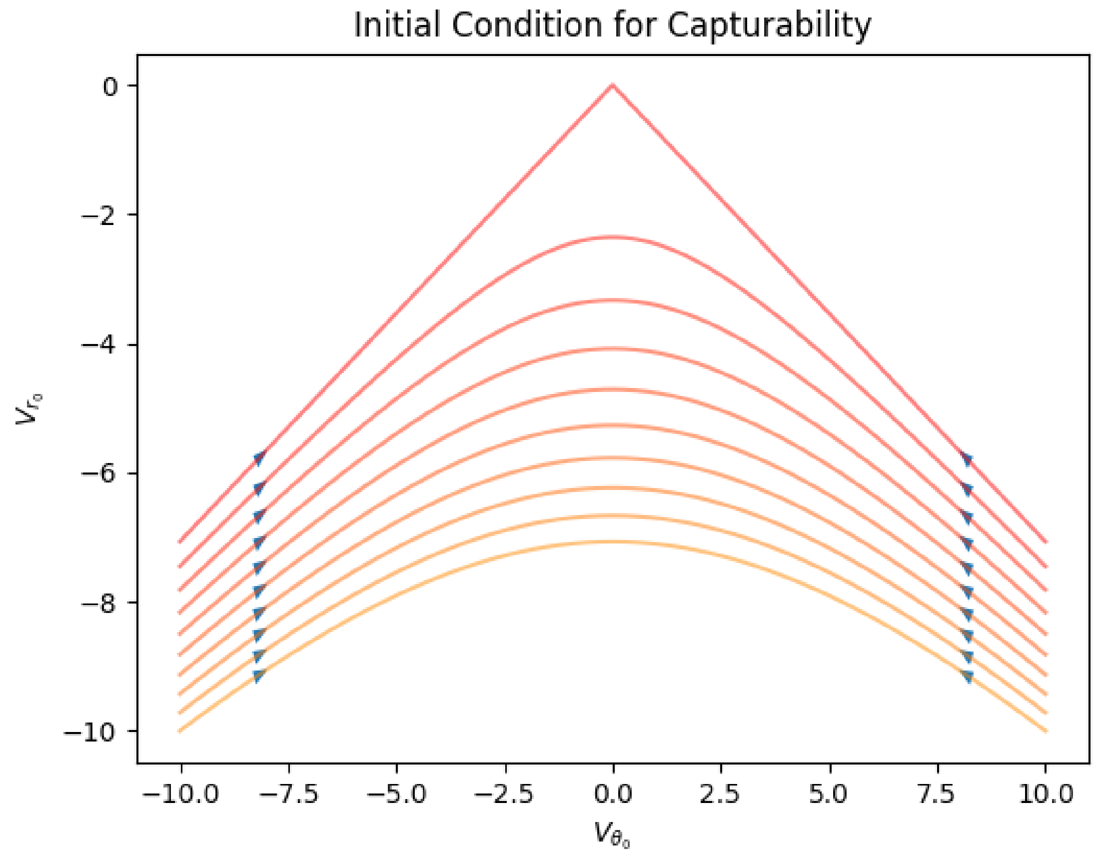

Figure 3: Arrows show the direction of movement

3 Question 3

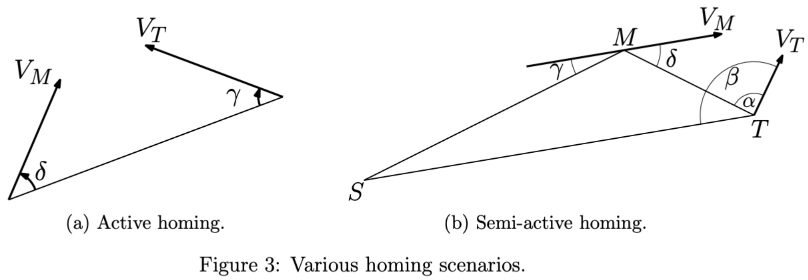

Figure 4

Doppler Shift Frequency is expressed as:

$$f_D = \frac{2V_r f_0}{c} = \frac{2V_r}{\lambda}$$

3.1 Active Homing

$$V_r = V_M cos\delta + V_T cos\gamma$$

$$\boxed{\implies f_D = \frac{2f_0(V_M cos\delta + V_T cos\gamma)}{c} = \frac{2(V_M cos\delta + V_T cos\gamma)}{\lambda}}$$

3.2 Semi-Active Homing

Between S and T:

$$V_r = -V_T cos(\pi - \beta) = V_T cos\beta$$

$$\implies f_{D_{S-T}} = \frac{2V_T cos\beta}{\lambda}$$

Target receives a frequency of $f_S + f_{D_{S-T}} = f_S + \frac{V_T cos\beta}{\lambda}$ from source.

Between M and T:

$$V_r = V_M cos\delta + V_T cos\alpha$$

$$f_{D_{T-M}} = \frac{2(V_M cos\delta + V_T cos\alpha)}{\lambda}$$

Missile receives a frequency of $f_M + f_{D_{T-M}} = f_S + f_{D_{S-T}} + \frac{2(V_M cos\delta + V_T cos\alpha)}{\lambda} = f_S + \frac{2(V_M cos\delta + V_T cos\alpha + V_T cos\beta)}{\lambda}$ from target echo.

Between M and S:

$$V_r = -V_M cos\gamma$$

$$f_{D_{S-M}} = \frac{2(-V_M cos\gamma)}{\lambda}$$

So the net Doppler Shift (f_D) for the Missile will be: $f_{D_{S-T}} + f_{D_{T-M}} - f_{D_{S-M}}$

$$\boxed{\implies f_D = \frac{2(V_M(cos\delta + cos\gamma) + V_T(cos\alpha + cos\beta))}{\lambda}}$$

3.3 Range of Frequencies

For Active Homing:

$$\implies f_D = \frac{2(V_M cos\delta + V_T cos\gamma)}{\lambda}$$

$$f_{D_{max}} = \frac{2(V_M + V_T)}{\lambda}$$

$$\implies f_{D_{max}} = \frac{2(600 + 300)}{30 \times 10{-}3}$$

$$\implies f_{D_{max}} = 60kHz$$

Similarly,

$$f_{D_{min}} = \frac{2(V_M - V_T)}{\lambda}$$

$$\implies f_{D_{min}} = \frac{2(600 - 300)}{30 \times 10{-}3}$$

$$\implies f_{D_{min}} = 20kHz$$

So, Range of Doppler Shift Frequency:

$$\boxed{f_{D_{max}} - f_{D_{min}} = 40kHz}$$

For Semi-Active Homing:

$$f_D = \frac{2(V_M(cos\delta + cos\gamma) + V_T(cos\alpha + cos\beta))}{\lambda}$$

$$f_{D_{max}} = \frac{2(V_M + V_T)}{\lambda}$$

$$\implies f_{D_{max}} = \frac{2(600 + 300)}{30 \times 10{-}3}$$

$$\implies f_{D_{max}} = 60kHz$$

Similarly (Minimum practical value)

$$f_{D_{min}} = \frac{2(V_M - V_T)}{\lambda}$$

$$\implies f_{D_{min}} = \frac{2(600 - 300)}{30 \times 10{-}3}$$

$$\implies f_{D_{min}} = 20kHz$$

So, Range of Doppler Shift Frequency:

$$\boxed{f_{D_{max}} - f_{D_{min}} = 40kHz}$$

4 Question 4

4.1 Lateral Acceleration Proof

$$\dot{\gamma_M} = \frac{a_M}{V_M}$$

$$a_M = V_M \dot{\gamma_M}$$

For pursuit guidance:

$$a_M = V_M \dot{\theta}$$

$$a_M = V_M \frac{V_T sin\theta_T - V_M sin\delta}{r}$$

From Engagement Dynamics:

$$\dot{r} = V_T cos\theta - V_M cos\delta$$

$$\dot{\theta_T} = -\dot{\theta_T} = \frac{-V_T sin\theta_T + V_M sin\delta}{r}$$

$$\frac{dr}{r} = \frac{r(cos\theta_T - K cos\delta)}{-sin\theta_T + K sin\delta}$$

where $K = V_M/V_T$ and. $|\nu sin\delta| < 1$, we get $r(\theta_T)$ (after integrating) as follows:

$$r(\theta_T) = C\frac{sin^{\mu-1}\left(\frac{\theta_T-\beta}{2}\right)}{cos^{\mu+1}\left(\frac{\theta_T+\beta}{2}\right)}$$

where: $\beta = sin^{-1}[Ksin\delta]$, $\mu = K\frac{cos\delta}{cos\beta} = \frac{\mu cos\delta}{\sqrt{1-\mu^2 sin^2\delta}}$
Substituting this expression of r in a_M, we get:

$$a_M = CV_MV_T(sin\theta_T - \nu sin\delta)\frac{cos^{\mu+1}\left(\frac{\theta_T+\beta}{2}\right)}{sin^{\mu-1}\left(\frac{\theta_T-\beta}{2}\right)}$$

$$\implies a_M = CV_MV_T(sin\theta_T - sin\beta)\frac{cos^{\mu+1}\left(\frac{\theta_T+\beta}{2}\right)}{sin^{\mu-1}\left(\frac{\theta_T-\beta}{2}\right)}$$

$$\implies a_M = 2CV_MV_T\frac{cos^{\mu+2}\left(\frac{\theta_T+\beta}{2}\right)}{sin^{\mu-2}\left(\frac{\theta_T-\beta}{2}\right)}$$

As $\theta_T = \gamma_T - \theta$

$$\implies a_M = 2CV_MV_T\frac{cos^{\mu+2}\left(\frac{\gamma_T-\theta+\beta}{2}\right)}{sin^{\mu-2}\left(\frac{\gamma_T-\theta-\beta}{2}\right)}$$

The above highlighted expression is obtained for $K > 0$ and and $|Ksin\delta| < 1$

For small K and δ (Using $sin\theta \approx \theta$ and $cos\theta \approx 1$ for small angles), we get the following:

$$\beta \approx K\delta$$

$$\mu = K$$

For Pure pursuit, $\delta = 0 \implies \beta = 0, \mu = K$ lateral acceleration is expressed as:

$$\implies a_{M,pure} = 2CV_MV_T\frac{cos^{K+2}\left(\frac{\gamma_T-\theta}{2}\right)}{sin^{K-2}\left(\frac{\gamma_T-\theta}{2}\right)}$$

$$\implies a_{M,deviated} = 2CV_MV_T\frac{cos^{\mu+2}\left(\frac{\gamma_T-\theta+K\delta}{2}\right)}{sin^{\mu-2}\left(\frac{\gamma_T-\theta-K\delta}{2}\right)}$$

Under the given conditions: $\delta > 0$. Hence:

$$cos\Big(\frac{\gamma_T-\theta}{2}\Big) > cos\Big(\frac{\gamma_T-\theta+K\delta}{2}\Big)$$

$$sin\Big(\frac{\gamma_T-\theta}{2}\Big) > sin\Big(\frac{\gamma_T-\theta-K\delta}{2}\Big)$$

So:

$$\implies \frac{a_{M,pure}}{a_{M,deviated}} = \frac{cos^{K+2}\left(\frac{\gamma_T-\theta}{2}\right)}{cos^{K+2}\left(\frac{\gamma_T-\theta+K\delta}{2}\right)} \frac{sin^{2-K}\left(\frac{\gamma_T-\theta}{2}\right)}{sin^{2-K}\left(\frac{\gamma_T-\theta-K\delta}{2}\right)}$$

We have already established that both the fractions under power are greater than 1 and as $K+2 > 1, 2-K > 1$, we can definitely say that:

$$\boxed{\implies \frac{a_{M,pure}}{a_{M,deviated}} > 1}$$

4.2 Trajectories

$V_T = 200m/s, V_M = 100m/s(K = 0.5), 200m/s(K = 1), 400m/s(K = 2)$. Proportionality Constant is taken as 50 ($a_M = V_M\dot{\theta} - 50(\gamma_M - \theta)$). r is taken as 5km.

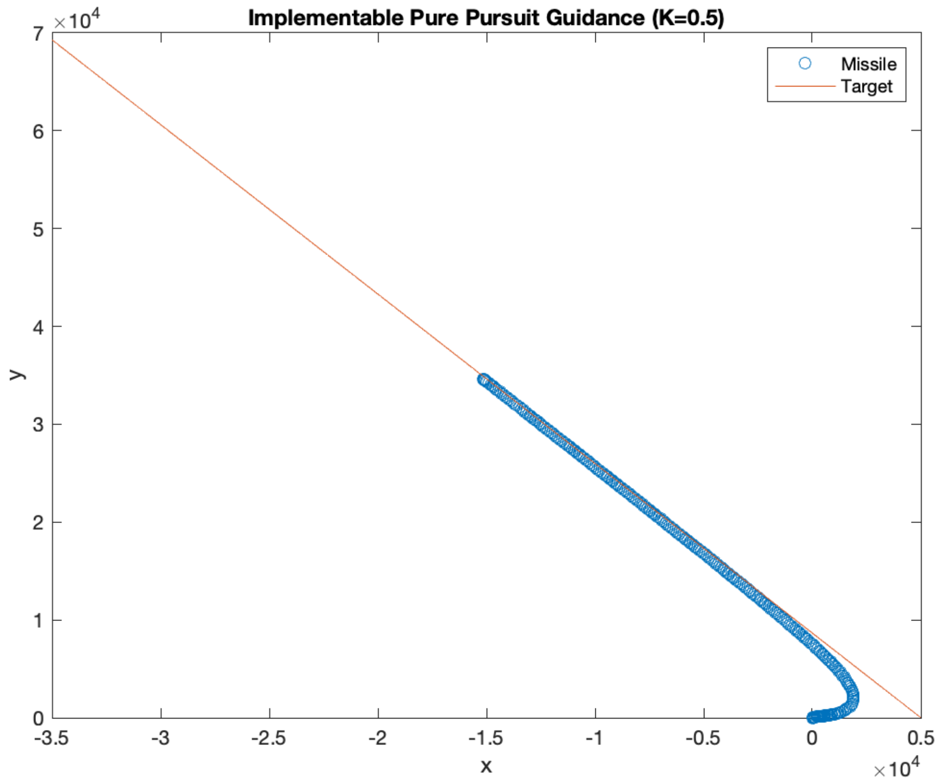

Figure 5: Simulated for 400s

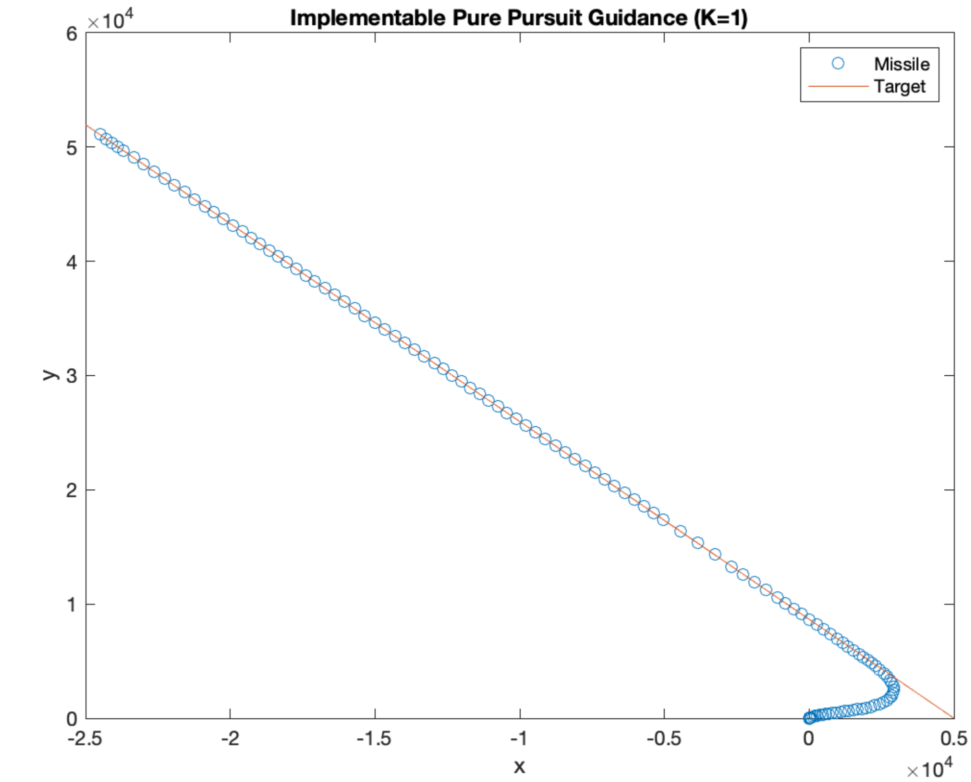

Figure 6: Simulated for 300s. Again Target seems to get away.

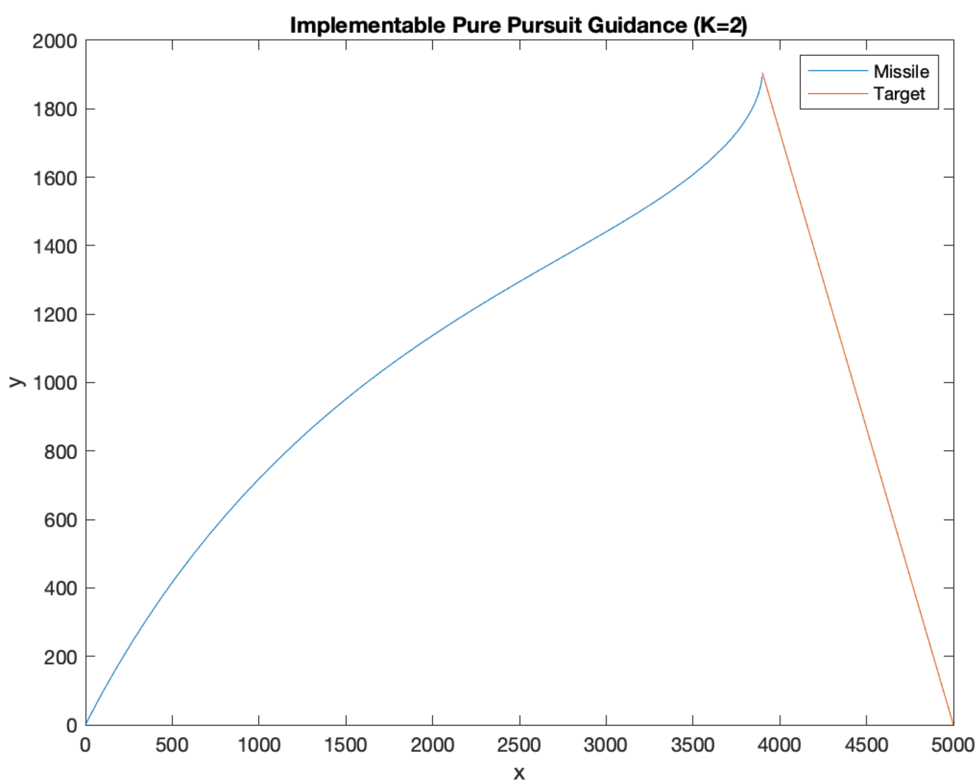

Figure 7: Simulated for 11s.

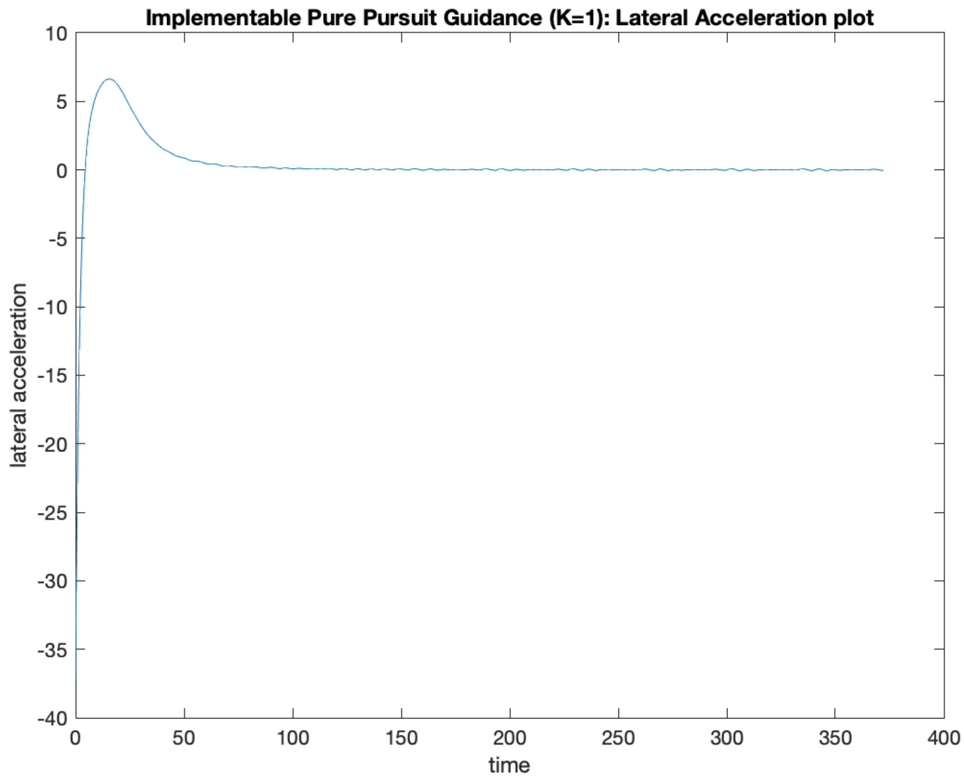

Figure 8: Simulated for 400s

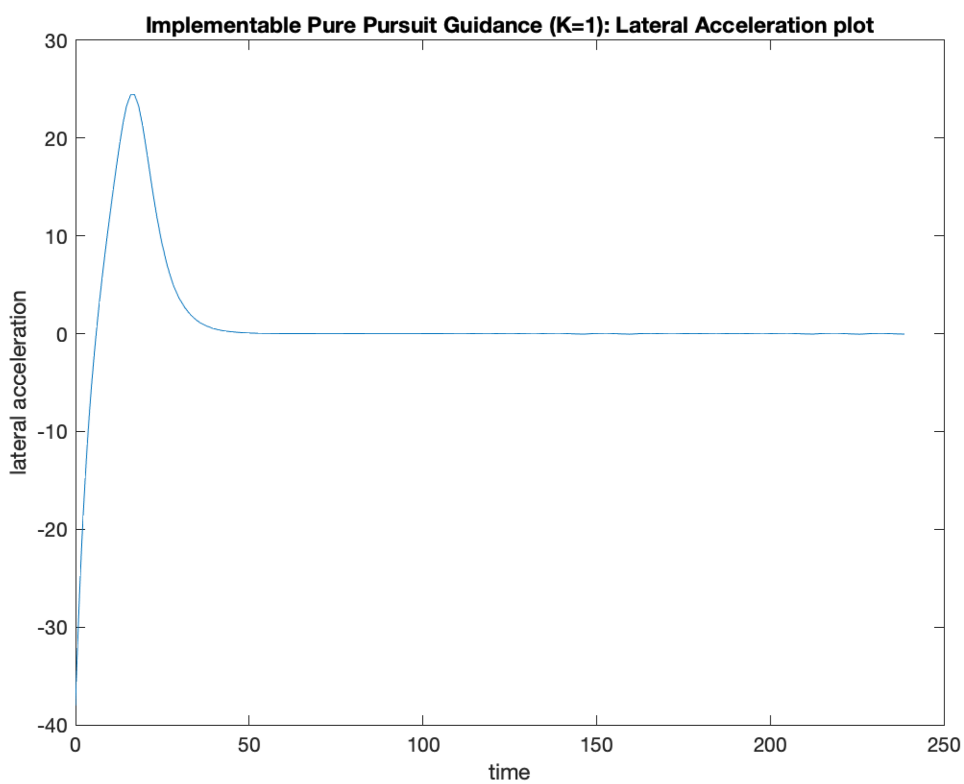

Figure 9: Simulated for 300s. Again Target seems to get away.

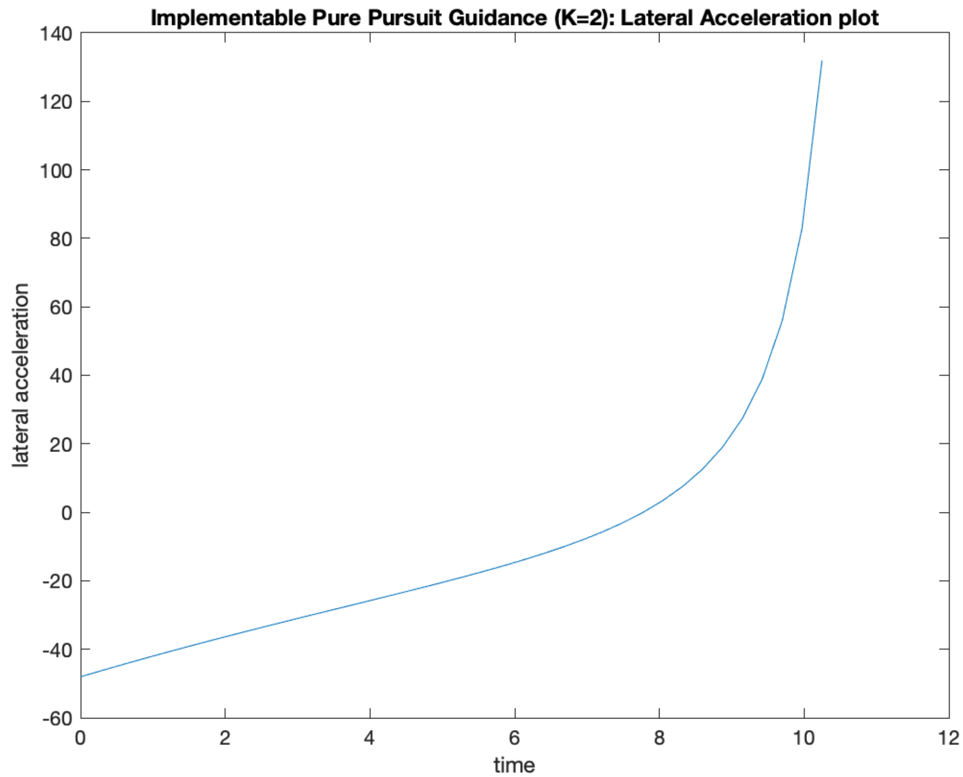

Figure 10: Simulated for 11s.

4.3 Inference

For K = 0.5, Missile does not seem to be getting close enough to the target for interception. For K = 1, again the target seems to get away. For K=2, we see a definite interception.

5 Question 5

5.1 Time-to-go condition

Given (constant δ):

$$t_{go} = \frac{r[V_r(t) + 2V_m cos\delta - V_\theta tan\delta]}{V_M^2 - V_T^2}$$

$$r = 0 \implies t_{go} = 0 \qquad \dots 1$$

For $t_{go} = 0$:

$$r[V_r(t) + 2V_m cos\delta - V_\theta tan\delta] = 0$$

As $V_r(t) = V_t cos(\gamma_t - \theta) - V_m cos\delta$ and $V_\theta(t) = V_t sin(\gamma_t - \theta) - V_m sin\delta$

$$\implies r[V_t cos(\gamma_t - \theta) - V_m cos\delta + 2V_m cos\delta - (V_t sin(\gamma_t - \theta) - V_m sin\delta)tan\delta] = 0$$

$$\implies r[V_t(cos(\gamma_t - \theta) - sin(\gamma_t - \theta)tan\delta) + V_m(cos\delta + sin\delta tan\delta)] = 0$$

$$\implies r\Big[\frac{V_t(cos(\gamma_t-\theta+\delta)}{cos\delta}+\frac{V_m}{cos\delta}\Big]=0$$

$$\implies r\Big[\frac{V_t(cos(\gamma_t-\theta+\delta)+V_m}{cos\delta}\Big]=0$$

Since the term inside the square bracket cannot be zero,

$$t_{go}=0 \implies r=0 \qquad \ldots\ 2$$

From 1 and 2, $t_{go}=0$ is both necessary and sufficient condition for a successful interception ($r=0$)

5.2 Deviated Pursuit Trajectories and Lateral Acceleration Plots

Initial separation was taken as 5 km while Proportionality constant was taken as 50.
($a_M = V_M\dot{\theta} - 50(\gamma_M - \theta - \delta)$)

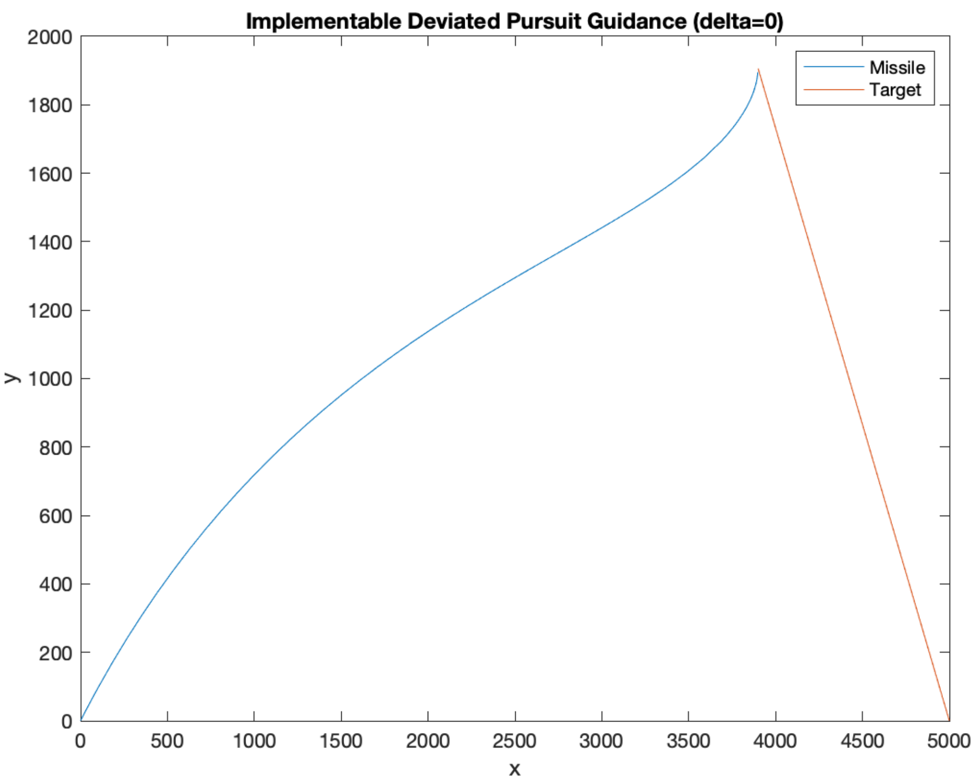

Figure 11: Simulated for 11s

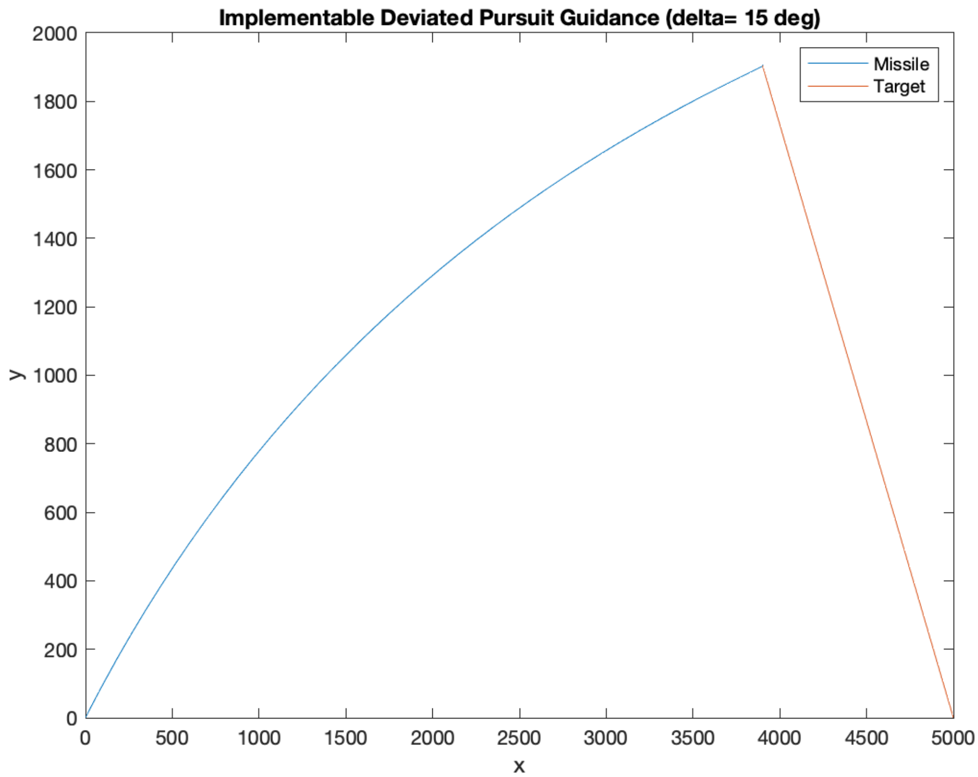

Figure 12: Simulated for 11s

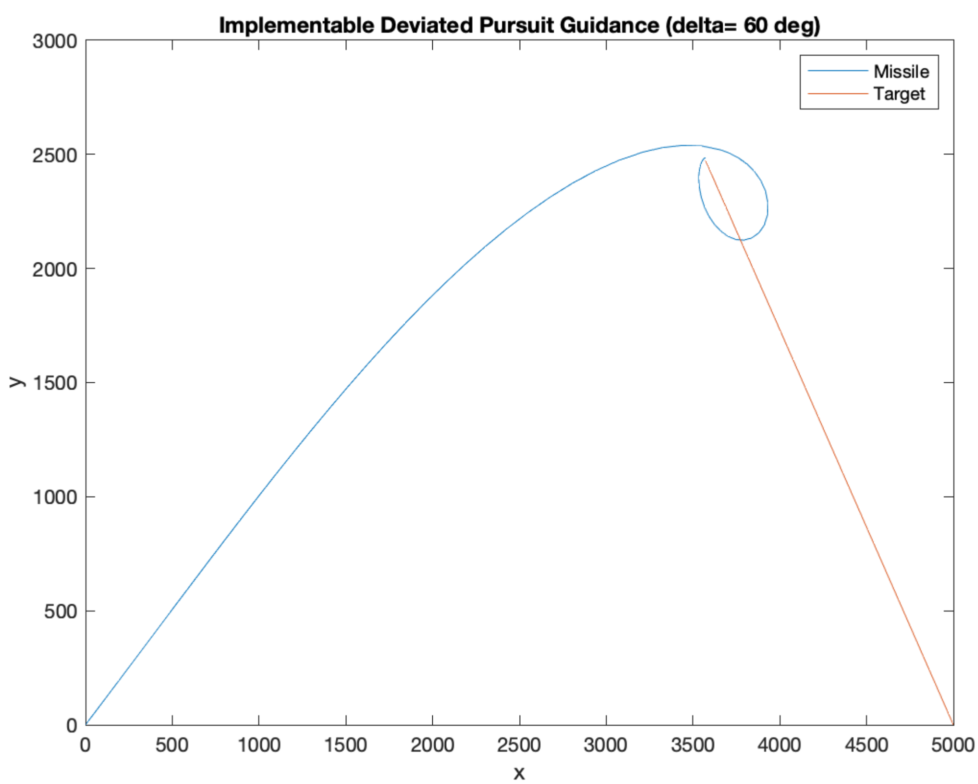

Figure 13: Simulated for 14.3s

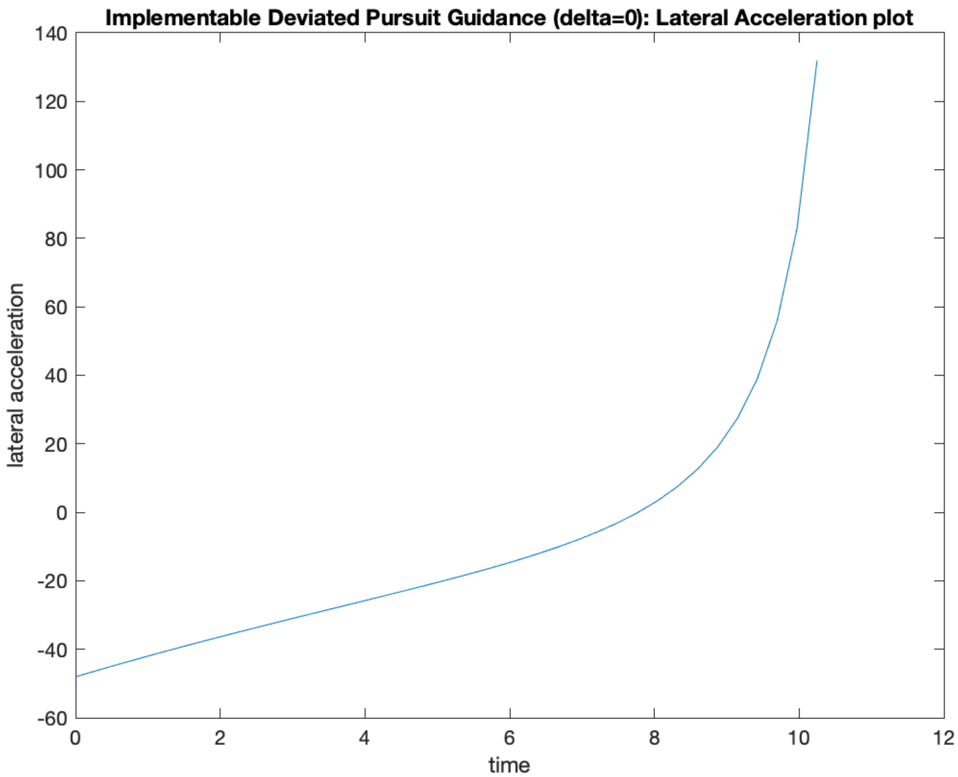

Figure 14: Simulated for 11s

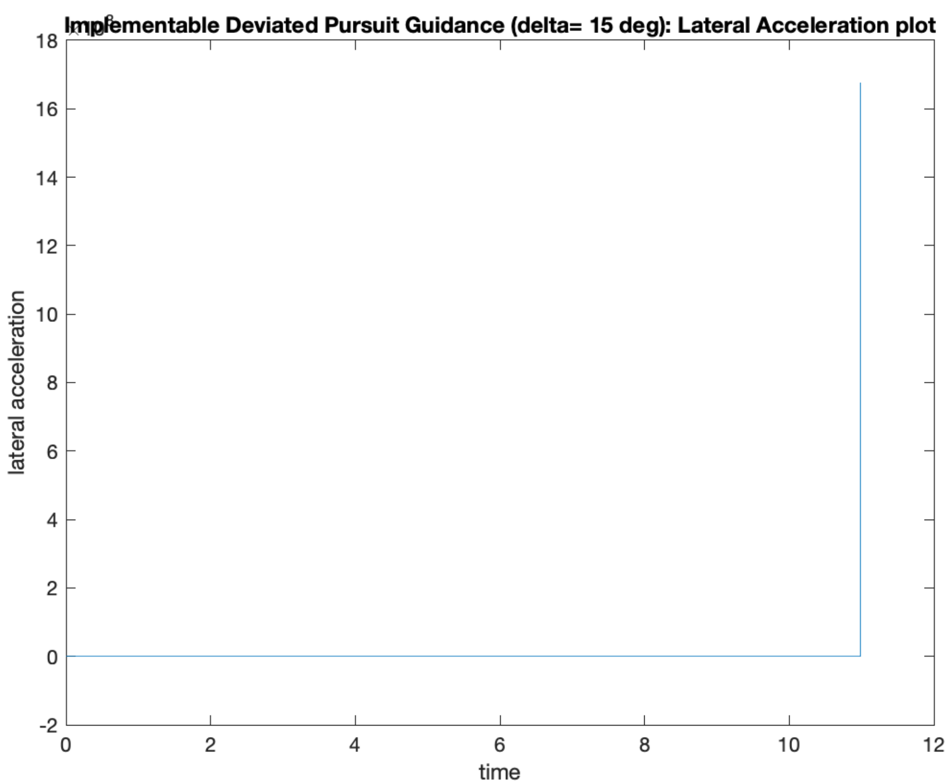

Figure 15: Simulated for 11s

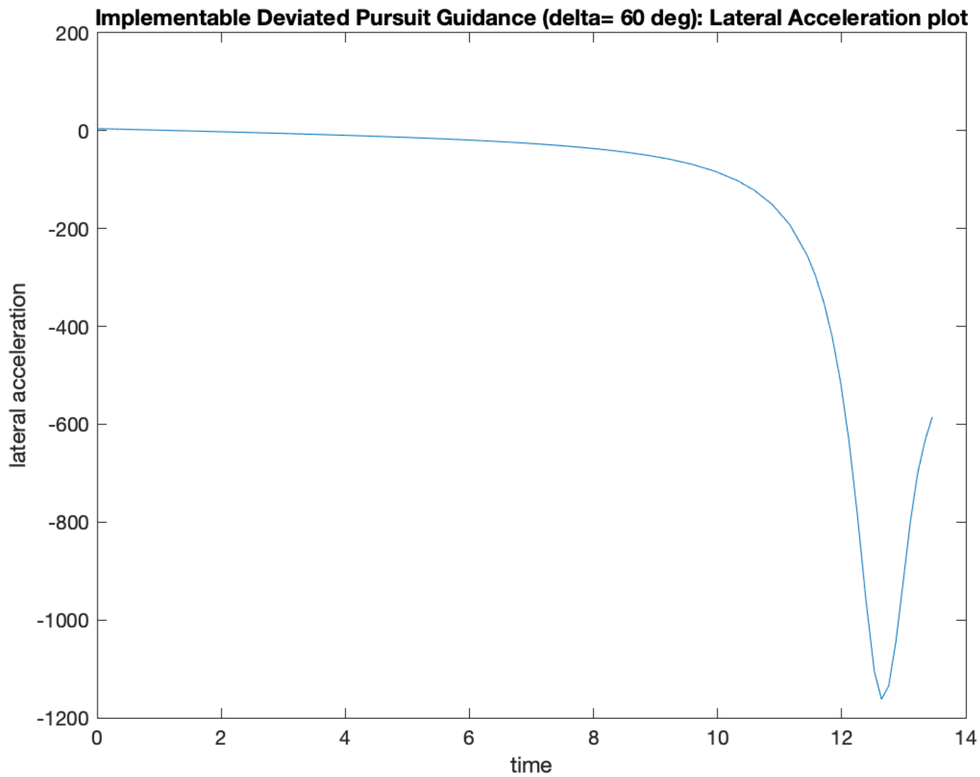

Figure 16: Simulated for 14.3s

5.3 Observation

There is a noticeable difference in curvature on changing δ. For $\delta = 0$, missile follows the missile going upward while for $\delta = 60°$, the missile loops around the missile.

6 Question 6

6.1 Effect of N on TPN

For True Proportion Navigation:

$$\dot{V}_\theta V_\theta + \dot{V}_r V_r + cV\dot{r} = 0$$

On integrating:

$$V_\theta^2 + V_r^2 + 2cVr = k$$

where $k = V_{\theta_0}^2 + V_{r_0}^2 + 2cVr_0$
For Capturability, $V_{\theta_0}^2 + V_{r_0}^2 + 2cVr_0 < 0$ and as $c = -NV_{r_0}$

$$V_{\theta_0}^2 < Vr_0{}^2(2N-1)$$

$$|V_{\theta_0}| < |Vr_0|\sqrt{(2N-1)}$$

So for existence of capture circle: We need $(2N-1) > 0$.

$N > 1/2$

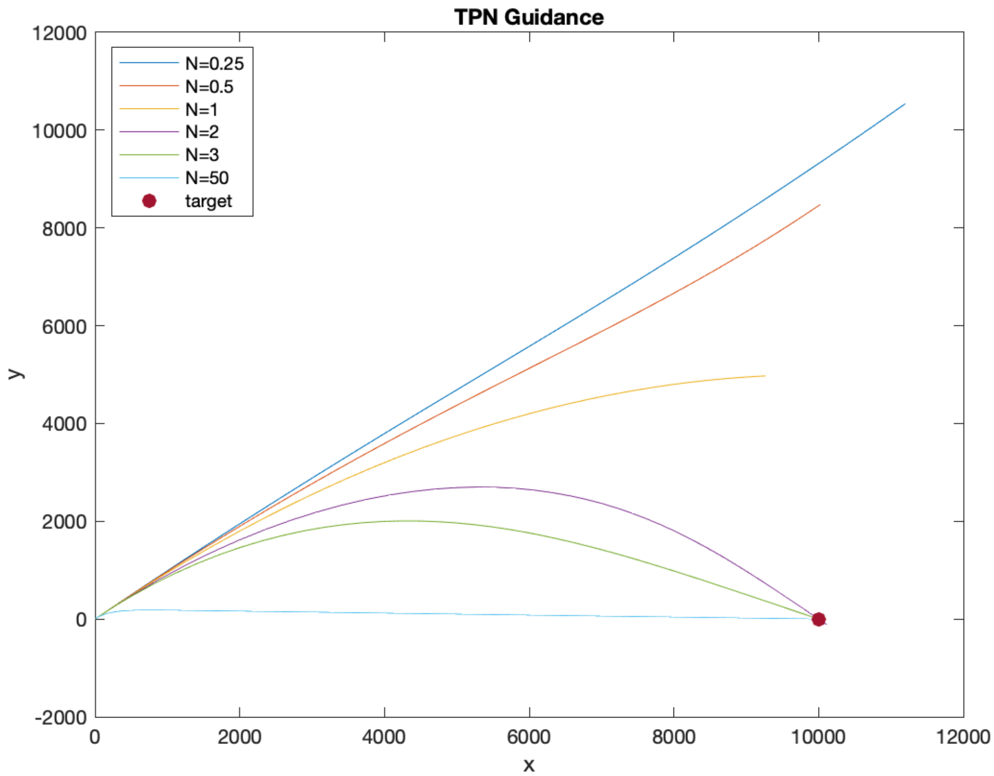

Figure 17: `N_values = [0.25; 0.5; 1; 2; 3; 50]; simulation_time_values = [45; 45; 55; 54; 44; 36];`

6.2 $N \to \infty$

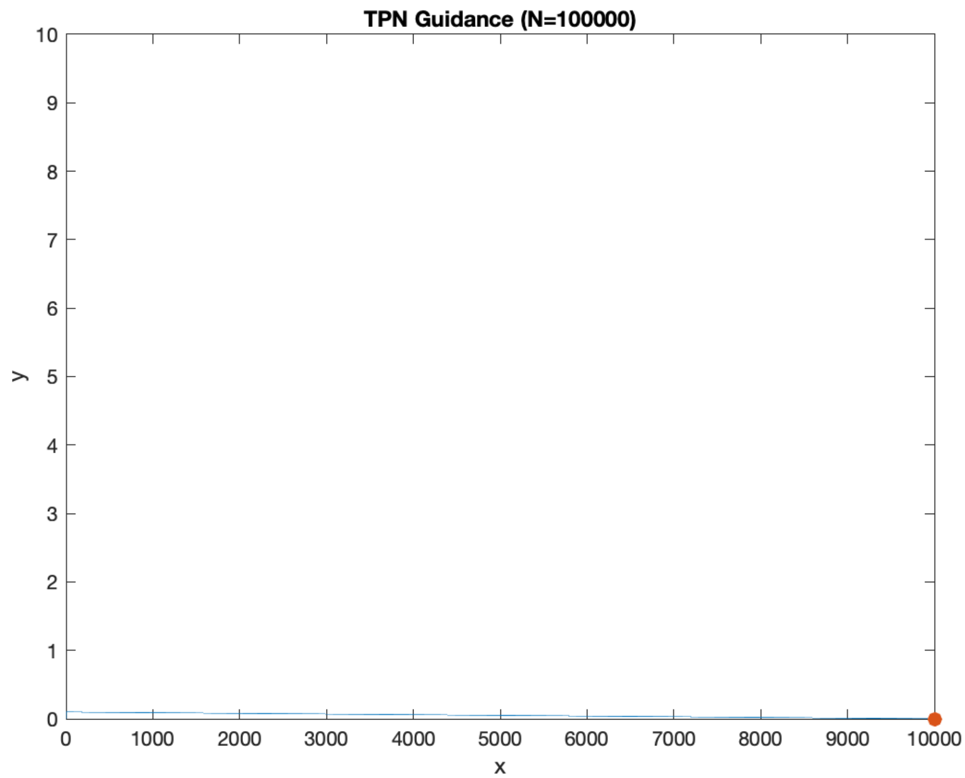

Figure 18: We can notice the very low $y-$axis values. For $N \to \infty$, we will have a straight line joining starting position of the missile and target.

7 Question 7

7.1 Proof of Relative Range

For a stationary target:

$$V_r = \dot{r} = -V_M cos\sigma$$
$$V_\theta = r\dot{\theta} = -V_M sin\sigma$$

On dividing the above two equations, we get:

$$\frac{dr}{r} = tan\sigma d\theta \qquad \dots\ 1$$

For PPN:

$$\dot{\gamma_M} = N\dot{\theta}$$
$$\implies d\gamma_M = Nd\theta$$
$$\implies d(\sigma + \theta) = Nd\theta$$
$$\implies d\theta(N-1) = d\sigma$$

Substituting the $d\theta$ in 1:

$$\frac{dr}{r} = \frac{tan\sigma}{N-1} d\sigma$$

Integrating this equation, we get:

$$ln\left(\frac{r}{r_0}\right) = \frac{1}{N-1} ln\left(\frac{sin\sigma}{sin\sigma_0}\right)$$

$$\boxed{\frac{r}{r_0} = \left(\frac{sin\sigma}{sin\sigma_0}\right)^{1/N-1}}$$

7.2 Trajectories of PPN guidance

$$\gamma_{M,d} = sin^{-1}(V_T sin(\gamma_T)/V_M)$$
$$\implies \gamma_{M,d} = 36.89°$$

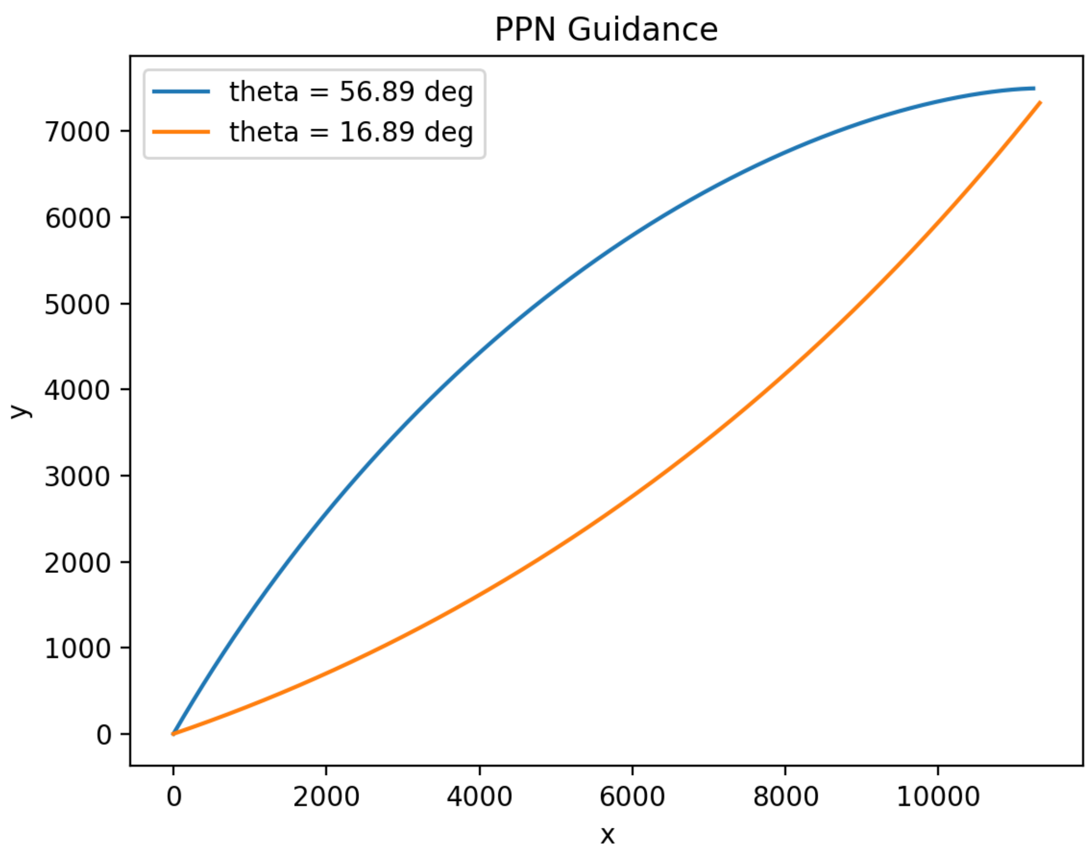

Figure 19: Simulated for 27.3s (for $\theta = 56.89°$) and 28s (for $\theta = 16.89°$)