MODULE - 4 4/11/17

MATRIX METHOD OF ANALYSIS

Stiffness matrix :-

Stiffness, WKT for a member as shown in the fig

$\delta = \frac{PL}{AE}$ where δ = Displacement,
P = force
L = span
A = cross sectional area
E = modulus of Elasticity.

But we also know that $K = \frac{P}{\delta} = \frac{P}{\frac{PL}{AE}} = \frac{AE}{L}$

where K = Element Stiffness

Generating stiffness matrices :-

The nxn stiffness matrix of a structure with a specified set of 'n' co-ordinates is determined by applying one unit displacement at a time & determining the forces at each co-ordinate to sustain the displacement.

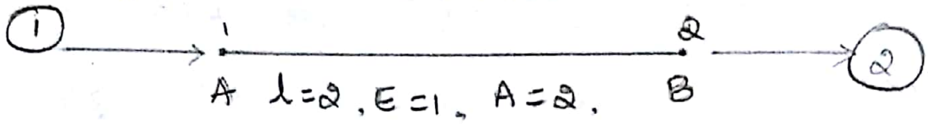
Eg:- If we want to determine a 3x3 stiffness matrix the following steps are applied.

Step ①:- Find the forces at 1, 2 & 3 when $\delta_1 = 1$, δ_2 & $\delta_3 = 0$ and find P_1, P_2 & P_3. These three forces constitute the first column of the stiffness matrix.

Step ②:- Find the three forces at 1, 2 & 3 when $\delta_2 = 1$ and δ_1 & $\delta_3 = 0$ these three forces constitute the second column of the stiffness matrix.

Step ③:- Find the three forces at 1, 2 & 3 when $\delta_3 = 1$ & δ_2 & $\delta_1 = 0$ these three forces constitute the third column of the stiffness matrix

Eg ①:- To find 2×2 stiffness matrix of 2 co-ordinate system.



$l=2, E=1, A=2,$

$I=1$

Stiffness matrix of AB is developed in two steps as shown below.

Step ①:- Apply $\delta_1=1$ $(\delta_2=0)$ at ①

P_1 & P_2

$$P_1 = \frac{AE}{L} = \frac{2\times1}{2} = 1 \qquad \delta_1 = \frac{P_1 L}{AE}, \quad P_1 = \frac{AE}{L} k_{11}$$

$$P_1 + P_2 = 0, \quad \boxed{P_2 = -1} \qquad P_2 = -\frac{AE}{L} \cdot k_{21}$$

$$K = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} k_{11} \\ k_{21} \end{Bmatrix}$$

Step ②: Apply $\delta_2 = 1$ $(\delta_1=0)$ at ②

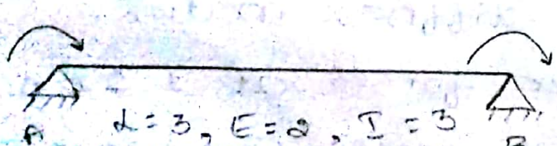
$$P_2 = \frac{AE}{L} = \frac{2\times1}{2} = 1 \qquad \delta_2 = \frac{P_2 L}{AE}$$

$$P_1 = -1 \quad P_2 = 1 \qquad P_2 = \frac{AE}{L} k_{22}$$

$$K = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} k_{12} \\ k_{22} \end{Bmatrix} \qquad P_1 = -\frac{AE}{L} k_{12}$$

$$\therefore K = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

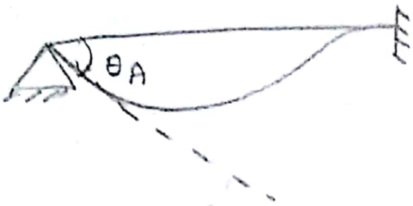
Eq ②:- To find 2×2 stiffness matrix



$L=3, E=2, I=3$

$A=1$

Step ①:-



$K = \frac{P}{\delta}$

$P = K\theta_A$

$P_1 = \frac{4EI}{L}\theta_A$

$P_1 = \frac{4EI\theta_A}{L}$ $[\because \theta_A = 1]$

$= \frac{4 \times 2 \times 3}{3} = \boxed{8 = P_1}$

Soln $K = \frac{M_1}{\theta_A}$

$P_2 = \frac{2EI}{L} \cdot \theta_A$ (or) $\frac{P_1}{2} = \boxed{4 = P_2}$

$\frac{4EI}{L} = \frac{M_1}{\theta_A}$

$\begin{Bmatrix} K_{11} \\ K_{21} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 4 \end{Bmatrix}$

$M_1 = \frac{4EI}{L}\cancel{\theta_A}^{(1)}$

Step ②:-

$\boxed{K_{11} = \frac{4EI}{L}}$

$K_{21} = M_2 = \frac{M_1}{2} = \frac{2EI}{L}$

$P_2 = \frac{4EI\theta_B}{L} = \frac{4 \times 2 \times 3}{3}$ $\boxed{P_2 = 8}$

$M_2 = \frac{4EI}{L} = K_{22}$

$P_1 = \frac{P_1}{2} = 4$ $\boxed{P_1 = 4}$

$M_1 = 2EI/L = K_{12}$

$\begin{Bmatrix} K_{12} \\ K_{22} \end{Bmatrix} = \begin{Bmatrix} P_2 \\ P_1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 8 \end{Bmatrix}$

$\therefore K = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$

$K = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Eq ③:- To find 2×2 stiffness matrix

Step ①:-

$M_{FAB} = \frac{-6EI\delta}{L^2}$

$M_{FBA} = \frac{-6EI\delta}{L^2}$

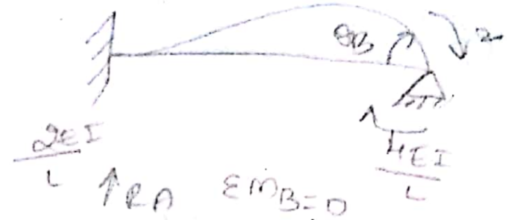
Becoz of settlement Anticlockwise

$\Sigma M_B = 0$

$(R \times L) - \frac{6EI\delta}{L^2} - \frac{6EI\delta}{L^2} = 0$

$\boxed{R = \frac{12EI}{L^3}}$ $[\delta = 1]$ $= K_{11}$

$P_1 = \frac{12EI}{L^3}$ $P_2 = \frac{-6EI}{L^3}$ $K_{21} = \frac{-6EI}{L^2}$



WKT the moment required at the near End to cause unit rotation the far End being fixed is $\frac{4EI}{L}$ & this will Induce a carry over moment of half the value and in the same direction i.e $\frac{2EI}{L}$.

$(R_A \times L) + \frac{4EI}{L} + \frac{2EI}{L} = 0$

$\therefore \boxed{R_{\downarrow} = \frac{-6EI}{L^2}}$

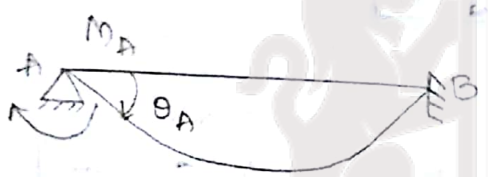
$R_A = \frac{-6EI}{L^2} = S_2$

$R_{\uparrow} = \frac{4EI}{L}$

$K = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}$

$$\begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{-6EI}{L^3} & \frac{4EI}{L} \end{bmatrix}$$

Eg: (A) To find stiffness matrix



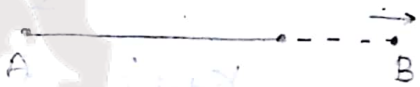
(1) $\theta_A = 1 \quad \theta_B = 0 \quad \delta = 0$

$K = \frac{M_1}{\theta_A} = \frac{4EI}{L} = K_{11}$

$K_{21} = \frac{2EI}{L}$

$K_{31} = 0$

(3)

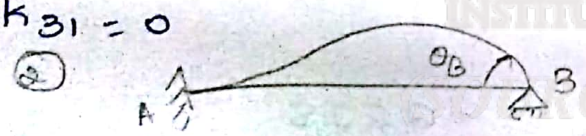


$K_{13} = 0$

$K_{23} = 0$

$K_{33} = \frac{AE}{L}$.

(2)

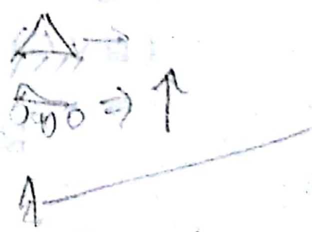


$\theta_B = 1 \quad \theta_A = 0$.

$K = \frac{4EI}{L} = K_{22}$

$K = \frac{2EI}{L} = K_{12}$.

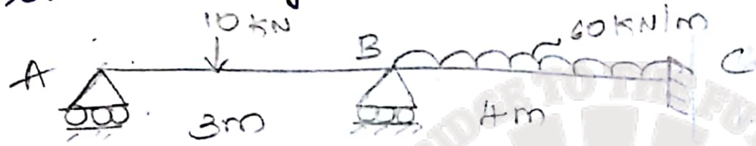
$0 = K_{32}$



Relation,

Det, K Indet, Static Indet.

① Using stiffness matrix method analyse the beam as shown in fig. Find the final moments & draw the BMD.



Step ①:- FEM's

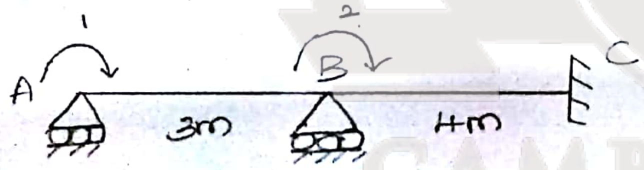
$M_{FAB} = -3.75$ kN-m

$M_{FBA} = 3.75$ kN-m

$M_{FBC} = -80$ kN-m

$M_{FCB} = 80$ kN-m

Step ②: Element stiffness matrix.

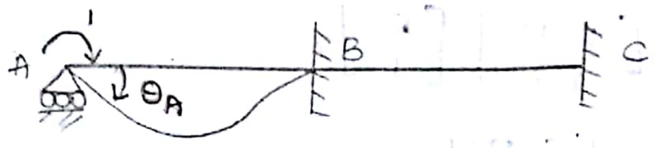


DOF = 2

size of Stiffness matrix = 2×2

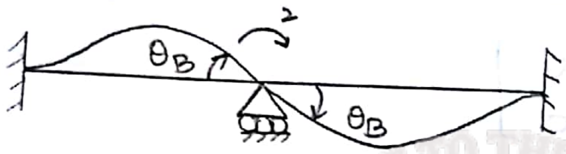
$$k = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

Step (2a) Consider $\theta_A = 1$ $\theta_B = 0$



$$K_{11} = \frac{4EI}{3} \qquad K_{21} = \frac{2EI}{3}$$

Step (2b) $\theta_A = 0$ $\theta_B = 1$



$$K_{12} = \frac{2EI}{3} \qquad K_{22} = \frac{4EI}{3} + \frac{4EI}{4} = \frac{7}{3}EI$$

$$\therefore [K] = \begin{bmatrix} 4EI/3 & 2EI/3 \\ 2EI/3 & 7EI/3 \end{bmatrix}$$

$$[K] = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \qquad K^{-1} = \frac{EI}{3}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}$$

Step (3):- To find unknown displacement θ_A & θ_B

$$[Fem] = \begin{bmatrix} -3.75 \\ \overset{BA + BC}{-3.75 + 80} = -76.25 \end{bmatrix}$$

Unknown displacement matrix $= [\Delta]$

$$K = \frac{F}{\Delta}$$

$$\Delta = \frac{F}{K} = K^{-1}[F]$$

$$\Delta = K^{-1}[Fem]$$

$[2\times2][2\times1] \rightarrow 2\times1$

$$[K]^{-1} = \frac{adj[K]}{|K|}$$

$$|K| = 8EI$$

$$adj|K| = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

$$= \frac{EI}{3}[28 - 4]$$

$adj|K| = 8EI$

$adj[K] = [K]^T$

$= \frac{EI}{3}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}$

$K^{-1} \times F$

$[\Delta] = \frac{1}{8EI} \cdot \frac{EI}{3}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -3.75 \\ -76.25 \end{bmatrix}$

$= \frac{1}{24}\begin{bmatrix} -15 + 152.5 \end{bmatrix} = \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$

$\Delta =$

$M'_{AB} = \frac{2EI}{L}\left[2\theta_A + \theta_B - \frac{3\cancel{EI}}{L}\right]$

$M_{AB} = M_{FAB} + M'_{AB}$

$[K^{-1}] = \frac{\frac{EI}{3}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}}{8EI}$

$K^{-1} = \frac{\begin{bmatrix} \frac{7EI}{3} & -\frac{2EI}{3} \\ -\frac{2EI}{3} & \frac{4EI}{3} \end{bmatrix}}{8EI}$

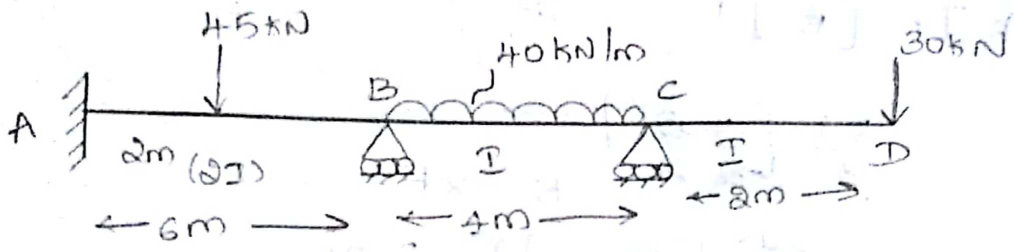
$K^{-1} = \frac{\begin{bmatrix} \frac{22EI^2}{9} - \frac{4EI}{9} \end{bmatrix}}{8EI}$

$K^{-1} = \frac{\frac{6}{9}\begin{bmatrix} 7 - 1 \end{bmatrix}}{28EI}$

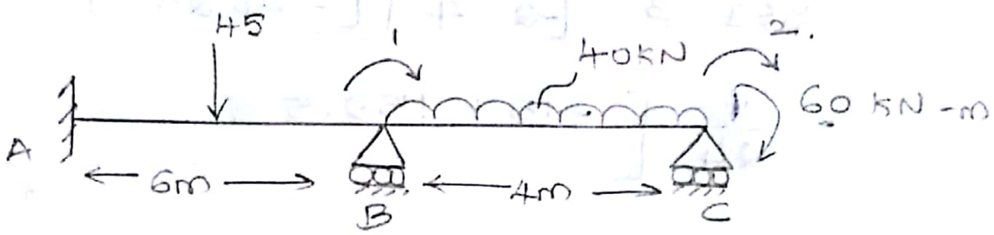
$K^{-1} = \frac{\frac{EI}{9}\begin{bmatrix} 6 \end{bmatrix}}{2}$

$K^{-1} = \frac{EI}{3}$

2



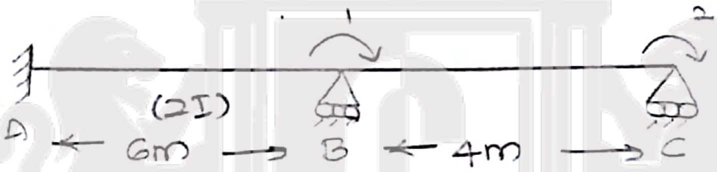
modified as



$M_{FAB} = -40 kN\text{-}m$ $\quad M_{FBC} = -53.33 kN\text{-}m$

$M_{FBA} = 20 kN\text{-}m$ $\quad M_{FCB} = 53.33 kN\text{-}m$

Step②: Element Stiffness matrix



DOF = 2

Size of stiffness matrix = 2×2

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

Force matrix:- step②

Consider at joints

Since its over hang -ing considerdz.

$$FEM = \begin{bmatrix} M_{FBA} + M_{FBC} \\ M_{FCB} \end{bmatrix} = \begin{bmatrix} 20 - 53.33 \\ 53.33 \end{bmatrix} = \begin{bmatrix} -33.33 \\ 53.33 \end{bmatrix} - \begin{bmatrix} 0 \\ 60 \end{bmatrix} = \begin{bmatrix} -33.33 \\ -6.67 \end{bmatrix}$$

$$FEM = \begin{bmatrix} 33.33 \\ 6.67 \end{bmatrix}$$

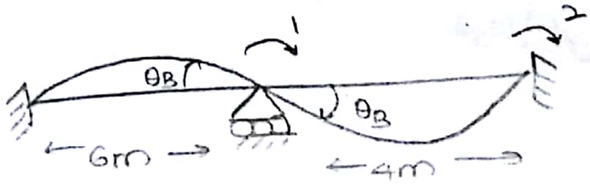
step③

Element stiffness matrix

$$K_{11} = \frac{4E\overset{(2I)}{I}}{6} + \frac{4EI}{4} = \frac{7}{3}EI$$

$$K_{21} = \frac{2EI}{L} = \frac{2E\overset{(2)}{I}}{4} = \frac{EI}{2}$$

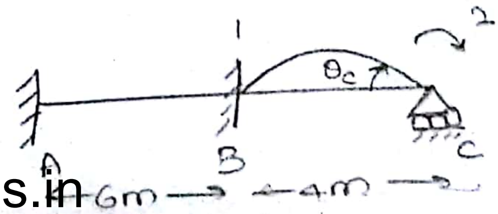
① $\theta_B = 1 \quad \theta_C = 0$



$$K_{12} = \frac{2EI}{L} = \frac{2EI}{4} = \frac{EI}{2}$$

$$K_{22} = \frac{4EI}{4} = EI$$

$\theta_B = 0 \quad \theta_C = 01$



$$K = \begin{bmatrix} 7/3\,EI & EI/2 \\ EI/2 & EI \end{bmatrix} = EI \begin{bmatrix} 7/3 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

unknown displacement matrix = [Δ]

$$K = \frac{F}{\Delta} \qquad [2\times 2][2\times 1] \;\; 2\times 1 \qquad K^{-1} = \frac{adj[K]}{|K|}$$

$$\Delta = K^{-1}[F]$$

$$adj|K| = [K]^T \qquad |K| = EI^2\left[7/3 - 1/4\right]$$

$$= EI\begin{bmatrix} +1 & -1/2 \\ -1/2 & +7/3 \end{bmatrix} \qquad \boxed{|K| = \frac{25EI^2}{12}}$$

$$[K^{-1}] = \frac{adj[K]}{|K|}$$

$$= \frac{EI\begin{bmatrix} 1 & -1/2 \\ -1/2 & 7/3 \end{bmatrix}}{\frac{25EI^2}{2}}$$

$$\Delta = K^{-1}[FEM]$$

$$= \begin{bmatrix} 2/25 & -1/25 \\ -1/25 & 14/75 \end{bmatrix}\begin{bmatrix} 33.33 \\ 6.67 \end{bmatrix}$$

$$[K^{-1}] = \begin{bmatrix} \frac{2}{25}EI & -\frac{1}{25}EI \\ -\frac{1}{25}EI & \frac{14}{75}EI \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 2.66 + 3.33 \\ 1.33 - 1.24 \end{bmatrix} \qquad \Delta = \frac{1}{25}\begin{bmatrix} 359.94 \\ -13.22 \end{bmatrix}$$

$$[K^{-1}] = \frac{1}{25}\begin{bmatrix} 12 & -6 \\ -6 & 28 \end{bmatrix} \qquad \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = \begin{bmatrix} 5.99 \\ 0.09 \end{bmatrix} \qquad \Delta = \begin{bmatrix} 14.4 \\ -0.53 \end{bmatrix}$$

K^{-1} $\qquad \theta_B = 14.4 \quad \theta_C = -0.53$

$$m'_{AB} = \frac{4EI}{6}[14.4] = 9.6 \text{ kN-m}$$

$$m'_{BA} = \frac{4EI}{6}[2\times 14.4] = 19.2 \text{ kN-m}$$

$$m'_{BC} = \frac{2EI}{4}[2\times 14.4 - 0.53] = 14.135 \text{ kN-m}$$

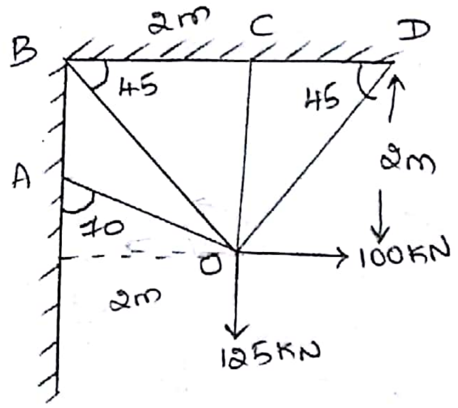
$$m'_{CB} = \frac{2EI}{4}[14.4 - 2\times 0.53] = 6.67 \text{ kN-m}$$

$$M_{AB} = M_{FAB} + m'_{AB}$$

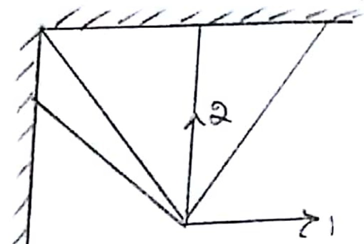
$M_{AB} = -30.4$ $\quad M_{BC} = -39.2$ $\quad M_{CD} = -60$

$M_{BA} = 39.2$ $\quad M_{CB} = 60$ $\quad M_{DC} = 0$

Analyse the pin Jointed plane frame as shown in the fig by stiffness matrix method. Cross sectional area of each member = 1000 mm^2 & modulus of elasticity of each member is 200 KN/mm^2



Step①:- Identifying DOF



$[K] = [2 \times 2]$

Step②:- Stiffness matrix

$\delta = \frac{PL}{AE}$ $\quad P = \frac{AE}{L}$

$$[K] = \begin{bmatrix} \Sigma \frac{AE}{L}\cos^2\theta & \Sigma \frac{AE}{L}\cos\theta \sin\theta \\ \Sigma \frac{AE}{L}\cos\theta\sin\theta & \Sigma \frac{AE}{L}\sin^2\theta \end{bmatrix}$$

A = 1000

E = 200

Member	$\left(\frac{AE}{L}\right)$	θ	$\frac{AE}{L}\cos^2\theta$	$\frac{AE}{L}\cos\theta\sin\theta$	$\frac{AE}{L}\sin^2\theta$
OA	94.33	160	83.29	-30.31	11.03
OB	70.92	135	35.46	-35.46	35.46
OC	100.00	90	0	0	100
OD	70.92	45	35.46	35.46	35.46
	336.17		154.21	-30.31	181.95

Δ^{le} OBC

$\tan 45 = \frac{2}{BC}$

BC = 2

$\cos 20 = \frac{2}{OA}$

OA = 2.12

Δ^{le} OBC

$OB = \sqrt{4+4}$

$OB = \sqrt{8}$

$\boxed{OB = 2.82}$

OD = 2.82

OC = 2

Mode ⑥

Mat A

② 2×2

Ac

Shift 4

Data

Mat B

⑥

Ac

Shift ④ . Mat A⁻¹

Shift ④ Mat B

Mat A⁻¹ × Mat B

= -8/5 -18/5

step ③ Force matrix

$$\{F^0\} = \begin{Bmatrix} 100 \\ -125 \end{Bmatrix}$$

$$K = \begin{bmatrix} 154.21 & -30.31 \\ -30.21 & 181.95 \end{bmatrix}$$

$$\{F^J\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{F^F\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 100 \\ -125 \end{Bmatrix} = \begin{Bmatrix} -100 \\ 125 \end{Bmatrix}$$

step ④ unknown Displacement matrix

$$[\Delta] = [K]^{-1}[F^F]$$

$$\begin{Bmatrix} \Delta_{ox} \\ \Delta_{oy} \end{Bmatrix} = \frac{\text{adj } K}{|K|} \times F^F$$

$$\cancel{\text{adj } K = EI \begin{bmatrix} 2/1 & -1/2 \\ -1/2 & 7/3 \end{bmatrix}}$$

$$\begin{bmatrix} \Delta_{ox} \\ \Delta_{oy} \end{bmatrix} = \frac{\begin{bmatrix} 181.7 & 30.32 \\ 30.32 & 154 \end{bmatrix}}{27062.5} \begin{bmatrix} -100 \\ 125 \end{bmatrix}$$

$$\cancel{|K| = EI^2}$$

$$\begin{bmatrix} \Delta_{ox} \\ \Delta_{oy} \end{bmatrix} = \frac{1}{27062.5} \begin{bmatrix} -14380 \\ 22282 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_{ox} \\ \Delta_{oy} \end{bmatrix} = \begin{bmatrix} -0.53 \\ 0.823 \end{bmatrix}$$

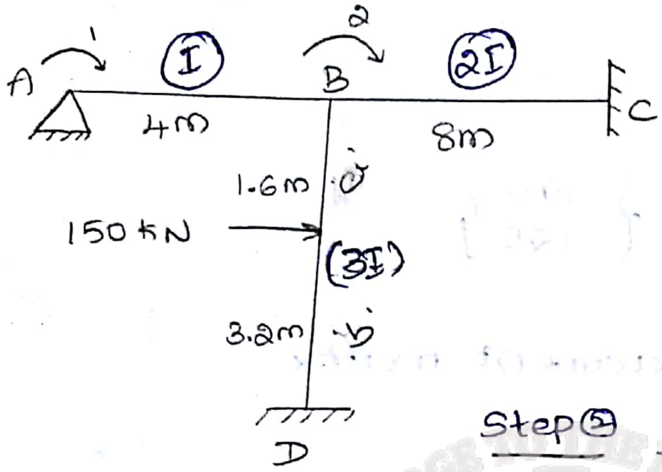
Final forces

$$F_{OA} = \frac{AE}{L}\left[(\cancel{\Delta_{Ax}}^{0} - \Delta_{ox})\cos\theta + (\cancel{\Delta_{Ay}}^{0} - \Delta_{oy})\sin\theta\right]$$

$$= 94.33\left[(0.53) \times (-0.939) + (-0.823)(0.342)\right]$$

Analysis of portal frames

1. Analyse the frame shown in the fig using stiffness matrix method.



Step ①

$M_{FAB} = M_{FBA} = M_{FBC} = M_{FCB} = 0$

$M_{FBB} = +53.33$ kN-m

$M_{FBB} = +106.66$.

Step ② Fem matrix

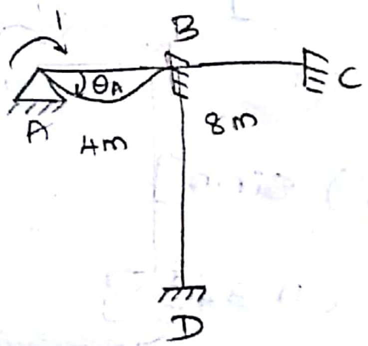
DOF = 2.

$$[Fem^o] = \begin{bmatrix} M_{FAB} \\ M_{FBA} + M_{FBD} + M_{FBC} \end{bmatrix} = \begin{bmatrix} 0 \\ +106.66 \end{bmatrix} \quad [F^J] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F^F = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ +106.67 \end{bmatrix} \quad \boxed{[F^F] = \begin{bmatrix} 0 \\ -106.67 \end{bmatrix}}$$

Step③: Element stiffness matrix

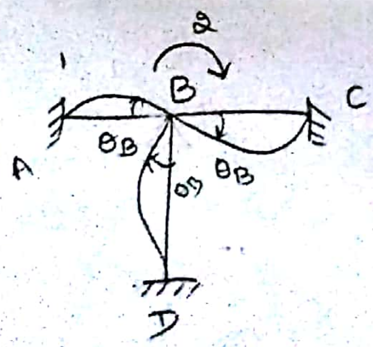
$[K] = 2 \times 2$



$\theta_A = 1 \quad \theta_B = 0$

$K_{11} = \frac{4EI}{4} = EI$

$K_{21} = \frac{2EI}{4} = \frac{EI}{2}$



$\theta_B = 01 \quad \theta_A = 0$

$K_{12} = \frac{2EI}{4}$

$K_{22} = \frac{4EI}{4} + \frac{4E(2I)}{8} + \frac{4E(3I)}{4.8} = \frac{9EI}{2}$

$$K = \begin{bmatrix} EI & 0.5EI \\ 0.5EI & 4.5EI \end{bmatrix}$$

Step ④: Unknown displacement matrix Δ

$$\Delta = [K^{-1}][F_{Em}]^F$$

$$K^{-1} = \frac{adj\,K}{|K|}$$

$$adj\,K = K^T \qquad |K| = [4.5EI^2 - 0.25EI^2]$$

$$= \begin{bmatrix} 4.5EI & -0.5EI \\ -0.5EI & EI \end{bmatrix} \qquad |K| = 4.25EI^2$$

$$\cancel{adj\,K = 4.5EI^2 - 0.25EI^2}$$

$$\boxed{\cancel{adj\,K = 4.25EI^2}}$$

$$K^{-1} = \begin{bmatrix} 1.05/EI & -0.117/EI \\ -0.117/EI & 0.23/EI \end{bmatrix}$$

$$\Delta = [K^{-1}]\begin{bmatrix} 0 \\ -106.66 \end{bmatrix}$$

$$= \begin{bmatrix} 1.05/EI & -0.117/EI \\ -0.117/EI & 0.23/EI \end{bmatrix}\begin{bmatrix} 0 \\ -106.66 \end{bmatrix}$$

$$\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = \begin{bmatrix} 0 + \frac{12.47}{EI} \\ -\frac{25.06}{EI} \end{bmatrix}$$

Step ⑤: SDE

$$m'_{AB} = \frac{2EI}{L}\left[2\theta_A + \theta_B - \frac{3\delta^{0}}{L}\right] = \frac{2EI}{4}\left[\frac{24.97}{EI} + \frac{25.06}{EI}\right]$$

$$m'_{AB} = 0 \text{ KN-m}$$

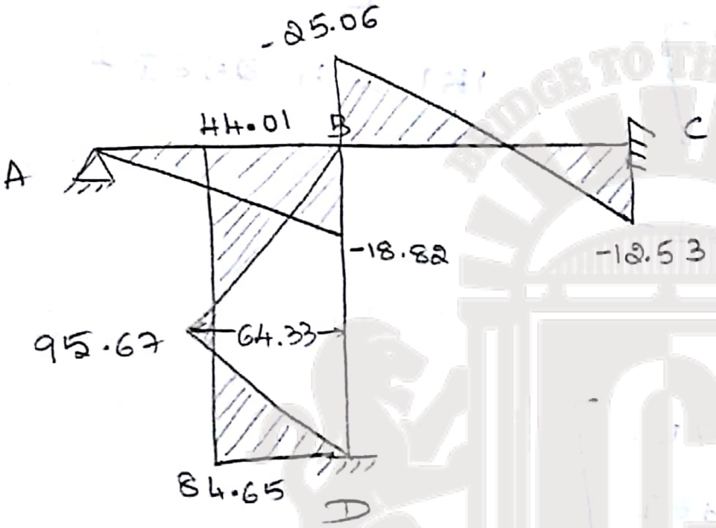
$$m'_{BA} = \frac{2EI}{4}\left[-\frac{50.12}{EI} + \frac{12.47}{EI}\right] = -18.82 \text{ KN-m}$$

$$M^*_{BC} = \frac{2E\overset{(2)}{I}}{8}\left[-\frac{50.12}{EI}\right] + 0 = -25.06 \text{ KN-m}$$

$$M_{CB} = \frac{2E(2I)}{8}\left[-25.06\right] = -12.53 \text{ KN-m}$$

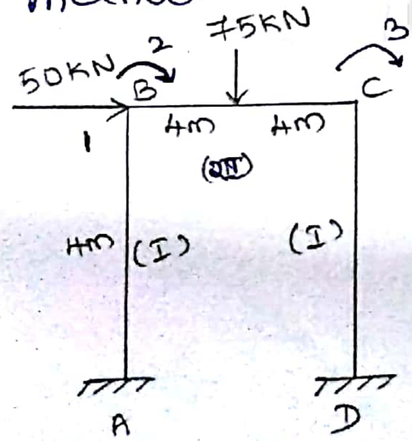
$$M_{BD} = \frac{2E(3I)}{4.8}\left[-\frac{50.12}{EI}\right] + 106.66 = 44.01 \text{ KN-m}$$

$$M_{DB} = \frac{2E(3I)}{4.8}\left[-25.06\right] - 53.33 = -84.65 \text{ KN-m}$$



Portal frame with sway

① Analyze the rigid jointed plane frame by stiffness matrix method.



Step ①:- FEM's

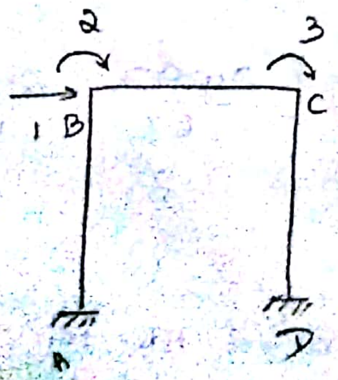
$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

$$M_{FBC} = -75 \text{ KN-m}$$

$$M_{FCB} = 75 \text{ KN-m}$$

Step ②: FEM matrix

$$[F_{EM}^{O}] = \begin{bmatrix} 0 \\ -75 \\ 75 \end{bmatrix} \qquad [F_{EM}^{J}] = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$



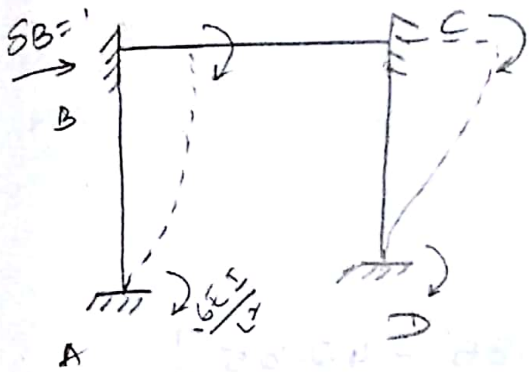
$$[F]^F = \overset{J}{\begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}} - \overset{O}{\begin{bmatrix} 0 \\ -75 \\ 75 \end{bmatrix}} \qquad [F^F] = \begin{bmatrix} 50 \\ 75 \\ -75 \end{bmatrix}$$

Step ③ :- Element stiffness matrix.

$[K] = 3 \times 3$

(3a)

$\delta_B = 1, \quad \theta_B = 0, \quad \theta_C = 0$



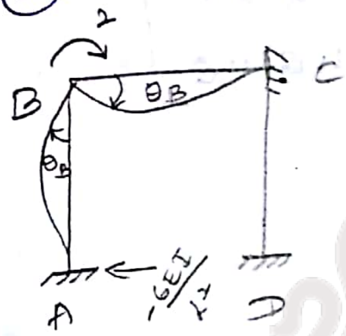
$$k_{11} = \frac{12EI}{L^3} + \frac{12EI}{L^3} = 0.375EI$$

$$k_{21} = \frac{-6EI}{L^2} = -0.375EI$$

$$k_{31} = \frac{-6E(2I)}{L^2} = \frac{-12EI}{(4)^2} = -0.375\,EI$$

(3b)

$\delta_B = 0 \quad \theta_B = 1 \quad \theta_C = 0$



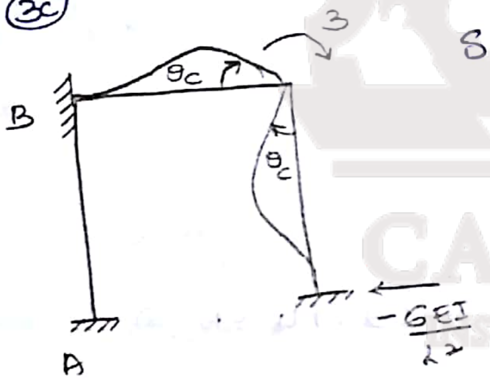
$$k_{12} = \frac{-6EI}{L^2} = -0.375$$

$$k_{22} = \frac{4EI}{4} + \frac{4E(2I)}{8} = 2EI$$

$$k_{32} = \frac{2E(2I)}{8} = 0.5EI$$

(3c)

$\delta_B = 0, \quad \theta_B = 0, \quad \theta_C = 1$



$$k_{13} = \frac{-6EI}{L^2} = -0.375\,EI$$

$$k_{23} = \frac{2E(2I)}{8} = 0.5EI$$

$$k_{33} = \frac{4E(2I)}{4} + \frac{4EI}{4} = 2EI$$

$$K = \begin{bmatrix} 0.375EI & -0.375EI & -0.375EI \\ -0.375EI & 2EI & 0.5EI \\ -0.375EI & 0.5EI & 2EI \end{bmatrix}$$

Step ④ :- Unknown displacement matrix Δ

$$\Delta = [K^{-1}]\,[F_{em}]^F$$

$$K^{-1} = \frac{\operatorname{adj} K}{|K|} =$$

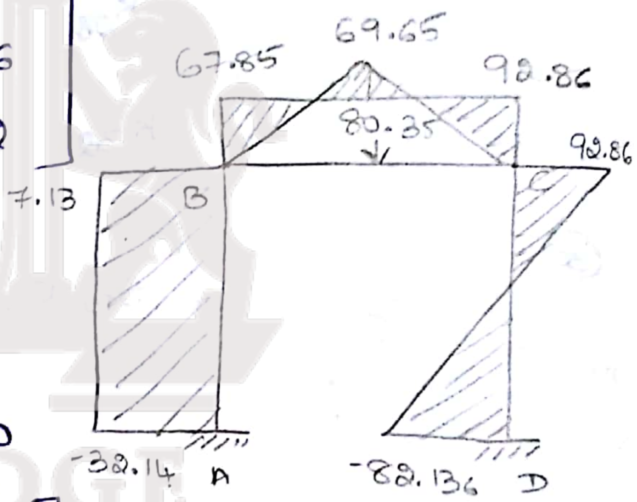
~~adj K = [K]^T~~

$$\text{adj}\,k^{-1} = \begin{bmatrix} \frac{80}{21} & \frac{4}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{13}{21} & -\frac{1}{21} \\ \frac{4}{7} & -\frac{1}{21} & \frac{13}{21} \end{bmatrix} \begin{bmatrix} 50 \\ 75 \\ -75 \end{bmatrix}$$

~~Δ = [K^{-1}][F_{EM}] - F~~

$$\begin{bmatrix} \Delta_3 \\ \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 190.47 + 42.85 - 42.85 \\ 28.57 + 46.42 + 3.57 \\ 28.57 - 3.57 - 46.42 \end{bmatrix}$$

$$\begin{bmatrix} \delta_3 \\ \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 190.47 \\ 78.56 \\ -21.42 \end{bmatrix}$$



Step ⑤:

$$M_{AB} = M'_{AB} + M_{FAB}$$

$$= \frac{2EI}{L}\left[2\cancel{\theta_A}^{\,0} + \theta_B - \frac{3\delta}{L}\right] + 0 \quad (\delta_3)$$

$$M_{AB} = \frac{2EI}{4}\left[78.56 - \frac{3(190.47)}{4}\right] = -32.14 \text{ kN-m}$$

$$2\theta_B + \cancel{\theta_A}^{\,0} - \frac{3\delta}{L}$$

$$M_{BA} = \frac{2EI}{4}\left[157.12 - \frac{3(190.47)}{4}\right] = 7.133 \text{ kN-m}$$

$$M_{BC} = \frac{2E\overset{[2I]}{I}}{8}\left[157.12 - 21.42\right] - 75 = 67.85 \text{ kN-m}$$

$$M_{CB} = \frac{2E\overset{(2I)}{I}}{8}\left[-42.84 + 78.56\right] + 75 = 92.86 \text{ kN-m}$$

$$M_{CD} = \frac{2EI}{4}\left[-42.84 - \frac{3(190.47)}{4}\right] = -92.84 \text{ kN-m}$$

$$M_{DC} = \frac{2EI}{4}\left[-21.42 - \frac{3(190.47)}{4}\right] = -82.136$$