Q6 - a). Given that $P = A^T A$.

To prove that for a vector y, $y^T P y \geq 0$

Consider $y^T P y$

$$y^T P y = y^T A^T A y = \cancel{(yAx)^T A}$$

$$= (Ay)^T \cdot Ay$$

let $Ay = u$, Hence

$y^T P y = u^T u$, which is a dot product, which can't be -ve

hence $u^T u \geq 0$, and

$$y^T P y \geq 0$$

similarly for part 2.

$$z^t Q z = z^t A A^t z = (A^t z)^T \cdot A^t z$$

$$= n^T n \geq 0 \qquad n = A^t z$$

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Eigenvalues of P . Q can't be -ve.

Consider λ, as eigenvalue of P

$Pv = \lambda v$, multiply (pre) by v^T

~~$A^T A v = \lambda v$.~~

$v^T P v = v^T \lambda v$

$= \lambda v^T v$.

$\lambda = \dfrac{v^T P v}{v^T v}$, but $v^T P v \geq 0$ and $v^T v \geq 0$

Hence $\lambda \geq 0$.

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b.) Given →

$$Pu = \lambda u.$$

Pre multiply by A, giving

$$APu = A\lambda u = \lambda Au.$$

$$AA^T(Au) = \lambda(Au)$$

$Q(Au) = \lambda(Au) \Rightarrow Au$ is eigenvector of Q.

Similarly, $Qv = \overset{\to \mu}{\lambda} v$; pre multiply by A^T

$\cancel{A^TAA^T}$

$$A^TAA^Tu = A^T\overset{\to \mu}{\lambda} u = \overset{\to \mu}{\lambda}(A^Tu)$$

$$Q(A^Tu) = \underset{\hookrightarrow \mu}{\lambda}(A^Tu)$$

$\Rightarrow A^Tu$ is an eigenvector of Q.

number of elements $= m$.

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c) v_i - eigenvector of Q

$Au_i = \gamma_i v_i$, substitute for u_i

$$\frac{A \cdot A^T v_i}{\|A^T v_i\|_2} = \gamma_i v_i$$

$$\frac{Q v_i}{\|A^T v_i\|_2} = \gamma_i v_i \quad \text{But } Q v_i = \lambda v_i.$$

Hence $$\frac{\lambda v_i}{\|A^T v_i\|_2} = \gamma_i v_i$$

Post multiply by v_i^{-1}, to get

$$\frac{\lambda}{\|A^T v_i\|_2} = \gamma_i,$$ ~~the expressed.~~

also, λ - eigenvector of Q, $\lambda > 0$

$\|A^T v_i\|_2 > 0$.

Hence $\gamma_i > 0$

d). $P.u_i = \lambda_i u_i$ $\qquad u_i^T u_j = 0, \; i \neq j$

$\cancel{Q v_i = \gamma_i v_i}$

$Q v_i = \gamma_i v_i$ similarly for v

$$u_i = \frac{A^T v_i}{\|A^T v_i\|_2}$$ — from above, (c).

$$(u_i)^T = \frac{v_i^T A}{\|A^T v_i\|_2}$$

We have,

$\cancel{(v_i)^T A v_j} = \cancel{A}$

(1) $\leftarrow v_j^T A u_i = v_j^T \frac{A \cdot A^T v_i}{\|A^T v_i\|_2}$.

But $AA^T v_i = \gamma_i v_i$; put this,

$$\frac{v_j^T \gamma_i v_i}{\|A^T v_i\|_2} = \frac{\gamma_i v_j^T v_i}{\|A^T v_i\|_2}$$

For $i = j$, $v_i^T v_i = \|v_i\|^2$.

else, $\underline{v_j^T v_i = 0}$

For all values $i, j \to 1$ to m,

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_m^T \end{bmatrix} A \, [u_1 | u_2 | \dots | u_m] = \operatorname{diag}(\Gamma)$$

matrix,

with $a_{ii} = \dfrac{y_i \|v_i\|^2}{\|A^T v_i\|}$

$a_{ij} = 0$. $i \neq j$

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But given $U = [v_1 | v_2 | \dots | v_m]$

$$V = [u_1 | u_2 | \dots | u_m]$$

Hence

$$U^T A V = \Gamma$$

But because U, V - made of orthogonal vectors, hence

$$UU^T = U^T U = V^T V = VV^T = I.$$

Hence

$$UU^T A V V^T = U \Gamma V^T$$

$$A = U \Gamma V^T$$

Hence proved

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