Computational Fluid Dynamics (AE 320)

Assignment 3

Submitted By

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Contents

1 Files Description

All the codes are written in python. This assignment submission contains the following files:

- **question1.py**: This code runs FTBS, FTCS and FTFS schemes for the initial and boundary conditions provided in the first question of the assignment. The code generates plots at time values of 0.2, 0.5, 0.7 and 1.0 second as asked in the question for all the schemes for CFL values of 0.4, 0.9 and 1.2. All the graphs are generated at a pause time of 1 second and stored in a directory "saved_graphs_question1/", which is created during the program execution.

- **question2.py**: This code runs FTBS, FTFS and FTCS scheme for the given initial and boundary conditions of $sin(2\pi x)$ in question 2. The code generates wave form plots at time values of 1, 1.5, 3, 4s (as asked in the question) for CFL values of 0.5, taking grid points as 100. All the graphs are generated at a pause time of 0.5 seconds and stored in a directory "saved_graphs_question2/", which is created during the program execution.

- **question2_part2.py**: This code runs FTBS and FTCS2 scheme for the given initial and boundary conditions of $cos^2(\pi x)$ in question 2. The code generates wave form plots at time values of 2, 4 and 6s (as asked in the question) for CFL values of 0.4 and 0.7, taking grid points as 80. All the graphs are generated at a pause time of 0.5 seconds and stored in a directory "saved_graphs_question2_part_2/", which is created during the program execution.

- **question3.py**: This code runs FTBS and FTCS2 scheme for the given initial and boundary conditions in question 3. The code generates wave form plots at time values of 4s and 40s for 40 and 600 grid points taking CFL value as 0.8. All the graphs are generated at a pause time of 0.5 seconds and stored in a directory "saved_graphs_question3/", which is created during the program execution.

- **report.pdf**: This report which discusses the results of the code files.

Note: Grid points interpretation For all the assignments, n grid points in the domain [0,1] have been interpreted as uniformly spaced points with $\Delta x = 1/n$. This creats n+1 points inn the domain including 0 and 1. These points have been generated using np.linspace() function in numpy. Likewise has been done for domain [-1, 1].

2 Numerical Schemes employed

This assignment comprises numerical solutions to the following advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

We use FTBS, FTCS, FTFS and FTCS2 schemes to iteratively solve this equation. On applying these schemes, we get following solutions (takingn usual notations as defined in class).

FTBS:

$$u_p^{q+1} = (1-\sigma)u_p^q + \sigma u_p^{q-1}$$

FTCS:

$$u_p^{q+1} = u_p^q - \sigma(u_p^{q+1} - u_p^{q-1})$$

FTFS:

$$u_p^{q+1} = (1+\sigma)u_p^q - \sigma u_p^{q+1}$$

FTCS2:

$$u_p^{q+1} = (1-\sigma^2)u_p^q - 0.5\sigma(1-\sigma)u_{p+1}^q + 0.5\sigma(1+\sigma)u_{p-1}^q$$

3 Question 1

Given:

$$a = 1$$

$$\Delta x = 1/50 = 0.02$$

The equation has been simulated for the unit domain $[0, 1]$ with initial and boundary conditions as follows:

$$u(0, t) = 1, t > 0$$

$$u(x, 0) = 0$$

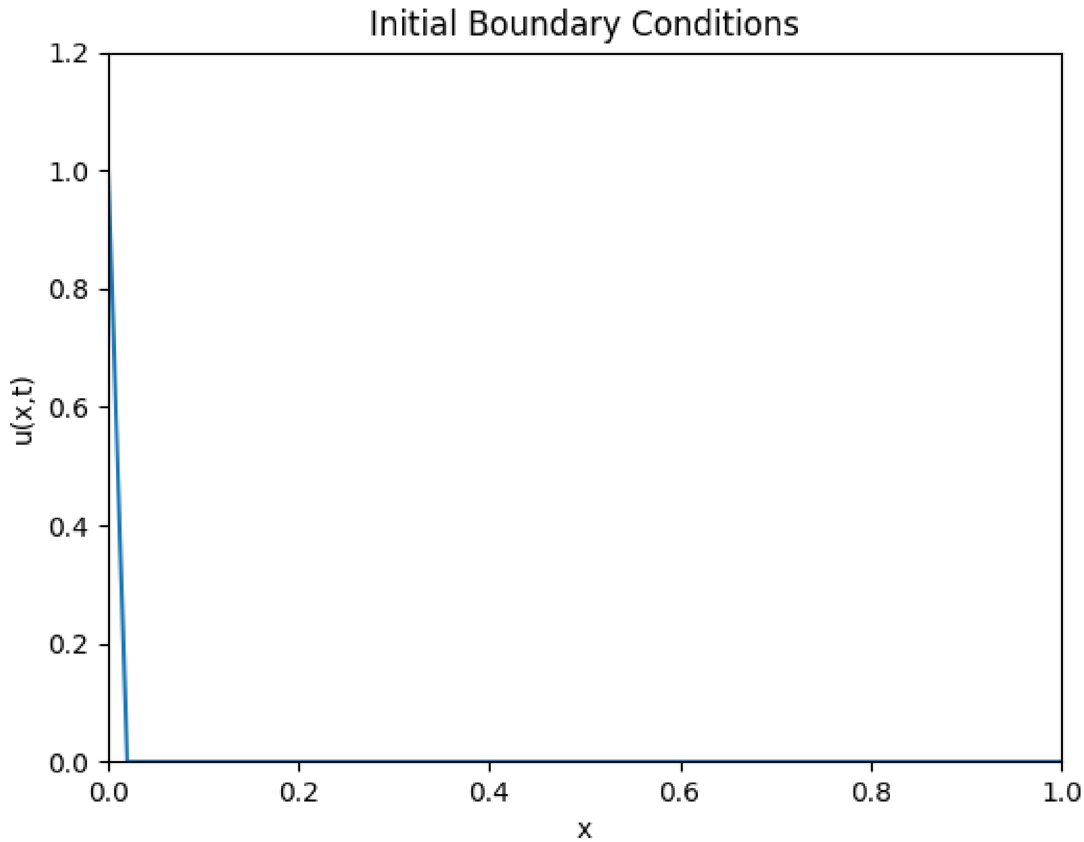
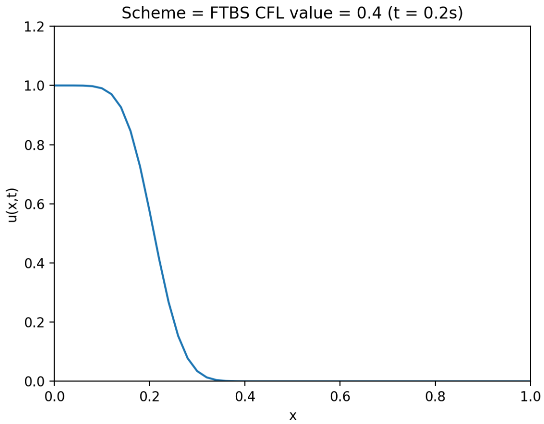
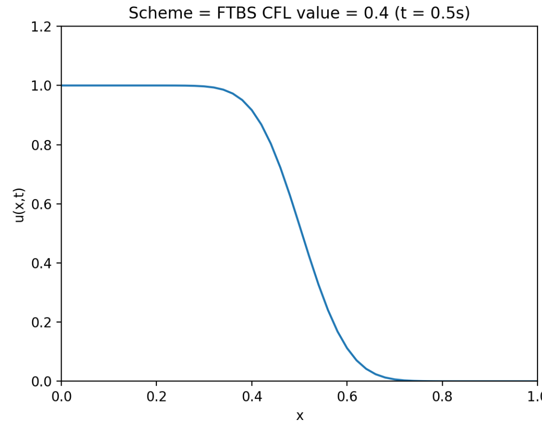


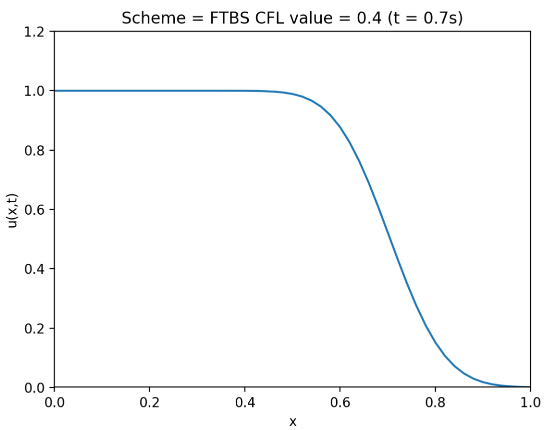
Figure 1: Initial and Boundary conditions

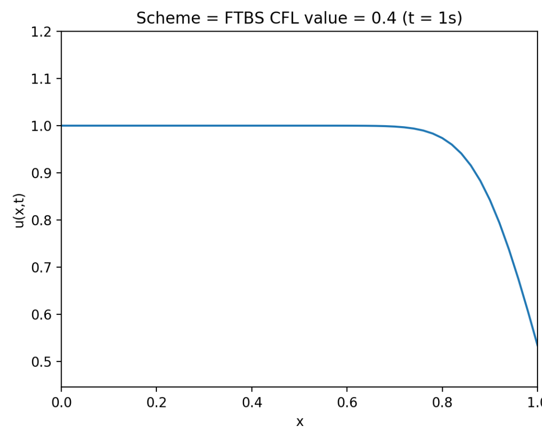
3.1 FTBS

3.1.1 CFL Value = 0.4



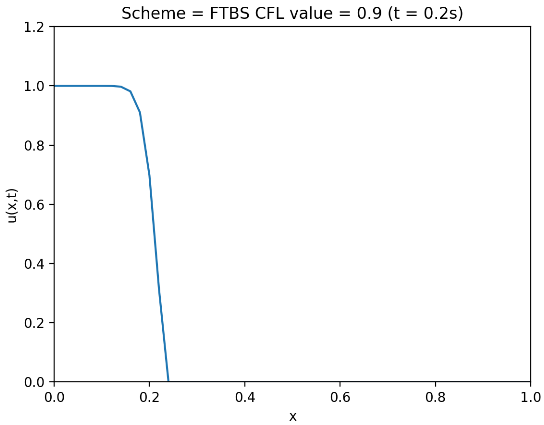




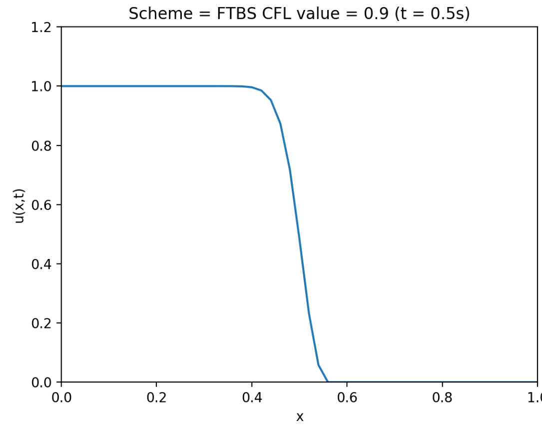


The above plots show the wave propagation of wave in FTBS scheme, CFL value = 0.4. We can see that the wave is indeed travelling forward in space with time. For simulated time of 1s, the wave tends to converge to $u(x, t) = 1$ from $x = 0$ to $x = 1$. As FTBS is consistent and stable for CFL value between 0 and 1, this propagation behaviour was expected.

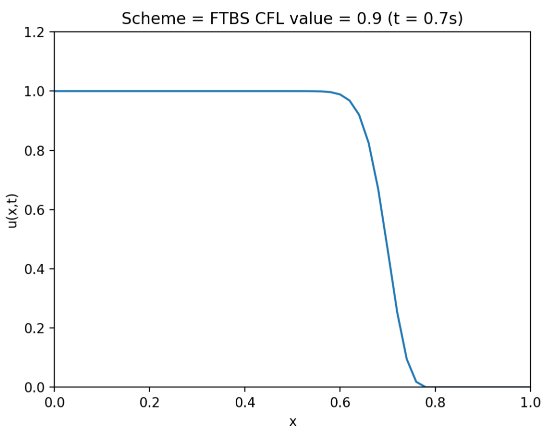
3.1.2 CFL Value = 0.9



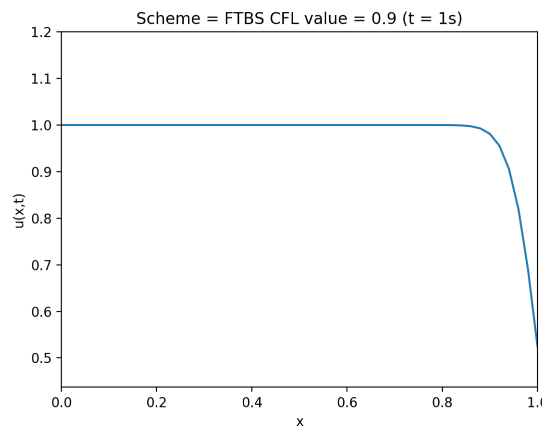
(e) t = 0.2s



(f) t = 0.5s



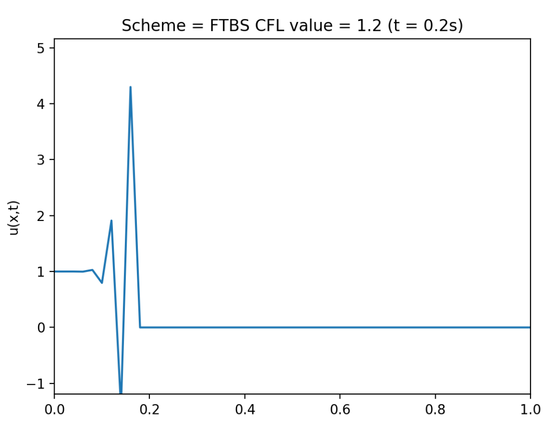
(g) t = 0.7s

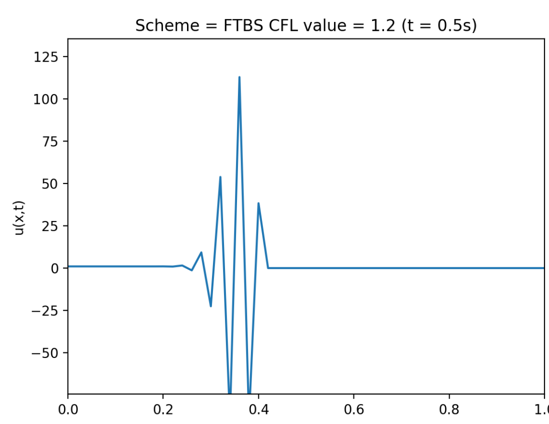


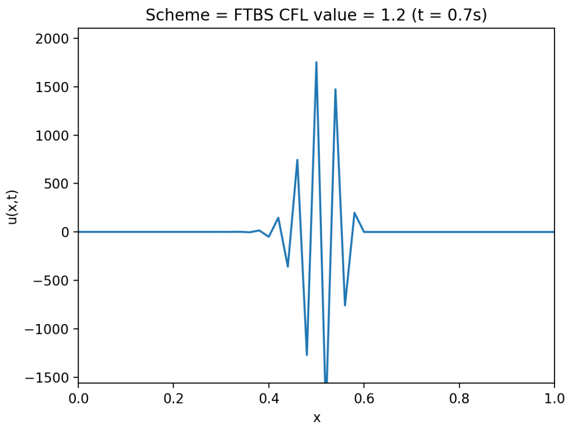
(h) t = 1s

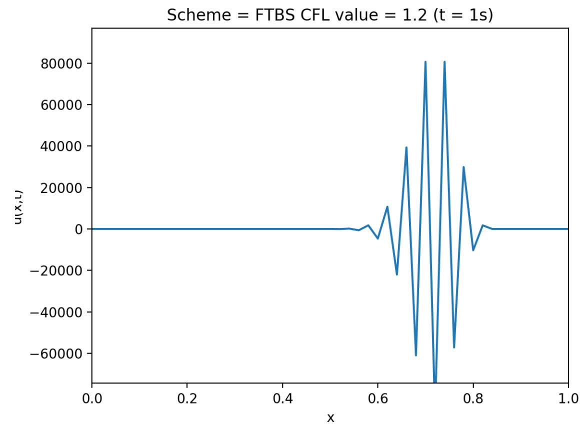
As the value of CFL is between 0 and 1, we expected convergance to desired solution as we did in case of CFL value of 0.4. But in this case, the convergence is faster. CFL denotes the ratio of physical speed to grid speed. So for a greater CFL value, the propagation is expected to be faster.

3.1.3 CFL Value = 1.2







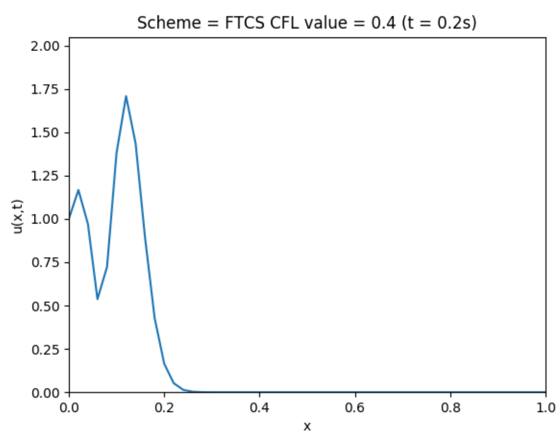


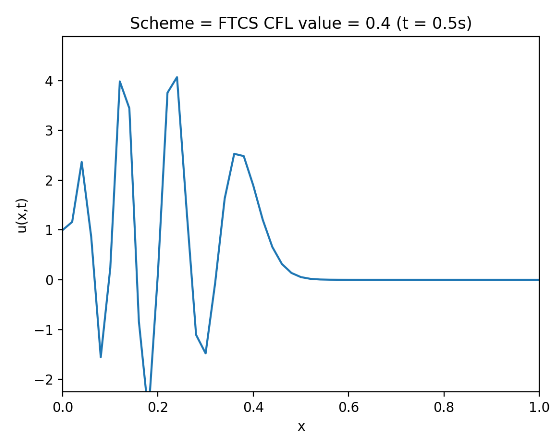
FTBS scheme is not stable for CFL value greater than 1 and hence we see that our solution blows up in the middle although the disturbance tends to travel forward.

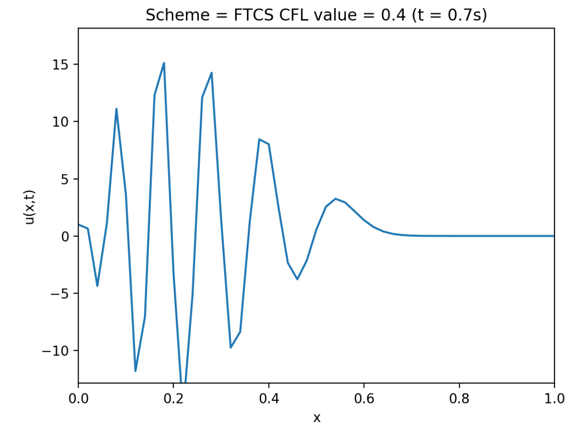
3.2 FTCS

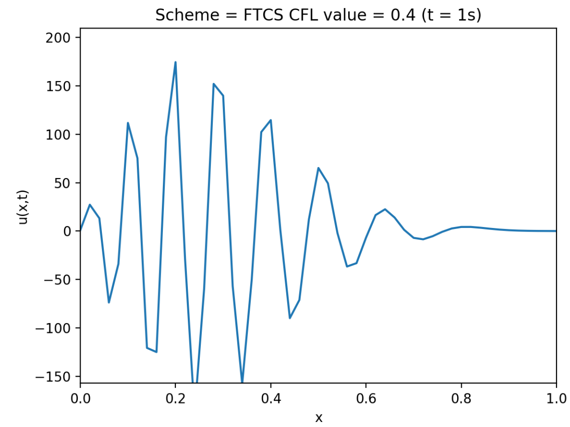
As FTCS is unconditionally unstable, we see that $u(x,t)$ blows up drastically. The disturbance propagates forward but the blow up is huge as CFL value and simulated time increases as the instability increases. We saw sharper spikes of blow ups in case of FTFS but we see a greater spread of spikes in case of FTCS. This is possibly due to dispersive nature of FTCS scheme and it taking both forward and backward points into consideration for calculating the updated value at next timestep. Blow up increase on increasing CFL value.

3.2.1 CFL Value = 0.4

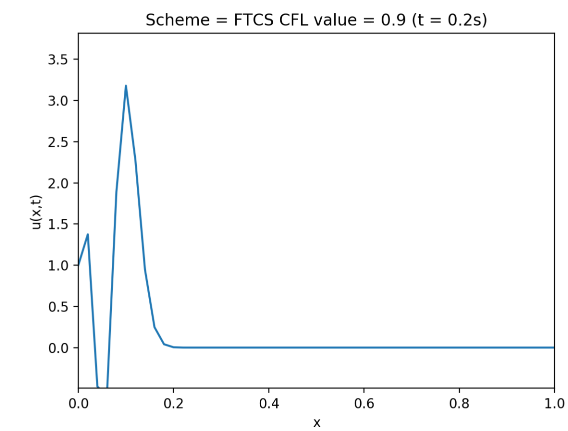


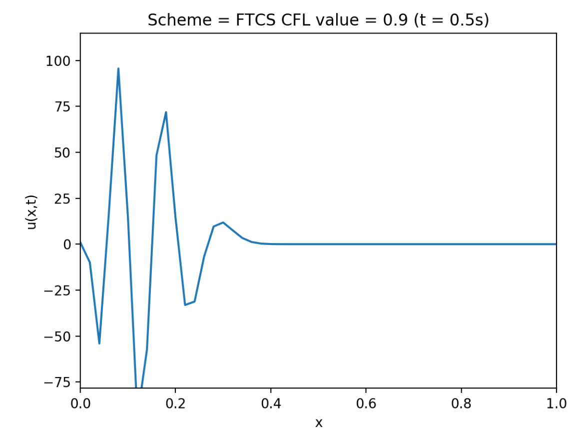


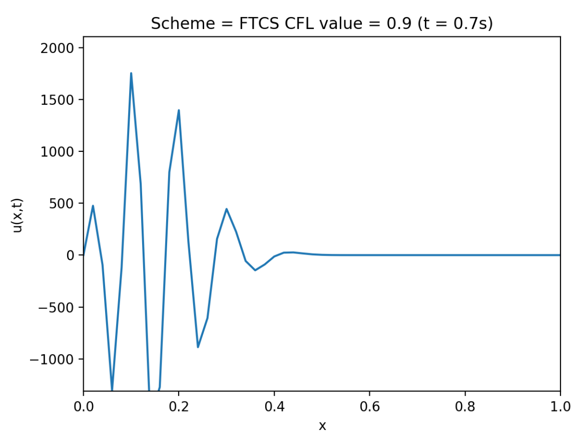


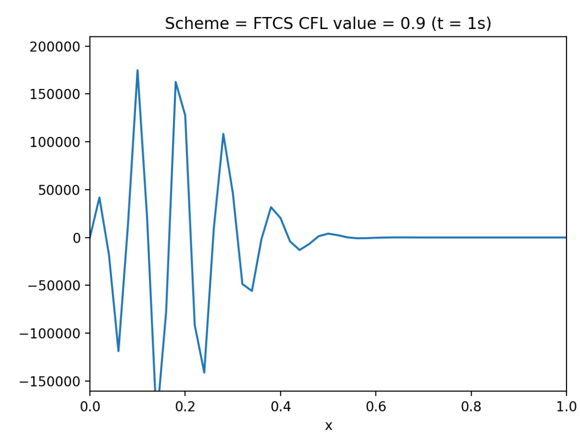


3.2.2 CFL Value = 0.9

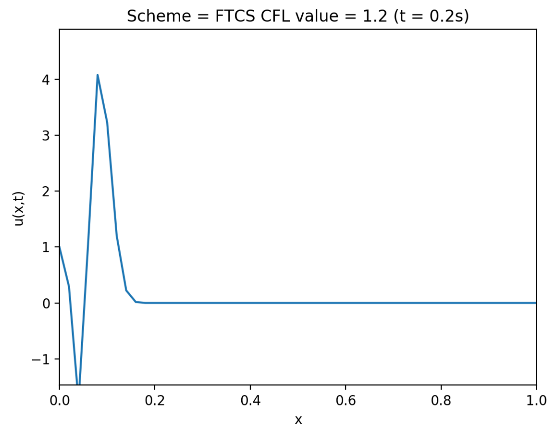


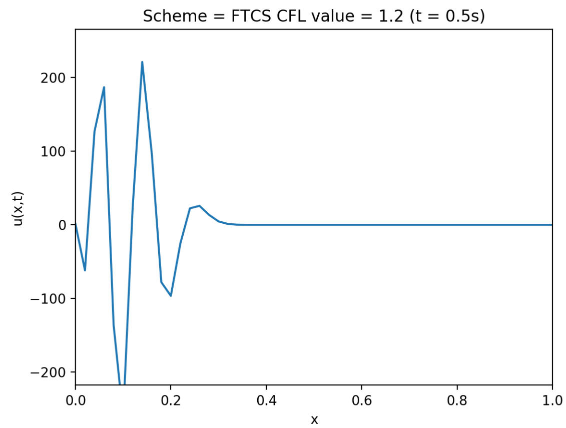


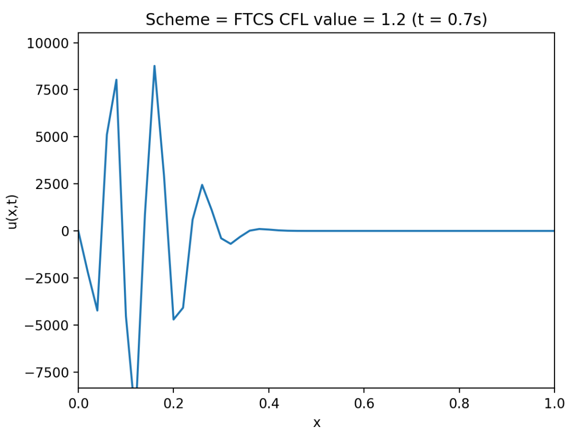


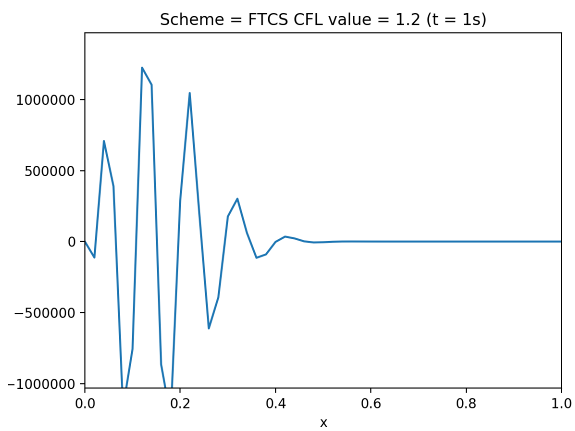


3.2.3 CFL Value = 1.2





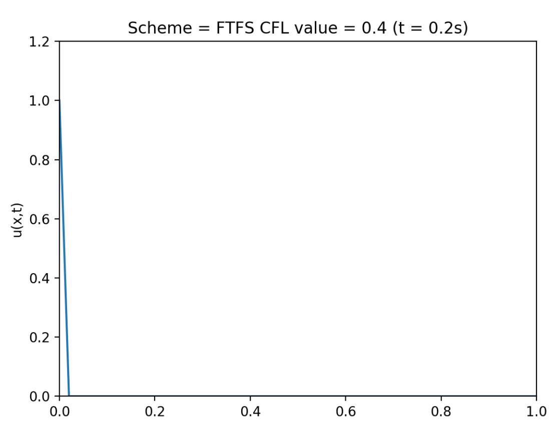


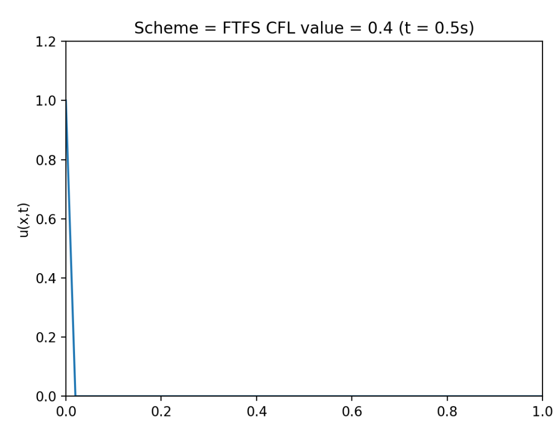


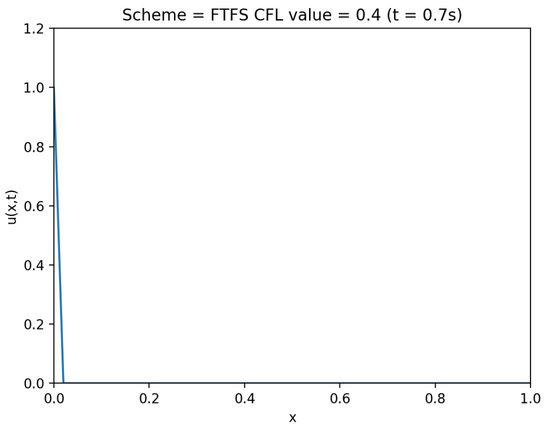
3.3 FTFS

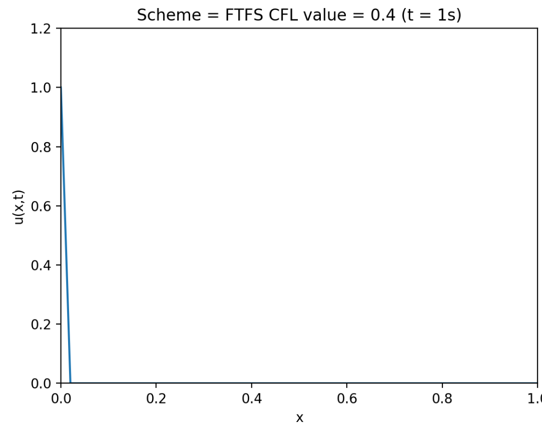
For FTFS we see no propagation of wave at all as the initial disturbance is at the starting point of the domain and FTFS considers only value at those x values which are ahead of the given point (forward in space).

3.3.1 CFL Value = 0.4









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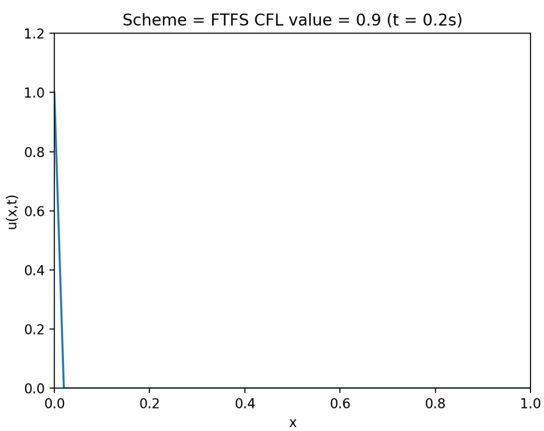
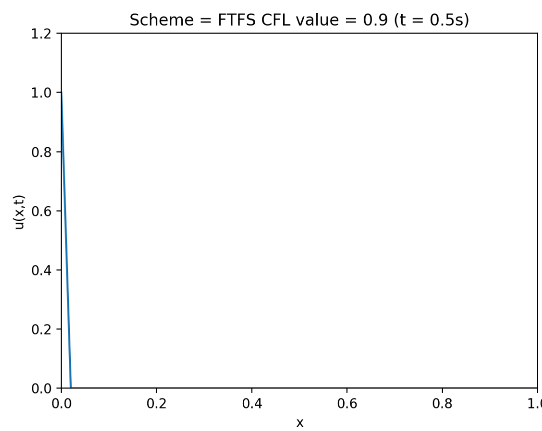
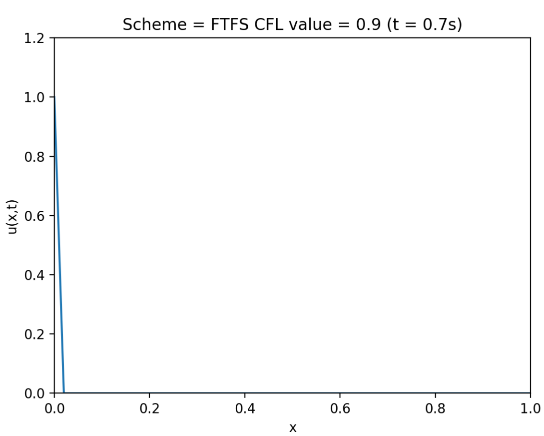
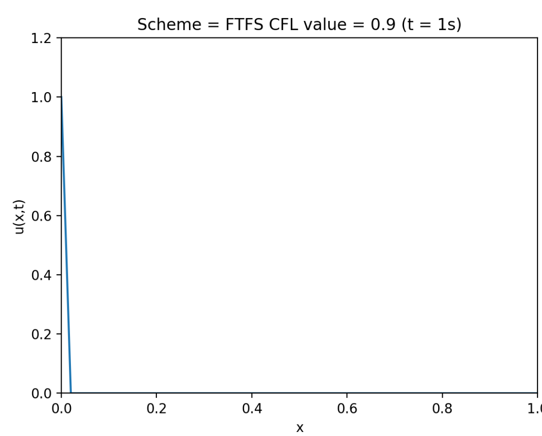
3.3.2 CFL Value = 0.9

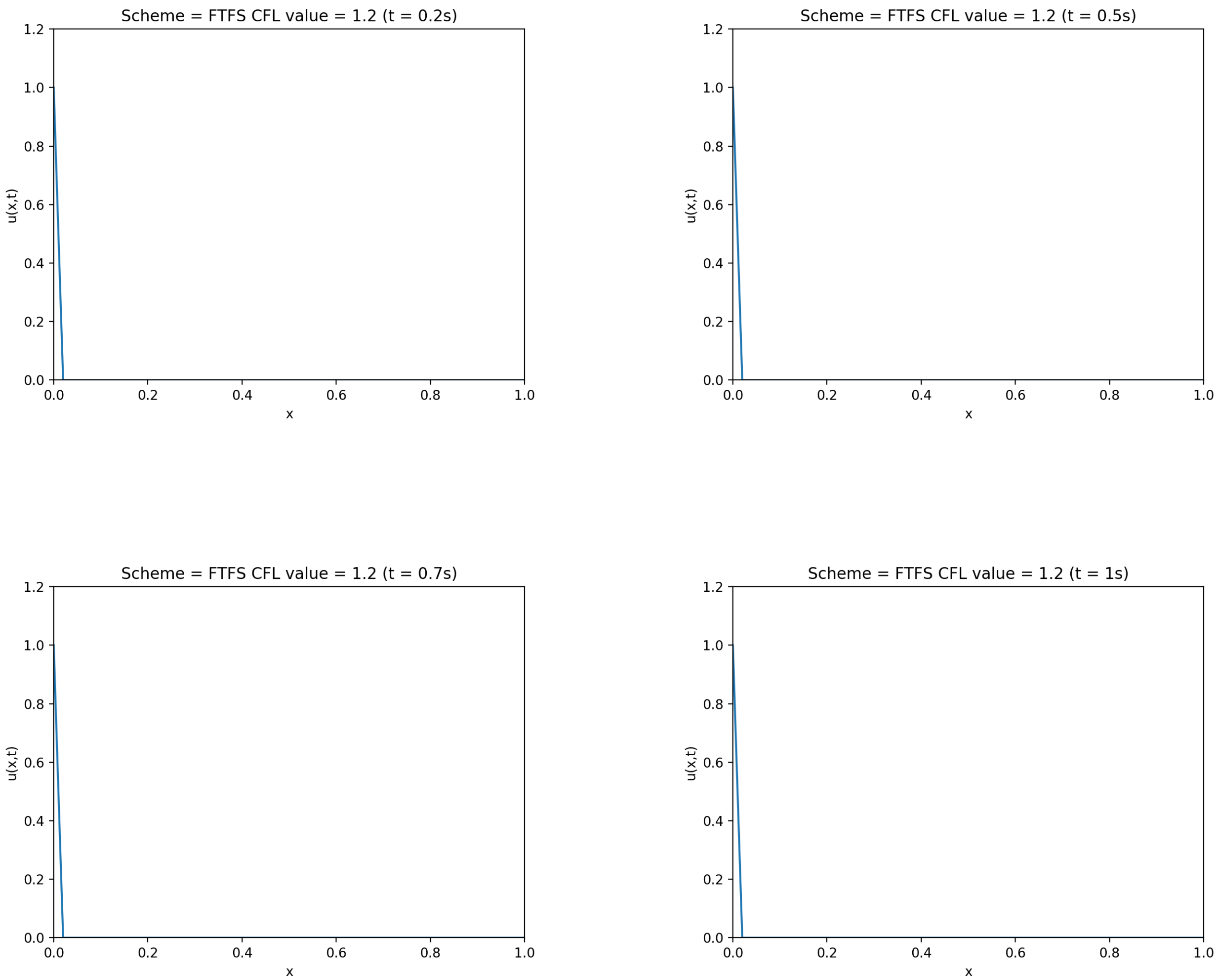








3.3.3 CFL Value = 1.2



4 Question 2

4.1 Part 1: $sin(2\pi x)$

$$a = 1$$

$$\Delta x = 1/100 = 0.01$$

Given initial and boundary condition:

$$u(0, t) = 0$$

$$u(x, 0) = sin(2\pi x)$$

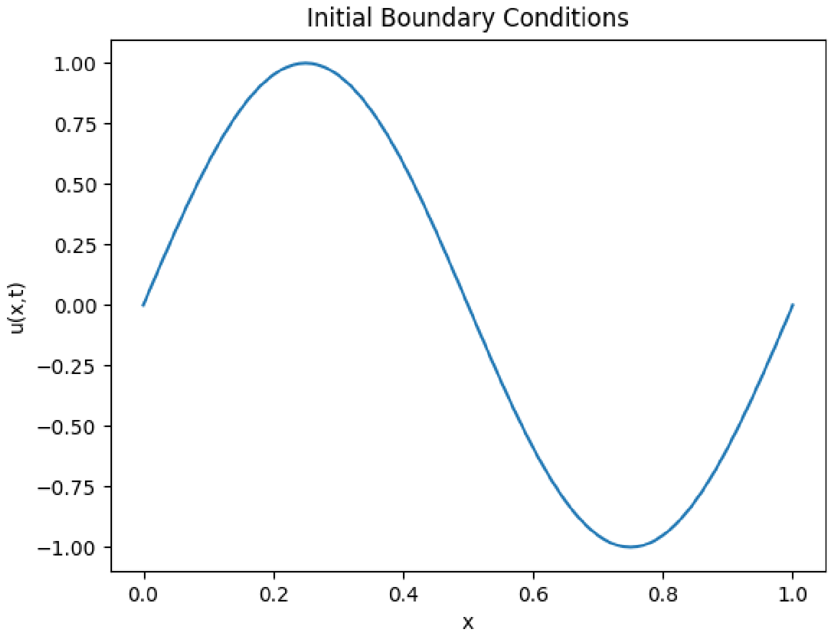
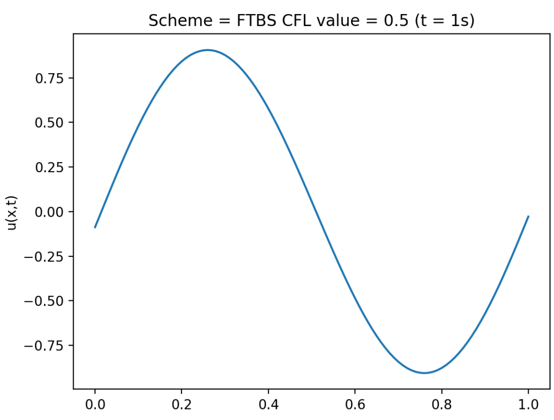
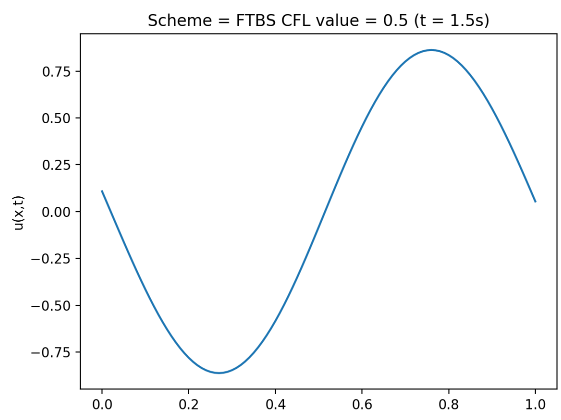


Figure 2: Wave form at time t = 0.

4.1.1 FTBS





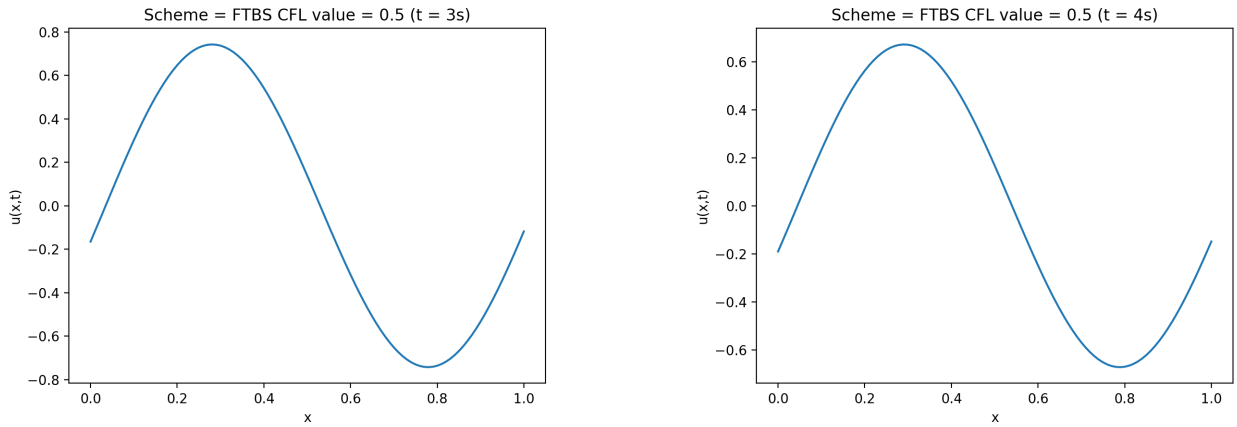


Figure 3: Wave travels forward and we see a nice waveform at every timestep as FTBS is stable for the given CFL value (less than 1). As gain of FTBS is less than 1, we notice that the magnitude of wave decreases as the wave propagates in time.

4.1.2 FTCS

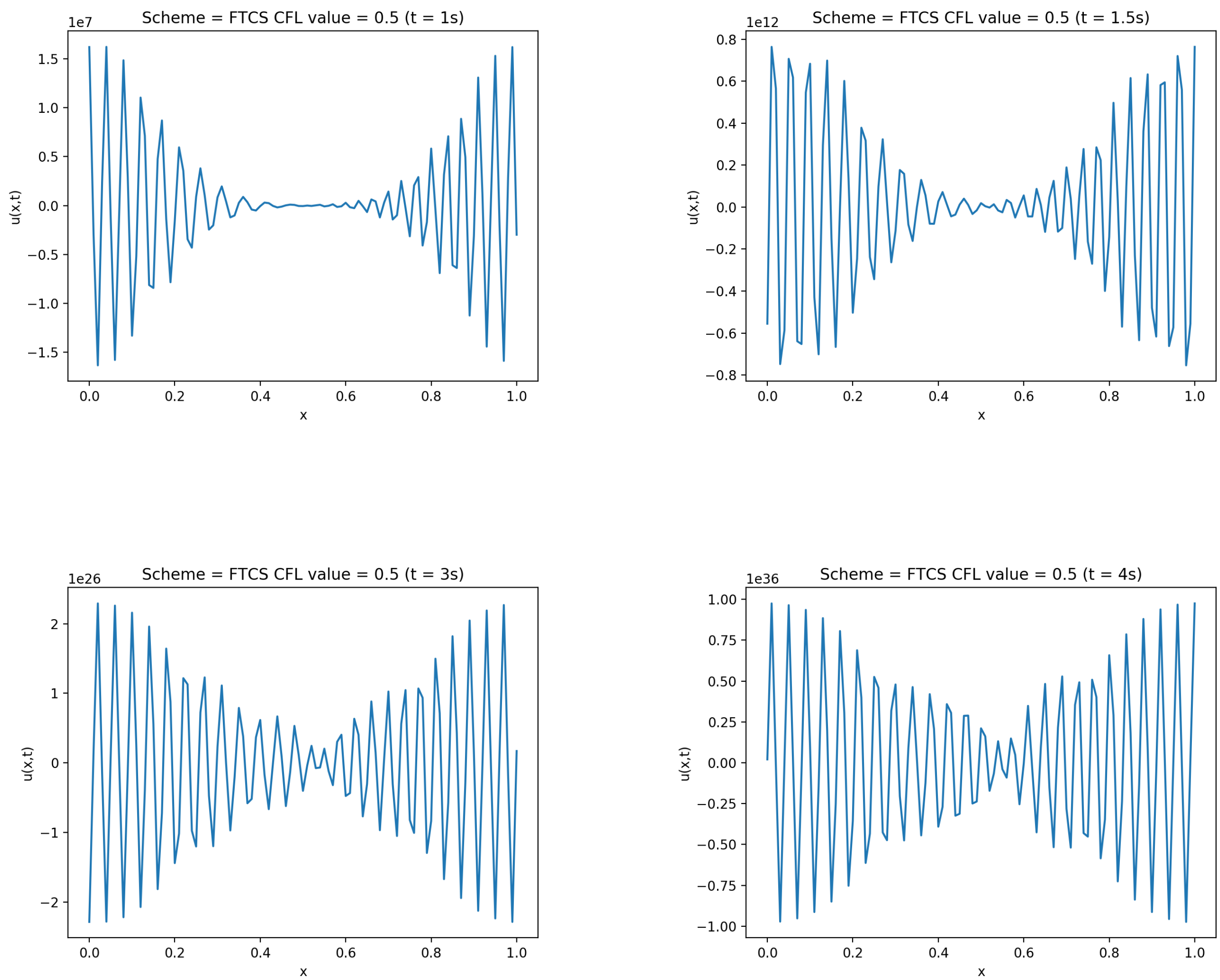


Figure 4: FTCS Scheme is unconditionally unstable and moreover also has a dispersive effect which can be seen in contrast to FTBS and FTFS solution schemes. The wave form gets more distorted with time which is expected from an unstable scheme.

4.1.3 FTFS

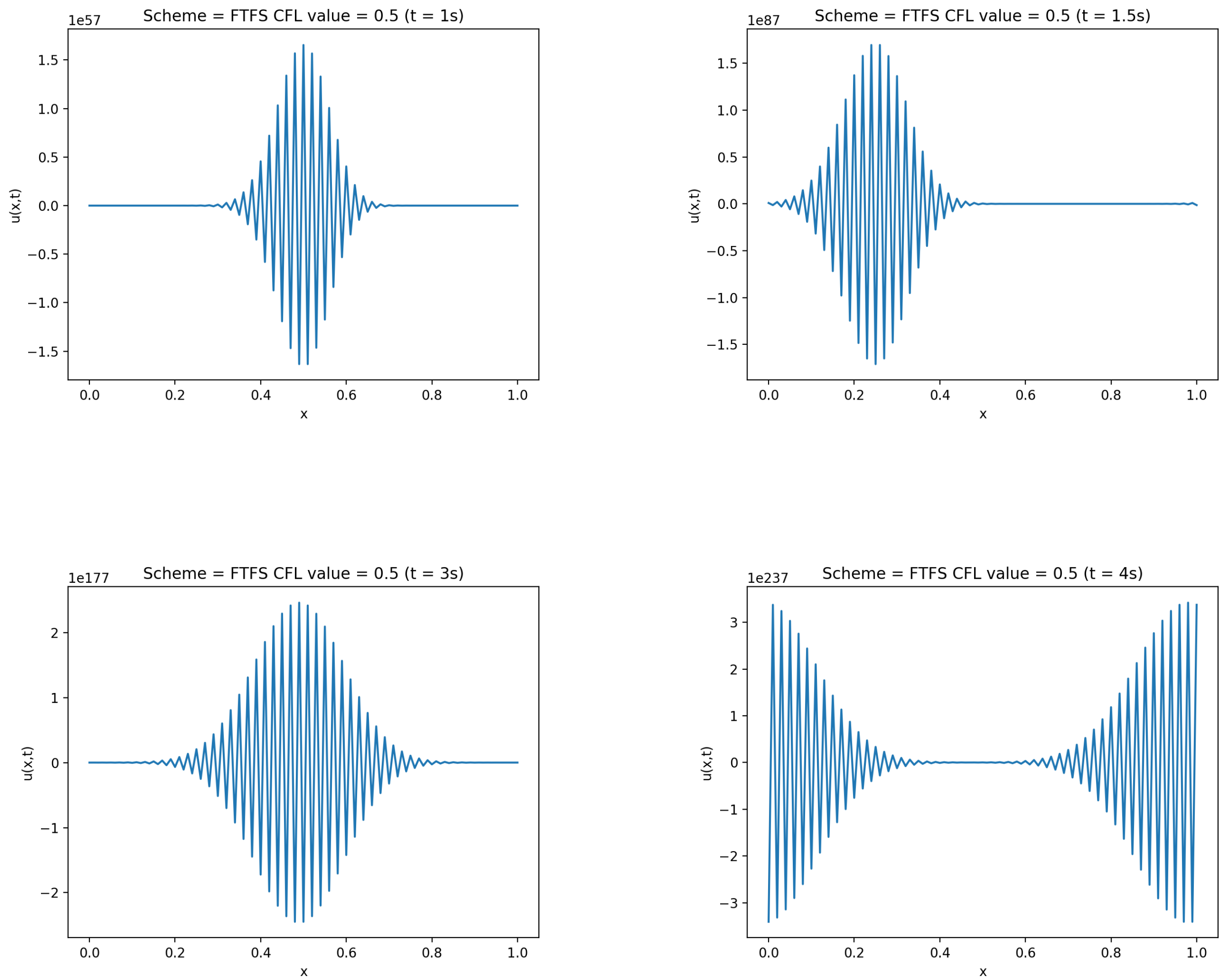


Figure 5: For the given CFL values, FTFS is unstable and as the dissipation of FTFS is also high, we get a blowup of solution in the middle of the grid. This blowup seems to travel forward with time.

4.2 Part 2: $cos^2(\pi x)$

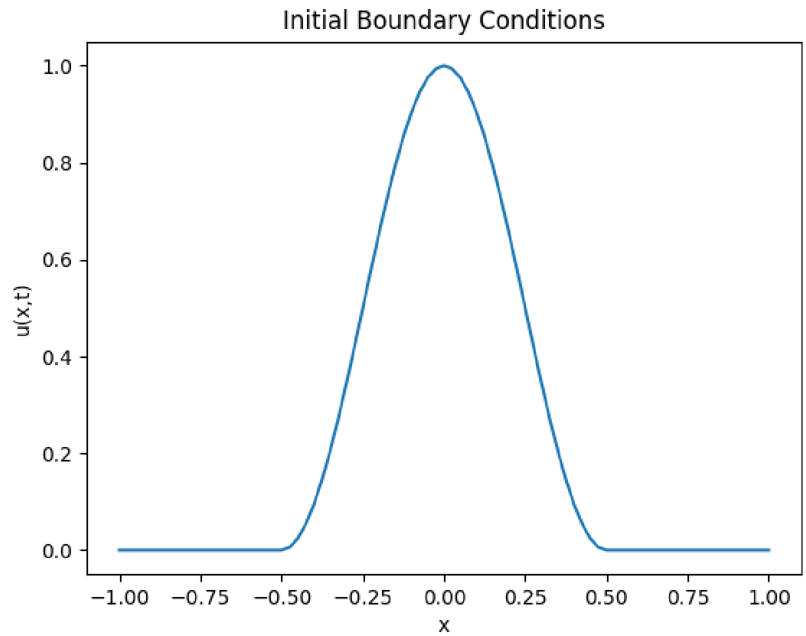
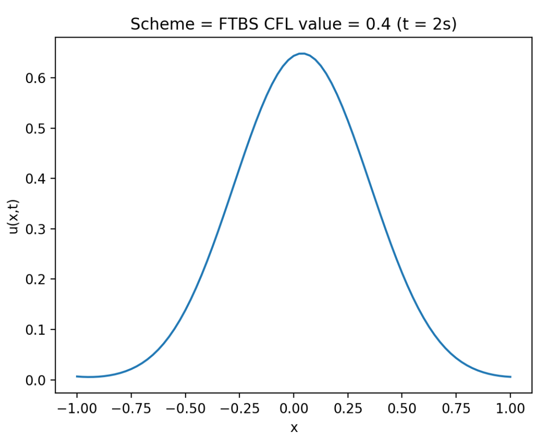
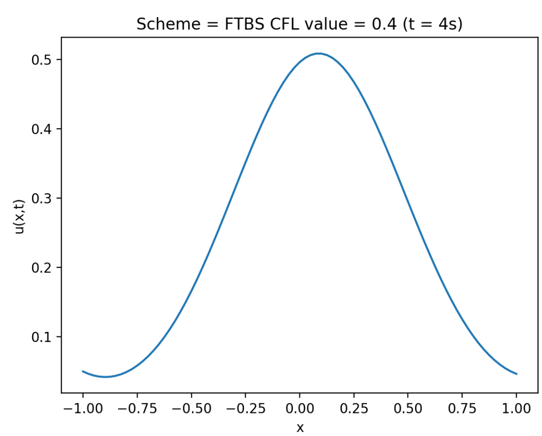


Figure 6: Initial and boundary condition for question 2 part 2

4.2.1 FTBS





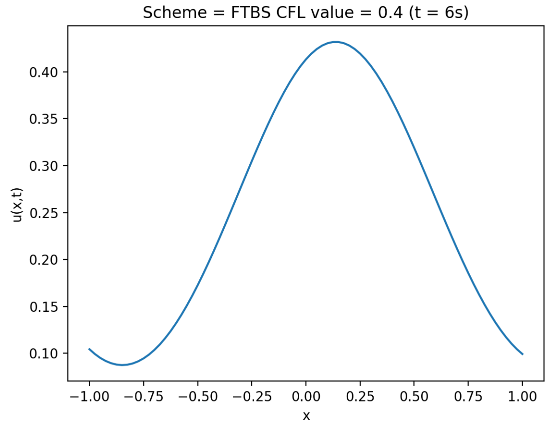


Figure 7: We see a stable wave propagation as wave retains its shape but the magnitude of cosine wave reduces as the wave travels in time as we saw in case of sin wave. Due to gain less than 1 for the given CFL value and dissipation (due to second order term in the modified wave equation), we see that wave smoothes out and its magnitude reduces.

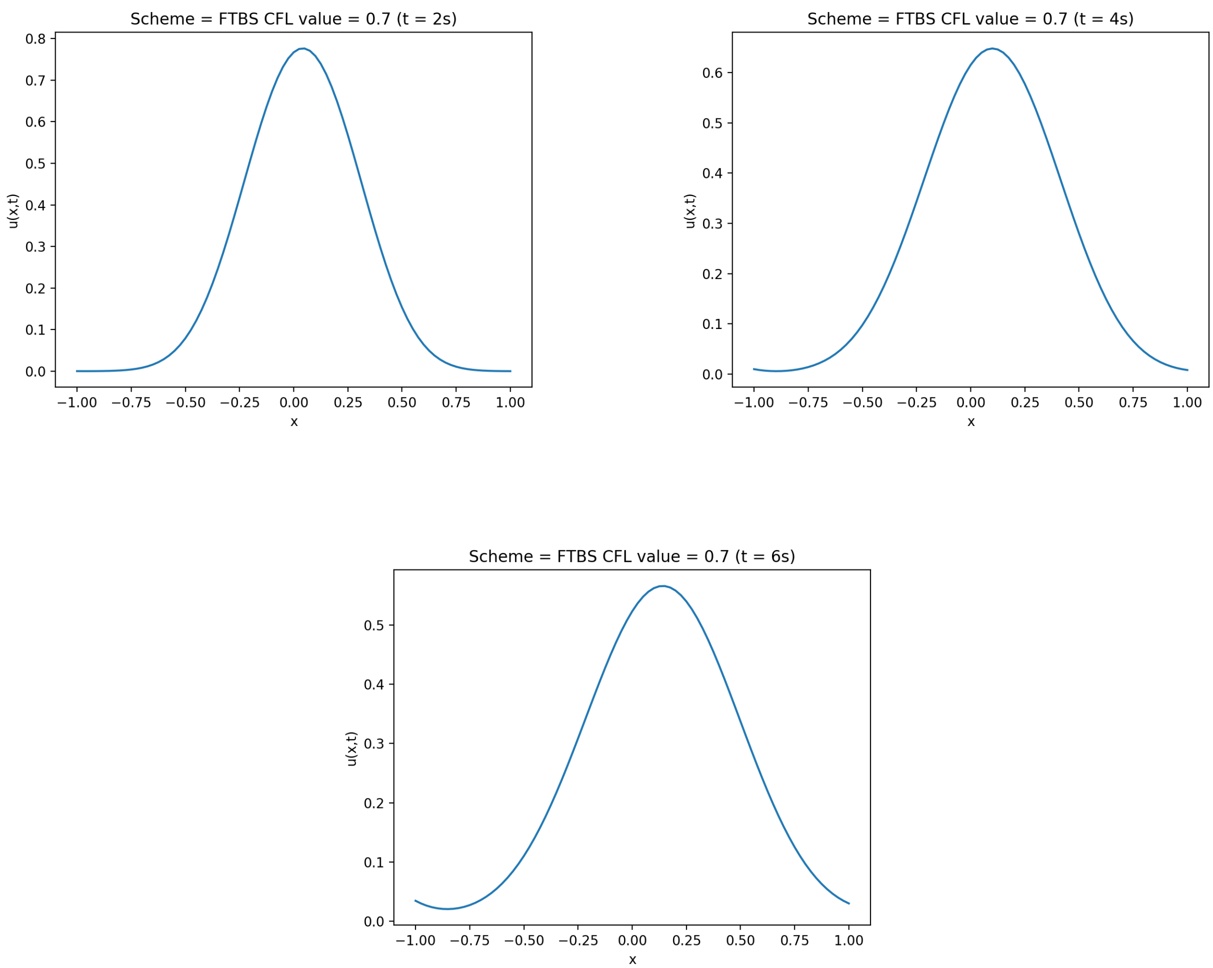


Figure 8: As CFL value increases, propagation speed increases and moreover, the reduction of peak occurs at a slower rate. This is because gain and dissipation term reduces on increasing CFL value and hence, the magnitude reduces at a slower rate.

4.2.2 FTCS2

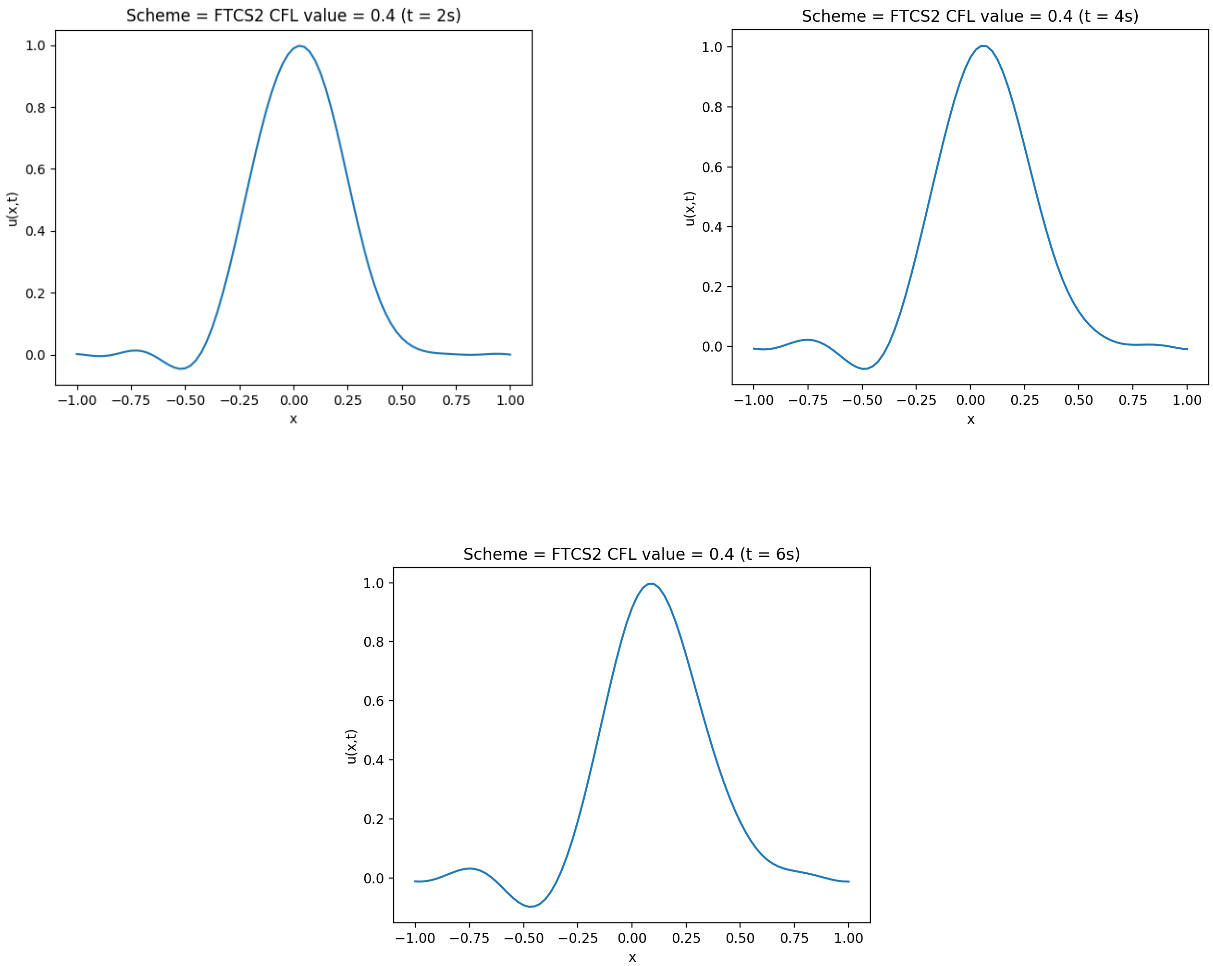
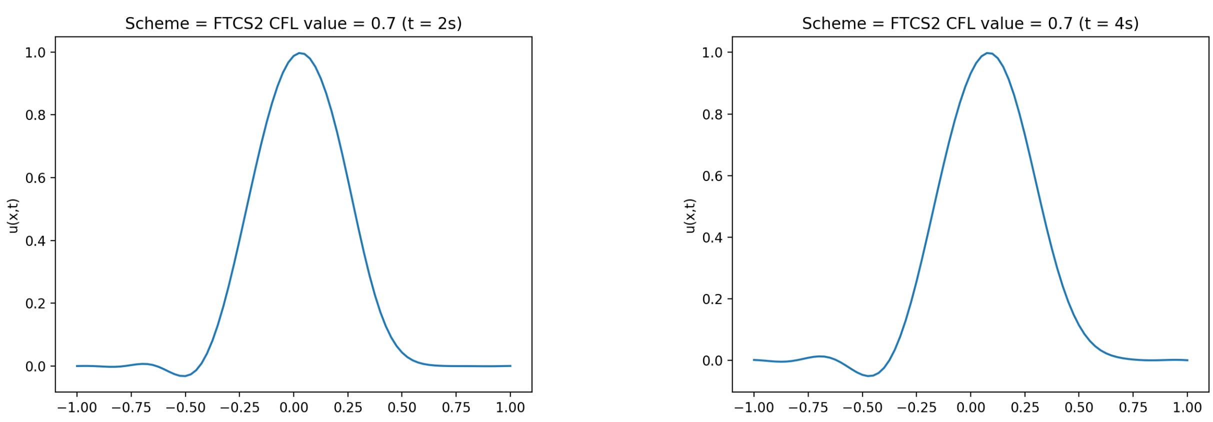


Figure 9: This scheme shows much better shaped waveforms without blow ups as in case of traditional FTCS scheme as this scheme is stable. The magnitude does not reduce at all but we see some dispersion due to presence of 3rd order term in the modified wave equation for this scheme. The dispersion manifests itself in the form of ripples near $|x| = 0.5$



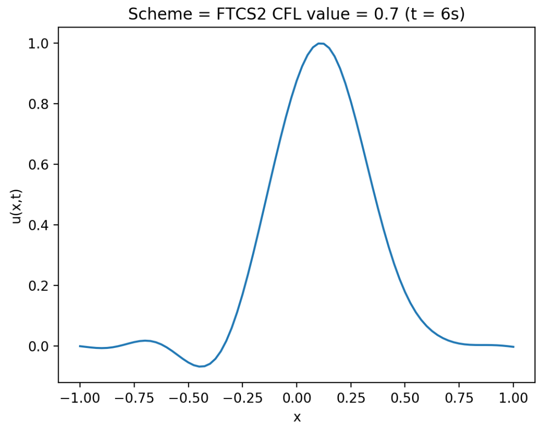


Figure 10: For higher CFL value, propagation speed is higher. Wave displays a consistent shape (with some dispersion) and retains its peak magnitude.

5 Question 3

Given:

$$a = 1$$

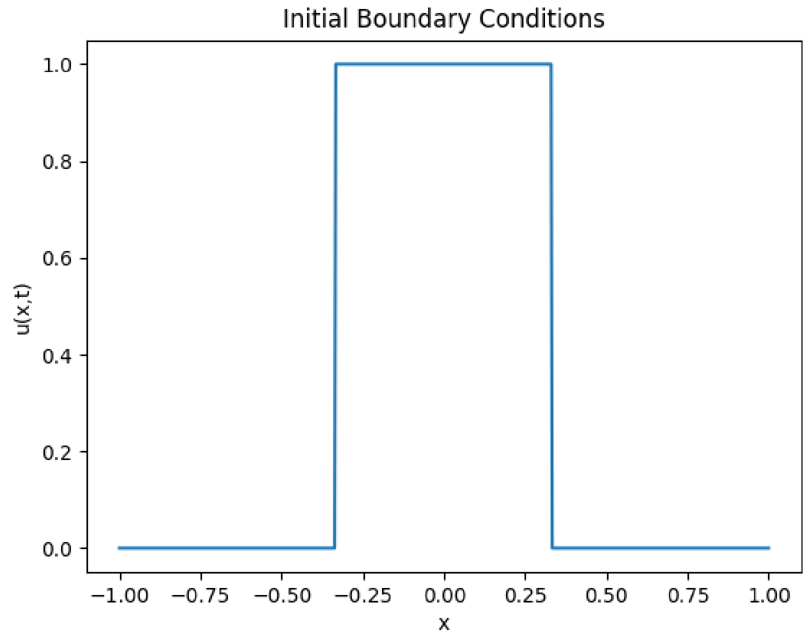


Figure 11: Initial and boundary condition for question 3

5.1 FTBS

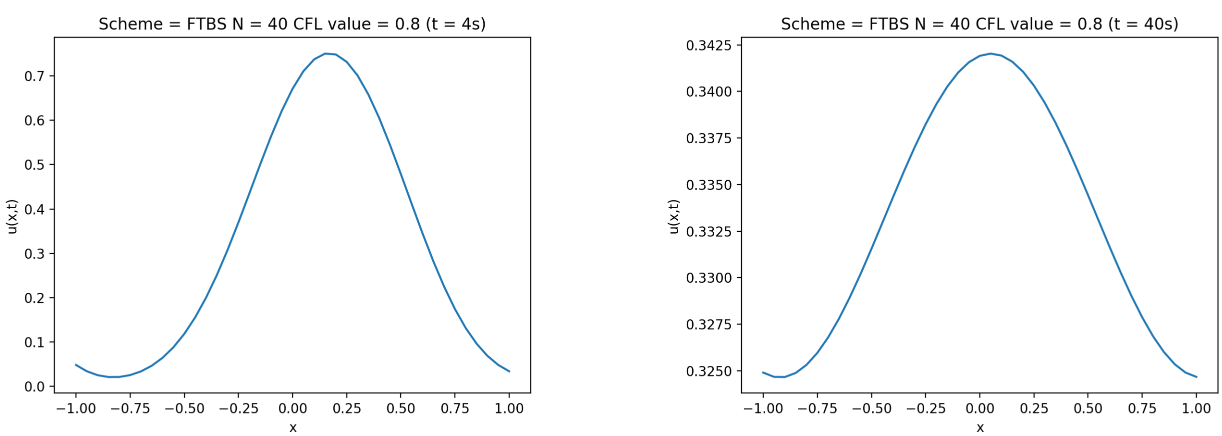


Figure 12: Wave propagates forward with FTBS scheme. We see that the flat peak smooths out due to dissipation and the peak reduces as time increases as we saw in the previous question as well. The modified heat equation of FTBS equation has $a(1 - \sigma)$ term. So FTBS is stable for $\sigma < 1$ but is also quite dissipative as it will dissipate higher frequencies faster than lower frequencies. Since dissipation is rather rapid, we don't observe the dispersion in the FTBS scheme.

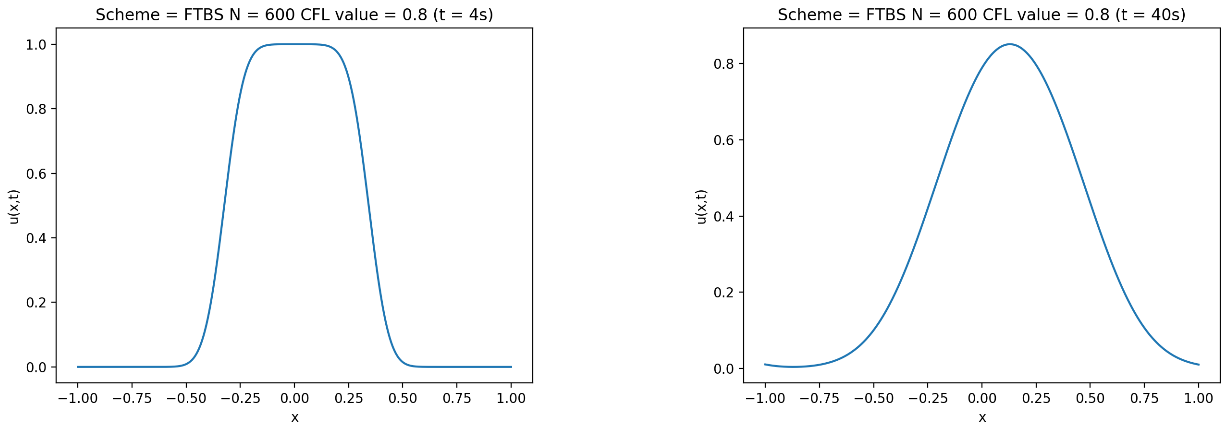


Figure 13: The smoothing of peak is more gradually observed in this case as number of grid points are more.

5.2 FTCS2

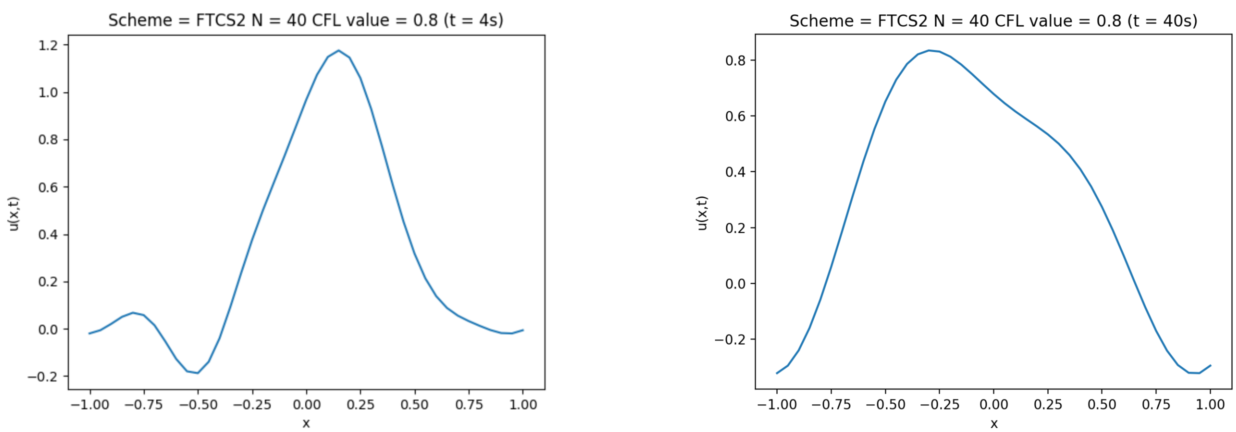


Figure 14: We expect to see dispersion in this due to third derivative term in the modified wave equation but as the number of grid points are very less, we don's see clearly dispersive effect but rather smoothing of the peak and also development of dips around the peaks which we did not see in FTBS.

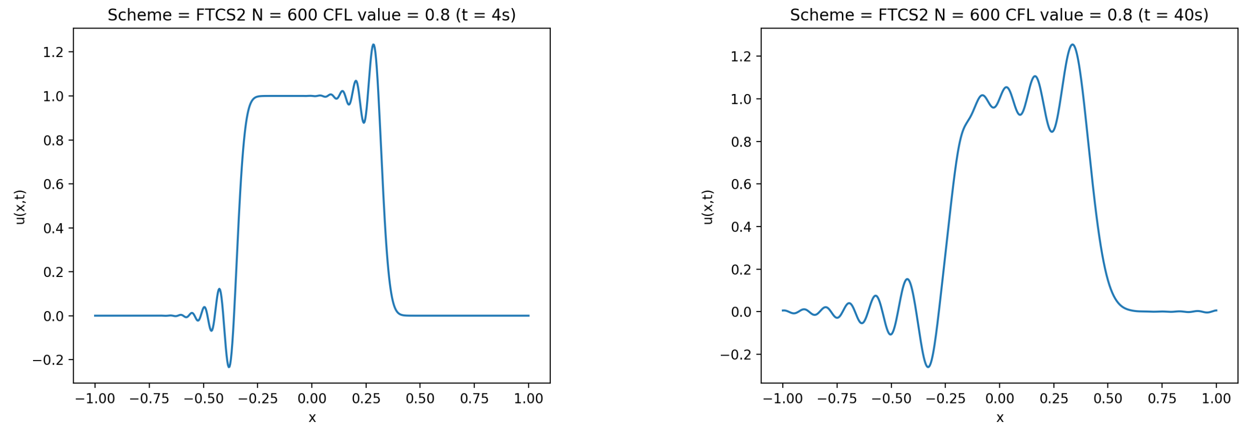


Figure 15: A distinct feature in contrast to FTBS solution is that the magnitude of the peak does not reduce as the wave propagates with time. We see distinct dispersion in case of more number of grid points.