STRUCTURAL ANALYSIS

A Matrix Approach

Second Edition

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FOREWORD

It is the prime responsibility of the structural engineer to ensure that his structures transmit the service loads safely and efficiently. In order to discharge this responsibility effectively, a clear understanding of the structural response is essential. The response of a structure under the action of a given system of forces is characterized by the force–displacement relationship. For a system with a single degree of freedom, the force–displacement relationship is expressed by a number known as the stiffness or its reciprocal, the flexibility. For structures which usually have multiple degrees of freedom, the relationship between the forces and the resulting displacements is expressed by the stiffness matrix or its reciprocal, the flexibility matrix. The elements of these matrices connect systematically the force components with the resulting displacement components. The matrix approach, therefore, is rightly described as the systematic analysis of structures. The matrix approach provides a clear and systematic picture of the forces on the one hand and the displacements on the other.

In recognition of the vital role played by the matrix approach towards a clear understanding of the structural action, the subject of matrix analysis of structures has now been introduced by most of the universities in India and abroad. At the junior undergraduate level, a first introduction to the matrix methods as applied to elementary skeletal structures is usually given. The discussion of complex and large skeletal structures as well as non-skeletal structures is usually included at the senior undergraduate and postgraduate levels. Apart from the importance of the matrix approach towards a clear understanding of the structural action, the matrix methods have assumed vital significance with the advent of the digital computer. The use of digital computers for structural analysis and design is increasing day-by-day. We have already reached a stage where the analysis and design of large and important structures are invariably handled with the help of digital computers. As the matrix methods are indispensable for an automatic computer analysis of a structure, the significance of matrix methods is self-evident.

The existing books on the matrix methods of structural analysis have generally been written with the assumption that the reader possesses a reasonably high understanding of structural mechanics. An uninitiated student, with only an elementary knowledge of structural mechanics, therefore, finds these books beyond his comprehension. At present there is a clear need to bring out a book which presents the matrix approach in its most simple form so that even an ordinary undergraduate student can read it without much difficulty. It is my firm conviction that the present book would meet this

The fundamental concepts and basic theorems of Structural Mechanics and their applications which form the prerequisite for the development of the matrix approach are discussed in the first two chapters. In the third chapter, the necessary background material on determinants and matrices is provided. The development of the flexibility and the stiffness matrices and the first introduction to the two main methods of matrix analysis are presented in Chapter 4. A thorough treatment of the three types of structures, viz., the beams, rigid-jointed frames and pin-jointed frames by both the methods of matrix analysis are presented in chapters 5, 6 and 7. Only planar structures have been considered in these chapters. The material included in chapters 1 to 7 gives a thorough coverage of the syllabus on matrix methods generally prescribed at the undergraduate level.

Chapters 8 to 12 cover the syllabus generally prescribed for an advanced course on matrix methods of structural analysis. Chapters 8 and 9 deal with space frames—while the rigid-jointed space frames are discussed in Chapter 8, Chapter 9 deals with the pin-jointed space frames. A critical reappraisal and comparative study of the two main methods of matrix analysis are presented in Chapter 10. The element approach, which is particularly suitable for the automatic analysis of structure by a digital computer, is discussed in Chapter 11. Several possible variations of the two main methods and special techniques aimed at simplicity, greater precision and lesser computational effort form the subject matter of the last chapter.

The authors are grateful to several colleagues and friends who have helped directly or indirectly in the preparation of this book. They are particularly indebted to Prof. O P Jain, Director, IIT Delhi, for writing the Foreword. The authors would feel obliged if any errors in this book are brought to their notice. Constructive suggestions from the readers for further improvement are also most welcome.

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LIST OF SYMBOLS

D_s	Degree of static indeterminacy
D_k	Degree of kinematic indeterminacy
E	Young's modulus
G	Shear modulus
I	Moment of inertia
i, j	Typical coordinates
j	Number of joints
K	Torsion constant
k_{ij}	Typical element of the stiffness matrix $[k]$
	Force at coordinate i due to unit displacement at coordinate j
M	Bending moment
m	Number of members
m_j	Bending moment due to a unit force at coordinate j
P_j	Force at coordinate j
	Typical element of the force matrix $[P]$
P_j'	Force at coordinate j in the restrained structure due to the loads other than those acting at the coordinates
	Typical element of the matrix $[P']$
Q	Shear force
r	Number of external reaction components
S	Axial force
s_j	Axial force in the members of a pin-jointed frame due to a unit force at coordinate j
T	Twisting moment
U	Strain energy
U^*	Complementary energy
u, v	Principal axes
x, y, z	Cartesian coordinate axes
Δ_j	Displacement at coordinate j
	Typical element of the displacement matrix $[\Delta]$

Δ_{jL}	Displacement at coordinate j in the released structure due to the applied loads
	Typical element of the matrix $[\Delta_L]$
δ_{ij}	Displacement at coordinate i due to a unit force at coordinate j
	Typical element of the flexibility matrix $[\delta]$
ϕ	Member rotation
θ	Joint rotation
	End slope of a member
Σ	Sigma
$1, 2, \ldots, n$	System coordinates
$1^*, 2^*, \ldots, m^*$	Element coordinates

1 BASIC CONCEPTS

1.1 INTRODUCTION

The primary function of a structure is to receive loads (usually known as service loads) at certain points and transmit them safely to some other points. For instance, a building frame receives occupancy loads of the building besides the self-weight of the structural components of the building and transfers them safely to the foundations. Similarly, a highway bridge has to support the live load due to the traffic and the dead load of the bridge itself besides several other loads. The structural system of the bridge has to be designed so as to transmit these loads safely through the supporting piers and abutments to the foundations. In performing this primary function of receiving service loads at certain points and transferring them safely to some other points, the structure develops internal forces in its component members known as *structural elements*. It is the responsibility of the structural engineer to design all the structural elements of a *structural system* in such a way that they perform their functions adequately. The inadequacy of one or more structural elements may lead to the malfunctioning or even collapse of the entire structure. The object of structural analysis is to determine the internal forces and the corresponding displacements of all the structural elements as well as those of the entire structural system. The safety and proper functioning of the structure can be ensured only through a thorough structural analysis. The importance of a correct structural analysis for the proper functioning and safety of the structure cannot, therefore, be over-emphasized. A systematic analysis of structural systems can be carried out by using matrices. The matrix approach for the solution of structural problems is also eminently suitable for a solution using modern digital computers. Hence, the advantage of using the matrix approach for large structural problems is evident.

1.2 CLASSIFICATION OF STRUCTURES

The history of development of structural forms is as old as the history of civilization itself. It is, therefore, natural that a very large variety of structural forms and systems are in use today. Hence, it is not easy to classify these structures so as to include all of them. While several systems of classification have been suggested, the following system of classification appears to be helpful for developing basic concepts:

(i) skeletal structures
(ii) surface structures
(iii) solid structures

Skeletal structures are those which can be idealized to a series of straight or curved lines. As the name suggests, the structure looks like a skeleton. The common examples of skeletal structures are roof trusses, lattice girders and building frames. *Surface structures* are those which can be idealized to plane or curved surfaces. Slabs and shells belong to this category. *Solid structures* are those which can neither be idealized to a skeleton nor to a plane or curved surface.

In general, only the skeletal structures can be analysed by the elementary methods of structural mechanics. The stress analysis of surface and solid structures usually involves higher mathematics and the theory of elasticity or plasticity. Fortunately, the majority of structural systems in common use can be considered as skeletal structures. A systematic analysis of skeletal structures can be carried out by using the matrix approach.

The skeletal structures can be further classified into the following two types:

(i) pin-jointed frames
(ii) rigid-jointed frames

As the name suggests, the members of pin-jointed frames are connected by means of pin-joints. These frames support the applied loads by developing only axial forces in the constituent members if the external forces act at the joints and the members are straight. Unless otherwise stated, it will be assumed throughout that in the case of a pin-jointed frame, the external forces act at the joints and the members are straight. On the other hand, the joints of the rigid-jointed frames are assumed to be rigid so that the angles between the members meeting at a joint remain unchanged. These frames resist external forces by developing bending moments, shear forces, axial forces and twisting moments in the members of the frame.

Skeletal structures may also be classified as:

(i) plane frames
(ii) space frames

All members of the plane frame as well as the external loads are assumed to be in one plane. If these frames are pin-jointed, the members carry only axial forces. On the other hand, if the frames are rigid-jointed, the members are subjected to axial forces, shear forces and bending moments. In the case of space frames, all the members of the frame do not lie in one plane. Very often, space frames are formed by combining a series of plane frames. The members

of a space frame are subjected to axial forces only, if the joints are pin-connected. On the other hand, the members of a rigid-jointed space frame are subjected to axial forces, shear forces, bending moments and twisting moments.

1.3 EQUATIONS OF STATIC EQUILIBRIUM

Using the Cartesian system of coordinates as the reference frame, the equations of static equilibrium may be written as

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = 0 \tag{1.1}$$

$$\Sigma M_x = \Sigma M_y = \Sigma M_z = 0 \tag{1.2}$$

where ΣF_x, ΣF_y and ΣF_z are algebraic sums of the components of all external forces, including reactive forces, along x-, y- and z-axes respectively and ΣM_x, ΣM_y and ΣM_z are the algebraic sums of the moments of all external forces, including reactive forces, about x- y- and z-axes respectively.

The external forces can be divided into the following two systems:

(i) applied loads
(ii) reactive forces

For static equilibrium, the resultant of all applied loads is equal in magnitude and opposite in sign to the resultant of the reactive forces. Thus the applied loads and the reactive forces may be looked upon as constituting two systems of forces which oppose each other and keep the structure in equilibrium. It must be mentioned that if the entire structure is in static equilibrium, every part of it, however small, must also be in equilibrium. Hence, the equations of static equilibrium apply not only to the structure as a whole but also to every part of it. In particular, they apply to all the members and joints of the structure.

In case of plane frames subjected to in-plane external forces, only three equations are sufficient for static equilibrium. Assuming that the frame and all external forces lie in the x-y plane, the equations of static equilibrium may be expressed as

$$\Sigma F_x = \Sigma F_y = 0 \tag{1.3}$$

$$\Sigma M_z = 0 \tag{1.4}$$

In this case the remaining three equations, viz., $\Sigma F_z = \Sigma M_x = \Sigma M_y = 0$, are identically satisfied. If the x-axis is horizontal and the y-axis is vertical. Eqs (1.3) and (1.4) may be rewritten as

$$\Sigma H = \Sigma V = 0 \tag{1.3a}$$

$$\Sigma M = 0 \tag{1.4a}$$

where ΣH and ΣV are the algebraic sums of the components of all external forces, including reactive forces, along horizontal and vertical directions respectively, and ΣM is the algebraic sum of the moments of all external forces, including reactive forces, about any point in the plane of the structure.

These frames are free from torsion unlike those in which some of the members or the external forces do not lie in the x-y plane.

1.4 INTERNAL FORCES

The resultant internal force acting at any cross-section of a member of a skeletal structure is equivalent to a force P passing through the centroid of the cross-section and a couple M as shown in Fig. 1.1. The force P can be resolved into a component S, known as the axial force along the axis of the member and two components Q_x and Q_y along the two principal axes of the cross-section. The forces Q_x and Q_y acting along the principal axes x and y are known as the components of the total shear force Q acting on the cross-section. Similarly, the couple M can be resolved into three components. The component M_z known as the twisting moment T acts about the axis of the member. The components M_x and M_y acting about the principal axes x and y respectively are known as the biaxial bending moments. Hence, in general, at any cross-section of a member of skeletal space structure there are six internal force components, viz., the axial force S, the biaxial shear force components Q_x and Q_y, the twisting moment T and the biaxial bending moments M_x and M_y.

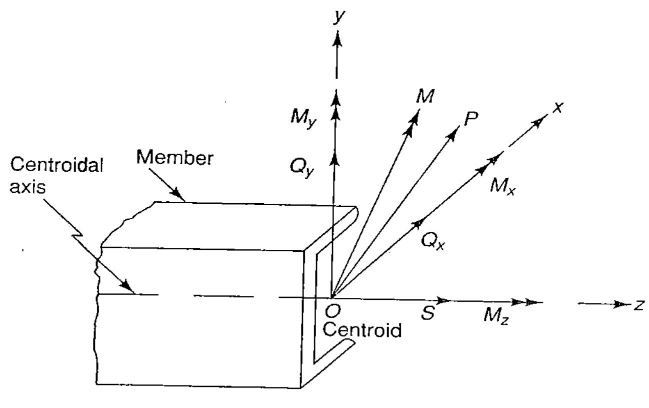


Fig. 1.1

In the case of a plane structure loaded in its own plane, the twisting moments are absent. Besides, the bending moments and shear forces act in the plane of the structure. Hence, at any cross-section of a member of a rigid-jointed plane skeletal structure loaded in its own plane, there are only three internal forces, viz., the axial force S, the shear force Q and the bending moment M, all of which act in the plane of the structure. If the members of the structure are pin-connected, the shear force Q and the bending moment M are absent. Hence at

any cross-section of a member of a pin-jointed skeletal structure, there is only one internal force, viz., the axial force S. In Fig. 1.1, the forces are represented by single-headed arrows and the couples by double-headed arrows. For defining the sense of the couple, the vector notation and the right-handed screw system may be adopted. Accordingly, the couple M_x represented by a double-headed arrow pointing in the positive direction of the x-axis is clockwise when looking towards the positive direction of the x-axis.

1.5 FREE-BODY DIAGRAMS

As stated in Sec. 1.3, the equations of static equilibrium apply not only to the structural system as a whole but also to all its members or elements. The free-body diagram of the entire structure or that of any part of it shows all forces acting on it which are required to maintain its equilibrium. The free-body diagrams of different parts of a structure clearly show the manner in which internal forces must develop in order to maintain equilibrium with the external forces. Hence, free-body diagrams are extremely important for a clear understanding of the distribution of internal forces in any structure. The following examples illustrate how the free-body diagrams are drawn.

Example 1.1

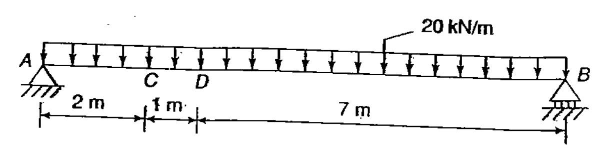
A simply supported beam AB, shown in Fig. 1.2(a), has a span of 10 m and supports a transverse load of 20 kN/m. Draw the free-body diagrams of portions AC, DB and CD.

Solution

Total load is $20 \times 10 = 200$ kN and the reactions at the supports A and B, $R_A = R_B =$ 100 kN.

(i) *Free-body diagram of AC*

This portion is acted upon by a downward force of $2 \times 20 = 40$ kN uniformly spread over this length and an upward force of 100 kN at A. Hence, in order to satisfy the equation of equilibrium $\Sigma V = 0$, a force of 60 kN must act at C in the downward direction. Besides, a counter-clockwise couple of 160 kN-m must act at C in order to satisfy the equation of equilibrium $\Sigma M = 0$. There are no horizontal forces. Hence, the equation $\Sigma H = 0$ is identically satisfied. The free-body diagram of AC is shown in Fig. 1.2(b). The force of 60 kN and the couple of 160 kN-m acting at the cross-section C are the reactions of the portion CB on the portion AC. They are shear force and bending moment respectively at the cross-section C.



(a)

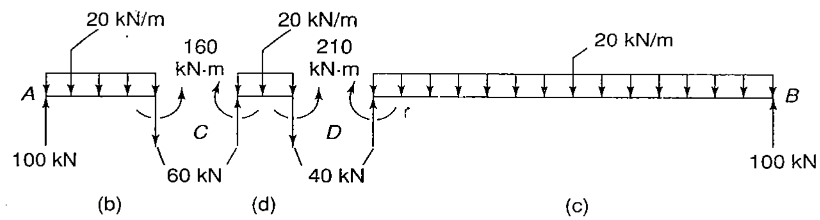
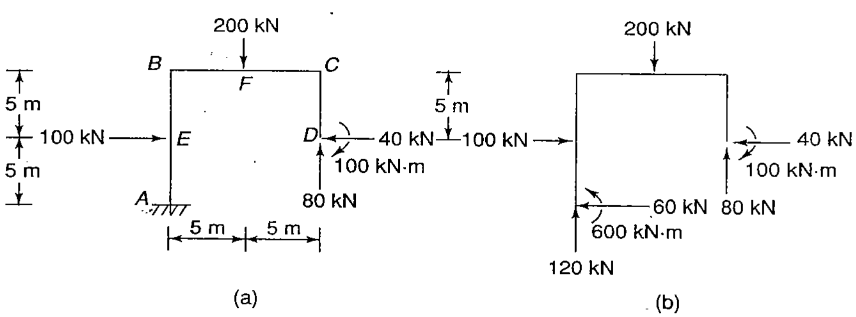


Fig. 1.2

(ii) *Free-body diagram of DB*
The total downward load on this portion is 140 kN and an upward force of 100 kN. Hence, in order to satisfy the equations of equilibrium $\Sigma V = 0$ and $\Sigma M = 0$, an upward force of 40 kN and a clockwise couple of 210 kN·m must act at cross-section *D*. These are evidently the shear force and bending moment at cross-section *D*. The free-body diagram of *DB* is shown in Fig. 1.2(c).
(iii) *Free-body diagram of CD*
As action and reaction are equal and opposite, the internal forces acting at cross-sections *C* and *D* of the portion *CD* must be equal to magnitude and opposite in direction to the forces acting at cross-section *C* of portion *AC* and cross-section *D* of portion *DB*. Hence, an upward force of 60 kN and a clockwise couple of 160 kN·m must act at *C*. Similarly, a downward force of 40 kN and a counter-clockwise couple of 210 kN·m must act at *D*. The free-body diagram of *CD* is shown in Fig. 1.2(d). It is evident that the axial forces and twisting moments are not present in this case.

Example 1.2

The rectangular frame ABCD, shown in Fig. 1.3(a), is fixed at A and free at D. The joints B and C are rigid. Sketch the free-body diagrams of the entire frame, the three members of the frame and the joints B and C.



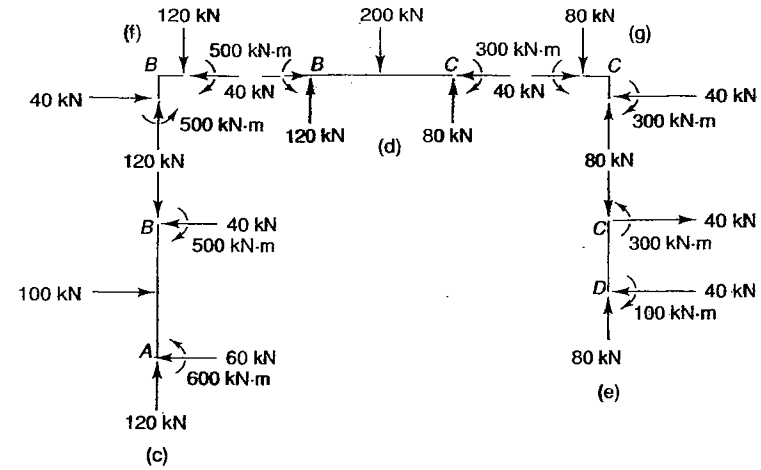


Fig. 1.3

Solution

(i) *Free-body diagram of entire frame*
Considering the equilibrium of the entire frame, the reactive forces required at *A* to satisfy the equations of static equilibrium are:
(a) vertical upward force of 120 kN,
(b) horizontal force of 60 kN, and
(c) counter-clockwise couple of 600 kN·m.
The free-body diagram of the entire frame is shown in Fig. 1.3(b).
(ii) *Free-body diagram of AB*
As the reactive forces acting at the cross-section *A* have already been computed, the internal reactive forces which must act at cross-section B in order to maintain equilibrium of portion *AB* are:
(a) vertical downward force of 120 kN,
(b) horizontal force of 40 kN to the left, and
(c) clockwise couple of 500 kN·m.
The free-body diagram of portion *AB* is shown in Fig. 1.3(c).
(iii) *Free-body diagram of BC*
As action and reaction are equal and opposite, the internal reactive forces acting at cross-section *B* of portion *BC* are equal in magnitude and opposite in direction to those acting at the same cross-section of the portion *AB*, i.e.,
(a) vertical upward force of 120 kN,
(b) horizontal force of 40 kN to the right, and
(c) counter-clockwise couple of 500 kN·m.

Thus the internal reactive forces required at cross-section C to maintain equilibrium of portion BC are

(a) vertical upward force of 80 kN

(b) horizontal force of 40 kN to the left, and

(c) clockwise couple of 300 kN·m.

The free-body diagram of portion BC is shown in Fig. 1.3(d).

(iv) *Free-body diagram of CD*

Considering the interaction of portions BC and CD, the internal reactive forces at cross-section C for portion CD comprise

(a) vertical downward force of 80 kN,

(b) horizontal force of 40 kN to the right, and

(c) counter-clockwise couple of 300 kN·m.

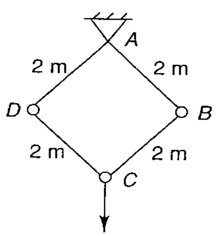
It may be noted that these reactive forces maintain equilibrium of portion CD. The free-body diagram of portion CD is shown in Fig. 1.3(e).

(v) *Free-body diagrams of joints B and C*

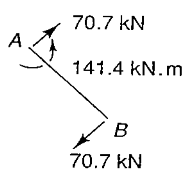
The free-body diagrams of joints B and C are shown in Fig. 1.3(f) and (g). It should be noted that the forces acting on the joints are equal in magnitude and opposite in sense to those acting at the ends of the members converging at the joint under consideration.

Example 1.3

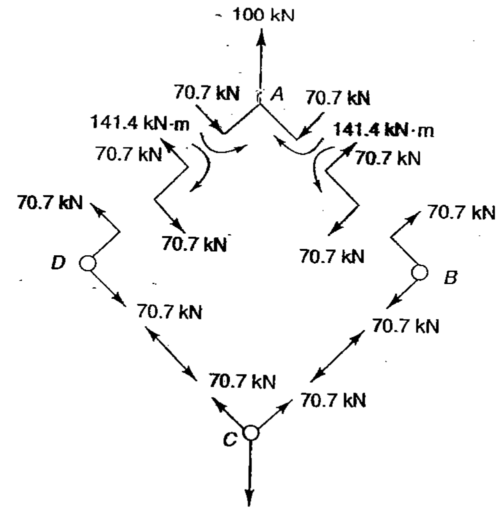
A square frame ABCD, shown in Fig. 1.4(a), is suspended from joint A and supports a vertical load of 100 kN at C. The length of each member is 2 m. Joint A is rigid and the remaining joints are pin-connected. Draw the free-body diagrams of all the joints and members.



(a)



(b)



(c)

Fig. 1.4

Solution

Considering the equilibrium of joint C, the tensile force in member DC as well as in CB is equal to 70.7 kN. Considering the equilibrium of portion AB, the reactive forces acting on it must be as shown in Fig. 1.4(b). The free-body diagrams of all the members and joints are shown in Fig. 1.4(c).

1.6 DEGREE OF STATIC INDETERMINACY

Statically indeterminate structures are those structures which cannot be analysed with the help of equations of static equilibrium alone. These structures are also known as *hyperstatic* structures. For the analysis of these structures it becomes necessary to consider the deformation of the structure because the equations of statics alone are not sufficient for the solution of the problem. In the case of statically indeterminate structures, the number of unknowns is greater than the number of independent equations derived from the conditions of static equilibrium. Additional equations, based on the compatibility of deformations, must be written in order to obtain a sufficient number of equations for the determination of all the unknowns. The number of these additional equations, necessary for the solution of the problem, is known as the *degree of static indeterminacy* or *the degree of redundancy* of the structure. The total degree of static indeterminacy of the structure D, may be considered as the sum of the following two types of indeterminacies:

(i) degree of external indeterminacy, D_{se}
(ii) degree of internal indeterminacy, D_{si}

Thus,
$$D_s = D_{se} + D_{si} \tag{1.5}$$

The external indeterminacy is related to the support-system of the structure. It has been pointed out in Sec. 1.3 that for static equilibrium there are six independent equations to be satisfied in the case of a space structure and three equations for a plane structure. Hence, the reactions of a support-system are statically determinate if it gives rise to six independent reaction components in the case of a space structure and three for a plane structure. If the number of independent reaction components is more, the structure is externally indeterminate to that extent. If the number of independent external reaction components is r, the degree of external indeterminacy D_{se} for space structures is given by the equation

$$D_{se} = (r - 6) \tag{1.6}$$

and for plane structures it is given by the equation

$$D_{se} = (r - 3) \tag{1.7}$$

In developing a clear understanding of the degree of internal indeterminacy of a skeletal structure, it is convenient to consider pin-jointed and rigid-jointed frames separately.

A pin-jointed frame is statically determinate internally if it has just the minimum number of members m' required to preserve its geometry. If the number of members is more, the pin-jointed frame is internally indeterminate to that extent.

Considering that there are j joints in a pin-jointed plane frame, three members are required to connect the first three joints. Two more members are required for connecting each additional joint to the triangular frame already formed. Thus, the number of members required to connect the remaining $(j - 3)$ joints is $2(j - 3)$. Hence, the total number of members in a pin-jointed *plane frame* with j joints is given by the equation

$$m' = 2(j - 3) + 3 = (2j - 3) \tag{1.8}$$

In the case of pin-jointed space frame, the most elementary frame is a tetrahedron having four joints and six members. Besides, three additional members are required to connect each of the remaining $(j - 4)$ joints. Thus, the total number of members required in a pin-jointed *space frame* is given by the equation

$$m' = 3(j - 4) + 6 = (3j - 6) \tag{1.9}$$

In general, it may be stated that a pin-jointed frame is statically indeterminate internally if the number of members is more than $(2j - 3)$ in the case of a plane frame and $(3j - 6)$ in the case of a space frame. If the number fo members is less than the requirement as per Eq. (1.8) or (1.9), the frame is internally ***unstable*** or ***deficient***. On the other hand, if the actual number of members m is more than the requirement as per Eq. (1.8) or (1.9), the frame is *over stiff* and consequently it is *statically indeterminate*. The degree of internal indeterminacy D_{si} for a plane frame is given by the equation

$$D_{si} = m - (2j - 3) \tag{1.10}$$

and for a space frame it is given by the equation

$$D_{si} = m - (3j - 6) \tag{1.11}$$

It may be pointed out that although the condition regarding the number of members represented by Eq. (1.8) or (1.9) is necessary, it is not sufficient for internal determinacy. In other words, the frame may not be statically determinate internally even when Eq. (1.8) or (1.9) is satisfied. For example, the pin-jointed plane frame shown in Fig. 1.5 has six joints and nine members as required by Eq. (1.8) but the frame is not statically determinate

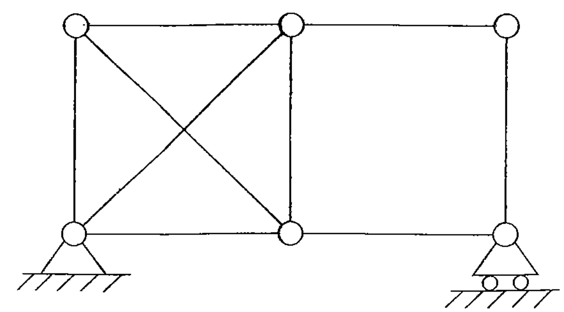


Fig. 1.5

internally. In this pin-jointed frame the left panel has more than the required number of members and is consequently over-stiff. The right panel has one member less than the minimum required to preserve the geometry of the panel and is consequently unstable. Thus, the frame represents a combination of an overstiff panel and an unstable panel. As a whole, it cannot be considered statically determinate internally. The above concepts have been summarized in Table 1.1.

Table 1.1

Pin-jointed plane frames			*Pin-jointed space frames*		
$m < (2j-3)$ unstable internally	$m = (2j-3)$ stable and statically determinate internally*	$m > (2j-3)$ overstiff and statically indeterminate internally	$m < (3j-6)$ unstable internally	$m = (3j-6)$ stable and statically determinate internally*	$m > (3j-6)$ overstiff and statically indeterminate internally

* Not necessarily true. A further check should be applied to see that Eq. (1.8) or (1.9) applies not only to the structure as a whole but also to all the panels separately.

A rigid-jointed frame is statically determinate internally if its members form an open configuration resembling the structure of a tree as shown in Fig. 1.6. An open configuration here means that there are no loops or closed cells. Any one of the following two checks may be applied to verify that the structural configuration is open:

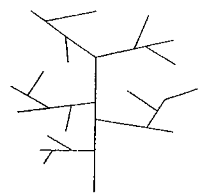


Fig. 1.6

(i) Starting from any point on the structure and proceeding along any route, it is impossible to return to the same point without retracing the path.

(ii) It is impossible to make a cut anywhere in the structure without splitting the structure into two separate parts.

If a rigid-jointed structure does not have an open configuration, it is statically indeterminate internally. A statically indeterminate structure may be converted into a statically determinate structure by making sufficient number of cuts so that the resulting configuration is open. At each cut three reaction components (two forces and one couple) are released in the case of a plane structure and six reaction components (three forces and three couples) in the case of a space structure. Therefore, the degree of internal indeterminacy D_{si} for a rigid-jointed plane frame is given by the equation

$$D_{si} = 3c \tag{1.12}$$

and for a rigid-jointed space frame it is given by the equation

$$D_{si} = 6c \tag{1.13}$$

where c = number of cuts required for obtaining an open configuration.

Consider the rigid-jointed plane frame shown in Fig. 1.7(a). An open configuration can be obtained in this case by introducing five cuts as shown in Fig. 1.7(b).

Skeletal structures, having some of the joints as pin connected and others as rigid, are not very common. To determine the degree of internal indeterminacy of these *hybrid* type of structures, the pin joints may in the first instance be replaced by rigid joints. Considering a plane frame, the number of independent reaction components that must be added to convert a pin joint

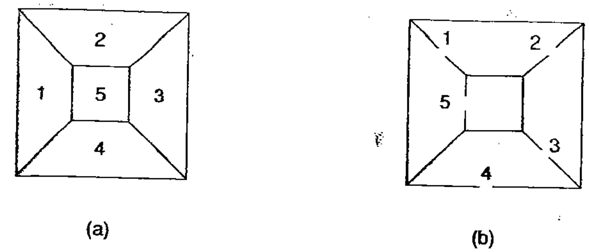


Fig. 1.7

into a rigid joint is $(m-1)$, where m is the number of members meeting at that joint. The is so because m bending couples must be applied, one to each of the m members converging at the joint under consideration. As the sum of all these bending couples must be zero, the number of independent reaction components required to convert a pin joint into a rigid joint is $(m-1)$. Similarly, in the case of space structures, the number of independent reaction components which must be added to convert a pin joint into a rigid joint is $3(m-1)$ because three couples (two bending couples and one twisting couple) must be applied to each member to obtain a rigid connection. As the sum of the couples about the three coordinate axes at the joint must be separately zero, the number of independent reaction components required in this case is $3(m-1)$. Hence, the number of independent reaction components required to replace a plane structure with hybrid joints by a rigid-jointed structure is $\Sigma(m-1)$, where summation has to be carried out for all the pin joints in the hybrid structure. As the degree of internal indeterminacy of a rigid-jointed plane frame is $3c$, the degree of ~~internal~~ indeterminacy of a plane frame with hybrid joints is given by the equation

$$D_{si} = 3c - \Sigma(m-1) \tag{1.14}$$

Similarly, the degree of internal indeterminacy of a space frame with hybrid joints is given by the equation

$$D_{si} = 6c - \Sigma 3(m-1) \tag{1.15}$$

An alternative approach for the determination of the degree of indeterminacy of a structure is to take a unified view without considering external and internal indeterminacies separately. In the case of a pin-jointed plane frame, there are m unknown member forces and r unknown reaction components. Thus, the total number of unknowns is $(m + r)$. The conditions of static equilibrium provide two equations, Eq. (1.3), at each of the j joints giving a total of $2j$ independent equations. Hence, the degree of static indeterminacy of a pin-jointed plane frame may be written as

$$D_s = (m + r) - 2j \tag{1.16}$$

In the case of a pin-jointed space frame, the conditions of static equilibrium provide three equations, Eq. (1.1), at each of the j joints giving a total of $3j$

independent equations. Hence, the degree of static indeterminacy of a pin-jointed space frame may be written as

$$D_s = (m + r) - 3j \tag{1.17}$$

Consider next the rigid-jointed structures. Every member of a rigid-jointed plane frame carries three unknown internal forces, viz., an axial force, a shear force and a bending moment. Thus, including the r reaction components, the total number of unknown forces is $(3m + r)$. The conditions of static equilibrium provide three equations, Eqs (1.3) and (1.4), at each of j joints giving a total of $3j$ independent equations. Hence, the degree of static indeterminacy of a rigid-jointed plane frame may be written as

$$D_s = (3m + r) - 3j \tag{1.18}$$

In the case of a rigid-jointed space frame, every member carries six unknown internal forces, viz., three forces and three couples. Thus including the r reaction components, the total number of unknown forces is $(6m + r)$. The conditions of static equilibrium provide six equations, Eqs (1.1) and (1.2), at each of the j joints giving a total of $6j$ independent equations. Hence, the degree of static indeterminacy of a rigid-jointed space frame may be written as

$$D_s = (6m + r) - 6j \tag{1.19}$$

Consider next, a hybrid structure having a combination of pin joints and rigid joints. The number of unknown internal forces in a member of a plane frame or a space frame depends upon its end conditions as shown in Table 1.2. Hence, the total number of unknowns, which is equal to the sum of the unknown member forces and the external reaction components, can be calculated. The number of equations of static equilibrium at each joint of the frame are also shown in Table 1.2. Hence, the total number of equations of static equilibrium of the entire frame may be calculated. The degree of static indeterminacy of the structure is equal to the difference between the total number of unknowns and the total number of equations of static equilibrium.

Table 1.2

S. No.	End conditions of the member or type of joint	Number of unknown member forces		Number of equations of static equilibrium	
		Plane frame	Space frame	Plane frame	Space frame
1.	Rigid joints at both ends	3	6		
2.	Rigid joint at one end and pin joint at the other	2	3		
3.	Pin joints at both ends	1	1		
4.	Rigid joint			3	6
5.	Pin joint			2	3

Example 1.4

Determine the degree of static indeterminacy of the pin-jointed plane frame shown in Fig. 1.8.

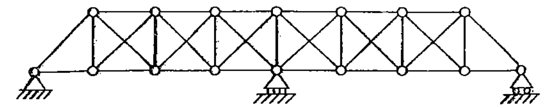


Fig. 1.8

Solution

Total number of independent external reaction components,

$$r = 2 + 1 + 1 = 4$$

Using Eq. (1.7), degree of external indeterminacy,

$$D_{se} = 4 - 3 = 1$$

Number of joints, $j = 16$

Actual number of members, $m = 35$

Using Eq. (1.8), minimum number of members required to preserve geometry of the frame,

$$m' = 2 \times 16 - 3 = 29$$

Using Eq. (1.10), degree of internal indeterminacy,

$$D_{si} = 35 - 29 = 6$$

Hence, degree of static indeterminacy

$$D_s = D_{se} + D_{si} = 1 + 6 = 7$$

Alternatively, the degree of static indeterminacy may be computed using Eq. (1.16). Substituting

$$m = 35 \quad r = 4 \quad j = 16$$

into Eq. (1.16)

$$D_s = 35 + 4 - 2 \times 16 = 7$$

Example 1.5

Determine the degree of static indeterminacy of the rigid-jointed plane frame shown in Fig. 1.9.

Solution

Total number of independent external reaction components,

$$r = 2 \times 3 + 2 + 1 = 9$$

Using Eq. (1.7), degree of external indeterminacy,

$$D_{se} = 9 - 3 = 6$$

The number of cuts required to obtain an open configuration, $c = 12$. For instance, cuts may be made in all the beams except in the topmost beams. Using Eq. (1.12), degree of internal indeterminacy

$$D_{si} = 3 \times 12 = 36$$

Hence, degree of static indeterminacy,

$$D_s = D_{se} + D_{si}$$
$$= 6 + 36 = 42$$

Alternatively, the degree of static indeterminacy may be computed using Eq. (1.18). Substituting

$$m = 35$$
$$r = 9$$
$$j = 24$$

into Eq. (1.18),

$$D_s = 3 \times 35 + 9 - 3 \times 24 = 42$$

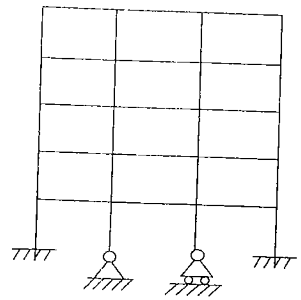


Fig. 1.9

Example 1.6

Determine the degree of static indeterminacy of the bow-string girder shown in Fig. 1.10. Assume all joints to be rigid.

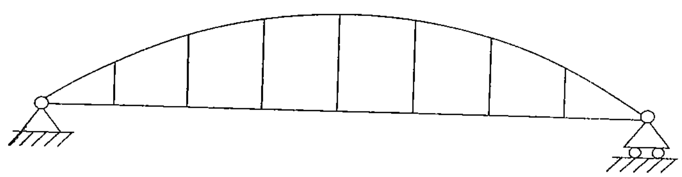


Fig. 1.10

Solution

Total number of independent external reaction components, $r = 3$. Degree of external indeterminacy,

$$D_{se} = 3 - 3 = 0$$

The number of cuts required to obtain an open configuration, $c = 8$. For instance, a cut may be made in the horizontal member in each cell. Using Eq. (1.12), degree of internal indeterminacy,

$$D_{si} = 3 \times 8 = 24$$

Hence, degree of static indeterminacy,

$$D_s = D_{se} + D_{si} = 0 + 24 = 24$$

Alternatively, the degree of static indeterminacy may be computed using Eq. (1.18). Substuting

$$m = 23 \quad r = 3 \quad j = 16$$

into Eq. (1.18),

$$D_s = 3 \times 23 + 3 - 3 \times 16 = 24$$

$h(m-1) = 5(3-1) + 2(4-1)$

Example 1.7

Determine the degree of static indeterminacy of the hybrid plane frame shown in Fig. 1.11.

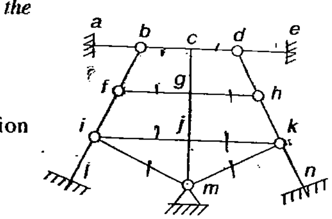


Fig. 1.11

7

Solution

Total number of independent external reaction components,

$$r = 4 \times 3 + 2 = 14$$

Degree of external indeterminacy,

$$D_{se} = 14 - 3 = 11$$

The number of pin joints is 7. The number of members meeting at each of the pin joints b, d, f, h and m is 3. The number of members meeting at each of the pin joints i and k is 4. Therefore,

$$\Sigma(m-1) = 5(3-1) + 2(4-1) = 16$$

The number of cuts required to obtain an open configuration, $c = 6$. Using Eq. (1.14), degree of internal indeterminacy,

$$D_{si} = 3 \times 6 - 16 = 2$$

Hence, degree of static indeterminacy

$$D_s = D_{se} + D_{si} = 11 + 2 = 13$$

Alternatively, the number of members with rigid joints at both ends is 2. The number of members with a rigid joint at one end and pin joint at the other end is 11. The number of members with pin joints at both ends is 6. Hence, the total number of unknown member forces is $2 \times 3 + 11 \times 2 + 6 \times 1 = 34$. The total number of external reaction components is $4 \times 3 + 1 \times 2 = 14$. Hence, the total number of unknowns is $34 + 14 = 48$. The number of rigid joints is 7. The number of pin joints is also 7. Hence, the total number of equations of static equilibrium is $7 \times 3 + 7 \times 2 = 35$. Therefore, the degree of static indeterminacy of the structure is $48 - 35 = 13$.

Example 1.8

Determine the degree of static indeterminacy of the pin-jointed space frame shown in Fig. 1.12. All the vertical and horizontal panels are cross-braced.

Solution

Total number of independent external reaction components,

$$r = 4 \times 3 = 12$$

Using Eq. (1.6), degree of external indeterminacy,

$$D_{se} = 12 - 6 = 6$$

Number of joints, $j = 12$

Actual number of members,

$$m = 36$$

Using Eq. (1.9), minimum number of members required to preserve the geometry of the structure,

$$m' = 3 \times 12 - 6 = 30$$

Using Eq. (1.11), degree of internal indeterminacy,

$$D_{si} = 36 - 30 = 6$$

Hence, degree of static indeterminacy,

$$D_s = D_{se} + D_{si} = 6 + 6 = 12$$

Alternatively, the degree of static indeterminacy may be computed using Eq. (1.17). Substituting

$$m = 36 \quad r = 12 \quad j = 12$$

into Eq. (1.17),

$$D_s = 36 + 12 - 3 \times 12 = 12$$

P.S.

8 m+r-3j

6j+r 6m+r-6j

6×36 +12 - 6×12.

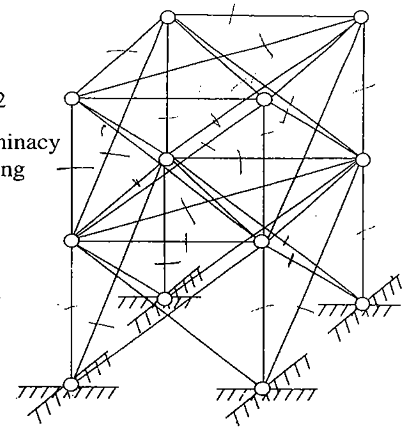


Fig. 1.12

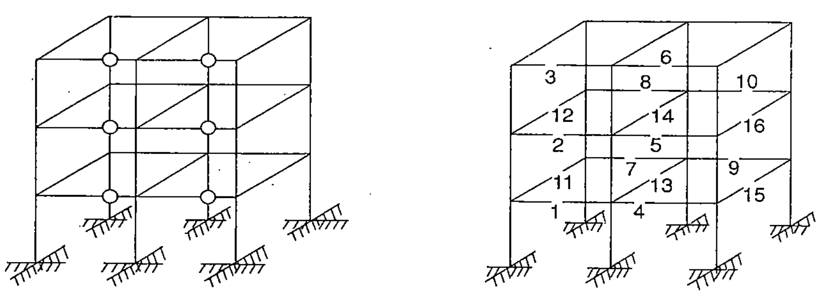
Example 1.9

Determine the degree of static indeterminacy of the rigid-jointed building frame shown in Fig. 1.13(a).

Solution

Total number of independent external reaction components,

$$r = 6 \times 6 = 36$$



(a) (b)

Fig. 1.13

Degree of external indeterminacy,

$$D_{se} = 36 - 6 = 30$$

Number of cuts required to obtain an open configuration, $c = 16$ [Fig. 1.13(b)]. Using Eq. (1.13), degree of internal indeterminacy,

$$D_{si} = 6 \times 16 = 96$$

Hence, degree of static indeterminacy of the frame,

$$D_s = D_{se} + D_{si} = 30 + 96 = 126$$

Alternatively, the degree of static indeterminacy may be computed using Eq. (1.19). Substituting

$$m = 39 \quad r = 36 \quad j = 24$$

into Eq. (1.19),

$$D_s = 6 \times 39 + 36 - 6 \times 24 = 126$$

1.7 DEGREE OF KINEMATIC INDETERMINACY

A skeletal structure is said to be kinematically indeterminate if the displacement components of its joints cannot be determined by compatibility equations alone. In order to evaluate the displacement components at the joints of these structures, it is necessary to consider the equations of static equilibrium. In the case of kinematically indeterminate structure, the number of unknown displacement components is greater than the number of compatibility equations. For these structures, additional equations based on equilibrium must be written in order to obtain a sufficient number of equations for the determination of all the unknown displacement components. The number of these additional equations necessary for the determination of all the independent displacement components is known as the *degree of kinematic indeterminacy* or the *degree of freedom* of the structure. In accordance with the foregoing definitions, a fixed beam is kinematically determinate and a simply supported beam is kinematically indeterminate.

Consider first the pin-jointed frames. Each joint of a pin-jointed plane frame has two independent displacement components because it can move in any two orthogonal directions in the plane of the frame. Similarly, each joint of a pin-jointed space frame has three independent displacement components. Hence the degree of kinematic indeterminacy of a pin-jointed plane frame is given by the equation

$$D_k = 2j - e \tag{1.20a}$$

Similarly, for a pin-jointed space frame the degree of kinematic indeterminacy is given by the equation

$$D_k = 3j - e \tag{1.20b}$$

where j = number of joints and
e = number of equations of compatibility.

The number of equations of compatibility is equal to the number of constraints imposed by the support conditions. As each independent external reaction component provides a constraint against a linear movement in its own direction, the number of equations of compatibility is equal to the number of independent external reaction components. Consequently, the degree of kinematic indeterminacy of a pin-jointed plane frame may be expressed by the equation

$$D_k = 2j - r \tag{1.20c}$$

Similarly, the degree of kinematic indeterminacy of a pin-jointed space frame may be expressed by the equation

$$D_k = 3j - r \tag{1.20d}$$

where r = number of independent external reaction components.

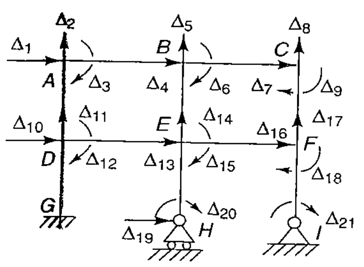
Consider next the rigid-jointed frames. Each joint of a rigid-jointed plane frame has three independent displacement components because it has two linear movements and one rotation. Similarly, each joint of a rigid-jointed space frame has six independent displacement components. Hence, the degree of kinematic indeterminacy of a rigid-jointed plane frame is given by the equation

$$D_k = 3j - e \tag{1.21a}$$

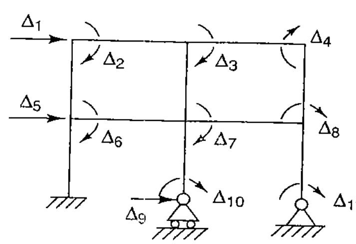
Similarly, for a rigid-jointed space frame the degree of kinematic indeterminacy is given by the equation

$$D_k = 6j - e \tag{1.21b}$$

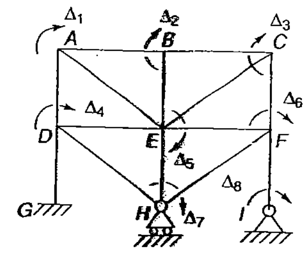
The number of equations of compatibility e is equal to the number of constraints imposed by the support conditions and other factors such as the inextensibility of members. Consider, for example, the rigid-jointed plane frame of a double-storeyed building shown in Fig. 1.14(a). The frame has nine joints. There are three external reaction components at fixed support G, one at roller support H and two at hinge support I. Each external reaction component imposes a constraint on the structure because the displacement component in the direction of a reaction component is zero.



(a)



(b)



(c)

Fig. 1.14

Hence the number of compatibility equations due to the constraints at the supports is equal to the number of external reaction components, $e = 3 + 1 + 2 = 6$. Thus, using Eq. (1.21a), $D_k = 3 \times 9 - 6 = 21$. The directions of the 21 independent displacement components Δ_1 to Δ_{21} are identified in Fig. 1.14(a).

In the analysis of rigid-jointed frames it is commonly assumed that the members are inextensible. This assumption regarding the inextensibility of members imposes additional constraints on the displacements of the structure. Referring to Fig. 1.14(a), it is evident that due to the inextensibility of the columns, joints *A*, *B*, *C*, *D*, *E* and *F* cannot have any vertical displacement. These constraints provide six compatibility equations. The inextensibility of the beams provides four equations of compatibility because the horizontal displacements of joints *B* and *C* are the same as that of joint *A* and the horizontal displacements of joints *E* and *F* are the same as that of joint *D*. Hence, the inextensibility of the members provides ten compatibility equations. It follows that if the members are assumed to be inextensibile, the degree of kinematic indeterminacy of the structure shown in Fig. 1.14(a) is $D_k = 3 \times 9 - (6 + 10) = 11$. The directions of 11 independent displacement components Δ_1 to Δ_{11} are identified in Fig. 1.14(b).

It may be noted that for the frame shown in Fig. 1.14(a), the number of constraints or the number of compatibility equations due to the inextensibility of the members is equal to the number of members m. This is true for typical unbraced building frames. Hence, for rigid-jointed unbraced plane frames the degree of kinematic indeterminacy may be expressed as

$$D_k = 3j - (r + m) \tag{1.21c}$$

In the case of rigid-jointed unbraced space frames,

$$D_k = 6j - (r + m) \tag{1.21d}$$

In the case of a rigid-jointed plane frame with fixed column bases representing a typical building frame having B bays and S storeys, the total number of joints excluding the column bases is $S(B + 1)$. As each of these joints can have a rotation and each floor can sway independent of the other floors, the total number of independent displacement components is given by the equation

$$D_k = S(B + 1) + S = S(B + 2) \qquad (1.22)$$

If some of the column bases have a certain degree of freedom, the degree of freedom of the frame is correspondingly increased.

Consider next, the rigid-jointed braced frame shown in Fig. 1.14(c) in which diagonal braces *AE, EC, DH* and *HF* have been added. If the members of the frame are taken to be extensible, the independent displacement components of the joints are the same as in Fig. 1.14(a). However, if the members are assumed to be inextensible, it is evident that none of the joints can have a linear displacement. Consequently, there are only eight independent displacement components whose directions have been identified in Fig. 1.14(c). In general, it may be stated that if the common assumption of the inextensibility of the members is adopted, the joints of triangulated rigid-jointed frames cannot have linear displacements.

Example 1.10

Determine the degree of kinematic indeterminacy of the pin-jointed plane frame shown in Fig. 1.15.

Solution

Number of joints, $j = 6$

Number of independent external reaction components,

$$r = 2 + 1 = 3$$

Using Eq. (1.20c),

$$D_k = 2 \times 6 - 3 = 9$$

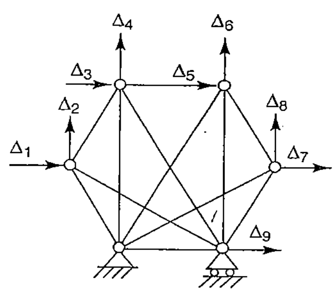


Fig. 1.15

The directions of the nine independent displacement components Δ_1 to Δ_9 are identified in the figure.

Example 1.11

Determine the degree of kinematic indeterminacy of the tripod shown in Fig. 1.16. Identify the independent displacement components.

Solution

Number of joints, $j = 4$

Number of independent external reaction components,

$$r = 3 + 2 + 1 = 6$$

DKI.
P.P. 2j-r
P.S. 3j-r
R.P. 3j-r
R.S 6j-r

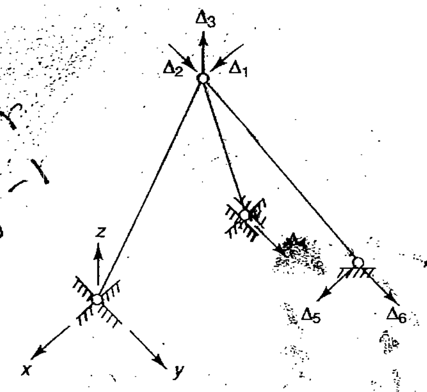


Fig. 1.16

Using Eq. (1.20d),

$$D_k = 3 \times 4 - 6 = 6$$

The directions of the six independent displacement components Δ_1 to Δ_6 are identified in the figure.

Example 1.12

Determine the degree of freedom of the continuous beam shown in Fig. 1.17. Assume that the beam is inextensible.

Solution

Number of joints, $j = 4$

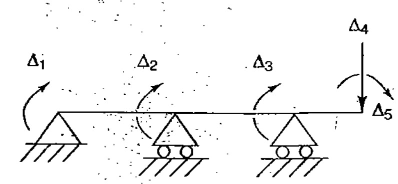


Fig. 1.17

Number of members, $m = 3$

Number of independent external reaction components, $r = 4$

Using Eq. (1.21c),

$$D_k = 3 \times 4 - (4 + 3) = 5$$

The directions of the five independent displacement components are identified in the figure.

Example 1.13

Determine the degree of kinematic indeterminacy of the space frame shown in Fig. 1.18. Joint O is rigid. Also calculate the degree of kinematic indeterminacy if the members are assumed to be inextensible.

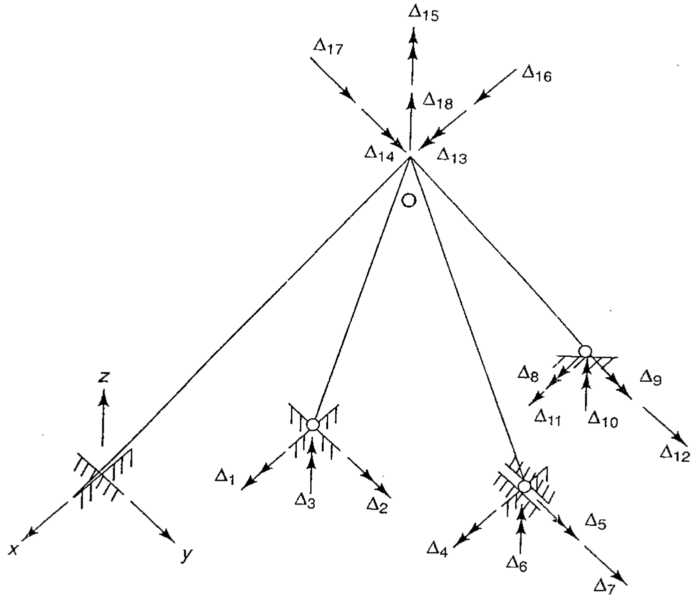


Fig. 1.18

Solution

Number of joints, $j = 5$

Number of constraints $= r = 6 + 3 + 2 + 1 = 12$

Using Eq. (1.21b),

$$D_k = 6 \times 5 - 12 = 18$$

The directions of the eighteen independent displacement components Δ_1 to Δ_{18} are identified in the figure.

If the members are assumed to be inextensible, the linear displacement components at the joint O are zero. Consequently, the degree of kinematic indeterminacy is reduced to fifteen.

1.8 STABILITY

As structural stability is the prime concern of the structural engineer, he must ensure that the structure is supported adequately so that it develops a strong stability against all kinds of destabilising forces. It is convenient to divide the overall stability of the structure into:

(i) external stability

(ii) internal stability

A structure is ***externally stable*** if the supports are capable of providing the required number of independent reaction components for static equilibrium of the structure. The static equilibrium of a plane structure requires that the sum of the components of all forces along any two orthogonal axes in the plane of the structure be zero and the sum of the moments of all forces about any axis perpendicular to the plane of the structure is also zero. Thus a support-system is stable only if it can develop non-trivial reaction forces along any two orthogonal axes in the plane of the structure and a non-trival couple about any axis perpendicular to the plane of the structure. The support-systems shown in Fig. 1.19 are not stable. The support-system shown in Fig. 1.19(a) can provide only three parallel reaction components. It cannot, therefore, resist a force perpendicular to the direction of the reactive forces. In Fig. 1.19(b), the three reactive forces are concurrent. Hence, the support-system is incapable of resisting a couple about the point O. It follows that for stability, the three reactive forces should be: (i) non-parallel and (ii) non-concurrent. The static equilibrium of a space structure requires that Eqs (1.1) and (1.2) be satisfied. Thus, a support-system is stable only if it can develop non-trivial reactive forces along any three ~~orthogonal axes~~ and non-trivial couples about these axes. It follows that for stability, the six reactive forces of a space structure should be (i) non-parallel, (ii) non-coplanar and (iii) non-concurrent.

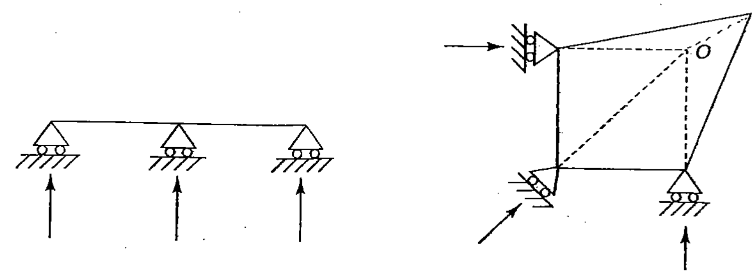


Fig. 1.19

A structural system is *internally stable* if it can preserve its geometry under the action of all kinds of forces tending to deform it. Significant internal forces are produced in the members of a structural system as a result of even small changes in the geometry. On the other hand, the geometry of unstable systems, known as *mechanisms,* can change substantially without generating appreciable internal forces. A pin-jointed frame is internally stable if the number of members is sufficient to preserve its geometry. The minimum number of members required in pin-jointed plane and space frames for this purpose is shown in Table 1.1. In general, if the number of members is less than the minimum requirement, an unstable system, known as a mechanism, is obtained. On the other hand, an overstiff statically indeterminate system is obtained if the number of members is more than the minimum required for preserving the geometry of the system. A rigid-jointed frame is internally stable and statically determinate if it has an open configuration as defined in Sec. 1.6. On the other hand, rigid-jointed frames without open configuration are, in general, overstiff and consequently statically indeterminate internally. In order to see whether a frame with hybrid joints is internally stable, a check must be applied to see if it can preserve its geometry under the action of all possible combinations of external loads.

An alternative approach to the problem of stability is to consider the structural system as a whole without distinguishing between external and internal stabilities. From the derivation of Eqs (1.16) to (1.19), it follows that a pin-jointed plane frame is (a) unstable if $(m + r) < 2j$, (b) stable and statically determinate if $(m + r) = 2j$, and (c) stable and statically indeterminate if $(m + r) > 2j$.

Similarly, a pin-jointed space frame is (a) unstable if $(m + r) < 3j$, (b) stable and statically determinate if $(m + r) = 3j$, and (c) stable and statically indeterminate if $(m + r) > 3j$.

A rigid-jointed plane frame is (a) unstable if $(3m + r) < 3j$, (b) stable and statically determinate if $(3m + r) = 3j$, and (c) stable and statically indeterminate if $(3m + r) > 3j$.

Similarly, a rigid-jointed space frame is (a) unstable if $(6m + r) < 6j$, (b) stable and statically determinate if $(6m + r) = 6j$, and (c) stable and statically indeterminate if $(6m + r) > 6j$.

It may be noted that a certain degree of exchange may take place between the required number of members and the reaction components in order to achieve overall stability. For instance, the deficiency in respect of the number of members may be made good by introducing additional reaction components. Consider, for example, the pin-jointed plane frame shown in Fig. 1.20(a). It has eight joints and consequently it needs $(2j - 3) = 13$ members to be internally stable. Since the frame has only eight members, it is internally deficient to the fifth degree. Internal stability can be achieved by adding five members as

shown by the broken lines in Fig. 1.20(b). Alternatively, a stable system can also be obtained by introducing five additional reaction components in addition to the three needed for static equilibrium, as shown in Fig. 1.20(c).

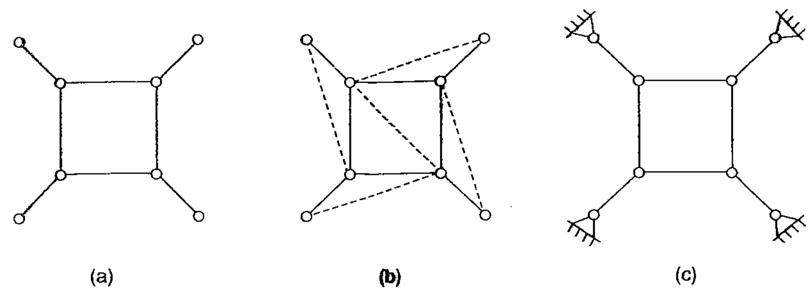


Fig. 1.20

Example 1.14

Comment on the stability of the pin-jointed plane frame shown in Fig. 1.21.

Solution

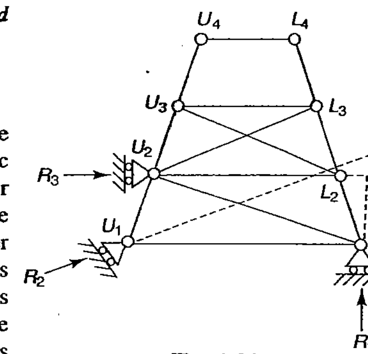


Fig. 1.21

First consider the external stability. Three reaction components required for static equilibrium are supplied by the roller supports at L_1, U_1 and U_2. As the three reactive forces R_1, R_2 and R_3 are neither parallel nor concurrent, the frame is externally stable, although the stability is very weak. This is so because the three reactive forces are nearly concurrent. It is evident that, if the resultant of external forces has a larger lever-arm about the point O, the reaction R_2 will have to be very large to satisfy the condition $\Sigma M = 0$ at point O.

Consider next, the internal stability. The frame has 8 joints and consequently requires 13 members as per Eq. (1.8). The frame does have 13 members but even then it is not stable. Actually, the frame is a combination of a stable panel $U_1U_2L_2L_1$, an overstiff panel $U_2U_3L_3L_2$ and an unstable panel $U_3U_4L_4L_3$. When the internal stability of the frame as a whole is considered, the frame will have to be designated as *unstable.*

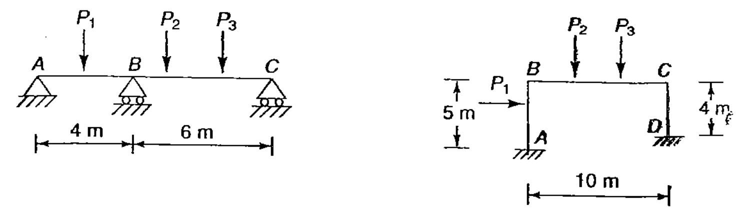
1.9 SETTLEMENT OF SUPPORTS

In actual practice all supports yield under the action of the forces exerted on them. However, if the settlements of the supports are very small, they may be ignored without causing any appreciable error. In this case, the supports may be considered as unyielding.

In the case of statically determinate structures, the settlements of the supports do not produce internal forces because these structures undergo only rigid-body motion without any deformation. It has been seen in Sec. 1.8 that a three-dimensional structure requires six independent external reaction components for stability and external determinacy. Hence, if the structure is statically determinate externally, the maximum number of displacement components due to the settlement of supports is six. In a rigid-body motion of a space structure there are six independent displacement components, viz., three linear movements and three rotations along and about three mutually orthogonal axes. Consequently, the settlement of supports in the case of a determinate support-system can produce only a rigid-body motion of the structure. In the case of plane structures, only three independent external reaction components are necessary for stability and external determinacy. As the rigid-body motion in a plane also involves only three independent displacement components, viz., two orthogonal linear movements and a rotation, it is evident that the settlements of supports of an externally determinate plane structure cannot produce internal forces in the structure.

In the case of externally indeterminate structures, the settlements of the supports generally induce internal forces. If the number of external reaction components due to the supports is r, the degree of external indeterminacy is $(r-6)$ in the case of a space structure and $(r-3)$ in the case of a plane structure. If the number of displacement components is less than or equal to the degree of external indeterminacy, the rigid-body motion of the structure is generally not possible. On the other hand, if the number of displacement components due to the settlements of supports is more than the degree of external indeterminacy, a rigid-body motion of the structure occurs. In this case, the internal forces in the structure are induced on account of only the net displacement components which may be computed by substracting the rigid-body displacement components from the gross displacement components.

Consider, for example, the two span continuous beam shown in Fig. 1.22(a). Let the downward settlements of the supports A, B and C be 0.02 m, 0.06 m and 0.03 m respectively. As the degree of static indeterminacy of the structure is one and the number of support movements is three, a rigid-body motion of the structure is involved. If the vertical movement of 0.02 m at A is taken as a rigid-body motion, the net vertical downward displacements at B and C are 0.04 m and 0.01 m respectively. Next, the beam may be given a rigid-body rotation about the point A so that the net vertical displacement at C becomes zero. To achieve this, a clockwise rotation of $\dfrac{0.01}{4+6} = 0.001$ radian is necessary. As this rigid-body rotation produces a downward movement at B equal to



(a) (b)

Fig. 1.22

$4 \times 0.001 = 0.004$ m, the net vertical movement of the support B may be taken as $(0.04 - 0.004) = 0.036$ m. It is this movement at B which produces internal forces in the beam. Hence for the analysis of the continuous beam shown in Fig. 1.22(a), only the net displacement of 0.036 m downwards at B without any displacement at A and C need be considered.

Consider next, the portal frame shown in Fig. 1.22(b). Due to the settlement of the support at A, the three displacement components are 0.01 m towards right, 0.02 m downwards and 0.001 radian clockwise. Similarly, the settlement of the support at D gives rise to displacement components of 0.015 m towards right, 0.04 m downwards and 0.0015 radian clockwise. As the number of displacement components is more than the degree of external indeterminacy, a rigid-body motion occurs. Treating the linear displacements at A as rigid-body movements, the net linear movements at D are $(0.015 - 0.01) = 0.005$ m towards right and $(0.04 - 0.02) = 0.02$ m downwards. Next, the frame may be rotated clockwise as a rigid body about the point A through an angle equal to 0.01 radian. This rotation produces at D a vertical movement equal to $10 \times 0.001 = 0.01$ m downwards, a horizontal movement equal to $1 \times 0.001 = 0.001$ m towards right and a clockwise rotation of 0.001 radian. Hence, treating the displacement components at A as rigid-body movements, the net displacements at D which are responsible for inducing internal forces in the structure comprise a horizontal movement of $(0.005 - 0.001) = 0.004$ m towards right, a vertical movement of $(0.02 - 0.01) = 0.01$ m downwards and a clockwise rotation of $(0.0015 - 0.001) = 0.0005$ radian.

From the foregoing examples it may be noted that the number of displacement components due to the settlement of supports in excess of the degree of external indeterminacy of the structure gives rise to only rigid-body displacements. Throughout this book, in the discussion of yielding supports, only the net displacements of the structure which induce internal forces in the structure will be considered. It will be presumed that the rigid-body displacements have been eliminated as explained in the foregoing examples. It should also be seen that if the rigid body displacements are eliminated, the number of displacement components due to the settlements of supports cannot be greater than the degree of external indeterminacy of the structure.

PROBLEMS

1.1 Draw the free-body diagrams for the members *AB*, *BC* and *CD* of the rigid-jointed frame shown in Fig. 1.23. Hence determine the axial force, the shear force and the bending moment at *D*.

1.2 If the self-weight of the members of the frame shown in Fig. 1.23 is 2 kN/m, draw the free-body diagrams for the members *AB*, *BC* and *CD* considering the loads in the figure as well as the self-weight. Hence determine the axial force, the shear force and the bending moment at *D*.

1.3 If the frame of Fig. 1.23 lies in the horizontal plane, draw the free-body diagrams for the members *AB*, *BC* and *CD*. Consider only the self-weight of the members which is given as 2 kN/m. Hence determine the shear force, the bending moment and the twisting moment at *D*.

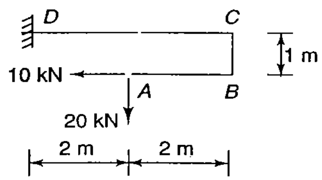


Fig. 1.23

1.4 Check the stability of the beams shown in Fig. 1.24 and indicate which of them are unstable.

1.5 Which of the beams shown in Fig. 1.24 are statically determinate? For these beams, calculate the degrees of kinematic indeterminacies.

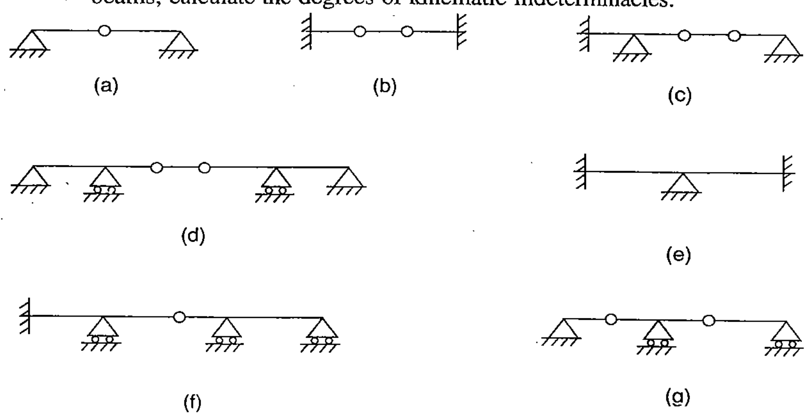


Fig. 1.24

1.6 Which of the beams shown in Fig. 1.24 are statically indeterminate? Determine the degrees of static and kinematic indeterminacies of these beams.

1.7 Check the stability of the pin-jointed plane frames shown in Fig. 1.25 and indicate which of them are unstable.

1.8 Which of the pin-jointed plane frames shown in Fig. 1.25 are statically determinate? For these frames, calculate the degrees of kinematic indeterminacies.

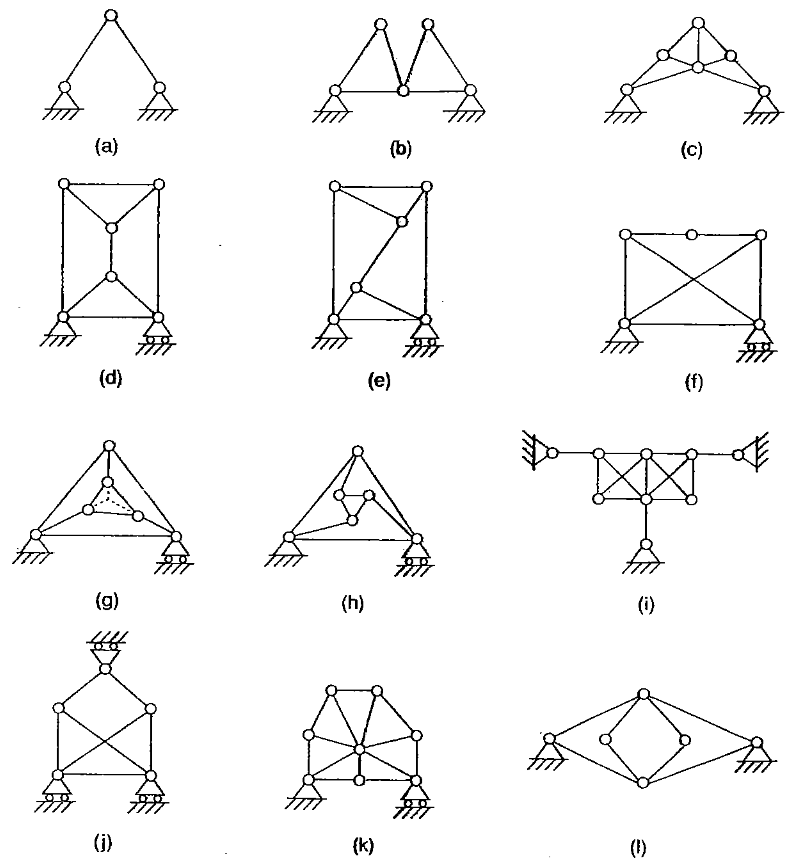


Fig. 1.25 (Contd)

1.9 Which of the pin-jointed plane frames shown in Fig. 1.25 are statically indeterminate? Determine the degrees of static and kinematic indeterminacies of these frames.

1.10 Check the stability of the plane frames shown in Fig. 1.26 and indicate which of them are unstable.

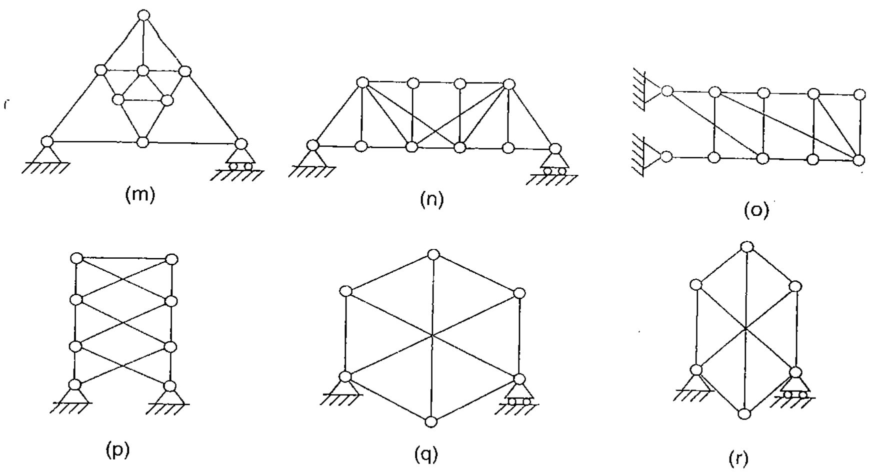


Fig. 1.25

1.11 Which of the plane frames shown in Fig. 1.26 are statically determinate? For these frames calculate the degrees of kinematic indeterminacies.

1.12 Which of the plane frames shown in Fig. 1.26 are statically indeterminate? Determine the degrees of static and kinematic indetrminacies of these frames.

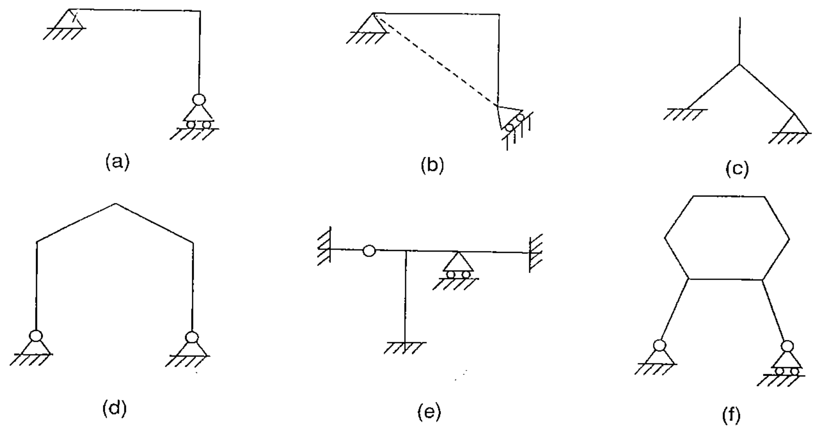


Fig. 1.26 (Contd)

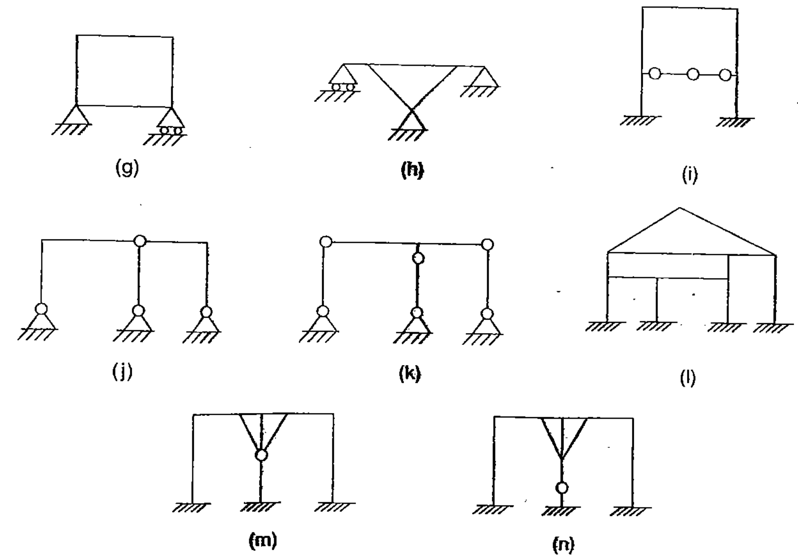


Fig. 1.26

1.13 Determine the degrees of static and kinematic indeterminacies of the pin-jointed space frames shown in Fig. 1.27.

1.14 Calculate the degrees of static and kinematic indeterminacies of the frames shown in Fig. 1.27 if all the pin-joints are replaced by the rigid-joints. The support conditions remain unchanged.

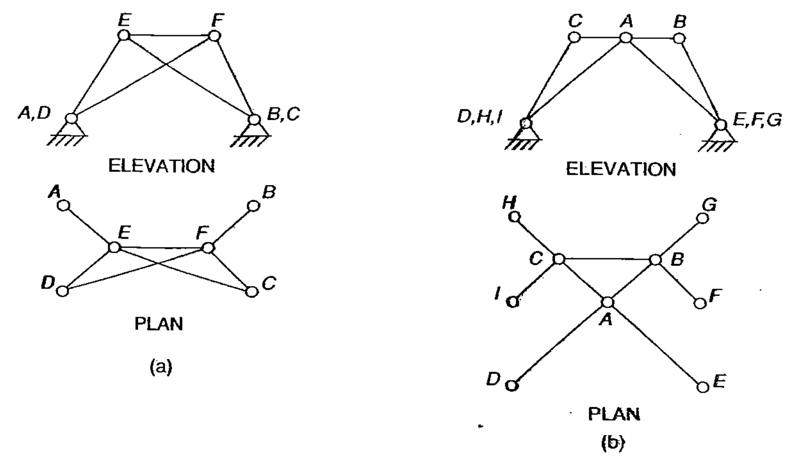
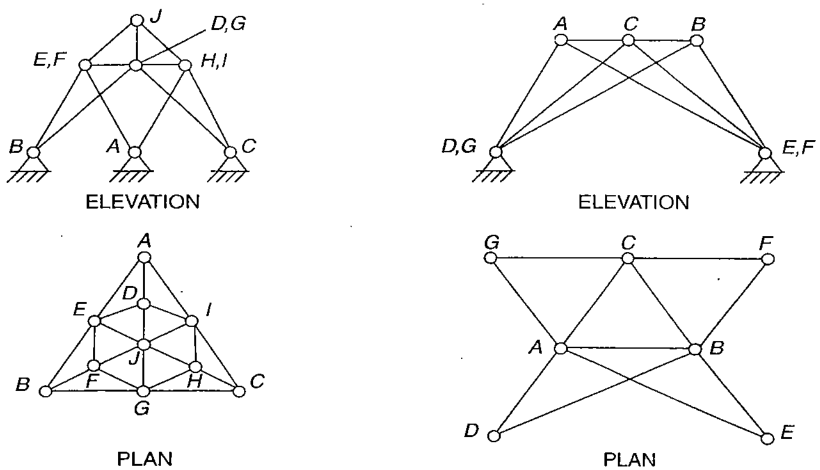


Fig. 1.27 (Contd)



(c) (d)

Fig. 1.27

1.15 A rigid-jointed building frame has ten storeys. It has five bays in one direction and eight bays in the other direction. Determine the degrees of static and kinematic indeterminacies if all the column bases are (i) fixed and (ii) hinged.

2 BASIC METHODS OF STRUCTURAL ANALYSIS

2.1 STRUCTURAL RESPONSE

Structural systems subjected to static loading exhibit their response in the form of induced internal stresses and consequent displacements. In general, the members or elements of a structure are subjected to four types of internal forces, viz., an axial force, a shear force, a bending moment and a twisting moment. The resulting internal stresses give rise to linear displacements and rotations. Thus the structural system as a whole undergoes a set of displacements. The external forces acting on the structure undergo these displacements and consequently lose their potential energy. In accordance with the *Law of conservation of energy*, the loss of potential energy of the external forces is compensated by an equal amount of energy stored in the structure in the form of *strain energy*. The main object of structural analysis is to evaluate the response of the structure exhibited by way of induced internal stresses and resulting displacements because these are directly related to the safety and serviceability of the structural system.

Consider a structure acted upon by a series of loads $P_1, P_2, \ldots, P_j, \ldots, P_n$ producing displacements $\Delta_1, \Delta_2, \ldots, \Delta_j, \ldots, \Delta_n$ along their lines of action respectively. Figure 2.1 shows the load-displacement characteristic for one of the loads P_j acting on the structure as P_j increases gradually from zero to its full value. The work done by the load P_j in undergoing a small displacement $\delta\Delta_j$, represented by the shaded strip $A_1A_2B_2B_1$, is given by the equation

$$\delta U_j = P_j\,\delta\Delta_j \tag{a}$$

The total Work done by the load P_j in undergoing the total displacement Δ_j is obtained by the integration of Eq. (a).

$$U_j = \int \delta U_j = \int P_j \delta\Delta_j = \text{Area } OAB \tag{b}$$

The work done by P_j, represented by area *OAB*, is stored in the form of strain energy. Hence, area *OAB* represents the strain energy stored in the structure as load P_j increases monotonously from zero to its full value. The increase in strain energy, as load P_j undergoes a small displacement $\delta\Delta_j$, is given by Eq. (a).

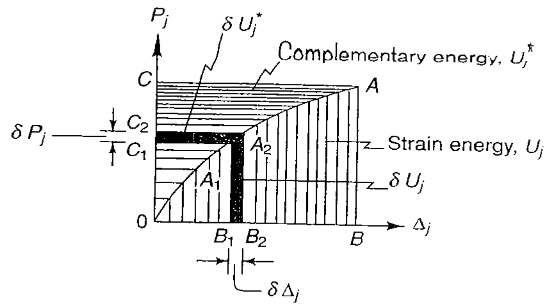


Fig. 2.1

Hence, the increment in strain energy, when all the loads acting on the structure undergo small displacements, is given by the equation

$$\delta U = \Sigma \delta U_j = \Sigma P_j \, \delta \Delta_j$$

or
$$\delta U = P_1 \, \delta\Delta_1 + P_2 \, \delta\Delta_2 + \ldots + P_j \, \delta\Delta_j + \ldots + P_n \, \delta\Delta_n \tag{2.1}$$

The total strain energy of a structure, when all the loads undergo their total displacements, may be obtained by the integration of Eq. (2.1).

The shaded strip $C_1C_2A_2A_1$ is represented by the equation

$$\delta U_j^* = \Delta_j \delta P_j = \text{area of shaded strip } C_1C_2A_2A_1 \tag{c}$$

The quantity U_j^*, commonly known as *complementary energy,* will be obtained by the integration of Eq. (c). Evidently the complementary energy U_j^*, due to the load P_j alone, is represented by area OAC. If all the loads are given small increments, the increase in complementary energy is given by the equation

$$\delta U^* = \Sigma \delta U_j^* = \Sigma \Delta_j \delta P_j$$

or
$$\delta U^* = \Delta_1 \delta P_1 + \Delta_2 \delta P_2 + \ldots + \Delta_j \delta P_j + \ldots + \Delta_n \delta P_n \tag{2.2}$$

Total complementary energy U^* of the structure may be obtained by the integration of Eq. (2.2). It may be noted that if the response is linear, curve OA of Fig. 2.1 becomes a straight line and the strain energy and the complementary energy are equal

$$U = U^* \tag{2.3}$$

2.2 FUNDAMENTAL ASSUMPTIONS

The fundamental assumption in the analysis of structures is the *linearity* of the structural response. It follows that the internal stresses and the resulting displacements increase in proportion to the external forces. In Fig. 2.2 the external load P acting on the structure and the resulting displacement Δ at any

point of the structure are plotted along the vertical and horizontal axes respectively. The structure is said to behave linearly if the load displacement relationship is represented by the straight line OA. The response of a structure can be linear, only if the *principle of superposition* is valid. According to this principle, the total response of a structure on account of the combined action of any two systems of external forces P_I and P_{II} is equal to the sum of the responses due to the two systems of forces acting separately. Thus, referring to Fig. 2.2,

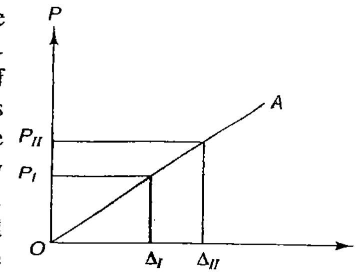


Fig. 2.2

$$\Delta_{I+II} = \Delta_{II+I} = \Delta_I + \Delta_{II} \tag{2.4}$$

where Δ_{I+II} = total displacement due to the combined action of P_I and P_{II} applied in sequence of P_I and P_{II}

Δ_{II+I} = total displacement due to the combined action of P_I and P_{II} applied in the sequence of P_{II} and P_I

Δ_I = displacement due to the action of P_I alone

Δ_{II} = displacement due to the action of P_{II} alone.

Equation (2.4) shows that the total displacement due to any two systems of loads may be obtained by the summation of the displacements caused by the two systems acting separately and that the sequence in which the loads are applied is immaterial. The structural response is linear and the principle of superposition holds if the following fundamental assumptions are satisfied:

(i) The structure is in a condition of static equilibrium.
(ii) The material of the structure behaves linearly. It is implied that the material is homogeneous, isotropic and elastic, and follows Hooke's law. A material is homogeneous if it has identical properties at all points. The property of isotropy indicates identical behaviour in all directions. The material is said to be elastic if the strain disappears completely on the removal of stress. According to Hooke's law, the stress-strain curve of the material is a straight line. A majority of engineering materials, particularly metals, behave linearly.
(iii) The supports are unyielding. In the case of yielding supports, the structural response is, generally, non-linear and the principle of superposition is not valid.
(iv) The displacements are small. Thus, large displacements, generally covered under the large displacement theory, are excluded from consideration here. In general, the principle of superposition is valid only if the displacements are small because in the case of small displacements the forces and displacements in the transverse direction

are independent of the forces and displacements in the axial direction. Consider, for example, the beam-column shown in Fig. 2.3. The bending moment M at any point C is given by the equation

$$M = R_A \times d - W_1 \times d_1 - W_2 \times d_2 + P \times \Delta_c$$

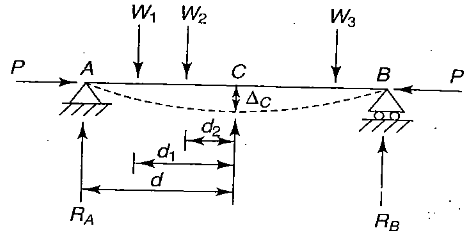


Fig. 2.3

If the displacement Δ_c is small, the last term in the expression for bending moment is small compared to the first three terms. As the displacement is calculated by the double integration of the bending-moment expression, it follows that the transverse displacement is independent of the axial loads and is a function of transverse loads, only if the displacement is small. In other words, if the displacement is assumed to be small, the transverse displacement is caused by only transverse loads and likewise axial displacement is caused only by axial loads. It is evident that this type of uncoupling of the axial and transverse loads and resultant displacements is necessary for the linearity of structural response and the validity of the principle of superposition.

(v) There is no self-straining of the structure. In other words, the internal force in every member of the structure is zero if no external load acts on the structure. It follows that the factors such as thermal changes, mismatch due to fabrication errors and prestress are absent. A statically indeterminate structure develops internal stresses if it undergoes a thermal change. Similarly, internal self-equilibrating systems of forces are set up on account of mismatch due to fabrication errors. Self-straining of the structure may also be caused by fabrication processes such as welding and prestressing. If these internal stresses due to the self-straining of the structure are appreciable, the response of the structure to external loads may become non-linear.

It will be assumed throughout that the foregoing assumptions hold, so that the structural response is linear and the principle of superposition is valid. In particular, the assumptions are applicable in the statement and derivation of the basic theorems to be discussed in the following sections. These assumptions,

which form the basis of the important energy theorems, may be summed up as:

(i) The structure is in a condition of static equilibrium.
(ii) The response of the structural system is linear so that all effects tending to cause the non-linearity of the structural response are absent.

2.3 SIGN CONVENTION

The following sign conventions, which are most common in structural analysis, will be adopted for the internal forces in structural members.

2.3.1 Axial Force

The axial force will be taken to be positive if tensile. The forces acting at ends A and B and at some intermediate cross-section C of structural element AB in tension are shown in Fig. 2.4(a).

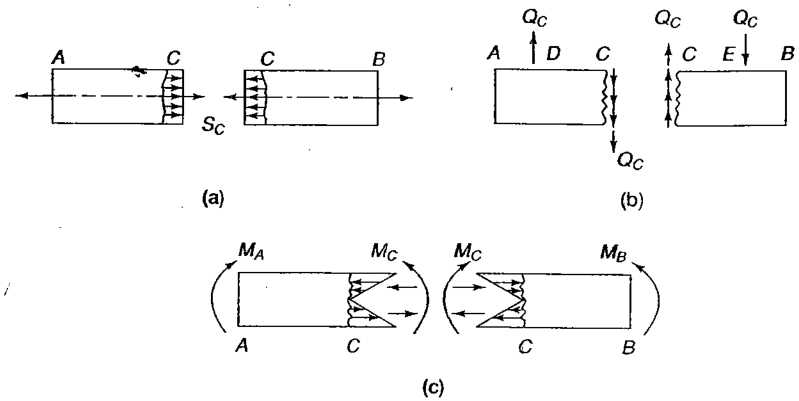


Fig. 2.4

2.3.2 Shear Force

The shear force at a cross-section in a structural element will be taken to be positive if the resultant transverse forces on either side of the cross-section form a clockwise couple. In Fig. 2.4(b), force Q_C at D is the resultant of all forces acting on the portion CA normal to the axis of element AB. Similarly, force Q_C at E is the resultant of all forces acting on the portion CB normal to the axis of element AB. As forces Q_C at D and E form a clockwise couple, the shear force at C is positive. Considering the free bodies of portions AC and CB it is evident that the resultant transverse forces Q_C at cross-section C should be directed as indicated in Fig. 2.4(b). The two transverse forces at C together constitute the positive shear force.

2.3.3 Bending Moment

In a horizontal member, commonly known as a beam, an end couple which tends to create tension at the bottom fibres, is taken to be positive. This sign convention, known as *beam-convention*, is unsuitable for a frame which comprise horizontal and vertical members. In the sign convention adopted for frames, known as *frame-convention*, a bending couple is taken to be positive if it is clockwise. Figure 2.4(c) shows a structural element AB acted upon by a clockwise couple M_A at A and a counter-clockwise couple M_B at B. Both the couples are positive according to the beam convention because they produce tension at the bottom fibres. However, in accordance with the frame-convention, the couple at A is positive and the couple at B is negative.

The bending moment at an intermediate cross-section of a structural element comprises a pair of bending couples of opposite sense, one clockwise and the other counter-clockwise. The bending moment is called *sagging* if the bending couples, constituting the bending moment, create tension at the bottom fibres of a horizontal member. If the two bending couples, constituting the bending moment, create tension at the top fibres, the bending moment is known as *hogging*. According to the beam-convention, the sagging bending moment is positive and hogging bending moment is negative. At an intermediate cross-section C, bending moment M_C is positive according to the beam convention if the two bending couples at C which constitute the bending moment at C, are directed as shown in Fig. 2.4(c) because these bending couples create tension at the bottom fibres in portions CA and CB. It is not possible to use the frame convention for the bending moment because a bending moment invariably comprises a pair of couples of opposite sense.

2.4 GENERALIZED SYSTEM OF COORDINATES

It is convenient to express the forces and the corresponding displacements by means of the generalized system of coordinates 1, 2, ..., n introduced in Fig. 2.5. The directions of the coordinates can be chosen arbitrarily. Once the directions of the coordinates have been chosen, a positive force is one which acts in the positive direction of the coordinate. Otherwise it is taken to be negative. Thus when a statement is made that a force of 100 kN acts at coordinate j, it means that a force of 100 kN acts vertically downwards at point E. Similarly, a force of 200 kN·m acting at coordinate $(j + 1)$ indicates that a clockwise couple of 200 kN·m acts at point E. If it is stated that the displacements at coordinates j and $(j + 1)$ are 2 mm and 0.002 radian respectively, it means that the vertical downward displacement at point E is 2 mm and the clockwise rotation at point

E is 0.002 radian. It should be noted that in this generalized notation, a couple is also designated as a force. The main advantage of the generalized system of coordinates introduced here is that it makes the description of forces and displacements very concise and unambiguous. Besides, the generalized notation is eminently suitable for the matrix approach to structural analysis.

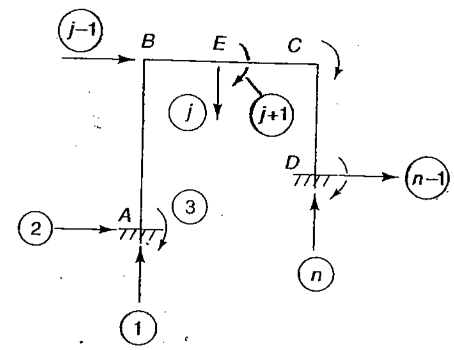


Fig. 2.5

Using the generalized system of coordinates, the following symbols may now be introduced:

P_j = force at coordinate j
Δ_j = displacement at coordinate j
k_{ij} = force at coordinate i due to a unit displacement at coordinate j
δ_{ij} = displacement at coordinate i due to a unit force at coordinate j.

If a structure is acted upon by a system of forces $P_1, P_2, ..., P_j, ..., P_n$, then, using the notation introduced above, the displacement Δ_j at coordinate j due to all forces of the system is given by the equation

$$\Delta_j = \delta_{j1}P_1 + \delta_{j2}P_2 + ... + \delta_{jj}P_j + ... + \delta_{jn}P_n \tag{2.5}$$

Similarly, the force, P_j at coordinate j due to a set of displacements $\Delta_1, \Delta_2, ..., \Delta_j, ..., \Delta_n$ is given by the equation

$$P_j = k_{j1}\Delta_1 + k_{j1}\Delta_2 + ... + k_{jj}\Delta_j + ... + k_{jn}\Delta_n \tag{2.6}$$

In the analysis of a skeletal structure, which is statically indeterminate internally, it becomes necessary to release some of the internal forces to make it statistically determinate. The internal member forces are represented by pairs of forces, equal in magnitude and opposite in direction. The internal member forces at an intermediate cross-section C of member AB are shown in Fig. 2.4. When a coordinate has to be assigned to an internal member force or to its corresponding internal displacement, a pair of arrows has to be employed. Thus, the coordinate to be assigned to an axial force or an axial displacement may comprise a pair of straight arrows directed towards each other or away from each other along the axis of the member. The coordinate to be assigned to a shear force or a shear displacement may comprise a pair of straight arrows parallel to each other, pointing in opposite directions and perpendicular to the axis of the member. Similarly, the coordinate to be assigned to a bending moment or a bending displacement may comprise a pair of circular arrows of opposite sense.

Consider, for example, the pin-jointed plane frame shown in Fig. 2.6(a). A cut in redundant member U_1L_2 is equivalent to the release of two equal and opposite forces P_j where P_j is the force in the redundant member. If coordinate j has to be assigned to this force, two straight arrows along the axis of the member are employed. If the arrows are directed towards each other, as shown in Fig. 2.6(b), the coordinate j corresponds to a tensile force in member U_1L_2. On the other hand, if the two arrows point away from each other, coordinate j corresponds to a compressive force in the member.

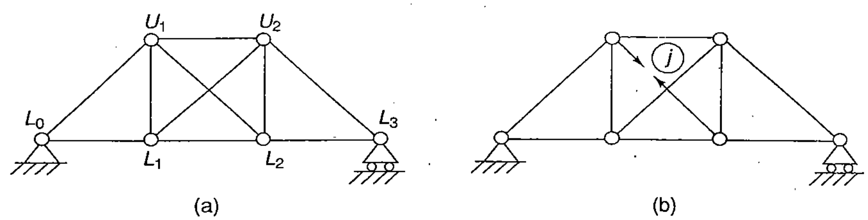


Fig. 2.6

Consider next, a member AB of a rigid-jointed plane frame. In Fig. 2.7(a), coordinates 1, 2 and 3 have been assigned to the internal member forces at cross-section C. Coordinate 1 corresponds to a ~~positive~~ (tensile) axial force S_C at C. Coordinate 2 corresponds to a positive shear force Q_C at C. Coordinate 3 corresponds to a positive (sagging) bending moment M_C at C. Figure 2.7(b) shows the free-body diagrams of portions AC and CB in which S_C, Q_C and M_C, are all positive. If a cut is introduced at cross-section C so as to release the internal forces S_C, Q_C and M_C, the displacements at C caused by the applied loads are shown by the broken lines in Fig. 2.7(c). Using the notations of Fig. 2.7(c), the displacements at coordinates 1, 2 and 3 due to the applied loads are given by the equations

Axial displacement,	$\Delta'_1 = -x'_L - x'_R$	(2.7a)
Shear displacement,	$\Delta'_2 = -y'_L - y'_R$	(2.7b)
Bending displacement,	$\Delta'_3 = -\theta'_L - \theta'_R$	(2.7c)

If the cut at C does not exist, i.e., the member is continuous, the displacements Δ'_1, Δ'_2 and Δ'_3 shown in Fig. 2.7(c) cannot occur. In order to eliminate these displacements, internal forces S_C, Q_C and M_C at cross-section C, are called into play. The displacements caused by these internal forces are equal and opposite to those caused by the applied loads. The displacements

at the cross-section C, due to the internal forces alone, are shown in Fig. 2.7(d). Using the notations of Fig. 2.7(d), these displacements are given by the equations

Axial displacement,	$\Delta''_1 = -x''_L - x''_R$	(2.8a)
Shear displacement,	$\Delta''_2 = -y''_L - y''_R$	(2.8b)
Bending displacement,	$\Delta''_3 = -\theta''_L - \theta''_R$	(2.8c)

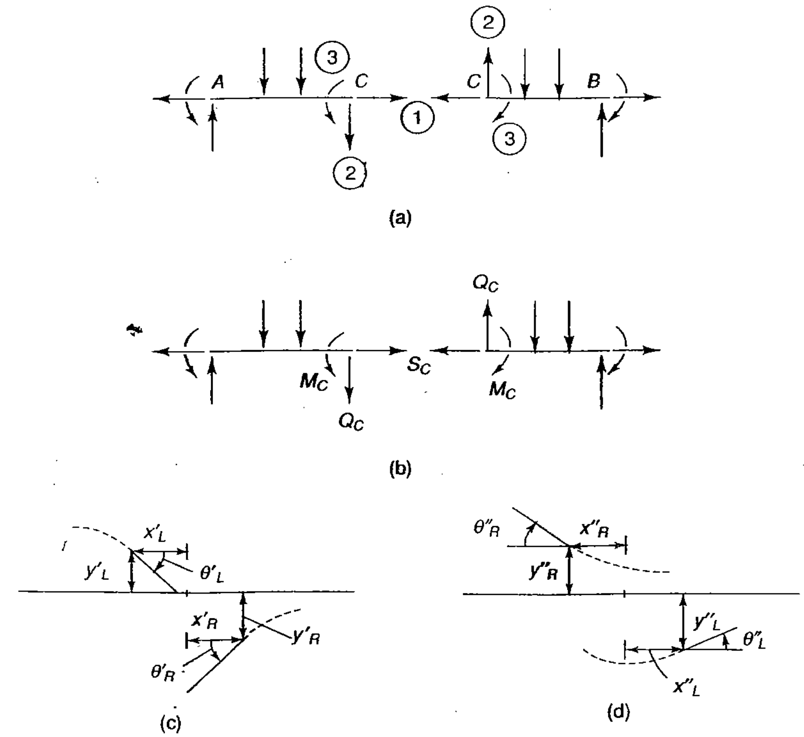


Fig. 2.7

As the displacements at the cut at C due to the applied loads and the internal forces are equal and opposite, the net displacements at coordinates 1, 2 and 3 vanish. This condition, which follows from the compatibility of deformations, is necessary for the continuity at C in member AB. Combining Eqs (2.7) and (2.8), the net displacements at coordinates 1, 2 and 3 are given by the equations

Axial displacement,	$\Delta_1 = \Delta'_1 + \Delta''_1 = 0$	(2.9a)
Shear displacement,	$\Delta_2 = \Delta'_2 + \Delta''_2 = 0$	(2.9b)
Bending displacement,	$\Delta_3 = \Delta'_3 + \Delta''_3 = 0$	(2.9c)

The following points should be noted in expressing loads and displacements with the help of generalized system of coordinates:

(i) In order to distinguish the numbers representing the magnitudes of the forces from the coordinate numbers, the latter are encircled.

(ii) If a force acts in a direction opposite to that of the coordinate, it will be treated as a negative force. Similarly, if a displacement at a coordinate is in a direction opposite to that of the coordinate, it will be treated as a negative displacement.

Example 2.1

Express the loads and displacements shown in Fig. 2.8(a) and (b) with the help of the generalized system of coordinates.

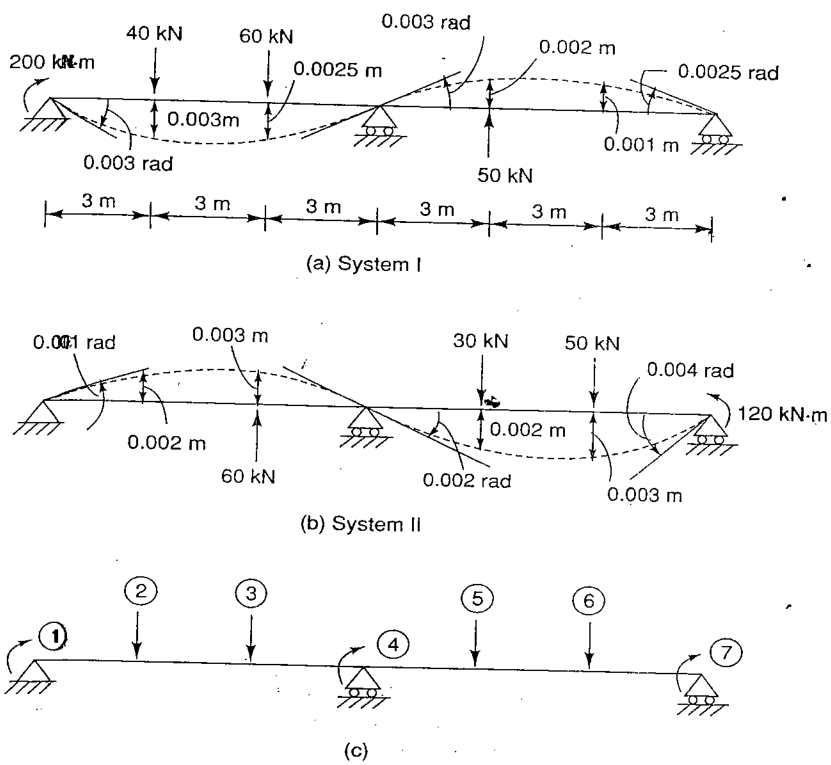


Fig. 2.8

Solution

All the forces and displacements shown in Fig. 2.8(a) and (b) can be expressed by selecting seven coordinates as shown in Fig. 2.7(c). Using these coordinates, the forces and displacements for the two systems can be expressed as shown in Table 2.1.

Table 2.1

System coordinates →	*1*	*2*	*3*	*4*	*5*	*6*	*7*
I Loads, P_I	200	40	60	0	–50	0	0
Displacements, Δ_I	0.003	0.003	0.0025	–0.003	–0.002	–0.001	0.0025
II Loads, P_{II}	0	0	–60	0	30	50	–120
Displacement, Δ_{II}	–0.001	–0.002	–0.003	0.002	0.002	0.003	–0.004

2.5 STRAIN ENERGY EXPRESSIONS

It has been shown in Sec. 1.4 that the members of structural systems may be subjected to axial forces, shear forces, bending moments and twisting moments. These internal forces produce displacements and consequently do work. This work is stored in the structure in the form of strain energy. The expressions for the strain energy may be derived by computing the work done by the internal forces in going through the corresponding displacements.

2.5.1 Strain Energy Due to Axial Force

If S is the axial force acting on an element ds of a structural member, the work done by S in undergoing the axial displacement of the element ~~ds is given~~ by the equation

$$dU = \frac{1}{2} S \text{ (axial displacement of element } ds)$$

$$= \frac{1}{2} S \left(\frac{S ds}{AE} \right)$$

$$= \frac{S^2 ds}{2AE} \tag{2.10}$$

where A = area of cross-section of the member and

E = modulus of elasticity

Hence, the total strain energy of the member on account of axial force S is given by the equation

$$U = \int \frac{S^2 ds}{2AE} \tag{2.11}$$

where integration has to be carried out for the entire length of the member.

For a straight prismatic member of length L and subjected to a constant axial force S, the expression for total strain energy becomes

$$U = \frac{S^2 L}{2AE} \tag{2.12}$$

2.5.2 Strain Energy Due to Shear Force

If Q is the shear force acting on an element ds of a structural member, the work done by Q in undergoing the shear displacement of element ds is given by the equation

$$\begin{aligned} dU &= \frac{1}{2}\, Q \text{ (shear displacement of element } ds) \\ &= \frac{1}{2}\, Q\left(\frac{Q\,ds}{A_r G}\right) \\ &= \left(\frac{Q^2 ds}{2A_r G}\right) \end{aligned} \tag{2.13}$$

where A_r = reduced area of cross-section. It depends on the shape of the cross-section. Values of A_r for some common shapes are given in column 2 of Table 2.2.

G = shear modulus of elasticity.

Hence, the total strain energy of the member on account of shear force Q is given by the equation.

$$U = \int \frac{Q^2 ds}{2A_r\, G} \tag{2.14}$$

where integration has to be carried out for the entire length of the member.

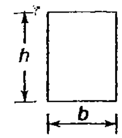
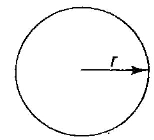
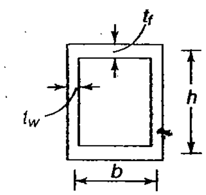
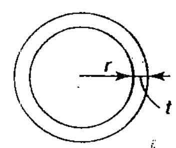
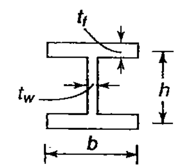
2.5.3 Strain Energy Due to Bending Moment (Flexure)

If M is the bending moment acting on an element ds of a structural member, the work done by M in undergoing the bending displacement of element ds is given by the equation

$$\begin{aligned} dU &= \frac{1}{2}\, M \text{ (bending displacement of element } ds) \\ &= \frac{1}{2}\, M\left(\frac{M\,ds}{EI}\right) \\ &= \frac{M^2 ds}{2\,EI} \end{aligned} \tag{2.15}$$

where I = moment of inertia of the cross-section of the member about the neutral axis.

Table 2.2

Section (1)	A_r (2)	K (3)
	$\frac{A}{1.2}$	$\approx hb^3\left[\frac{1}{3} - 0.21\frac{b}{h}\left(1 - \frac{b^4}{12h^4}\right)\right]$
	$0.9\,A$	$\frac{\pi r^4}{2}$
	$2ht_w$	$\approx 2b^2h^2\,\frac{t_f t_w}{bt_w + ht_f}$
	$\frac{A}{2}$	$\approx 2\pi r^3 t$
	ht_w	$\approx \frac{1}{3}(h t_w^3 + 2bt_f^3)$

Hence, the total strain energy of the member on account of bending moment M is given by the equation

$$U = \int \frac{M^2 ds}{2\,EI} \tag{2.16}$$

where integration has to be carried out for the entire length of the member.

2.5.4 Strain Energy Due to Twisting Moment

If T is the twisting moment acting on an element ds of a structural member, the work done by T in undergoing the torsional displacement of element ds is given by the equation

$$dU = \frac{1}{2}\, T \text{ (torsional displacement of element } ds)$$

$$= \frac{1}{2}\, T \left(\frac{Tds}{GK}\right)$$

$$= \frac{T^2 ds}{2GK} \qquad (2.17)$$

where K = torsion constant of the section. It depends exclusively on the shape of the cross-section. Values of K for some common shapes are given in column 3 of Table 2.2.

Hence, the total strain energy of the member on account of twisting moment T is given by the equation

$$U = \int \frac{T^2 ds}{2GK} \qquad (2.18)$$

where integration has to be carried out for the entire length of the member.

In general, at any cross-section of a structural member there can be six internal force components comprising the axial force S, the biaxial shear forces Q_x and Q_y, the biaxial bending couples M_x and M_y and twisting moment T as discussed in Sec. 1.4. Hence, the strain energy of a member of a rigid-jointed space frame may be expressed by the equation

$$U = \int \frac{S^2 ds}{2AE} + \int \frac{Q_x^2 ds}{2A_{rx}G} + \int \frac{Q_y^2 ds}{2A_{ry}G} + \int \frac{M_x^2 ds}{2EI_x} + \int \frac{M_y^2 ds}{2EI_y} + \int \frac{T^2 ds}{2GK} \qquad (2.19)$$

where A_{rx}, A_{ry} = reduced areas of the cross-section when the shearing occurs in the xz- and yz-planes respectively.

The integration has to be carried out for the entire length of the member.

In the case of a rigid-jointed plane frame loaded in its own plane, while the twisting moments are absent, shear forces Q and bending moments M act in the plane of the frame. Consequently, the strain energy of a member of a rigid-jointed plane frame loaded in its own plane is expressed by the equation

$$U = \int \frac{S^2 ds}{2AE} + \int \frac{Q^2 ds}{2A_r G} + \int \frac{M^2 ds}{2EI} \qquad (2.20)$$

The strain energy due to the axial forces and shear forces is generally small compared to the strain energy due to bending moments. Hence it is a common practice in the analysis of structures to ignore the strain energy due to the axial forces and the shear forces. Consequently, the strain energy of a member of a rigid-jointed frame may be written as

$$U = \int \frac{M^2 ds}{2EI} \qquad (2.21)$$

In pin-jointed frames only axial forces are present. Hence, using Eq. (2.12), the total strain energy of a pin-jointed frame with straight members may be expressed as

$$U = \Sigma \frac{S^2 L}{2AE} \qquad (2.22)$$

Example 2.2

The L-shaped bar shown in Fig. 2.9(a) is of uniform rectangular cross-section, 60 mm × 120 mm. Calculate the total strain energy. Take E = 2 × 10⁵ MPa and G = 0.8 × 10⁵ MPa.

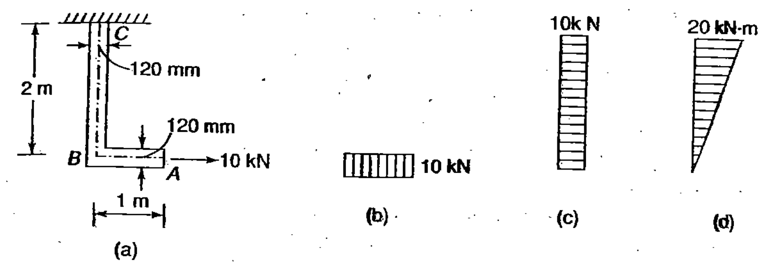


Fig. 2.9

Solution

Considering the free-body diagram of portion AB, it is evident that it carries a constant tensile force of 10 kN. The axial force diagram for portion AB is shown in Fig. 2.9(b). Portion AB is free from shear force, bending moment and twisting moment. Hence, the strain energy of portion AB is given by the equation

$$U_{AB} = \frac{S^2 L}{2AE} = \frac{10^2 \times 1}{2 \times 0.06 \times 0.12 \times 2 \times 10^8} = 3.47 \times 10^{-5} \text{ kN·m}$$

Next, considering the free-body diagram of portion BC, it is evident that this portion of the bar is subjected to shear force and bending moment. The shear force and bending-moment diagrams are shown in Fig. 2.9(c) and (d) respectively. Portion BC does not have any axial force or twisting moment. The strain energy of portion BC due to shear force is given by the equation

$$U_{BC}\text{ (shear)} = \int_B^C \frac{Q^2 ds}{2A_r G} = \int_0^2 \frac{10^2 dx}{2 \times \frac{0.06 \times 0.12}{1.2} \times 0.8 \times 10^8}$$

$$= 20.83 \times 10^{-5} \text{ kN.m}$$

The strain energy of portion BC due to flexure is given by the equation

$$U_{BC}\,(\text{flexure}) = \int_B^C \frac{M^2 ds}{2EI} = \int_0^2 \frac{(10x)^2 dx}{2\times 2\times 10^8 \times \frac{1}{12}\times 0.06\times 0.12^3}$$

$$= 7716.05 \times 10^{-5}\ \text{kN·m}$$

Hence, the total strain energy of the L-shaped bar may be computed as

$$U = (3.47 + 20.83 + 7716.05) \times 10^{-5} = 0.0774035\ \text{kN·m}$$

It may be noted that the strain energies due to axial force and shear force are small as compared to the strain energy due to bending moment (flexure). It is for this reason that in analysing beams and rigid-jointed plane frames, the strain energies due to axial force and shear force are ignored and consequently the total strain energy is taken equal to the strain energy due to flexure.

2.6 PRINCIPLE OF CONSERVATION OF ENERGY

According to the principle of conservation of energy, the total energy of a system remains constant. Energy can neither be produced nor destroyed. It can only be converted from one form to another.

In general, all structures are deformable. Consequently, when external loads act on structures, they deform. The work done by the external loads in undergoing the consequent displacements is stored in the structure in the form of strain energy. As the external loads generally lose their potential energy in undergoing their displacements, the work done by the external loads may be considered as negative and the equal amount of strain energy stored in the structure may be considered as positive. Hence, the total energy of ~~the system~~ system remains constant in accordance with the principle of conservation of energy.

2.7 MAXWELL'S RECIPROCAL THEOREM

Clerk Maxwell's reciprocal theorem is the most important and fundamental theorem based directly on the principle of conservation of energy and the principle of superposition. Most of the important theorems of structural mechanics can be derived from Maxwell's reciprocal theorem. Using the generalized system of coordinates introduced in Sec. 2.4, Maxwell's reciprocal theorem, for any structure which satisfies the basic assumptions enumerated in Sec. 2.2, may be stated as follows :

In a linearly elastic structure in static equilibrium, the displacement at coordinate i due to a unit force acting at coordinate j is equal to the displacement at coordinate j due to a unit force at coordinate i. The theorem may be expressed symbolically as

$$\delta_{ij} = \delta_{ji} \tag{2.23}$$

PROOF

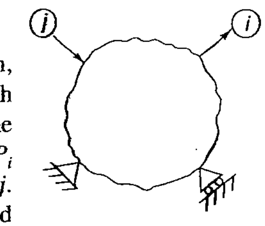


Fig. 2.10

In order to derive Maxwell's reciprocal theorem, consider the structure shown in Fig. 2.10 with coordinates i and j indicated on the figure. Apply the force P_i at coordinate i producing displacement $\delta_{ii}P_i$ at coordinate i and displacement $\delta_{ji}P_i$ at coordinate j. Hence, the work done, when force P_i is applied gradually to the structure, is given by the equation

$$U_1 = \frac{1}{2} P_i(\delta_{ii}P_i)$$

$$= \frac{1}{2} P_i^2\delta_{ii} \tag{a}$$

Next, force P_j is applied to the structure producing displacement $\delta_{ij}P_j$ at coordinate i and displacement $\delta_{jj}P_j$ at coordinate j. Hence, the work done when force P_j is applied gradually to the structure is given by the equation

$$U_2 = \frac{1}{2} P_j(\delta_{jj}P_j) + P_i(\delta_{ij}P_j)$$

$$= \frac{1}{2} P_j^2\delta_{jj} + P_iP_j\delta_{ij} \tag{b}$$

Adding Eqs (a) and (b), the total work done when the forces P_i and P_j are applied gradually in that order is given by the equation

$$U = U_1 + U_2 = \frac{1}{2} P_i^2\delta_{ii} + \frac{1}{2} P_j^2\delta_{jj} + P_iP_j\delta_{ij} \tag{c}$$

This work is stored in the structure in the form of strain energy. If the response of the structure is linear and the principle of superposition is valid, the sequence of loading is immaterial. Hence, the strain energy should remain unchanged if the sequence of loading is reversed. Thus, applying force P_j first, the work done is given by the equation

$$\bar{U}_1 = \frac{1}{2} P_j(\delta_{jj}P_j) = \frac{1}{2} P_j^2\delta_{jj} \tag{d}$$

Next apply force P_i. The additional work done is given by the equation

$$\bar{U}_2 = \frac{1}{2} P_i(\delta_{ii}P_i) + P_j(\delta_{ji}P_i)$$

$$= \frac{1}{2} P_i^2\delta_{ii} + P_iP_j\delta_{ji} \tag{e}$$

Adding Eqs (d) and (e), the total work done when the forces P_j and P_i are applied gradually in that order, is given by the equation

$$U = \bar{U}_1 + \bar{U}_2 = \frac{1}{2}P_i^2\delta_{ii} + \frac{1}{2}P_j^2\delta_{jj} + P_iP_j\delta_{ji} \tag{f}$$

As the total energy does into depend on the sequence of loading, it follows that the right hand sides of Eqs (c) and (f) should be equal. Hence,

$$\delta_{ij} = \delta_{ji} \tag{2.23}$$

The reciprocal theorem, expressed symbolically by Eq. (2.23), establishes the reciprocity of the displacements caused by unit forces. A similar reciprocity exists with regard to forces caused by unit displacements. A reciprocal theorem in terms of the forces may be expressed as follows:

In a linearly elastic structure in static equilibrium, the force at coordinate i due to a unit displacement at coordinate j is equal to the force at coordinate j due to a unit displacement at coordinate i. The theorem may be expressed symbolically as

$$k_{ij} = k_{ji} \tag{2.24}$$

PROOF

Consider the structure shown in Fig. 2.10 with coordinates i and j indicated on the figure. Give a displacement Δ_i at coordinate i without any displacement at coordinate j. Evidently, the forces at coordinates i and j are $k_{ii}\Delta_i$ and $k_{ji}\Delta_i$. Hence, the work done due to displacement Δ_i may be written as

$$U_1 = \frac{1}{2}(k_{ii}\Delta_i)\Delta_i = \frac{1}{2}k_{ii}\Delta_i^2 \tag{g}$$

Next, give a displacement Δ_j at coordinate j without any displacement at coordinate i. The additional forces at coordinates i and j are $k_{ij}\Delta_j$ and $k_{jj}\Delta_j$ respectively. As in this step the displacement is given only at coordinate j, no additional work is done at coordinate i. In computing the work done at coordinate j, it may be noted that the force $k_{ji}\Delta_i$ is already present at coordinate j. The work done by this force in undergoing displacement Δ_j is $(k_{ji}\Delta_i)\Delta_j$. The work done by additional force $k_{jj}\Delta_j$ at coordinate j in undergoing displacement Δ_j is $\frac{1}{2}(k_{jj}\Delta_j)\Delta_j$. Thus, the total additional work done when displacement Δ_j is given at coordinate j is given by the equation

$$\begin{aligned} U_2 &= (k_{ji}\Delta_i)\Delta_j + \frac{1}{2}(k_{jj}\Delta_j)\Delta_j \\ &= k_{ji}\Delta_i\Delta_j + \frac{1}{2}k_{jj}\Delta_j^2 \end{aligned} \tag{h}$$

Adding Eqs. (g) and (h), the total work done on account of displacements Δ_i and Δ_j applied in that sequence is given by the equation

$$U = U_1 + U_2 = \frac{1}{2}k_{ii}\Delta_i^2 + \frac{1}{2}k_{jj}\Delta_j^2 + k_{ji}\Delta_i\Delta_j \tag{i}$$

This work is stored in the structure in the form of strain energy. Next, the displacements Δ_i and Δ_j may be given in reverse order. Giving Δ_j first, without any displacement at coordinate i, the work done is given by the equation

$$\bar{U}_1 = \frac{1}{2}k_{jj}\Delta_j^2 \tag{j}$$

Next, give the displacement Δ_i at coordinate i without any displacement at coordinate j. The additional work done at this stage is given by the equation

$$\bar{U}_2 = (k_{ij}\Delta_j)\Delta_i + \frac{1}{2}k_{ii}\Delta_i^2 \tag{k}$$

Adding Eqs (j) and (k), the total work done is given by the equation

$$U = \bar{U}_1 + \bar{U}_2 = \frac{1}{2}k_{jj}\Delta_j^2 + \frac{1}{2}k_{ii}\Delta_i^2 + k_{ij}\Delta_j\Delta_i \tag{l}$$

As the energy stored is independent of the sequence in which the displacements are given, it follows that the right hand sides of Eqs (i) and (l) should be equal. Hence,

$$k_{ij} = k_{ji} \tag{2.24}$$

It may be noted that the words "force" and "displacement" have been used here in a generalized sense so as to include couple and rotation also. For instance, applying the Maxwell's reciprocal theorem to the beam shown in Fig. 2.11, it follows that the deflection at coordinate i due to a unit couple at coordinate j is equal to the rotation at coordinate j due to a unit load at coordinate i. Particular care must also be taken in respect of the units when the Maxwell's reciprocal theorem is stated in the mixed form which involves deflection as well as rotation. Common units should be used for force, couple and deflection. For instance, if the couple is expressed in kN·m units then the force must be expressed in kN and deflection in m. The rotation should be expressed in radians.

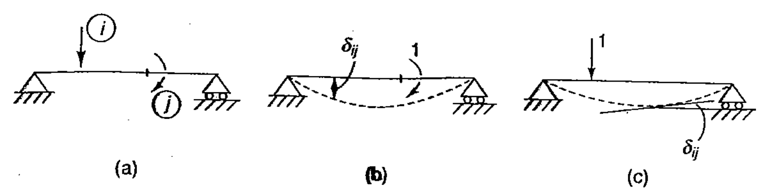


Fig. 2.11

Example 2.3

Calculate the displacement at coordinate 2 of the two-hinged arch shown in Fig. 2.12 due to a load of 100 kN acting at coordinate 1 if a load of 500 kN·m at coordinate 2 produces a displacement of 0.005 m at coordinate 1.

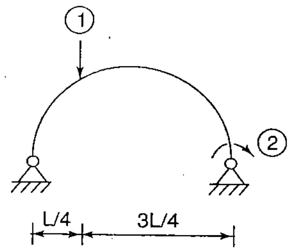


Fig. 2.12

Solution

Displacement at coordinate 1 due to a load of 1 kN·m at coordinate 2,

$$\delta_{12} = \frac{0.005}{500} = 0.00001 \text{ m}$$

Using Maxwell's reciprocal theorem, displacement at coordinate 2 due to a load of 1 kN at coordinate 1,

$$\delta_{21} = \delta_{12} = 0.00001 \text{ radian}$$

Hence, the displacement at coordinate 2 due to a load of 100 kN at coordinate 1,

$$\Delta_2 = \delta_{21} \times 100 = 0.001 \text{ radian}$$

2.8 GENERALIZED RECIPROCAL THEOREM

The generalized reciprocal theorem, commonly known as Betti's theorem, states that *if a linearly elastic structure in static equilibrium is subjected to two systems of forces, then the virtual work done by the first system of forces in undergoing the displacements caused by the second system of forces is equal to the virtual work done by the second system of forces in undergoing the displacements caused by thefirst system offorces.* In the statement of the Betti's theorem, the work has been termed as *virtual* (imaginary). When a set of forces undergo displacements caused by themselves, they produce real work. On the other hand, when a set of forces undergo displacements caused by another set of forces, which may be real or imaginary, the work is termed as virtual. Betti's generalized reciprocal theorem can be derived readily from Maxwell's reciprocal theorem as follows:

Let P_1, P_2 and P_3 be the forces of the first system and Δ_1, Δ_2 and Δ_3 be the corresponding displacements at coordinates 1, 2 and 3. Similarly, let P_1', P_2' and P_3' be the forces of the second system and Δ_1', Δ_2' and Δ_3' be the corresponding displacements. The virtual work done by the first system of forces in undergoing the displacements caused by the second system of forces is given by the equation

$$U = P_1\Delta_1' + P_2\Delta_2' + P_3\Delta_3' \tag{a}$$

Using Eq. (2.5), the displacements Δ_1', Δ_2' and Δ_3' which are caused by forces P_1', P_2' and P_3', may be expressed by the equations

$$\begin{aligned} \Delta_1' &= \delta_{11} P_1' + \delta_{12} P_2' + \delta_{13} P_3' \\ \Delta_2' &= \delta_{21} P_1' + \delta_{22} P_2' + \delta_{23} P_3' \\ \Delta_3' &= \delta_{31} P_1' + \delta_{32} P_2' + \delta_{33} P_3' \end{aligned} \tag{b}$$

Substituting from Eq. (b) into Eq. (a),

$$\begin{aligned} U = {} & P_1(\delta_{11} P_1' + \delta_{12} P_2' + \delta_{13} P_3') \\ & + P_2(\delta_{21} P_1' + \delta_{22} P_2' + \delta_{23} P_3') \\ & + P_3(\delta_{31} P_1' + \delta_{32} P_2' + \delta_{33} P_3') \end{aligned} \tag{c}$$

Similarly, the virtual work done by the second system of forces in undergoing the displacements caused by the first system of forces is given by the equation

$$\begin{aligned} U &= P_1' \Delta_1 + P_2' \Delta_2 + P_3' \Delta_3 \\ &= P_1'(\delta_{11} P_1 + \delta_{12} P_2 + \delta_{13} P_3) \\ &\quad + P_2'(\delta_{21} P_1 + \delta_{22} P_2 + \delta_{23} P_3) \\ &\quad + P_3'(\delta_{31} P_1 + \delta_{32} P_2 + \delta_{33} P_3) \end{aligned} \tag{d}$$

The term-by term comparison of the right hand sides of Eqs (c) and (d) shows that they are identical provided $\delta_{ij} = \delta_{ji}$. Hence, the theorem is proved for the case in which each of the two systems comprises three forces. The theorem can similarly be extended to any number of forces. The generalized reciprocal theorem may be expressed by the equation

$$\Sigma P\Delta' = \Sigma P'\Delta \tag{2.25}$$

Example 2.4

A three-span continuous beam is subjected to two systems of forces as shown in Fig. 2.13(a) and (b). Calculate the displacement under the force of 90 kN in System II.

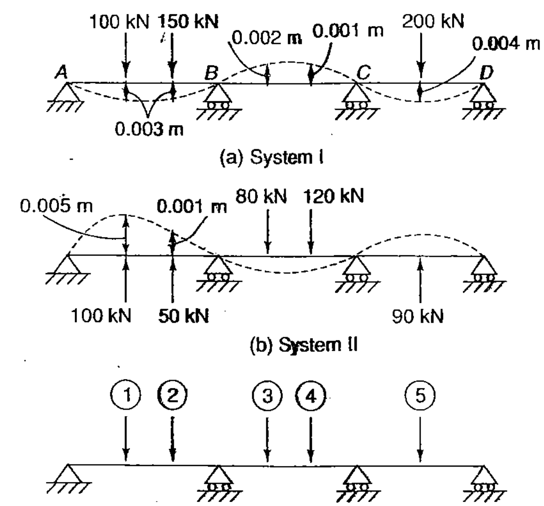


Fig. 2.13

Solution

Choosing the generalized system of coordinates as shown in Fig. 2.13(c), the given forces and displacements for the two systems are shown in Table 2.3.

Table 2.3

System ↓	Coordinates	1	2	3	4	5
I	P	100	150	0	0	200
	Δ	0.003	0.003	– 0.002	– 0.001	0.004
II	P'	– 100	– 50	80	120	– 90
	Δ'	– 0.005	– 0.001	–	–	?

$$\Sigma P\Delta' = 100(-0.005) + 150(-0.001) + 200(\Delta'_5)$$
$$= -0.65 + 200\Delta'_5$$
$$\Sigma P'\Delta = -100(0.003) - 50(0.003) + 80(-0.002) + 120(-0.001) - 90(0.004)$$
$$= -1.09$$

Using Eq. (2.25),

$$-0.65 + 200\,\Delta'_5 = -1.09$$

or

$$\Delta'_5 = -0.0022 \text{ m}$$

Example 2.5

For the portal frame shown in Fig. 2.14 with generalized coordinates indicated on it, forces and displacements for the two systems of forces are given in Table 2.4. Calculate the displacement Δ'_9 under the second system of forces.

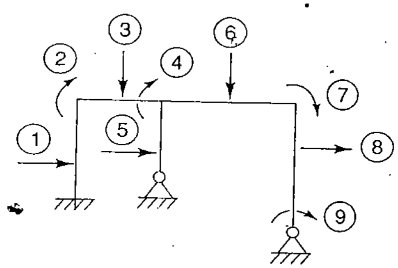


Fig. 2.14

Solution

$$\Sigma P\Delta' = 50(0.003) + 30(0.001) + 10(0.002) - 40(-0.001) + 20\Delta'_9$$
$$= 0.24 + 20\Delta'_9$$
$$\Sigma P'\Delta = 100(0.001) + 40(0.002) + 50(0.002) + 20(0.001) + 10(0.002)$$
$$= 0.32$$

Using Eq. (2.25),

$$0.24 + 20\Delta'_9 = 0.32$$

or

$$\Delta'_9 = 0.004 \text{ radian}$$

Table 2.4

System ↓	Coordinates→	1	2	3	4	5	6	7	8	9
I	P	50 kN	0	30 kN	0	0	10 kN	0	–40 kN	20 kN·m
	Δ	–	0.001 rad	0.002 m	0.002 rad	0.001 m	–	0.002 rad	–	–
II	P'	0	100 kN·m	40 kN	50 kN·m	20 kN	0	10 kN·m	0	0
	Δ'	0.003 m	–	0.001 m	–	–	0.002 m	–	–0.001 m	?

Table 2.5

System ↓	Coordinates→	1	2	3	4	5	6	7	8	9
I	P	20 kN	40 kN	30 kN	0	0	10 kN	–	–	–
	Δ	–	–	0.002 m	0.002 m	0.003 m	–	0	0	0
II	P'	0	0	20 kN	40 kN	30 kN	0	–	–	–
	Δ'	0.001 m	0.002 m	0.003 m	–	–	?	0	0	0

Example 2.6

For the pin-jointed plane frame shown in Fig. 2.15 with generalized coordinates indicated on it, the forces and displacements for the two systems of forces are given in Table 2.5. Calculate the displacement Δ'_6 under the second system of forces.

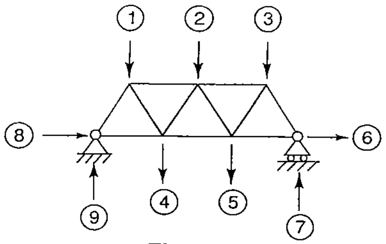


Fig. 2.15

Solution

$$\Sigma P\Delta' = 20(0.001) + 40(0.002) + 30(0.003) + 10(\Delta'_6)$$
$$= 0.19 + 10\,\Delta'_6$$
$$\Sigma P'\Delta = 20(0.002) + 40(0.002) + 30(0.003)$$
$$= 0.21$$

Using Eq. (2.25),

$$0.19 + 10\,\Delta'_6 = 0.21$$
$$\Delta'_6 = 0.002 \text{ m}$$

2.9 PRINCIPLE OF VIRTUAL WORK

Figure 2.16(a) shows a structure acted upon by a system of external loads $P_1, P_2, \ldots, P_j \ldots, P_n$ producing displacements $\Delta_1, \Delta_2, \ldots, \Delta_j, \ldots, \Delta_n$ at coordinates 1, 2, ..., j, ..., n. These forces produce internal principal stresses σ_x, σ_y and σ_z and the principal strains, ε_x, ε_y and ε_z on an infinitesimal element of volume, $dv = dx\,dy\,dz$ as shown in the figure. In Fig. 2.16 (b) the structure is acted upon by another system of external loads P'_1, P'_2, ..., P'_j, ..., P'_n producing displacements $\Delta'_1, \Delta'_2, \ldots, \Delta'_j, \ldots, \Delta'_n$ at coordinates 1, 2, ..., j, ..., n. These forces produce internal principal stresses σ'_x, σ'_y and σ'_z and the principal strains ε'_x, ε'_y and ε'_z on the infinitesimal element. In accordance with the principle of conservation of energy, the external work is equal to the strain energy. Hence,

$$\Sigma P\Delta = \int_v (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)\,dv \qquad \text{(a)}$$

$$\Sigma P'\Delta' = \int_v (\sigma'_x\varepsilon'_x + \sigma'_y\varepsilon'_y + \sigma'_z\varepsilon'_z)\,dv \qquad \text{(b)}$$

where the summation on the left side of the equation should include all the external loads and the integration on the right side should cover the entire volume of the structure.

If the system of forces and displacements shown in Fig. 2.16(a) are considered real and those shown in Fig. 2.16(b) are considered virtual (imaginary), then in accordance with the principal of virtual work,

$$\Sigma P\Delta' = \int_v (\sigma_x\varepsilon'_x + \sigma_y\varepsilon'_y + \sigma_z\varepsilon'_z)\,dv \qquad (2.26\text{a})$$

The left side of the equation represents the external virtual work done by real loads P in undergoing virtual displacements Δ'. The right side of the equation represents the internal virtual work done by real principal stresses σ_x, σ_y and σ_z in undergoing virtual principal strains ε'_x, ε'_y, ε'_z, respectively. Consequently, the principle of virtual work may be stated as follows:

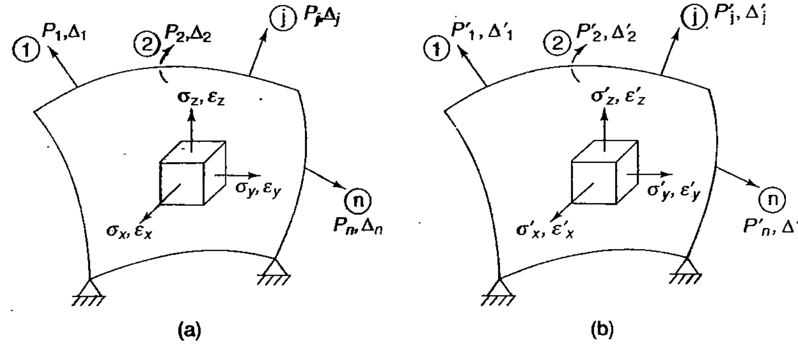


Fig. 2.16

In a linearly elastic structure in static equilibrium, the external virtual work done by the set of real loads P in undergoing virtual displacements Δ' is equal to the internal virtual work done by real principal stresses, σ_x, σ_y, and σ_z, in undergoing virtual principal strains ε'_y, ε'_y, and ε'_z. Alternatively, the principle of virtual work may also be expressed by the equation

$$\Sigma P'\Delta = \int_v (\sigma'_x\varepsilon_x + \sigma'_y\varepsilon_y + \sigma'_z\varepsilon_z)\,dv \qquad (2.26\text{b})$$

In the case of skeletal structures, the internal stresses at any cross-section give rise to internal forces comprising axial force, the biaxial shear forces, the biaxial bending moments and the twisting moment as described in Sec. 1.4. Let the real internal forces at any cross-section in Fig. 2.16(a) be represented by F producing the corresponding internal displacements D. Similarly, let F' and D' represent respectively the virtual internal forces and the corresponding internal displacements in Fig. 2.16(b). Then the principle of virtual work expressed by Eqs. (2.26a) and (2.26b) may now be written in terms of internal forces and displacements as follows:

$$\Sigma P\Delta' = \int FD'\,ds \qquad (2.26\text{c})$$

$$\Sigma P'\Delta = \int F'D\,ds \qquad (2.26\text{d})$$

where the integration on the right side should be carried out to cover the entire structure.

It may be noted that Betti's theorem (Sec. 2.8) when applied to external forces gives

$$\Sigma P\Delta' = \Sigma P'\Delta$$

Similarly, Betti's theorem when applied to internal forces gives

$$\int FD'\,ds = \int F'D\,ds$$

Thus combining Betti's theorem with the principle of virtual work,

$$\Sigma P\Delta' = \Sigma P'\Delta = \int FD'\,ds = \int F'D\,ds$$

2.10 CASTIGLIANO'S THEOREM

Castigliano's theorem provides a powerful tool for the analysis of statically indeterminate structures. The theorem is based on the energy concept and can be derived readily from Betti's generalized reciprocal theorem.

Castigliano's Theorem (Part I)

This theorem, also known as Castiglàno's first theorem, states that *the partial derivative of the strain energy of a linearly elastic structure expressed in terms of displacements with respect to any displacement Δ_j at coordinate j is equal to the force P_j at coordinate j. This theorem may be expressed symbolically as*

$$\frac{\partial U}{\partial \Delta_j} = P_j \qquad (2.27a)$$

PROOF

Consider a series of forces $P_1, P_2, ..., P_j ..., P_n$ acting on a structure at coordinates 1, 2, ..., *j*, ..., *n* producing displacements $\Delta_1, \Delta_2, ..., \Delta_j, ..., \Delta_n$. These loads and displacements may be assumed to constitute the first system in the generalized reciprocal theorem as shown in Table 2.6. Now impose a small increment $\delta\Delta_j$ to the displacement at coordinate *j* keeping the displacements at all other coordinates unchanged. As a consequence the increments in the force are $\delta P_1, \delta P_2, ..., \delta P_j ..., \delta P_n$. The increment in displacement at coordinate *j* and the consequent increments in loads at all the coordinates may be assumed to constitute the second system of the generalized reciprocal theorem as shown in Table 2.6. Applying the generalized reciprocal theorem, Eq. (2.25), to the two systems shown in the table,

$$P_j\,\delta\Delta_j = \Delta_1\delta P_1 + \Delta_2\delta P_2 + ... + \Delta_j\delta P_j + ... + \Delta_n\delta P_n$$

Table 2.6

System ↓	Coordinates→	1	2	...	...	j	...	...	n
I	P	P_1	P_2	...	...	P_j	...	...	P_n
	Δ	Δ_1	Δ_2	...	...	Δ_j	...	...	Δ_n
II	P'	δP_1	δP_2	...	...	δP_j	...	...	δP_n
	Δ'	0	0	0	0	$\delta\Delta_j$	0	0	0

Using Eqs (2.2) and (2.3), the above equation may be rewritten as

$$\frac{\delta U}{\delta \Delta_j} = P_j$$

In the limit $\delta\Delta_j \to 0$, the above equation becomes

$$\frac{\partial U}{\partial \Delta_j} = P_j \qquad (2.27a)$$

In Eq. (2.27a), the partial derivative of strain energy has been used because the change of strain energy due to an increment in displacement at coordinate *j* only has been considered keeping all other displacements unchanged.

Castigliano's Theorem (Part II)

This theorem, also known as Castigliano's second theorem, states that the *partial derivative of the strain energy of a linearly elastic structure expressed in terms of forces with respect to any force P_j at coordinate j is equal to the displacement Δ_j at coordinate j. This theorem may be expressed symbolically as*

$$\frac{\partial U}{\partial P_j} = \Delta_j \qquad (2.27b)$$

PROOF

Consider a series of loads $P_1, P_2, ..., P_j, ..., P_n$ acting on a structure at coordinates 1, 2, ..., *j*, ..., *n* producing displacements $\Delta_1, \Delta_2, ..., \Delta_j ..., \Delta_n$. These forces and displacements may be assumed to constitute the first system in the generalized reciprocal theorem as shown in Table 2.7.

Table 2.7

System ↓	Coordinates→	1	2	...	...	j	...	...	n
I	P	P_1	P_2	...	...	P_j	....	...	P_n
	Δ	Δ_1	Δ_2	...	...	Δ_j	...	...	Δ_n
II	P'	0	0	0	0	δP_j	0	0	0
	Δ'	$\delta\Delta_1$	$\delta\Delta_2$	...	...	$\delta\Delta_j$	...	...	$\delta\Delta_n$

Now impose a small increment δP_j to the load at coordinate *j* keeping all other forces unchanged. As a consequence, the increments in the displacements are $\delta\Delta_1, \delta\Delta_2, ..., \delta\Delta_j, ..., \delta\Delta_n$. The increment in the load at coordinate *j* and consequent increments in displacements at all the coordinates may be assumed to constitute the second system of the generalized reciprocal theorem as shown in Table 2.7. Applying the generalized reciprocal theorem, Eq. (2.25), to the two systems shown in the table,

$$P_1\delta\Delta_1 + P_2\delta\Delta_2 + ... + P_j\delta\Delta_j + ... + P_n\delta\Delta_n = \Delta_j\delta P_j$$

Using Eq. (2.1), the above equation may be rewritten as

$$\frac{\delta U}{\delta P_j} = \Delta_j$$

In the limit $\delta P_j \to 0$, the above equation becomes

$$\frac{\partial U}{\partial P_j} = \Delta_j \qquad (2.27b)$$

In Eq. (2.27b), the partial derivative of strain energy has been used because the change of strain energy due to an increment in the load at coordinate j only has been considered keeping all other loads unchanged.

As discussed in Sec. 2.5, the strain energy in the case of rigid-jointed plane frames is predominantly due to bending moment and is given by Eq. (2.21). Substituting Eq. (2.21) into Eq. (2.27b), Castiglino's theorem (Part II) may be rewritten as

$$\int \frac{M \dfrac{\partial M}{\partial P_j} ds}{EI} = \Delta_j \qquad (2.28)$$

Similarly, in the case of pin-jointed frames, Castigliano's theorem (Part II) may be rewritten as

$$\int \frac{S \dfrac{\partial S}{\partial P_j} ds}{AE} = \Delta_j \qquad (2.29)$$

Equations (2.28) and (2.29) have been extensively used for the determination of internal forces in statically determinate structures.

If there is no load acting at coordinate j, at which the displacement is required, an imaginary or *dummy load* P_j may be assumed to act at coordinate j in addition to the real loads already acting on the structure. The expression for bending moment M may be written down and differentiated to obtain $\partial M/\partial P_j$. While substituting for M in Eq. (2.28), P_j should be put equal to zero because the load at coordinate j is zero. This method is known as *dummy load method.*

Example 2.7

For the simply supported beam shown in Fig. 2.17(a), calculate the displacements at coordinates 1, 2 and 3 using Castigliano's theorem.

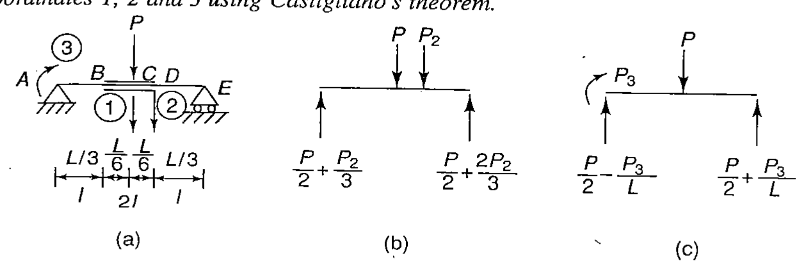


Fig. 2.17

Solution

Referring to coordinates in Fig. 2.17(a),

$$P_1 = P$$
$$P_2 = P_3 = 0$$

(i)
$$\Delta_1 = \int \frac{M \dfrac{\partial M}{\partial P_1} dx}{EI} = \int \frac{M \dfrac{\partial M}{\partial P} dx}{EI}$$

Using Table 2.8,

$$\Delta_1 = \int_0^{L/3} \frac{\dfrac{Px}{2}\cdot\dfrac{x}{2}dx}{EI} + \int_{L/3}^{L/2} \frac{\dfrac{Px}{2}\cdot\dfrac{x}{2}dx}{2EI} + \int_{L/3}^{L/2} \frac{\dfrac{Px}{2}\cdot\dfrac{x}{2}dx}{2EI} + \int_0^{L/3} \frac{\dfrac{Px}{2}\cdot\dfrac{x}{2}dx}{EI}$$

$$= \frac{35}{2592}\frac{PL^3}{EI}$$

Table 2.8

Portion of beam→	*AB*	*BC*	*CD*	*DE*
I	I	$2I$	$2I$	I
Origin	A	A	E	E
Limits	0 to $L/3$	$L/3$ to $L/2$	$L/3$ to $L/2$	0 to $L/3$
M	$(P/2)x$	$(P/2)x$	$(P/2)x$	$(P/2)x$
$\dfrac{\partial M}{\partial P}$	$\dfrac{x}{2}$	$\dfrac{x}{2}$	$\dfrac{x}{2}$	$\dfrac{x}{2}$

(ii)
$$\Delta_2 = \int \frac{M \dfrac{\partial M}{\partial P_2} dx}{EI}$$

As there is no load at coordinate 2, apply a dummy load P_2 at coordinate 2 as shown in Fig. 2.17(b).

Table 2.9

Portion of beam→	*AB*	*BC*	*CD*	*DE*
I	I	$2I$	$2I$	I
Origin	A	A	E	E
Limits	0 to $L/3$	$L/3$ to $L/2$	$L/3$ to $L/2$	0 to $L/3$
M	$\dfrac{P}{2}x + \dfrac{P_2}{3}x$	$\dfrac{P}{2}x + \dfrac{P_2}{3}x$	$\dfrac{P}{2}x + \dfrac{2}{3}P_2x - P_2\left(x - \dfrac{L}{3}\right)$	$\dfrac{P}{2}x + \dfrac{2}{3}P_2x$

(*Contd*)

(Contd)

Portion of beam→	AB	BC	CD	DE
$\dfrac{\partial M}{\partial P_2}$	$\dfrac{x}{3}$	$\dfrac{x}{3}$	$\left(\dfrac{L-x}{3}\right)$	$\dfrac{2}{3}x$
M when $P_2 = 0$	$\dfrac{P}{2}x$	$\dfrac{P}{2}x$	$\dfrac{P}{2}x$	$\dfrac{P}{2}x$

Using Table 2.9,

$$\Delta_2 = \int_0^{L/3} \frac{\frac{Px}{2}\cdot\frac{x}{3}\,dx}{EI} + \int_0^{L/2} \frac{\frac{Px}{2}\cdot\frac{x}{3}\,dx}{2EI} + \int_{L/3}^{L/2} \frac{\frac{Px}{2}\cdot\left(\frac{L-x}{3}\right)dx}{2EI} + \int_0^{L/3} \frac{\frac{Px}{2}\cdot\frac{2x}{3}\,dx}{EI}$$

$$= \frac{31}{2592} = \frac{PL^3}{EI}$$

(iii) $$\Delta_3 = \int \frac{M\dfrac{\partial M}{\partial P_3}dx}{EI}$$

As there is no load at coordinate 3, apply a dummy load P_3 at coordinate 3 as shown in Fig. 2.17(c).

Table 2.10

Portion of beam→	AB	BC	CD	DE
I	I	$2I$	$2I$	I
Origin	A	A	E	E
Limits	0 to $L/3$	$L/3$ to $L/2$	$L/3$ to $L/2$	0 to $L/3$
M	$\left(\dfrac{P}{2+P_3} - \dfrac{P_3}{L}\right)x$	$\left(\dfrac{P}{2+P_3} - \dfrac{P_3}{L}\right)x$	$\left(\dfrac{P}{2+P_3} + \dfrac{P_3}{L}\right)x$	$\left(\dfrac{P}{2+P_3} + \dfrac{P_3}{L}\right)x$
$\dfrac{\partial M}{\partial P_3}$	$\left(1-\dfrac{x}{L}\right)$	$\left(1-\dfrac{x}{L}\right)$	$\dfrac{x}{L}$	$\dfrac{x}{L}$
M when $P_3 = 0$	$\dfrac{P}{2}x$	$\dfrac{P}{2}x$	$\dfrac{P}{2}x$	$\dfrac{P}{2}x$

Using Table 2.10,

$$\Delta_3 = \int_0^{L/3} \frac{\frac{P}{2}x\left(1-\frac{x}{L}\right)dx}{EI} + \int_{L/3}^{L/2} \frac{\frac{P}{2}x\left(1-\frac{x}{L}\right)dx}{2EI} + \int_{L/3}^{L/2} \frac{\frac{P}{2}x\cdot\frac{x}{L}\,dx}{2EI} + \int_0^{L/3} \frac{\frac{P}{2}x\cdot\frac{x}{L}\,dx}{EI}$$

$$= \frac{13}{288}\frac{PL^2}{EI}$$

2.11 MINIMUM ENERGY THEOREM

An infinite number of statically admissible solutions are possible for any statically indeterminate structure. Of these infinite solutions, the correct solution is the one which makes the strain energy of the structure minimum.

This is known as the theorem of minimum energy.

Consider a redundant reaction R_j in a statically indeterminate structure. The displacement Δ_j is equal to zero in the case of an unyielding support. Consequently, applying Castigliano's theorem (Part II),

$$\frac{\partial U}{\partial R_j} = \Delta_j = 0$$

It follows that, in order to make the deformation at coordinate j consistent with the support condition, the strain energy U must assume an extreme value. It can be shown that this extreme value is the minimum value. Thus, the correct solution of a statically indeterminate structure is statically admissible, kinematically consistent and makes the strain energy of the structure minimum. The theorem of minimum energy may, therefore, be restated as follows:

The correct distribution of internal forces in a statically indeterminate structure is the one which is both statically admissible and kinematically consistent and makes the strain energy of the structure minimum.

2.12 UNIT-LOAD METHOD

An elegant procedure for the determination of displacements in structures is provided by the unit-load method. In the case of linear response, the bending moment at any cross-section of a rigid-joined plane frame increases proportionately with the applied loads. The linear relationship between bending moment and any load P_j is represented by the straight line OA in Fig. 2.18. The slope of the straight line OA is evidently equal to $\partial M/\partial P_j$. From the figure it is evident that

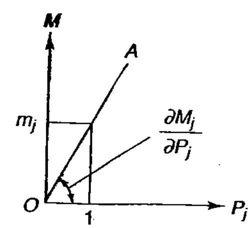


Fig. 2.18

$$\frac{\partial M}{\partial P_j} = m_j \tag{2.30}$$

where m_j = bending moment due to a unit force at coordinate j.

Hence, Eq. (2.28) may be rewritten as

$$\Delta_j = \int \frac{Mm_j\,ds}{EI} \tag{2.31}$$

In the case of a pin-jointed frame, the axial force S in any member increases proportionately with the applied load P_j. Hence

$$\frac{\partial S}{\partial P_j} = s_j \tag{2.32}$$

where s_j = force in the member due to a unit force at coordinate j.

Using this equation, Eq. (2.29) can be rewritten as

$$\Delta_j = \int \frac{S s_j \, ds}{AE} \tag{2.33}$$

As the axial force in any member of a pin-jointed frame is constant throughout its length, Eq. (2.33) can be further written as

$$\Delta_j = \sum \frac{S \, s_j \, L}{AE} \tag{2.34}$$

Using Eq. (2.31), the displacement at coordinate i due to a unit force at coordinate j in a rigid-jointed plane frame can be written as

$$\delta_{ij} = \int \frac{m_i m_j \, ds}{EI} \tag{2.35}$$

Similarly, in the case of a pin-jointed frame, using Eq. (2.34), the displacement at coordinate i due to a unit force at coordinate j can be written as

$$\delta_{ij} = \sum \frac{s_i s_j L}{AE} \tag{2.36}$$

The displacement Δ_j and flexibility coefficient δ_{ij} for the beams and rigid-jointed frames can be calculated by Eqs (2.31) and (2.35).

Equations (2.31) and (2.33) can be derived readily with the help of the principle of virtual work. Let the system of virtual loads shown in Fig. 2.16(b) comprise only a unit load at coordinate j so that $P_j = 1$. In this case, Eq. (2.26d) which is based on the principle of virtual work can be written as

$$1 \times \Delta_j = \int F'D \, ds \tag{a}$$

In the case of a structure, such as a rigid-jointed plane frame loaded in its own plane, in which the strain energy due to flexure alone is significant,

$$F' = m_j \tag{b}$$

and

$$D = \frac{M \, ds}{EI} \tag{c}$$

To establish Eq. (b) it may be noted that virtual internal force F' is bending moment m_j due to a unit virtual load at coordinate j. Also, the real displacement D is caused by internal force F. As the strain energy due to flexure is predominant, only flexural displacement Md_s/EI should be taken as D. Equation (2.31) follows from substituting Eqs (b) and (c) into Eq. (a). In a similar manner Eq. (2.33) for the pin-jointed frames can be established.

An alternative approach is to use the semi-graphical procedure known as *method of diagram-multiplication.* In Eq. (2.31), the term $M\,ds$ represents a small element of the M-diagram and m_j represents the corresponding ordinate of the m_j bending-moment diagram. Hence, displacement Δ_j can be expressed as

$$\Delta_j = \int \frac{M \, m_j \, ds}{EI} = \frac{A_M \, \bar{m}_j}{EI} \tag{2.37}$$

where A_M = area of the M-diagram
$\bar{m}_j$ = ordinate of m_j bending-moment diagram located at the centroid of M-diagram.

In case the m_j bending-moment diagram is discontinuous or EI is not constant, the structure should be divided into a sufficient number of parts so that m_j is continuous and EI is constant in each part of the structure. Consequently, the M-diagram should also be divided correspondingly. Displacement Δ_j is then obtained by using the equation

$$\Delta_j = \sum \frac{A_M \, \bar{m}_j}{EI} \tag{2.38}$$

where the summation should be carried out to include all the parts. The products of A_M and $\bar{m}_j$ for the common shapes of bending-moment diagrams are shown in Table 2.11. This table can be readily used for the determination of displacement Δ_j. With the help of this table, product $A_M \bar{m}_j$ may be read directly for any given combination of the shapes of M and m_j bending-moment diagrams. Similarly, Eq. (2.35) can be rewritten as

$$\delta_{ij} = \int \frac{m_i m_j \, ds}{EI} = \frac{A_{m_i} \bar{m}_j}{EI} \tag{2.39}$$

where A_{m_i} = area of the m_i-diagram.

Example 2.8

For the simply supported beam shown in Fig. 2.19(a) with generalized coordinates indicated on it, calculate δ_{11}, δ_{21} and δ_{31}. Hence, calculate the displacements Δ_1, Δ_2 and Δ_3 if a load P acts at the centre of the beam.

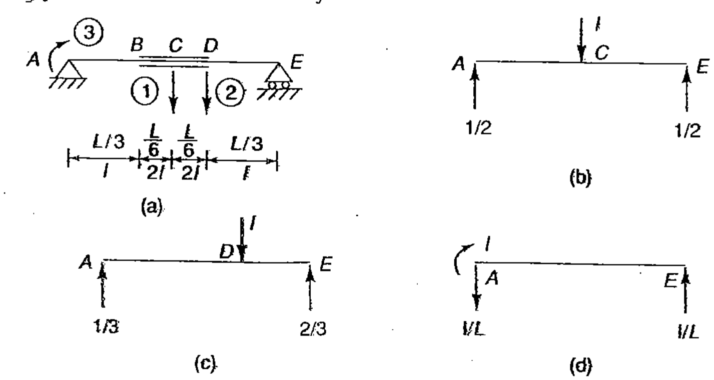
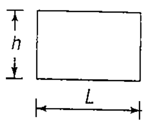
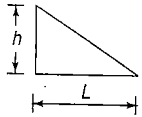
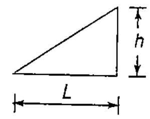
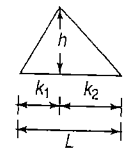
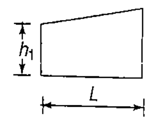
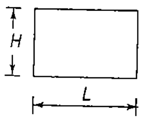
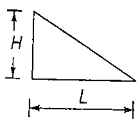
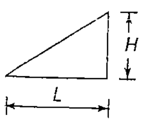


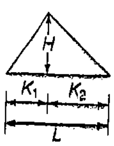
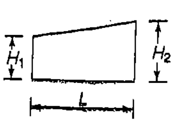
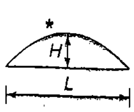
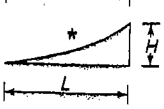
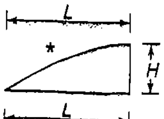
Fig. 2.19

Table 2.11

m_j-diagram → / M-diagram or m_i-diagram ↓	 (1)	 (2)	 (3)	 (4)	 (5)
 (6)	hHL	$\frac{1}{2}hHL$	$\frac{1}{2}hHL$	$\frac{1}{2}hHL$	$\frac{HL}{2}(h_1 + h_2)$
 (7)	$\frac{1}{2}hHL$	$\frac{1}{3}hHL$	$\frac{1}{6}hHL$	$\frac{hH(L + k_2)}{6}$	$\frac{HL}{6}(2h_1 + h_2)$
 (8)	$\frac{1}{2}hHL$	$\frac{1}{6}hHL$	$\frac{1}{3}hHL$	$\frac{hH(L + k_1)}{6}$	$\frac{HL}{6}(h_1 + 2h_2)$

(Contd)

Table 2.11 (*Contd*)

	$\frac{1}{2}hHL$	$\frac{hH(L + K_2)}{6}$	$\frac{hH(L + K_1)}{6}$	$k_1 < K_1$: $\frac{hHL}{6k_2K_1}(L^2 - k_1^2 - K_2^2)$; $k_1 > K_1$: $\frac{hHL}{6k_1K_2}(L^2 - k_2^2 - K_1^2)$	$\frac{H}{6}[h_1(L + K_2) + h_2(L + K_1)]$
	$\frac{hL}{2}(H_1 + H_2)$	$\frac{hL}{6}(2H_1 + H_2)$	$\frac{hL}{6}(H_1 + 2H_2)$	$\frac{h}{6}[H_1(L + k_2) + H_2(L + k_1)]$	$\frac{L}{6}(2h_1H_1 + 2h_2H_2 + h_1H_2 + h_2H_1)$
 *	$\frac{2}{3}hHL$	$\frac{1}{3}hHL$	$\frac{1}{3}hHL$	$\frac{Hh}{3L}(L^2 + k_1k_2)$	$\frac{1}{3}HL(h_1 + h_2)$
 *	$\frac{1}{3}hHL$	$\frac{1}{12}hHL$	$\frac{1}{4}hHL$	$\frac{hH}{12L}(L^2 + k_1L + k_1^2)$	$\frac{HL}{12}(h_1 + 3h_2)$
 *	$\frac{2}{3}hHL$	$\frac{1}{4}hHL$	$\frac{5}{12}hHL$	$\frac{hH}{12L}(5L^2 - k_2L - k_2^2)$	$\frac{HL}{12}(3h_1 + 5h_2)$

* Second degree parabola.

Table 2.12

Portion of the beam→	*AB*	*BC*	*CD*	*DE*
I	I	$2I$	$2I$	I
Origin	A	A	E	E
Limits	0 to $L/3$	$L/3$ to $L/2$	$L/3$ to $L/2$	0 to $L/3$
m_1	$\frac{x}{2}$	$\frac{x}{2}$	$\frac{x}{2}$	$\frac{x}{2}$
m_2	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{2}{3}x - \left(x - \frac{L}{3}\right)$	$\frac{2}{3}x$
m_3	$\left(1 - \frac{x}{L}\right)$	$\left(1 - \frac{x}{L}\right)$	$\frac{x}{L}$	$\frac{x}{L}$

Substituting for m_1, m_2 and m_3 from Table 2.12 into Eq. (2.35),

$$\delta_{11} = \int \frac{m_1 m_1 ds}{EI} = \int_0^{L/3} \frac{\frac{x}{2}\cdot\frac{x}{2}\,dx}{EI} + \int_{L/3}^{L/2} \frac{\frac{x}{2}\cdot\frac{x}{2}\,dx}{2EI}$$

$$+ \int_{L/3}^{L/2} \frac{\frac{x}{2}\cdot\frac{x}{2}\,dx}{2EI} + \int_0^{L/3} \frac{\frac{x}{2}\cdot\frac{x}{2}\,dx}{EI}$$

$$= \frac{35}{2592}\frac{L^3}{EI}$$

$$\delta_{21} = \int \frac{m_1 m_2 ds}{EI} = \int_0^{L/3} \frac{\frac{x}{2}\cdot\frac{x}{3}\,dx}{EI} + \int_{L/3}^{L/2} \frac{\frac{x}{2}\cdot\frac{x}{3}\,dx}{2EI}$$

$$+ \int_{L/3}^{L/2} \frac{\frac{x}{2}\left\{\frac{2}{3}x - \left(x - \frac{L}{3}\right)\right\}dx}{2EI} + \int_0^{L/3} \frac{\frac{x}{2}\cdot\frac{2}{3}x\,dx}{EI}$$

$$= \frac{31}{2592}\frac{L^3}{EI}$$

$$\delta_{31} = \int \frac{m_1 m_3 ds}{EI} = \int_0^{L/3} \frac{\frac{x}{2}\left(1 - \frac{x}{L}\right)dx}{EI} + \int_{L/3}^{L/2} \frac{\frac{x}{2}\left(1 - \frac{x}{L}\right)dx}{2EI}$$

$$+ \int_{L/3}^{L/2} \frac{\frac{x}{2}\cdot\frac{x}{L}\,dx}{2EI} + \int_0^{L/3} \frac{\frac{x}{2}\cdot\frac{x}{L}\,dx}{EI}$$

$$= \frac{13}{288}\frac{L^2}{EI}$$

With reference to the coordinates indicated in Fig. 2.19(a), $P_1 = P$ and $P_2 = P_3 = 0$. Hence,

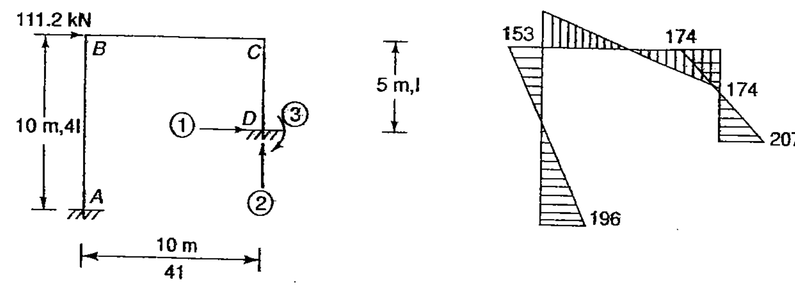
$$\Delta_1 = \delta_{11}P_1 + \delta_{12}P_2 + \delta_{13}P_3 = \delta_{11}P = \frac{35}{2592}\frac{PL^3}{EI}$$

$$\Delta_2 = \delta_{21}P_1 + \delta_{22}P_2 + \delta_{23}P_3 = \delta_{21}P = \frac{31}{2592}\frac{PL^3}{EI}$$

$$\Delta_3 = \delta_{31}P_1 + \delta_{32}P_2 + \delta_{33}P_3 = \delta_{31}P = \frac{13}{288}\frac{PL^2}{EI}$$

Example 2.9

Using the unit-load method, analyse the portal frame shown in Fig. 2.20(a) and hence draw the bending-moment diagram.



(a) (b)

Fig. 2.20

Solution

Let the three reaction components at D be treated as the redundants. Hence, choosing the coordinates as shown in the figure, $\Delta_1 = \Delta_2 = \Delta_3 = 0$ because support D is unyielding. Using the unit-load method,

$$\Delta_1 = \int \frac{Mm_1 ds}{EI} = 0 \qquad \text{(a)}$$

$$\Delta_2 = \int \frac{Mm_2 ds}{EI} = 0 \qquad \text{(b)}$$

$$\Delta_3 = \int \frac{Mm_3 ds}{EI} = 0 \qquad \text{(c)}$$

Substituting for M, m_1, m_2 and m_3 from Table 2.13 into Eqs (a), (b) and (c).

Table 2.13

Portion of the beam→	DC	CB	BA
I	I	$4I$	$4I$
Origin	D	C	B
Limits	0 to 5	0 to 10	0 to 10
M	$P_1x - P_3$	$5P_1 + P_2x - P_3$	$10P_2 - P_3 + P_1(5 - x) - 111.2x$
m_1	x	5	$(5 - x)$
m_2	0	x	10
m_3	– 1	– 1	– 1

Note: Bending moment producing compression on outer fibres has been taken positive.

$$\Delta_1 = \int_0^5 \frac{(P_1x - P_3)\, x\, dx}{EI} + \int_0^{10} \frac{(5P_1 + P_2x - P_3)5dx}{4EI}$$

$$+ \int_0^{10} \frac{\{10P_2 - P_3 + P_1(5 - x) - 111.2\, x\}\,(5 - x)\, dx}{4EI} = 0$$

or $\quad 150\, P_1 + 75\, P_2 - 30\, P_3 + 2780 = 0 \qquad$ (d)

$$\Delta_2 = \int_0^{10} \frac{(5P_1 + P_2x - P_3)\, xdx}{4EI}$$

$$+ \int_0^{10} \frac{\{10P_2 - P_3 + P_1(5 - x) - 111.2x\}\, dx}{4EI} = 0$$

or $\quad 75\, P_1 + 400\, P_2 - 45\, P_3 - 16680 = 0 \qquad$ (e)

$$\Delta_3 = \int_0^5 \frac{(P_1x - P_3)(-1)dx}{EI} + \int_0^{10} \frac{(5P_1 + P_2x - P_3)(-1)dx}{4EI}$$

$$+ \int_0^{10} \frac{\{10P_2 - P_3 + P_1(5 - x) - 111.2x\}(-1)dx}{4EI} = 0$$

or $\quad 25\, P_1 + 37.5\, P_2 - 10\, P_3 - 1390 = 0 \qquad$ (f)

Solving Eqs (d), (e) and (f) as simultaneous equations,

$P_1 = -76.3$ kN

$P_2 = 32.7$ kN

$P_3 = -207.1$ kN·m

After the redundants are known, the bending moment at any point can be calculated. The bending-moment diagram, drawn on the compression side, is shown in Fig. 2.20(b).

Example 2.10

For the pin-jointed plane frame shown in Fig. 2.21, calculate the displacements at coordinates 1 and 2. The axial stiffness of each member is 42.5 kN/mm.

Solution

The forces in the members of the pin-jointed frame due to the given load are given in column 2 of Table 2.14. The forces in the various members due to a unit load at coordinate 1 and due to a unit load at coordinate 2 are given in columns 3 and 4 of Table 2.14

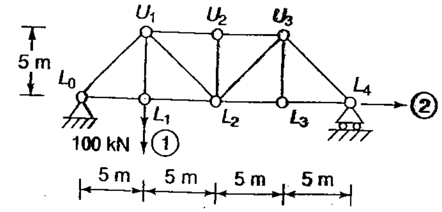


Fig. 2.21

Table 2.14

Member	S kN	s_1	s_2	Ss_1 kN	Ss_2 kN
(1)	(2)	(3)	(4)	(5)	(6)
U_1U_2	– 50	– 0.5	0	25	0
U_2U_3	– 50	– 0.5	0	25	0
L_0L_1	75	0.75	1	56.25	75
L_1L_2	75	0.75	1	56.25	75
L_2L_3	25	0.25	1	6.25	25
L_3L_4	25	0.25	1	6.25	25
U_1L_1	100	1	0	100	0
U_2L_2	0	0	0	0	0
U_3L_3	0	0	0	0	0
U_1L_0	$-75\sqrt{2}$	$-0.75\sqrt{2}$	0	112.5	0
U_1L_2	$-25\sqrt{2}$	$-0.25\sqrt{2}$	0	12.5	0
U_3L_4	$-25\sqrt{2}$	$-0.25\sqrt{2}$	0	12.5	0
U_3L_2	$25\sqrt{2}$	$0.25\sqrt{2}$	0	12.5	0
				Σ 425	200

Tension +
Compression –

Substituting for ΣSs_1 and ΣSs_2 from Table 2.14 into Eq. (2.34),

$$\Delta_1 = \sum \frac{Ss_1 L}{AE} = \frac{L}{AE} \sum Ss_1 = \frac{1}{42.5} \times 425 = 10 \text{ mm}$$

$$\Delta_2 = \sum \frac{Ss_2 L}{AE} = \frac{L}{AE} \sum Ss_2 = \frac{1}{42.5} \times 200 = 4.7 \text{ mm}$$

Example 2.11

For the non-prismatic beam shown in Fig. 2.22(a), determine the displacements at coordinates 1, 2 and 3 using the method of diagram multiplication.

Solution

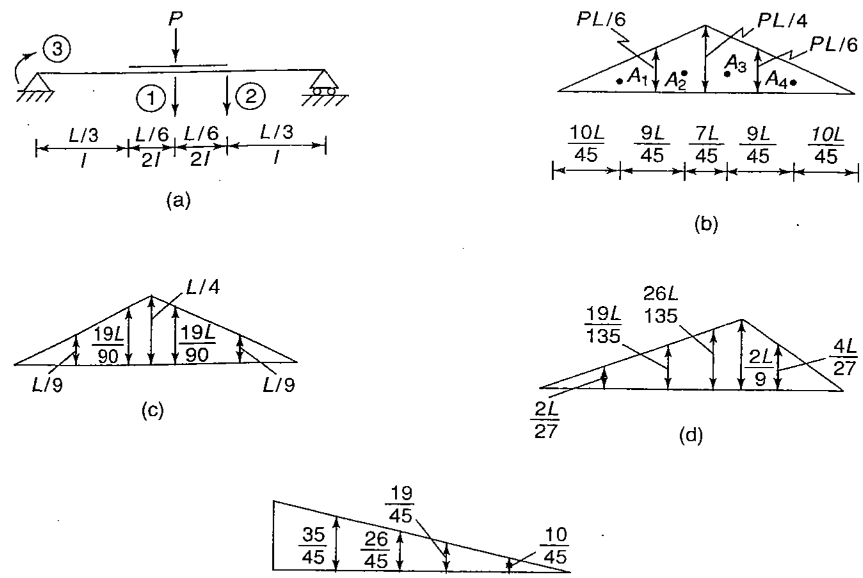


Fig. 2.22

The bending-moment diagram due to load P, designated as M- diagram, is shown in Fig. 2.22(b). Keeping in view the changes in EI, the diagram can be divided into four parts with areas A_1, A_2, A_3 and A_4 with their respective centroids located as shown in the figure.

$$A_1 = A_4 = \frac{1}{2} \times \frac{L}{3} \times \frac{PL}{6} = \frac{PL^2}{36}$$

$$A_2 = A_3 = \frac{L}{6} \times \frac{PL/6 + PL/4}{2} = \frac{5PL^2}{144}$$

The m_1-, m_2- and m_3-diagrams obtained by applying a unit load successively at coordinates 1, 2 and 3 are shown in Fig. 2.22(c), (d) and (e) respectively.

To calculate the displacement Δ_1 at coordinate 1, consider area A_1 and the ordinate of m_1-diagram located at the centroid of area A_1.

$$A_M \bar{m}_1 = \frac{PL^2}{36} \times \frac{L}{9} = \frac{PL^3}{324}$$

Alternatively, using Table 2.11, third line and third column,

$$A_M \bar{m}_1 = \frac{1}{3} hHL = \frac{1}{3} \times \frac{L}{6} \times \frac{PL}{6} \times \frac{L}{3} = \frac{PL^3}{324}$$

Similarly, considering areas A_2, A_3 and A_4 and using Eq. (2.38),

$$\Delta_1 = \sum \frac{A_M \bar{m}_1}{EI} = \frac{1}{EI} \times \frac{PL^2}{36} \times \frac{L}{9} + \frac{1}{2EI} \times \frac{5PL^2}{144} \times \frac{19L}{90}$$

$$+ \frac{1}{2EI} \times \frac{5PL^2}{144} \times \frac{19L}{90} + \frac{1}{EI} \times \frac{PL^2}{36} \times \frac{L}{9}$$

$$= \frac{35PL^3}{2592\,EI} \text{ (downwards)}$$

To compute displacement Δ_2 at coordinate 2, multiply areas A_1, A_2, A_3 and A_4 by the respective ordinates of m_2-diagram located at their centroids and take the sum of the products.

$$\Delta_2 = \frac{1}{EI} \times \frac{PL^2}{36} \times \frac{2L}{27} + \frac{1}{2EI} \times \frac{5PL^2}{144} \times \frac{19L}{135}$$

$$+ \frac{1}{2EI} \times \frac{5PL^2}{144} \times \frac{26L}{135} + \frac{1}{EI} \times \frac{PL^2}{36} \times \frac{4L}{27}$$

$$= \frac{31\,PL^3}{2592\,EI} \text{ (downwards)}$$

To compute displacement Δ_3 at coordinate 3, multiply areas A_1, A_2, A_3 and A_4 by the respective ordinates of the m_3-diagram located at their centroids and take the sum of the products.

$$\Delta_3 = \frac{1}{EI} \times \frac{PL^2}{36} \times \frac{35}{45} + \frac{1}{2EI} \times \frac{5PL^2}{144} \times \frac{26}{45}$$

$$+ \frac{1}{2EI} \times \frac{5PL^2}{144} \times \frac{19}{45} + \frac{1}{EI} \times \frac{PL^2}{36} \times \frac{10}{45}$$

$$= \frac{13PL^2}{288EI} \text{ (clockwise)}$$

Example 2.12

Figure 2.23(a) shows beam ABC and the loads acting on it. Calculate the displacements due to the applied loads at coordinates 1 and 2 shown in Fig. 2.23(b). Also calculate displacements δ_{11}, δ_{21} and δ_{22}.

Solution

The M-diagram due to applied loads is shown in Fig. 2.23(c). The m_1- diagram due to a unit force at coordinate 1 and the m_2-diagram due to a unit force at coordinate 2 are shown in Fig. 2.23(d) and (e) respectively.

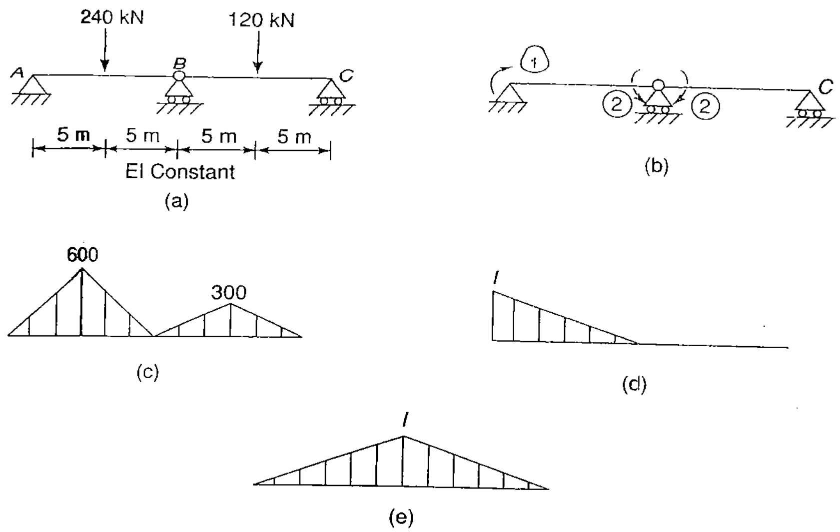


Fig. 2.23

The displacement at coordinate 1 due to the applied loads is given by the equation

$$\Delta_1 = \sum \frac{A_M \bar{m}_1}{EI} = \frac{1}{EI} \times \frac{1}{2} \times 10 \times 600 \times \frac{1}{2} = \frac{1500}{EI}$$

The displacement at coordinate 2 due to the applied load is given by the equation

$$\Delta_2 = \sum \frac{A_M \bar{m}_2}{EI} = \frac{1}{EI} \times \frac{1}{2} \times 10 \times 600 \times \frac{1}{2} + \frac{1}{EI} \times \frac{1}{2} \times 10 \times 300 \times \frac{1}{2} = \frac{2250}{EI}$$

The displacement at coordinate 1 due to a unit force at coordinate 1 may be computed as

$$\delta_{11} = \sum \frac{A_M \bar{m}_1}{EI} = \frac{1}{EI} \times \frac{1}{2} \times 10 \times 1 \times \frac{2}{3} = \frac{10}{3EI}$$

The displacement at coordinate 2 due to a unit force at coordinate 2 may be computed as

$$\delta_{21} = \sum \frac{A_{m_1} \bar{m}_2}{EI} = \frac{1}{EI} \times \frac{1}{2} \times 10 \times 1 \times \frac{1}{3} = \frac{5}{3EI}$$

The displacement at coordinate 2 due to a unit force at coordinate 2 is given by the equation

$$\delta_{22} = \sum \frac{A_{m_2} \bar{m}_2}{EI} = \frac{1}{EI} \times \frac{1}{2} \times 10 \times 1 \times \frac{2}{3} + \frac{1}{EI} \times \frac{1}{2} \times 10 \times 1 \times \frac{2}{3} = \frac{20}{3EI}$$

Example 2.13

Using the method of diagram-multiplication, analyse the portal frame shown in Fig. 2.24(a).

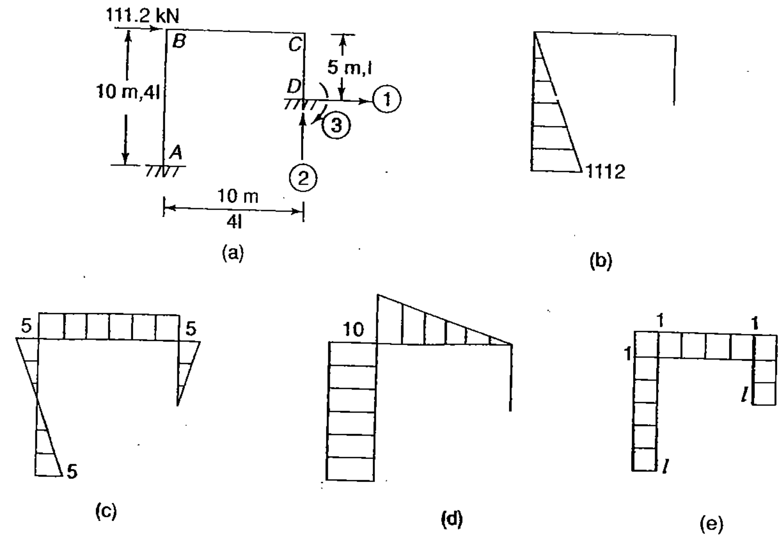


Fig. 2.24

Solution

Let the three reaction components at D be treated as redundants. Hence, choosing the coordinates as shown in the figure, $\Delta_1 = \Delta_2 = \Delta_3 = 0$ because the support at D is unyielding. Let Δ_{1L}, Δ_{2L} and Δ_{3L} be the displacements at coordinates 1, 2 and 3 respectively when the support at D is removed, i.e., the support reactions at D, $P_1 = P_2 = P_3 = 0$. Then, it folows that

$$\Delta_1 = \Delta_{1L} + \delta_{11}P_1 + \delta_{12}P_2 + \delta_{13}P_3 = 0 \qquad \text{(a)}$$

$$\Delta_2 = \Delta_{2L} + \delta_{21}P_1 + \delta_{22}P_2 + \delta_{23}P_3 = 0 \qquad \text{(b)}$$

$$\Delta_3 = \Delta_{3L} + \delta_{31}P_1 + \delta_{32}P_2 + \delta_{33}P_3 = 0 \qquad \text{(c)}$$

The unknown reaction components P_1, P_2, and P_3 can be determined by solving Eqs (a), (b) and (c) as simultaneous equations. The displacements Δ_{1L}, Δ_{2L}, Δ_{3L}, δ_{11} etc. can be computed by using the method of diagram-multiplication. For this purpose M, m_1-, m_2-, and m_3-diagrams can be drawn as shown in Fig. 2.24 (b), (c), (d) and (e) respectively. In drawing these diagrams it is assumed that the support at D has been removed.

$$\Delta_{1L} = \sum \frac{A_M \bar{m}_1}{EI} = \frac{1}{4EI} \times \frac{1112 \times 10}{2} \times \frac{5}{3} = \frac{6950}{3EI}$$

$$\Delta_{2L} = \sum \frac{A_M \bar{m}_2}{EI} = \frac{1112 \times 10}{2} (-10) = -\frac{13900}{EI}$$

$$\Delta_{3L} = \sum \frac{A_M \bar{m}_3}{EI} = \frac{1}{4EI} \times \frac{1112 \times 10}{2} (1) = \frac{1390}{EI}$$

$$\delta_{11} = \sum \frac{A_{m_1} \bar{m}_1}{EI} = \frac{1}{4EI} \times \frac{5 \times 5}{2} \times \frac{10}{3} + \frac{1}{4EI} \times \frac{5 \times 5}{2} \times \frac{10}{3}$$

$$+ \frac{1}{4EI} \times 10 \times 5 \times 5 + \frac{1}{EI} \times \frac{5 \times 5}{2} \times \frac{10}{3} = \frac{125}{EI}$$

$$\delta_{12} = \delta_{21} = \sum \frac{A_{m_1} \bar{m}_2}{EI} = \frac{1}{4EI} \times 10 \times 5 \times 5 = \frac{125}{2EI}$$

$$\delta_{13} = \delta_{31} = \sum \frac{A_{m_1} \bar{m}_3}{EI} = \frac{1}{4EI} \times 10 \times 5 \, (-1) + \frac{1}{EI} \times \frac{5 \times 5}{2} (-1) = \frac{-25}{EI}$$

$$\delta_{22} = \sum \frac{A_{m_2} \bar{m}_2}{EI} = \frac{1}{4EI} \times 10 \times 10 \times 10$$

$$+ \frac{1}{4EI} \times \frac{1}{2} \times 10 \times 10 \times \frac{20}{3} = \frac{1000}{3EI}$$

$$\delta_{23} = \delta_{32} = \sum \frac{A_{m_2} \bar{m}_3}{EI} = \frac{1}{4EI} \times 10 \times 10(-1)$$

$$+ \frac{1}{4EI} \times \frac{1}{2} \times 10 \times 10(-1) = -\frac{75}{2EI}$$

$$\delta_{33} = \sum \frac{A_{m_3} \bar{m}_3}{EI} = \frac{1}{4EI} \times 10 \times 1 \times 1 + \frac{1}{4EI} \times 10 \times 1 + \frac{1}{EI} \times 5 \times 1 \times 1 = \frac{10}{EI}$$

Substituting the values of the above displacements in Eqs (a), (b) and (c) and simplifying,

$$150\, P_1 + 75\, P_2 - 30\, P_3 + 2780 = 0 \qquad \text{(d)}$$

$$75\, P_1 + 400\, P_2 - 45\, P_3 - 16680 = 0 \qquad \text{(e)}$$

$$-50\, P_1 - 75\, P_2 + 20\, P_3 + 2780 = 0 \qquad \text{(f)}$$

Solving Eqs (d), (e) and (f) as simultaneous equations,

$$P_1 = -76.3 \text{ kN}$$

$$P_2 = 32.7 \text{ kN}$$

$$P_3 = -207.1 \text{ kN·m}$$

These values coincide with those computed earlier by the unit-load method in Ex. 2.9.

2.13 CONJUGATE-BEAM METHOD

The conjugate-beam method is based on the two theorems of moment-area which may be stated as follows:

According to the first theorem of moment-area, the difference of slopes at any two points A and B of a flexural member is equal to the area A_{AB} of the M/EI diagram between two points.

$$\theta_A - \theta_B = A_{AB} \qquad (2.40)$$

According to the second theorem of moment-area, the deflection Δ of any point A measured in a direction perpendicular to the axis of the member from the tangent to the deflection curve at any other point B is equal to the moment of the area of the M/EI diagram between the points A and B about the point A.

$$\Delta = A_{AB} \cdot \bar{x} \qquad (2.41)$$

where $\bar{x}$ = distance of centroid of the area A_{AB} from the point A.

The concept of the conjugate beam provides a simple method for the determination of slope and deflection at any point of a beam. It can be readily shown that the expressions for shear force, bending moment, slope and deflection in a beam can be derived by successive integration of the expression for the intensity of load. It follows that, if the load acting on a beam is replaced by the *M/EI*-diagram, the shear force at any point in this beam, known as *conjugate beam*, is equal to the slope at the same point in the actual beam. Similarly, the bending moment at any point in the conjugate beam is equal to the deflection at the same point in the actual beam. Consequently, the shear-force and the bending-moment diagrams of the conjugate beam are respectively identical to the rotation and deflection diagrams of the actual beam. As the slope and the deflection in the actual beam are analogous to the shear force and bending moment in the conjugate beam, it is necessary to select the support conditions for the conjugate beam so as to maintain the analogy between the two beams. The slope and deflection at a fixed support are zero. Hence, there should be no shear force and bending moment at that point in the conjugate beam. It follows that a fixed end in the actual beam should be replaced by a free end in the conjugate beam. It can similarly be shown that a free end in the actual beam should be replaced by a fixed end in the conjugate beam. Also, a simple end support (hinge or roller) continues to remain so in the conjugate beam. A simple interior support in the actual beam should be replaced by an interior hinge in the conjugate and an interior hinge in the actual beam should be replaced by a simple interior support in the conjugate beam. Figure 2.25 shows how a support in the actual beam has to be replaced in the conjugate beam together with the appropriate justification for the change. In this figure no distinction has been made between a roller-support and a hinge-support. As beams do not carry axial forces, there is no difference between a hinge-support and a roller-support. Consequently, a hinge-support may be replaced by a roller-support or *vice versa*, both in the actual beam as well as in the conjugate beam.

It may be noted that if the actual beam is statically determinate, the conjugate beam is also statically determinate. On the other hand, if the actual beam is statically indeterminate, the conjugate beam is unstable and is held in equilibrium by means of elastic loading.

To distinguish between the internal forces and deformations in the actual beam from those in the conjugate beam, a prime will be attached to the symbols representing the internal forces and deformation in the conjugate beam. The corresponding terms in the actual beam will be represented by unprimed symbols. Thus, R_A, Q_A, M_A, θ_A, and Δ_A represent respectively the reaction, shear force, bending moment, slope (rotation) and deflection at pint A in the actual beam. In the conjugate beam, the reaction, shear force and bending moment at A will be represented by the symbols R'_A, Q'_A and M'_A respectively. Hence Q'_A and M'_A are analogous to θ_A and Δ_A respectively. Thus,

$$\theta_A = Q'_A$$

$$\Delta_A = M'_A \tag{2.42}$$

S.No.	Actual Beam	Conjugate Beam
1.	Fixed end $\theta_A = 0$ $\Delta_A = 0$	Free end $\theta'_A = 0$ $M'_A = 0$
2.	Free end $\theta_A \neq 0$ $\Delta_A \neq 0$	Fixed end $Q'_A \neq 0$ $M'_A \neq 0$
3.	Simple end support $\theta_A \neq 0$ $\Delta_A = 0$	Simple end support $Q'_A \neq 0$ $M'_A = 0$
4.	Simple Interior support $\theta_A \neq 0$ $\Delta_A = 0$	Interior hinge $Q'_A \neq 0$ $M'_A = 0$
5.	Interior hinge $\theta_A \neq 0$ $\Delta_A \neq 0$	Simple Interior support $Q'_A \neq 0$ $M'_A \neq 0$

Fig. 2.25

The elegant conjugate-beam method has the advantage of having a simple and unambiguous sign convention. If the bending moment in the actual beam is positive (sagging), the corresponding elastic load M/EI in the conjugate beam should be considered to be acting downwards. On the other hand, if the bending moment in the actual beam is negative (hogging), the corresponding elastic load in the conjugate beam should act upward. If the shear force and bending moment in the conjugate beam are positive, the analogous rotation and deflection in the actual beam are clockwise and downward respectively. On the other hand, if the shear force and bending moment in the conjugate beam are found to be negative, the analogous rotation and deflection in the actual beam are counter-clockwise and upward respectively.

Example 2.14

Analyse the fixed-ended beam shown in Fig. 2.26 (a) using the conjugate-beam method, Hence, draw the shear-force and bending-moment diagrams.

Solution

The bending-moment diagram due to load P for simple support conditions is shown in Fig. 2.26(b). The bending-moment diagram due to end moments M_A and M_B is shown in Fig. 2.26(c). The conjugate beam with its elastic load is shown in Fig. 2.26(d). As ends A and B in the actual beam are fixed, they become free in the conjugate beam. Thus the conjugate beam is an unsupported floating beam held in equilibrium by the elastic load.

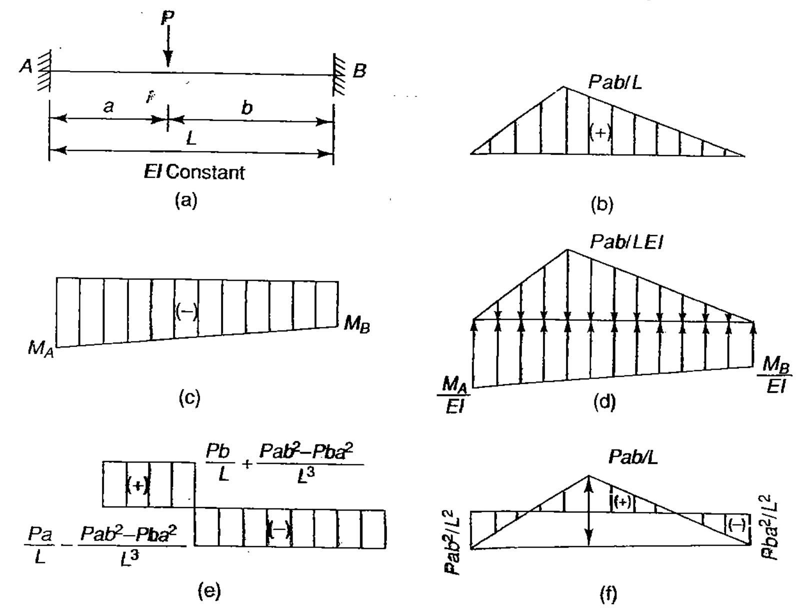


Fig. 2.26

Taking moments about A,

$$\frac{1}{2}\times L\times\frac{Pab}{LEI}\left(\frac{L+a}{3}\right)-\frac{M_A/EI+M_B/EI}{2}\times L\times\frac{L}{3}\times\frac{M_A+2M_B}{M_A+M_B}=0 \qquad \text{(a)}$$

Similarly, taking moments about B,

$$\frac{1}{2}\times L\times\frac{Pab}{LEI}\left(\frac{L+b}{3}\right)-\frac{M_A/EI+M_B/EI}{2}\times L\times\frac{L}{3}\times\frac{M_B+2M_A}{M_A+M_B}=0 \qquad \text{(b)}$$

Solving Eqs (a) and (b),

$$M_A=\frac{Pab^2}{L^2}$$

$$M_B=\frac{Pba^2}{L^2}$$

The shear-force and bending-moment diagrams are shown in Fig. 2.26 (e) and (f) respectively.

Example 2.15

For the propped cantilever shown in Fig. 2.27(a), determine the slopes at B and C and the deflection at B.

Solution

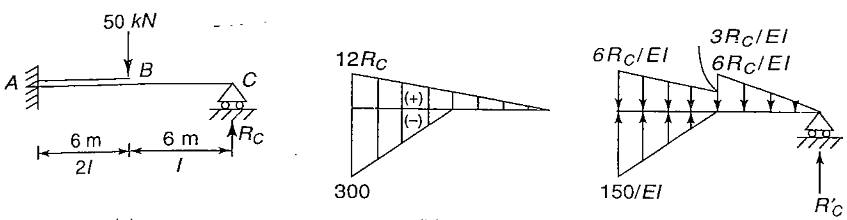
The bending-moment diagrams due to the load of 50 kN and prop reaction R_C acting separately are shown in Fig. 2.27(b). Consequently, the conjugate beam, shown in Fig. 2.27(c), should carry the elastic loading indicated on the figure. To determine prop-reaction R_C, the bending moment at C in the conjugate beam should be equated to zero.

$$\Delta_C=M'_C=\frac{1}{2}\times 6\times\frac{150}{EI}\left(6+\frac{2}{3}\times 6\right)-6\times\frac{6R_C/EI+3R_C/EI}{2}$$

$$\times\left(6+\frac{3+12}{3+6}\times\frac{6}{3}\right)-\frac{1}{2}\times 6\times\frac{6R_C}{EI}\times\frac{2\times 6}{3}=0$$

Hence,

$$R_C=\frac{250}{18}\text{ kN}$$



(a) (b) (c)

Fig. 2.27

Using Eq. (2.42)

$$\theta_C=Q'_C=-R'_C=\frac{1}{2}\times 6\times\frac{150}{EI}-6\times\frac{250/3EI+250/6EI}{2}$$

$$-\frac{1}{2}\times 6\times\frac{250}{3EI}=-\frac{175}{EI}$$

or

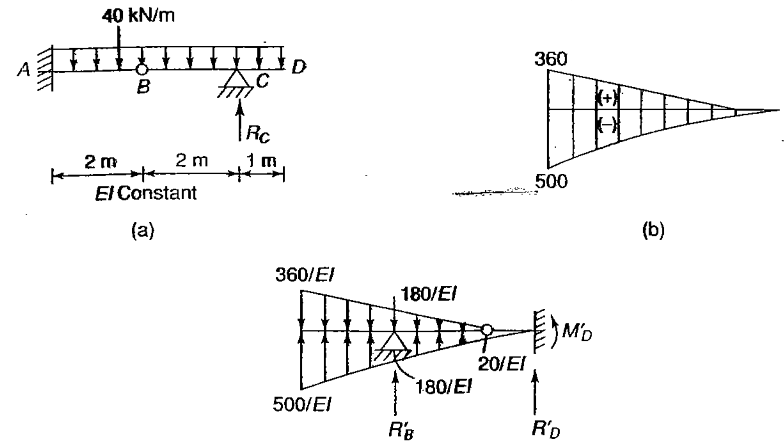
$$R'_C=\frac{175}{EI}$$

$$\theta_B=Q'_B=\frac{1}{2}\times 6\times\frac{250}{3EI}-\frac{175}{EI}=\frac{75}{EI}$$

$$\Delta_B=M'_B=6R'_C-\frac{1}{2}\times 6\times\frac{6R_C}{EI}\times 2=\frac{550}{EI}$$

Example 2.16

In the beam shown in Fig. 2.28(a), calculate the deflection and the change of slope at B.



(a) (b)

(c)

Fig. 2.28

Solution

As the beam is statically determinate, the support reactions can be computed by using the equations of static equilibrium.

$$R_C=90\text{ kN}$$

The triangular bending-moment diagram due to R_C, being positive, has been plotted above the base line in Fig. 2.28(b). Similarly, the parabolic bending-moment diagram due to the uniformly distributed load, being negative, has been plotted below the base line. The conjugate beam has been shown in Fig. 2.28(c). The support conditions for the conjugate beam have been chosen in accordance with the guidelines provide in Fig. 2.25. As the conjugate beam is statically determinate, the support reactions can be calculated by statics.

$$R'_B = -\frac{560}{3EI} \quad R'_D = \frac{220}{3EI} \quad M'_D = -\frac{75}{EI}$$

The slope at B to its right,

$$\theta_B \text{ (right)} = Q'_B\text{(right)} = \frac{1}{2}\times 2\times\frac{180}{EI} - \frac{1}{3}\times 3$$

$$\times\frac{180}{EI} - \frac{220}{EI} = -\frac{220}{3EI} \text{ (counter-clockwise)}$$

The slope at B to its left,

$$\theta_B \text{ (left)} = Q'_B \text{ (left)} = Q'_B \text{ (right)} - R'_B$$

$$= \frac{340}{3EI} \text{ (clockwise)}$$

Sudden change in slope at B,

$$\theta_B \text{ (left)} - \theta_B\text{(right)} = \frac{560}{3EI}$$

$$\Delta_B = M'_B = \frac{160}{EI} \text{ (downward)}$$

2.14 STIFFNESS OF A PRISMATIC MEMBER

In the analysis of continuous beams and rigid-jointed frames, the resistance offered by a prismatic member to the rotations at its ends and the transverse displacement of one end relative to the other end is of importance. In this section the resistance offered by a flexural member to these deformations is discussed.

2.14.1 Rotation Without Transverse Displacement of One End of a Prismatic Member, with the Other End Hinged

Figure 2.29(a) shows a prismatic member AB, simply supported at its near end A and far end B, so that transverse displacement of one end relative to the other end is not possible. The couple required to produce a unit rotation at end A is known as *flexural stiffness* of the member at end A.

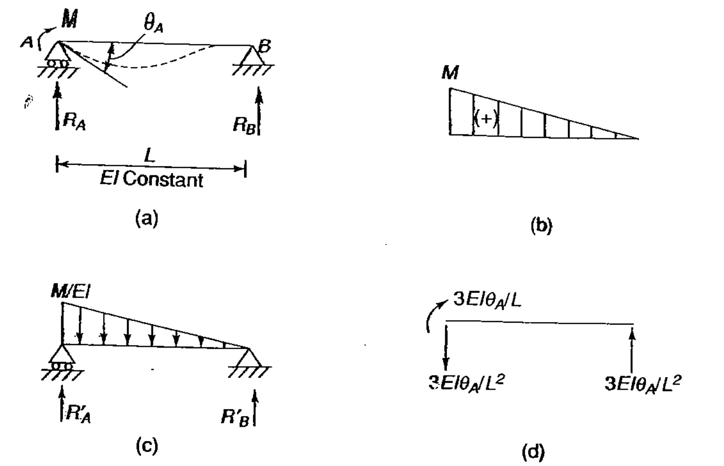


Fig. 2.29

Apply a clockwise moment M at near end A. The bending-moment diagram for the member is shown in Fig. 2.29(b). Hence, the conjugate beam is as shown in Fig. 2.29(c). The support reactions in the conjugate beam are

$$R'_A = \frac{ML}{3EI}$$

$$R'_B = \frac{ML}{6EI}$$

Using Eq. (2.42),

$$\theta_A = Q'_A = R'_A = \frac{ML}{3EI}$$

or

$$M = \frac{3EI\theta_A}{L} \tag{2.43a}$$

For the equilibrium of member AB,

$$R_A + R_B = 0$$

and

$$R_A \times L + M = 0$$

Hence,

$$-R_A = R_B = \frac{M}{L} = \frac{3EI\theta_A}{L^2} \tag{2.43b}$$

The free-body diagram of the member is shown in Fig. 2.29(d). As flexural stiffness is the couple required for a unit rotation, putting $\theta_A = 1$ in Eq. (2.43a),

$$\text{flexural stiffness at } A \text{ (far end } B \text{ hinged)} = \frac{3EI}{L} \tag{2.43c}$$

and

$$-R_A = R_B = \frac{3EI}{L^2} \tag{2.43d}$$

2.14.2 Rotation without Transverse Displacement of One End of a Prismatic Member, with the Other End Fixed

Figure 2.30(a) shows a prismatic member AB, simply supported at its near end A and fixed at far end B, so that the transverse displacement of one end relative to the other end is not possible. The couple required to produce a unit rotation at end A is known as the *flexural stiffness* of the member at end A.

Apply a clockwise moment M at near end A. Let R_A be the upward reaction at roller support A. The bending-moment diagram for the member is shown in Fig. 2.30(b). Hence, the conjugate beam is as shown in Fig. 2.30(c). For the equilibrium of the conjugate beam,

$$M'_A = 0 = L \times \frac{M}{EI} \times \frac{L}{2} + \frac{L}{2} \times \frac{R_A L}{EI} \times \frac{2L}{3}$$

or

$$R_A = -\frac{1.5\,M}{L} \tag{a}$$

and

$$R'_A = \frac{\dfrac{M}{EI} + \dfrac{R_A L + M}{EI}}{2} \times L = \frac{ML}{4EI}$$

Using Eq. (2.42),

$$\theta_A = Q'_A = R'_A = \frac{ML}{4EI}$$

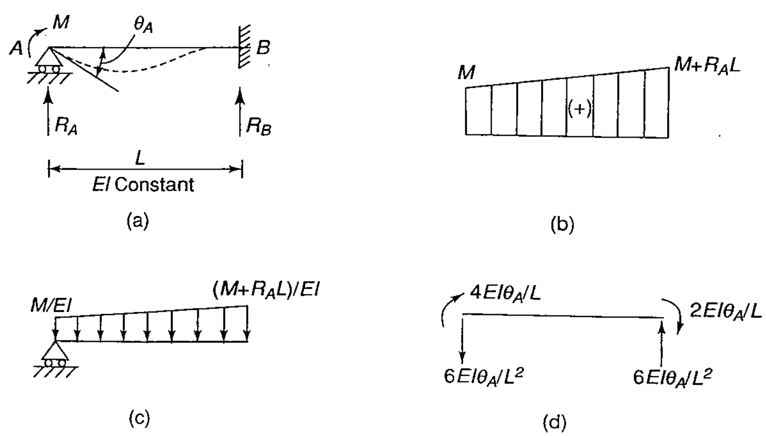


Fig. 2.30

or

$$M = \frac{4EI\theta_A}{L} \tag{2.44a}$$

Substituting into Eq. (a),

$$R_A = -\frac{6EI\theta_A}{L^2} \tag{2.44b}$$

For the equilibrium of member AB,

$$R_B = -R_A = \frac{6EI\theta_A}{L^2} \tag{2.44c}$$

and

$$M_B = \frac{2EI\theta_A}{L} \tag{2.44d}$$

The free-body diagram of the member is shown in Fig. 2.30(d). From the free-body diagram, it is seen that when a clockwise couple is applied at near end A, a clockwise couple of half the magnitude is set up at far end B. Thus, the *carry-over factor,* defined as the ratio of the moments at the far and near ends, is 1/2 in the case of straight prismatic member according to the frame convention. As flexural stiffness is the couple required for unit rotation, putting $\theta_A = 1$ in Eq. (2.44a),

$$\text{flexural stiffness at } A = \frac{4EI}{L} \tag{2.44e}$$

(far end B fixed)

Also,

$$M_B = \frac{2EI}{L} \tag{2.44f}$$

and

$$R_B = -R_A = \frac{6EI}{L^2} \tag{2.44g}$$

2.14.3 Transverse Displacement without Rotation of One End of a Prismatic Member, with the Other End Hinged

Figure 2.31(a) shows a prismatic member AB simply supported at its near end A and far end B. Let the support at A be given a transverse displacement Δ relative to the support at B. While giving the transverse displacement, no rotation is permitted at end A as shown in Fig. 2.31(a). The transverse force required at A to produce a unit transverse displacement without rotation of end A is known as *transverse stiffness* of the member at end A.

The counter-clockwise couple M required at end A to maintain a zero slope at this point may be computed by using the conjugate-beam method. The bending-moment diagram for the actual beam is shown in Fig. 2.31(b). The conjugate beam with its elastic load is shown in Fig. 2.31(c). It may be noted that as the deflection at A in the actual beam is Δ upwards, a counter-clockwise couple of magnitude Δ must act at A in the conjugate beam. As no rotation has been permitted at A in the actual beam, the shear force at A in the conjugate beam should be zero.

$$Q_A = \frac{\Delta}{L} - \frac{ML}{3EI} = 0$$

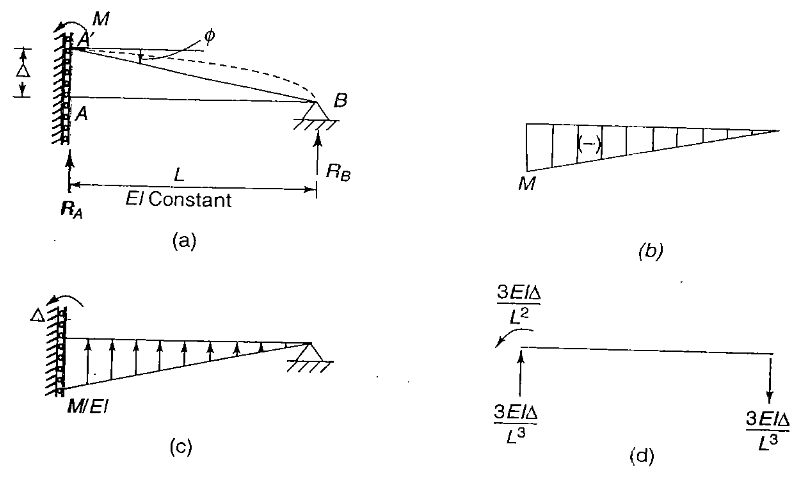


Fig. 2.31

Hence, $$M = \frac{3EI\Delta}{L^2} = \frac{3EI\phi}{L} \text{ (counter-clockwise)} \tag{2.45a}$$

where ϕ = clockwise rotation of the member.

For the equilibrium of member AB,

$$R_A = -R_B = \frac{M}{L} = \frac{3EI\Delta}{L^3} = \frac{3EI\phi}{L^2} \tag{2.45b}$$

The free-body diagram is shown in Fig. 2.31(d). As the transverse stiffness is the transverse force required for a unit transverse displacement without rotation of near and A, putting $\Delta = 1$ in Eq. (2.45b),

$$\text{transverse stiffness at } A = \frac{3EI}{L^3} \tag{2.45c}$$
(far end B hinged)

Also, $$M = \frac{3EI}{L^2} \text{ (counter-clockwise).} \tag{2.45d}$$

2.14.4 Transverse Displacement without Rotation of One End of a Prismatic Member, with the Other End Fixed

Figure 2.32(a) shows a prismatic member AB simply supported at its near end A and fixed at its far end B. Let the support at A be given a transverse displacement Δ relative to the support at B. While giving the transverse displacement, no rotation is permitted at end A as shown in Fig. 2.32(a). The transverse force required at A to produce unit transverse displacement without rotation of end A is known as the *transverse stiffness* of the member at end A.

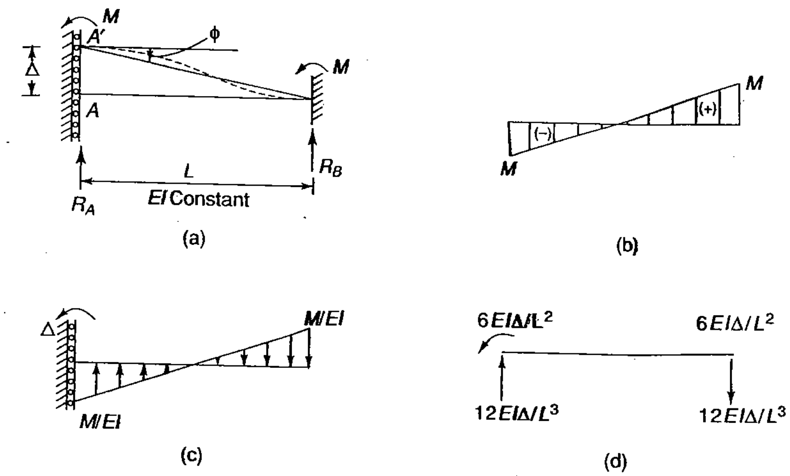


Fig. 2.32

It is evident that counter-clockwise bending couples will be set up at both ends of the beam. It is also clear that these bending couples should be equal from the consideration of symmetry. Consequently, the bending-moment diagram shown in Fig. 2.32(b) can be drawn. The conjugate beam with its elastic load is shown in Fig. 2.32(c). It may be noted that as the deflection at A in the actual beam is Δ upwards, a counter-clockwise couple of a magnitude Δ must act at A in the conjugate beam. Taking moments about A,

$$\Delta + \frac{1}{2} \times \frac{M}{EI} \times \frac{L}{2} \times \frac{L}{6} - \frac{1}{2} \times \frac{M}{EI} \times \frac{L}{2} \times \frac{5L}{6} = 0$$

or $$M = \frac{6EI\Delta}{L^2} = \frac{6EI\phi}{L} \text{ (counter-clockwise)} \tag{2.46a}$$

where ϕ = clockwise rotation of the member.

For the equilibrium of the actual member,

$$R_A = -R_B = \frac{2M}{L} = \frac{12EI\Delta}{L^3} = \frac{12EI\phi}{L^2} \tag{2.46b}$$

The free-body diagram is shown in Fig. 2.32(d). As the transverse stiffness is the transverse force required for a unit transverse displacement without rotation of near end A, putting $\Delta = 1$ in Eq. (2.46b),

$$\text{transverse stiffness at } A \text{ (far end } B \text{ fixed)} = \frac{12EI}{L^3} \tag{2.46c}$$

Also,
$$M = \frac{6EI}{L^2} \text{ (counter-clockwise)} \tag{2.46d}$$

Example 2.17

For the non-prismatic beam AB shown in Fig. 2.33(A), determine the flexural stiffness and carry-over factors at both ends.

Solution

(*i*) *Flexural Stiffness and Carry-over Factor at A*

To determine these, end A, to be treated as the near end, should be simply supported and far end B should be fixed as shown in Fig. 2.33(b). Also, a clockwise couple M should be applied at near end A. The bending-moment diagrams due to couple M and prop reaction R_A are shown in Fig. 2.33(c) and (d) respectively. The conjugate beam with its elastic load is shown in Fig. 2.33(e). Reaction R_A may be computed by noting that the deflection at A, $\Delta_A = 0$.

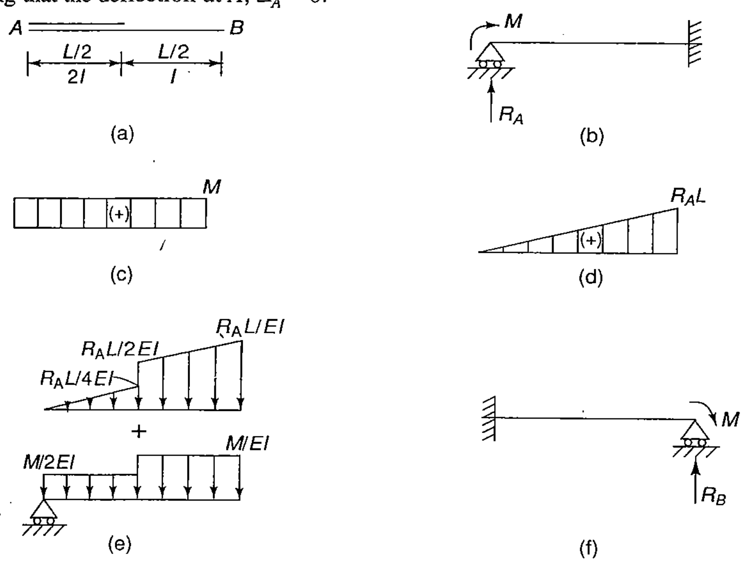


Fig. 2.33

$$\Delta_A = M'_A = \frac{L}{2}\times\frac{M}{2EI}\times\frac{L}{4}+\frac{L}{2}\times\frac{M}{EI}\times\frac{3L}{4}+$$

$$\frac{1}{2}\times\frac{L}{2}\times\frac{R_AL}{4EI}\times\frac{L}{3}+\frac{L}{2}\times\frac{R_AL/2+R_AL}{2EI}\times\left(\frac{L}{2}+\frac{L}{6}\times\frac{1+4}{1+2}\right)$$

$$= 0$$

Therefore, $R_A = -1.4\dfrac{M}{L}$ (downward)

Using Eq. (2.42),

$$\theta_A = Q'_A = R'_A = \frac{L}{2}\times\frac{M}{2EI}+\frac{L}{2}\times\frac{M}{EI}+\frac{1}{2}\times\frac{L}{2}\times\frac{R_AL}{4EI}+\frac{L}{2}\times\frac{R_AL/2+R_AL}{2EI}$$

$$= \frac{3}{4}\frac{ML}{EI}+\frac{7}{16}\frac{R_AL^2}{EI}$$

$$= 0.1375\,\frac{ML}{EI}$$

$$\text{Flexural stiffness at } A = \frac{M}{\theta_A} = \frac{M}{0.1375\,ML/EI}$$

$$= \frac{7.273\,EI}{L}$$

For equilibrium of the member,

$$M_B = R_AL + M = -0.4\,M$$

The minus sign shows that the bending moment at B is hogging. Consequently, the fixed-end moment at B is clockwise.

$$\text{Hence, the carry-over factor from } A \text{ to } B = \frac{\text{Couple carried over to } B}{\text{couple applied at } A}$$

$= \dfrac{-0.4\,M}{M} = -0.4$ in accordance with the beam convention. The carry-over factor is evidently 0.4 according to the frame convention.

(*ii*) *Flexural Stiffness and Carry-over Factor at B*

To determine these, end B, to be treated as the near end, should be simply supported and far end A should be fixed as shown in Fig. 2.33(f). Also, a clockwise couple M should be applied at near end B. Proceeding as in (i),

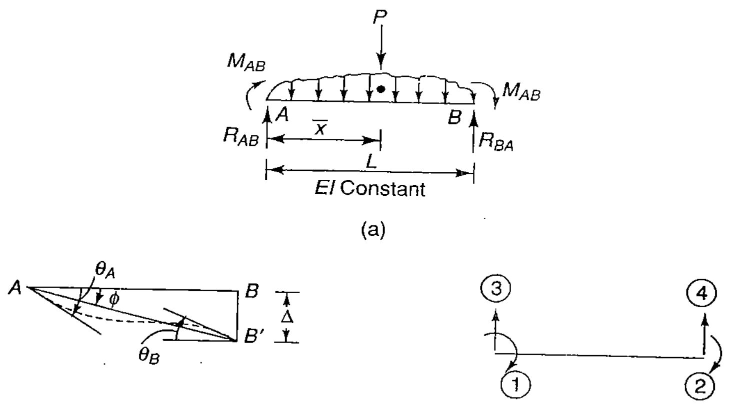
$$\text{Flexural stiffness at } B = \frac{4.364EI}{L}$$

$$\text{Carry-over factor from } B \text{ to } A = \frac{-2}{3} \text{ (according to beam convention)}$$

$$= \frac{2}{3} \text{ (according to frame convention)}$$

2.15 SLOPE-DEFLECTION EQUATIONS

The slope-deflection equations give the relationships between the bending moments acting on a structural member and the displacements of the member at its ends. Consider a straight prismatic member AB carrying an arbitrary transverse loading as shown in Fig. 2.34(a). The resultant P of the total load acts at a distance $\bar{x}$ from end A. Let M_{AB} and M_{BA} be the bending couples at ends A and B respectively. Figure 2.34(b) shows the deflected shape of the member. The deflected shape has been drawn in such a manner that the rotations θ_A and θ_B at ends A and B, and the rotation of the member ϕ are clockwise. This deflected shape of the member can be obtained by giving the displacements θ_A, θ_B, and Δ successively, as indicated in Table 2.15, leading to the derivation of slope-deflection equations.



(a) (b) (c)

Fig. 2.34

Table 2.15

Step	Operation	Moment at A	Moment at B
(1)	(2)	(3)	(4)
1.	Apply the given transverse load to the member keeping ends A and B fixed.	M^F_{AB}	M^F_{BA}
2.	Give a clockwise rotation θ_A at A.	$\frac{4EI\theta_A}{L}$ (Eq. 2.44a)	$\frac{2EI\theta_A}{L}$ (Eq. 2.44d)
3.	Give a clockwise rotation θ_B at B.	$\frac{2EI\theta_B}{L}$	$\frac{4EI\theta_B}{L}$
4.	Give a transverse downward displacement Δ at end B so that the number rotates clockwise through an angle ϕ.	$-\frac{6EI\Delta}{L^2}$ (Eq. 2.46a)	$-\frac{6EI\Delta}{L^2}$ (Eq. 2.46a)
	Total	M_{AB}	M_{BA}

The net moments M_{AB} and M_{BA} at the ends of member AB are obtained by the summation of the moments listed in columns 3 and 4 of Table 2.15.

$$M_{AB} = M^F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right) \tag{2.47a}$$

$$M_{BA} = M^F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_B - \frac{3\Delta}{L}\right) \tag{2.47b}$$

Equations (2.47) are the well known *slope-deflection equations*.

Alternatively, Eq. (2.47) may also be written as

$$M_{AB} = M^F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B - 3\phi) \tag{2.48a}$$

$$M_{BA} = M^F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A - 3\phi) \tag{2.48b}$$

Knowing end couples M_{AB} and M_{BA} reactions R_{AB} and R_{BA} can be computed by considering the free-body of member AB. Taking moments about B,

$$R_{AB}L - P(L - \bar{x}) + M_{AB} + M_{BA} = 0$$

or

$$R_{AB} = \frac{P(L-\bar{x})}{L} - \frac{M_{AB} + M_{BA}}{L} \tag{a}$$

Similarly, taking moments about A,

$$R_{BA} = \frac{P\bar{x}}{L} + \frac{M_{AB} + M_{BA}}{L} \tag{b}$$

In Eqs (a) and (b), the first terms on the right hand sides are the simple support reactions R^S_{AB} and R^S_{BA}. Consequently, the equations may be rewritten as

$$R_{AB} = R^S_{AB} - \frac{M_{AB} + M_{BA}}{L} \tag{2.49a}$$

$$R_{BA} = R^S_{BA} + \frac{M_{AB} + M_{BA}}{L} \tag{2.49b}$$

With reference to the coordinates shown in Fig. 2.34(c), Eq. (2.47) may be rewritten as

$$P_1 = P^F_1 + \frac{2EI}{L}\left[2\Delta_1 + \Delta_2 - 3\left(\frac{\Delta_3 - \Delta_4}{L}\right)\right] \tag{2.50a}$$

$$P_2 = P^F_2 + \frac{2EI}{L}\left[2\Delta_2 + \Delta_1 - 3\left(\frac{\Delta_3 - \Delta_4}{L}\right)\right] \tag{2.50b}$$

where $P^F_1 = M^F_{AB}$ = Fixed-end moment at A.

$P^F_2 = M^F_{BA}$ = Fixed-end moment at B.

Example 2.18

At left end of a member with partial fixity, shown in Fig. 2.35, the rotation is 0.01 radian clockwise and the settlement is 20 mm. At the right end of the member, the rotation is 0.0075 radian counter-clockwise and the settlement is 15 mm. If the moment of inertia, $I = 180 \times 10^6$ mm^4 and Young's modulus, $E = 2 \times 10^5$ MP$_a$, calculate the support reactions.

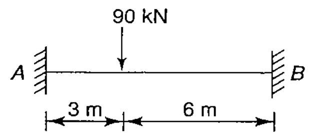


Fig. 2.35

Solution

From the given data,

$$\theta_A = 0.01 \text{ radian}$$
$$\theta_B = -0.0075 \text{ radian}$$
$$\Delta = 15 - 20 = -5 \text{ mm} = -0.005 \text{ m}$$
$$E = 2 \times 10^5 \text{ MPa} = 2 \times 10^8 \text{ kN/m}^2$$
$$I = 180 \times 10^6 \text{ mm}^4 = 180 \times 10^{-6} \text{ m}^4$$

Using Eq. (2.47),

$$M_{AB} = -\frac{90 \times 3 \times 6 \times 6}{9 \times 9} + \frac{2 \times 2 \times 10^8 \times 180 \times 10^{-6}}{9} \times \left[2(0.01) - 0.0075 - 3\left(\frac{-0.005}{9}\right)\right]$$
$$= -6.4 \text{ kN·m}$$

$$M_{BA} = \frac{90 \times 6 \times 3 \times 3}{9 \times 9} + \frac{2 \times 2 \times 10^8 \times 180 \times 10^{-6}}{9} \times \left[0.01 - 2(0.0075) - 3\left(\frac{-0.005}{9}\right)\right]$$
$$= 33.6 \text{ kN} \cdot \text{m}$$

Using Eq. (2.49),

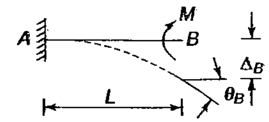
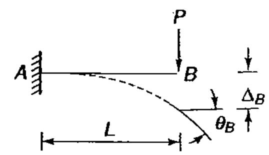
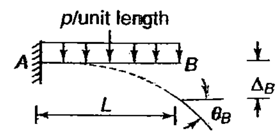
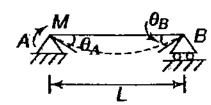
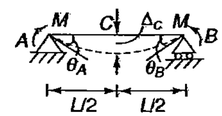
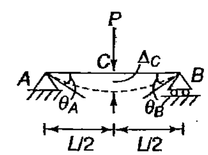
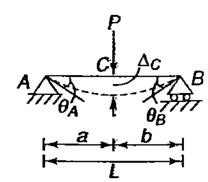
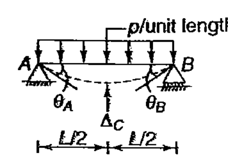
$$R_{AB} = \frac{90 \times 6}{9} - \frac{-6.4 + 33.6}{9} = 57 \text{ kN}$$

$$R_{BA} = \frac{90 \times 3}{9} + \frac{-6.4 + 33.6}{9} = 33 \text{ kN}$$

2.16 SOME STANDARD RESULTS

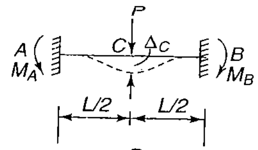
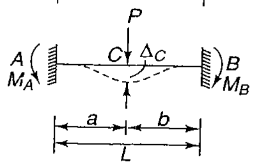
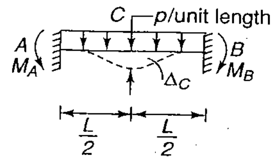
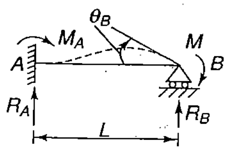
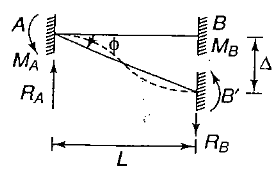
In the preceding sections, a variety of problems on the displacements of beams have been solved. Some of the standard cases which are frequently required in structural analysis are listed in Table 2.16. A more comprehensive collection of standard results has been presented in Appendix A.

Table 2.16

S.No.	Structure	Expressions	
1.		$\theta_B = \frac{ML}{EI}$	$\Delta_B = \frac{ML^2}{2EI}$
2.		$\theta_B = \frac{PL^2}{2EI}$	$\Delta_B = \frac{PL^3}{3EI}$
3.		$\theta_B = \frac{pL^3}{6EI}$	$\Delta_B = \frac{pL^4}{8EI}$
4.		$\theta_A = \frac{ML}{3EI}$ for $\theta_A = 1$	$\theta_B = \frac{ML}{6EI}$ $M = \frac{3EI}{L}$
5.		$\theta_A = \theta_B = \frac{ML}{2EI}$	$\Delta_C = \frac{ML^2}{8EI}$
6.		$\theta_A = \theta_B = \frac{PL^2}{16EI}$	$\Delta_C = \frac{PL^3}{48EI}$
7.		$\theta_A = \frac{Pab(2L - a)}{6LEI}$ $\theta_B = \frac{Pa(L^2 - a^2)}{6LEI}$	$\Delta_C = \frac{Pa^2b^2}{3LEI}$
8.		$\theta_A = \theta_B = \frac{pL^3}{24EI}$	$\Delta_C = \frac{5pL^4}{384EI}$

(Contd)

(Contd)

S.No.	Structure	Expressions
9.		$M_A = M_B = \frac{PL}{8}$ $\quad \Delta_C = \frac{PL^3}{192EI}$
10.		$M_A = \frac{Pab^2}{L^2}$ $\quad M_B = \frac{Pba^2}{L^2}$ $\Delta_C = \frac{Pa^3b^3}{3L^3EI}$
11.		$M_A = M_B = \frac{pL^2}{12EI}$ $\quad \Delta_C = \frac{PL^4}{384EI}$
12.		$M_A = \frac{M}{2}$ $\quad \theta_B = \frac{ML}{4EI}$ $R_A = R_B = 1.5\,\frac{M}{L}$ For $\theta_B = 1$ $M = \frac{4EI}{L}$ $\quad M_A = \frac{2EI}{L}$ $R_A = R_B = \frac{6EI}{L^2}$
13.		$M_A = M_B = \frac{6EI\Delta}{L^2} = \frac{6EI\phi}{L}$ $R_A = R_B = \frac{12EI\Delta}{L^3} = \frac{12EI\phi}{L^2}$ For $\Delta = 1$ or $\phi = \frac{1}{L}$ $M_A = M_B = \frac{6EI}{L^2}$ $R_A = R_B = \frac{12EI}{L^3}$

PROBLEMS

2.1 Calculate the support reactions for the structures shown in Fig. 2.36 using the principal of virtual work. Verify the result using the equations of the static equilibrium.

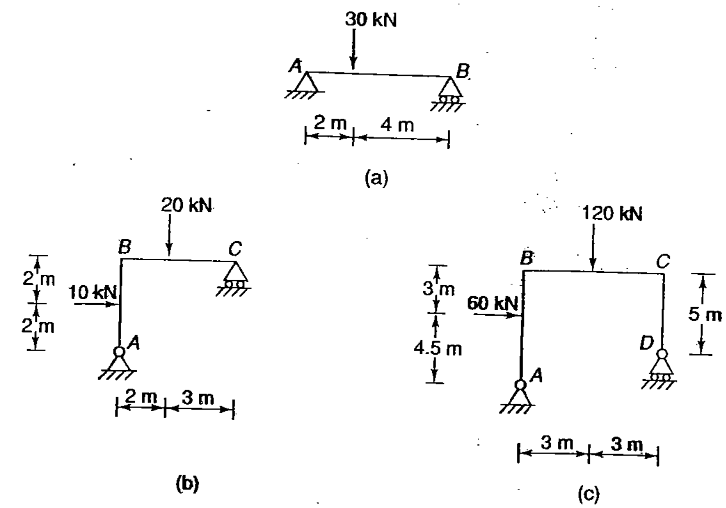


Fig. 2.36

2.2 Using the principle of virtual work, calculate the deflection at coordinate 1 for the beams shown in Fig. 2.37 due to (i) bending moment and (ii) shear force. The beams are of rectangular cross-section having width b and depth d.

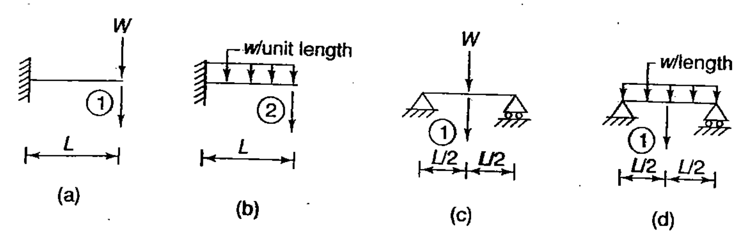


Fig. 2.37

2.3 Using the principle of virtual work, calculate the displacements in the structures of Fig. 2.38 at the coordinates shown in the figure due to (i) bending moment (ii) shear force and (iii) axial force. The structures are of uniform cross-section.

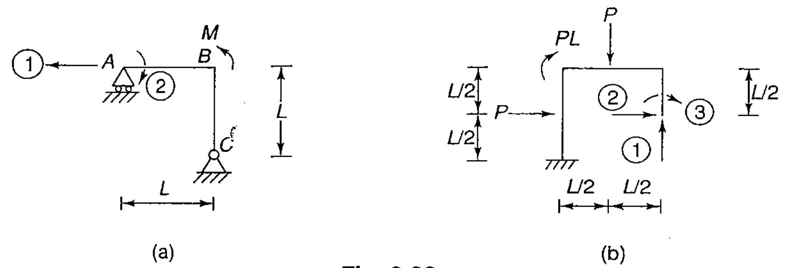


Fig. 2.38

2.4 In the continuous beam shown in Fig. 2.39, a vertical downward load of 50 kN at B produces an anti-clockwise rotation of 0.02 radian at C. Calculate the deflection at B due to a clockwise couple of 20 kN·m at C.

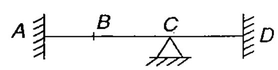


Fig. 2.39

2.5 Table 2.17 shows the loads and displacements for two systems of loads in a particular structure. Using the reciprocal theorem, calculate the displacement at coordinate 4 in system II.

Table 2.17

System	*Force and displacement*	*Coordinates* 1	2	3	4	5	6
I	P	5.0	2.5	1.0	3.0	0	0
	Δ	–	–	0.004	0.002	0.015	0.001
II	P'	0	0	2.0	3.5	4.0	3.0
	Δ'	0.010	0.002	0.005	?	–	–

2.6 Figure 2.40 shows two systems of loads and displacements. Calculate the horizontal displacement at B in system I using the reciprocal theorem.

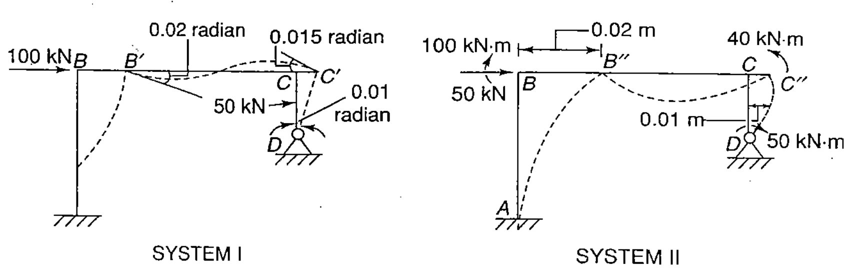


Fig. 2.40

2.7 Figure 2.41 shows three systems of loads and displacements. Using the reciprocal theorem, calculate the support reactions in system III.

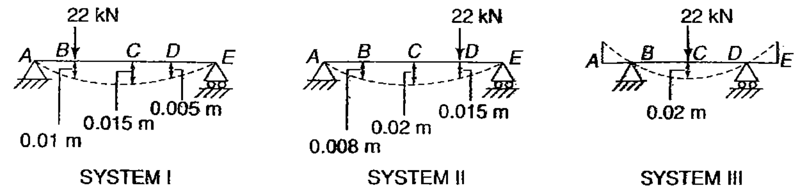


Fig. 2.41

2.8 Figure 2.42 shows three systems of loads and displacements. Using the reciprocal theorem, calculate the fixed-end moment M_A in system III.

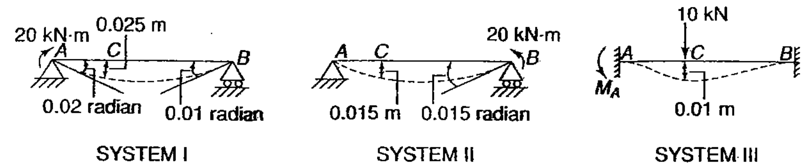


Fig. 2.42

2.9 For the rigid-jointed frames of Fig. 2.43, calculate the displacements at the coordinates shown in the figure using Castigliano's theorem. Verify the result by the unit-load method.

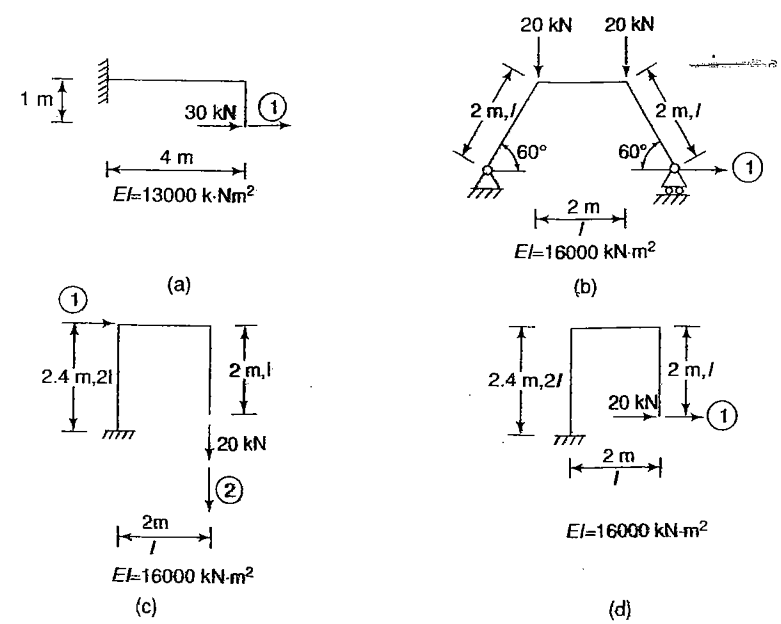


Fig. 2.43

2.10 For the pin-jointed frames of Fig. 2.44, calculate the displacements at the coordinates shown in the figure using Castigliano's theorem. Verify the result by the unit-load method.

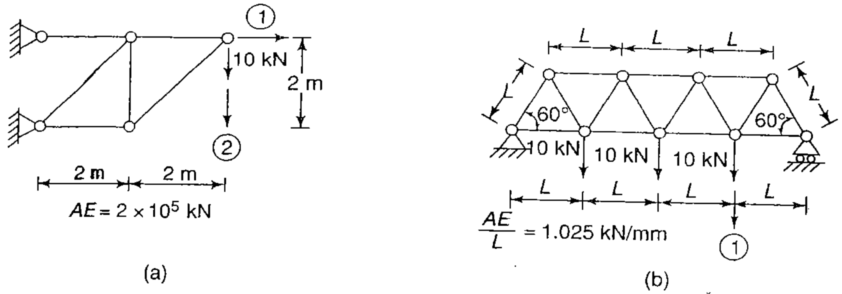


Fig. 2.44

2.11 For the simply supported beam shown in Fig. 2.45, determine the position of the point at which maximum deflection occurs. Also, calculate the maximum deflection. Use the conjugate-beam method.

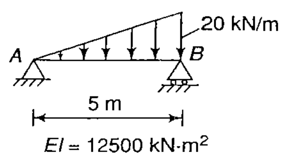


Fig. 2.45

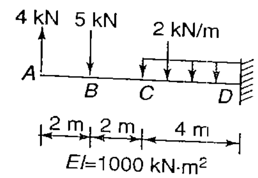


Fig. 2.46

2.12 In the cantilever beam of Fig. 2.46, determine the position and value of maximum deflection.

2.13 Figure 2.47 shows the bending-moment diagram for an intermediate span of a continuous beam. Calculate the deflection at B.

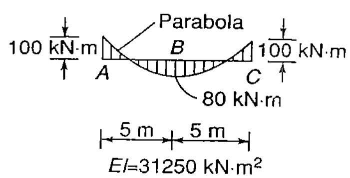


Fig. 2.47

2.14 For the beams of Fig. 2.48, calculate the displacements at the coordinates shown in the figure. Use any one of the following three methods:
(i) Castigliano's theorem
(ii) Unit-load method
(iii) Conjugate-beam method.
Verify the result by an alternative method.

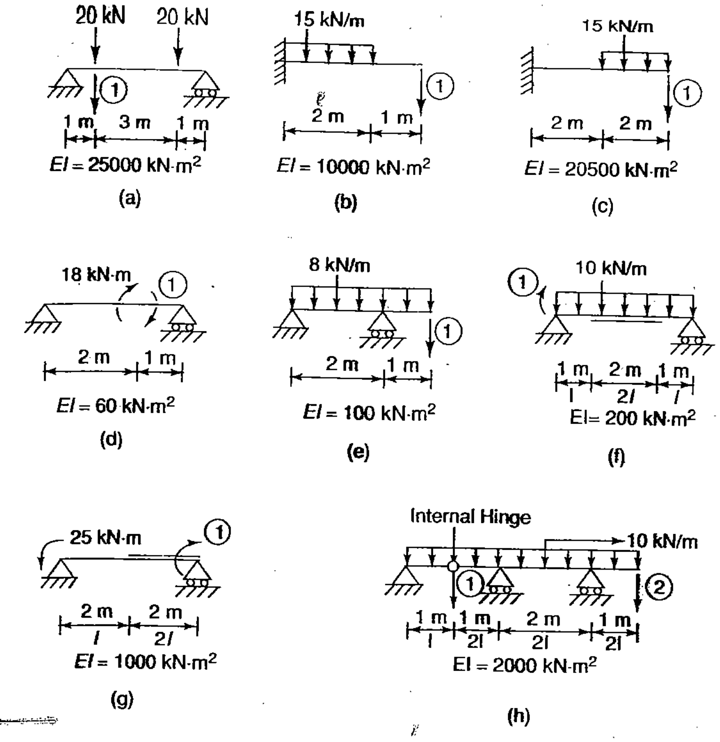


Fig. 2.48

2.15 Using Castigliano's theorem, analyse the structures shown in Fig. 2.49. Hence determine the support reaction at A. Verify the result by the unit-load method.

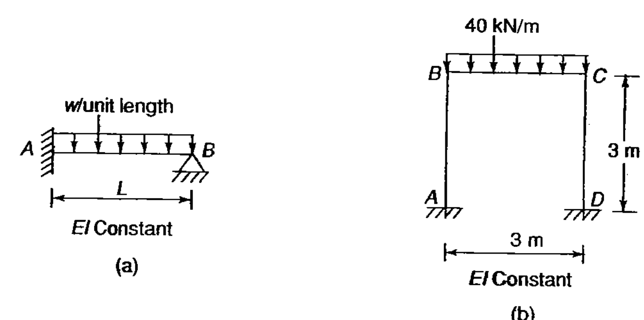


Fig. 2.49

2.16 In the fixed beam shown in Fig. 2.50, calculate the support reaction at A using the conjugate-beam method.

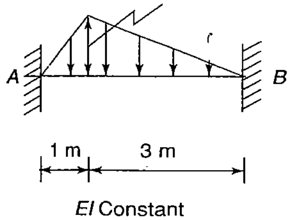


Fig. 2.50

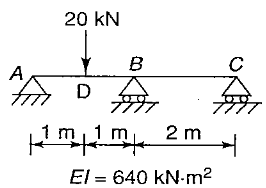


Fig. 2.51

2.17 Using the conjugate-beam method, analyse the two-span continuous beam shown in Fig. 2.51. Hence determine the deflection at D and rotation at A.

2.18 Using the slope-deflection equations, determine the bending moments at B and C in the two-span continous beam of Fig. 2.52 if the clockwise rotations at A and B are $2.22/EI$ radian and $1.19/EI$ radian respectively.

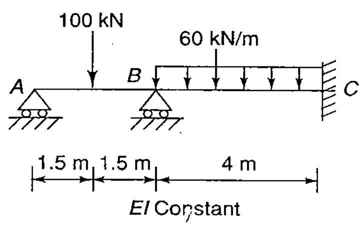


Fig. 2.52

2.19 In the portal frame of Fig. 2.53, the displacements with reference to the coordinates shown in figure are as follows:

$$\Delta_1 = \frac{3.56}{EI} \qquad \Delta_2 = \frac{2.67}{EI} \qquad \Delta_3 = 0$$

Using the slope-deflection equations, calculate the support reactions at A and D. Verify the result using the equations of static equilibrium. EI is constant.

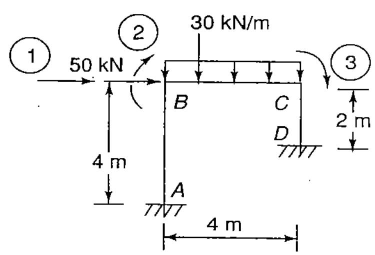


Fig. 2.53

2.20 Derive all the expressions given in Table 2.16 using the following methods:
(i) Castigliano's theorem
(ii) Unit-load method
(iii) Conjugate-beam method.

2.21 Using the reciprocal theorem, derive Muller-Breslau's principle according to which the influence line for the reaction component in a structure, whether statically determinate or otherwise, is the same as the deflection curve of the structure when a unit displacement is given to the reaction component for which the influence line is required.

2.22 Using the conjugate-beam method, derive Clapeyron's three-moments theorem for any two consecutive spans of continuous beam:

$$M_A\left(\frac{L_1}{I_1}\right) + 2MB\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_C\left(\frac{L_2}{I_2}\right)$$

$$= -\frac{6A_1\bar{x}_1}{L_1I_1} - \frac{6A_2\bar{x}_2}{L_2I_2} + 6E\left(\frac{\Delta_1}{L_1} + \frac{\Delta_2}{L_2}\right)$$

where referring to Fig. 2.54, M_A, M_B and M_C are the bending moments at supports A, B and C respectively, taking sagging bending moment as positive. Δ_1 and Δ_2 are the vertical distances by which supports A and C respectively are higher than middle support B. Also, A_1 and A_2 are the areas of the bending-moment diagrams for spans AB and BC considered as simply supported beams and $\bar{x}_1$ and $\bar{x}_2$ are the distances of centroids of these areas from supports A and C respectively.

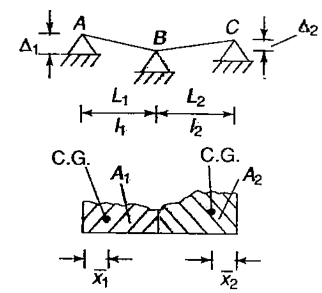


Fig. 2.54

3 DETERMINANTS AND MATRICES

3.1 DETERMINANTS

A square array of number is known as a *determinant*. The order of the determinant is n if it has n rows and n columns. A determinant of order n can be expressed by equation

$$|a| = \begin{vmatrix} a_{11} & a_{12}\dots a_{1j}\dots a_{1n} \\ a_{21} & a_{22}\dots a_{2j}\dots a_{2n} \\ \vdots & \\ a_{i1} & a_{i2}\dots a_{ij}\dots a_{in} \\ \vdots & \\ a_{n1} & a_{n2}\dots a_{nj}\dots a_{nn} \end{vmatrix} \tag{3.1}$$

In Eq. (3.1), the symbol a_{ij} denotes the element of the determinant lying in the ith row and jth column. The determinant of order $(n-1)$ obtained by erasing the ith row and jth column is known as *minor* of element a_{ij}. The *cofactor* of element a_{ij} is defined as the product of $(-1)^{i+j}$ and the minor of element a_{ij}.

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{3.2}$$

where C_{ij} = cofactor of element a_{ij}
M_{ij} = minor of element a_{ij}.

As Eq. (3.2) indicates, the cofactor is equal to the minor with a positive or negative sign attached to it depending upon whether the sum $(i+j)$ is even or odd. Hence, the cofactor is also known as *signed minor.*

Every determinant has a definite numerical value. The value of the determinant may be computed by using the Laplace expansion as indicated by Eq. (3.3).

$$|a| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in} \tag{3.3}$$
$$= \sum_{j=1}^{j=n} a_{ij}C_{ij}$$

In Eq. (3.3), the value of the determinant has been computed by multiplying each element of the ith row by its cofactor and taking the sum of the products.

Alternatively, the value of the determinant may be computed by multiplying each element of the jth column by its cofactor and taking the sum of the products.

$$|a| = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$$
$$= \sum_{i=1}^{i=n} a_{ij}C_{ij}$$

A determinant is known as non-singular if its value is non-zero.

The following important properties of determinants may be verified:

(i) If all the elements in one row or one column are zero, the determinant is zero.
(ii) When any two rows or two columns are interchanged, the sign of the determinant is changed.
(iii) If the elements in a row or column are multiplied by a constant and the result added to the corresponding elements in another row or column, the determinant is not changed.
(iv) If one row or column can be generated by a linear combination of other rows or columns, the determinant is zero. From this it follows that if two rows or two columns are identical, the determinant is zero.
(v) The sum of the products of the elements in any row i with the corresponding cofactors of another row m is zero.

Example 3.1

Compute the value of the determinant

$$|a| = \begin{vmatrix} 4 & 2 & 1 \\ 3 & 1 & 4 \\ 6 & 4 & 5 \end{vmatrix}$$

Solution

Using Eq. (3.2) and selecting the third row for the Laplace expansion,

$$|a| = 6(-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} + 4(-1)^{3+2}\begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix} + 5(-1)^{3+3}\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix} + 5\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix}$$

Now taking the first row for the Laplace expansion in each of the second order determinants,

$$|a| = 6\{2(-1)^{1+1}|4| + 1(-1)^{1+2}|1|\} - 4\{4(-1)^{1+1}|4| + 1(-1)^{1+2}|3|\}$$
$$+ 5\{4(-1)^{1+1}|1| + 2(-1)^{1+2}|3|\}$$
$$= 6(2\times 4 - 1\times 1) - 4(4\times 4 - 1\times 3) + 5(4\times 1 - 2\times 3) = -20.$$

Alternatively, selecting the second column for the Laplace expansion,

$$|a| = 2(-1)^{1+2}\begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} 4 & 1 \\ 6 & 5 \end{vmatrix} + 4(-1)^{3+2}\begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= -2\begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 6 & 5 \end{vmatrix} - 4\begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix}$$

Now taking the first column for the Laplace expansion in each of the second order determinant,

$$\begin{aligned} |a| &= -2\{3(-1)^{1+1}|5| + 6(-1)^{2+1}|4|\} \\ &\quad + 1\{4(-1)^{1+1}|5| + 6(-1)^{2+1}|1|\} \\ &\quad - 4\{4(-1)^{1+1}|4| + 3(-1)^{2+1}|1|\} \\ &= -2(3 \times 5 - 6 \times 4) + (4 \times 5 - 6 \times 1) - 4(4 \times 4 - 3 \times 1) \\ &= -20. \end{aligned}$$

3.2 MATRICES

A rectangular array of numbers is called a *matrix.* The *order of a matrix* is said to be $m \times n$ if it has m rows and n columns. If a matrix has only one row ($m = 1$), it is known as *row matrix.* On the other hand, if a matrix has only one column ($n = 1$), it is known as a *column matrix.* A matrix of order $m \times n$ may be expressed as

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj} & \cdots & a_{jn} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \tag{3.4}$$

The symbol a_{ij} represents an element lying in the ith row and jth column of matrix $[a]$. A row matrix of order $1 \times n$ may be expressed as

$$[a] = [a_{11}\ a_{12} \cdots a_{ij} \cdots a_{1n}] \tag{3.5}$$

Similarly, a column matrix of order $m \times 1$ may be expressed as

$$[a] = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{bmatrix} \tag{3.6}$$

A matrix is known as a *null or zero matrix* if all its elements are zero. It is denoted by the symbol [0]. A null matrix, in matrix algebra, corresponds to zero in ordinary algebra. If the rows and columns of matrix $[a]$ are interchanged, the resulting matrix is known as the *transpose* of matrix $[a]$ and is denoted by the symbol $[a]^T$. For instance,

if $$[a] = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 2 \end{bmatrix}$$

then $$[a]^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 5 & 2 \end{bmatrix}.$$

A matrix is known as a *square matrix* if the number of rows and columns is the same, i.e., $m = n$. A square matrix may be expressed as

$$[a] = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1j} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2j} \cdots a_{2n} \\ \vdots \\ a_{i1} & a_{i2} \cdots a_{ij} \cdots a_{in} \\ \vdots \\ a_{n1} & a_{n2} \cdots a_{nj} \cdots a_{nn} \end{bmatrix} \tag{3.7}$$

The elements $a_{11}, a_{22}, \ldots a_{nn}$ form the *leading* or *main diagonal* of the square matrix $[a]$. It is evident that an element a_{ij} lies on the leading diagonal if $i = j$. A *diagonal matrix* is square matrix which has zero elements everywhere except on the leading diagonal. A *unit* or *identity matrix* is a special case of a diagonal matrix in which all elements lying on the leading diagonal are unity and all other elements are zero. A unit matrix may be expressed as

$$[I] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{3.8}$$

A unit matrix, in matrix algebra, corresponds to *unity* in ordinary algebra. A *symmetrical* matrix is a square matrix whose elements are symmetrical about its main diagonal. Thus, $a_{ij} = a_{ji}$ in a symmetrical matrix. A symmetrical matrix remains unchanged if its columns and rows are interchanged, i.e., $[a]^T = [a]$. An *antisymmetrical* or *skew symmetrical matrix* is a square matrix whose elements are antisymmetrical about its main diagonal and all elements on the main diagonal are zero, i.e., $a_{ij} = -a_{ji}$ and $a_{ii} = 0$. It may be noted that if $[a]$ is antisymmetrical, $[a]^T = -[a]$.

$$[a] = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 8 & 7 \\ 4 & 7 & 5 \end{bmatrix} \text{ is a symmetrical matrix}$$

and $$[b] = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 8 \\ 3 & -8 & 0 \end{bmatrix} \text{ is an antisymmetrical matrix.}$$

The *determinant of a square matrix* has the same elements as the matrix. It is evident that only a square matrix can have its determinant because determinants are always square. It may be verified that the determinant of a matrix is the same as the determinant of its transpose. The elements on the main diagonal and all the elements above the main diagonal comprise the *upper triangle* of a square matrix. Similarly, the elements on the main diagonal and all the elements below the main diagonal comprise the *lower triangle* of a square matrix. A square matrix is said to be an *upper triangular matrix* if all its non-zero elements are confined to the upper triangle. Similarly, a square matrix is said to be a *lower triangular matrix* if all its non-zero elements are confined to the lower triangle.

A matrix is said to be *banded matrix,* if all its non-zero elements are located in the vicinity of the main diagonal. Thus, the non-zero elements of a banded matrix appear like a band along the main diagonal of the matrix. For instance, the matrix $[a]$ of order ten given by Eq. (3.9) is a banded matrix.

$$[a] = \begin{bmatrix} 2 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 9 & 6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 3 \end{bmatrix} \qquad (3.9)$$

In this matrix all the non-zero elements are located in the neighbourhood of the main diagonal. The half *band-width* of a banded matrix is governed by the non-zero element located farthest from the main diagonal. Thus, if the farthest non-zero element is the sth element counted from the main diagonal, the half band-width is s. In counting the position of the farthest non-zero element, the element on the main diagonal should be counted as one. The band-width of banded matrix is given by the equation

$$\text{Band-width} = (2s - 1) \qquad (3.10)$$

For instance, the farthest non-zero element in the matrix of Eq. (3.9) is third from the main diagonal. Hence, the half band-width is 3 and the band-width is $2 \times 3 - 1 = 5$. It may be noted that in the matrix of Eq. (3.9), the non-zero elements above the main diagonal control the band-width because the farthest non-zero element is located above the main diagonal. It may be noted that a *diagonal matrix* is a banded matrix with the band-width equal to 1. A banded matrix is said to be *tri-diagonal* if its band-width is three. Thus, in a tri-diagonal matrix all the non-zero elements are confined to the main diagonal and its immediate neighbours.

In carrying out matrix operations, it is sometimes convenient to partition them into submatrices and then deal with them separately. To do so, vertical and horizontal lines are drawn as shown below:

$$[a] = \left[\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ \hdashline a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{array}\right] = \left[\begin{array}{c:c} [A_{11}] & [A_{12}] \\ \hdashline [A_{21}] & [A_{22}] \end{array}\right] \qquad (3.11)$$

where

$$[A_{11}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad [A_{12}] = \begin{bmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \\ a_{34} & a_{35} \end{bmatrix}$$

$$[A_{21}] = \begin{bmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \qquad [A_{22}] = \begin{bmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{bmatrix}$$

In Eq. (3.11), $[A_{11}]$, $[A_{12}]$, $[A_{21}]$ and $[A_{22}]$ are known as the submatrices. The original matrix in the partitioned form is called a *partitioned matrix.* The submatrices may be treated as if they were elements of the partitioned matrix.

3.3 MATRIX ADDITION AND SUBTRACTION

Two matrices are equal only if each element of the first matrix is equal to the corresponding element of the second matrix. It follows that the two matrices are equal only if they are of the same order. Thus, $[a] = [b]$ if $a_{ij} = b_{ij}$. Addition and subtraction are possible only for matrices of the same order. Thus,

$$[a] + [b] = [c] \tag{3.12}$$

if $\quad c_{ij} = a_{ij} + b_{ij}$

and $\quad [a] - [b] = [d]$

if $$d_{ij} = a_{ij} - b_{ij} \tag{3.13}$$

For example,

if $$[a] = \begin{bmatrix} 7 & 7 \\ 2 & 5 \\ 6 & 9 \end{bmatrix} \text{ and } [b] = \begin{bmatrix} 5 & 4 \\ 1 & 2 \\ 7 & 6 \end{bmatrix}$$

then their sum, $\quad [a] + [b] = [c]$

or $$\begin{bmatrix} 7 & 7 \\ 2 & 5 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ 1 & 2 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 3 & 7 \\ 13 & 15 \end{bmatrix}$$

and their difference, $[a] - [b] = [d]$

or $$\begin{bmatrix} 7 & 7 \\ 2 & 5 \\ 6 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 2 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ -1 & 3 \end{bmatrix}$$

It may be noted that the associative and commutative laws are applicable in the case of matrix addition.

$$[a] + \{[b] + [c]\} = \{[a] + [b]\} + [c]$$
$$= [a] + [b] + [c]$$
$$[a] + [b] = [b] + [a] \tag{3.14}$$

3.4 MATRIX MULTIPLICATION

A matrix may be multiplied by a constant or by another matrix. In the multiplication of a matrix by a constant, every element of the matrix is multiplied by the constant. For example,

$$12\begin{bmatrix} 3 & 8 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 36 & 96 \\ 24 & 48 \end{bmatrix}$$

The product of two matrices may be defined as follows:

A matrix $[c]$ is known as the product of two matrices $[a]$ and $[b]$ if the element c_{ij} belonging to the ith row and jth column of matrix $[c]$ is obtained by the summation of the term-by-term product of the elements of the ith row of matrix $[a]$ and the jth column of matrix $[b]$. The multiplication of two matrices may be expressed by the equation

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2q} \\ \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{iq} \\ \vdots \\ a_{p1} & a_{p2} & \dots & a_{pj} & \dots & a_{pq} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2r} \\ \vdots \\ b_{i1} & b_{i2} & \dots & b_{ij} & \dots & b_{ir} \\ \vdots \\ b_{q1} & b_{q2} & \dots & b_{qj} & \dots & b_{qr} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1r} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2r} \\ \vdots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{ir} \\ \vdots \\ c_{p1} & c_{p2} & \dots & c_{pj} & \dots & c_{pr} \end{bmatrix} \tag{3.15}$$

where $\quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{iq}b_{qj}$

Equation (3.15) may be written in a compact form:

$$[a]\,[b] = [c]$$

where $$c_{ij} = \sum_{k=1}^{k=q} a_{ik}b_{kj} \tag{3.16}$$

In the multiplication of two matrices defined by Eqs. (3.15) and (3.16), the order in which the matrices are multiplied is important. In Eqs. (3.15) and (3.16), matrix $[b]$ is *premultiplied* by matrix $[a]$ or in other words matrix $[a]$ is *postmultiplied* by matrix $[b]$ in order to obtain matrix $[c]$.

The multiplication of partitioned matrices may be carried out in the usual manner if each submatrix is treated as an element of the partitioned matrix. Consider two partitioned matrices $[a]$ and $[b]$ given by the equation

$$[a] = \left[\begin{array}{c|c} [A_{11}] & [A_{12}] \\ \hline [A_{21}] & [A_{22}] \end{array}\right] \qquad [b] = \left[\begin{array}{c|c} [B_{11}] & [B_{12}] \\ \hline [B_{21}] & [B_{22}] \end{array}\right] \tag{3.17a}$$

where $[A_{11}]$, $[A_{12}]$, $[A_{21}$ and $[A_{22}]$ are the submatrices of partitioned matrix $[a]$ and $[B_{11}]$, $[B_{12}]$, $[B_{21}]$ and $[B_{22}]$ are the submatrices of partitioned matrix $[b]$. The product of patritioned matrices $[a]$ and $[b]$ is given by the equation

$$[a][b] = [c] = \left[\begin{array}{c|c} [C_{11}] & [C_{12}] \\ \hline [C_{21}] & [C_{22}] \end{array}\right] \tag{3.17b}$$

where $[C_{11}]$, $[C_{12}]$, $[C_{21}]$ and $[C_{22}]$ are the submatrices of product matrix $[c]$ and are given by the equations

$$\begin{aligned} [C_{11}] &= [A_{11}]\,[B_{11}] + [A_{12}]\,[B_{21}] \\ [C_{12}] &= [A_{11}]\,[B_{12}] + [A_{12}]\,[B_{22}] \\ [C_{21}] &= [A_{21}]\,[B_{11}] + [A_{22}]\,[B_{21}] \\ [C_{22}] &= [A_{21}]\,[B_{12}] + [A_{22}]\,[B_{22}] \end{aligned} \tag{3.17c}$$

The following points are evident from the multiplication of the two matrices:

(i) For the multiplication of two matrices to be possible, the number of columns in the first matrix must be equal to the number of rows in the second matrix. In other words, two matrices are said to be conformable in the order $[a]\ [b]$ if the number of columns in $[a]$ is equal to the number of rows in $[b]$.

(ii) If a matrix of order $p \times q$ is postmultiplied by a matrix of order $q \times r$, the order of the product matrix is $p \times r$.

(iii) Even if two matrices are conformable in either order, multiplication in general is not commutative, i.e.,

$$[a]\ [b] \neq [b]\ [a] \tag{3.18}$$

(iv) The associative and distributive laws are applicable to matrix multiplication, provided the sequence of matrices is strictly maintained, i.e.,

$$\{[a]\ [b]\}\ [c] = [a]\ \{[b]\ [c]\}$$

and
$$[a]\ \{[b] + [c]\} = [a]\ [b] + [a]\ [c] \tag{3.19}$$

(v) Multiplication of any two partitioned matrices is possible, only if,
   (a) the two matrices before partitioning are conformable.
   (b) treating each submatrix as an element in the partitioned matrices, the partitioned matrices are conformable.
   (c) the submatrices, which are to be multiplied with each other, are also conformable.

Example 3.2

Postmultiply and premultiply [a] by [b], given that,

$$[a] = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 1 & 3 \\ 4 & 2 & 2 & 1 \end{bmatrix} \qquad [b] = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Solution

(i) Postmultiplication of $[a]$ by $[b]$

Let
$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 \cdots & 1 \cdots & 1 \cdots & 3 \\ 4 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ & & \vdots \\ 3 & 1 & 1 \\ & & \vdots \\ 2 & 1 & 3 \\ & & \vdots \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & \underline{c_{23}} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

In order to obtain any element such as c_{23}, belonging to the second row and third column of the product matrix, take the summation of the term-by-term product of the elements of the second row of matrix $[a]$ and third column of matrix $[b]$. Thus,

$$c_{23} = 2 \times 2 + 1 \times 1 + 1 \times 3 + 3 \times 2 = 14$$

The other elements of the product matrix can be similarly written. The result is expressed by the following equation:

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 1 & 3 \\ 4 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 17 & 21 \\ 10 & 19 & 14 \\ 15 & 23 & 18 \end{bmatrix}$$

(ii) Premultiplication of $[a]$ by $[b]$

Let
$$\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ 2 \cdots & 1 \cdots & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 & 2 \\ & \vdots & & \\ 2 & 1 & 1 & 3 \\ & \vdots & & \\ 4 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & \underline{c_{32}} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

In order to obtain any element such as c_{32} belonging to the second row and second column of the product matrix, take the summation of the term-by-term product of the elements of the third row of matrix $[b]$ and second column of matrix $[a]$. Thus,

$$c_{32} = 2 \times 3 + 1 \times 1 + 3 \times 2 = 13$$

The other elements of the product matrix may be similarly written. The result is expressed by the following equation:

$$\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 1 & 3 \\ 4 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 11 & 12 & 16 \\ 9 & 12 & 15 & 10 \\ 16 & 13 & 15 & 10 \\ 15 & 10 & 11 & 13 \end{bmatrix}$$

It may be noted that $[a][b] \neq [b][a]$ as mentioned in Eq. (3.18).

Example 3.3

Premultiply and postmultiply [a] by [b], given that,

$$[a] = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix} \qquad [b] = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix}$$

Solution

(i) Premultiplication of $[a]$ by $[b]$

$$\text{Let}\quad \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & \cdots\ 0.15 & \cdots\ 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix} \begin{bmatrix} 30 & -10 & 0 \\ & \vdots & \\ -10 & 15 & -5 \\ & \vdots & \\ 0 & -5 & 5 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & \underline{c_{22}} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

In order to obtain any element such as c_{22} belonging to the second row and second column of the product matrix, take the summation of the term-by-term product of the elements of second row of matrix $[b]$ and second column of matrix $[a]$. Thus,

$$c_{22} = (0.05)(-10) + (0.15)(15) + (0.15)(-5) = 1$$

The other elements of the product matrix may be similarly written. The result is expressed by the following equation:

$$\begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix} \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) Postmultiplication of $[a]$ by $[b]$

$$\text{Let}\quad \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0\ \cdots & -5\ \cdots & 5 \end{bmatrix} \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ & \vdots & \\ 0.05 & 0.15 & 0.15 \\ & \vdots & \\ 0.05 & 0.15 & 0.35 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & \underline{c_{32}} & c_{33} \end{bmatrix}$$

In order to obtain any element such as c_{32}, belonging to the third row and second column of the product matrix, take the summation of the term-by-term product of the elements of the third row of matrix $[a]$ and second column of matrix $[b]$. Thus,

$$c_{32} = (0)(0.05) + (-5)(0.15) + (5)(0.15) = 0$$

The other elements of the product matrix may be similarly written. The result is expressed by the following equation:

$$\begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3.4

Obtain the product of the partitioned matrices $[a]$ and $[b]$ given by the equations

$$[a] = \left[\begin{array}{ccc:cc} 2 & 5 & 3 & 1 & 6 \\ 1 & 2 & 4 & 3 & 2 \\ 3 & 0 & 2 & 7 & 1 \\ \hdashline 1 & 3 & 0 & 2 & 7 \\ 2 & 5 & 1 & 4 & 3 \end{array}\right] \qquad [b] = \begin{bmatrix} 2 \\ 3 \\ 1 \\ \cdots \\ 4 \\ 2 \end{bmatrix}$$

Solution

$$\text{Let}\qquad [a]\,[b] = \left[\begin{array}{c:c} [A_{11}] & [A_{12}] \\ \hdashline [A_{21}] & [A_{22}] \end{array}\right] \left[\begin{array}{c} [B_{11}] \\ \hdashline [B_{21}] \end{array}\right] = \left[\begin{array}{c} [C_{11}] \\ \hdashline [C_{21}] \end{array}\right]$$

Treating the submatrices as ordinary elements,

$$[C_{11}] = [A_{11}][B_{11}] + [A_{12}][B_{21}]$$

$$= \begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 4 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 3 & 2 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 28 \\ 28 \\ 38 \end{bmatrix}$$

$$[C_{21}] = [A_{21}][B_{11}] + [A_{22}][B_{21}]$$

$$= \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 33 \\ 42 \end{bmatrix}$$

Hence, the product matrix is

$$\begin{bmatrix} 38 \\ 28 \\ 38 \\ 33 \\ 42 \end{bmatrix}$$

3.5 MATRIX INVERSION

Two square matrices $[a]$ and $[b]$ of the same order are known as inverse or reciprocal of one another if their product in either order is an identity matrix. Thus, matrices $[a]$ and $[b]$ are the inverse of each other if,

$$[a]\,[b] = [b]\,[a] = [I] \tag{3.20}$$

For example, matrices $[a]$ and $[b]$ of Ex. 3.3 are the inverse of each other. As matrix $[b]$ is the inverse of matrix $[a]$, it may be expressed by the symbol $[a]^{-1}$. Thus, either the premultiplication or post multiplication of matrix $[a]$ by its inverse $[a]^{-1}$ gives the identity matrix. The operation of inverting a matrix is equivalent to matrix division. There are several methods for the determination of the inverse of a matrix. Four of them which are frequently used have been described here.

3.5.1 Adjoint Method of Matrix Inversion

This method may be expressed symbolically by the equation

$$[a]^{-1} = \frac{\text{Adj}\,[a]}{|\,a\,|} \tag{3.21}$$

where the adjoint of matrix $[a]$ denoted by the symbol Adj $[a]$ is obtained by transposing matrix $[a]$ and then replacing each element of the transposed matrix by its cofactor. In Eq. (3.21), the symbol $|a|$ denotes the determinant of matrix $[a]$. The elements of the determinant $|a|$ are the same as the elements of the matrix $[a]$.

From Eq. (3.21) it is evident that:

(i) the inverse of a matrix can be determined only if it is a square matrix because only a square matrix has its determinant.

(ii) the inverse of a matrix exists only if its determinant is non-singular, i.e., the value of determinant is non-zero.

The following are some of the additional properties of matrices. These properties are useful in matrix inversion.

(i) The inverse of a matrix is unique.

(ii) The inverse of the transpose of a matrix is equal to the transpose of the matrix. Thus,

$$\{[a]^T\}^{-1} = \{[a]^{-1}\}^T \tag{3.22}$$

(iii) The inverse of the product of two matrices is equal to the product of the inverse of two matrices in the reverse order. Thus,

$$\{[a][b]\}^{-1} = [b]^{-1}[a]^{-1} \tag{3.23}$$

(iv) The inverse of a symmetrical matrix is also a symmetrical matrix.

(v) The inverse of a triangular matrix is also a triangular matrix of the same type. Thus, the inverse of an upper triangular matrix is also an upper triangular matrix. Similarly, the inverse of a lower triangular matrix is also a lower triangular matrix.

(vi) If matrix $[a]$ is a triangular matrix, the elements on the main diagonal of $[a]^{-1}$ are the reciprocals of the elements on the main diagonal of $[a]$.

(vii) Any matrix, when multiplied by its transpose, results in symmetrical matrix.

3.5.2 Inversion by Linear Transformations (The Gauss-Jordan Method)

This is one of the fastest methods of inversion of a matrix by a computer. In this method, matrix $[a]$ of order n, which is to be inverted, is first premultiplied by a square matrix $[t_1]$ which causes the product $[t_1][a] = [b]$ to have the first column of the identity matrix of order n. If $a_{11}, a_{21}, \ldots, a_{n1}$ are the elements of the first column of matrix $[a]$, $\frac{1}{a_{11}}, -\frac{a_{21}}{a_{11}}, -\frac{a_{31}}{a_{11}}, \ldots, -\frac{a_{n1}}{a_{11}}$ constitute the elements of the first column of matrix $[t_1]$. All the remaining columns of matrix $[t_1]$ are similar to those of an identity matrix of order n. Matrix $[b]$ is then premultiplied by a square matrix $[t_2]$ which causes the product $[t_2][b] = [c]$ to have the first and second columns of identity matrix of order n. If $b_{12}, b_{22}, \ldots, b_{n2}$ are the elements of the second column of matrix $[b]$, then $-\frac{b_{12}}{b_{22}}, \frac{1}{b_{22}}, -\frac{b_{32}}{b_{22}}, \ldots -\frac{b_{n2}}{b_{22}}$ are the elements of the second column of matrix $[t_2]$. All the remaining columns of matrix $[t_2]$ are similar to those of an identity matrix of order n. This is repeated n times until the final product is an identity matrix of order n, i.e.,

$$[t_n][t_{n-1}] \ldots [t_2][t_1][a] = [I]$$

or

$$[a]^{-1} = [t_n][t_{n-1}] \ldots [t_2][t_1] \tag{3.24}$$

3.5.3 Inversion by Factorization (The Choleski Method)

In this method, the matrix $[a]$ to be inverted is premultiplied by its transpose. The product matrix $[a]^T[a] = [b]$ is a symmetrical matrix. From the matrix property given by Eq. (3.24),

$$[a]^{-1}\{[a]^T\}^{-1} = [b]^{-1}$$

Postmultiplying both sides by $[a]^T$,

$$[a]^{-1}\{[a]^T\}^{-1}[a]^T = [b]^{-1}[a]^T$$

or

$$[a]^{-1} = [b]^{-1}[a]^T \tag{a}$$

Thus, the problem of inverting matrix $[a]$ is reduced to the determination of the inverse of symmetrical matrix $[b]$ and then postmultiplying it by the transpose of matrix $[a]$. To invert symmetrical matrix $[b]$, it is first equated to the product of a lower triangular matrix $[l]$ and its transpose $[l]^T$.

$$[b] = [l][l]^T \tag{b}$$

Let

$$[b] = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{21} & b_{22} & b_{32} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

and

$$[l] = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

Equation (b) can now be written as

$$\begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{21} & b_{22} & b_{32} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \tag{c}$$

The elements of matrix $[l]$ can be obtained by comparing the corresponding elements on both sides of Eq. (c) taken in the following order:

$$\begin{aligned} b_{11} &= l^{1}_{11} \\ b_{21} &= l_{11}l_{21} \\ b_{31} &= l_{11}l_{31} \\ b_{22} &= l^{2}_{21} + l^{2}_{22} \\ b_{32} &= l_{21}l_{31} + l_{22}l_{32} \\ b_{33} &= l^{2}_{31} + l^{2}_{32} + l^{2}_{33} \end{aligned} \tag{d}$$

Inverting both sides of Eq. (b) and postmultiplying by $[l]$,

$$[b]^{-1}[l] = \{[l]^T\}^{-1} \tag{e}$$

Let

$$[b]^{-1} = [c] = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{21} & c_{22} & c_{32} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Using the matrix property that the elements on the main diagonal of $\{[l]^T\}^{-1}$ are the reciprocal of the elements on the main diagonal of $[l]^T$, Eq. (e) may be written as

$$\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{21} & c_{22} & c_{32} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{l_{11}} & ? & ? \\ 0 & \dfrac{1}{l_{22}} & ? \\ 0 & 0 & \dfrac{1}{l_{33}} \end{bmatrix} \tag{f}$$

The elements of matrix $[c]$ may be obtained by comparing the corresponding elements on both sides of Eq. (f) taken in the following order:

$$\begin{aligned} & c_{33}l_{33} = \frac{1}{l_{33}} \\ & c_{32}l_{22} + c_{33}l_{32} = 0 \\ & c_{31}l_{11} + c_{32}\,l_{21} + c_{33}l_{31} = 0 \\ & c_{22}l_{22} + c_{32}l_{32} = \frac{1}{l_{22}} \\ & c_{21}l_{11} + c_{22}l_{21} + c_{32}l_{31} = 0 \\ & c_{11}l_{11} + c_{21}l_{21} + c_{31}l_{31} = \frac{1}{l_{11}} \end{aligned} \tag{g}$$

Thus all the elements of matrix $[c] = [b]^{-1}$ are known. Substituting these values in Eq. (a), the inverse of matrix $[a]$ is obtained. If the given matrix is symmetrical, then Eq. (b) is the starting point. Though the method has been explained by taking a matrix of order 3, it is applicable to a matrix of any size.

3.5.4 Inversion by Partitioning

If matrix $[b]$ is the inverse of matrix $[a]$, then

$$[a][b] = [I] \tag{h}$$

Partitioning the matrices of Eq. (h) into four sub-matrices, the following equation is obtained:

$$\left[\begin{array}{c:c} [A_{11}] & [A_{12}] \\ \hdashline [A_{21}] & [A_{22}] \end{array}\right] \left[\begin{array}{c:c} [B_{11}] & [B_{12}] \\ \hdashline [B_{21}] & [B_{22}] \end{array}\right] = \left[\begin{array}{c:c} [I] & [0] \\ \hdashline [0] & [I] \end{array}\right] \tag{i}$$

Comparing the corresponding terms on both sides of Eq. (i),

$$\begin{aligned} [A_{11}][B_{11}] + [A_{12}][B_{21}] &= [I] \\ [A_{11}][B_{12}] + [A_{12}][B_{22}] &= [0] \\ [A_{21}][B_{11}] + [A_{22}][B_{21}] &= [0] \\ [A_{21}][B_{12}] + [A_{22}][B_{22}] &= [I] \end{aligned} \tag{j}$$

Solving Eq. (j),

$$\begin{aligned} [B_{22}] &= \{[A_{22}] - [A_{21}][A_{11}]^{-1}[A_{12}]\}^{-1} \\ [B_{12}] &= -[A_{11}]^{-1}[A_{12}][B_{22}] \\ [B_{21}] &= -[B_{22}][A_{21}]\,[A_{11}]^{-1} \\ [B_{11}] &= [A_{11}]^{-1} - [A_{11}]^{-1}[A_{12}][B_{21}] \end{aligned} \tag{3.25}$$

Thus all the elements of $[b] = [a]^{-1}$ are obtained.

Example 3.5

Invert matrix [a] given by the equation

$$[a] = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

Solution

(i) *Adjoint Method*

Since the given matrix is a symmetrical matrix, its transpose remains the same. The determinant of the transposed matrix may be written as

$$\begin{vmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{vmatrix}$$

The cofactors of the elements of the determinant are determined as follows:

$$C_{11} = \text{cofactor of } a_{11} = \begin{vmatrix} 15 & -5 \\ -5 & 5 \end{vmatrix} (-1)^{1+1} = 50$$

$$C_{12} = \text{cofactor of } a_{12} = \begin{vmatrix} -10 & -5 \\ 0 & 5 \end{vmatrix} (-1)^{1+2} = 50$$

$$C_{13} = \text{cofactor of } a_{13} = \begin{vmatrix} -10 & 15 \\ 0 & -5 \end{vmatrix} (-1)^{1+3} = 50$$

$$C_{21} = \text{cofactor of } a_{21} = \begin{vmatrix} -10 & 0 \\ -5 & 5 \end{vmatrix} (-1)^{2+1} = 50$$

$$C_{22} = \text{cofactor of } a_{22} = \begin{vmatrix} 30 & 0 \\ 0 & 5 \end{vmatrix} (-1)^{2+2} = 150$$

$$C_{23} = \text{cofactor of } a_{23} = \begin{vmatrix} 30 & -10 \\ 0 & -5 \end{vmatrix} (-1)^{2+3} = 150$$

$$C_{31} = \text{cofactor of } a_{31} = \begin{vmatrix} -10 & 0 \\ 15 & -5 \end{vmatrix} (-1)^{3+1} = 50$$

$$C_{32} = \text{cofactor of } a_{32} = \begin{vmatrix} 30 & 0 \\ -10 & -5 \end{vmatrix} (-1)^{3+2} = 150$$

$$C_{33} \text{ cofactor of } a_{33} = \begin{vmatrix} 30 & -10 \\ -10 & 15 \end{vmatrix} (-1)^{3+3} = 350$$

The adjoint of the given matrix is

$$\begin{bmatrix} 50 & 50 & 50 \\ 50 & 150 & 150 \\ 50 & 150 & 350 \end{bmatrix}$$

Using Eq. (3.3), the determinant of the given matrix,

$$\begin{aligned} |a| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 30 \times 50 + (-10) \times 50 + 0 \times 50 \\ &= 1000 \end{aligned}$$

The inverse of the given matrix is

$$[a]^{-1} = \frac{1}{1000}\begin{bmatrix} 50 & 50 & 50 \\ 50 & 150 & 150 \\ 50 & 150 & 350 \end{bmatrix}$$

$$= \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix}$$

(ii) *Inversion by Linear Transformations*

$$[a] = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

$$[t_1] = \begin{bmatrix} \frac{1}{30} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[b] = [t_1][a] = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & \frac{35}{3} & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

$$[t_2] = \begin{bmatrix} 1 & \frac{1}{35} & 0 \\ 0 & \frac{3}{35} & 0 \\ 0 & \frac{15}{35} & 1 \end{bmatrix}$$

$$[c] = [t_2][b] = \begin{bmatrix} 1 & 0 & -\frac{1}{7} \\ 0 & 1 & -\frac{3}{7} \\ 0 & 0 & \frac{20}{7} \end{bmatrix}$$

$$[t_3] = \begin{bmatrix} 1 & 0 & \frac{1}{20} \\ 0 & 1 & \frac{3}{20} \\ 0 & 0 & \frac{7}{20} \end{bmatrix}$$

$$[a]^{-1} = [t_3][t_2][t_1] = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix}$$

(iii) ***Inversion by Factorization***

The given matrix is a symmetrical matrix. In the first instance it is equated to the product of a lower triangular matrix and its transpose.

$$\text{Let}\quad \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \tag{a}$$

Comparing the corresponding terms on both sides of Eq. (a),

$$\begin{aligned} 30 &= l_{11}^2 \\ -10 &= l_{11}l_{21} \\ 0 &= l_{11}l_{31} \\ 15 &= l_{21}^2 + l_{22}^2 \\ -5 &= l_{21}l_{31} + l_{22}l_{32} \\ 5 &= l_{31}^2 + l_{32}^2 + l_{33}^2 \end{aligned} \tag{b}$$

Solving Eq. (b),

$$\begin{aligned} l_{11} &= \sqrt{30} & l_{21} &= -\sqrt{\frac{10}{3}} & l_{31} &= 0 \\ l_{22} &= \sqrt{\frac{35}{3}} & l_{32} &= -\sqrt{\frac{15}{7}} & l_{33} &= \sqrt{\frac{20}{7}} \end{aligned} \tag{c}$$

Inverting both sides of Eq. (a) and postmultiplying by $[l]$,

$$\begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}^{-1} \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}^{-1} \tag{d}$$

$$\text{Let}\quad \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{21} & c_{22} & c_{32} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{e}$$

Substituting from Eqs (c) and (e) into Eq. (d),

$$\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{21} & c_{22} & c_{32} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ -\sqrt{\frac{10}{3}} & \sqrt{\frac{35}{3}} & 0 \\ 0 & -\sqrt{\frac{15}{7}} & \sqrt{\frac{20}{7}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{30}} & ? & ? \\ 0 & \sqrt{\frac{3}{35}} & ? \\ 0 & 0 & \sqrt{\frac{7}{20}} \end{bmatrix} \tag{f}$$

Comparing the corresponding terms on both sides of Eq. (f),

$$c_{33}\sqrt{\frac{20}{7}} = \sqrt{\frac{7}{20}}$$

$$c_{32}\sqrt{\frac{35}{3}} + c_{33}\left(-\sqrt{\frac{15}{7}}\right) = 0$$

$$c_{31}\sqrt{30} + c_{32}\left(-\sqrt{\frac{10}{3}}\right) + c_{33}(0) = 0$$

$$c_{22}\sqrt{\frac{35}{3}} + c_{32}\left(-\sqrt{\frac{15}{7}}\right) = \sqrt{\frac{3}{35}}$$

$$c_{21}\sqrt{30} + c_{22}\left(-\sqrt{\frac{10}{3}}\right) + c_{32}(0) = 0 \tag{g}$$

$$c_{11}\sqrt{30} + c_{21}\left(-\sqrt{\frac{10}{3}}\right) + c_{31}(0) = \frac{1}{\sqrt{30}}$$

Solving Eq. (g),

$$\begin{aligned} c_{11} &= 0.05 & c_{21} &= 0.05 & c_{31} &= 0.05 \\ c_{22} &= 0.15 & c_{32} &= 0.15 & c_{33} &= 0.35 \end{aligned} \tag{h}$$

Substituting from Eq. (h) into Eq. (e), the inverse of the given matrix is

$$[a]^{-1} = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix}$$

(iv) *Inversion by Partitioning*

Let the given matrix be partitioned as

$$[a] = \begin{bmatrix} [A_{11}] & [A_{12}] \\ [A_{21}] & [A_{22}] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 30 & -10 \\ -10 & 15 \end{bmatrix} & \begin{bmatrix} 0 \\ -5 \end{bmatrix} \\ \begin{bmatrix} 0 & -5 \end{bmatrix} & [5] \end{bmatrix}$$

If $[b]$ is the inverse of matrix $[a]$ and is partitioned as

$$[b] = \begin{bmatrix} [B_{11}] & [B_{12}] \\ [B_{21}] & [B_{22}] \end{bmatrix}$$

then using Eq. (3.26),

$$[B_{22}] = \left\{[5] - [0\ -5]\begin{bmatrix} 30 & -10 \\ -10 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -5 \end{bmatrix}\right\}^{-1} = [0.35]$$

$$[B_{12}] = -\begin{bmatrix} 30 & -10 \\ -10 & 15 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -5 \end{bmatrix}[0.35] = \begin{bmatrix} 0.05 \\ 0.15 \end{bmatrix}$$

$$[B_{21}] = -[0.35][0\ -5]\begin{bmatrix} 30 & -10 \\ -10 & 15 \end{bmatrix}^{-1} = [0.05\quad 0.15]$$

$$[B_{11}] = \begin{bmatrix} 30 & -10 \\ -10 & 15 \end{bmatrix}^{-1} - \begin{bmatrix} 30 & -10 \\ -10 & 15 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -5 \end{bmatrix}[0.05\quad 0.15]$$

$$= \begin{bmatrix} 0.05 & 0.05 \\ 0.05 & 0.15 \end{bmatrix}$$

Hence, the inverse of the given matrix is

$$[a]^{-1} = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix}$$

Example 3.6

Solve the matrix equation

$$\left[\begin{array}{cc|cc} 2 & 1 & 4 & 3 \\ 1 & 2 & 2 & 3 \\ \hline 3 & 1 & 4 & 2 \\ 1 & 5 & 2 & 4 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \hline 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ \hline x_3 \\ x_4 \end{bmatrix}$$

Solution

Partitioning the matrices as indicated by the dotted lines, the given matrix equation may be split up into the following two equations:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \qquad \text{(a)}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \qquad \text{(b)}$$

Equation (a) may be rewritten as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \qquad \text{(c)}$$

Substituting from Eq. (c) into Eq. (b),

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 21 \end{bmatrix} \qquad \text{(d)}$$

Thus the unknowns are

$$x_1 = -4 \qquad x_2 = 3$$
$$x_3 = -1 \qquad x_4 = 21$$

3.6 SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

Consider the matrix multiplication indicated by the equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{bmatrix} \qquad (3.26)$$

Using the rules for matrix multiplication, element c_i of the product matrix on the right hand side of the equation may be expressed as

$$c_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n$$

The other elements of the product matrix may be similarly determined. Hence, Eq. (3.26) is equivalent to the following set of simultaneous equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n &= c_i \qquad (3.27) \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nj}x_j + \ldots + a_{nn}x_n &= c_n \end{aligned}$$

It follows that the set of simultaneous equations [Eq. (3.27)] may be expressed in the matrix form by Eq. (3.26). It can be expressed in a compact matrix form by the equation

$$[a][x] = [c] \qquad (3.28)$$

In Eq. (3.28), $[a]$ is known as *coefficient matrix*. The solution of the set of simultaneous equations may be obtained by premultiplying both sides of Eq. (3.28) by the inverse of the coefficient matrix, i.e., $[a]^{-1}$. Thus,

$$[a]^{-1}[a][x] = [a]^{-1}[c]$$

or

$$[x] = [a]^{-1}[c] \qquad (3.29)$$

If the product of two matrices on the right hand side of Eq. (3.29) is matrix $[b]$, then

$$[x] = [b]$$

or

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_2 \\ &\vdots \\ x_i &= b_i \\ &\vdots \\ x_n &= b_n \end{aligned}$$

Example 3.7

Solve the following simultaneous equations:

$$\begin{aligned} 30x_1 - 10x_2 \quad &= 2 \\ -10x_1 + 15x_2 - 5x_3 &= 3 \\ -5x_2 + 5x_3 &= 5 \end{aligned}$$

Solution

The above set of simultaneous equations can be written in the matrix form

$$\begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Premultiplying both sides of the above equation by the inverse of the coefficient matrix,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 1.3 \\ 2.3 \end{bmatrix}$$

3.7 CONDITIONING OF MATRICES

A set of simultaneous equations is said to be *well-conditioned* if a small error in the coefficients of the variables does not make an appreciable change in the roots of the equations. On the other hand, the set of equations is said to be *ill-conditioned,* if a small error in the coefficients leads to large variations in the roots. Consider, for example, the following set of two simultaneous equations:

$$3x_1 + 10x_2 = 49 \qquad \text{(a)}$$
$$4x_1 - x_2 = 8 \qquad \text{(b)}$$

The roots of the equations are

$$x_1 = 3 \qquad x_2 = 4 \qquad \text{(c)}$$

Suppose an error of 1% occurs in one of the coefficients of Eq. (a) so that instead of Eqs (a) and (b), the following set of equations is obtained:

$$3x_1 + 9.9x_2 = 49 \qquad \text{(d)}$$
$$4x_1 - x_2 = 8 \qquad \text{(e)}$$

The roots of these equations are

$$x_1 = 3.009 \qquad x_2 = 4.037 \qquad \text{(f)}$$

Comparing the roots given by Eqs (c) and (f), it may be noted that the error in the roots does not exceed 1% when the error in one of the coefficients is 1%. It follows that a small change in the coefficients leads to a small variation in the roots. Hence, Eqs. (a) and (b) are well-conditioned. Consider next, the set of equations

$$10x_1 + x_2 = 12.0 \qquad \text{(g)}$$
$$9.9x_1 + x_2 = 11.9 \qquad \text{(h)}$$

The roots of the equations are

$$x_1 = 1.0 \qquad x_2 = 2.0 \qquad \text{(i)}$$

Suppose an error of 1% occurs in one of the coefficients so that instead of Eqs (g) and (h), the following set of equations is obtained:

$$10x_1 + x_2 = 12.0 \qquad \text{(j)}$$
$$9.8x_1 + x_2 = 11.9 \qquad \text{(k)}$$

The roots of these equations are

$$x_1 = 0.5 \qquad x_2 = 7.0 \qquad \text{(l)}$$

Comparing the roots given by Eqs (i) and (l), it is noted that a small change in one of the coefficients leads to a large variation in the roots. Hence, Eqs (g) and (h) are ill-conditioned.

It may be noted that the well-conditioned set of Eqs. (a) and (b) represent a pair of straight lines which are nearly orthogonal. In this case a slight shift in the orientation of the lines, caused by small errors in the evaluation of the coefficients, does not lead to a large shift of their point of intersection. Hence, the error in the roots of the equations due to small errors in the coefficients is itself small. Thus the equations are well-conditioned.

The ill-conditioned set of Eqs (g) and (h) represents a pair of straight lines which are nearly parallel. In this case a slight shift in the orientation of the lines caused by small errors in the evaluation of the coefficients leads to a large shift of their point of intersection. Hence, the error in the roots of the equations due to small errors in the coefficients is large. Thus the equations are well conditioned.

Consider next, a set of three linear simultaneous equations. Each of the three equations represents a plane in three dimensional space. The set of equations is well-conditioned if the three planes are nearly orthogonal to one another. The conditioning of the equations deteriorates as the planes become nearly parallel. The logic can be extended to a set of n linear simultaneous equations. In this case each equation represents a hyperplane in n-dimensional space. The set of equations represents a well-conditioned set if the n hyper-planes are nearly orthogonal.

It has been shown in Sec. 3.6 that a set of linear simultaneous equations may be expressed in the matrix form:

$$[a]\,[x] = [c] \qquad \text{(m)}$$

In Eq. (m), the coefficient matrix $[a]$ is said to be well conditioned if the corresponding set of simultaneous equations is well conditioned. For example, as the well conditioned set of Eqs. (a) and (b) can be expressed in the matrix form:

$$\begin{bmatrix} 3 & 10 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 49 \\ 8 \end{bmatrix}$$

it follows that the coefficient matrix

$$[a] = \begin{bmatrix} 3 & 10 \\ 4 & -1 \end{bmatrix}$$

is a well-conditioned matrix. On the other hand the coefficient matrix

$$[a] = \begin{bmatrix} 10.0 & 1.0 \\ 9.9 & 1.0 \end{bmatrix}$$

is an ill-conditioned matrix because it corresponds to a set of ill-conditioned equations.

The conditioning of a matrix can be gauged by finding the value of the determinant of its *normalized matrix*. The normalized matrix is obtained by dividing the elements of each row by the square root of the sum of the squares of all the elements in that row. The numerical value of the determinant of a normalized matrix can never exceed unity. The conditioning of a matrix is best when the numerical value of the determinant of its normalized matrix is equal to one. The conditioning of the matrix deteriorates as the value of the determinant of the normalized matrix approaches zero. In general, a banded matrix is a well conditioned matrix. It may also be stated that, in general, the conditioning of a matrix improves if the non-zero elements are brought closer to the main diagonal.

Example 3.8

Compare the conditioning of the following two matrices:

$$[a] = \begin{bmatrix} 3 & 10 \\ 4 & -1 \end{bmatrix} \qquad [b] = \begin{bmatrix} 1.5 & 2.0 \\ 1.4 & 2.1 \end{bmatrix}$$

Solution

The normalized matrix of matrix $[a]$ is

$$[a]_n = \begin{bmatrix} \dfrac{3}{\sqrt{109}} & \dfrac{10}{\sqrt{109}} \\ \dfrac{4}{\sqrt{17}} & \dfrac{-1}{\sqrt{17}} \end{bmatrix}$$

The value of the determinant of this matrix is

$$|a|_n = -0.99892$$

The normalized matrix of matrix $[b]$ is

$$[b]_n = \begin{bmatrix} \dfrac{1.5}{\sqrt{6.25}} & \dfrac{2.0}{\sqrt{6.25}} \\ \dfrac{1.4}{\sqrt{6.37}} & \dfrac{2.1}{\sqrt{6.37}} \end{bmatrix}$$

The value of the determinant of this matrix is

$$|b|_n = 0.05547$$

Hence, matrix $[a]$ is well conditioned and matrix $[b]$ is ill conditioned.

Example 3.9

Compare the conditioning of the following two matrices:

$$[a] = \begin{bmatrix} 5 & 0 & 4 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad [b] = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution

The normalized matrix of matrix $[a]$ is

$$[a]_n = \begin{bmatrix} \dfrac{5}{\sqrt{41}} & 0 & \dfrac{4}{\sqrt{41}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \dfrac{3}{\sqrt{9}} \end{bmatrix}$$

The value of the determinant of this matrix is

$$|a|_n = 0.816$$

The normalized matrix of matrix $[b]$ is

$$[b]_n = \begin{bmatrix} \frac{5}{\sqrt{26}} & 0 & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ 0 & 0 & \frac{3}{\sqrt{9}} \end{bmatrix}$$

The value of the determinant of this matrix is

$$|b|_n = 0.970$$

Hence the conditioning of matrix $[b]$ is better than that of matrix $[a]$. It may be noted that the two matrices are the same except that the non-zero elements 1 and 4 have been interchanged. In matrix $[b]$, the larger non-zero element 4 is located on the main diagonal whereas in matrix $[a]$ the smaller non-zero element 1 is located on the main diagonal.

PROBLEMS

3.1 Using the properties of the determinant given in Sec. 3.1, show that the following determinants have zero values:

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 4 & 0 & 5 \\ 5 & 0 & 3 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \\ 7 & 4 & 8 \end{vmatrix}$$

3.2 If the value of the determinant $|A|$ is 88, show that the value of the determinant $|B|$ is (-88). Use the properties of the determinant given in Sec 3.1.

$$|A| = \begin{vmatrix} 3 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix} \qquad |B| = \begin{vmatrix} 3 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{vmatrix}$$

3.3 Using the properties of the determinant given in Sec. 3.1, show that the values of the determinants $|A|$ and $|B|$ are the same.

$$|A| = \begin{vmatrix} 1.2 & 0.2 & 0 \\ 0.2 & 1.2 & 0.4 \\ 0 & 0.4 & 0.8 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1.2 & 0.2 & -0.4 \\ 0.2 & 1.2 & -2.0 \\ 0 & 0.4 & 0 \end{vmatrix}$$

3.4 Calculate the values of the following determinants:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1.2 & 0.2 & 0 \\ 0.2 & 1.2 & 0.4 \\ 0 & 0.4 & 0.8 \end{vmatrix}$$

$$|C| = \begin{vmatrix} 3 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix} \qquad |D| = \begin{vmatrix} 3 & -1 & -2 & 0 & -1 \\ -1 & 3 & 0 & 0 & 1 \\ -2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 & 3 \end{vmatrix}$$

3.5 Given that

$$[A] = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \qquad [B] = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

Execute the following matrix operations if possible:
(i) $[A] + [B]$, (ii) $[B]\,[A]$, (iii) $[A][B]$, (iv) $[A]^T[B]$, (v) $[A]^T\,[B][A]$ and (vi) $[A]\,[B]^T\,[A]^T$

3.6 Determine the matrix $[A]$ if

$$[A] = \begin{bmatrix} 24 \\ -12 \\ 232 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -12 & 5 & -1 \end{bmatrix} \begin{bmatrix} 11.4 \\ -7.4 \\ 50.1 \end{bmatrix}$$

3.7 Determine the adjoints of the following matrices:

$$[A] = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix} \qquad [B] = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

3.8 Determine the inverses of the following matrices using the four methods given in Sec. 3.5. Verify the result by multiplying the given matrices with their respective inverses.

$$[A] = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix} \qquad [B] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 2 & 2 & 4 & -2 \\ 1 & 3 & 2 & 1 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 4 & 2 \end{bmatrix} \qquad [D] = \begin{bmatrix} 1.64 & -0.48 & -1.00 & 0 \\ -0.48 & 1.36 & 0 & 0 \\ -1.00 & 0 & 1.64 & 0.48 \\ 0 & 0 & 0.48 & 1.36 \end{bmatrix}$$

3.9 Express the following sets of simultaneous equations in the matrix form. Hence obtain the solutions by the matrix inversion.

(i)
$$\begin{aligned} x + 2y - 3z &= 7 \\ 3x + 2y + 2z &= -5 \\ 4x - y - 5z &= 5 \end{aligned}$$

(ii)
$$\begin{aligned} 3x + 6y + 4z &= 27 \\ 4x - 2y + 3z &= 9 \\ 2x + 2y + 6z &= 24 \end{aligned}$$

(iii)
$$\begin{aligned} x + 2y + 3z &= 4 \\ 3x + 3y + 4z &= 5 \\ 3x + 4y + 5z &= 6 \end{aligned}$$

3.10 Obtain the solutions of the following two sets of simultaneous equations by the method of matrix inversion:

(i)
$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 + 4x_4 &= 40 \\ 6x_1 + 3x_2 - 2x_3 - 3x_4 &= -1 \\ 2x_1 - 2x_2 + 5x_3 - x_4 &= 2 \\ 4x_1 - 3x_2 - x_3 + 4x_4 &= 9 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 30 \\ 7x_1 + 2x_2 + x_3 + 2x_4 &= 22 \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= 20 \\ 5x_1 + 9x_2 + 4x_3 + 3x_4 &= 47 \end{aligned}$$

3.11 The relationship between forces P_1 and P_2 and displacements Δ_3 and Δ_4 is given by the equation

$$\begin{bmatrix} P_1 - 18 \\ P_2 - 6 \\ -15 \\ -15 \end{bmatrix} = \begin{bmatrix} 0.024 & -0.012 & 0 & -0.06 \\ -0.012 & 0.012 & 0.06 & 0.06 \\ 0 & 0.06 & 0.80 & 0.20 \\ -0.06 & 0.06 & 0.20 & 0.40 \end{bmatrix} \begin{bmatrix} -10P_1 \\ -25P_2 \\ EI\Delta_3 \\ EI\Delta_4 \end{bmatrix}$$

Determine the values of P_1, P_2, Δ_3 and Δ_4 by partitioning the matrices in an appropriate manner.

3.12 The following equation represents the relationship between the forces and displacements in a structure:

$$\begin{bmatrix} P_1 \\ P_2 \\ 11.12 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.096 & 0 & -0.096 & 0 & 0.24 \\ 0 & 0.048 & 0 & 0.24 & 0.24 \\ 0.096 & 0 & 0.144 & -0.24 & -0.24 \\ 0 & 0.24 & -0.24 & 3.20 & 0.80 \\ 0.24 & 0.24 & -0.24 & 0.80 & 2.40 \end{bmatrix} \begin{bmatrix} -10P_1 \\ -20P_2 \\ EI\Delta_3 \\ EI\Delta_4 \\ EI\Delta_5 \end{bmatrix}$$

Calculate the values of P_1, P_2, Δ_3, Δ_4, and Δ_5 by partitioning the matrices.

3.13 Determine the values of the determinants of the normalized matrices of the following matrices:

$$[A] = \begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix} \qquad [B] = \begin{bmatrix} 4 & -15 \\ -15 & 100 \end{bmatrix} \qquad [C] = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Indicate which of these matrices is (i) best conditioned and (ii) worst conditioned.

3.14 Indicate which of the following two matrices is better conditioned

$$[A] = \begin{bmatrix} 56 & 33 & -23 \\ 33 & 54 & -21 \\ -23 & -21 & 44 \end{bmatrix} \qquad [B] = \begin{bmatrix} 486 & 198 & 0 \\ 198 & 132 & 0 \\ 0 & 0 & 575 \end{bmatrix}$$

4 FLEXIBILITY AND STIFFNESS MATRICES

4.1 FLEXIBILITY AND STIFFNESS

Flexibility and its converse, known as *stiffness*, are important properties which characterize the response of a structure by means of the force-displacement relationship. In a general sense, the flexibility of a structure is defined as the displacement caused by a unit force and the stiffness is defined as the force required for a unit displacement. Consider first, a structural element with a single degree of freedom. The spring *AB*, shown in Fig. 4.1(a), is fixed at end *A* and has a single degree of freedom at end *B* along coordinate 1. The flexibility of the spring is defined as the displacement δ_{11} at coordinate 1 due to a unit force at coordinate 1. If a force P_1 produces a displacement Δ_1 at coordinate 1,

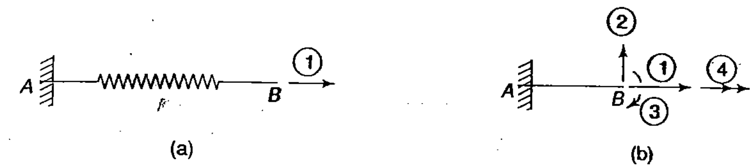


Fig. 4.1

$$\text{flexibility} = \frac{\Delta_1}{P_1} = \delta_{11} \tag{4.1}$$

Similarly, the stiffness of the spring is defined as the force k_{11} required for a unit displacement at coordinate 1.

$$\text{stiffness} = \frac{P_1}{\Delta_1} = k_{11} \tag{4.2}$$

Consider next, a structural element with multiple degrees of freedom. The structural member *AB* of uniform cross-section, shown in Fig. 4.1(b), is fixed at end *A*. End *B* can have the following four types of displacements:

(i) axial displacements Δ_1 at coordinate 1,
(ii) transverse displacement Δ_2 at coordinate 2,

(iii) bending or flexural displacement Δ_3 at coordinate 3 and
(iv) torsional displacement or twist Δ_4 at coordinate 4.

The flexibility and stiffness of structural member AB, with respect to each of the four types of displacements, may now be defined as follows:

4.1.1 Axial Displacement

If an axial force P_1 is applied at coordinate 1, displacement Δ_1 at coordinate 1 is given by the equation

$$\Delta_1 = \frac{P_1 L}{AE} \tag{4.3}$$

where L = length of the member
A = cross-sectional area of the member
E = modulus of elasticity.

As flexibility is the displacement caused by a unit force, the flexibility with respect to axial displacement is obtained by putting $P_1 = 1$ in Eq. (4.3).

$$\text{axial flexibility, } \delta_{11} = \frac{L}{AE} \tag{4.4}$$

By definition, the axial stiffness of the member is the force required for unit displacement along coordinate 1. Hence, putting $\Delta_1 = 1$ in Eq. (4.3).

$$\text{axial stiffness, } k_{11} = \frac{AE}{L} \tag{4.5}$$

The flexibility and stiffness with respect to axial displacement given by Eqs (4.4) and (4.5) are of relevance to members of pin-jointed frames which carry axial forces only. In the case of rigid-jointed frames, the axial displacements are small as compared to transverse displacements. Consequently, it is a common practice in the analysis of rigid-jointed frames to ignore the axial flexibility of the member. In other words, the members of the rigid-jointed frames are considered to be infinitely stiff with respect to axial displacements.

4.1.2 Transverse Displacement

It has been shown in Sec. 2.14 that force P_2 required at coordinate 2 for displacement Δ_2 at coordinate 2 without any displacement at coordinates 1, 3 and 4 is given by the equation

$$P_2 = \frac{12EI\Delta_2}{L^3} \tag{4.6}$$

Hence, by definition, the flexibility and stiffness with respect to transverse displacement may be written as

$$\text{transverse flexibility, } \delta_{22} = \frac{L^3}{12EI} \tag{4.7}$$

and

$$\text{transverse stiffness, } k_{22} = \frac{12EI}{L^3} \tag{4.8}$$

Equations (4.7) and (4.8) are based on the assumption that end A, known as the far-end, is fixed. If far-end A is hinged, the force P_2 required at coordinate 2 for a displacement Δ_2 at coordinate 2 without any displacement at coordinates 1, 3 and 4 is given by Eq. (2.45b)

$$P_2 = \frac{3EI\Delta_2}{L^3} \tag{4.9}$$

Hence, by definition, the flexibility and stiffness with respect to transverse displacement may be written as

$$\text{transverse flexibility, } \delta_{22} = \frac{L^3}{3EI} \tag{4.10}$$

and

$$\text{transverse stiffness, } k_{22} = \frac{3EI}{L^3} \tag{4.11}$$

4.1.3 Bending or Flexural Displacement

It has been shown in Sec. 2.14 that the force P_3 required at coordinate 3 for displacement Δ_3 at coordinate 3 without any displacement at coordinates 1, 2 and 4 is given by the equation

$$P_3 = \frac{4EI\Delta_3}{L} \tag{4.12}$$

Hence, by definition, the flexibility and stiffness with respect to flexural displacement may be written as

$$\text{flexural flexibility, } \delta_{33} = \frac{L}{4EI} \tag{4.13}$$

and

$$\text{flexural stiffness, } k_{33} = \frac{4EI}{L} \tag{4.14}$$

Equations (4.13) and (4.14) are based on the assumption that far-end A is fixed. If far-end A is hinged, the force P_2 required at coordinate 3 for a displacement Δ_3 at coordinate 3 without any displacement at coordinates 1, 2 and 4 is given by Eq. (2.43a).

$$P_3 = \frac{3EI\Delta_3}{L} \tag{4.15}$$

Hence, by definition, the flexibility and stiffness with respect to flexural displacement may be written as

$$\text{flexural flexibility, } \delta_{33} = \frac{L}{3EI} \tag{4.16}$$

and

$$\text{flexural stiffness, } k_{33} = \frac{3EI}{L} \tag{4.17}$$

4.1.4 Torsional Displacement or Twist

From the equation of torsion, the angle of twist Δ_4 due to the torque P_4 is given by the equation

$$\Delta_4 = \frac{P_4 L}{GK} \tag{4.18}$$

where G = shear modulus of elasticity
K = torsion constant (Table 2.2).

By definition, the torsional flexibility of the member is the angle of twist caused by a unit torque along coordinate 4. Hence, putting $P_4 = 1$ in Eq. (4.18),

$$\text{torsional flexibility, } \delta_{44} = \frac{L}{GK} \tag{4.19}$$

Similarly, torsional stiffness, which is defined as the torque required for a unit angle of twist, is obtained by putting $\Delta_4 = 1$ in Eq. (4.18). Hence, torsional stiffness is given by the equation

$$\text{torsional stiffness, } k_{44} = \frac{GK}{L} \tag{4.20}$$

Table 4.1 shows the values of flexibility and stiffness of a prismatic member with respect to the four types of displacements. It may be noted that in each case the flexibility and stiffness are reciprocal of each other.

Table 4.1

S.No.	*Type of displacement, Δ*	*Flexibility, δ*	*Stiffness, k*
1.	Axial	$\frac{L}{AE}$	$\frac{AE}{L}$
2.	Transverse		
	(a) Far-end fixed	$\frac{L^3}{12EI}$	$\frac{12EI}{L^3}$
	(b) Far-end hinged	$\frac{L^3}{3EI}$	$\frac{3EI}{L^3}$
3.	Bending or flexural		
	(a) Far-end fixed	$\frac{L}{4EI}$	$\frac{4EI}{L}$
	(b) Far-end hinged	$\frac{L}{3EI}$	$\frac{3EI}{L}$
4.	Torsional	$\frac{L}{GK}$	$\frac{GK}{L}$

In this section a structural member with only four degrees of freedom has been considered. The concepts of flexibility and stiffness developed in this section have been generalized for a structural system with n degrees of freedom in the following sections.

Example 4.1

A steel bar AB of uniform circular cross-section has a diameter of 20 mm and a length of 1 m. Calculate the maximum values of the displacements Δ_1, Δ_2, Δ_3 and Δ_4 which can be given separately at coordinates 1, 2, 3 and 4 as shown in Fig. 4.1(b), if the maximum direct stress is limited to 100 MPa. Take E = 200 kN/mm² and G = 80 kN/mm².

Solution

Cross-sectional area, $A = \frac{\pi}{4} \times 20^2 = 314.3 \text{ mm}^2$

Moment of inertia, $I = \frac{\pi}{64} \times 20^4 = 7860 \text{ mm}^4$

Section modulus, $Z = 786 \text{ mm}^3$

For a circular cross-section, the torsion constant,

$$K = \frac{\pi}{32} \times 20^4 = 15720 \text{ mm}^4$$

The flexibilities with respect to the four types of displacements may now be computed. Using Table 4.1.

$$\delta_{11} = \frac{1000}{314.3 \times 200} = 0.015908 \text{ mm/kN}$$

$$\delta_{22} = \frac{1000^3}{12 \times 200 \times 7860} = 53 \text{ mm/kN}$$

$$\delta_{33} = \frac{1000}{4 \times 200 \times 7860} = 1.5908 \times 10^{-4} \text{ radian/kN·mm}$$

$$\delta_{44} = \frac{1000}{80 \times 15720} = 7.95165 \times 10^{-4} \text{ radian/kN·mm}$$

Forces P_1, P_2, P_3, and P_4 at the coordinates may now be computed by using the condition that the maximum direct stress is not to exceed 100 N/mm².

$$P_1 = 100 \times 314.3 = 31430 \text{ N} = 31.43 \text{ kN}$$

When displacement Δ_2 is given at coordinate 2 without any displacement at coordinate 3, the bending couple at coordinate 3 is $P_2L/2$. This is evident from the free-body diagram shown in Fig. 2.32(d). Hence,

$$\frac{P_2 L/2}{Z} = 100$$

or $$P_2 = \frac{100 \times 786 \times 2}{1000} = 157.2 \text{ N} = 0.1572 \text{ kN}$$

For a circular cross-section,

$$\text{Maximum shear stress} = \frac{4}{3} \times \text{average shear stress}$$

$$= \frac{4}{3} \times \frac{157.2}{314.3}$$

$$= 0.667 \text{ N/mm}^2$$

Here the maximum permissible direct stress of 100 N/mm^2 has been equated to the maximum bending stress, because the shear stress is small in comparison to the bending stress. The same situation arises when the displacement is given at coordinate 3 without any displacement at coordinate 2. The bending stress dominates and the shear stress is negligibly small. Hence, in the computation of P_3, the maximum permissible direct stress may be equated to the maximum bending stress.

$$P_3 = 100 \times 786 = 78600 \text{ N.mm} = 78.6 \text{ kN·mm}$$

As torsion produces a state of pure shear, the maximum direct stress is equal to the maximum shear stress. Hence, using the equation of torsion,

$$P_4 = \frac{\text{maximum shear stress} \times K}{\text{radius}} = \frac{100 \times 15720}{10}$$

$$= 157200 \text{ N·mm} = 157.20 \text{ kN·mm}$$

The displacements at the coordinates may now be computed by multiplying forces P_1, P_2, P_3 and P_4 by the respective flexibilities.

$$\Delta_1 = 31.43 \times 0.015908 = 0.5 \text{ mm}$$

$$\Delta_2 = 0.1572 \times 53 = 8.33 \text{ mm}$$

$$\Delta_3 = 78.6 \times 1.5908 \times 10^{-4} = 0.0125 \text{ radians}$$

$$\Delta_4 = 157.2 \times 7.95165 \times 10^{-4} = 0.125 \text{ radian}$$

It may be noted that axial displacement Δ_1 is small in comparison to transverse displacement Δ_2. It is for this reason that axial displacements are usually ignored in comparison to transverse displacement in the analysis of rigid-jointed frames.

4.2 FLEXIBILITY MATRIX

Consider a structure which satisfies the basic assumptions enumerated in Sec. 2.2. Let the system of forces $P_1, P_2, ..., P_n$ act on the structure. The word 'forces' has been used here in the generalized sense so as to include couples and reaction components. The system of forces $P_1, P_2, ..., P_n$ may include all or some of the forces acting on the structure. Let the system of forces $P_1, P_2, ...,$

P_n produce displacements $\Delta_1, \Delta_2, ..., \Delta_n$ at coordinates 1, 2, ..., n. Using the principle of superposition discussed in Sec. 2.2, displacements $\Delta_1, \Delta_2, ..., \Delta_n$ may be expressed by the equations

$$\begin{aligned}
\Delta_1 &= \delta_{11}P_1 + \delta_{12}P_2 + ... + \delta_{1j}P_j + ... + \delta_{1n}P_n \\
\Delta_2 &= \delta_{21}P_1 + \delta_{22}P_2 + ... + \delta_{2j}P_j + ... + \delta_{2n}P_n \\
&\vdots \\
\Delta_i &= \delta_{i1}P_1 + \delta_{i2}P_2 + ... + \delta_{ij}P_j + ... + \delta_{in}P_n \\
&\vdots \\
\Delta_n &= \delta_{n1}P_1 + \delta_{n2}P_2 + ... + \delta_{nj}P_j + ... + \delta_{nn}P_n
\end{aligned} \tag{4.21}$$

In Eq. (4.21), δ_{ij} is the displacement at coordinate i due to a unit force at coordinate j. Hence, $\delta_{i1}P_1$ is the displacement at coordinate i due to P_1. Similarly, $\delta_{i2}P_2$ is the displacement at coordinate i due to P_2. Hence, the total displacement at coordinate i due to all the forces may be expressed as

$$\Delta_i = \delta_{i1}P_1 + \delta_{i2}P_2 + ... + \delta_{in}P_n$$

This equation is the same as Eq. (2.5). This explains how Eq. (4.21) have been written down. As explained in Sec. 3.6, the set of simultaneous Eq. (4.21), representing the *force-displacement relationship* may be expressed in the following matrix form:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_i \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} \dots \delta_{1j} \dots \delta_{1n} \\ \delta_{21} & \delta_{22} \dots \delta_{2j} \dots \delta_{2n} \\ \vdots & \\ \delta_{i1} & \delta_{i2} \dots \delta_{ij} \dots \delta_{in} \\ \vdots & \\ \delta_{n1} & \delta_{n2} \dots \delta_{nj} \dots \delta_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \\ \vdots \\ P_n \end{bmatrix} \tag{4.22}$$

Equation (4.22) may be written in the compact form

$$[\Delta] = [\delta][P] \tag{4.23}$$

where $[\Delta]$ = a column matrix of order $n \times 1$, known as *displacement matrix*

$[P]$ = a column matrix of order $n \times 1$, known as *force matrix*

$[\delta]$ = a square matrix of order n, known as *flexibility matrix*.

From Eq. (4.22) it may be noted that the elements of the jth column of the flexibility matrix are the displacements at coordinates 1, 2, ..., n due to a unit force at coordinate j. Hence, in order to *generate the jth column of the flexibility matrix, a unit force should be applied at coordinate j and the displacement at all the coordinates determined. These displacements constitute the elements of the jth column of the flexibility matrix*. Hence, in order to develop the flexibility matrix, a unit force should be applied successively at coordinates 1, 2, ..., n and the displacements at all the coordinates computed.

Example 4.2

Three springs A, B, and C are connected in series as shown in Fig. 4.2. The stiffnesses of the springs are 20, 10 and 5 N/mm respectively. Develop the flexibility matrix for the system of springs with reference to coordinates 1, 2 and 3 as shown in the figure.

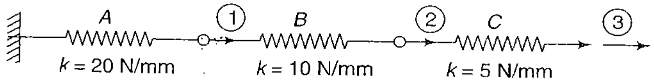


Fig. 4.2

Solution

The flexibility matrix of the system can be developed by applying a unit force successively at coordinates 1, 2 and 3 and evaluating displacements at all the coordinates. To generate the first column of the flexiblity matrix, apply a unit force at coordinate 1. From the figure it is clear that spring *A* is subjected to a tensile force of 1 N whereas the forces in springs *B* and *C* are zero. Consequently, the displacement at coordinate 1, $\delta_{11} = 1/20 = 0.05$ mm. As there is no force in springs *B* and *C*, they move towards the right as rigid bodies. Consequently, the displacements at coordinates 2 and 3 are the same as the displacement at coordinate 1, i.e., $\delta_{21} = \delta_{31} = 0.05$ mm. Thus, the elements of the first column of the flexibility matrix are 0.05, 0.05 and 0.05.

To generate the second column of the flexibility matrix, apply a unit force at coordinate 2. From the figure it is clear that springs *A* and *B* have a tensile force of 1 N each and the force is spring *C* is zero. Hence, the displacement at coordinate 1, $\delta_{12} = 1/20 = 0.05$ mm and the displacement at coordinate 2, $\delta_{22} = 1/20 + 1/10 = 0.15$ mm. As there is no force in spring *C*, it moves towards the right as rigid body. Consequently, the displacement at coordinate 3, $\delta_{32} = 0.15$ mm. Thus the elements of the second column of the flexibility matrix are 0.05, 0.15 and 0.15.

To generate the third column of the flexibility matrix, apply a unit force at coordinate 3. From the figure it is clear that all the three springs carry a tensile force of 1 N each. Hence, the displacement at coordinate 1, $\delta_{13} = 1/20 = 0.05$ mm the displacement at coordinate 2, $\delta_{23} = 1/20 + 1/10 = 0.15$ mm and the displacement at coordinate 3, $\delta_{33} = 1/20 + 1/10 + 1/5 = 0.35$ mm. Thus the elements of the third column of the flexibility matrix are 0.05, 0.15 and 0.35.

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.15 & 0.15 \\ 0.05 & 0.15 & 0.35 \end{bmatrix}$$

Example 4.3

Develop the flexibility matrix for the simply supported beam AB with reference to the coordinates shown in Fig. 4.3.

Solution

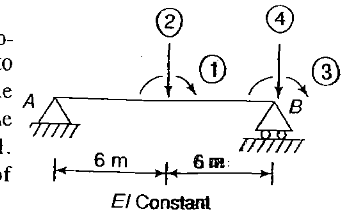


Fig. 4.3

The flexibility matrix can be developed by applying a unit force successively at coordinates 1 to 4 and evaluating the displacements at all the coordinates. To generate the first column of the flexiblity matrix, apply a unit force at coordinate 1.

Using Eqs (A.48), (A.49) and (A.50) of Appendix A,

$$\delta_{11} = \frac{12}{12EI} = \frac{1}{EI}$$
$$\delta_{21} = 0$$
$$\delta_{31} = -\frac{12}{24EI} = -\frac{0.5}{EI}$$
$$\delta_{41} = 0$$

To generate the second column of the flexibility matrix, apply a unit force at coordinate 2. Using Eqs (A.43), (A.44) and (A.45) of Appendix A,

$$\delta_{12} = 0$$
$$\delta_{22} = \frac{12^3}{48EI} = \frac{36}{EI}$$
$$\delta_{32} = \frac{-12^2}{16EI} = \frac{-9}{EI}$$
$$\delta_{42} = 0$$

To generate the third column of the flexibility matrix, apply a unit force at coordinate 3. Using Eqs (A.53), (A.54) and (A.55) of Appendix A,

$$\delta_{13} = -\frac{12}{24EI} = \frac{0.5}{EI}$$
$$\delta_{23} = \frac{-12^2}{16EI} = \frac{-9}{EI}$$
$$\delta_{33} = \frac{12}{3EI} = \frac{4}{EI}$$
$$\delta_{43} = 0$$

All the elements of the fourth column of the flexibility matrix are zero, since the beam remains undeflected when a unit force is applied at coordinate 4. Hence,

$$\delta_{14} = \delta_{24} = \delta_{34} = \delta_{44} = 0$$

The required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{EI}\begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 36 & -9 & 0 \\ -0.5 & -9 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It may be noted that in this case the determinant of the matrix is zero because all elements of the fourth column are zero. Hence, the inverse of the flexibility matrix does not exist in this case.

From the foregoing discussions the following properties of the flexibility matrix are evident:

(i) The flexibility matrix is a square matrix of order n, where n is the number of coordinates chosen for the solution of the problem at hand.

(ii) The flexibility matrix is a symmetrical matrix. This follows from the fact that $\delta_{ij} = \delta_{ji}$ in accordance with the Maxwells reciprocal theorem, Sec. 2.7.

(iii) While the other elements of the flexibility matrix may be positive or negative, the elements lying on the leading diagonal are always positive. This is so because the displacement at any coordinate j due to a unit force at coordinate j is always in the positive direction of coordinate j. Thus δ_{jj} is always positive.

(iv) As the elements of a flexibility matrix are displacements, they can be computed only if the structure is supported adequately and the support conditions are clearly specified. If the structure is unstable internally or externally, the displacements are infinitely large. Consequently, the flexibility matrix does not exist.

(v) If any coordinate j coincides with a reaction component at which no displacement is possible, all elements of the jth column are zero because the structure remains undeformed when a unit force is applied at coordinate j. Thus the displacements at all the coordinates due to a unit force at coordinate j are zero. It may also be noted that all elements of the jth row are zero. This follows from the symmetry of the flexibility matrix. The same conclusion can be arrived at by noting that the displacement at coordinate j is zero irrespctive of the position of the unit force. As all the elements of the jth column are zero, the value of the determinant of the flexibility matirix is zero. Consequently, the inverse of the flexibility matrix does not exist. However, if the reaction component at coordinate j is treated as redundant and released, the determinant of the flexibility matrix may be non-zero and the inverse of the flexibility matrix may, therefore, exist.

4.3 STIFFNESS MATRIX

Let 1, 2, ..., n be the system of the coordinates chosen to express the system of forces $P_1, P_2, ..., P_n$ producing displacements $\Delta_1, \Delta_2, ..., \Delta_n$. If a unit displacement is given at coordiante j without any displacement at other cooordinates, the forces required at coordinates 1, 2, ..., n may be represented by $k_{1j}, k_{2j}, ..., k_{nj}$ respectively. These are the forces which must act at coordinates 1, 2, ..., n to hold the structure in this specific deformed position in which $\Delta_j = 1$ and $\Delta_i (i \neq j) = 0$. In other words, $k_{1j}, k_{2j}, ..., k_{nj}$ are the forces required at coordinate 1, 2, ..., n respectively in order to produce a unit displacement at coordinate j and zero displacement at all other coordinates. Thus k_{ij} is the force at coordinate i due to a unit displacement at coordinate j only. The total force P_i at coordinate i due to displacements $\Delta_1, \Delta_2, ..., \Delta_n$ may be computed by using the principle of superposition, Sec. 2.2.

$$P_i = k_{i1}\Delta_1 + k_{i2}\Delta_2 + ... + k_{in}\Delta_n$$

This equation is the same as Eq. (2.6). Similar equations can be written for the forces at other coordinates resulting in the following set of simultaneous equations:

$$\left.\begin{aligned} P_1 &= k_{11}\Delta_1 + k_{12}\Delta_2 + ... + k_{1j}\Delta_j + ... + k_{1n}\Delta_n \\ P_2 &= k_{21}\Delta_1 + k_{22}\Delta_2 + ... + k_{2j}\Delta_j + ... + k_{2n}\Delta_n \\ &\vdots \\ P_i &= k_{i1}\Delta_1 + k_{i2}\Delta_2 + ... + k_{ij}\Delta_j + ... + k_{in}\Delta_n \\ &\vdots \\ P_2 &= k_{21}\Delta_1 + k_{n2}\Delta_2 + ... + k_{nj}\Delta_j + ... + k_{nn}\Delta_n \end{aligned}\right\} \tag{4.24}$$

Equation (4.24), representing the force-displacement relationship, may be expressed in the following matrix form

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_i \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12}...k_{1j}...k_{1n} \\ k_{21} & k_{22}...k_{2j}...k_{2n} \\ \vdots & \\ k_{i1} & k_{i2}...k_{ij}...k_{in} \\ \vdots & \\ k_{n1} & k_{n2}...k_{nj}...k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \vdots \\ \Delta_n \end{bmatrix} \tag{4.25}$$

Equation (4.24) may be written in the compact form

$$[P] = [k][\Delta] \tag{4.26}$$

where $[k]$ = a square matrix of order n, known as stiffnes matrix.

From Eq. (4.25) it may be noted that the elements of the jth column of the stiffness matrix are the forces at coordinates 1, 2, ..., n due to a unit displacement at coordinate j. Hence, *in order to generate the jth column of the stiffness matrix, a unit displacement must be given at coordinate j without any displacement at other coordinates and the forces required at all the coordinates determined. These forces constitute the elements of the jth column of the stiffness matrix.* Hence, in order to develop the stiffness matrix, unit displacement should be given successively at coordinates 1, 2, ..., n and forces at all the coordinates calculated.

Example 4.4

Develop the stiffness matrix for the set of springs shown in Fig. 4.2.

Solution

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1, 2 and 3 without any displacement at other coordinates and determining the forces required at all the coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at coordinates 2 and 3. Consequently, spring A will elongate by 1 mm, spring B will contract by 1 mm and spring C will have no deformation. To elongate spring A by 1 mm, a force of 20 N is required at coordinate 1. Similarly, a force of 10 N at coordinate 1 is required to contract spring B by 1 mm. Thus the total force required at coordinate 1, $k_{11} = 20 + 10 = 30$ N. A force of -10 N is required at coordinate 2 to contract spring B by 1 mm. The minus sign indicates that the force acts towards the left. Thus, $k_{21} = -10$ N. Also, $k_{31} = 0$ because spring C has no deformation. Hence, the elements of the first column of the stiffness matrix are 30, -10 and 0.

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at coordinates 1 and 3. This can be achieved only if spring A remains undeformed, spring B elongates by 1 mm and spring C contracts by 1 mm. To elongate spring B by 1 mm, a force of -10 N is required at coordinate 1. As spring A remains undeformed, the total force required at coordiante 1 is -10 N. Thus, $k_{12} = -10$ N. The total force required at coordinate 2, to elongate spring B by 1 mm and contract spring C by 1 mm is evidently $10 + 5 = 15$ N. Thus, $k_{22} = 15$ N. The force required at coordinate 3 to contract spring C by 1 mm is -5 N. Thus $k_{32} = -5$ N. The elements of the second column of the stiffness matrix are, therefore, -10, 15 and -5.

Similarly, to generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at coordinates 1 and 2. This can be achieved only if springs A and B remain undeformed and spring C elongates by 1 mm. It follows that the force at coordinate 1 is zero and the forces required at coordinates 2 and 3 to elongate the spring C by 1 mm are -5 N and 5 N respectively. Thus, $k_{13} = 0$, $k_{23} = -5$ N and $k_{33} = 5$ N. The elements of the third column of the stiffness matrix are, therefoore, 0, -5 and 5.

Hence, the required stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

Example 4.5

Develop the stiffness matrix for the end-loaded prismatic member AB with reference to the coordinates shown in Fig. 4.4(a). Comment on the relevance of the chosen coordinates. Examine the reciprocity of the stiffness matrix.

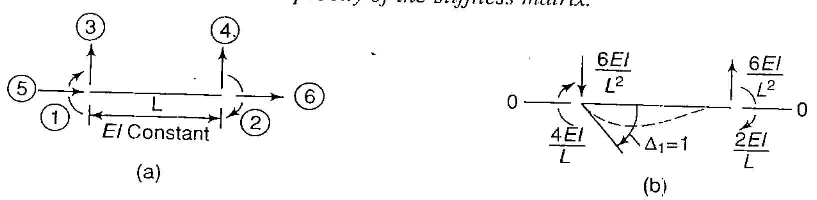


Fig. 4.4 (Contd)

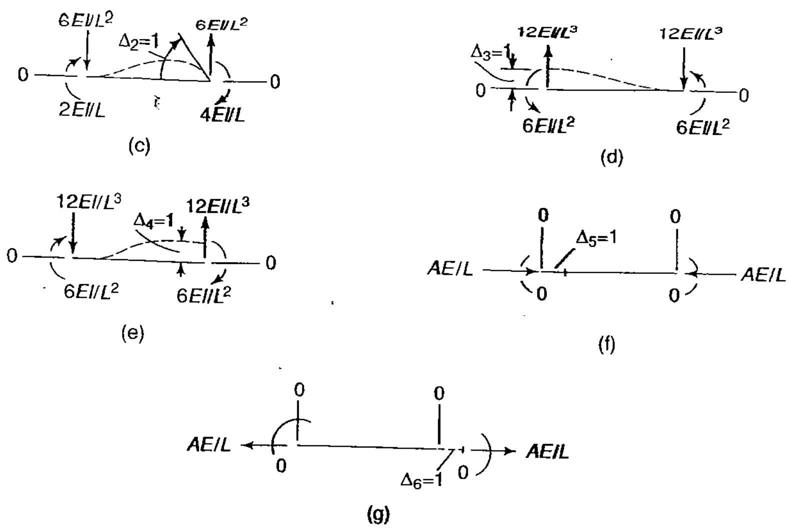


Fig. 4.4

Solution

The stiffness matrix of the member can be developed by giving a unit displacement successively at each coordinate without any displacement at other coordinates. The forces at coordinates 1 to 6, when a unit displacement is given successively at each of the coordinates 1 to 4, may be computed by using the equations given in Sec. 2.14. For example, when a unit displacement is given at coordinate 1, the forces at coordinates 1 to 6, which constitute the elements of the first column of the stiffnes matrix, are

$$k_{11} = \frac{4EI}{L} \qquad k_{21} = \frac{2EI}{L}$$

$$k_{31} = -\frac{6EI}{L^2} \qquad k_{41} = \frac{6EI}{L^2}$$

$$k_{51} = k_{61} = 0$$

Similarly, the elements of the second, third and fourth columns of the stiffness matrix can be determined.

When a unit displacement is given at coordinate 5 without any displacement at other coordinates, the forces evidently are

$$k_{15} = k_{25} = k_{35} = k_{45} = 0 \qquad k_{55} = \frac{AE}{L} \qquad k_{65} = -\frac{AE}{L}$$

These forces constitute the elements of the fifth column of the stiffness matrix. The sixth column of the stiffness matrix may be generated in a similar manner by giving a unit displacement at coordinate 6.

The deformed shape of the member, when unit displacement is given succesively at coordinates 1 to 6, together with the resulting forces required to sustain the deformed shape of the member, are shown in the free-boy diagrams in Fig. 4.4(b) to (g). Thus the stiffness matrix of member AB with reference to the chosen coordinates may be written as

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & 0 & 0 \\ \frac{2EI}{L} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & 0 & 0 \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{12EI}{L^3} & 0 & 0 \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{12EI}{L^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{AE}{L} & -\frac{AE}{L} \\ 0 & 0 & 0 & 0 & -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \tag{4.27}$$

where A = area of cross-section of the member
L = length of the member.

The stiffness matrix of Eq. (4.27) is relevant to a structural member belonging to a rigid-jointed plane frame because it carries the three types of internal forces, viz., the axial force, the shear force and the bending moment. If the member is assumed to be inextensible, as is commonly done in structural analysis, coordinates 5 and 6 are unnecessary. Similarly, in the case of beams which commonly carry transverse loading, the axial forces are absent. Consequently, coordinates 5 and 6 are unnecessary in the case of transversely loaded beams. If coordinates 5 and 6 are deleted as shown in Fig. 4.5, the stiffness matrix, which is of relevance to the members of rigid-jointed plane frames and transversely loaded beams, takes the form

Fig. 4.5

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{6EI}{L^2} \\ \frac{2EI}{L} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{12EI}{L^3} \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{12EI}{L^3} \end{bmatrix} \tag{4.28}$$

As the members of a pin-jointed frame carry only axial forces, only coordinates 5 and 6 are of relevance. Renumbering these coordinates as 1 and 2 as shown in Fig. 4.6, the stiffness matrix for the member of a pin-jointed frame may be written as

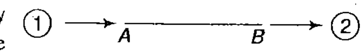


Fig. 4.6

$$[k] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \tag{4.29}$$

It is interesting to note that the stiffness matrices of Eqs (4.27), (4.28) and (4.29) do not have their inverse. This is evident if it is noted that the determinants of these matrices are zero due to the property of the determinant, viz., if one row or column can be generated by linear combination of other rows or columns, the determinant is zero. For example, column 3 of matrix of Eq. (4.27) can be obtained by multiplying the elements of column 4 by minus 1.

From the foregoing discussions the following properties of the stiffness matrix are evident:

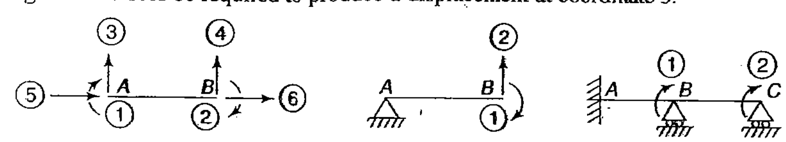
(i) The stiffness matrix is a square matrix of order n, where n is the number of coordinates chosen for the solution of the problem at hand.

(ii) The stiffness matrix is a symmetrical matrix. This follows from Eq. (2.24), viz., $k_{ij} = k_{ji}$.

(iii) While the other elements may be positive or negative, the elements lying on the leading diagonal are always positive. This is so because the force at coordinate j due to a unit displacement at coordinate j is always in the positive direction of coordiante j. Thus k_{jj} is always positive.

(iv) The stiffness matrix can be developed only if the ~~structure~~ is stable and restrained adequately to prevent rigid body motion. This can be achieved by a proper choice of the coordinates and the support reactions or a combination of both. If a displacement at a coordinate is impossible, the stiffness matrix does not exist. To explain the foregoing properties the following cases may be considered.

In Fig. 4.7(a) the coordinates have been chosen in such a way that the rigid body motion of member AB is not possible when a unit displacement is given at any one of the coordinates. Hence, the stiffness matrix with reference to coordinates 1 and 2 can be developed without any mention of the support condition. However, if member AB is assumed to be inextensible, the stiffness matrix does not exist because an infinitely large force would be required to produce a displacement at coordinate 5.



(a) (b) (c)

Fig. 4.7

In Fig. 4.7(b) the rigid body motion is prevented by a combination of support reactions and the chosen coordinates. Hence, the stiffness matrix for member AB can be developed with reference to the coordinates 1 and 2. It may be noted that member AB is unstable if support reactions alone are considered, but the combination of support reactions and the chosen coordinates makes it stable and determinate.

In Fig. 4.7(c) continuous beam ABC is stable with the specified support conditions. As rotations at roller supports B and C are not prevented, the stiffness matrix with reference to coordinates 1 and 2 can be developed. However, the stiffness matrix will not exist if a third coordinate in the vertical direction at B is added because no vertical displacement at coordinate 3 would be possiible.

Example 4.6

Develop the stiffness matrix for the portal frame with reference to the coordinates shown in Fig. 4.8(a).

Solution

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1, 2 and 3 without any displacement at other coordinates and determining the forces required at all the coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at coordinates 2 and 3 as shown in Fig. 4.8(b). Forces k_{11}, k_{21} and k_{31} are required at coordinates 1, 2 and 3 to hold the frame in the deformed shape shown in Fig. 4.8(b). The free-body diagrams of the three members and joints B and C are shown in Fig. 4.8(c). In these diagrams S_1, S_2 and S_3 are the axial forces in members AB, BC, and CD respectively. These forces have been assumed to be tensile.

Member AB undergoes a clockwise rotation, $\phi_{AB} = \dfrac{1}{h_{AB}}$. Hence, using Table 2.16, the counter-clockwise couple at each end of the member is $\dfrac{6EI_{AB}\phi_{AB}}{h_{AB}} = \dfrac{6EI_{AB}}{h_{AB}^2}$ and the transverse force at each end is $\dfrac{12EI_{AB}}{h_{AB}^3}$. The directions of the transverse forces must, evidently, be such as to prduce a clockwise couple so that the counter-clockwise end couples are balanced. Member BC does not bend. Consequently, it carries only an axial foce. Member CD undergoes a clockwise rotation, $\phi_{CD} = \dfrac{1}{h_{CD}}$. Hence using Table 2.16, the counter-clockwise couple at each end of the member is

$$\frac{6EI_{CD}\phi_{CD}}{h_{CD}} = \frac{6EI_{CD}}{h_{CD}^2}$$

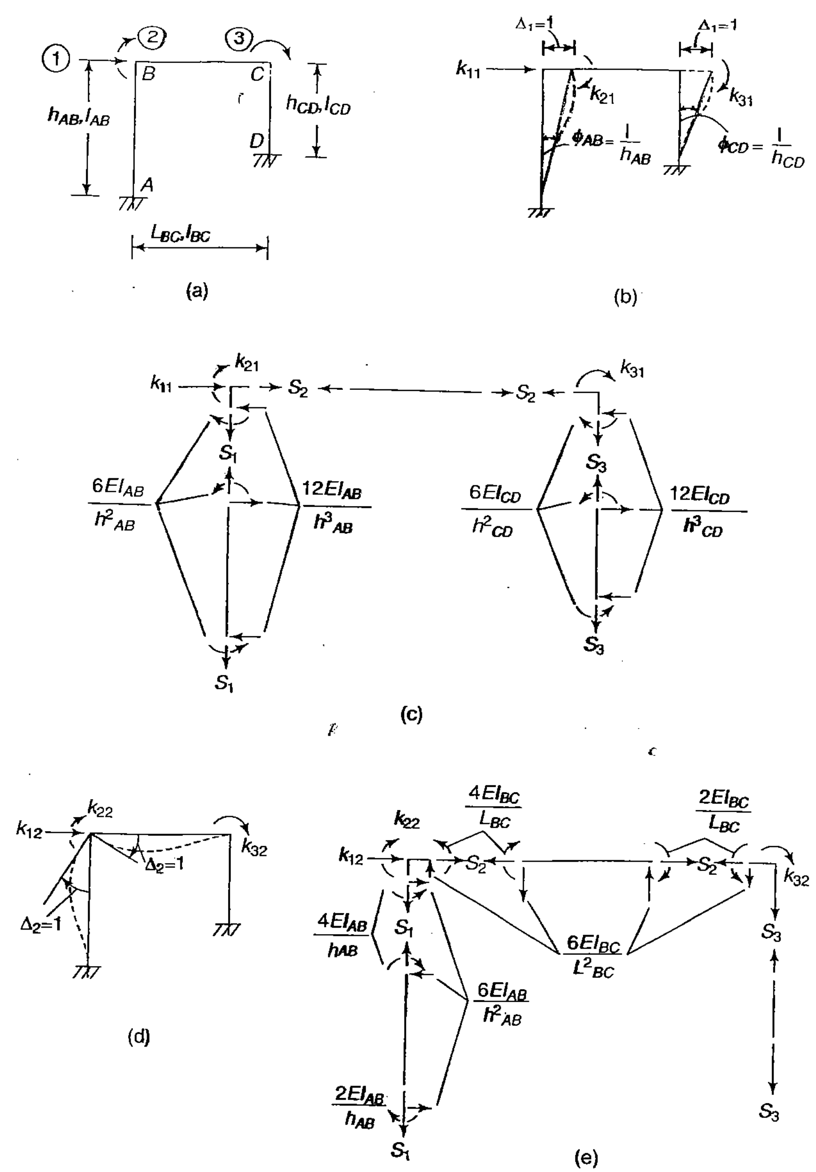
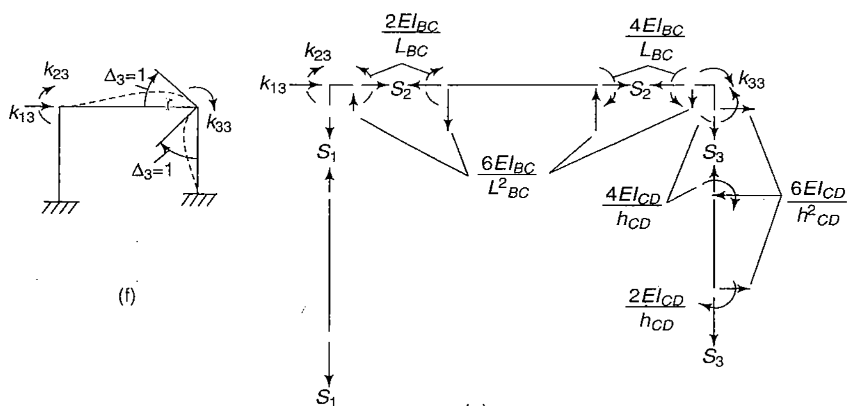


Fig. 4.8 (contd)



(f) (g)

Fig. 4.8

and the transverse force at each end is $\dfrac{12EI_{CD}}{h_{CD}^3}$

The directions of the transverse forces must, evidently, be such as to produce a clockwise couple so that the counter-clockwise end couples are balanced.

In the free-body diagram of joint B, k_{11} is the horizontal force required to produce a unit horizontal displacement of joint B. Couple k_{21} is required at joint B to prevent its rotation. The other forces acting at the joint are equal in magnitude and opposite in sense to those acting at end B of members AB and BC. In the free-body diagram of joint C, k_{31} is the couple required at joint C to prevent its rotation. The other forces acting at the joint are equal in magnitude and opposite in sense to those acting at end C of members BC and CD.

For the equilibrium of joint C,

$$\left.\begin{aligned} S_2 + \frac{12EI_{CD}}{h_{CD}^3} &= 0 \\ S_3 &= 0 \\ k_{31} + \frac{6EI_{CD}}{h_{CD}^2} &= 0 \end{aligned}\right\} \quad \text{(a)}$$

For the equilibrium of joint B,

$$\left.\begin{aligned} k_{11} + S_2 - \frac{12EI_{AB}}{h_{AB}^2} &= 0 \\ S_1 &= 0 \\ k_{21} + \frac{6EI_{AB}}{h_{AB}^2} &= 0 \end{aligned}\right\} \quad \text{(b)}$$

Solving Eqs (a) and (b),

$$\left.\begin{aligned} k_{11} &= \frac{12EI_{AB}}{h_{AB}^3} + \frac{12EI_{CD}}{h_{CD}^3} \\ k_{21} &= -\frac{6EI_{AB}}{h_{AB}^2} \\ k_{31} &= -\frac{6EI_{CD}}{h_{CD}^2} \\ S_1 &= 0 \\ S_2 &= -\frac{12EI_{CD}}{h^3_{CD}} \\ S_3 &= 0. \end{aligned}\right\} \quad \text{(c)}$$

The negative sign in the expression for S_2 indicates the member BC is in compression.

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at coordinates 1 and 3 as shown in Fig. 4.8(d). In this case the members do not rotate because the horizontal displacement of the frame is prvented. The free-body diagrams of the three members and joints B and C are shown in Fig. 4.8(e). Table 2.16 has been used in obtaining the end couples and the transverse forces at the ends of members AB and BC. Member CD does not bend. It carries only an axial force.

For the equilibrium of the joint C,

$$\left.\begin{aligned} S_3 + \frac{6EI_{BC}}{L_{BC}^2} &= 0 \\ S_2 &= 0 \\ k_{32} - \frac{2EI_{BC}}{L_{BC}} &= 0 \end{aligned}\right\} \quad \text{(d)}$$

For the equilibrium of joint B,

$$\left.\begin{aligned} k_{12} + S_2 + \frac{6EI_{AB}}{h_{AB}^2} &= 0 \\ S_1 - \frac{6EI_{BC}}{L_{BC}^2} &= 0 \\ k_{22} - \frac{4EI_{AB}}{h_{AB}} - \frac{4EI_{BC}}{L_{BC}} &= 0 \end{aligned}\right\} \quad \text{(e)}$$

Solving Eqs (d) and (e),

$$k_{12} = -\frac{6EI_{AB}}{h_{AB}^2}$$

$$k_{22} = \frac{4EI_{AB}}{h_{AB}} + \frac{4EI_{BC}}{L_{BC}}$$

$$k_{32} = \frac{2EI_{BC}}{L_{BC}} \tag{f}$$

$$S_1 = \frac{6EI_{BC}}{L_{BC}^2}$$

$$S_2 = 0$$

$$S_3 = -\frac{6EI_{BC}}{L_{BC}^2}$$

The negative sign in the expression for S_3 indicates that member CD is in compression.

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at coordinates 1 and 2 as shown in Fig. 4.8(f). In this case the members do not rotate because the horizontal displacement of the frame is prevented. The free-body diagrams of the three members and joints B and C are shown in Fig. 4.8(g). Table 2.16 has been used in obtaining the end couples and transverse forces at the ends of members BC and CD. Member AD does not bend. It carries only an axial force.

For the equilibrium of joint C,

$$\left.\begin{aligned} S_2 - \frac{6EI_{CD}}{h_{CD}^2} &= 0 \\ S_3 + \frac{6EI_{BC}}{L_{BC}^2} &= 0 \\ k_{33} - \frac{4EI_{BC}}{L_{BC}} - \frac{4EI_{CD}}{h_{CD}} &= 0 \end{aligned}\right\} \tag{g}$$

For the equilibrium of joint B,

$$\left.\begin{aligned} k_{13} + S_2 &= 0 \\ S_1 - \frac{6EI_{BC}}{L_{BC}^2} &= 0 \\ k_{23} - \frac{2EI_{BC}}{L_{BC}} &= 0 \end{aligned}\right\} \tag{h}$$

Solving Eqs (g) and (h),

$$\left.\begin{aligned} k_{13} &= -\frac{6EI_{CD}}{h_{CD}^2} \\ k_{23} &= \frac{2EI_{BC}}{L_{BC}} \\ k_{33} &= \frac{4EI_{BC}}{L_{BC}} + \frac{4EI_{CD}}{h_{CD}} \\ S_1 &= \frac{6EI_{BC}}{L_{BC}^2} \end{aligned}\right\} \tag{i}$$

$$\left.\begin{aligned} S_2 &= \frac{6EI_{CD}}{h_{CD}^2} \\ S_3 &= -\frac{6EI_{BC}}{L_{BC}^2} \end{aligned}\right\} \tag{j}$$

The negative sign in the expression for S_3 indicates that member CD is in compression.

The final expressions for the elements of the stiffness matrix are

$$k_{11} = \frac{12EI_{AB}}{h_{AB}^3} + \frac{12EI_{CD}}{h_{CD}^3} \tag{4.30a}$$

$$k_{22} = \frac{4EI_{AB}}{h_{AB}} + \frac{4EI_{BC}}{L_{BC}} \tag{4.30b}$$

$$k_{33} = \frac{4EI_{BC}}{L_{BC}} + \frac{4EI_{CD}}{h_{CD}} \tag{4.30c}$$

$$k_{12} = k_{21} = -\frac{6EI_{AB}}{h_{AB}^2} \tag{4.30d}$$

$$k_{23} = k_{32} = \frac{2EI_{BC}}{L_{BC}} \tag{4.30e}$$

$$k_{31} = k_{13} = -\frac{6EI_{CD}}{h_{CD}^2} \tag{4.30f}$$

4.4 RELATIONSHIP BETWEEN FLEXIBILITY MATRIX AND STIFFNESS MATRIX

It was shown in Table 4.1 that the flexibility and stiffness of a structural member are reciprocal of each other for each type of displacement discussed in Sec. 4.1. It is interesting to note that the property of reciprocity applies also to the flexibility and stiffness matrices. The reciprocity is subject to the condition that the flexibility and stiffness matrices have a common system of coordinates. It follows that the product of the flexibility and stiffness matrices is a unit matrix.

$$[\delta][k] = [I] \tag{4.31}$$

Proof. It was shown in Sec. 4.2, Eq. (4.23), that the force-displacement relationship may be expressed by the matrix equation

$$[\Delta] = [\delta][P]$$

Premultiplying both sides of this equation by $[\delta]^{-1}$,

$$[\delta]^{-1}[\Delta] = [\delta]^{-1}[\delta][P]$$

or

$$[\delta]^{-1}[\Delta] = [P]$$

Comparing this equation with Eq. (4.26),

$$[\delta]^{-1} = [k]$$

Premultiplying both sides by $[\delta]$

$$[\delta][\delta]^{-1} = [\delta][k]$$

or

$$[I] = [\delta][k]$$

From this equation it follows that the flexibility and stiffness matrices are reciprocal of each other. The same result could also be obtained by starting with Eq. (4.26) and premultiplying both sides by $[k]^{-1}$.

A more rigorous proof may be obtained by the use of the generalized reciprocal theorem as follows:

Let $[c]$ be the product of flexibility matrix $[\delta]$ and stiffness matrix $[k]$ so that

$$[\delta][k] = [c]$$

By the rules of matrix multiplication, element c_{ii} lying on the leading diagonal of the product matrix may be written as

$$c_{ii} = \delta_{i1}k_{1i} + \delta_{i2}k_{2i} + \dots + \delta_{in}k_{ni}$$

Using Maxwells reciprocal theorem,

$$c_{ii} = \delta_{1i}k_{1i} + \delta_{2i}k_{2i} + \dots + \delta_{ni}k_{ni} \qquad \text{(a)}$$

Consider now the two systems shown in Table 4.2. In system I a unit force has been applied at coordinate i producing displacements $\delta_{1i}, \delta_{2i}, \dots, \delta_{ni}$. In system II a unit displacement is given at coordinate i producing forces $k_{1i}, k_{2i}, \dots, k_{ni}$. Applying the generalized reciprocal theorem,

$$1 = \delta_{1i}k_{1i} + \delta_{2i}k_{2i} + \dots + \delta_{ni}k_{ni} \qquad \text{(b)}$$

Table 4.2

System	Force and displacement	Coordinates										
		1	*2*	.	.	*i*	.	.	*j*	.	.	*n*
I	P	0	0	0	0	1	0	0	0	0	0	0
	Δ	δ_{1i}	δ_{2i}	.	.	δ_{ii}	.	.	δ_{ji}	.	.	δ_{ni}
II	P'	k_{1i}	k_{2i}	.	.	k_{ii}	.		k_{ji}	.	.	k_{ni}
	Δ'	0	0	0	0	1	0	0	0	0	0	0

From Eqs (a) and (b),

$$c_{ii} = 1 \qquad \text{(c)}$$

This shows that all elements on the leading diagonal of matrix $[c]$ are equal to unity.

Similarly, element c_{ij} $(i \neq j)$ can be written as

$$c_{ij}\ (i \neq j) = \delta_{i1}k_{1j} + \delta_{i2}k_{2j} + \dots + \delta_{in}k_{nj}$$

Using the reciprocal theorem,

$$c_{ij}\ (i \neq j) = \delta_{1i}k_{1j} + \delta_{2i}k_{2j} + \dots + \delta_{in}k_{nj} \qquad \text{(d)}$$

Now consider the two systems shown in Table 4.3. In system I a unit force has been applied at coordinate i producing displacements $\delta_{1i}, \delta_{2i}, \dots \delta_{ni}$. In system II a unit displacement is given at coordinate j producing forces $k_{1j}, k_{2j}, \dots, k_{nj}$. Applying the generalizeed reciprocal theorem,

$$0 = \delta_{1i}k_{1j} + \delta_{2i}k_{2j} + \dots + \delta_{ni}k_{nj} \qquad \text{(e)}$$

Table 4.3

System	Force and displacement	Coordinates										
		1	*2*	.	.	*i*	.	.	*j*	.	.	*n*
I	P	0	0	0	0	1	0	0	0	0	0	0
	Δ	δ_{1i}	δ_{2i}	.	.	δ_{ii}	.	.	δ_{ji}	.	.	δ_{ni}
II	P'	k_{1j}	k_{2j}	.	.	k_{ij}	.		k_{jj}	.	.	k_{nj}
	Δ'	0	0	0	0	0	0	0	1	0	0	0

From Eqs. (d) and (e),

$$c_{ij}(i \neq j) = 0 \qquad \text{(f)}$$

Equation (f) shows that all elements, except those on the leading diagonal of matrix $[c]$, are zero. It follows from Eqs (c) and (f) that matrix $[c]$ is a unit matrix. Hence, Eq. (4.31) is established. As the product of the flexibility and stiffness matrices is a unit matrix, they are reciprocal or inverse of each other.

Consider, for example, a straight prismatic member AB with simple supports as shown in Fig. 4.9(a). Using Table 2.16, the flexibility matrix $[\delta]$ with reference to coordinates 1 and 2, shown in Fig. 4.9(a), can be written as

$$[\delta] = \begin{bmatrix} \dfrac{L}{3EI} & -\dfrac{L}{6EI} \\ -\dfrac{L}{6EI} & \dfrac{L}{3EI} \end{bmatrix} \qquad (4.32)$$

Fig. 4.9

Similarly, using Table 2.16, the stiffness matrix with reference to coordinates 1 and 2 may be written as

$$[k] = \begin{bmatrix} \dfrac{4EI}{L} & \dfrac{2EI}{L} \\ \dfrac{2EI}{L} & \dfrac{4EI}{L} \end{bmatrix} \tag{4.33}$$

It may be readily verified that the matrices of Eqs (4.32) and (4.33) are the reciprocal of each other.

Consider next, a straight prismatic member AB with cross-sectional area A. It is fixed at A and free at B as shown in Fig. 4.9(b). Using Table 2.16, the flexibility matrix with reference to coordinates 1, 2 and 3, shown in Fig. 4.9(b), may be written as

$$[\delta] = \begin{bmatrix} \dfrac{L}{EI} & \dfrac{L^2}{2EI} & 0 \\ \dfrac{L^2}{2EI} & \dfrac{L^3}{3EI} & 0 \\ 0 & 0 & \dfrac{L}{AE} \end{bmatrix} \tag{4.34}$$

Similarly, using Table 2.16, the stiffness matrix with reference to coordinate 1, 2 and 3 may be written as

$$[k] = \begin{bmatrix} \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} & 0 \\ -\dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} & 0 \\ 0 & 0 & \dfrac{AE}{L} \end{bmatrix} \tag{4.35}$$

It may be readily verified that the matrices of Eqs (4.34) and (4.35) are the reciprocal of each other.

If the axial deformation of the member is ignored, coordinate 3 is unnecessary. In this case the flexibility and stiffness matrices with reference to coordinates 1 and 2 may be written as

$$[\delta] = \begin{bmatrix} \dfrac{L}{EI} & \dfrac{L^2}{2EI} \\ \dfrac{L^2}{2EI} & \dfrac{L^3}{3EI} \end{bmatrix} \tag{4.36}$$

$$[k] = \begin{bmatrix} \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} \\ -\dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} \end{bmatrix} \tag{4.37}$$

It may again be checked that the matrices of Eqs (4.36) and (4.37) are the reciprocal of each other.

Exaxmple 4.7

Two steel bars AB and BC, each having a cross-sectional area of 20 mm^2, are connected in series as shown in Fig. 4.10. Develop the flexibility and stiffness matrices with reference to coordinates 1 and 2 shown in the figure. Verify that the two matrices are the invervse of eachy other. Take E = 200 kN/mm^2.

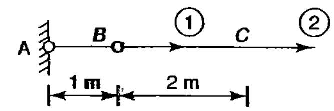


Fig. 4.10

Solution

$$\text{Axial flexibility of bar } AB = \frac{L}{AE} = \frac{1000}{20 \times 200} = 0.25 \text{ mm/kN}$$

$$\text{Axial stiffness of bar } AB = \frac{AE}{L} = 4 \text{ kN/mm}$$

$$\text{Axial flexibility of bar } BC = \frac{L}{AE} = \frac{2000}{20 \times 200} = 0.5 \text{ mm/kN}$$

$$\text{Axial stiffness of bar } BC = \frac{AE}{L} = 2 \text{ kN/mm}$$

The flexibility matrix can be developed by applying a unit force successively at coordinates 1 and 2 and evaluating the displacements at coordinates 1 and 2. To generate the first column of the flexibility matrix, apply a unit force at coordinate 1. The displacements at coordinates 1 and 2 are

$$\delta_{11} = \delta_{21} = 0.25 \text{ mm}$$

Similarly, to generate the second column of the flexibility matrix, apply a unit force at coordinate 2. The displacements at coordinates 1 and 2 are

$$\delta_{12} = 0.25 \text{ mm}$$

$$\delta_{22} = 0.25 + 0.5 = 0.75 \text{ mm}$$

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 and 2 without any displacement at the other coordinate and determining the forces required at coordinates 1 and 2. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1. The forces required at coordinates 1 and 2 are

$$k_{11} = 4 + 2 = 6 \text{ kN}$$

$$k_{21} = -2 \text{ kN}$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2. The forces required at coordinates 1 and 2 are

$$k_{12} = -2 \text{ kN}$$

$$k_{22} = 2\text{kN}$$

Hence, the required stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}$$

Multiplying the flexibility and stiffness matrices,

$$[\delta][k] = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As the product of the two matrices is a unit matrix, the two matrices are the inverse of each other.

Example 4.8

Develop the flexibility and stiffness matrices for prismatic member AB with reference to the coordinates shown in Fig. 4.11 (a) for the following support conditons:

(i) hinged support at A and roller support at B

(ii) fixed supports at A and B

(iii) fixed support at A and roller support at B.

Verify in each case that the flexibility and stiffness matrices are the inverse of each other.

Solution

(i) The support conditions are shown in Fig. 4.11(b). The flexibility matrix can be developed by applying a unit force successively at coordinates 1 and 2 and evaluating displacements at coordiantes 1 and 2. To generate the first column of the flexibility matrix, apply a unit force at coordinate 1. Using Eqs (A. 71) and (A.72) of Appendix A, the displacement at coordinates 1 and 2 are

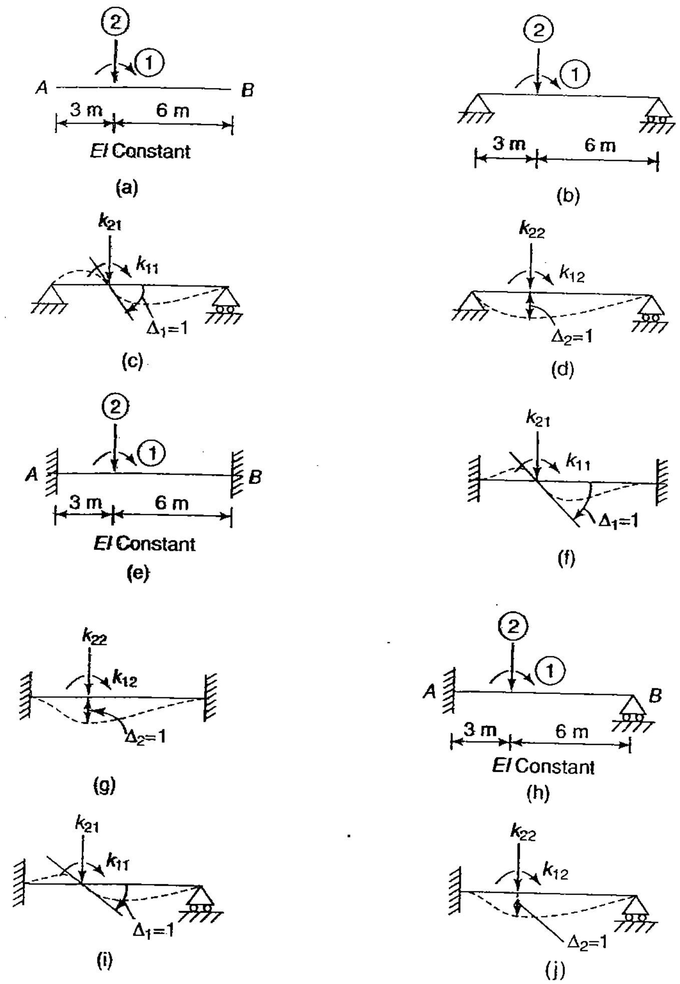


Fig. 4.11

$$\delta_{11} = \frac{1}{3 \times 9EI}[3 \times 3^2 - 3 \times 3 \times 9 + 9^2] = \frac{1}{EI}$$

$$\delta_{21} = \frac{3(9-3)(9-6)}{3 \times 9EI} = \frac{2}{EI}$$

To generate the second column of the flexibility matrix, apply a unit force at coordinate 2. Using Eqs (A.63) and (A.64) of Appendix A, the displacements at coordinates 1 and 2 are

$$\delta_{12} = \frac{3(9-3)(9-6)}{3 \times 9EI} = \frac{2}{EI}$$

$$\delta_{22} = \frac{3^2 \times 6^2}{3 \times 9EI} = \frac{12}{EI}$$

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{EI}\begin{bmatrix} 1 & 2 \\ 2 & 12 \end{bmatrix}$$

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 and 2 without any displacement at the other coordinate and determining the forces required at coordinates 1 and 2. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 4.11(c). The forces required at the coordinates are

$$k_{11} = \frac{3EI}{3} + \frac{3EI}{6} = 1.5EI$$

$$k_{21} = -\frac{3EI}{3^2} + \frac{3EI}{6^2} = -0.25EI$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown in Fig. 4.11(d). The forces required at coordinates 1 and 2 are

$$k_{12} = -\frac{3EI}{3^2} + \frac{3EI}{6^2} = -0.25EI$$

$$k_{22} = \frac{3EI}{3^3} + \frac{3EI}{6^3} = 0.125EI$$

Hence, the required stiffness matrix $[k]$ is given by the equation

$$[k] = EI\begin{bmatrix} 1.500 & -0.250 \\ -0.250 & 0.125 \end{bmatrix}$$

Multiplying the flexibility and stiffness matrices,

$$[\delta][k] = \frac{1}{EI}\begin{bmatrix} 1 & 2 \\ 2 & 12 \end{bmatrix} EI \begin{bmatrix} 1.500 & -0.250 \\ -0.250 & 0.125 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As the product is a unit matrix, the two matrices are the inverse of the each other.

(ii) The support conditions are shown in Fig. 4.11(e). The flexibility matrix can be developed by applying a unit force successively at coordinates 1 and 2 and evaluating the displacement at coordinates 1 and 2. To generate the first column of the flexibility matrix, apply a unit force at coordinate 1. Using Eqs (A.113) and (A.114) of Appendix *A*, the displacements at coordinates 1 and 2 are

$$\delta_{11} = \frac{3(9-3)(9^2 - 3 \times 3 \times 9 + 3 \times 3^2)}{9^3 EI} = \frac{2}{3EI}$$

$$\delta_{21} = \frac{3^2}{2 \times 9^3 EI} \times (9-3)^2(9-6) = \frac{2}{3EI}$$

To generate the second column of the flexibility matrix, apply a unit force at coordinate 2. Using Eqs. (A.104) and (A.105) of Appendix A, the displacements at coordinates 1 and 2 are

$$\delta_{12} = \frac{3^2}{2 \times 9^3 EI}(9-3)^2(9-6) = \frac{2}{3EI}$$

$$\delta_{22} = \frac{3^3(9-3)^3}{3 \times 9^3 EI} = \frac{8}{3EI}$$

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{2}{3EI}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 and 2 without any displacement at the other coordinate and determining the forces required at coordinates 1 and 2. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 4.11 (f). The forces required at coordinates 1 and 2 are

$$k_{11} = \frac{4EI}{3} + \frac{4EI}{6} = 2EI$$

$$k_{21} = -\frac{6EI}{3^2} + \frac{6EI}{6^2} = -0.5EI$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown in Fig. 4.11(g). The forces required at coordinates 1 and 2 are

$$k_{12} = -\frac{6EI}{3^2} + \frac{6EI}{6^2} = -0.5EI$$

$$k_{22} = \frac{12EI}{3^3} + \frac{12EI}{6^3} = 0.5EI$$

Hence, the required stiffness matrix $[k]$ is given by the equation

$$[k] = EI\begin{bmatrix} 2.0 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

Multiplying the flexibility and stiffness matrices,

$$[\delta][k] = \frac{2}{3EI}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} EI \begin{bmatrix} 2 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As the product is a unit matrix, the two matrices are the inverse of each other.

(iii) The support conditions are shown in Fig. 4.11(h). The flexibility matrix can be developed by applying a unit force successively at coordinates 1 and 2 and evaluating the displacements at coordinates 1 and 2. To generate the first column of the flexibility matrix, apply a unit force at coordinate 1. Using Eqs (A.35) and (A.36) of Appendix A, the displacements at coordinates 1 and 2 are

$$\delta_{11} = \frac{3}{4\times 9^3 EI}\left[4\times 9^3 - 12\times 9^2\times 3 + 12\times 9\times 3^2 - 3\times 3^3\right]$$

$$= \frac{11}{12EI}$$

$$\delta_{21} = \frac{3^2}{4\times 9^3 EI}[2\times 9^3 - 6\times 9^2\times 3 + 5\times 9\times 3^2 - 3^3]$$

$$= \frac{7}{6EI}$$

To generate the second column of the flexibility matrix, apply a unit force at coordinate 2. Using Eqs (A.30) and (A.31) of Appendix A, the displacements at coordinates 1 and 2 are

$$\delta_{12} = \frac{3^2}{4\times 9^3 EI}\left[2\times 9^3 - 6\times 9^2\times 3 + 5\times 9\times 3^2 - 3^3\right]$$

$$= \frac{7}{6EI}$$

$$\delta_{22} = \frac{3^3}{12\times 9^3 EI}\left[4\times 9^3 - 9\times 9^2\times 3 + 6\times 9\times 3^2 - 3^3\right]$$

$$= \frac{11}{3EI}$$

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{12EI}\begin{bmatrix} 11 & 14 \\ 14 & 44 \end{bmatrix}$$

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 and 2 without any displacement at the other coordinate and determining the forces required at coordinates 1 and 2. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 4.11(i). The forces required at coordinates 1 and 2 are

$$k_{11} = \frac{4EI}{3} + \frac{3EI}{6} = \frac{11EI}{6}$$

$$k_{21} = \frac{-6EI}{3^2} + \frac{3EI}{6^2} = \frac{-7EI}{12}$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown in Fig. 4.11(j). The forces required at coordinates 1 and 2 are

$$k_{12} = -\frac{6EI}{3^2} + \frac{3EI}{6^2} = \frac{-7EI}{12}$$

$$k_{22} = \frac{12EI}{3^3} + \frac{3EI}{6^3} = \frac{11EI}{24}$$

Hence, the required stiffness matrix $[k]$ is given by the equation

$$[k] = \frac{EI}{24}\begin{bmatrix} 44 & -14 \\ -14 & 11 \end{bmatrix}$$

Multiplying the flexibility and stiffness matrices,

$$[\delta][k] = \frac{1}{12EI}\begin{bmatrix} 11 & 14 \\ 14 & 44 \end{bmatrix}\frac{EI}{24}\begin{bmatrix} 44 & -14 \\ -14 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As the product is a unit matrix, the two matrices are the inverse of each other.

As the flexibility and stiffness matrices are the reciprocal of each other, any one of them can be derived by inverting the other. While in some cases both the matrices can be written down with approximately equal computational effort, it frequently happens that the calculations required for developing one of the two types of matrices are much more as compared to those for the other. In such cases, either the flexibility matrix or stiffness matrix, whichever is easier, should be developed. The other matrix can then be evaluated by the process of matrix inversion. The three examples given below illustrate this point.

Example 4.9

Develop the flexibility and stiffness matrices for beam AB with reference to the coordinates shown in Fig. 4.12(a).

Solution

The flexibility matrix can be developed by applying a unit force successively at the coordinates and evaluating the displacements at all the coordianates. To generate the first column of the flexibility matrix, apply a unit force at coordinate 1. Using Eqs (A.14), (A.15) and (A.16) of Appendix A, the displacements at the coordinates are

$$\delta_{11} = \frac{10}{EI}$$

$$\delta_{21} = \frac{10\times 10}{2EI} = \frac{50}{EI}$$

$$\delta_{31} = \frac{10}{EI}$$

$$\delta_{41} = \frac{10(2\times 20 - 10)}{6EI} = \frac{150}{EI}$$

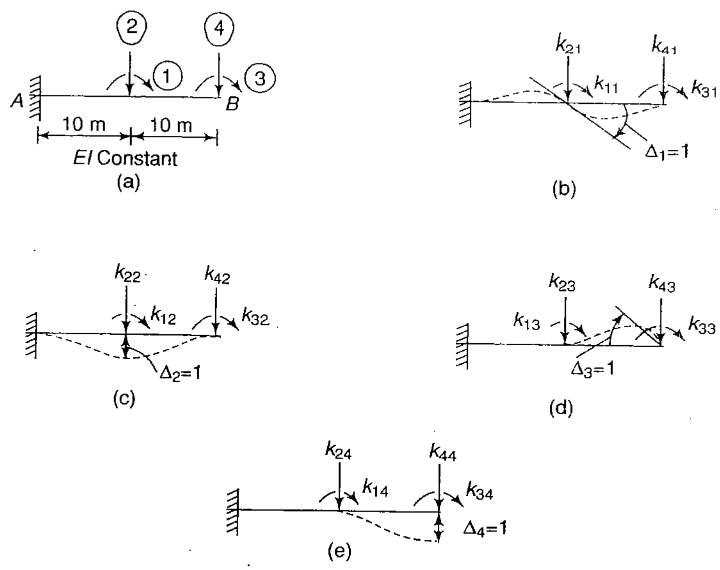


Fig. 4.12

To generate the second column of the flexibility matrix, aply a unit force at coordinate 2. Using Eqs. (A.9), (A.10) and (A.11) of Appendix A, the displacements at the coordinates are

$$\delta_{12} = \frac{10 \times 10}{2EI} = \frac{50}{EI}$$

$$\delta_{22} = \frac{10^3}{3EI} = \frac{1000}{3EI}$$

$$\delta_{32} = \frac{10 \times 10}{2EI} = \frac{50}{EI}$$

$$\delta_{42} = \frac{10^2(3 \times 20 - 10)}{6EI} = \frac{2500}{3EI}$$

To generate the third column of the flexibility matrix, apply a unit force at coordinate 3. Using Eqs (A.5) to (A.8) of Appendix A, the displacements at the coordinates are

$$\delta_{13} = \frac{10}{EI} \qquad \delta_{23} = \frac{10^2}{2EI} = \frac{50}{EI}$$

$$\delta_{33} = \frac{20}{EI} \qquad \delta_{43} = \frac{20^2}{2EI} = \frac{200}{EI}$$

To generate the fourth column of the flexibility matrix, apply a unit force at coordinate 4. Using Eqs (A.1) to (A.4) of Appendix A, the displacements at the coordinates are

$$\delta_{14} = \frac{10(2 \times 20 - 10)}{2EI} = \frac{150}{EI}$$

$$\delta_{24} = \frac{10^2(3 \times 20 - 10)}{6EI} = \frac{2500}{3EI}$$

$$\delta_{34} = \frac{20^2}{2EI} = \frac{200}{EI}$$

$$\delta_{44} = \frac{20^3}{3EI} = \frac{8000}{3EI}$$

Hence, the required flexibility matrix $[\delta]$ is given by equation

$$[\delta] = \frac{1}{3EI}\begin{bmatrix} 30 & 150 & 30 & 450 \\ 150 & 1000 & 150 & 2500 \\ 30 & 150 & 60 & 600 \\ 450 & 2500 & 600 & 8000 \end{bmatrix}$$

The stiffness matrix can be developed by giving a unit displacement successively at each coordinate without any displacement at the other coordinates and determining the forces required at all the coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 4.12(b). The forces required at the coordinates are

$$k_{11} = \frac{4EI}{10} + \frac{4EI}{10} = 0.8EI$$

$$k_{21} = \frac{6EI}{10^2} - \frac{6EI}{10^2} = 0$$

$$k_{31} = \frac{2EI}{10} = 0.2EI$$

$$k_{41} = -\frac{6EI}{10^2} = -0.06EI$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown in Fig. 4.12(c). The forces required at the coordinates are

$$k_{12} = \frac{6EI}{10^2} - \frac{6EI}{10^2} = 0$$

$$k_{22} = \frac{12EI}{10^3} + \frac{12EI}{10^3} = 0.024EI$$

$$k_{32} = \frac{6EI}{10^2} = 0.06EI$$

$$k_{42} = -\frac{12EI}{10^3} = -0.012EI$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 as shown in Fig. 4.12(d). The forces required at the coordinates are

$$k_{13} = \frac{2EI}{10} = 0.2EI$$

$$k_{23} = \frac{6EI}{10^2} = 0.06EI$$

$$k_{33} = \frac{4EI}{10} = 0.4EI$$

$$k_{43} = \frac{-6EI}{10^2} = -0.06EI$$

To generate the fourth column of the stiffness matrix, give a unit displacement at coordinate 4 as shown in Fig. 4.12(e). The forces required at the coordinates are

$$k_{14} = \frac{-6EI}{10^2} = -0.06EI$$

$$k_{24} = \frac{-12EI}{10^3} = -0.012EI$$

$$k_{34} = \frac{-6EI}{10^2} = -0.06EI$$

$$k_{44} = \frac{12EI}{10^3} = 0.012EI$$

Hence, the required stiffness matrix $[k]$ is given by the equation

$$[k] = EI\begin{bmatrix} 0.800 & 0 & 0.200 & -0.060 \\ 0 & 0.024 & 0.060 & -0.012 \\ 0.200 & 0.060 & 0.400 & -0.060 \\ -0.060 & -0.012 & -0.060 & 0.012 \end{bmatrix}$$

In this example the computational effort required for developing the flexibility matrix is approximately the same as that for the stiffness matrix.

Example 4.10

Develop the flexibility and stiffness matrices for frame ABCD with reference to the coordinates shown in Fig. 4.13.

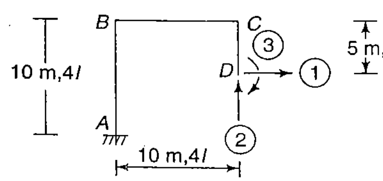


Fig. 4.13

Soluton

The flexibility matrix can be developed by

applying a unit force successively at coordinates 1, 2 and 3 and evaluating the displacements at all the coordinates. These displacements can be computed easily by applying the unit load method discussed in Sec. 2.12. The necessary computational details are shown in Table 4.4. Bending moment producing compression on outer fibres has been taken positive.

Table 4.4

Portion	*DC*	*CB*	*BA*
I	*I*	*4I*	*4I*
Origin	*D*	*C*	*B*
Limits	0 to 5	0 to 10	0 to 10
m_1	x	5	$(5-x)$
m_2	0	x	10
m_3	-1	-1	-1

Using Table 4.4,

$$\delta_{11} = \int \frac{m_1 m_1 dx}{EI} = \int_0^5 \frac{x.xdx}{EI} + \int_0^{10} \frac{5\times 5dx}{4EI} + \int_0^{10} \frac{(5-x)(5-x)dx}{4EI}$$

$$= \frac{125}{EI}$$

$$\delta_{21} = \delta_{12} = \int \frac{m_2 m_1 dx}{EI} = \int_0^{10} \frac{5xdx}{4EI} + \int_0^{10} \frac{10(5-x)dx}{4EI} = \frac{125}{2EI}$$

$$\delta_{31} = \delta_{13} = \int \frac{m_3 m_1 dx}{EI} = \int_0^5 \frac{-xdx}{EI} + \int_0^{10} \frac{-5dx}{4EI} + \int_0^{10} \frac{-(5-x)dx}{4EI} = -\frac{25}{EI}$$

$$\delta_{22} = \int \frac{m_2 m_2 dx}{EI} = \int_0^{10} \frac{x.xdx}{4EI} + \int_0^{10} \frac{10\times 10dx}{4EI} = \frac{1000}{3EI}$$

$$\delta_{32} = \delta_{23} = \int \frac{m_3 m_2 dx}{EI} = \int_0^{10} \frac{-xdx}{4EI} + \int_0^{10} \frac{-10dx}{4EI} = -\frac{75}{2EI}$$

$$\delta_{33} = \int \frac{m_3 m_3 dx}{EI} = \int_0^5 \frac{(-1)(-1)dx}{EI} + \int_0^{10} \frac{(-1)(-1)dx}{4EI}$$

$$+ \int_0^{10} \frac{(-1)(-1)dx}{4EI} = \frac{10}{EI}$$

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{6EI}\begin{bmatrix} 750 & 375 & -150 \\ 375 & 2000 & -225 \\ -150 & -225 & 60 \end{bmatrix}$$

The stiffness matrix can be developed by giving a unit displacement succesively at coordinates 1, 2 and 3 without any displacement at the other coordinates and determining the forces required at all the coordinates. For instance, to generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at coordinates 2 and 3. This type of displacement can occur if a support is provided at end D which permits horizontal displacement but prevents vertical displacement and rotation. Hence, in order to determine stiffnes elements k_{11}, k_{21} and k_{31}, it is necessary to solve a portal frame with second degree of indeterminacy. To generate each of the remaining two columns of the stiffness matrix, it becomes necessary to solve a portal frame with second degree of indeterminacy. Thus, a portal frame with second degree of indeterminacy will have to be solved three times for the development of the stiffness matrix. It would, therefore, appear much simpler to obtain the stiffness matrix in this case by inverting the flexibility matrix. Hence, inverting the flexibility matrix, the required stiffness matrix $[k]$ is given by the equation

$$[k] = EI\begin{bmatrix} 0.01741 & 0.00282 & 0.05412 \\ 0.00282 & 0.00565 & 0.02824 \\ 0.05412 & 0.02824 & 0.34118 \end{bmatrix}$$

Example 4.11

Develop the flexibility and stiffness matrices for portal frame ABCD with reference to the coordinates shown in Fig. 4.14(a).

Solution

The flexibility matrix can be developed by applying a unit force successively at coordinates 1, 2 and 3 and evaluating the displacements at all the coordinates. It will be observed that the portal frame, which is indeterminate to the third degree, will have to be analysed three times by the slope-deflection method or otherwise. In this instance, lesser computational effort will be needed if the flexibility matrix is derived by inverting the stiffness matrix.

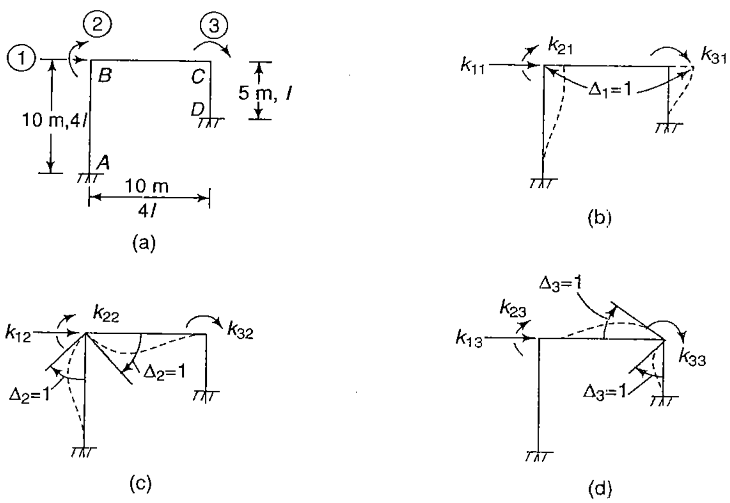


Fig. 4.14

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1, 2 and 3 without any displacement at other coordinates and determining the forces required at all the coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 4.14(b). Using Table 4.1, forces required at the coordinates are

$$k_{11} = \frac{12E(4I)}{10^3} + \frac{12E(I)}{5^3} = 0.144EI$$

$$k_{21} = -\frac{6E(4I)}{10^2} = -0.24EI$$

$$k_{31} = -\frac{6E(I)}{5^2} = -0.24EI$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 3 as shown in Fig. 4.14(c). Using Table 4.1, the forces required at the coordinates are

$$k_{12} = -\frac{6E(4I)}{10^2} = -0.24EI$$

$$k_{22} = \frac{4E(4I)}{10} + \frac{4E(4I)}{10} = 3.2EI$$

$$k_{32} = \frac{2E(4I)}{10} = 0.8EI$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 as shown in Fig. 4.14(d). Using Table 4.1, the forces required at the coordinates are

$$k_{13} = -\frac{6E(I)}{5^2} = -0.24EI$$

$$k_{23} = \frac{2E(4I)}{10} = 0.8EI$$

$$k_{33} = \frac{4E(4I)}{10} + \frac{4EI}{5} = 2.4EI$$

Hence, the required stiffness mateix $[k]$ is given by the equation

$$[k] = EI\begin{bmatrix} 0.144 & -0.240 & -0.240 \\ -0.240 & 3.200 & 0.800 \\ -0.240 & 0.800 & 2.400 \end{bmatrix}$$

Alternatively, the elements of the stiffness matrix can be determined directly by using Eq. (4.30).

To obtain the flexibility matrix, invert the stiffness matrix. The required flexibility matrix $[\delta]$ is given y the equation

$$[\delta] = \frac{1}{EI}\begin{bmatrix} 8.9869 & 0.4902 & 0.7353 \\ 0.4902 & 0.3676 & -0.0735 \\ 0.7353 & -0.0735 & 0.5147 \end{bmatrix}$$

4.5 FORCE AND DISPLACEMENT METHODS

The use of flexibility and stiffness matrices provides systematic method for the analysis of large structures with a high degree of static or kinematic indeterminacy. The matrix approach is particularly amenable to a computer oriented solution of the problem. The solution using the matrix approach usually involves the inversion of either the flexibility or the stiffness matrix which is a common subroutine of modern digital computers. There are two main methods of structural analysis using the matrix approach.

4.5.1 Force Method

This method is also known as *flexibility method* or *compatibility method.* In this method the degree of static indeterminacy of the structure is determined and the redundants are identified. A coordinate is assigned to each redundant. Thus P_1, P_2, ..., P_n are the redundants at coordinates 1, 2, ..., n. If all the redundants are removed, the resulting structure, known as the *released structure,* is statically determinate. This released structure is also known as *basic determinate structure*. From the principle of superposition, the net displacement at ~~any point~~ in a statically determinate structure is the sum of the displacements in the basic determinate structure due to the applied loads and the redundants. This condition, known as *compatibility condition*, may be expressed by the equations

$$\begin{aligned} \Delta_1 &= \Delta_{1L} + \Delta_{1R} \\ \Delta_2 &= \Delta_{2L} + \Delta_{2R} \\ &\vdots \\ \Delta_j &= \Delta_{jL} + \Delta_{jR} \\ &\vdots \\ \Delta_n &= \Delta_{nL} + \Delta_{nR} \end{aligned} \tag{4.38}$$

where Δ_j = displacement at coordinate j in the statically indeterminate structure

Δ_{jL} = displacement at coordinate j in the released structure due to the applied loads

Δ_{jR} = displacement at coordinate j in the released structure due to the redundants

Equation (4.38) may be expressed in the matrix form

$$[\Delta] = [\Delta_L] + [\Delta_R] \tag{4.39}$$

Substituting from Eq. (2.5) into Eq. (4.38),

$$\begin{aligned} \Delta_1 &= \Delta_{1L} + \delta_{11} P_1 + \delta_{12} P_2 + \ldots \quad \delta_{1n} P_n \\ \Delta_2 &= \Delta_{2L} + \delta_{21} P_1 + \delta_{22} P_2 + \ldots \quad \delta_{2n} P_n \\ &\vdots \\ \Delta_n &= \Delta_{nL} + \delta_{n1} P_1 + \delta_{n2} P_2 + \ldots + \delta_{nn} P_n \end{aligned} \tag{4.40}$$

Equation (4.40) may be written in the matrix form

$$[\Delta] = [\Delta_L] + [\delta][P] \tag{4.41}$$

Solving Eq. (4.41) for the redundants,

$$[P] = [\delta]^{-1}\{[\Delta] - [\Delta_L]\} \tag{4.42}$$

If the net displacements at the redundants are zero,

$$\Delta_1 = \Delta_2 = \ldots = \Delta_n = 0$$

In this case Eq. (4.42) takes the form

$$[P] = -[\delta]^{-1}[\Delta_L] \tag{4.43}$$

It may be noted that the force method is actually the *method of consistent deformation* expressed in a systematic manner through the use of matrices. The method is known as the force method because the compatibility equations are derived by the superposition of two systems of forces, one constituting the applied loads and the other constituting the redundants. Another reason for calling it the force method is that in this approach the forces are treated as unknowns. The method is also known as the flexibility method because the flexibility matrix has to be developed to establish the compatibility of displacements. As the condition of compatibility of displacements is utilized in this method, it is also named as compatibility method.

4.5.2 Displacement Method

This method is also known as *stiffness method* or *equilbrium method.* In this method the degree of kinematic indeterminacy (degree of freedom) of the structure is determined and a coordinate is assigned to each independent displacement component. In general, the displacement components at the supports and joints are treated as independent displacement components. Let 1, 2, ..., n be the coordinates assigned to these independent displacement components Δ_1, Δ_2 ..., Δ_n. In the first instance, lock all the supports and joints to obtain the *restrained strucure* in which no displacement is possible at the coordinates. Let P'_1, P'_2,..., P'_n be the forces required at coordinates 1, 2, ..., n in the restrained structure in which the displacements Δ_1, Δ_2, ..., Δ_n are zero. Next, let the supports and joints be unlocked permitting displacements Δ_1, Δ_2, ..., Δ_n at the coordinates. These displacements require forces $P_{1\Delta}$, $P_{2\Delta}$, ..., $P_{n\Delta}$ at coordinates 1, 2, ..., n respectively. If P_1, P_2, ..., P_n are the external forces at coordinates 1, 2, ...,n, then the *condition of equilibrium* of the structure may be expressed by the equations

$$\begin{aligned} P_1 &= P_1' + P_{1\Delta} \\ P_2 &= P_2' + P_{2\Delta} \\ &\vdots \\ P_n &= P_n' + P_{n\Delta} \end{aligned} \tag{4.44}$$

Equation (4.44) may be expressed in the matrix form

$$[P] = [P'] + [P_\Delta] \tag{4.45}$$

Substituting from Eq. (2.6) into Eq. (4.44),

$$\begin{aligned} P_1 &= P_1' + k_{11}\Delta_1 + k_{12}\Delta_2 + \ldots + k_{1n}\Delta_n \\ P_2 &= P_2' + k_{21}\Delta_1 + k_{22}\Delta_2 + \ldots + k_{2n}\Delta_n \\ &\vdots \\ P_n &= P_n' + k_{n1}\Delta_1 + k_{n2}\Delta_2 + \ldots + k_{nn}\Delta_n \end{aligned} \tag{4.46}$$

Equation (4.46) may be expressed in the matrix form

$$[P] = [P'] + [k][\Delta] \tag{4.47}$$

Solving Eq. (4.47) for the unknown independent displacement components,

$$[\Delta] = [k]^{-1}\{[P] - [P']\} \tag{4.48}$$

It is interesting to note that Eq. (4.48) is similar to Eq. (4.42) derived earlier except that forces and displacements have been interchanged. If the external forces act only at the coordinates, the terms P_1', P_2', ..., P_n' vanish. In this case Eq. (4.48) takes the form

$$[\Delta] = [k]^{-1}[P] \tag{4.49}$$

On the other hand, if there are no external forces at the coordinates,

$$P_1 = P_2 = \ldots = P_n = 0$$

In this case Eq. (4.42) takes the form

$$[\Delta] = -[k]^{-1}[P'] \tag{4.50}$$

After the computation of displacements, the bending moments in the members of the structure may be calculated by using the slope-deflection Eq. (2.47).

It may be noted that in the displacement method it is not necessary to identify the redundants. The method is applicable to statically determinate as well as indeterminate structures. The method is known as the displacement method because in this approach the displacements are treated as unknowns. To calculate the displacements and to establish the equations of equilibrium, it is necessary to develop the stiffness matrix for the structure. The method is, therefore, also known as the stiffness method. As the condition of equilibrium of the structure is utilized for the determination of displacements, the method is also named as the equilibrium method.

It may be noted that the two matrix methods show striking similarites in respect of the steps to be followed in the solution of the problem. The steps in the two methods proceed on almost parallel lines. Yet, there are also equally striking differences in the two methods. For example, whereas the statically indeterminate structure to be analysed is released to obtain the basic determinate structure by removing the redundants in the force method, it is restrained by the addition of restraining forces so as to obtain the restrained structure in the displacement method. The similarities and dis-similarities of the two methods are brought out clearly in Table 4.5.

Table 4.5

Step	*Force method (flexibility or compatibility method)*	*Displacement method (stiffness or equilibrium method)*
1.	Determine the degree of static indeterminacy (degree of redundancy), n.	Determine the degree of kinematic indeterminacy, (degree of freedom), n.
2.	Choose the redundants.	Identify the independent displacement components.
3.	Assign coordinates 1, 2, ..., n to the redundants.	Assign coordinates 1, 2, ..., n to the independent displacement components.
4.	Remove all the redundants to obtain the released structure.	Prevent all the independent displacement components to obtain the restrained structure.
5.	Determine $[\Delta_L]$, the displacements at the coordinates due to the applied loads acting on the released structure.	Determine $[P']$, the forces required at the coordinates in the restrained structure due to the loads other than those acting at the coordinates.
6.	Determine $[\Delta_R]$, the displacements at the coordinates due to the redundants acting on the released structure.	Determine $[P_\Delta]$, the forces required at the coordinates in the unrestrained structure to cause the independent displacement components $[\Delta]$.
7.	Compute the net displacements at the coordiantes. $[\Delta] = [\Delta_L] + [\Delta_R]$	Compute the net forces at the coordinates. $[P] = [P'] + [P_\Delta]$
8.	Use the condtions of compatiblility of displacements to compute the reduntands. $[P] = [\delta]^{-1}\{[\Delta]-[\Delta_L]\}$	Use the conditions of equilibrium of forces to compute the displacements. $[\Delta] = [k]^{-1}\{[P] - [P']\}$
9.	Knowing the redundants, compute the internal member forces by using equations of statics.	Knowing the displacements, compute the internal member forces by using slope-deflection equations.

Example 4.12

Analyse the continuous beam shown in Fig. 4.15(a) using (i) force method and (ii) displacement method.

Solution

(i) *Force Method*

The degree of static indeterminacy of the beam is two. Let the basic determinate structure be obtained by releasing the bending moments at B and C. This can be done by inserting hinges at B and C so that the spans AB and BC behave like simply supported beams. The released structure and coordinates 1 and 2 assigned to the redundant bending moments at B and C are shown in Fig. 4.15(b). The displacements at coordinates 1 and 2 may be computed as explained in Sec. 2.4.

To develop the flexibility matrix, a unit force should be applied successively at coordinates 1 and 2. Thus to generate the first column of the flexibility matrix, apply a unit force at coordinate 1. The deflection curve due to a unit force at coordinate 1 is shown in Fig. 4.15(c). Using Table 2.16,

$$\delta_{11} = \frac{4}{3EI} + \frac{3}{3EI} = \frac{7}{3EI}$$

$$\delta_{21} = -\frac{4}{6EI} = -\frac{2}{3EI}$$

To generate the second column of the flexibility matrix, apply a unit force at coordinate 2. The deflection curve due to unit force at coordinate 2 is shown in Fig. 4.15(d). Using Table 2.16,

$$\delta_{12} = -\frac{4}{6EI} = -\frac{2}{3EI}$$

$$\delta_{22} = \frac{4}{3EI}$$

Hence, the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{3EI}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix} \tag{a}$$

Figure 4.15(e) shows the deflection curve due to the applied loads. Using Table 2.16,

$$\Delta_{1L} = \frac{100 \times 3^2}{16EI} + \frac{60 \times 4^3}{24EI} = \frac{216.25}{EI}$$

$$\Delta_{2L} = -\frac{60 \times 4^3}{24EI} = -\frac{160}{EI} \tag{b}$$

For the continuity of the beam, the net flexural displacements at coordinates 1 and 2 must be zero. Hence,

$$\Delta_1 = \Delta_2 = 0$$

Substituting from Eqs (a) and (b) into Eq. (4.43),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -3EI\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{216.25}{EI} \\ -\dfrac{160}{EI} \end{bmatrix}$$

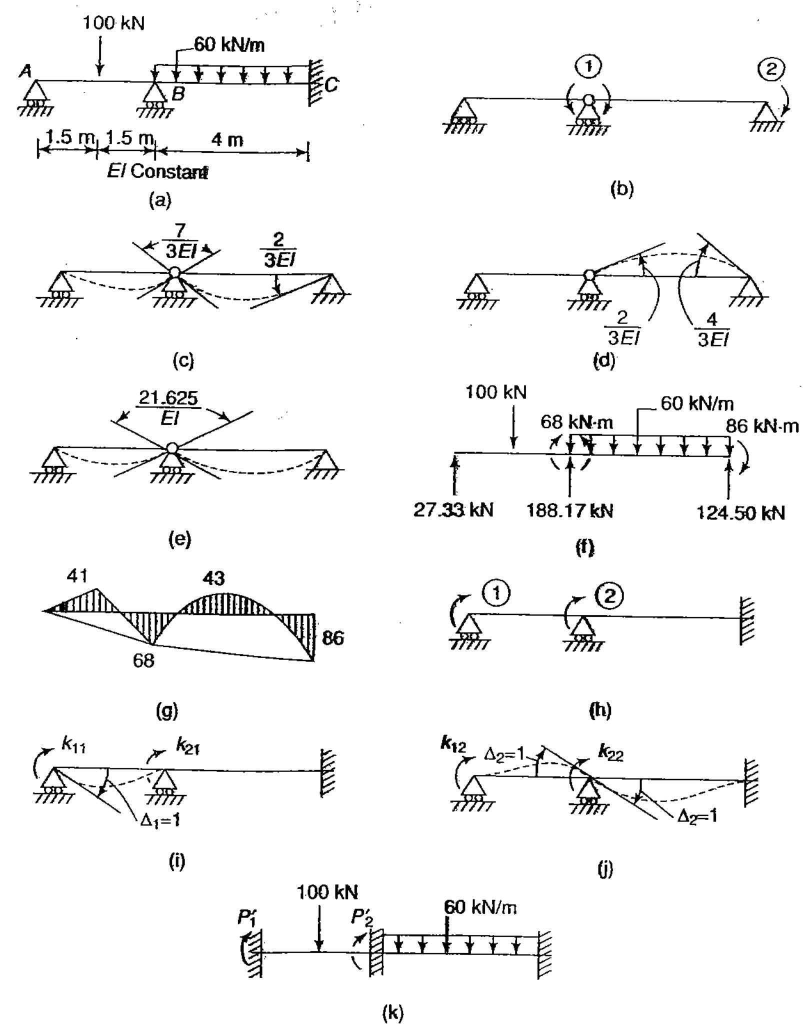


Fig. 4.15

Hence,

$$P_1 = 68 \text{ kN·m}$$

$$P_2 = 86 \text{ kN·m}$$

Thus the hogging moments at B and C are 68 kN·m and 86 kN·m respectively. The free-body diagram and the bending-moment diagram are shown in Fig. 4.15(f) and (g) respectively.

(ii) *Displacement Method*

The degree of freedom of the beam is two. The rotations at A and B are the two independent displacement components. Assigning coordinates 1 and 2 to these displacement components as shown in Fig. 4.15(h), the stiffness matrix with reference to the chosen coordinates may be developed by giving a unit displacement successively at coordinates 1 and 2. Thus, to generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown by the deflection curve in Fig. 4.15(i). Using Table 2.16,

$$k_{11} = \frac{4EI}{3} \qquad k_{21} = \frac{2EI}{3}$$

Similarly, to generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown by the deflection curve in Fig. 4.15(j). Using Table 2.16,

$$k_{12} = \frac{2EI}{3}$$

$$k_{22} = \frac{4EI}{3} + \frac{4EI}{4} = \frac{7EI}{3}$$

Hence, the stiffness matrix $[k]$ is given by the equation

$$[k] = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \tag{a}$$

As there are no external loads at the coordinates,

$$P_1 = P_2 = 0$$

The restraining forces P_1' and P_2' in the restrained structure due to the applied loads may be computed by considering spans AB and BC as fixed-ended beams as shown in Fig. 4.15(k). Using Table 2.16,

$$P_1' = -\frac{100 \times 3}{8} = -37.5 \text{ kN·m}$$

$$P_2' = \frac{100 \times 3}{8} - \frac{60 \times 4^2}{12} = -42.5 \text{ kN·m}$$

Substituting from Eqs (a) and (b) into Eq. (4.50),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = -\frac{3}{EI}\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}^{-1}\begin{bmatrix} -37.5 \\ -42.5 \end{bmatrix}$$

$$\Delta_1 = \frac{22.2}{EI} \qquad \Delta_2 = \frac{11.9}{EI}$$

The bending moments may now be computed by using slope-deflection Eq. (2.47).

$$M_{AB} = -37.5 + \frac{2EI}{3}\left(2 \times \frac{22.2}{EI} + \frac{11.9}{EI}\right) = 0$$

$$M_{BA} = 37.5 + \frac{2EI}{3}\left(2 \times \frac{11.9}{EI} + \frac{22.2}{EI}\right) = 68 \text{ kN·m}$$

$$M_{BC} = -80 + \frac{2EI}{4}\left(2 \times \frac{11.9}{EI}\right) = -68 \text{ kN·m}$$

$$M_{CB} = 80 + \frac{2EI}{4}\left(\frac{11.9}{EI}\right) = 86 \text{ kN·m}$$

Thus the hogging bending moments at B and C are 68 kN·m and 86 kN·m respectively. These bending moments are the same as obtained in flexibility method, part (i).

Example 4.13

Analyse the portal frame ABCD *shown in Fig. 4.16(a) using (i) force method and (ii) displacement method.*

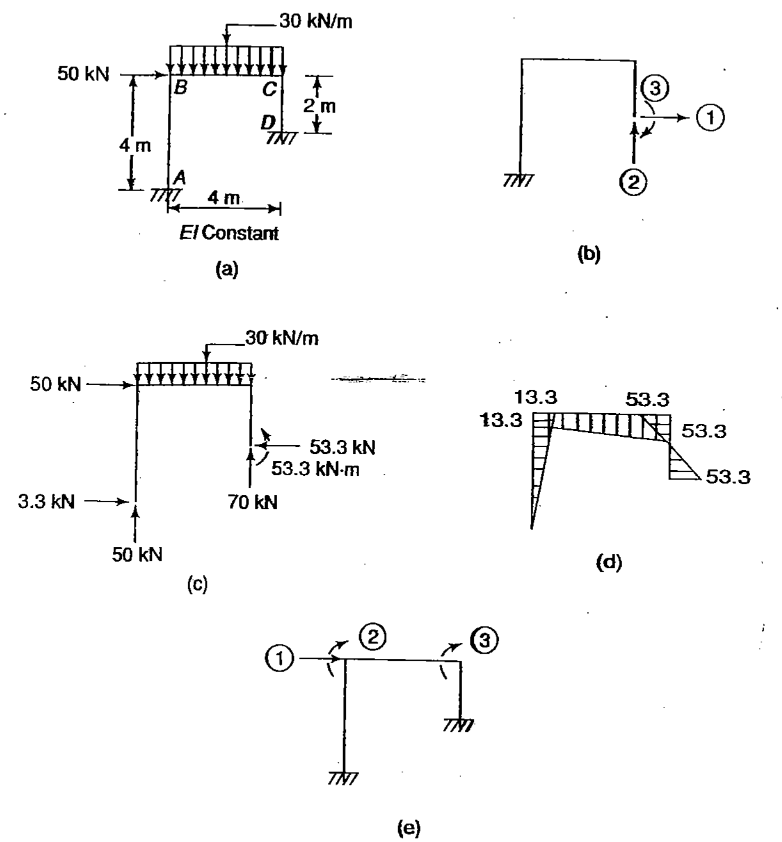


Fig. 4.16

Solution

(i) *Force Method*

The portal frame is statically indeterminate to the third degree. The three reaction components, viz., a horizontal reaction P_1, a vertical reaction P_2 and a bending couple P_3 at end D may be treated as redundants. Hence, the basic determinate structure may be obtained by releasing redundants P_1, P_2 and P_3. The coordinates 1, 2 and 3 are assigned to each of the three redundants as shown in Fig. 4.16(b).

The displacements Δ_{1L}, Δ_{2L} and Δ_{3L} due to the applied loading for the basic determinate structure may be computed by applyig the unit-load method. The necessary details for the computation are shown in Table 4.6. Bending moment producing compression on outer fibres has been taken positive.

Table 4.6

Portion	Dc	CB	BA
I	I	I	I
Origin	D	C	B
Limits	0 to 2	0 to 4	0 to 4
M	0	$-15x^2$	$-50x-240$
m_1	x	2	$(2-x)$
m_2	0	x	4
m_3	-1	-1	-1

$$\Delta_{1L} = \int \frac{Mm_1dx}{EI} = \int_0^4 \frac{-30x^2dx}{EI} + \int_0^4 \frac{(-50x-240)(2-x)dx}{EI} = -\frac{1120}{EI}$$

$$\Delta_{2L} = \int \frac{Mm_2dx}{EI} = \int_0^4 \frac{15x^3dx}{EI} + \int_0^4 \frac{-4(50x+240)dx}{EI} = -\frac{6400}{EI} \qquad \text{(a)}$$

$$\Delta_{3L} = \int \frac{Mm_3dx}{EI} = \int_0^4 \frac{15x^2dx}{EI} + \int_0^4 \frac{(50x+240)dx}{EI} = \frac{1680}{EI}$$

The flexibility matrix can be obtained by applying a unit force successively at coordinates 1, 2 and 3 and evaluating the displacements at all the coordinates.

$$\delta_{11} = \int \frac{m_1^2dx}{EI} = \int_0^2 \frac{x^2dx}{EI} + \int_0^4 \frac{4dx}{EI} + \int_0^4 \frac{(2-x)^2dx}{EI} = \frac{24}{EI}$$

$$\delta_{22} = \int \frac{m_2^2dx}{EI} = \int_0^4 \frac{x^2dx}{EI} + \int_0^4 \frac{16dx}{EI} = \frac{256}{3EI}$$

$$\delta_{33} = \int \frac{m_3^2dx}{EI} = \int_0^2 \frac{dx}{EI} + \int_0^4 \frac{dx}{EI} + \int_0^4 \frac{dx}{EI} = \frac{10}{EI}$$

$$\delta_{12} = \delta_{21} = \int \frac{m_1m_2dx}{EI} = \int_0^4 \frac{2xdx}{EI} + \int_0^4 \frac{4(2-x)dx}{EI} = \frac{16}{EI}$$

$$\delta_{23} = \delta_{32} = \int \frac{m_2m_3dx}{EI} = \int_0^4 \frac{-xdx}{EI} + \int_0^4 \frac{-4dx}{EI} = -\frac{24}{EI}$$

$$\delta_{31} = \delta_{13} = \int \frac{m_1m_3dx}{EI} = \int_0^2 \frac{-xdx}{EI} + \int_0^4 \frac{-2dx}{EI} + \int_0^4 \frac{-(2-x)dx}{EI}$$

$$= -\frac{10}{EI}$$

Thus, the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{3EI}\begin{bmatrix} 72 & 48 & -30 \\ 48 & 256 & -72 \\ -30 & -72 & 30 \end{bmatrix} \qquad \text{(b)}$$

As the support at D is unyielding, the net displacements,

$$\Delta_1 = \Delta_2 = \Delta_3 = 0$$

Substituting from Eq. (a) and (b) into Eq. (4.43),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -3EI \begin{bmatrix} 72 & 48 & -30 \\ 48 & 256 & -72 \\ -30 & -72 & 30 \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{1120}{EI} \\ -\dfrac{6400}{EI} \\ \dfrac{1680}{EI} \end{bmatrix}$$

Hence,

$$P_1 = -53.3 \text{ kN}$$

$$P_2 = 70 \text{ kN}$$

$$P_3 = -53.3 \text{ kN-m}$$

The free-body diagram and bending-moment diagram are shown in Fig. 4.16(c) and (d) respectively.

(ii) *Displacement Method*

As ends A and D are fixed, the horizontal displacement at B and the rotations at joints B and C are the three independent displacement components. Hence, the degree of freedom of the portal frame is three. The coordinates 1, 2 and 3 are assigned to the three independent displacement components as shown in Fig. 4.16(e).

The stiffness matrix for the portal frame can be developed by giving a unit displacement successively at coordinates 1, 2 and 3 and determining the forces required at the coordinates. Using Eq. (4.30), the elements of the stiffness matrix are

$$k_{11} = \frac{12EI}{4^3} + \frac{12EI}{2^3} = \frac{27}{64}EI$$

$$k_{22} = \frac{4EI}{4} + \frac{4EI}{4} = 2EI$$

$$k_{33} = \frac{4EI}{4} + \frac{4EI}{2} = 3EI$$

$$k_{12} = k_{21} = -\frac{6EI}{4^2} = -\frac{3}{8}EI$$

$$k_{23} = k_{32} = \frac{2EI}{4} = \frac{EI}{2}$$

$$k_{31} = k_{13} = -\frac{6EI}{2^2} = -1.5EI$$

Hence, the required stiffness matrix $[k]$ is given by the equation

$$[k] = \frac{EI}{16}\begin{bmatrix} 27 & -6 & -24 \\ -6 & 32 & 8 \\ -24 & 8 & 48 \end{bmatrix}$$

The external loads at the coordinates are

$$P_1 = 50 \text{ kN}$$
$$P_2 = P_3 = 0$$

The uniformly distributed load of 30 kN/m does not act at any one of the three coordinates. The restraining forces at the coordinates in the restrained structure due to the uniformly distributed load are

$$P_1' = 0$$

$$P_2' = -\frac{30 \times 4^2}{12} = -40 \text{ kN} \cdot \text{m}$$

$$P_3' = \frac{30 \times 4^2}{12} = 40 \text{ kN} \cdot \text{m}$$

The displacements Δ_1, Δ_2 and Δ_3 may be obtained by substituting into Eq. (4.48).

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{16}{EI}\begin{bmatrix} 27 & -6 & -24 \\ -6 & 32 & 8 \\ -24 & 8 & 48 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -40 \\ 40 \end{bmatrix} \right\}$$

$$= \frac{1}{9EI}\begin{bmatrix} 320 \\ 240 \\ 0 \end{bmatrix}$$

i.e.,

$$\Delta_1 = \frac{35.6}{EI}$$

$$\Delta_2 = \frac{26.7}{EI}$$

$$\Delta_3 = 0$$

The end moments in the members are obtained by using slope-deflection Eq. (2.47)

$$M_{AB} = 0$$
$$M_{BA} = -M_{BC} = 13.3 \text{ kN·m}$$
$$M_{CB} = -M_{CD} = 53.3 \text{ kN.m}$$
$$M_{DC} = -53.3 \text{ kN·m}$$

The free-body diagram and bending-moment diagram are shown in Fig. 4.16(c) and (d) respectively.

PROBLEMS

4.1 Show that the flexibility matrix for the simply supported beam of Fig. 4.17, with reference to the coordinates shown in the figure, is given by the equation

$$[\delta] = \frac{L}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

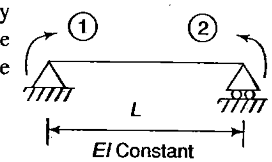


Fig. 4.17

Develop the stiffness matrix and verify that it is the reciprocal of the flexibility matrix.

4.2 For the simply supported beam of uniform cross-section shown in Fig. 4.18, develop the flexibility matrix with reference to the coordinates shown in the figure. Does the stiffness matrix exist?

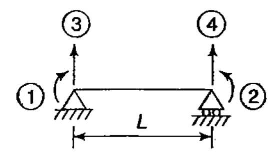


Fig. 4.18

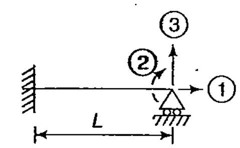


Fig. 4.19

4.3 Develop the flexibility matrix for the propped cantilever beam of uniform cross-section shown in Fig. 4.19 with reference to the coordinates shown in the figure. Does the stiffness matrix exist?

4.4 For the cantilever beam of uniform cross-section shown in Fig. 4.20, develop the flexibility and stiffness matrices with reference to the coordinates shown in the figure. Verify that the two matrices are the reciprocal of each other.

4.5 Develop the stiffness matrix for beam *AB* of uniform cross-section shown in Fig. 4.21 with reference to the coordinates shown in the figure. End *A* is hinged and end *B* is free. Discuss why the flexibility matrix does not exist.

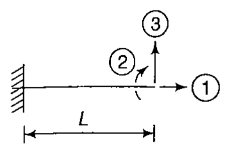


Fig. 4.20

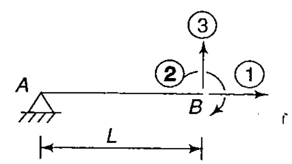


Fig. 4.21

4.6 Develop the flexibility and stiffness matrices for the beams shown in Fig. 4.22 with reference to the coordinates shown in the figure. Verify that the two matrices are reciprocal of each other. *EI* is constant unless shown otherwise in the figure.

4.7 For the structures shown in Fig. 4.23, develop the flexibility or stiffness matrix, whichever is easier, with reference to the coordinates indicated in the figure. Hence obtain the other matrix by inversion. *EI* is constant unless shown otherwise in the figure.

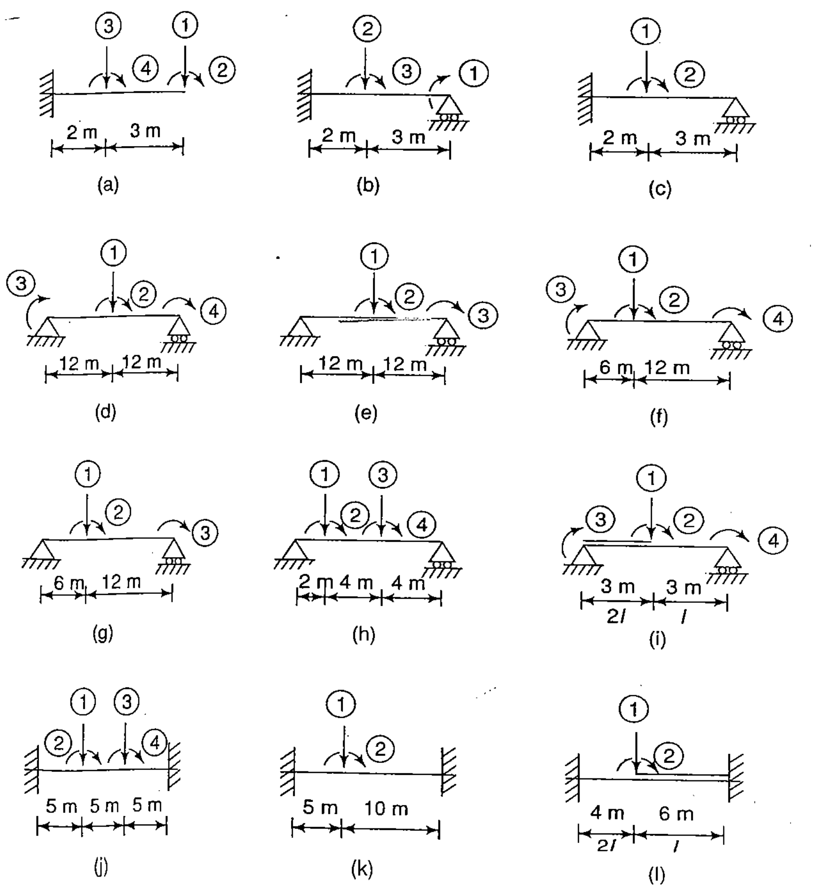


Fig. 4.22

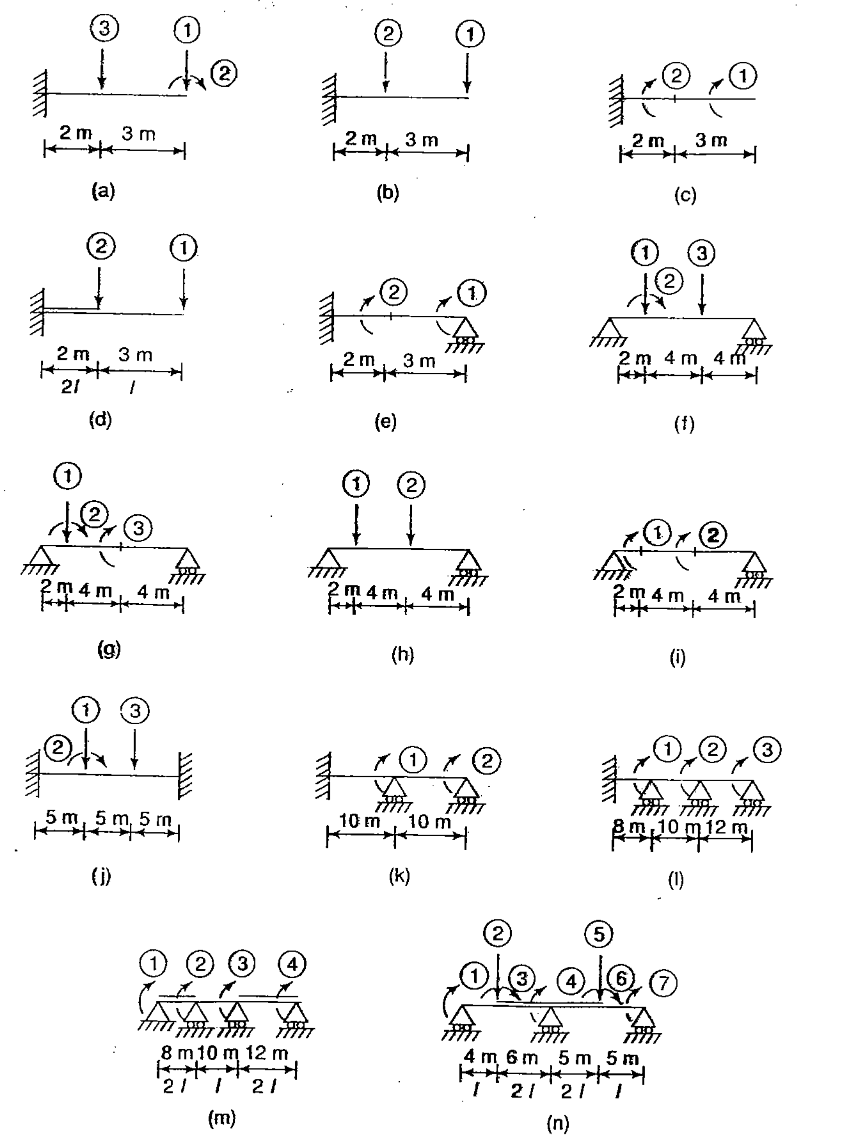


Fig. 4.23 (Contd)

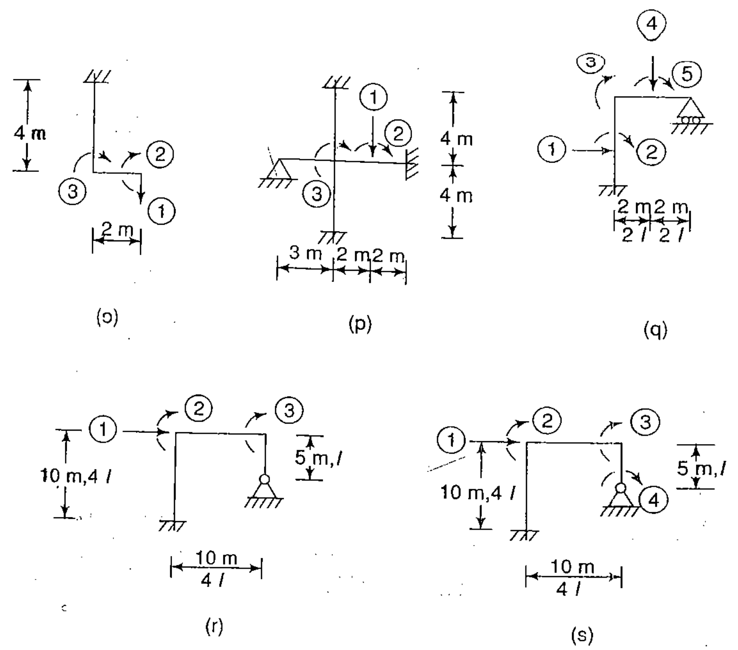


Fig. 4.23

4.8 Analyse the two-span continuous beam shown in Fig. 4.24 by the force method. Hence determine the bending moments at B and C. Verify the result by the displacement method.

4.9 Analyse the right-angled bent shown in Fig. 4.25 by the force method. Hence determine the bending moments at B and C. Verify the result by the displacement method.

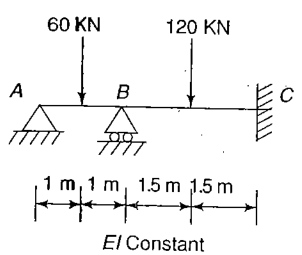


Fig. 4.24

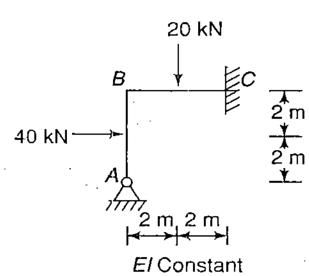


Fig. 4.25

5 CONTINUOUS BEAMS

5.1 INTRODUCTION

This chapter is devoted to the analysis of continuous beams using the two principal methods, viz., the force method and the displacement method. Beams are commonly subjected to transverse loads. Consequently, they do not carry axial forces. In general, at any cross-section of a beam there is a shear force and a bending moment which produce rotations and deflectios in the beam. Hence, the stiffnesses to transverse and flexural (rotational) displacements are of relevance in developing the stiffness matrix of a beam. These have been discussed in Sec. 4.1. Similarly, for developing the flexibility matrix of a beam, the rotations and deflections have to be computed. For this purpose any one of the methods discussed in Chapter 2 may be used.

5.2 FORCE METHOD

The force method for the analysis of beams begins with the determination of the degree of static indeterminacy and identification of the redundants. The degree of static indeterminacy has been discussed in Sec. 1.6. As a beam has an open configuration, the degree of internal indeterminacy is zero. Hence, the degree of static indeterminacy,

$$D_s = r - 3$$

In case the beam has internal hinges, the degree of static indeterminacy,

$$D_s = r - 3 - h$$

where r = number of independent external reaction components

h = number of internal hinges

The basic determinate structure is obtained by releasing a sufficient number of internal forces and external reaction components. Care should be exercised in selecting the internal and external redundants so that the released structure is statically determinate, stable and as simple as possible. Coordinates 1, 2, ..., n are assigned to all the redundants, internal as well as external.

To illustrate the manner in which coordinates are assigned in the force method, consider the three-span continuous beam shown in Fig. 5.1(a). The

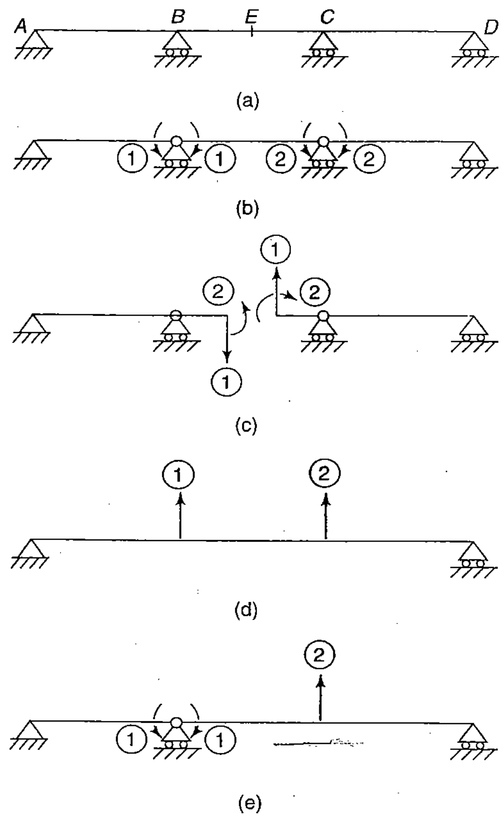


Fig. 5.1

beam is statically indeterminate to the second degree. Thus to make the beam statically determinate, two reaction components, either internal or external, have to be released. Among several possible released structures, only four have been selected for consideration and are shown in Fig. 5.1(b) to (e). The reaction components to be released are both internal in Fig. 5.1(b) and (c), both external in Fig. 5.1(d), and one internal and one external in Fig. 5.1(e). In Fig. 5.1 (b) hinges have been inserted at B and C, thereby releasing the bending moments at these cross-sections. The released structure thus obtained comprises a series of three simply supported beams. In Fig. 5.1(c) a cut has been introduced at some point E in the central span, thereby releasing the bending moment and shear force at E. In this case the released structure comprises two overhanging beams. In Fig. 5.1(d) the supports at B and C have been removed. The released structure in this case is a simply supported beam. In Fig. 5.1(e) a hinge has

been inserted at B and the support at C has been removed, thereby releasing the bending moment at B and the vertical reaction at C. The released structure in this case comprises two simply supported beams AB and BD. Coordinates 1 and 2, appropriate in each case, are also shown in Fig. 5.1(b) to (e).

The net displacement at a coordiante in the actual structure is evidently the sum of the displacement caused by the applied loads and the displacement due to the redundants in the released structure. Hence,

$$\begin{aligned}\Delta_1 &= \Delta_{1L} + \Delta_{1R} = \Delta_{1L} + \delta_{11}P_1 + \delta_{12}P_2 + \ldots + \delta_{1n}P_n \\ \Delta_2 &= \Delta_{2L} + \Delta_{2R} = \Delta_{2L} + \delta_{21}P_1 + \delta_{22}P_2 + \ldots + \delta_{2n}P_n \\ &\vdots \\ \Delta_n &= \Delta_{nL} + \Delta_{nR} = \Delta_{nL} + \delta_{n1}P_1 + \delta_{n2}P_2 + \ldots + \delta_{nn}P_n\end{aligned} \tag{5.1}$$

Where $\Delta_1, \Delta_2, \ldots, \Delta_n$ = net displacements at coordinates 1, 2, ..., n

$\Delta_{1L}, \Delta_{2L}, \ldots, \Delta_{nL}$ = displacements at coordinates 1, 2, ..., n due to applied loads in the released structure

$\Delta_{1R}, \Delta_{2R}, \ldots, \Delta_{nR}$ = displacements at coordiantes 1, 2, ..., n due redundants $P_1, P_2, \ldots, P_n$ in the released structure

Equation (5.1), which represents the compatibility conditions, is sufficient for the determination of all the redundants. Equation (5.1) may be written in the matrix from

$$\begin{bmatrix}\Delta_1\\ \Delta_2\\ \vdots\\ \Delta_n\end{bmatrix} = \begin{bmatrix}\Delta_{1L}\\ \Delta_{2L}\\ \vdots\\ \Delta_{nL}\end{bmatrix} + \begin{bmatrix}\delta_{11} & \delta_{12} \ldots \delta_{1n}\\ \delta_{21} & \delta_{22} \ldots \delta_{2n}\\ \vdots & \\ \delta_{n1} & \delta_{n2} \ldots \delta_{nn}\end{bmatrix}^{-1} \begin{bmatrix}P_1\\ P_2\\ \vdots\\ P_n\end{bmatrix} \tag{a}$$

Solving Eq. (a) for the redundants,

$$\begin{bmatrix}P_1\\ P_2\\ \vdots\\ P_n\end{bmatrix} = \begin{bmatrix}\delta_{11} & \delta_{12} \ldots \delta_{1n}\\ \delta_{21} & \delta_{22} \ldots \delta_{2n}\\ \vdots & \\ \delta_{n1} & \delta_{n2} \ldots \delta_{nn}\end{bmatrix}^{-1} \left\{\begin{bmatrix}\Delta_1\\ \Delta_2\\ \vdots\\ \Delta_n\end{bmatrix} - \begin{bmatrix}\Delta_{1L}\\ \Delta_{2L}\\ \vdots\\ \Delta_{nL}\end{bmatrix}\right\} \tag{5.2a}$$

Equation (5.2a) may be written in the compact form

$$[P] = [\delta]^{-1}\{[\Delta] - [\Delta_L]\} \tag{5.2b}$$

In Equation (5.2) the elements of the matrix $[\Delta]$, viz., $\Delta_1, \Delta_2, \ldots, \Delta_n$ are the net displacements at the coordinates. To maintain the continuity of the structure, the net displacements at those coordinates which are assigned to the internal redundants are zero. The net displacements at the coordinates which are assigned to the external redundants depend upon the types of the supports.

5.2.1 Unyielding Supports

Let coordinates 1, 2, ..., j be assigned to the internal redundants. For the continuity of the structure,

$$\Delta_1 = \Delta_2 = \cdots = \Delta_j = 0 \tag{b}$$

Let the remaining coordinates $j+1, j+2, \ldots, n$ be assigned to the external redundants. If these redundant forces are produced by unyielding supports, then

$$\Delta_{j+1} = \Delta_{j+2} = \cdots = \Delta_n = 0 \tag{c}$$

From Eqs. (b) and (c) it is evident that matrix $[\Delta]$ becomes a null matrix. Hence in the case of unyielding supports, Eq. (5.2) may be written as

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = -\begin{bmatrix} \delta_{11} & \delta_{12} \ldots \delta_{1n} \\ \delta_{21} & \delta_{22} \ldots \delta_{2n} \\ \vdots & \\ \delta_{n1} & \delta_{n2} \ldots \delta_{nn} \end{bmatrix}^{-1} \begin{bmatrix} \Delta_{1L} \\ \Delta_{2L} \\ \vdots \\ \Delta_{nL} \end{bmatrix} \tag{5.3a}$$

$$[P] = -[\delta]^{-1}[\Delta_L] \tag{5.3b}$$

5.2.2 Yielding Supports

The displacements at coordiantes 1, 2, ..., j assigned to the internal redundants are zero for the continuity of the structure as expressed by Eq. (b). The displacements at coordinates $j+1$, $j+2$,..., n assigned to the external redundants are known quantities because the settlements at the supports are specified. Hence in the case of yielding supports, matrix $[\Delta]$ may be written as

$$[\Delta] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Delta_{j+1} \\ \Delta_{j+2} \\ \vdots \\ \Delta_n \end{bmatrix} \tag{d}$$

Equation (d) may be substituted into Eq. (5.2) for computing the redundants in the case of yielding supports with specified settlements.

Example 5.1

Analyse the continuous beam shown in Fig. 5.2(a).

Solution

The total number of reaction components is five. As the conditions of static equilibrium provide three independent equations, the degree of static indeterminacy of the structure is two. Herein three alternative solutions with three different released structures are given.

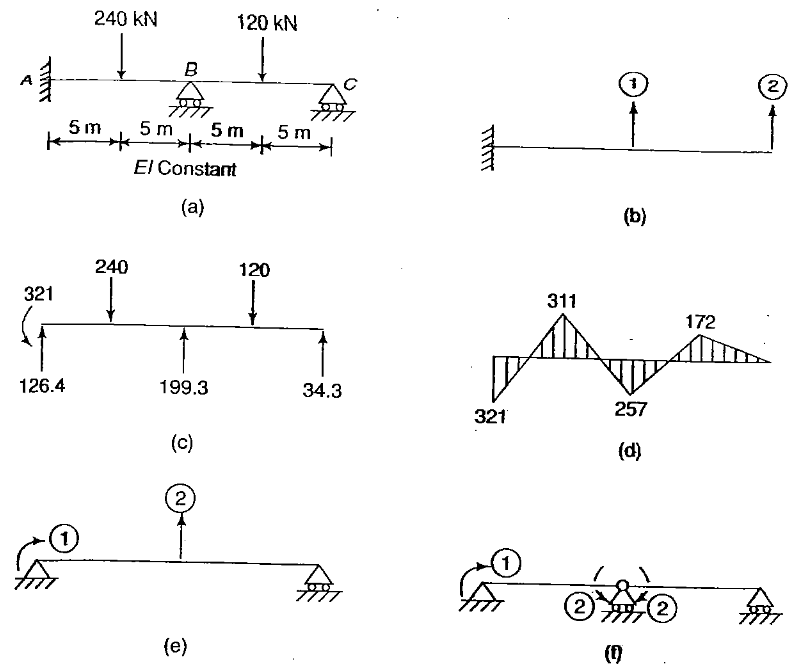


Fig. 5.2

(i) The released structure and the chosen coordinates are shown in Fig. 5.2(b). In this case supports B and C have been removed ~~so that the~~ released structure is a cantilever. The displacements at coordinates 1 and 2 due to the given external loads may be computed by any one of the methods discussed in Chapter 2. The displacement at coordinate 1,

$$\Delta_{1L} = -\frac{95000}{EI}$$

The minus sign shows that the deflection is downward, i.e., in the direction opposite to that of the coordinate 1. Similarly, the displacement at coordinate 2,

$$\Delta_{2L} = -\frac{257500}{EI}$$

The flexibility matrix with reference to coordinates 1 and 2 may be developed by applying a unit force successively at coordinates 1 and 2 and computing the displacements at these coordinates. For this purpose also, any one of the methods discussed in Chapter 2 may be used.

$$\delta_{11} = \frac{1000}{3EI}$$

$$\delta_{12} = \delta_{21} = \frac{2500}{3EI}$$

$$\delta_{22} = \frac{8000}{3EI}$$

Hence, the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \begin{bmatrix} \frac{1000}{3EI} & \frac{2500}{3EI} \\ \frac{2500}{3EI} & \frac{8000}{3EI} \end{bmatrix}$$

Substituting into Eq. (5.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} \frac{1000}{3EI} & \frac{2500}{3EI} \\ \frac{2500}{3EI} & \frac{8000}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{95000}{EI} \\ -\frac{257500}{EI} \end{bmatrix} = \begin{bmatrix} 199.3 \\ 34.3 \end{bmatrix}$$

Hence, $P_1 = 199.3$ kN and $P_2 = 34.3$ kN. Knowing the reaction components at B and C, the other reaction components may be calculated by using the equations of static equilibrium. The reaction components are shown in the free-body diagram of Fig. 5.2(c). Knowing the reaction components, the bending-moment diagram as shown in Fig. 5.2(d) can be drawn. It has been drawn on the compression side.

(ii) The released structure and the chosen coordinates are shown in Fig. 5.2(e). In this case the fixed-end moment at A and the vertical reaction at B have been removed so that the released structure is a simply supported beam. The displacements at coordinates 1 and 2 due to the applied loads are

$$\Delta_{1L} = \frac{14250}{2EI}$$

$$\Delta_{2L} = -\frac{41250}{EI}$$

The elements of the flexibility matrix with reference to the chosen coordinates may be computed by applying a unit force successively at coordinates 1 and 2.

$$\delta_{11} = \frac{20}{3EI}$$

$$\delta_{12} = \delta_{21} = -\frac{25}{EI}$$

$$\delta_{22} = \frac{500}{3EI}$$

Substituting into Eq. (5.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} \frac{20}{3EI} & -\frac{25}{EI} \\ -\frac{25}{EI} & \frac{500}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} \frac{14250}{2EI} \\ -\frac{41250}{EI} \end{bmatrix} = \begin{bmatrix} -321 \\ 199.3 \end{bmatrix}$$

Hence, $P_1 = -321$ kN·m and $P_2 = 199.3$ kN. The other reaction components may be calculated by using the equations of the static equilibrium. It may be checked that the reaction components are the same as those computed in solution (i).

(iii) The released structure and chosen coordinates are shown in Fig. 5.2(f). In this case the fixed-end moment at A and the bending moment at B have been released so that the released structure comprises simply supported beams AB and BC. The displacements at coordiantes 1 and 2 due to the applied loads and the elements of the flexibility matrix have already been computed in Ex. 2.12. These displacements are

$$\Delta_{1L} = \frac{1500}{EI}$$

$$\Delta_{2L} = \frac{2250}{EI}$$

$$\delta_{11} = \frac{10}{3EI}$$

$$\delta_{12} = \delta_{21} = \frac{5}{3EI}$$

$$\delta_{22} = \frac{20}{3EI}$$

Substituting into Eq. (5.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} \frac{10}{3EI} & \frac{5}{3EI} \\ \frac{5}{3EI} & \frac{20}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1500}{EI} \\ \frac{2250}{EI} \end{bmatrix} = \begin{bmatrix} -321 \\ -257 \end{bmatrix}$$

Hence, $P_1 = 321$ kN·m and $P_2 = -257$ kN·m. The other reaction components may be calculated by using the equations of the static equilibrium. It may be checked that the reaction components are the same as those computed in solution (i).

Example 5.2

Analyse the beam shown in Fig. 5.3(a) if the downward settlements of supports B and C in kN-m units are $\frac{2000}{EI}$ *and* $\frac{1000}{EI}$ *respectively.*

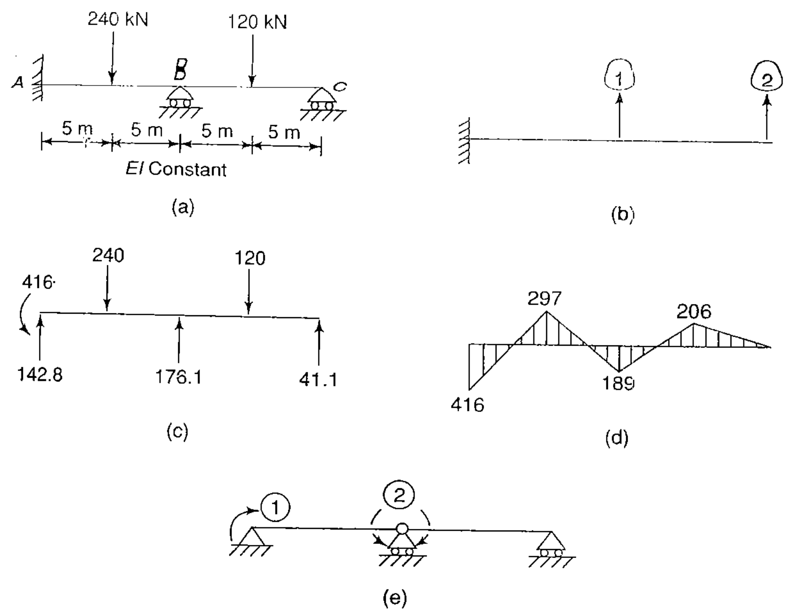


Fig. 5.3

Solution

Herein two alternative solutions of the problem are given. In the first solution, a coordinate has also been assigned to each of the specified settlement at the supports. However, this is not necessary as illustrated by the second solution.

(i) The released structure and chosen coordinates are shown in Fig. 5.3(b). It may be noted that this released structure is the same as the one in Ex. 5.1(i). Using the values computed in Ex. 5.1(i),

$$\Delta_{1L} = -\frac{95000}{EI}$$

$$\Delta_{2L} = -\frac{257500}{Ei}$$

$$\delta_{11} = \frac{1000}{3EI}$$

$$\delta_{12} = \delta_{21} = \frac{2500}{3EI}$$

$$\delta_{22} = \frac{8000}{3EI}$$

From the given data,

$$\Delta_1 = -\frac{2000}{EI}$$

$$\Delta_2 = -\frac{1000}{EI}$$

Substituting into Eq. (5.2),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1000}{3EI} & \dfrac{2500}{3EI} \\ \dfrac{2500}{3EI} & \dfrac{8000}{3EI} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\dfrac{2000}{EI} \\ -\dfrac{1000}{EI} \end{bmatrix} - \begin{bmatrix} -\dfrac{95000}{EI} \\ -\dfrac{257500}{EI} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 176.1 \\ 41.1 \end{bmatrix}$$

Hence, $P_1 = 176.1$ kN and $P_2 = 41.1$ kN. Knowing the reaction components at B and C, the other reaction components may be calculated by using the equations of static equilibrium. The free-body diagram and the bending-moment diagram can now be drawn as shown in Fig. 5.3(c) and (d) respectively. The bending-moment diagram has been drawn on the compression side.

(ii) The released structure and chosen coordinates are shown in Fig. 5.3 (e). The elements of flexibility matrix have already been computed in Ex. 2.12. These elements are

$$\delta_{11} = \frac{10}{3EI} \qquad \delta_{12} = \delta_{21} = \frac{5}{3EI} \qquad \delta_{22} = \frac{20}{3EI}$$

The displacements at the coordinates due to the applied loads and the settlements of supports are

$$\Delta_{1L} = \frac{240 \times 10^2}{16EI} + \frac{2000}{EI} \times \frac{1}{10} = \frac{1700}{EI}$$

$$\Delta_{2L} = \frac{240 \times 10^2}{16EI} + \frac{120 \times 10^2}{16EI} - \frac{2000}{EI} \times \frac{1}{10} - \left(\frac{2000}{EI} - \frac{1000}{EI}\right) \times \frac{1}{10}$$

$$= \frac{1950}{EI}$$

As the net displacements at coordinates 1 and 2 are zero, matrix $[\Delta]$ is a null matrix. Hence substituting into Eq. (5.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} \dfrac{10}{3EI} & \dfrac{5}{3EI} \\ \dfrac{5}{3EI} & \dfrac{20}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{1700}{EI} \\ \dfrac{1950}{EI} \end{bmatrix} = \begin{bmatrix} -415.7 \\ -188.6 \end{bmatrix}$$

The other reaction components may be calculated by using the equations of static equilibrium. It may be checked that the reaction components are the same as in solution (i).

Example 5.3

Analyse the continuous beam shown in Fig. 5.4(a).

Solution

The beam is statically indeterminate to the second degree. The released structure may be obtained by inserting hinges at B and C as shown in Fig. 5.4(b), so that the released structure comprises a series of three simply supported beams. The chosen coordinates 1 and 2 correspond to the released bending moments at B and C respectively. The displacements in the released structure at coordinates 1 and 2 due to the applied loads may be computed by using Table 2.16.

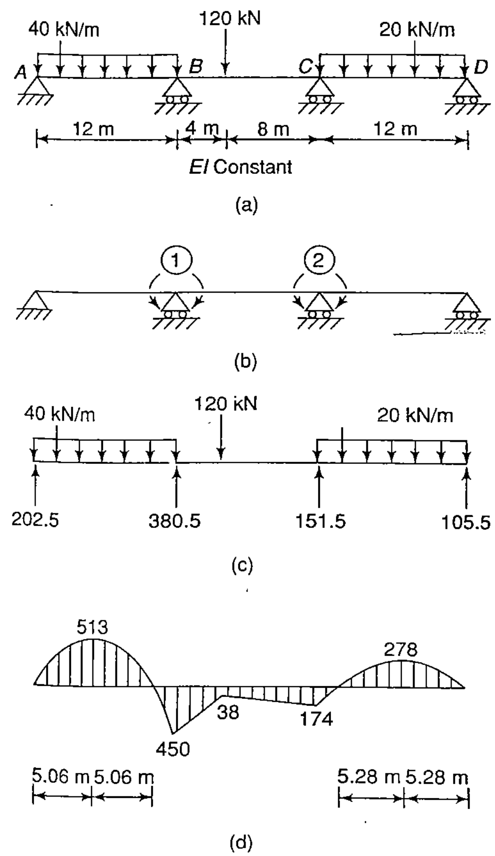


Fig. 5.4

The rotation at B in span $AB = \dfrac{40 \times 12^3}{24EI} = \dfrac{2880}{EI}$ (counter-clockwise).

The rotation at B in span $BC = \dfrac{120 \times 4 \times 8 \times 20}{6 \times 12EI}$

$$= \frac{3200}{3EI} \text{ (clockwise)}$$

Hence, the displacement at coordinate 1 due to the applied loads,

$$\Delta_{1L} = \frac{2880}{EI} + \frac{3200}{3EI} = \frac{11840}{3EI}$$

The rotation at C in span $BC = \dfrac{120 \times 4}{6 \times 12EI}\left(12^2 - 4^2\right) = \dfrac{2560}{3EI}$

(counter-clockwise)

The rotation at C in span $CD = \dfrac{20 \times 12^3}{24EI} = \dfrac{1440}{EI}$ (clockwise)

Hence, the displacement at coordinate 2 due to the applied loads,

$$\Delta_{2L} = \frac{2560}{3EI} + \frac{1440}{EI} = \frac{6880}{3EI}$$

The flexibility matrix may be developed by applying a unit force successively at coordinates 1 and 2 and using Table 2.16.

$$\delta_{11} = \frac{12}{3EI} + \frac{12}{3EI} = \frac{8}{EI}$$

$$\delta_{12} = \delta_{21} = \frac{12}{6EI} = \frac{2}{EI}$$

$$\delta_{22} = \frac{12}{3EI} + \frac{12}{3EI} = \frac{8}{EI}$$

Substituting into Eq. (5.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} \dfrac{8}{EI} & \dfrac{2}{EI} \\ \dfrac{2}{EI} & \dfrac{8}{EI} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{11840}{3EI} \\ \dfrac{6880}{3EI} \end{bmatrix} = \begin{bmatrix} -450 \\ -174 \end{bmatrix}$$

Hence, $P_1 = -450$ kN·m and $P_2 = -174$ kN·m. All the reaction components may now be computed by using the equations of static equilibrium. Hence, the free-body diagram shown in Fig. 5.4(c) can be drawn. The bending-moment diagram for the continuous beam drawn on the compression side is shown in Fig. 5.4(d).

Example 5.4

Analyse the continuous beam shown in Fig. 5.5(a). The downward settlement of supports B and C in kN-m units are $\frac{1500}{EI}$ and $\frac{750}{EI}$ respectively.

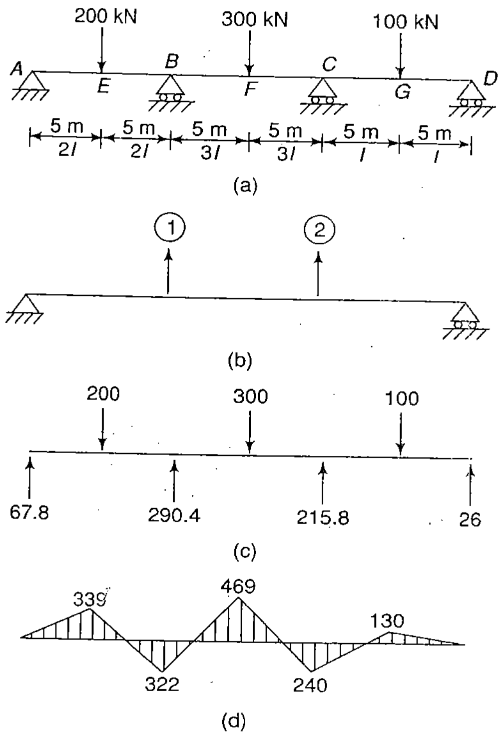


Fig. 5.5

Solution

The basic determinate structure may be obtained by treating the reactions at B and C as redundants and releasing them. Consequently, coordinates 1 and 2 may be assigned to the redundants as shown in Fig. 5.5(b). The displacements at coordinates 1 and 2 due to the applied loads and elements of the flexibility matrix may be computed by using any one of the methods discussed in Chapter 2. They are found to be

$$\Delta_{1L} = -\frac{100160}{EI}$$

$$\Delta_{2L} = -\frac{110940}{EI}$$

$$\delta_{11} = \frac{197.53}{EI}$$

$$\delta_{12} = \delta_{21} = \frac{191.36}{EI}$$

$$\delta_{22} = \frac{253.09}{EI}$$

From the given data

$$\Delta_1 = -\frac{1500}{EI}$$

$$\Delta_2 = -\frac{750}{EI}$$

Substituting into Eq. (5.2),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \frac{197.53}{EI} & \frac{191.36}{EI} \\ \frac{191.36}{EI} & \frac{253.09}{EI} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\frac{1500}{EI} \\ -\frac{750}{EI} \end{bmatrix} - \begin{bmatrix} -\frac{100160}{EI} \\ -\frac{110940}{EI} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 290.4 \\ 215.8 \end{bmatrix}$$

Hence, $P_1 = 290.4$ kN and $P_2 = 215.8$ kN. Knowing the reaction components at B and C, the other reaction components may be calculated from statics. The free-body diagram and bending-moment diagram drawn on the compression side are shown in Fig. 5.5(c) and (d) respectively.

Example 5.5

Analyse of the continuous beam ABCD shown in Fig. 5.6(a). The beam has an internal hinge at B.

Solution

The degree of static indeterminacy of the beam is two. The released structure shown in Fig. 5.6(b) has been obtained by introducing a cut at E, the centre of span CD, thereby releasing the shear force and bending moment at E. Coordinates 1 and 2 have been assigned to the redundant shear force and bending moment at E as shown in Fig. 5.6(b). The displacements in the released structure at coordinates 1 and 2 due to the applied loads and the elements of the flexibility matrix can be obtained by applying the unit-load method. The necessary details for the computation are shown in Table 5.1. Sagging bending moment has been taken positive.

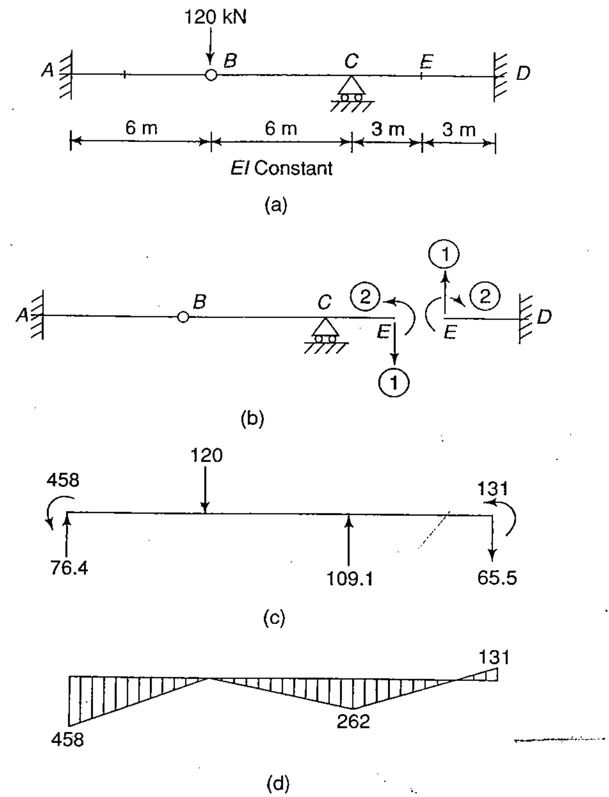


Fig. 5.6

Table 5.1

Portion	*AB*	*BC*	*CE*	*ED*
I	I	I	I	I
Origin	B	C	E	E
Limits	0 to 6	0 to 6	0 to 3	0 to 3
M	$-120x$	0	0	0
m_1	$-(x+9)$ $+1.5(x+6)$	$1.5x-(x+3)$	$-x$	x
m_2	$1-\frac{1}{6}(x+6)$	$1-\frac{x}{6}$	1	1

Using Table 5.1,

$$\Delta_{1L} = \int \frac{Mm_1 dx}{EI} = \int_0^6 \frac{(-120x)(0.5x)dx}{EI} = -\frac{4320}{EI}$$

$$\Delta_{2L} = \int \frac{Mm_2 dx}{EI} = \int_0^6 \frac{(-120x)\left(-\frac{x}{6}\right)dx}{EI} = \frac{1440}{EI}$$

$$\delta_{11} = \int \frac{m_1^2 dx}{EI} = \int_0^6 \frac{(0.5x)^2 dx}{EI} + \int_0^6 \frac{(0.5x-3)^2 dx}{EI}$$

$$+ \int_0^3 \frac{(-x)^2 dx}{EI} + \int_0^3 \frac{x^2 dx}{EI} = \frac{54}{EI}$$

$$\delta_{22} = \int \frac{m_2^2 dx}{EI} = \int_0^6 \frac{\left(-\frac{x}{6}\right)^2 dx}{EI} + \int_0^6 \frac{\left(1-\frac{x}{6}\right)^2 dx}{EI} + \int_0^3 \frac{dx}{EI} + \int_0^3 \frac{dx}{EI} = \frac{10}{EI}$$

$$\delta_{12} = \delta_{21} = \int \frac{m_1 m_2 dx}{EI} = \int_0^6 \frac{0.5x\left(-\frac{x}{6}\right)dx}{EI} + \int_0^6 \frac{(0.5x-3)\left(1-\frac{x}{6}\right)dx}{EI}$$

$$+ \int_0^3 \frac{-x\,dx}{EI} + \int_0^3 \frac{x\,dx}{EI} = -\frac{12}{EI}$$

As the supports are unyielding, substituting into Eq. (5.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} \frac{54}{EI} & -\frac{12}{EI} \\ -\frac{12}{EI} & \frac{10}{EI} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{4320}{EI} \\ \frac{1440}{EI} \end{bmatrix} = \begin{bmatrix} 65.454 \\ -65.454 \end{bmatrix}$$

Knowing the redundants P_1 and P_2, the other reaction components can be calculated from statics. The free-body diagram and bending-moment diagram drawn on the compression side are shown in Fig. 5.6(c) and (d) respectively.

5.3 DISPLACEMENT METHOD

The displacement method of analysis has been briefly discussed in Sec. 4.5. In the case of continuous beams this method may be described by the following steps:

(i) Determine the degree of freedom of the beam. The degree of freedom has been discussed in Sec. 1.7. For the stability of beams, at least one of the supports must be either fixed or hinged so as to prevent rigid body movement of the beam along its longitudinal axis. The axial displacements of the beam are very small in comparison to its transverse displacements (deflections). Hence, it is assumed that the beam is inextensible. Thus there can be only two displacements at any point of a beam, viz., a rotation and a deflection. At an unyielding hinge or a roller support, the degree of freedom is one and at a fixed support the degree of freedom is zero. Thus the degree of freedom of a continuous beam is equal to the number of hinge or roller supports.

(ii) If n is the degree of freedom of the beam, n independent displacement components exist. Assign one coordinate to each of the independent displacement components as shown in Fig. 5.7. As no rotation is possible at a fixed support, no coordinate need be assigned there. It may be noted that fixed supports cannot be provided at intermediate points because that would destroy the continuity of the beam.

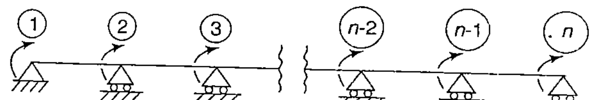


Fig. 5.7

(iii) Lock all the joints as shown in Fig. 5.8 so that each span behaves like a fixed ended beam. Calculate the fixed-end moments due to the applied loads for each span using the standard formulae given in Appendix A. The fixed-end moments have been denoted by single primes in Fig. 5.8. Thus M'_{AB}, M'_{BA}, M'_{BC} etc. are the fixed-end moments due to the applied loads.

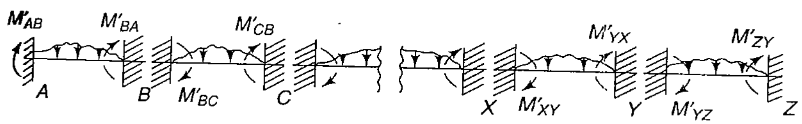


Fig. 5.8

(iv) Calculate the additional fixed-end moments produced on account of yielding of supports, if any. These additional bending moments denoted by double primes in Fig. 5.9 may also be calculated by using the standard formulae given in Appendix A. Thus M''_{AB}, M''_{BA}, M''_{BC} etc. are the fixed-end moments due to the settlement of supports.

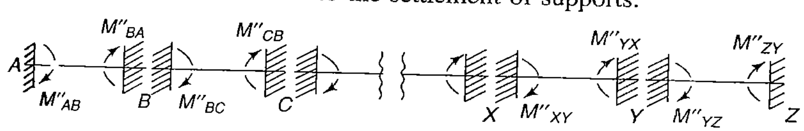


Fig. 5.9

(v) Compute the forces (moments) $P'_1, P'_2, \ldots, P'_j, \ldots P'_n$ at coordinates 1, 2, ..., j, ..., n. by adding the moments computed in steps (iii) and (iv). Referring to Fig. 5.10, force P'_j at coordinate j is given by the equation

$$P'_j = M'_{QP} + M''_{QP} + M'_{QR} + M''_{QR}$$

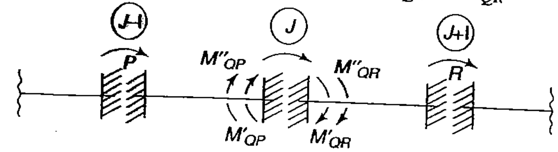


Fig. 5.10

(vi) Develop the stiffness matrix with reference to the chosen coordinates. To generate the jth column of the stiffness matrix, give a unit displacement at coordinate j only and determine the forces $k_{1j}, k_{2j}, \ldots, k_{jj}, \ldots, k_{nj}$ at coordinates 1, 2, ..., j, ..., n. Referring to Fig. 5.11 and using Table 2.16, the forces at the coordiantes are

$$k_{j-1,\,j} = \frac{2EI_{PQ}}{L_{PQ}}$$

$$k_{jj} = \frac{4EI_{PQ}}{L_{PQ}} + \frac{4EI_{QR}}{L_{QR}} \qquad \text{(a)}$$

$$k_{j+1,\,j} = \frac{2EI_{QR}}{L_{QR}}$$

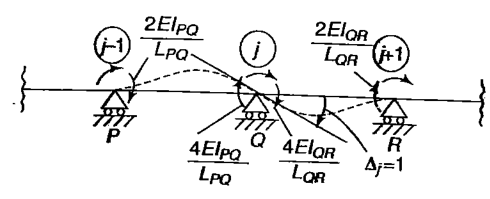


Fig. 5.11

The forces at all other coordinates are zero. Thus,

$$k_{1j} = k_{2j} = \ldots = k_{j-2,\,j} = k_{j+2,\,j} = \cdots = k_{nj} = 0$$

(vii) Now unlock the joints permitting displacements (rotations) $\Delta_1, \Delta_2, \ldots, \Delta_j, \ldots, \Delta_n$ at coordinates 1, 2, ..., j, ..., n. Using the equations derived in Sec. 4.3, forces $P_{1\Delta}, P_{2\Delta}, \ldots, P_{j\Delta}, \ldots, P_{n\Delta}$ at coordinates 1, 2, ..., j, ..., n

due to displacements $\Delta_1, \Delta_2, ..., \Delta_j, ..., \Delta_n$ are given by the matrix equation

$$\begin{bmatrix} P_{1\Delta} \\ P_{2\Delta} \\ \vdots \\ P_{j\Delta} \\ \vdots \\ P_{n\Delta} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \;..\; k_{1j} \;...\; k_{1n} \\ k_{21} & k_{22} \;...\; k_{2j} \;...\; k_{2n} \\ \vdots & \\ k_{j1} & k_{j2} \;...\; k_{jj} \;...\; k_{jn} \\ \vdots & \\ k_{n1} & k_{n2} \;...\; k_{nj} \;...\; k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \vdots \\ \Delta_n \end{bmatrix} \tag{b}$$

(viii) The net forces $P_1, P_2, ..., P_j, ..., P_n$ at coordinates 1, 2, ..., j, ..., n are obtained by adding the forces computed in steps (v) and (vii). For the equilibrium of joints, forces $P_1, P_2, ..., P_j, ..., P_n$ must be equal to the external forces acting at coordinates 1, 2, ..., j, ..., n. Usually these forces are zero because couples are rarely applied at the supports. The net force P_j at coordinate j is given by the equation

$$P_j = P'_j + P_{j\Delta}$$

Similar equations can be written at all the other coordinates. Thus the resulting set of equations can be written in the matrix form

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} P'_1 \\ P'_2 \\ \vdots \\ P'_j \\ \vdots \\ P'_n \end{bmatrix} + \begin{bmatrix} P_{1\Delta} \\ P_{2\Delta} \\ \vdots \\ P_{j\Delta} \\ \vdots \\ P_{n\Delta} \end{bmatrix} \tag{c}$$

Substituting from Eq. (b) into Eq. (c),

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} P'_1 \\ P'_2 \\ \vdots \\ P'_j \\ \vdots \\ P'_n \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \;...\; k_{1j} \;...\; k_{1n} \\ k_{21} & k_{22} \;...\; k_{2j} \;...\; k_{2n} \\ \vdots & \\ k_{j1} & k_{j2} \;...\; k_{jj} \;...\; k_{jn} \\ \vdots & \\ k_{n1} & k_{n2} \;...\; k_{nj} \;...\; k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \vdots \\ \Delta_n \end{bmatrix} \tag{d}$$

Solving Eq. (d) for the independent displacement components,

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \;...\; k_{1j} \;...\; k_{1n} \\ k_{21} & k_{22} \;...\; k_{2j} \;...\; k_{2n} \\ \vdots & \\ k_{j1} & k_{j2} \;...\; k_{jj} \;...\; k_{jn} \\ \vdots & \\ k_{n1} & k_{n2} \;...\; k_{nj} \;...\; k_{nn} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \\ \vdots \\ P_n \end{bmatrix} - \begin{bmatrix} P'_1 \\ P'_2 \\ \vdots \\ P'_j \\ \vdots \\ P'_n \end{bmatrix} \right\} \tag{5.4a}$$

Equation (5.4a) may be written in the compact form

$$[\Delta] = [k]^{-1}\{[P] - [P']\} \tag{5.4b}$$

It may be noted that Eq. (5.4) is similar to Eq. (5.2) in the force method except that the forces and displacements have been interchanged. If external forces do not act at the coordinates, i.e., external couples are not applied at the supports, forces $P_1, P_2, ..., P_j, ..., P_n$ vanish. Hence Eq. (5.4) takes the form

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \vdots \\ \Delta_n \end{bmatrix} = -\begin{bmatrix} k_{11} & k_{12} \;...\; k_{1j} \;...\; k_{1n} \\ k_{21} & k_{22} \;...\; k_{2j} \;...\; k_{2n} \\ \vdots & \\ k_{j1} & k_{j2} \;...\; k_{jj} \;...\; k_{jn} \\ \vdots & \\ k_{n1} & k_{n2} \;...\; k_{nj} \;...\; k_{nn} \end{bmatrix}^{-1} \begin{bmatrix} P'_1 \\ P'_2 \\ \vdots \\ P'_j \\ \vdots \\ P'_n \end{bmatrix} \tag{5.5a}$$

Equation (5.5a) may be written in the compact form

$$[\Delta] = -[k]^{-1}[P'] \tag{5.5b}$$

(ix) Knowing the independent displacement comoponents, the end moments may be computed by using the slope-deflection Eq.(2.47).

If a continuous beam has internal hinges, the degree of freedom increases depending upon the number and positions of internal hinges. Consider a span *AB* of continuous beam with an internal hinge at some intermediate point *C* as shown in Fig. 5.12(a). The deflection curve shown by the broken line in Fig. 5.12(b) is discontinuous at internal hinge *C*. The rotations of the two tangents to the deflection curve drawn at *C*, viz., θ_1 and θ_2 are the independent displacement components at the hinge in addition to the defleciton Δ_C. It follows that the degree of freedom of an internal hinge is three. Figure 5.12(c) shows coordinates 1, 2 and 3 which are assigned to the three independent displacement components at an internal hinge. With these coordinates, portions *AC* and *CB* may be considered as separate spans and the displacement method applied in the usual manner. The procedure is illustrated by Ex. 5.11.

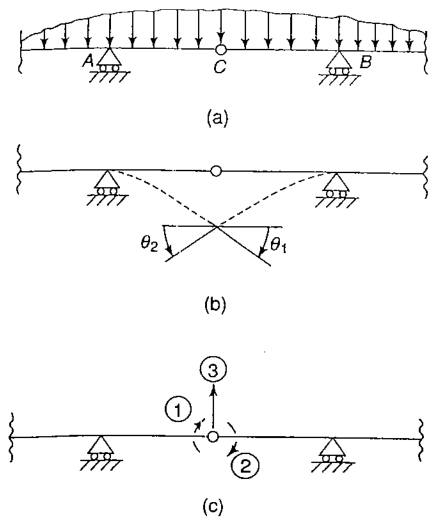


Fig. 5.12

If the internal hinge is located at an intermediate support C as shown in Fig. 5.13(a), there are only two independent displacement components because the deflection is zero. Figure 5.13(b) shows the deflection curve and the rotations θ_1 and θ_2 which constitute the two independent displacement components. Coordinates 1 and 2 may be assigned to these independent displacement components as shown in Fig. 5.13 (c).

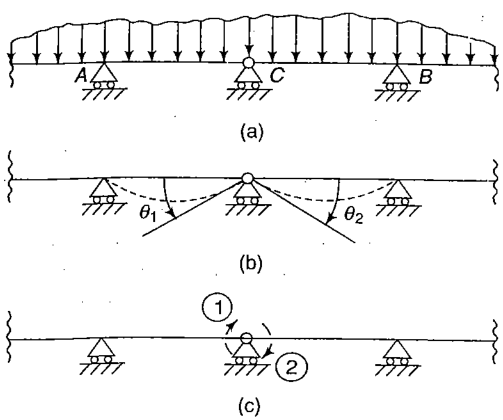


Fig. 5.13

To illustrate the manner in which coordinates should be selected for a continuous beam with internal hinges, consider the beam shown in Fig. 5.14. The beam has internal hinges at B and D. The degrees of freedom of the internal hinges at B and D are three and two respectively. In addition, the degree of freedom at each of the simple supports C and E is one. Consequently, the degree of freedom of the beam is seven. Coordinates 1 to 7 may be assigned to the seven independent displacement components as shown in the figure.

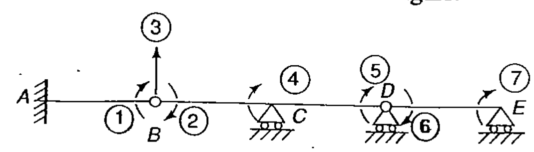


Fig. 5.14

Example 5.6

Analyse the continuous beam shown in Fig. 5.15(a).

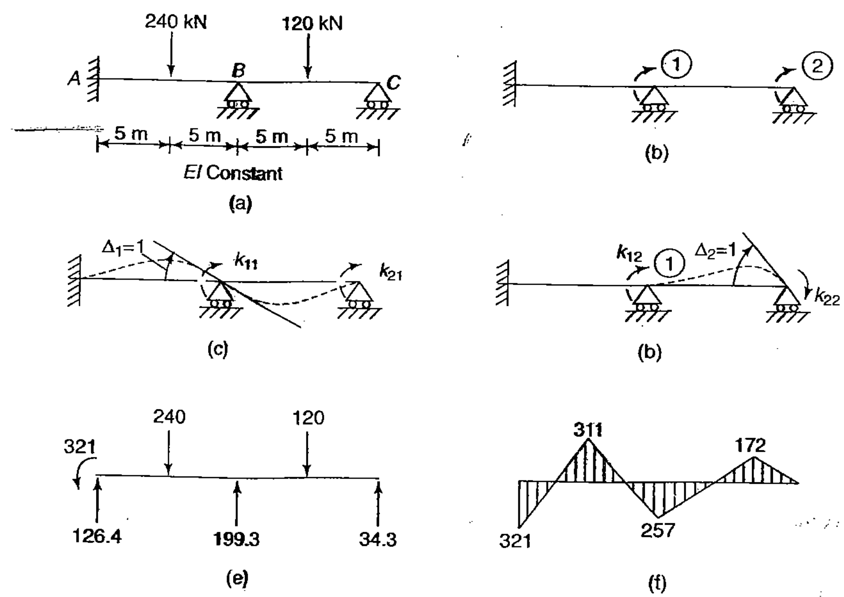


Fig. 5.15

Solution

In this problem the only two independent displacement components are the rotations at B and C. Hence the degree of freedom is two. Coordinates 1 and 2 may be assigned to the rotations at B and C as shown in Fig. 5.15(b). Locking joints B and C, the fixed-end moments due to the applied loads are

$$M'_{AB} = -\frac{240 \times 5 \times 5^2}{10^2} = -300 \text{ kN·m}$$

$$M'_{BA} = \frac{240 \times 5 \times 5^2}{10^2} = 300 \text{ kN·m}$$

$$M'_{BC} = -\frac{120 \times 5 \times 5^2}{10^2} = -150 \text{ kN·m}$$

$$M'_{CB} = \frac{120 \times 5 \times 5^2}{10^2} = 150 \text{ kN·m}$$

As the supports are unyielding, there are no additional fixed-end moments due to the settlement of supports. Hence,

$$M''_{AB} = M''_{BA} = M''_{BC} = M''_{CB} = 0$$

Therefore forces P'_1 and P'_2, at coordinates 1 and 2 for the fixed-end conditions are

$$P'_1 = 300 - 150 = 150 \text{ kN·m}$$
$$P'_2 = 150 \text{ kN·m}$$

Next, the stiffness matrix with reference to coordinates 1 and 2 may be developed. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 5.15(c).

$$k_{11} = \frac{4EI}{10} + \frac{4EI}{10} = 0.8\ EI$$

$$k_{21} = \frac{2EI}{10} = 0.2\ EI$$

Similarly, to generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown in Fig. 5.15(d).

$$k_{12} = \frac{2EI}{10} = 0.2EI$$

$$k_{22} = \frac{4EI}{10} = 0.4EI$$

As there are no external loads at coordinates 1 and 2,

$$P_1 = P_2 = 0$$

Substituting into Eq. (5.5),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = -\begin{bmatrix} 0.8EI & 0.2EI \\ 0.2EI & 0.4EI \end{bmatrix}^{-1} \begin{bmatrix} 150 \\ 150 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -107.14 \\ -321.43 \end{bmatrix}$$

Knowing the displacements, the end moments may be calculated by using the slope-deflection Eq. (2.47).

$$M_{AB} = 300 + \frac{2EI}{10}\left[-\frac{107.14}{EI}\right] = -321 \text{ kN·m}$$

$$M_{BA} = 300 + \frac{2EI}{10}\left[2\left(-\frac{107.14}{EI}\right)\right] = 257 \text{ kN·m}$$

$$M_{BC} = -150 + \frac{2EI}{10}\left[2\left(-\frac{107.14}{EI}\right) + \left(-\frac{321.43}{EI}\right)\right] = -257 \text{ kN·m}$$

$$M_{CB} = 150 + \frac{2EI}{10}\left[2\left(-\frac{321.43}{EI}\right) + \left(-\frac{107.14}{EI}\right)\right] = 0$$

The free-body diagram and the bending-moment diagram drawn on the compression side are shown in Fig. 5.15(e) and (f) respectively.

Example 5.7

Analyse the continuous beam shown in Fig. 5.16(a) if the downward settlement of supports B and C in kN-m units are 2000/EI and 1000/EI respectively.

Solution

The beam in this example is the same as in Ex. 5.6 except that supports B and C undergo pre-specified settlements. Hence, coordinates 1 and 2 as shown in Fig. 5.16(b) may be chosen. The fixed-end moments due to the applied loads as computed in Ex. 5.6 are

$$M'_{AB} = -300 \text{ kN·m}$$
$$M'_{BA} = 300 \text{ kN·m}$$
$$M'_{BC} = -150 \text{ kN·m}$$
$$M'_{CB} = 150 \text{ kN·m}$$

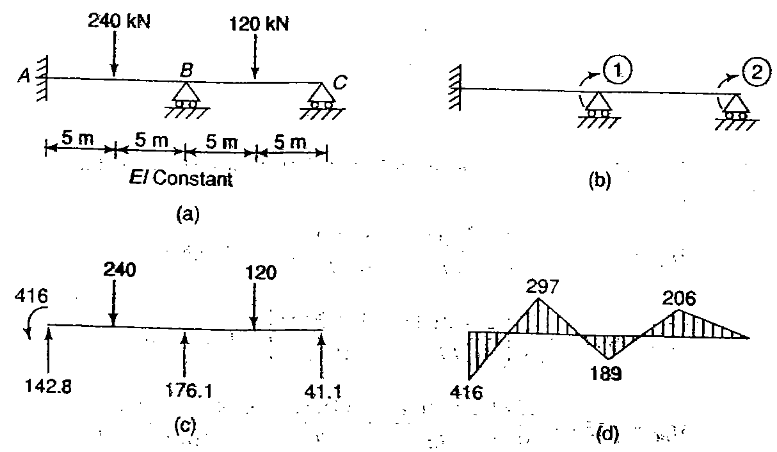


Fig. 5.16

The additional fixed-end moments due to the settlement of supports are

$$M''_{AB} = -\frac{6EI}{10^2}\left(\frac{2000}{EI}\right) = 120 \text{ kN·m}$$

$$M''_{BA} = -\frac{6EI}{10^2}\left(\frac{2000}{EI}\right) = 120 \text{ kN·m}$$

$$M''_{BC} = \frac{6EI}{10^2}\left(\frac{2000}{EI} - \frac{1000}{EI}\right) = 60 \text{ kN·m}$$

$$M''_{CB} = \frac{6EI}{10^2}\left(\frac{2000}{EI} - \frac{1000}{EI}\right) = 60 \text{ kN·m}$$

Therefore forces P'_1 and P'_2 at coordinates 1 and 2 for the fixed-end conditions are

$$P'_1 = 300 + (-150) + (-120) + 60 = 90 \text{ kN·m}$$

$$P'_2 = 150 + 60 = 210 \text{ kN·m}$$

The elements of the stiffness matrix with reference to coordinates 1 and 2 have been computed in Ex. 5.6. The stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 0.8EI & 0.2EI \\ 0.2EI & 0.4EI \end{bmatrix}$$

As there are no external loads at coordinates 1 and 2,

$$P_1 = P_2 = 0$$

Substituting into Eq. (5.5),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = -\begin{bmatrix} 0.8EI & 0.2EI \\ 0.2EI & 0.4EI \end{bmatrix}^{-1} \begin{bmatrix} 90 \\ 210 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{21.43}{EI} \\ -\dfrac{535.71}{EI} \end{bmatrix}$$

Knowing the displacements, the end moments may be calculated by using the slope-deflection Eq. (2.47).

$$M_{AB} = -300 + \frac{2EI}{10}\left[\frac{21.43}{EI} - \frac{3 \times 2000}{10EI}\right] = -416 \text{ kN·m}$$

$$M_{BA} = 300 + \frac{2EI}{10}\left[\frac{2 \times 21.43}{EI} - \frac{3 \times 2000}{10EI}\right] = 189 \text{ kN·m}$$

$$M_{BC} = -150 + \frac{2EI}{10}\left[\frac{2 \times 21.43}{EI} - \frac{535.71}{EI} - \frac{3(1000 - 2000)}{10EI}\right]$$

$$= -189 \text{ kN·m}$$

$$M_{CB} = 150 + \frac{2EI}{10}\left[2\left(-\frac{535.71}{EI}\right) + \frac{21.43}{EI} - \frac{3(1000 - 2000)}{10EI}\right] = 0$$

The free-body diagram and the bending-moment diagram drawn on the compression side are shown in Fig. 5.16(c) and (d) respectively.

Example 5.8

Analyse the continuous beam shown in Fig. 5.17(a).

Solution

In this problem the independent displacement components are the rotations at B and C. Hence, the degree of freedom is two. Coordinates 1 and 2 may be assigned to the rotations at B and C as shown in Fig. 5.17(b). Locking joints B and C, the fixed-end moments due to the applied loads are

$$M'_{AB} = -\frac{100 \times 2 \times 1^2}{3^2} = 22.2 \text{ kN·m}$$

$$M'_{BA} = \frac{100 \times 1 \times 2^2}{3^2} = 44.4 \text{ kN·m}$$

$$M'_{BC} = -\frac{200 \times 2 \times 2^2}{4^2} = -100 \text{ kN·m}$$

$$M'_{CB} = \frac{200 \times 2 \times 2^2}{4^2} = 100 \text{ kN·m}$$

$$M'_{CD} = -\frac{150 \times 1 \times 2^2}{3^2} = -66.7 \text{ kN·m}$$

$$M'_{DC} = \frac{150 \times 2 \times 1^2}{3^2} = 33.3 \text{ kN·m}$$

As the supports are unyielding, there are no additional fixed-end moments due to the settlement of supports. Hence,

$$M''_{AB} = M''_{BA} = M''_{BC} = M''_{CB} = M''_{CD} = M''_{DC} = 0$$

Therefore forces P'_1 and P'_2 at coordinates 1 and 2 for the fixed-end condtions are

$$P'_1 = 44.4 - 100 = 55.6 \text{ kN·m}$$

$$P'_2 = 100 - 66.7 = 33.3 \text{ kN·m}$$

Next, the stiffness matrix with reference to coordinates 1 and 2 may be developed. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 5.17(c).

$$k_{11} = \frac{4EI}{3} + \frac{4EI}{4} = 2.33EI$$

$$k_{21} = \frac{2EI}{4} = 0.5EI$$

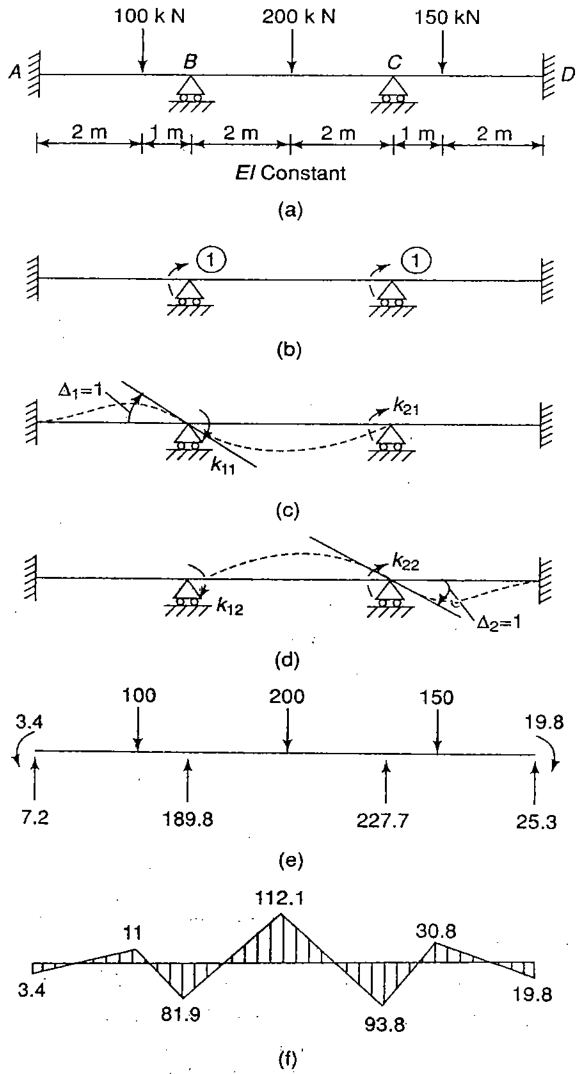


Fig. 5.17

Similarly, to generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown in Fig. 5.17(d).

$$k_{12} = \frac{2EI}{4} = 0.5EI$$

$$k_{22} = \frac{4EI}{3} + \frac{4EI}{4} = 2.33EI$$

As there are no external loads at coordinates 1 and 2,

$$P_1 = P_2 = 0$$

Similarly into Eq. (5.5),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = -\begin{bmatrix} 2.33EI & 0.50EI \\ 0.50EI & 2.33EI \end{bmatrix}^{-1} \begin{bmatrix} -55.6 \\ 33.3 \end{bmatrix} = \begin{bmatrix} \dfrac{28.16}{EI} \\ -\dfrac{20.32}{EI} \end{bmatrix}$$

Knowing the displacements, the end moments may be calculated by using the slope deflection Eq. (2.47).

$$M_{AB} = -22.2 + \frac{2EI}{3}\left[\frac{28.16}{EI}\right] = -3.4 \text{ kN·m}$$

$$M_{BA} = 44.4 + \frac{2EI}{3}\left[\frac{2 \times 28.16}{EI}\right] = 81.9 \text{ kN·m}$$

$$M_{BC} = -100 + \frac{2EI}{4}\left[\frac{2 \times 28.16}{EI} - \frac{20.32}{EI}\right] = -81.9 \text{ kN·m}$$

$$M_{CB} = 100 + \frac{2EI}{4}\left[2\left(-\frac{20.32}{EI}\right) + \frac{28.16}{EI}\right] = 93.8 \text{ kN·m}$$

$$M_{CD} = -66.7 + \frac{2EI}{3}\left[2\left(-\frac{20.32}{EI}\right)\right] = -93.8 \text{ kN·m}$$

$$M_{DC} = 33.3 + \frac{2EI}{3}\left[-\frac{20.32}{EI}\right] = 19.8 \text{ kN·m}$$

The free-body diagram and bending-moment diagram drawn on the compression side are shown in Fig. 5.17(e) and (f) respectively.

Example 5.9

Analyse the continuous beam shown in Fig. 5.18(a).

Solution

The given structure is equivalent to the one shown in Fig. 5.18(b). The independent displacement components are the rotations at *B*, *C* and *D*. Hence, the degree of freedom is three. Coordinates 1, 2 and 3 may be assigned to the rotations at supports *B*, *C* and *D* as shown in Fig. 5.18(c). Locking joints *B*, *C* and *D*, the fixed-end moments due to the applied loads are

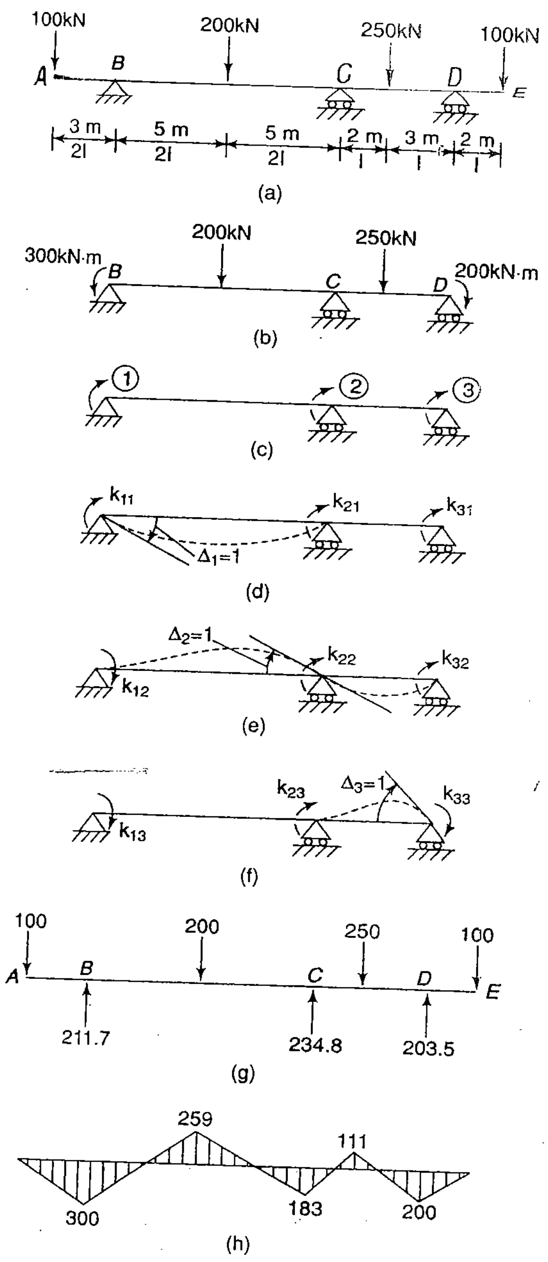


Fig. 5.18

$$M'_{BC} = -\frac{200 \times 5 \times 5^2}{10^2} = -250 \text{ kN·m}$$

$$M'_{CB} = \frac{200 \times 5 \times 5^2}{10^2} = 250 \text{ kN·m}$$

$$M'_{CD} = -\frac{250 \times 2 \times 3^2}{5^2} = -180 \text{ kN·m}$$

$$M'_{DC} = \frac{250 \times 3 \times 2^2}{5^2} = 120 \text{ kN·m}$$

As the supports are unyielding, there are no additional fixed-end moments due to the settlement of supports. Hence,

$$M''_{BC} = M''_{CB} = M''_{CD} = M''_{DC} = 0$$

Therefore, forces P'_1, P'_2, and P'_3 at coordinates **1, 2** and 3 for the fixed-end condition are

$$P'_1 = -250 \text{ kN·m}$$
$$P'_2 = 250 - 180 = 70 \text{ kN·m}$$
$$P'_3 = 120 \text{ kN·m}$$

Next, the stiffness matrix with reference to coordinates 1, 2 and 3 may be developed. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 as shown in Fig. 5.18(d).

$$k_{11} = \frac{4E(2I)}{10} = 0.8EI$$

$$k_{21} = \frac{2E(2I)}{10} = 0.4EI$$

$$k_{31} = 0$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 as shown in Fig. 5.18(e).

$$k_{12} = \frac{2E(2I)}{10} = 0.4EI$$

$$k_{22} = \frac{4E(2I)}{10} + \frac{4EI}{5} = 1.6EI$$

$$k_{32} = \frac{2EI}{5} = 0.4EI$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 as shown in Fig. 5.18(f).

$$k_{13} = 0$$

$$k_{23} = \frac{2EI}{5} = 0.4EI$$

$$k_{33} = \frac{4EI}{5} = 0.8EI$$

From Fig. 5.18(b),

$$P_1 = -300 \text{ kN·m} \qquad P_2 = 0 \qquad P_3 = 200 \text{ kN·m}$$

Substituting into Eq. (5.4),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 0.8EI & 0.4EI & 0 \\ 0.4EI & 1.6EI & 0.4EI \\ 0 & 0.4EI & 0.8EI \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -300 \\ 0 \\ 200 \end{bmatrix} - \begin{bmatrix} -250 \\ 70 \\ 120 \end{bmatrix} \right\}$$

$$= \frac{1}{EI} \begin{bmatrix} -27.08 \\ -70.83 \\ 135.42 \end{bmatrix}$$

Knowing the displacements, the end moments may be calculated by using the slope-deflection Eq. (2.47).

$$M_{BA} = 300 \text{ kN·m}$$

$$M_{BC} = -250 + \frac{2E(2I)}{10}\left[2\left(-\frac{27.08}{EI}\right) - \frac{70.83}{EI}\right] = -300 \text{ kN·m}$$

$$M_{CB} = 250 + \frac{2E(2I)}{10}\left[2\left(-\frac{70.83}{EI}\right) - \frac{27.08}{EI}\right] = 182.5 \text{ kN·m}$$

$$M_{CD} = -180 + \frac{2EI}{5}\left[2\left(-\frac{70.83}{EI}\right) + \frac{135.42}{EI}\right] = -182.5 \text{ kN·m}$$

$$M_{DC} = 120 + \frac{2EI}{5}\left[\frac{2 \times 135.42}{EI} - \frac{70.83}{EI}\right] = 200 \text{ kN·m}$$

$$M_{DE} = -200 \text{ kN·m}$$

The free-body diagram and the bending-moment diagram drawn on the compression side are shown in Fig. 5.18(g) and (h) respectively.

If the displacements at A and E are also of interest, coordinates may also be assigned to the rotations and deflections at A and E in addition to the coordinates shown in Fig. 5.18(c). This procedure, however, increases the order of stiffness matrix to seven. As an alternative, spans AB and DE may be considered as cantilevers with due allowance made for the rotations Δ_1 and Δ_3 at B and D respectively which have already been computed.

Example 5.10

Analyse the continuous beam shown in Fig. 5.19(a). The downward settlements of supports B and C in kN-m units are $\frac{1500}{EI}$ and $\frac{750}{EI}$ respectively.

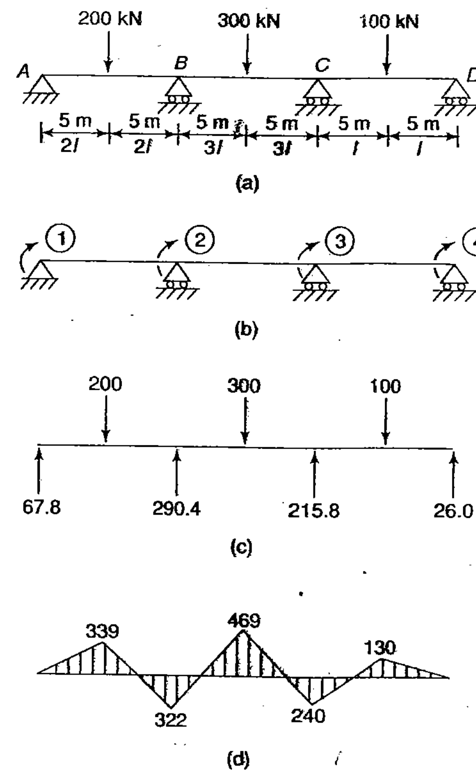


Fig. 5.19

Solution

The degree of freedom of the beam is four because the rotations at supports A, B, C and D are the independent displacement components. Coordinates 1 to 4 may be chosen as shown in Fig. 5.19(b). Locking all the joints, the fixed-end moments due to applied loads are

$$M'_{AB} = -\frac{200 \times 5 \times 5^2}{10^2} = -250 \text{ kN·m}$$

$$M'_{BA} = \frac{200 \times 5 \times 5^2}{10^2} = 250 \text{ kN·m}$$

$$M'_{BC} = -\frac{300 \times 5 \times 5^2}{10^2} = -375 \text{ kN·m}$$

$$M'_{CB} = \frac{300 \times 5 \times 5^2}{10^2} = 375 \text{ kN·m}$$

$$M'_{CD} = -\frac{100 \times 5 \times 5^2}{10^2} = -125 \text{ kN·m}$$

$$M'_{DC} = \frac{100 \times 5 \times 5^2}{10^2} = 125 \text{ kN·m}$$

Additional fixed-end moments due to the yielding of supports are

$$M''_{AB} = M''_{BA} = -\frac{6E(2I)}{10^2}\left(\frac{1500}{EI}\right) = -180 \text{ kN·m}$$

$$M''_{BC} = M''_{CB} = \frac{6E(3I)}{10^2}\left(\frac{750}{EI}\right) = 135 \text{ kN·m}$$

$$M''_{CD} = M''_{DC} = \frac{6EI}{10^2}\left(\frac{750}{EI}\right) = 45 \text{ kN·m}$$

Hence,

$$P'_1 = -250 - 180 = -430 \text{ kN·m}$$
$$P'_2 = 250 - 375 - 180 + 135 = -170 \text{ kN·m}$$
$$P'_3 = 375 - 125 + 135 + 45 = 430 \text{ kN·m}$$
$$P'_4 = 125 + 45 = 170 \text{ kN·m}$$

To develop the stiffness matrix, a unit displacement may be given successively at coordinates 1 to 4. The stiffness matrix $[k]$ with reference to the chosen coordinates is found to be

$$[k] = \begin{bmatrix} 0.8EI & 0.4EI & 0 & 0 \\ 0.4EI & 2.0EI & 0.6EI & 0 \\ 0 & \cancel{0.6EI} & 1.6EI & 0.2EI \\ 0 & 0 & 0.2EI & 0.4EI \end{bmatrix}$$

As there are no external loads at coordinates 1 to 4,

$$P_1 = P_2 = P_3 = P_4 = 0$$

Substituting into Eq. (5.5)

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix} = -\begin{bmatrix} 0.8EI & 0.4EI & 0 & 0 \\ 0.4EI & 2EI & 0.6EI & 0 \\ 0 & 0.6EI & 1.6EI & 0.2EI \\ 0 & 0 & 0.2EI & 0.4EI \end{bmatrix}^{-1} \begin{bmatrix} -430 \\ -170 \\ 430 \\ 170 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{507.6}{EI} \\ \dfrac{59.3}{EI} \\ -\dfrac{253.7}{EI} \\ -\dfrac{297.9}{EI} \end{bmatrix}$$

Knowing the displacements, the end moments may be calculated by using the slope-deflection Eq. (2.47).

$$M_{AB} = -250 + \frac{2E(2I)}{10}\left[\frac{2 \times 507.6}{EI} + \frac{59.3}{EI} - \frac{3 \times 1500}{10EI}\right] = 0$$

$$M_{BA} = 250 + \frac{2E(2I)}{10}\left[\frac{2 \times 59.3}{EI} + \frac{507.6}{EI} - \frac{3 \times 1500}{10EI}\right] = 322 \text{ kN·m}$$

$$M_{BC} = -375 + \frac{2E(3I)}{10}\left[\frac{2 \times 59.3}{EI} - \frac{253.7}{EI} - \frac{3(750 - 1500)}{10EI}\right]$$
$$= -322 \text{ kN·m}$$

$$M_{CB} = 375 + \frac{2E(3I)}{10}\left[2\left(-\frac{253.7}{EI}\right) + \frac{59.3}{EI} - \frac{3(750 - 1500)}{10EI}\right]$$
$$= 240 \text{ kN·m}$$

$$M_{CD} = -125 + \frac{2EI}{10}\left[2\left(-\frac{253.7}{EI}\right) - \frac{297.9}{EI} - \frac{3(750 - 1500)}{10EI}\right]$$
$$= -240 \text{ kN·m}$$

$$M_{DC} = 125 + \frac{2EI}{10}\left[2\left(-\frac{297.9}{EI}\right) - \frac{253.7}{EI} - \frac{3(-750)}{10EI}\right] = 0$$

The free-body diagram and bending-moment diagram drawn on the compression side are shown in Fig. 5.19(c) and (d) respectively.

Example 5.11

Analyse the continuous beam ABCD shown in Fig. 5.20(a). The beam has an internal hinge at B.

Solution

The rorations and deflection at the internal hinge B and the rotation at C are the four independent displacement components. Coordinates 1 to 4 have been assigned to these displacements as shown in Fig. 5.20(b). With the chosen coordinates, portions AB and BC may be treated as separate spans. The fixed-end moments in the restrained structure due to the applied loads other than those acting at the coordinates are zero. Because the supports are unyielding, there are no additional fixed-end moments due to the settlement of supports. Hence,

$$P'_1 = P'_2 = P'_3 = P'_4 = 0$$

From the given data, the applied loads acting at the coordinates are

$$P_1 = P_2 = P_4 = 0 \qquad P_3 = -120 \text{ kN}$$

Next, the stiffness matrix with reference to coordinates 1 to 4 may be developed. To generate the first column of the stiffness matrix give a unit displacement at coordinate 1 without any displacement at other coordinates as shown in Fig. 5.20(c).

$$k_{11} = \frac{4EI}{6} \qquad k_{21} = 0 \qquad k_{31} = \frac{EI}{6} \qquad k_{41} = 0$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at other coordinates as shown in Fig. 5.20(d).

$$k_{12} = 0 \qquad k_{22} = \frac{4EI}{6} \qquad k_{32} = -\frac{EI}{6} \qquad k_{42} = \frac{2EI}{6}$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at other coordinates as shown in Fig. 5.20(e).

$$k_{13} = \frac{EI}{6} \qquad k_{23} = -\frac{EI}{6} \qquad k_{33} = \frac{EI}{9} \qquad k_{43} = -\frac{EI}{6}$$

To generate the fourth column of the stiffness matrix, give a unit displacement at coordinate 4 without any displacement at other coordinates as shown in Fig. 5.20(f).

$$k_{14} = 0 \qquad k_{24} = \frac{2EI}{6} \qquad k_{34} = -\frac{EI}{6} \qquad k_{44} = \frac{4EI}{3}$$

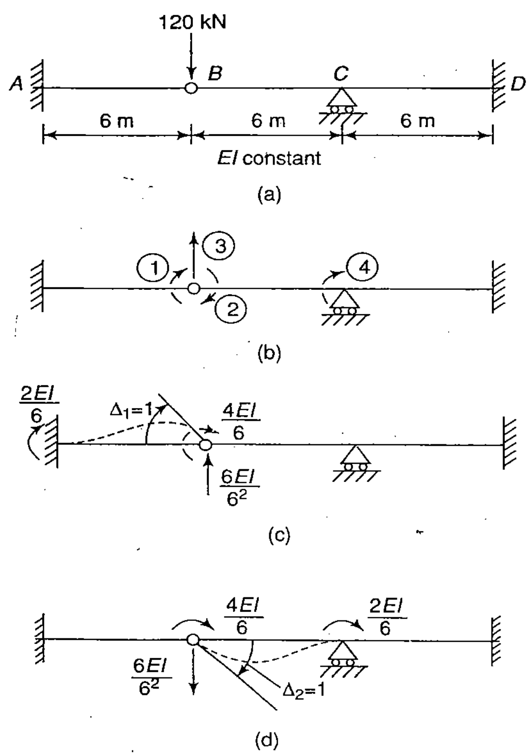


Fig. 5.20 (Contd)

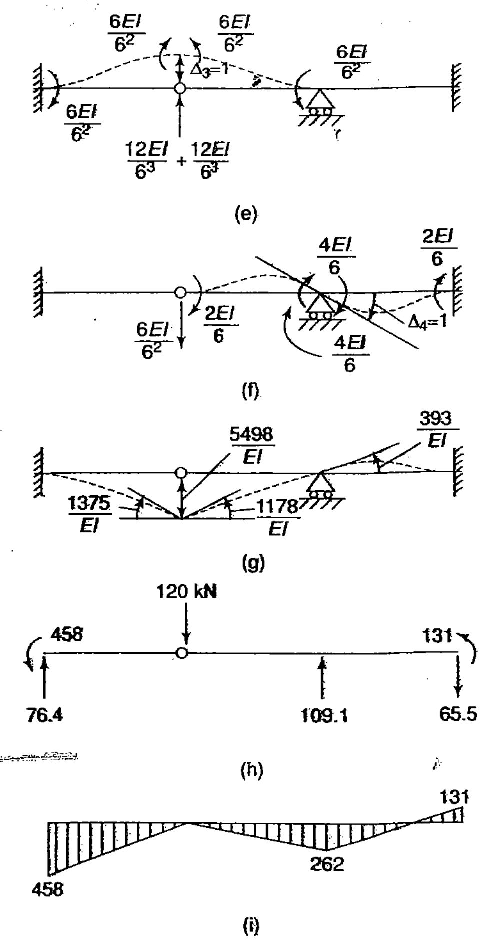


Fig. 5.20

Substituting into Eq. (5.4),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix} = \begin{bmatrix} \frac{4EI}{6} & 0 & \frac{EI}{6} & 0 \\ 0 & \frac{4EI}{6} & -\frac{EI}{6} & \frac{2EI}{6} \\ \frac{EI}{6} & -\frac{EI}{6} & \frac{EI}{9} & -\frac{EI}{6} \\ 0 & \frac{2EI}{6} & -\frac{EI}{6} & \frac{4EI}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -120 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 1374.545 \\ -1178.181 \\ -5498.181 \\ -392.727 \end{bmatrix}$$

Knowing the displacements, the end moments may be calculated by uinsg slope-deflection Eq. (2.47).

$$M_{AB} = \frac{2EI}{6}\left[\frac{1374.545}{EI} - \frac{3 \times 5498.181}{6EI}\right] = -458.1 \text{ kN·m}$$

$$M_{BA} = \frac{2EI}{6}\left[\frac{2 \times 1374.545}{EI} - \frac{3 \times 5498.181}{6EI}\right] = 0$$

$$M_{BC} = \frac{2EI}{6}\left[\frac{2(-1178.181)}{EI} - \frac{392.727}{EI} - \frac{3(-5498.181)}{6EI}\right] = 0$$

$$M_{CB} = \frac{2EI}{6}\left[\frac{2(-392.727)}{EI} - \frac{1178.181}{EI} - \frac{3(-5498.181)}{6EI}\right]$$

$$= 261.8 \text{ kN·m}$$

$$M_{CD} = \frac{2EI}{6}\left[\frac{2(-392.727)}{EI}\right] = -261.8 \text{ kN·m}$$

$$M_{DC} = \frac{2EI}{6}\left[-\frac{392.727}{EI}\right] = -130.9 \text{ kN·m}$$

The deflection curve, free-body diagram for the entire frame and bending-moment diagram drawn on the compression side are shown in Fig. 5.20(g), (h) and (i) respectively.

Example 5.12

Calculate the flexural stiffness at point D of the three-span continuous beam ABCD shown in Fig. 5.21.

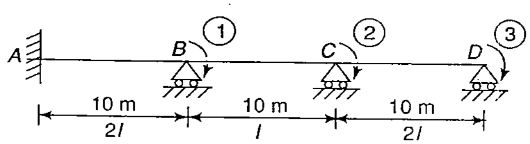


Fig. 5.21

Solution

By definition, the flexural stiffness at point D is the moment required at D to cause unit rotation at D. To find out its value, a unit couple is applied at D and the rotation at D is computed. Select coordinates 1, 2 and 3 as shown in Fig. 5.21. In the present problem, $P_1 = P_2 = 0$ and $P_3 = 1$ since unit couple is applied only at coordinate 3 and there are no external couples at coordinates 1 and 2. Also, $P'_1 = P'_2 = P'_3 = 0$ since there are no intermediate loads and no settlement of supports to produce fixed-end moments.

The stiffness matrix can be developed by giving at unit displacement successively at coordinates 1, 2 and 3 and calculating the forces. For the present problem the stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 1.2EI & 0.2EI & 0 \\ 0.2EI & 1.2EI & 0.4EI \\ 0 & 0.4EI & 0.8EI \end{bmatrix}$$

Substituting into Eq. (5.4),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 1.2EI & 0.2EI & 0 \\ 0.2EI & 1.2EI & 0.4EI \\ 0 & 0.4EI & 0.8EI \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{0.086}{EI} \\ -\dfrac{0.517}{EI} \\ \dfrac{1.509}{EI} \end{bmatrix}$$

Thus the rotation at D, $\Delta_3 = \dfrac{1.509}{EI}$

Hence, the moment required for unit rotation at $D = \dfrac{1}{\dfrac{1.509}{EI}} = 0.663EI$

Therefore the flexural stiffness at $D = 0.663EI$

5.4 COMPARISON OF METHODS

The force and displacement methods for the analysis of continuous beams have been discussed in the preceding sections. It may be noted that whereas the order of the matrix in the force method is equal to the degree of static indeterminacy, its order is equal to the degree of freedom in the displacement method. As the computational effort increases with the order of the matrix, the choice of the method may depend upon the relative values of the degrees of static and kinematic indeterminacies. For the continuous beams with only a few spans, the degrees of static and kinematic indeterminacies may differ considerably. For instance, the degree of static and kinematic indeterminacies of a three-span continuous beam are 4 and 2 respectively if the end supports are fixed. Consequently, the displacement method may appear to be preferable in this case. On the other hand, the degrees of static and kinematic indeterminacies are 2 and 4 respectively if the same continuous beam rests on simple supports. For this problem the force method may be preferable. As the number of spans increases, the difference between the static and kinematic indeterminacies tends to decrease. For a very large number of spans, the difference, if any, is negligible. Consequently, either of the two methods may be chosen. It may, however, be noted that for the same order of the matrix, lesser computaitional effort is required for the development of the stiffness matrix as compared to the development of the flexibility matrix. Hence the displacement method may eventually lead to lesser computational effort even if the order of the stiffness matrix is higher than that of the flexibility matrix. A more detailed discussion regarding the choice of the method is given in Chapter 10.

PROBLEMS

5.1 Analyse the propped cantilever shown in Fig. 5.22 by the force method treating the prop reaction as the redundant. Alternatively, solve the problem by treating the support moment at *A* as the redundant. Hence calculate the bending moment at *C* and the prop reaction. Verify the result by the displacement method.

5.2 Analyse the continuous beam shown in Fig. 5.23 by the force method treating the support reaction at *B* as the redundant. Hence calculate the bending moment at *B*.

5.3 Analyse the continuous beam of Fig. 5.23 by the force method treating the support reaction at *C* as the redundant. Hence calculate the support reaction at *A*.

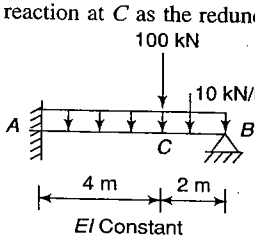


Fig. 5.22

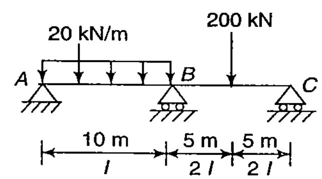


Fig. 5.23

5.4 Analyse the continuous beam of Fig. 5.23 by the force method treating the bending moment at *B* as the redundant. Hence calculate the support reactions at *A* and C.

5.5 Using the displacement method, analyse the continuous beam shown in Fig. 5.23. Hence calculate the support reaction at *B*.

5.6 Using the force method, analyse the continuous beam shown in Fig. 5.24 treating the support reaction at *C* as the redundant. Hence calculate the support reaction at *B*.

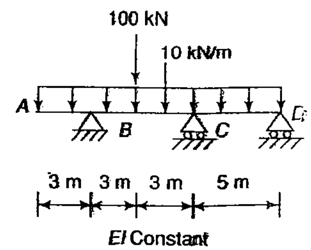


Fig. 5.24

5.7 Analyse the continuous beam shown in Fig. 5.24 using the force method in which the bending moment at *C* is taken as the redundant. Hence compute the support reaction at *C* and *D*.

5.8 Analyse the continuous beam of Fig. 5.24 by the displacement method. Hence calculate the bending moment at *C*.

5.9 Using the force method, compute the fixed-end moments for the beam in Fig. 5.25. Treat the support moments at *A* and *B* as the redundants. Verify the result by treating the reactive forces at *B* as the redundants.

5.10 Using the force method, calculate the fixed-end moments for the beam shown in Fig. 5.26.

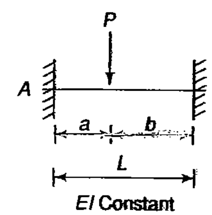


Fig. 5.25

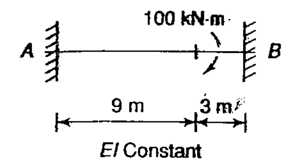


Fig. 5.26

5.11 Analyse the fixed beam shown in Fig. 5.27 by the force method. Hence calculate the bending moment at *C*.

5.12 Analyse the continuous beam shown in Fig. 5.28 by the force method in which support reactions at *A* and *B* are treated as the redundants. Hence calculate the bending moment at *B*. Verify the result by the displacement method.

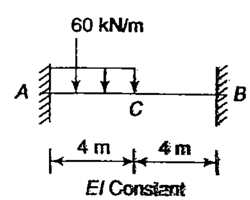


Fig. 5.27

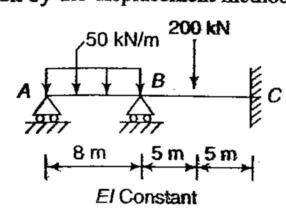


Fig. 5.28

5.13 Using the force method, analyse the continuous beam of Fig. 5.28 treating the bending moments at B and C as redundants. Hence calculate the support reactions.

5.14 Show that the downward reaction at support B in the continuous beam in Fig. 5.29 is $24EI/175\ L^2$ if the support at B settles downwards by a distance, $\Delta = L/100$. Use the displacement method. Verify the result by the force method.

5.15 In the beam of Fig. 5.29, if the support at A permits an anti-clockwise rotation, $\theta = 0.004$ radian and the support at B settles downwards by a distance $\Delta = L/100$, show that the upward reaction at support C is $81\ EI/1750\ L^2$.

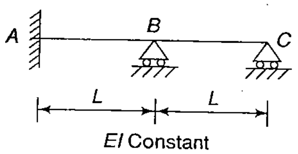


Fig. 5.29

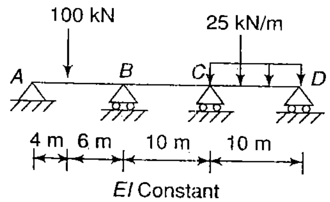


Fig. 5.30

5.16 Figure 5.30 shows a three-span continuous beam resting on simple supports. Analyse the beam by the force method treating the support reactions at B and C as the rdundants. Hence compute the bending moments at B and C.

5.17 Analyse the continuous beam shown in Fig. 5.30 by the force method treating the bending moments at B and C as the redundants. Hence calculate the support reactions at B and C.

5.18 Analyse the continuous beam of Fig. 5.30 by the force method in which the shear force and the bending moment at the centre of the central span are treated as the redundants. Hence calculate the support reactions at A and D.

5.19 What is the degree of freedom of the continuous beam shown in Fig. 5.30? Analyse the beam by the displacement method. Hence verify the result of Prob. 5.16.

5.20 Analyse the continuous beam of Fig. 5.31 by the displacement method. Hence calculate the fixed-end moments at A and C.

5.21 Analyse the continuous beam shown in Fig. 5.31 by the force method adopting the following released structures:

(i) Releasing the bending moments at A, B and C so that the released structure comprises a series of two simply supported beams.

(ii) Removing the support at B and releasing the bending moments at A and C so that the resulting structure is a simply supported beam.

(iii) Removing the supports at A and B so that the released structure is a cantilever.

Hence calculate the support reaction and bending moment at B.

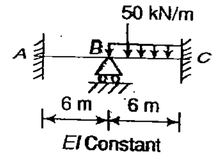


Fig. 5.31

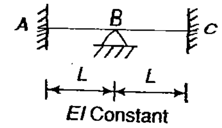


Fig. 5.32

5.22 Using the displacement method, compute the bending moment at B in the continuous beam shown in Fig. 5.32 if the support at C settles downward by a distance $\Delta = L/100$. Verify the result by the force method.

5.23 If the continuous beam shown in Fig. 5.33 carries a uniformly distributed load of intensity p/unit length, analyse the beam by the force method adopting the following alternatives regarding the choice of the redundants:

(i) the bending moment at B, C and D as redundants.

(ii) the support reactions at B, C and D as redundants.

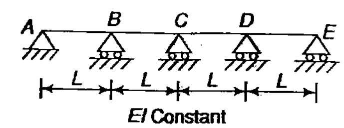


Fig. 5.33

Hence compute the bending moments at B, C and D. Verify the result by the displacement method.

5.24 If the support at B in the continuous beam of Fig. 5.33 settles downwards by unit distance, calculate the support reactions at B, C and D.

5.25 Using the force method, analyse the continuous beam shown in Fig. 5.34. Hence calculate the support reactions at A. Verify the result by the displacement method.

5.26 Support C of the continuous beam shown in Fig. 5.35 has a downward settlement of 30 mm. Calculate the support reactions at D by the force method. Verify the result by the displacement method.

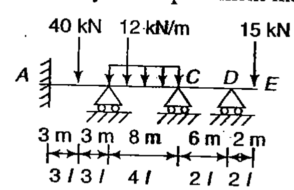


Fig. 5.34

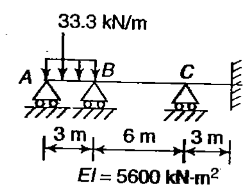
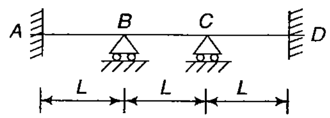


Fig. 5.35

5.27 Using the displacement method, analyse the continuous beam shown in Fig. 5.36 if spans AB and BC carry a uniformly distributed load, p/unit length. Hence calculate the bending moments at B and C. Verify the result by the force method.

5.28 Analyse the continuous beam of Fig. 5.36 by the displacement method if the support at A sinks downwards by a distance equal to Δ. Hence calculate the fixed-end moments at A and D. Verify the result by the force method.



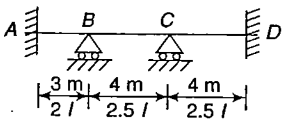
EI Constant

Fig. 5.36

5.29 Calculate the couple M required to produce a unit rotation at B in the continuous beam of Fig. 5.36. Use the displacement method.

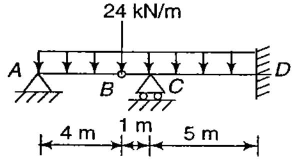
5.30 Analyse the continuous beam of Fig. 5.37 if the support at B undergoes a downward settlement of 10 mm. Use the displacement method. Hence calculate the bending moments at B and C. Verify the result by the force method.

5.31 The continuous beam of Fig. 5.38 has an internal hinge at B. Analyse the beam by the force method. Hence calculate the fixed-end moment at D. Verify the result by the displacement method



EI = 6000 kN·m²

Fig. 5.37



EI Constant

Fig. 5.38

5.32 Using the coordinates shown in Fig. 5.39, analyse the continuous beam by the force method for the following effects:

(i) a concentrated load of 240 kN at the centre of span AB and a uniformly distributed load of 40 kN over span BC

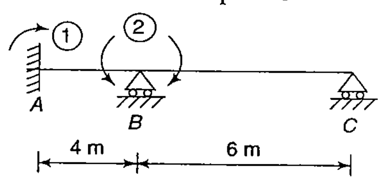


Fig. 5.39

(ii) downward settlements of $\frac{120}{EI}$ and $\frac{360}{EI}$ at A and B respectively

(iii) clockwise rotation of $\frac{60}{EI}$ at A.

Consider each effect separately. Verify the result by considering the three effects simultaneously. Hence calculate the bending moments at A and B.

6 RIGID-JOINTED PLANE FRAMES

6.1 INTRODUCTION

A rigid-jointed frame is a skeletal structure formed by an assembly of members which meet at rigid joints. The joints are said to be rigid if the angles between the members remain unchanged during the deformation of the structure. The members of a rigid-jointed plane frame carry axial forces, shear forces and bending moments. During deformation the members of a rigid-jointed frame assume a curved shape. As the axial displacements of the members of a rigid-jointed frame are much smaller than the transverse displacements, it is a common practice in structural analysis to ignore the axial displacements of the members. Thus all members of a rigid-jointed frame are considered *inextensible*. This basic assumption will be utilized in the development of the force and displacement methods of matrix analysis of rigid-jointed frame discussed in the following sections.

6.2 FORCE METHOD

The force method for the analysis of rigid-jointed plane frames begins with the determination of the degree of static indeterminacy and identification of the redundants. The degree of static indeterminacy has been discussed in detail in Sec. 1.6. The rigid-jointed plane frames may be statically indeterminate both internally as well as externally. The basic determinate structure is obtained by releasing a sufficient number of internal forces and external reaction components. Care should be exercised in selecting the internal and external redundants so that the released structure is statically determinate, stable and as simple as possible. Coordinates are assigned to all the redundants, internal as well as external.

To illustrate the manner in which coordinates are assigned in the force method of analysis of rigid-jointed plane frames, consider the structure shown in Fig. 6.1(a). The frame is statically indeterminate internally to the third degree and externally to the second degree. Thus to make the frame statically determinate internally, a cut may be made at a point G in member BE. Three member forces, viz., an axial force, a shear force and a bending moment are released at cut G. Coordinates 1, 2 and 3 are assigned to these internal forces as shown in Fig. 6.1(b). To make the frame statically determinate externally, the hinge support at F may be removed, thereby releasing the vertical and horizontal reaction components. Coordinates 4 and 5 are assigned to these external reaction components as shown in Fig. 6.1(b).

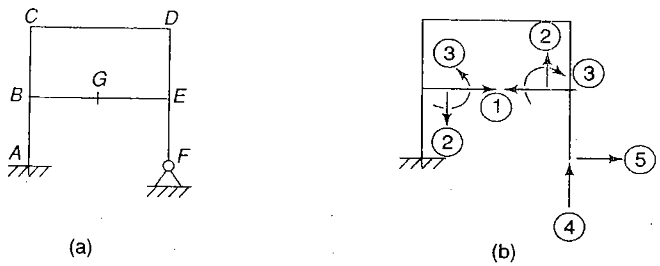


Fig. 6.1

After the selection of the released structure and the system of coordinates, the force method for the analysis of rigid-jointed plane frames is the same as for continuous beams discussed in Sec. 5.2. The chosen redundants may be determined from the compatibility conditions which lead to the equation

$$[P] = [\delta]^{-1}\{[\Delta] - [\Delta_L]\} \tag{6.1}$$

In the case of unyielding supports, Eq. (6.1) takes the form

$$[P] = -[\delta]^{-1}[\Delta_L] \tag{6.2}$$

In the case of yielding supports with prespecified settlements, Eq. (6.1) may be used in which the appropriate values of the prespecified settlements may be substituted into matrix $[\Delta]$.

Example 6.1

Analyse the portal frame shown in Fig. 6.2(a).

Solution

The frame is statically indeterminate to the third degree. The released structure may be obtained by removing the support at D and thereby releasing three reaction components. Coordinates 1, 2 and 3 may be assigned to these reaction components as shown in Fig. 6.2(b). The displacements at the chosen coordinates in the released structure due to the applied loads may be calculated by using any one of the methods discussed in Chapter 2. These displacements have been computed in Ex. 2.13 by the method of diagram multiplication.

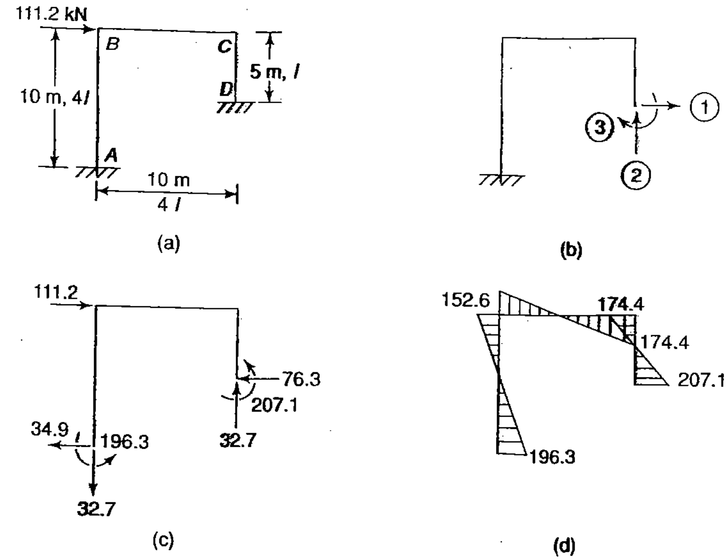


Fig. 6.2

$$\Delta_{1L} = \frac{6950}{3EI}$$

$$\Delta_{2L} = -\frac{13900}{EI}$$

$$\Delta_{3L} = \frac{1390}{EI}$$

Next, the flexibility matrix for the frame with reference to the chosen coordinates may be developed. This has been done in Ex. 4.10. The flexibility matrix has been found to be

$$[\delta] = \frac{1}{6EI}\begin{bmatrix} 750 & 375 & -150 \\ 375 & 2000 & -225 \\ -150 & -225 & 60 \end{bmatrix}$$

As the supports are unyielding, substituting into Eq. (6.2),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -6EI\begin{bmatrix} 750 & 375 & -150 \\ 375 & 2000 & -225 \\ -150 & -225 & 60 \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{6950}{3EI} \\ -\dfrac{13900}{EI} \\ \dfrac{1390}{EI} \end{bmatrix} = \begin{bmatrix} -76.3 \\ 32.7 \\ -207.1 \end{bmatrix}$$

Knowing the reactive forces at D, the reactive forces at A can be calculated by statics. Hence the free-body diagram of the entire frame as shown in Fig. 6.2(c) may be drawn. Figure 6.2(d) shows the bending-moment diagram for the frame drawn on the compression side.

Example 6.2

Analyse the portal frame of Fig. 6.2(a) if the settlements of support D to the right and downwards in kN-m units are 200/EI and 500/EI respectively.

Solution

For the solution of this problem, the same coordinates may be adopted as in Ex. 6.1. These coordinates are shown in Fig. 6.2(b). The displacements in the released structure due to the applied loads and the flexibility matrix with reference to the chosen coordinates are given in Ex. 6.1.

Also, from the data given in this example

$$\Delta_1 = \frac{200}{EI}$$

$$\Delta_2 = -\frac{500}{EI}$$

$$\Delta_3 = 0$$

Substituting into Eq. (6.1),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 6EI \begin{bmatrix} 750 & 375 & -150 \\ 375 & 2000 & -225 \\ -150 & -225 & 60 \end{bmatrix}^{-1}$$

$$\times \left\{ \begin{bmatrix} \dfrac{200}{Ei} \\ -\dfrac{500}{EI} \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{6950}{3EI} \\ \dfrac{-13900}{EI} \\ \dfrac{1390}{EI} \end{bmatrix} \right\} = \begin{bmatrix} -74.3 \\ 30.5 \\ -210.4 \end{bmatrix}$$

Knowing the reaction components at D, the reaction components at A can be calculated from statics. Hence the free-body diagram of the entire frame, as shown in Fig. 6.3(a) may be drawn. Fig. 6.3(b) shows the bending-moment diagram drawn on the compression side.

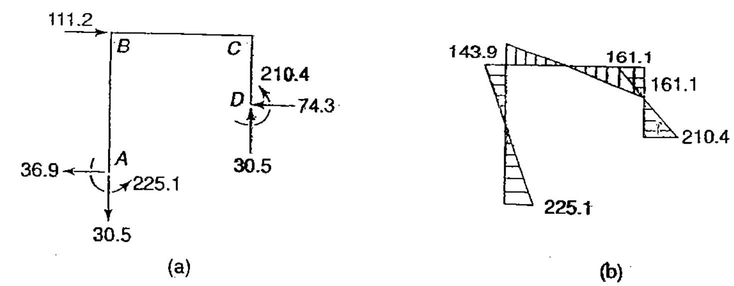


Fig. 6.3

Example 6.3

Analyse the bent shown in Fig. 6.4(a).

Solution

The degree of static indeterminacy of the frame is three. The released structure may be obtained by inserting hinges at A, B and C. Coordinates 1, 2 and 3 may be assigned to the released bending moments at A, B and C as shown in Fig. 6.4(b). The displacements at the chosen coordinates in the released structure due to the applied loads and the elements of the flexibility matrix may be determined by using any one of the methods discussed in Chapter 2. The M-diagram due to the applied loads for the released structure is shown in Fig. 6.4(c). The m_1-, m_2- and m_3-diagrams for the released structure due to a unit force at coordinates 1, 2 and 3 respectively are shown in Fig.6.4(d), (e) and (f). Using the method of diagram multiplication, Table 2.11,

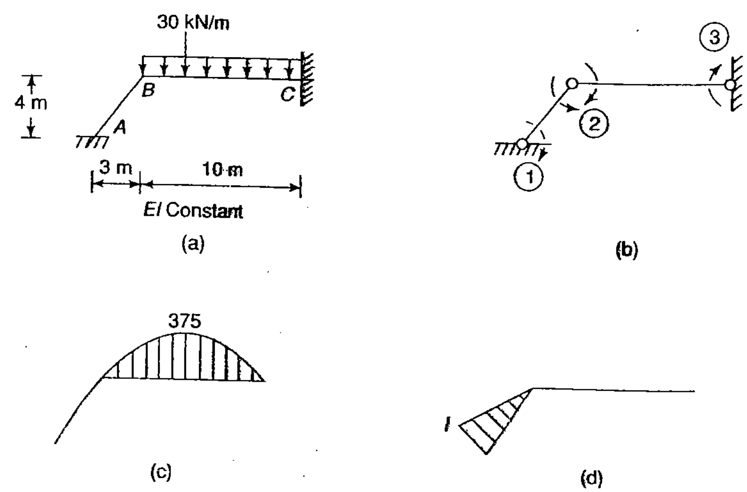


Fig. 6.4 (Contd)

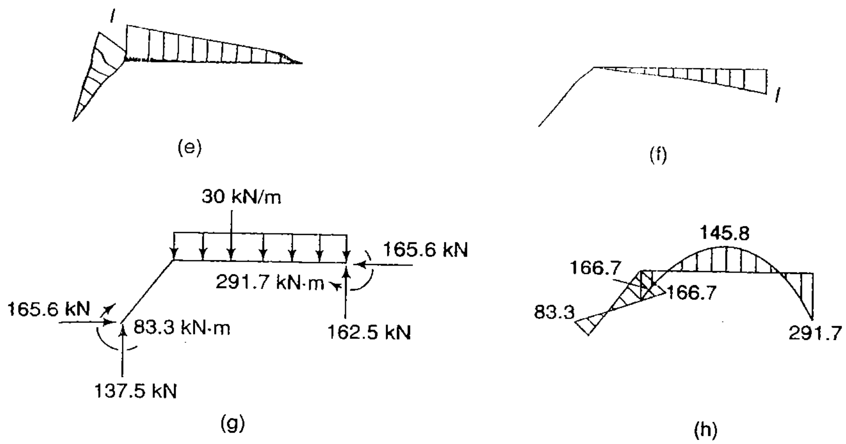


Fig. 6.4

$$\Delta_{1L} = 0$$

$$\Delta_{2L} = \frac{1}{EI} \times \frac{1}{3} \times 1 \times 375 \times 10 = \frac{1250}{EI}$$

$$\Delta_{3L} = -\frac{1}{EI} \times \frac{1}{3} \times 1 \times 375 \times 10 = -\frac{1250}{EI}$$

$$\delta_{11} = \frac{1}{EI} \times \frac{1}{3} \times 1 \times 1 \times 5 = \frac{5}{3EI}$$

$$\delta_{21} = \delta_{12} = \frac{1}{EI} \times \frac{1}{6} \times 1 \times 1 \times 5 = \frac{5}{6EI}$$

$$\delta_{31} = \delta_{13} = 0$$

$$\delta_{22} = \frac{1}{EI} \times \frac{1}{3} \times 1 \times 1 \times 5 + \frac{1}{EI} \times \frac{1}{3} \times 1 \times 1 \times 10 = \frac{5}{EI}$$

$$\delta_{32} = \delta_{23} = -\frac{1}{EI} \times \frac{1}{6} \times 1 \times 1 \times 10 = -\frac{5}{3EI}$$

$$\delta_{33} = \frac{1}{EI} \times \frac{1}{3} \times 1 \times 1 \times 10 = \frac{10}{3EI}$$

As the supports are unyielding, substituting into Eq. (6.2),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -1.2EI \begin{bmatrix} 2 & 1 & 0 \\ 1 & 6 & -2 \\ 0 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \dfrac{1250}{EI} \\ -\dfrac{1250}{EI} \end{bmatrix} = \begin{bmatrix} 83.33 \\ -166.67 \\ 291.67 \end{bmatrix}$$

Knowing the redundants P_1, P_2 and P_3 at A, B and C, the reactive forces at A and C may be calculated by statics. Hence the free-body diagram of the entire frame as shown in Fig. 6.4(g) may be drawn. Figure 6.4(h) shows the bending-moment diagram for the frame drawn on the compression side.

Example 6.4

Analyse the portal frame shown in Fig. 6.5(a).

Solution

The frame is statically indeterminate to the third degree. The released structure, shown in Fig. 6.5(b), has been obtained by making a cut at E, the centre of member BC and thereby releasing three internal forces. Coordinates 1, 2 and 3 may be assigned to these internal forces as shown in Fig. 6.5(b). The displacements at the chosen coordinates in the released structure due to the applied loads and the elements of the flexibility matrix may be determined by using any one of the methods discussed in Chapter 2. The ***M***-diagram due to the applied loads for the released structure is shown in Fig. 6.5(c). The m_1-, m_2- and m_3-diagrams for the released structure due to a unit force at coordinates 1, 2 and 3 respectively are shown in Fig. 6.5(d), (e) and (f). Using the method of diagram multiplication, Sec. 2.12,

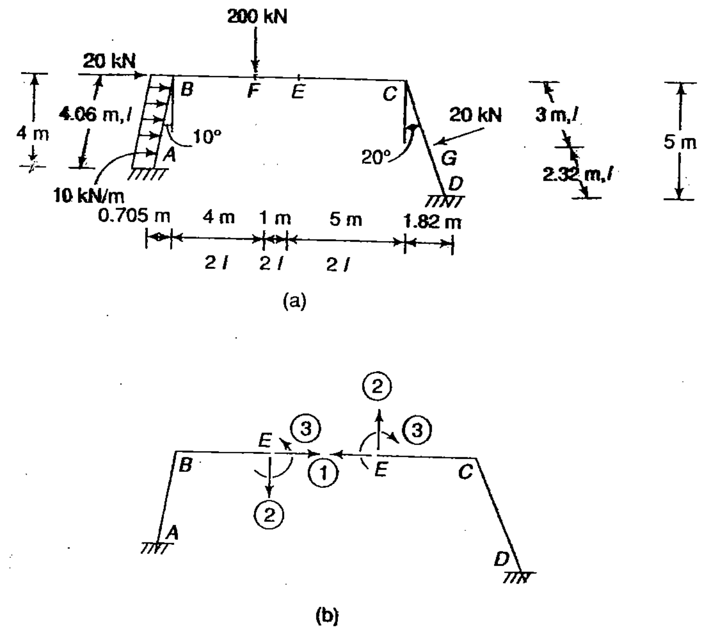


Fig. 6.5 (Contd)

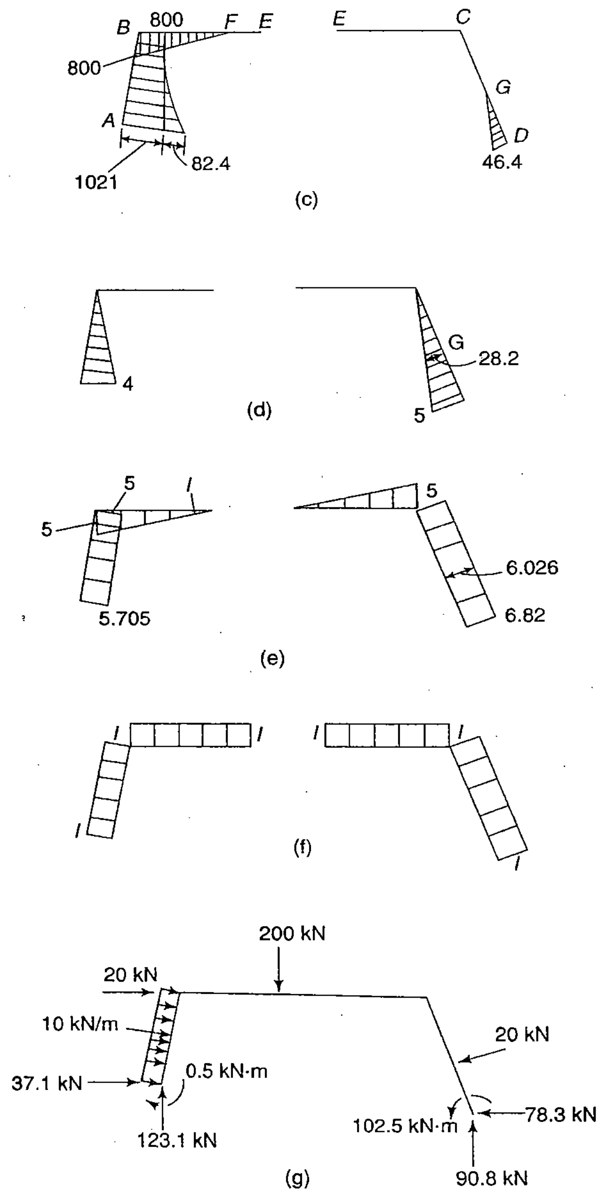


Fig. 6.5 (Contd)

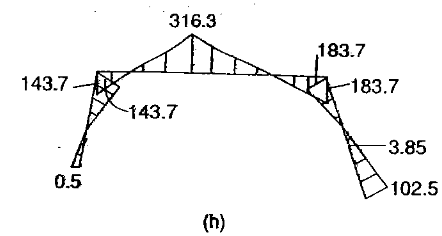


Fig. 6.5

$$\Delta_{1L} = \frac{1}{EI} \times \frac{4 \times 4.06}{6}(2 \times 1021 + 800) + \frac{1}{EI} \times \frac{82.4 \times 4.06 \times 4}{4} + \frac{1}{EI} \times \frac{46.4 \times 2.32}{6}(2 \times 5 + 2.82)$$

$$= \frac{8257}{EI}$$

$$\Delta_{2L} = \frac{1}{2EI} \times \frac{800 \times 4}{6}(2 \times 5 + 1) + \frac{1}{EI} \times \frac{4.06}{6} \times (2 \times 800 \times 5 + 2 \times 1021 \times 5.705 + 5 \times 1021 + 800 \times 5.705) + \frac{1}{EI} \times \frac{82.4 \times 4.06}{12}(5 + 3 \times 5.705) - \frac{1}{EI} \times \frac{46.4 \times 2.32}{6}(2 \times 6.82 + 6.026)$$

$$= \frac{23036}{EI}$$

$$\Delta_{3L} = -\frac{1}{2EI} \times \frac{800 \times 1 \times 4}{2} - \frac{1}{EI} \times 1 \times 4.06\,(800 + 1021) - \frac{1}{EI} \times \frac{82.4 \times 1 \times 4.06}{3} - \frac{1}{EI} \times \frac{46.4 \times 1 \times 2.32}{2}$$

$$= -\frac{4662}{EI}$$

$$\delta_{11} = \frac{1}{EI} \times \frac{1}{3} \times 4 \times 4 \times 4.06 + \frac{1}{EI} \times \frac{1}{3} \times 5 \times 5 \times 5.32$$

$$= \frac{65.99}{EI}$$

$$\delta_{21} = \delta_{12} = \frac{1}{EI} \times \frac{4 \times 4.06}{6}(2 \times 5.705 + 5) - \frac{1}{EI} \times \frac{5 \times 5.32}{6}$$

$$\times (2 \times 6.82 + 5) = -\frac{38.22}{EI}$$

$$\delta_{31} = \delta_{13} = -\frac{1}{EI} \times \frac{1}{2} \times 4 \times 1 \times 4.06 - \frac{1}{EI} \times \frac{1}{2} \times 5 \times 1 \times 5.32$$

$$= -\frac{21.42}{EI}$$

$$\delta_{22} = \frac{1}{2EI} \times \frac{1}{3} \times 5 \times 5 \times 5 + \frac{1}{EI} \times \frac{4.06}{6}$$

$$\times (2 \times 5 \times 5 + 2 \times 5.705 \times 5.705 + 5 \times 5.705 + 5 \times 5.705)$$

$$+ \frac{1}{2EI} \times \frac{1}{3} \times 5 \times 5 \times 5 + \frac{1}{EI} \times \frac{5.32}{6}$$

$$\times (2 \times 5 \times 5 + 2 \times 6.82 \times 6.82 + 5 \times 6.82 + 5 \times 6.82)$$

$$= \frac{345.44}{EI}$$

$$\delta_{32} = \delta_{23} = -\frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times 5 \times -\frac{1}{EI} \times \frac{1 \times 4.06}{2}$$

$$\times (5 + 5.705) + \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times 5$$

$$+ \frac{1}{EI} \times \frac{5.32}{2}(5 + 6.82)$$

$$= \frac{9.71}{EI}$$

$$\delta_{33} = \frac{1}{EI} \times 1 \times 1 \times 5 + \frac{1}{EI} \times 1 \times 1 \times 4.06 + \frac{1}{2EI} \times 1 \times 1 \times 5$$

$$+ \frac{1}{EI} \times 1 \times 1 \times 5.32$$

$$= \frac{14.38}{EI}$$

As the supports are unyielding, substituting into Eq. (6.2),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -EI \begin{bmatrix} 65.99 & -38.22 & -21.42 \\ -38.22 & 345.44 & 9.71 \\ -21.42 & 9.71 & 14.38 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{8257}{EI} \\ \dfrac{23036}{EI} \\ -\dfrac{4662}{EI} \end{bmatrix}$$

$$= \begin{bmatrix} -97.1 \\ -84 \\ 236.3 \end{bmatrix}$$

Knowing the internal forces P_1, P_2 and P_3 at E, the reactive forces at A and D may be calculated by statics. Hence the free-body diagram of the entire frame as shown in Fig. 6.5(g) may be drawn. Figure 6.5(h) shows the bending-moment diagram for the frame drawn on the compression side.

Example 6.5

Analyse the portal frame shown in Fig. 6.6(a) if the downward settlements at C and E in kN-m units are 1000/EI and 500/EI respectively.

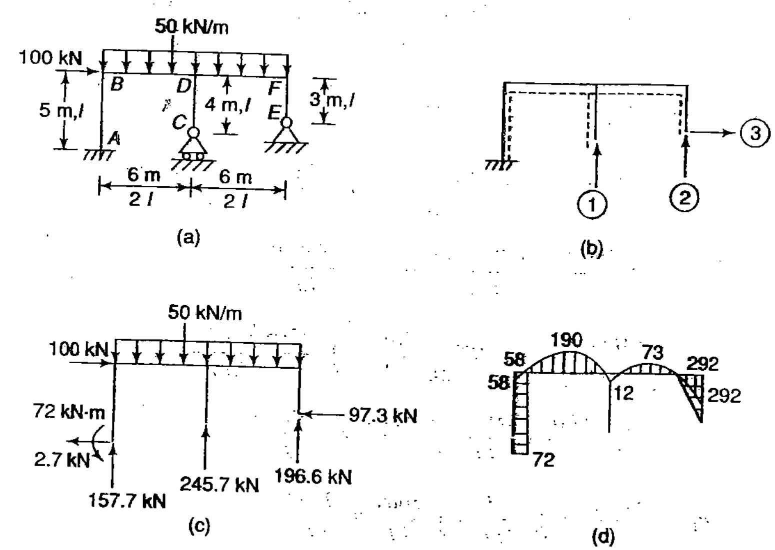


Fig. 6.6

Solution

The frame is statically determinate internally. There are six external reaction components. Hence the degree of external indeterminacy is three. The released structure may be obtained by removing the supports at C and E. The released structure and the chosen coordinates are shown in Fig. 6.6 (b). The displacements at the chosen coordinates in the released structure due to the applied loads and the elements of the flexibility matrix may be determined by applying the unit load method, Sec. 2.12. The necessary details for the computation are shown in Table 6.1. Bending moment producing tension on dotted side in Fig. (b) has been taken positive.

Table 6.1

Portion→	*EF*	*FD*	*CD*	*DB*	*BA*
I	I	$2I$	I	$2I$	I
Origin	E	F	C	D	B
Limits	0 to 3	0 to 6	0 to 4	0 to 6	0 to 5
M	0	$-25\,x^2$	0	$-25\,(6+x)^2$	$-(3600+100x)$
m_1	0	0	0	x	6
m_2	0	x	0	$(6+x)$	12
m_3	x	3	0	3	$-(x-3)$

Using Table 6.1,

$$\Delta_{1L} = \int \frac{Mm_1dx}{EI} = -\int_0^6 \frac{25\,(6+x)^2\,x dx}{2EI} - \int_0^5 \frac{(3600+100x)\times 6dx}{EI}$$

$$= -\frac{138450}{EI}$$

$$\Delta_{2L} = \int \frac{Mm_2dx}{EI} = -\int_0^6 \frac{25x^2\cdot x dx}{2EI} - \int_0^6 \frac{25\,(6+x)^2(6+x)dx}{2EI}$$

$$-\int_0^5 \frac{(3600+100\,x)\times 12dx}{EI} = -\frac{295800}{EI}$$

$$\Delta_{3L} = \int \frac{Mm_3dx}{EI} = -\int_0^6 \frac{25x^2\times 3dx}{2EI} - \int_0^6 \frac{25(6+x)^2\times 3dx}{2EI}$$

$$+\int_0^5 \frac{(3600+100x)(x-3)\,dx}{EI} = -\frac{30183}{EI}$$

$$\delta_{11} = \int \frac{m_1^2dx}{EI} = \int_0^6 \frac{x^2dx}{2EI} + \int_0^5 \frac{36dx}{EI} = \frac{216}{EI}$$

$$\delta_{21} = \delta_{12} = \int \frac{m_1m_2dx}{EI} = \int_0^6 \frac{x(6+x)dx}{2EI} + \int_0^5 \frac{6\times 12dx}{EI} = \frac{450}{EI}$$

$$\delta_{31} = \delta_{13} = \int \frac{m_1m_3dx}{EI} = \int_0^6 \frac{3xdx}{2EI} - \int_0^5 \frac{6(x-3)dx}{EI} = \frac{42}{EI}$$

$$\delta_{22} = \int \frac{m_2^2dx}{EI} = \int_0^6 \frac{x^2dx}{2EI} + \int_0^6 \frac{(6+x)^2\,dx}{2EI} + \int_0^5 \frac{144dx}{EI} = \frac{1008}{EI}$$

$$\delta_{32} = \delta_{23} = \int \frac{m_2m_3dx}{EI} = \int_0^6 \frac{(-x)(-3)dx}{2EI} + \int_0^6 \frac{3(6+x)dx}{2EI}$$

$$-\int_0^5 \frac{12(x-3)dx}{EI} = \frac{138}{EI}$$

$$\delta_{33} = \int \frac{m_3^2dx}{EI} = \int_0^3 \frac{x^2dx}{EI} + \int_0^6 \frac{9dx}{2EI} + \int_0^6 \frac{9dx}{2EI} + \int_0^5 \frac{(x-3)^2\,dx}{EI}$$

$$= \frac{74.66}{EI}$$

From the given data,

$$\Delta_1 = -\frac{1000}{EI}$$

$$\Delta_2 = -\frac{500}{EI}$$

$$\Delta_3 = 0$$

Substituting into Eq. (6.1),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} \frac{216}{EI} & \frac{450}{EI} & \frac{42}{EI} \\ \frac{450}{EI} & \frac{1008}{EI} & \frac{138}{EI} \\ \frac{42}{EI} & \frac{138}{EI} & \frac{74.66}{EI} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\frac{1000}{EI} \\ -\frac{500}{EI} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{138450}{EI} \\ -\frac{295800}{EI} \\ -\frac{30183}{EI} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 245.7 \\ 196.6 \\ -97.3 \end{bmatrix}$$

Knowing the redundant forces P_1, P_2 and P_3, the three reaction components at A may now be calculated by statics. Hence the free-body diagram of the entire frame as shown in Fig. 6.6(c) may be drawn. Figure 6.6(d) shows the bending-moment diagram for the frame drawn on the compression side.

Example 6.6

Analyse the portal frame shown in Fig. 6.7(a).

Solution

As there are only three external reaction components, the structure is statically determinate externally. Regarding internal indeterminacy, it may be noted that the structure has one closed cell. The open configuration may be obtained by making one cut anywhere in the closed cell, thereby releasing three internal forces. Hence the degree of static indeterminacy of the structure is three. There are several possible ways in which the released structure can be obtained. In the following, three alternative solutions using different released structures are given.

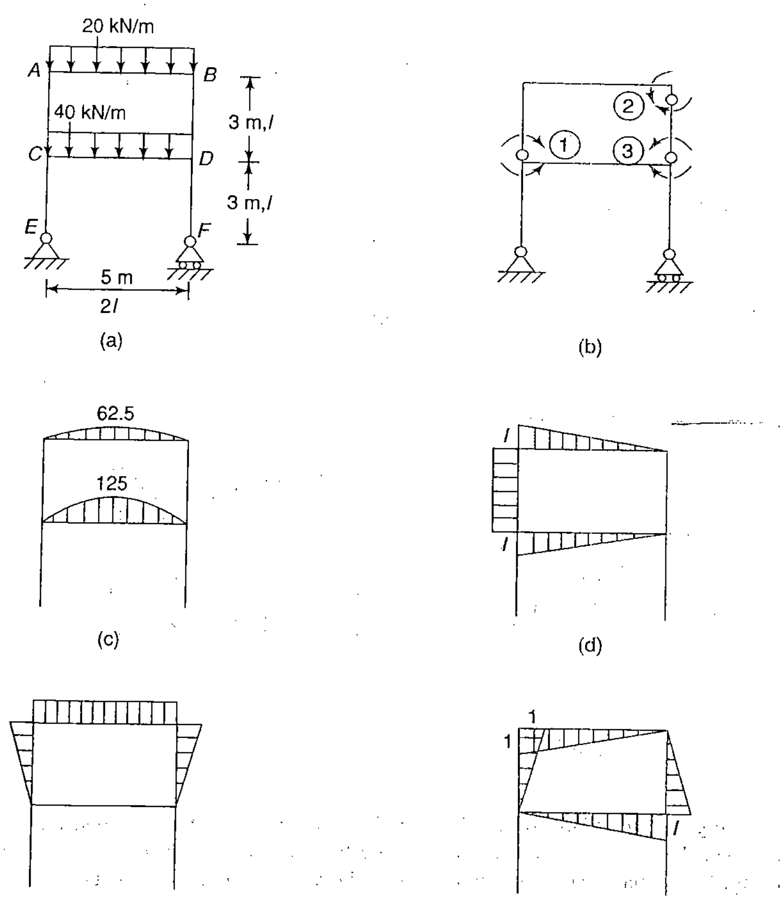


Fig. 6.7 (Contd)

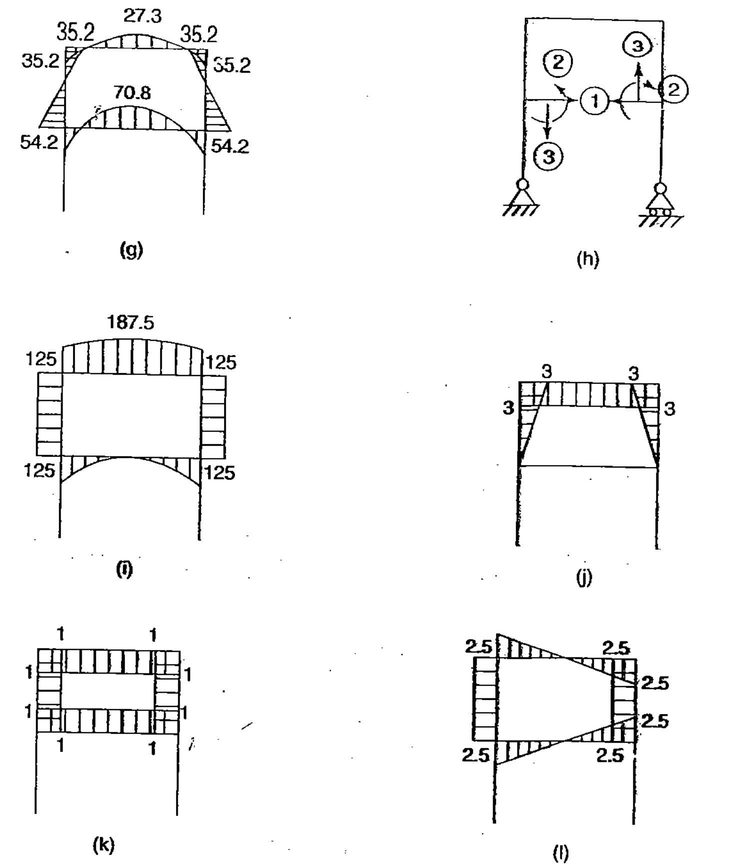


Fig. 6.7

(i) The released structure shown in Fig. 6.7(b) has been obtained by inserting three internal hinges. Coordinates 1, 2 and 3 corresponding to the three moment releases are also shown in Fig. 6.7(b). The M-diagram due to the applied loads for the released structure is shown in Fig. 6.7(c). The m_1-, m_2- and m_3-diagrams for the released structure due to a unit force at coordinates 1, 2 and 3 respectively are shown in Fig. 6.7(d), (e) and (f) respectively. Using the method of diagram multiplication, Sec. 2.12,

$$\Delta_{1L} = \frac{1}{2EI} \times \frac{2}{3} \times 5 \times 62.5 \times \frac{1}{2} - \frac{1}{2EI} \times \frac{2}{3} \times 5 \times 125 \times \frac{1}{2}$$

$$= -\frac{625}{12EI}$$

$$\Delta_{2L} = \frac{1}{2EI} \times \frac{2}{3} \times 5 \times 62.5 \times 1 = \frac{625}{6EI}$$

$$\Delta_{3L} = \frac{1}{2EI} \times \frac{2}{3} \times 5 \times 62.5 \left(-\frac{1}{2}\right) + \frac{1}{2EI} \times \frac{2}{3} \times 125 \left(-\frac{1}{2}\right)$$

$$= -\frac{625}{4EI}$$

$$\delta_{11} = \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times \frac{2}{3} + \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times \frac{2}{3}$$

$$+ \frac{1}{EI} \times 3 \times 1 \times 1 = \frac{14}{3EI}$$

$$\delta_{21} = \delta_{12} = \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times 1 + \frac{1}{EI} \times 3 \times 1 \times \frac{1}{2} = \frac{11}{4EI}$$

$$\delta_{31} = \delta_{13} = \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times \frac{1}{3} + \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \left(-\frac{2}{3}\right)$$

$$+ \frac{1}{EI} \times 3 \times 1 \left(-\frac{1}{2}\right) = -\frac{23}{12EI}$$

$$\delta_{22} = \frac{1}{2EI} \times 5 \times 1 \times 1 + \frac{1}{EI} \times 3 \times 1 \times \frac{2}{3}$$

$$+ \frac{1}{EI} \times \frac{1}{2} \times 3 \times 1 \times \frac{2}{3} = \frac{9}{2EI}$$

$$\delta_{32} = \delta_{23} = \frac{1}{2EI} \times 5 \times 1 \left(-\frac{1}{2}\right) + \frac{1}{EI} \times \frac{1}{2} \times 3 \times 1 \left(-\frac{2}{3}\right)$$

$$+ \frac{1}{EI} \times \frac{1}{2} \times 3 \times 1 \times \frac{1}{3} = -\frac{7}{4EI}$$

$$\delta_{33} = \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times \frac{2}{3} + \frac{1}{EI} \times \frac{1}{2} \times 3 \times 1 \times \frac{2}{3}$$

$$+ \frac{1}{2EI} \times \frac{1}{2} \times 5 \times 1 \times \frac{2}{3} + \frac{1}{EI} \times \frac{1}{2} \times 3 \times 1 \times \frac{2}{3}$$

$$= \frac{11}{3EI}$$

As the supports are unyielding, substituting in Eq. (6.2),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -\begin{bmatrix} \frac{14}{3EI} & \frac{11}{4EI} & -\frac{23}{12EI} \\ \frac{11}{4EI} & \frac{9}{2EI} & -\frac{7}{4EI} \\ -\frac{23}{12EI} & -\frac{7}{4EI} & \frac{11}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{625}{12EI} \\ \frac{625}{6EI} \\ -\frac{625}{4EI} \end{bmatrix}$$

$$= \begin{bmatrix} 54.1 \\ -35.2 \\ 54.1 \end{bmatrix}$$

The bending-moment diagram may be obtained by using the equation

$$\text{Bending moment} = M + m_1 P_1 + m_2 P_2 + m_3 P_3 \qquad \text{(a)}$$

Hence the bending moment at any point in the frame may be computed by multiplying the ordinates of the m_1-, m_2- and m_3-diagrams by P_1, P_2 and P_3 respectively and adding them to the ordinates of *M*-diagram. Figure 6.7(g) shows the bending-moment diagram for the frame drawn on the compression side.

(ii) The released structure shown in Fig. 6.7(h) has been obtained by making a cut at the centre of the lower beam. Coordinates 1, 2 and 3 corresponding to the three released internal forces are also shown in Fig. 6.7(h). The *M*-diagram due to the applied loads for the released structure is shown in Fig. 6.7(i). the m_1-, m_2- and m_3-diagrams for the released structure due to a unit force at coordinates 1, 2 and 3 respectively are shown in Fig. 6.7(j), (k) and (l) respectively. Using the method of diagram multiplication

$$\Delta_{1L} = \frac{1}{2EI}\left(125 \times 5 + \frac{2}{3} \times 5 \times 62.5\right)(-3)$$

$$+ \frac{1}{EI} \times 3 \times 125\left(-\frac{3}{2}\right) + \frac{1}{EI} \times 3 \times 125\left(-\frac{3}{2}\right)$$

$$= -\frac{2375}{EI}$$

$$\Delta_{2L} = \frac{1}{2EI}\left(125 \times 5 + \frac{2}{3} \times 5 \times 62.5\right)(-1)$$

$$+ \frac{1}{2EI} \times \frac{1}{3} \times 2.5 \times 125(-1)$$

$$+ \frac{1}{3} \times 2.5 \times 125(-1) + \frac{1}{EI} \times 3 \times 125(-1)$$

$$+ \frac{1}{EI} \times 3 \times 125(-1) = -\frac{7625}{6EI}$$

$$\Delta_{3L} = 0$$

$$\delta_{11} = \frac{1}{2EI} \times 5 \times 3 \times 3 + \frac{1}{EI} \times \frac{1}{2} \times 3 \times 3 \times 2$$

$$+ \frac{1}{EI} \times \frac{1}{2} \times 3 \times 3 \times 2 = \frac{81}{2EI}$$

$$\delta_{21} = \delta_{12} = \frac{1}{2EI} \times 5 \times 3 \times 1 = \frac{1}{EI} \times \frac{1}{2} \times 3 \times 3 \times 1$$

$$+ \frac{1}{EI} \times \frac{1}{2} \times 3 \times 3 \times 1 = \frac{33}{2EI}$$

$$\delta_{31} = \delta_{13} = 0$$

$$\delta_{22} = \frac{1}{2EI} \times 5 \times 1 \times 1 + \frac{1}{EI} \times 3 \times 1 \times 1 + \frac{1}{2EI} \times 5 \times 1 \times 1$$

$$+ \frac{1}{EI} \times 3 \times 1 \times 1 = \frac{11}{EI}$$

$$\delta_{32} = \delta_{23} = 0$$

$$\delta_{33} = \left(\frac{1}{2EI} \times \frac{1}{2} \times 2.5 \times 2.5 \times \frac{5}{3}\right) \times 4$$

$$+ \left(\frac{1}{EI} \times 3 \times 2.5 \times 2.5\right) \times 2 = \frac{575}{12EI}$$

Substituting into Eq. (6.2),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -\begin{bmatrix} \frac{81}{2EI} & \frac{33}{2EI} & 0 \\ \frac{33}{2EI} & \frac{11}{EI} & 0 \\ 0 & 0 & \frac{575}{12EI} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{2375}{EI} \\ -\frac{7625}{6EI} \\ 0 \end{bmatrix} = \begin{bmatrix} 29.8 \\ 70.9 \\ 0 \end{bmatrix}$$

The bending-moment diagram as shown in Fig. 6.7(g) may be obtained by using Eq. (a) as in Solution (i).

(iii) In solution (ii), shear force P_3 has been found to be zero. This could have been anticipated from the symmetry of the structure and the applied loads. Thus in Fig. 6.7(h), coordinate 3 could be dropped and the solution obtained in terms of coordinates 1 and 2 only, resulting in considerable simplication. Hence,

$$\Delta_{1L} = -\frac{2375}{EI}$$

$$\Delta_{2L} = -\frac{7625}{6EI}$$

$$\delta_{11} = \frac{81}{2EI}$$

$$\delta_{21} = \delta_{12} = \frac{33}{2EI}$$

$$\delta_{22} = \frac{11}{EI}$$

Substituting into Eq. (6.2),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} \frac{81}{2EI} & \frac{33}{2EI} \\ \frac{33}{2EI} & \frac{11}{EI} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{2375}{EI} \\ -\frac{7625}{6EI} \end{bmatrix} = \begin{bmatrix} 29.8 \\ 70.9 \end{bmatrix}$$

These values are the same as obtained previously in Solution (ii).

6.3 SHEAR EQUATIONS FOR RIGID-JOINTED FRAMES

Rigid-jointed building frames sway either to the left or to the right on account of lateral (horizontal) forces and lack of symmetry of the frame or of the applied vertical loads. Whenever the building frames sway, the sum of the shear forces in all the columns at any level must be equal to the total lateral force applied to the frame above that level to satisfy the basic condition of static equilibrium, $\Sigma H = 0$. This condition gives rise to a set of equations known as ***shear equations***. Thus a shear equation expresses the condition of equilibrium between the sum of internal shear forces in all the columns at any level, expressed in terms of the column moments, and the total external horizontal force acting on the frame above the level under consideration. A shear equation can be written for each storey. Hence the number of shear equations is equal to the number of storeys in the building frame.

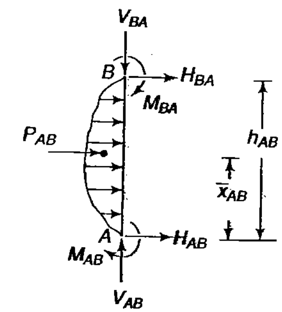


Fig. 6.8

For the derivation of the shear equation, consider the free-body diagram of typical column AB acted upon by end forces and a resultant horizontal force P_{AB} as shown in Fig. 6.8. Taking moments about point A,

$$M_{AB} + M_{BA} + P_{AB}\bar{x}_{AB} + H_{BA}h_{AB} = 0$$

Hence,

$$H_{BA} = -\left[\frac{M_{AB} + M_{BA} + P_{AB}\bar{x}_{AB}}{h_{AB}}\right]$$

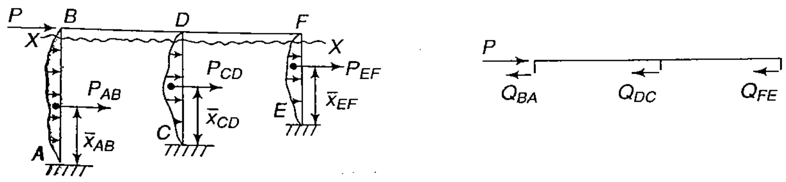
The shear force in the column AB at B,

$$Q_{BA} = H_{BA} = -\left[\frac{M_{AB} + M_{BA} + P_{AB}\bar{x}_{AB}}{h_{AB}}\right] \quad \text{(a)}$$

Similar equations can be derived for other columns of the frame.

Consider now the frame shown in Fig. 6.9(a). Taking a cross-section XX immediately below the top of the frame and considering the equilibrium of the horizontal forces shown in Fig. 6.9(b) acting on the part of the frame above cross-section XX,

$$P = Q_{BA} + Q_{DC} + Q_{FE} = \Sigma Q$$



(a) (b)

Fig. 6.9

Using Eq. (a), the shear equation for the frame shown in Fig. 6.9(a),

$$P = -\left[\frac{M_{AB} + M_{BA} + P_{AB}\bar{x}_{AB}}{h_{AB}} + \frac{M_{CD} + M_{DC} + P_{CD}\bar{x}_{CD}}{h_{CD}} + \frac{M_{EF} + M_{FE} + P_{EF}\bar{x}_{EF}}{h_{EF}}\right] \quad (6.3)$$

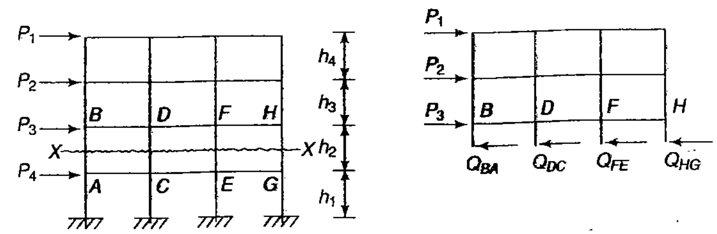
The columns usually do not carry transverse forces at intermediate points. The horizontal forces due to wind and other lateral effect are generally assumed to act at the joints. Hence the shear equation for the common case, in which there are no intermediate forces on the columns, may be written as

$$P = -\left[\frac{M_{AB} + M_{BA}}{h_{AB}} + \frac{M_{CD} + M_{DC}}{h_{CD}} + \frac{M_{EF} + M_{FE}}{h_{EF}}\right] \quad (6.4)$$

The minus sign on the right hand sides of Eqs (6.3) and (6.4) shows that when the horizontal forces act from left to right, thereby producing positive shear force in the columns of the frame, the end moments in the columns are counter-clockwise.

Next, consider the multistorey building frame shown in Fig. 6.10(a). To derive the shear equation for the second storey, consider the equilibrium of the part of the frame above section XX as shown in Fig. 6.10(b). To satisfy the

condition of static equilibrium, $\Sigma H = 0$, the sum of the shear forces in all the columns of the second storey must be equal to the sum of the horizontal forces above the section.



(a) (b)

Fig. 6.10

$$P_1 + P_2 + P_3 = Q_{BA} + Q_{DC} + Q_{FE} + Q_{HG} = \Sigma Q$$

Using Eq. (a),

$$P_1 + P_2 + P_3 = -\frac{1}{h_2}\left[M_{AB} + M_{BA} + M_{CD} + M_{DC} + M_{EF} + M_{FE} + M_{GH} + M_{HG}\right] \quad \text{(b)}$$

The minus sign on the right hand side shows that when the frame sways to the right, the column moments are counter-clockwise.

In general, the shear equation for jth storey may be written as

$$P_j = -\frac{\Sigma M_j}{h_j} \quad (6.5)$$

where P_j = storey shear of jth storey and is given by sum of the horizontal forces above the jth storey

ΣM_j = sum of end couples of all the columns of the jth storey

h_j = height of the jth storey.

6.4 STIFFNESS OF A RIGID JOINT

The stiffness of a structural member has been discussed in Sec. 4.1. The stiffness of a rigid joint depends upon the stiffnesses of the members meeting at the joint. In the context of the stiffness of a rigid joint, the following two types of stiffnesses have to be considered.

6.4.1 Rotational Stiffness

The rotational stiffness of a rigid joint may be defined as the couple required at the joint for unit rotation without translation. Consider, for example, a rigid joint O at which five members meet as shown in Fig. 6.11. Coordinates 1, 2 and 3 have been assigned to the rotations at O, B and D respectively as shown in Fig. 6.11(a). The rotational stiffness k_{11} of the rigid jonit O is equal to the sum of the flexural stiffnesses of the members meeting at joint O. When a unit rotation is given at coordinate 1 without any rotation at coordinates 2 and 3 as shown in Fig. 6.11(b), the rotational stiffness of joint O,

$$k_{11} = \frac{4EI_{OA}}{L_{OA}} + \frac{4EI_{OB}}{L_{OB}} + \frac{4EI_{OC}}{L_{OC}} + \frac{4EI_{OD}}{L_{OD}} + \frac{4EI_{OE}}{L_{OE}}$$

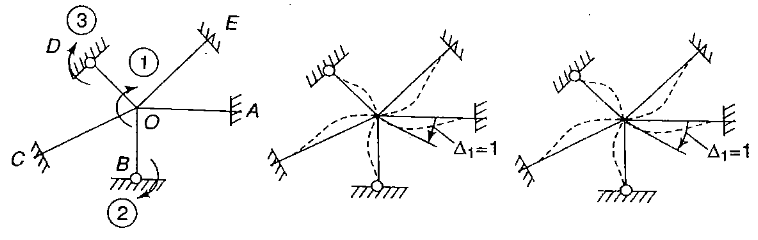


Fig. 6.11

If coordinates 2 and 3 are omitted, thereby permitting rotations at hinge supports B and D as shown in Fig. 6.11(c), the rotational stiffness of the joint O,

$$k_{11} = \frac{4EI_{OA}}{L_{OA}} + \frac{3EI_{OB}}{L_{OB}} + \frac{4EI_{OC}}{L_{OC}} + \frac{3EI_{OD}}{L_{OD}} + \frac{4EI_{OE}}{L_{OE}}$$

6.4.2 Translational Stiffness

The translational (sway) stiffness of a rigid joint in any direction is defined as the force required for a unit translation in the chosen direction without rotation of the joint. As the members are considered inextensible, it is evident that the translational stiffness of a joint will have a finite value only if translation of the joint is possible without change of the length of members meeting at the joint. If the translation of the joint in the chosen direction necessarily requires elongation or contraction of some of the members meeting at the joint, the translational stiffness of the joint in the chosen direction is infinity. It means that the sway along the chosen direction is not possible.

Example 6.7

Determine the rotational stiffness of the rigid joint O shown in Fig. 6.12. If a couple of 1000 kN·m is applied at coordinate 1, calculate the bending moment resisted by each member.

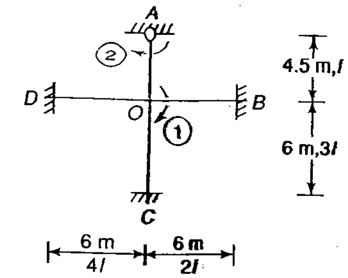


Fig. 6.12

Solution

(i) When a unit rotation is given at coordinate 1 without any rotation at coordinate 2, the rotational stiffness of joint O,

$$k_{11} = \frac{4EI}{4.5} + \frac{4E(2I)}{6} + \frac{4E(3I)}{6} + \frac{4E(4I)}{6}$$

$$= \frac{8}{9}EI + \frac{4}{3}EI + 2EI + \frac{8}{3}EI$$

$$= \frac{62}{9}EI$$

The couple applied at joint O will be shared by the members meeting at the joint in proportion to their flexural stiffnesses. It follows that

$$M_{OA} = \frac{8/9EI}{62/9EI} \times 1000 = 129 \text{ kN·m}$$

$$M_{OB} = \frac{4/3EI}{62/9EI} \times 1000 = 193.5 \text{ kN·m}$$

$$M_{OC} = \frac{2EI}{62/9EI} \times 1000 = 290.3 \text{ kN·m}$$

$$M_{OD} = \frac{8/3EI}{62/9EI} \times 1000 = 387.2 \text{ kN·m}$$

(ii) If coordinate 2 is omitted, thereby permitting rotation at hinge support A, the rotational stiffness of the joint O,

$$k_{11} = \frac{3EI}{4.5} + \frac{4E(2I)}{6} + \frac{4E(3I)}{6} + \frac{4E(4I)}{6}$$

$$= \frac{2}{3}\,EI + \frac{4}{3}\,EI + 2EI + \frac{8}{3}\,EI$$

$$= \frac{20}{3}\,EI$$

The couple applied at joint O will be shared by the members meeting at the joint in proportion to their flexural stiffnesses. It follows that

$$M_{OA} = \frac{2/3EI}{20/3EI} \times 1000 = 100 \text{ kN·m}$$

$$M_{OB} = \frac{4/3EI}{20/3EI} \times 1000 = 200 \text{ kN·m}$$

$$M_{OC} = \frac{2EI}{20/3EI} \times 1000 = 300 \text{ kN·m}$$

$$M_{OD} = \frac{8/3EI}{20/3EI} \times 1000 = 400 \text{ kN·m}$$

Example 6.8

Compute the translational stiffness of joint B in the horizontal direction for the rigid-jointed frame shown in Fig. 6.13(a).

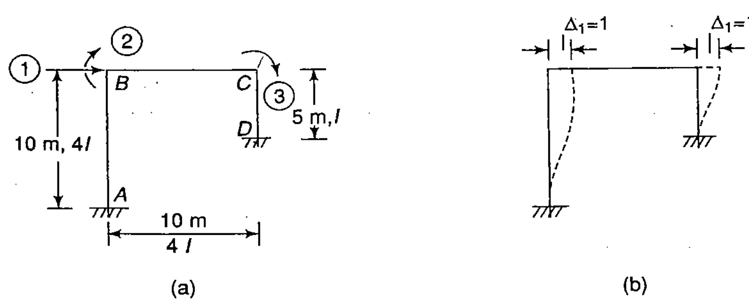


Fig. 6.13

Solution

It may be noted that joint B can translate only in the horizontal direction. The translation of joint B in the vertical direction is not possible because it would lead to a change in the length of member AB. When joint B is given a unit translation in the horizontal direction without rotation of joints B and C, i.e., $\Delta_1 = 1$ and $\Delta_2 = \Delta_3 = 0$, the frame deforms as shown in Fig. 6.13(b). It is evident that the translation of joint B is resisted by the transverse stiffnesses of members AB and DC. Using the value of the transverse stiffness given in Table 4.1, the translational stiffness of joint B is given by the equation

$$k_{11} = \frac{12E(4I)}{10^3} + \frac{12EI}{5^3} = 0.144EI$$

The same result may be obtained by using shear Eq. (6.4) in which the end moments in the columns may be computed by using Table 2.16. It may be noted that the translational stiffness of the joint C is the same as that of joint B.

Example 6.9

Compute the translational stiffness of joint B in the horizontal direction for the rigid-jointed frame shown in Fig. 6.14(a).

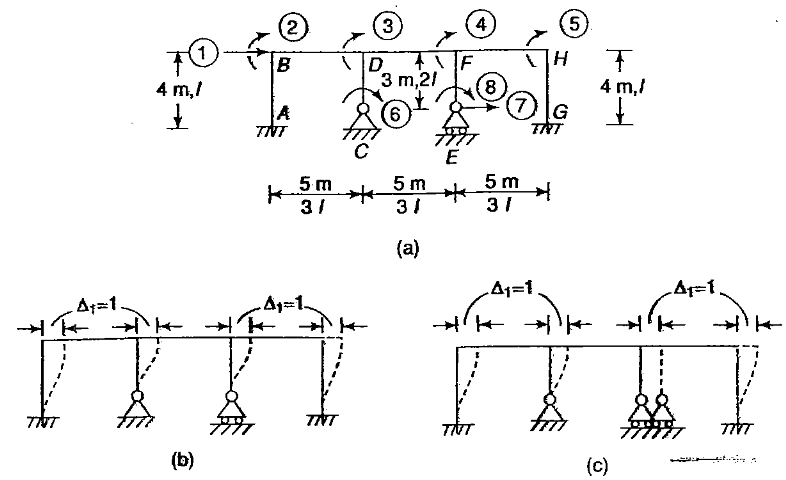


Fig. 6.14

Solution

(i) The translational stiffness of joint B, when a unit displacement is given at coordinate 1 without any displacement at other coordinates (Fig. 6.14(b)),

$$k_{11} = \frac{12EI}{4^3} + \frac{12E(2I)}{3^3} + \frac{12E(2I)}{3^3} + \frac{12EI}{4^3}$$

$$= 2.153EI$$

(ii) If coordinates 6, 7 and 8 are omitted, thereby permitting displacements at supports C and E as shown in Fig. 6.14(c), the translational stiffness of joint B,

$$k_{11} = \frac{12EI}{4^3} + \frac{3E(2I)}{3^3} + \frac{12EI}{4^3} = 0.597EI$$

It may be noted that in this case the sway of joint B is not resisted by member EF because this member moved as a rigid body on account of roller support at E.

Example 6.10

Compute the translational stiffness of joint A in the horizontal direction for the rigid-jointed frame shown in Fig. 6.15(a).

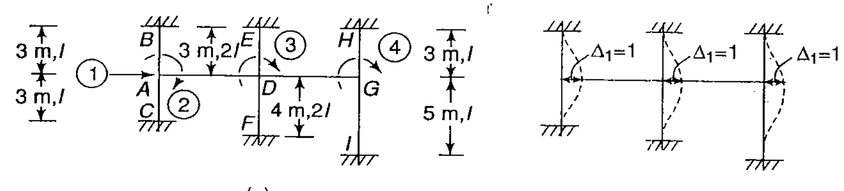


Fig. 6.15

Solution

When joint A is given a unit translation in the horizontal direction without rotation of joints A, D and G, the frame deforms as shown in Fig. 6.15(b). It is evident that the translation of the joint is resisted by the transverse stiffnesses of members AB, AC, DE, DF, GH and GI. Using the value of the transverse stiffness given in Table 4.1, the translational stiffness of joint A is given by the equation

$$k_{11} = \frac{12EI}{3^3} + \frac{12EI}{3^3} + \frac{12E(2I)}{3^3} + \frac{12E(2I)}{4^3} + \frac{12EI}{3^3} + \frac{12EI}{5^3} = 2.693EI$$

It may be noted that the translational stiffness of joint D or G is the same as that of A.

6.5 STIFFNESS MATRIX FOR RECTANGULAR FRAMES

In Chapter 4 the stiffness matrix for single bay single storey rectangular frames has been discusssed in Examples. 4.6, 4.10 and 4.13. In this section the stiffness matrix for multibay multistorey rectangular frames is discussed. Consider, for example, a rigid-jointed rectangualr frame having two bays and five storeys as shown in Fig. 6.16. As discussed in Sec. 1.7, the degree of freedom of the frame is 5(2 + 2) = 20. The independent displacement components are: five horizontal displacements, one at each floor level; and fifteen rotations, one at each joint. Consequently, twenty coordinates as shown in Fig. 6.16 have to be assigned for the development of the stiffness matrix. In order to develop the stiffness matrix, a unit displacement may be given successively at each coordinate without any displacement at other coordinates and evaluating the forces required at all the coordinates. In the following, unit displacements at some of the typical coordinates have been considered and the corresponding

elements of the stiffness matrix have been computed. The remaining elements of the stiffness matrix can be computed in a similar manner.

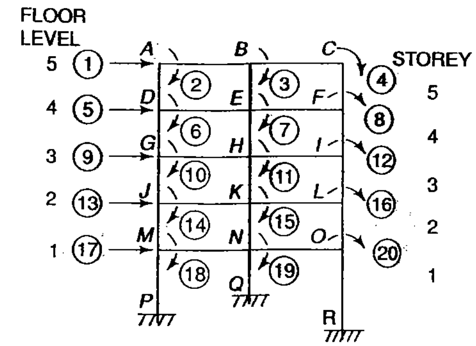
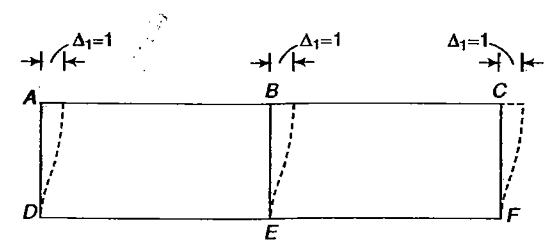


Fig. 6.16

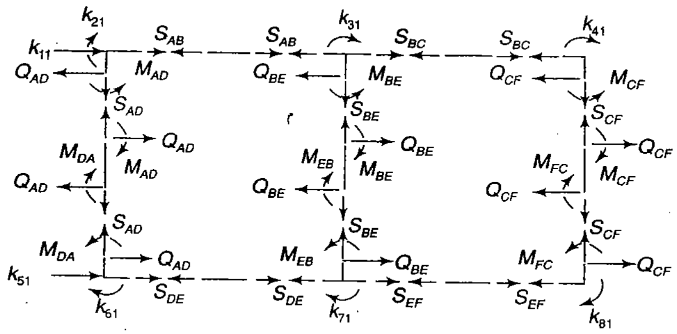
6.5.1 Unit Displacement at Coordinate 1

When a unit displacement is given at coordinate 1 without any displacement at other coordinates, the members of the frame belonging only to the fifth storey are deformed and consequently carry internal forces. The remaining members of the frame do not deform and are, therefore, free from internal forces. The deflected shape of the members of the fifth storey are shown in Fig. 6.17(a). It may be noted that the beams do not bend and, therefore, they carry only axial forces. The free-body diagrams of all the members and joints of the fifth storey are shown in Fig. 6.17(b). In the free-body diagram, the axial forces in the members have been assumed to be tensile. The bending couples and the transverse forces acting at the ends of the columns may be calculated by using Table 2.16.



(a)

Fig. 6.17 (Contd)



(b)

Fig. 6.17

$$M_{CF} = M_{FC} = -\frac{6EI_{CF}}{h_{CF}^2}$$

$$M_{BE} = M_{EB} = -\frac{6EI_{BE}}{h_{BE}^2}$$

$$M_{AD} = M_{DA} = -\frac{6EI_{AD}}{h_{AD}^2} \quad \text{(a)}$$

$$Q_{CF} = \frac{12EI_{CF}}{h_{CF}^3}$$

$$Q_{BE} = \frac{12EI_{BE}}{h_{BE}^3}$$

$$Q_{AD} = \frac{12EI_{AD}}{h_{AD}^3}$$

The axial forces in the members can be evaluated by considering the interaction of the column and the beams. The axial forces in the members are

$$S_{BC} = -Q_{CF} = -\frac{12EI_{CF}}{h_{CF}^3}$$

$$S_{EF} = Q_{CF} = \frac{12EI_{CF}}{h_{CF}^3}$$

$$S_{AB} = S_{BC} - Q_{BE} = -\left[\frac{12EI_{CF}}{h_{CF}^3} + \frac{12EI_{BE}}{h_{BE}^3}\right] \quad \text{(b)}$$

$$S_{DE} = S_{EF} + Q_{BE} = \left[\frac{12EI_{CF}}{h_{CF}^3} + \frac{12EI_{BE}}{h_{BE}^3}\right]$$

$$S_{AD} = S_{BE} = S_{CF} = 0$$

It should be noted that beams *AB* and *BC* are in compression and beams *DE* and *EF* are in tension. Also, the magnitude of the axial force in a beam is equal to the sum of the transverse forces in the columns to the right of it. The columns do not carry axial forces.

Equations (a) and (b) give the internal forces in all the members of the fifth storey. Now, the elements of the stiffness matrix can be evaluated by considering the equilibrium of joints *A* to *F*.

$$k_{11} = Q_{AD} - S_{AB} = \frac{12EI_{AD}}{h_{AD}^3} + \frac{12EI_{CF}}{h_{CF}^3} + \frac{12EI_{BE}}{h_{BE}^3}$$

$$k_{21} = -\frac{6EI_{AD}}{h_{AD}^2} \quad k_{31} = -\frac{6EI_{BE}}{h_{BE}^2} \quad k_{41} = -\frac{6EI_{CF}}{h_{CF}^2} \quad \text{(c)}$$

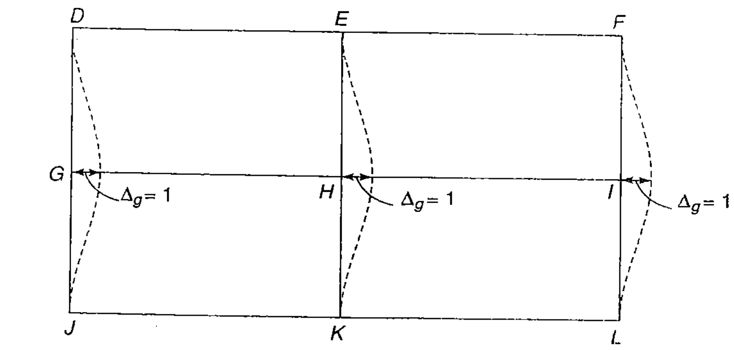
$$k_{51} = -Q_{AD} - S_{DE} = -\left[\frac{12EI_{AD}}{h_{AD}^3} + \frac{12EI_{CF}}{h_{CF}^3} + \frac{12EI_{BE}}{h_{BE}^3}\right]$$

$$k_{61} = -\frac{6EI_{AD}}{h_{AD}^2} \quad k_{71} = -\frac{6EI_{BE}}{h_{BE}^2} \quad k_{81} = -\frac{6EI_{CF}}{h_{CF}^2}$$

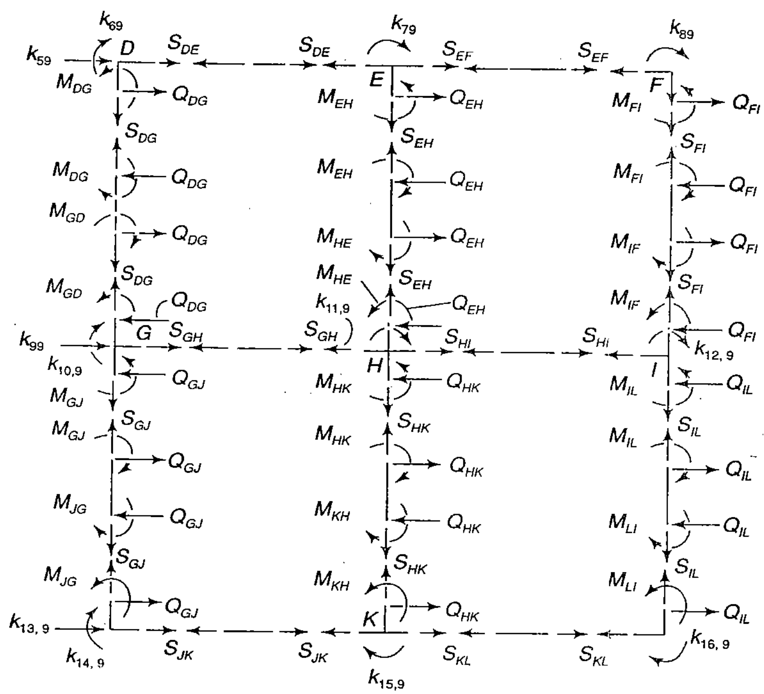
6.5.2 Unit Displacement at Coordinate 9

When a unit displacement is given at coordinate 9 without any displacement at other coordiantes, the members of the frame belonging to the third and fourth storeys only are deformed and consequently carry internal forces. The remaining members of the frame do not deform and are, thereforce, free from internal forces. The deflected shape of the members of the third and fourth storeys are shown in Fig. 6.18(a). The free-body diagrams of all the members and joints of the third and fourth storeys are shown in Fig. 6.18(b). The internal forces in the members of the third and fourth storeys may be evaluated as in Sec. 6.5.1. Therefore, the elements of the stiffness matrix may be computed by considering the equilibrium of joints *D* to *L*.

$$k_{59} = -\left[\frac{12EI_{DG}}{h_{DG}^3} + \frac{12EI_{EH}}{h_{EH}^3} + \frac{12EI_{FI}}{h_{FI}^3}\right]$$



(a)



(b)

Fig. 6.18

$$k_{69} = \frac{6EI_{DG}}{h_{DG}^2} \quad k_{79} = \frac{6EI_{EH}}{h_{EH}^2} \quad k_{89} = \frac{6EI_{FI}}{h_{FI}^2} \tag{d}$$

$$k_{99} = \left[\frac{12EI_{DG}}{h_{DG}^3} + \frac{12EI_{GJ}}{h_{GJ}^3} + \frac{12EI_{EH}}{h_{EH}^3} + \frac{12EI_{HK}}{h_{HK}^3} + \frac{12EI_{FI}}{h_{FI}^3} + \frac{12EI_{IL}}{h_{IL}^3}\right]$$

$$k_{10,9} = \frac{6EI_{DG}}{h_{DG}^2} - \frac{6EI_{GJ}}{h_{GJ}^2} \quad k_{11,9} = \frac{6EI_{EH}}{h_{EH}^2} - \frac{6EI_{HK}}{h_{HK}^2}$$

$$k_{12,9} = \frac{6EI_{FI}}{h_{FI}^2} - \frac{6EI_{IL}}{h_{IL}^2}$$

$$k_{13,9} = -\left[\frac{12EI_{GJ}}{h_{GJ}^3} + \frac{12EI_{HK}}{h_{HK}^3} + \frac{12EI_{IL}}{h_{IL}^3}\right]$$

$$k_{14,9} = -\frac{6EI_{GJ}}{h_{GJ}^2} \quad k_{15,9} = -\frac{6EI_{HK}}{h_{HK}^2}$$

$$k_{16,9} = -\frac{6EI_{IL}}{h_{IL}^2}$$

6.5.3 Unit Displacement at Coordinate 17

When a unit displacement is given at coordinate 17 without any displacement at other coordinates, the members of the frame belonging to the first and second storeys only are deformed and consequently carry internal forces. The remaining members of the frame do not deform and are, therefore, free from internal forces. The deflected shape of the members of the first and second storeys are shown in Fig. 6.19(a). The free-body diagrams of all the members and joints of the first and second storeys are shown in Fig. 6.19(b). The internal forces in the members of the first and second storeys may be evaluated as in Sec. 6.5.1. Therefore, the elements of the stiffness matrix may be computed by considering the equilibrium of joints J to O.

$$k_{13,17} = -\left[\frac{12EI_{JM}}{h_{JM}^3} + \frac{12EI_{KN}}{h_{KN}^3} + \frac{12EI_{LO}}{h_{LO}^3}\right]$$

$$k_{14,17} = \frac{6EI_{JM}}{h_{JM}^2} \quad k_{15,17} = \frac{6EI_{KN}}{h_{KN}^2} \quad k_{16,17} = \frac{6EI_{LO}}{h_{LO}^2}$$

$$k_{17,17} = \frac{12EI_{JM}}{h_{JM}^3} + \frac{12EI_{MP}}{h_{MP}^3} + \frac{12EI_{KN}}{h_{KN}^3} + \frac{12EI_{NQ}}{h_{NQ}^3} + \frac{12EI_{LO}}{h_{LO}^3} + \frac{12EI_{OR}}{h_{OR}^3}$$

$$k_{18,17} = \frac{6EI_{JM}}{h_{JM}^2} - \frac{6EI_{MP}}{h_{MP}^2} \quad k_{19,17} = \frac{6EI_{KN}}{h_{KN}^2} - \frac{6EI_{NQ}}{h_{NQ}^2}$$

$$k_{20,17} = \frac{6EI_{LO}}{h_{LO}^2} - \frac{6EI_{OR}}{h_{OR}^2}$$

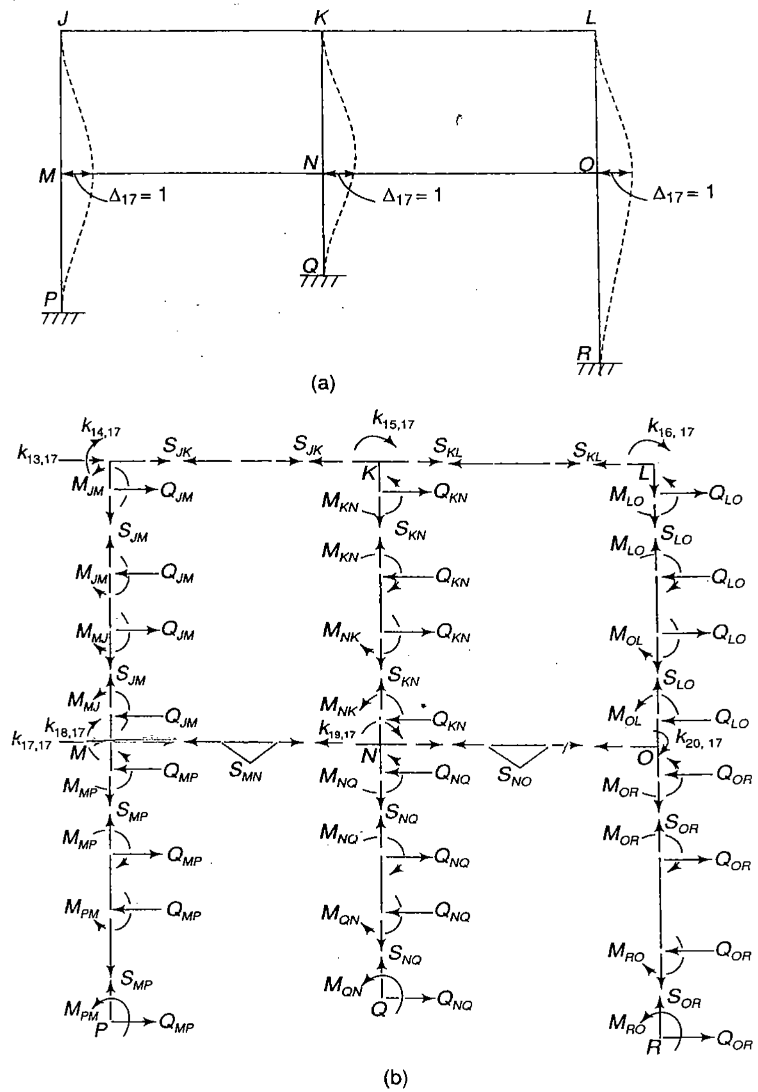


Fig. 6.19

6.5.4 Unit Displacement at Coordinate 11

When a unit displacement (rotation) is given at coordinate 11 without any displacement at other coordinates, only those members which meet at joint H are bent as shown in Fig. 6.20(a). The free-body diagrams of the members and joints belonging to the third and fourth storeys are shown in Fig. 6.20(b).

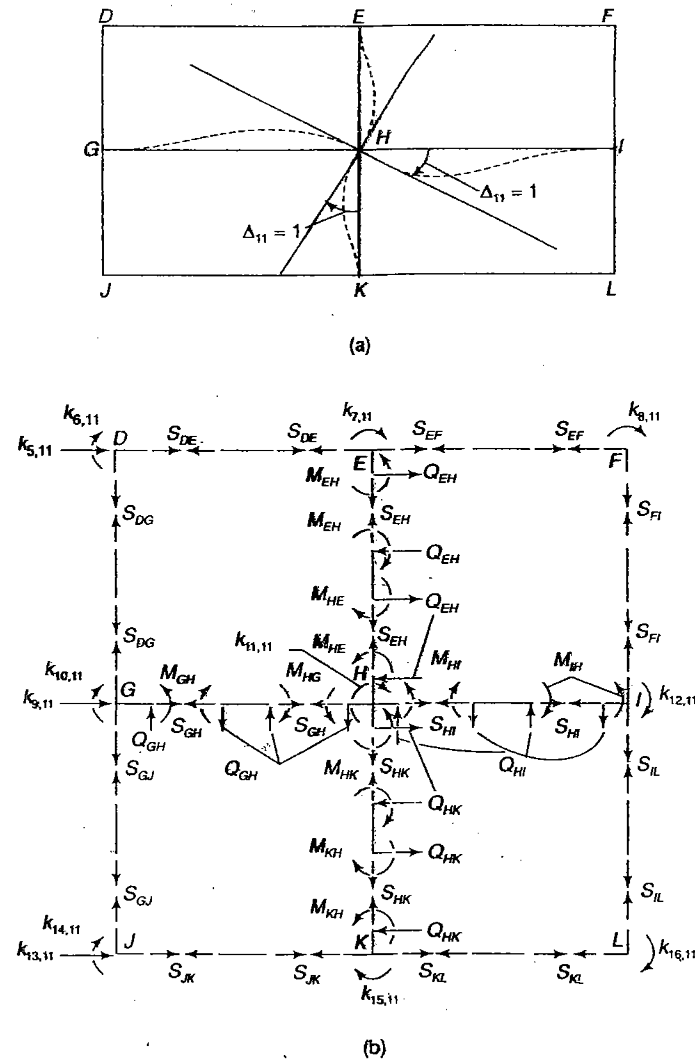


Fig. 6.20

The bending couples and the transverse forces acting at the ends of members GH, HI, EH and HK may be calculated by using Table 2.16,

$$M_{GH} = \frac{2EI_{GH}}{L_{GH}} \qquad M_{HG} = \frac{4EI_{GH}}{L_{GH}} \qquad Q_{GH} = \frac{6EI_{GH}}{L^2_{GH}}$$

$$M_{HI} = \frac{4EI_{HI}}{L_{HI}} \qquad M_{IH} = \frac{2EI_{HI}}{L_{HI}} \qquad Q_{HI} = \frac{6EI_{HI}}{L_{HI}}$$

$$M_{EH} = \frac{2EI_{EH}}{h_{EH}} \qquad M_{HE} = \frac{4EI_{EH}}{h_{EH}} \qquad Q_{EH} = \frac{6EI_{EH}}{h^2_{EH}}$$

$$M_{HK} = \frac{4EI_{HK}}{h_{HK}} \qquad M_{KH} = \frac{2EI_{HK}}{h_{HK}} \qquad Q_{HK} = \frac{6EI_{HK}}{h^2_{HK}}$$

Now the axial forces in the beams belonging to the third and fourth storeys may be computed by considering interaction of members. The axial forces in the columns need not be computed as they do not enter into the expression for the elements of the stiffness matrix.

$$S_{EF} = S_{HI} = S_{KL} = 0$$

$$S_{DE} = \frac{6EI_{EH}}{h^2_{EH}}$$

$$S_{GH} = \frac{6EI_{HK}}{h^2_{HK}} - \frac{6EI_{EH}}{h^2_{EH}}$$

$$S_{JK} = -\frac{6EI_{HK}}{h^2_{HK}}$$

Next, the elements of the stiffness matrix can be evaluated by considering the equilibrium of joints D to L.

$$k_{5,11} = -\frac{6EI_{EH}}{h^2_{EH}} \qquad k_{6,11} = k_{8,11} = 0$$

$$k_{7,11} = \frac{2EI_{EH}}{h_{EH}} \qquad k_{9,11} = \frac{6EI_{EH}}{h^2_{EH}} - \frac{6EI_{HK}}{h^2_{HK}}$$

$$k_{10,11} = \frac{2EI_{GH}}{L_{GH}} \tag{f}$$

$$k_{11,11} = \frac{4EI_{GH}}{L_{GH}} + \frac{4EI_{EH}}{h_{EH}} + \frac{4EI_{HI}}{L_{HI}} + \frac{4EI_{HK}}{h_{HK}}$$

$$k_{12,11} = \frac{2EI_{HI}}{L_{HI}}$$

$$k_{13,11} = \frac{6EI_{HK}}{h^2_{HK}} \qquad k_{14,11} = k_{16,11} = 0$$

$$k_{15,11} = \frac{2EI_{HK}}{h_{HK}} \tag{f}$$

In the preceding discussion, a rigid-jointed rectangular frame with two bays and five storeys has been discussed. It may be noted that there are two types of independent displacement components in rigid-jointed frames:

(a) horizontal (sway) displacements, one at each floor level, and

(b) rotations, one at each joint.

General expressions for the elements of the stiffness matrix can now be written down by generalising the expressions given by Eqs 1(c) to (f).

In Fig. 6.21, a part of a rigid-jointed plane frame with several bays and storeys is shown. Coordinates i, j and k corespond to horizontal (sway) displacements at the $(m + 1)$th, mth and $(m - 1)$th floores respectively. Coordinate q corresponds to the rotations at a typical intermediate joint lying on the mth floor.

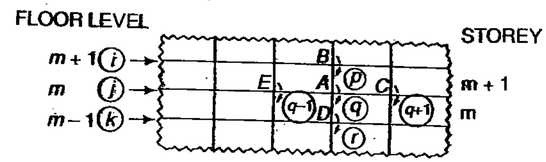


Fig. 6.21

6.5.5 Elements of the Stiffness Matrix due to a Sway

When a unit displacement is given at coordinate j without any displacement at the other coordinates, the columns belonging to the mth and $(m + 1)$th storeys are bent in the manner indicated in Fig. 6.18(a). Consequently, the elements of the stiffness matrix at the sway coordinates i, j and k due to a unit horizontal displacement at coordinate j may be written as

$$k_{jj} = \sum^{m\text{th storey}} \frac{12EI}{h^3} + \sum^{(m+1)\text{th storey}} \frac{12EI}{h^3}$$

$$k_{ij} = -\sum^{(m+1)\text{th storey}} \frac{12EI}{h^3} \tag{6.6}$$

$$k_{kj} = -\sum^{m\text{th storey}} \frac{12EI}{h^3}$$

where $12EI/h^3$ = transverse stiffness of a column.

The summation has to be carried out so as to include all columns belonging to the storey indicated over the summation sign. It may be noted that $(k_{ij} + k_{jj} + k_{kj}) = 0$ as required by the condition of static equilibrium. It may also be noted that the elements of the stiffness matrix at the remaining sway coordinates are zero.

The elements of the stiffness matrix at the rotational coordinates q, p and r due to a unit horizontal displacement at coordiate j may be written as

$$\begin{aligned} k_{qj} &= \frac{6EI_{AB}}{h_{AB}^2} - \frac{6EI_{AD}}{h_{AD}^2} \\ k_{pj} &= \frac{6EI_{AB}}{h_{AB}^2} \\ k_{rj} &= -\frac{6EI_{AD}}{h_{AD}^2} \end{aligned} \tag{6.7}$$

It may be noted that $k_{qj} = (k_{pj} + k_{rj})$. It should also be noted that the elements of the stiffness matrix at all the rotational coordinates lying on the $(m+1)$th, mth and $(m-1)$th floors are non-zero. The elements of the stiffness matrix at the remaining rotational coordinates are zero.

6.5.6 Elements of the Stiffness Matrix due to a Rotation

When a unit displacement is given at coordianate q without any displacement at the other coordinates, the members meeting at joint A are bent in the manner indicated in Fig. 6.20(a). Consequently, the elements of the stiffness matrix at the sway coordinates i, j and k due to a unit rotation at coordinate q may be written as

$$\begin{aligned} k_{jq} &= \frac{6EI_{AB}}{h_{AB}^2} - \frac{6EI_{AD}}{h_{AD}^2} \\ k_{iq} &= -\frac{6EI_{AB}}{h_{AB}^2} \\ k_{kq} &= \frac{6EI_{AD}}{h_{AD}^2} \end{aligned} \tag{6.8}$$

It may be noted that $(k_{iq} + k_{jq} + k_{kq}) = 0$ as required by the condition of static equilibrium. It may also be noted that the elements of the stiffness matrix at the remaining sway coordinates are zero.

The elements of the stiffness matrix at the rotational coordinates q, p, r, $(q-1)$ and $(q+1)$ due to a unit rotation at coordinate q may be written as

$$\begin{aligned} k_{qq} &= \frac{4EI_{AB}}{h_{AB}} + \frac{4EI_{AC}}{L_{AC}} + \frac{4EI_{AD}}{h_{AD}} + \frac{4EI_{AE}}{L_{AE}} \\ k_{pq} &= \frac{2EI_{AB}}{h_{AB}} \qquad k_{rq} = \frac{2EI_{AD}}{h_{AD}} \\ k_{(q-1),q} &= \frac{2EI_{AE}}{L_{AE}} \qquad k_{(q+1),q} = \frac{2EI_{AC}}{L_{AC}} \end{aligned} \tag{6.9}$$

It may be noted that the elements of the stiffness matrix at the remaining rotational coordinates are zero.

Using Eqs (6.6) to (6.9), the stiffness matrix for a rectangular rigid jointed plane frame having any number of bays and storeys can be developed. It may be noted that the stiffness matrix does not change if the sway coordinates 1, 5, 9, 13 and 17 in Fig. 6.16 are located at joints, C, F, I, L and O, respectively and are directed towards right. In fact, these coordinates can be located at any point on the respective floor level. For instance, coordinate 1 can be located at any point on line AC without any change in the stiffness matrix.

Example 6.11

Determine the elements of the stiffness matrix for the portal frame with reference to the coordinates shown in Fig. 6.22(a).

Solution

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 to 6 and determining the forces required at all the coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at other coordinates as shown in Fig. 6.22(b). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$\begin{aligned} k_{11} &= \frac{4E(4I)}{10} = 1.600EI \\ k_{21} &= -\frac{6E(4I)}{10^2} = -0.240EI \\ k_{31} &= \frac{2E(4I)}{10} = 0.800EI \\ k_{41} &= k_{51} = k_{61} = 0 \end{aligned}$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at other coordinates as shown in Fig. 6.22(c). Compute the forces at all the coordinates using Eqs (6.6) and (6.7).

$$\begin{aligned} k_{12} &= -\frac{6E(4I)}{10^2} = -0.240EI \\ k_{22} &= \frac{12E(4I)}{10^3} + \frac{12EI}{5^3} = 0.144EI \end{aligned}$$

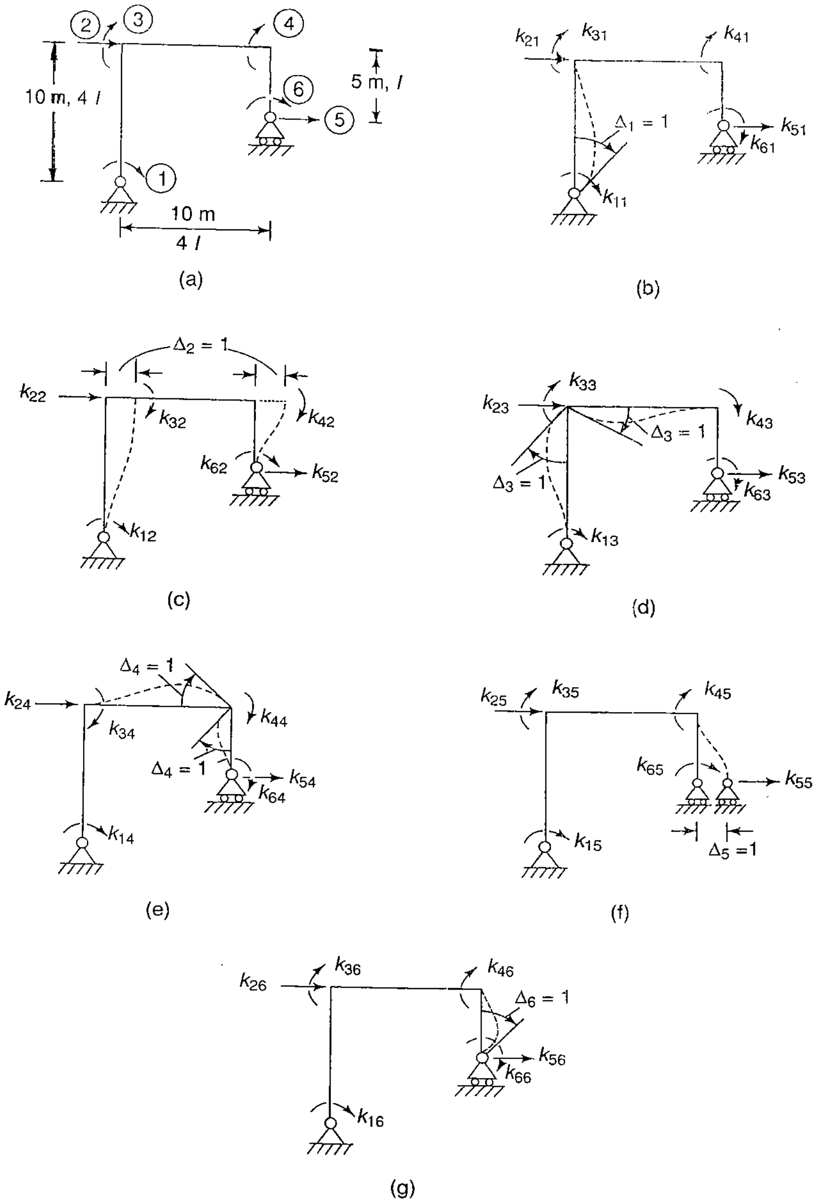


Fig. 6.22

$$k_{32} = -\frac{6E(4I)}{10^2} = -0.240EI$$

$$k_{42} = -\frac{6EI}{5^2} = -0.240EI$$

$$k_{52} = -\frac{12EI}{5^3} = -0.096EI$$

$$k_{62} = -\frac{6EI}{5^2} = 0.240EI$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at other coordinates as shown in Fig. 6.22(d). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{13} = \frac{2E(4I)}{10} = 0.800EI$$

$$k_{23} = -\frac{6E(4I)}{10^2} = -0.240EI$$

$$k_{33} = \frac{4E(4I)}{10} + \frac{4E(4I)}{10} = 3.200EI$$

$$k_{43} = \frac{2E(4I)}{10} = 0.800EI$$

$$k_{53} = k_{63} = 0$$

To generate the fourth column of stiffness matrix, give a unit displacement at coordinate 4 without any displacement at other coordinates as shown in Fig. 6.22(e). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{14} = 0$$

$$k_{24} = -\frac{6EI}{5^2} = -0.240EI$$

$$k_{34} = \frac{2E(4I)}{10} = 0.800EI$$

$$k_{44} = \frac{4E(4I)}{10} + \frac{4EI}{5} = 2.400EI$$

$$k_{54} = \frac{6EI}{5^2} = 0.240EI$$

$$k_{64} = \frac{2EI}{5} = 0.400EI$$

To generate the fifth column of the stiffness matrix, give a unit displacement at coordinate 5 without any displacement at other coordinates as shown in Fig. 6.22(f). Compute the forces at all the coordinates using Eqs (6.6) and (6.7).

$$k_{15} = 0$$

$$k_{25} = -\frac{12EI}{5^3} = -0.096EI$$

$$k_{35} = 0$$

$$k_{45} = \frac{6EI}{5^2} = 0.240EI$$

$$k_{55} = \frac{12EI}{5^3} = 0.096EI$$

$$k_{65} = \frac{6EI}{5^2} = 0.240EI$$

To generate the sixth column of the stiffness matrix, give a unit displacement at coordinate 6 without any displacement at other coordinates as shown in Fig. 6.22(g). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{16} = 0$$

$$k_{26} = -\frac{6EI}{5^2} = -0.240EI$$

$$k_{36} = 0$$

$$k_{46} = \frac{2EI}{5} = 0.400EI$$

$$k_{56} = \frac{6EI}{5^2} = 0.240EI$$

$$k_{66} = \frac{4EI}{5} = 0.800EI$$

Thus all the elements of the stiffness matrix have been determined. It may be noted that the resulting stiffness matrix is symmetrical.

Example 6.12

Determine the elements of the stiffness matrix for the portal frame with reference to the coordinates shown in Fig. 6.23(a).

Solution

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 to 5 and determining the forces required at all the coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at otehr coordinates as shown in Fig. 6.23(b). Compute the forces at all the coordinates using Eqs (6.6) and (6.7).

$$k_{11} = \frac{12E(4I)}{10^3} + \frac{12EI}{5^3} + \frac{12EI}{5^3} + \frac{12E(4I)}{10^3}$$
$$= 0.288EI$$

$$k_{21} = -\frac{6E(4I)}{10^2} = -0.240EI$$

$$k_{31} = -\frac{6EI}{5^2} = -0.240EI$$

$$k_{41} = -\frac{6EI}{5^2} = -0.240EI$$

$$k_{51} = -\frac{6EI(4I)}{10^2} = -0.240EI$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at other coordinates as shown in Fig. 6.23(c). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{12} = -\frac{6E(4I)}{10^2} = -0.240EI$$

$$k_{22} = \frac{4E(4I)}{10} + \frac{4EI}{5} = 2.400EI$$

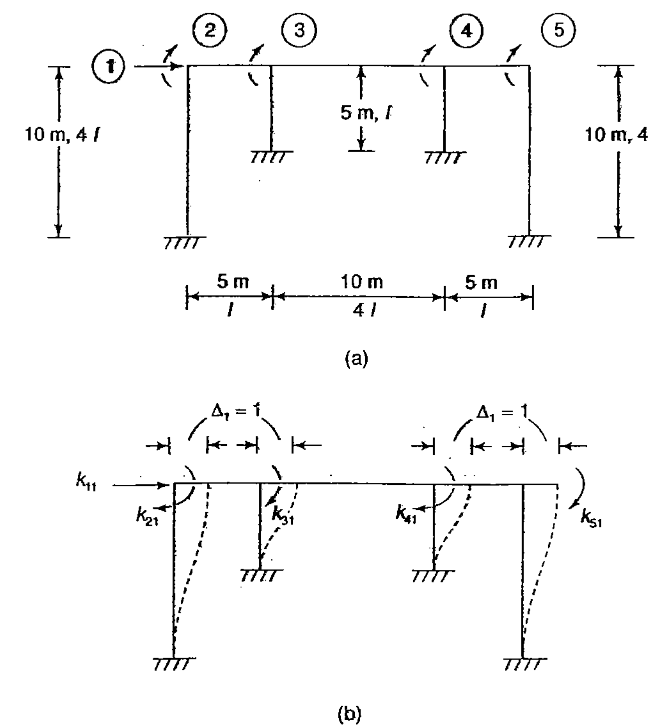


Fig. 6.23 (Contd)

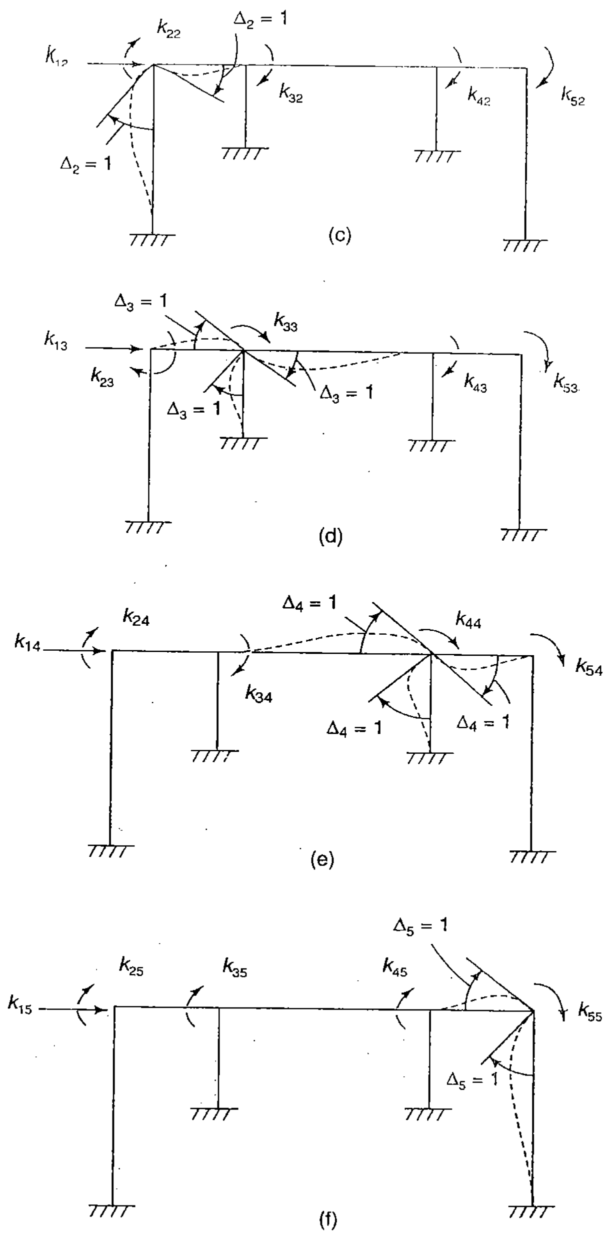


Fig. 6.23

$$k_{32} = \frac{2EI}{5} = 0.400EI$$

$$k_{42} = k_{52} = 0$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at other coordinates as shown in Fig. 6.23(d). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{13} = -\frac{6EI}{5^2} = -0.240EI$$

$$k_{23} = \frac{2EI}{5} = 0.400EI$$

$$k_{33} = \frac{4EI}{5} + \frac{4E(4I)}{10} + \frac{4EI}{5} = 3.200EI$$

$$k_{43} = \frac{2E(4I)}{10} = 0.800EI$$

$$k_{53} = 0$$

To generate the fourth column of the stiffness matrix, give a unit displacement at coordinate 4 without any displacement at other coordianates as shown in Fig. 6.23(e). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{14} = -\frac{6EI}{5^2} = -0.240EI$$

$$k_{24} = 0$$

$$k_{34} = \frac{2E(4I)}{10} = 0.800EI$$

$$k_{44} = \frac{4E(4I)}{10} + \frac{4EI}{5} + \frac{4EI}{5} = 3.200EI$$

$$k_{54} = \frac{2EI}{5} = 0.400EI$$

To generate the fifth column of the stiffness matrix, give a unit displacement at coordinate 5 without any displacement at other coordinates as shown in Fig. 6.23(f). Compute the forces at all the the coordinates using Eqs (6.8) and (6.9).

$$k_{15} = -\frac{6E(4I)}{10^2} = -0.240EI$$

$$k_{25} = 0$$

$$k_{35} = 0$$

$$k_{45} = \frac{2EI}{5} = 0.400EI$$

$$k_{55} = \frac{4EI}{5} + \frac{4EI}{10} = 2.400EI$$

Thus all elements of the stiffness matrix have been determined. It may be noted that the resulting stiffness matrix is symmetrical.

Example 6.13

Determine the elements of the stiffness matrix for the portal frame with reference to the coordinates shown in Fig. 6.24(a).

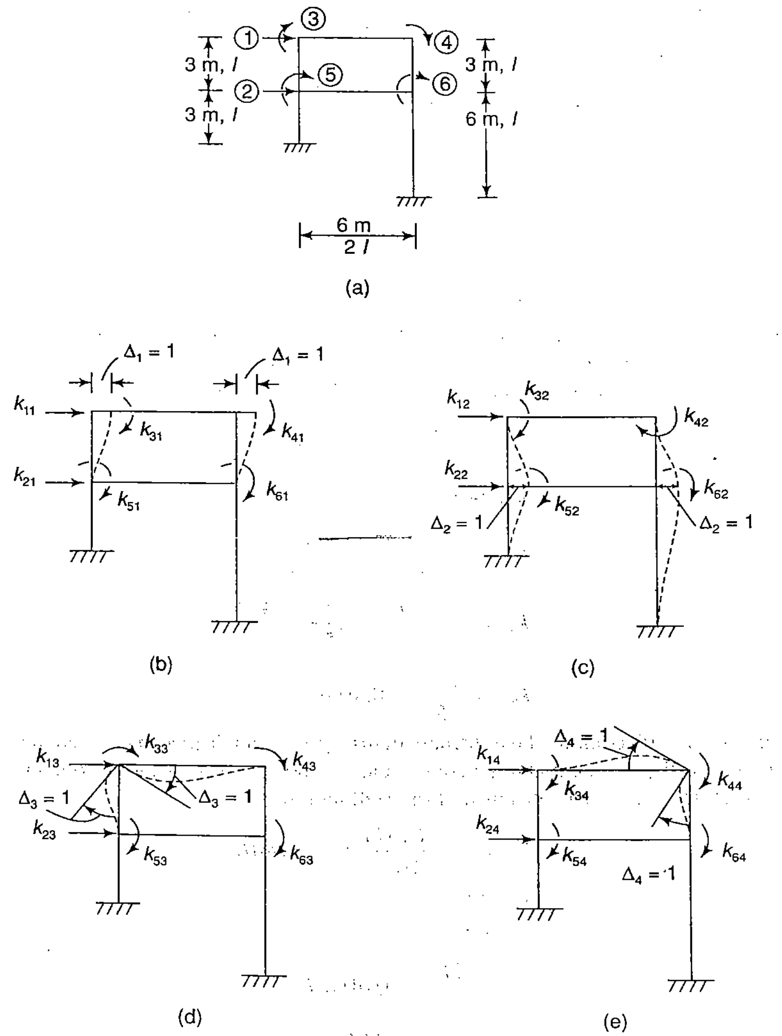


Fig. 6.24 (Contd)

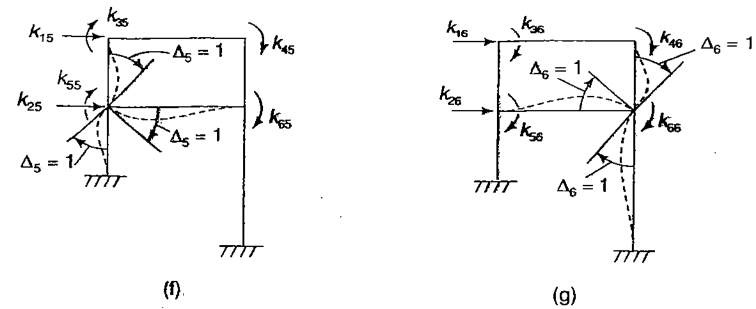


Fig. 6.24

Solution

The stiffness matrix can be developed giving a unit displacement successively at coordinates 1 to 6 and determining the forces required at all the coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at other coordinates as shown in Fig. 6.24(b) and compute the forces at all the coordinates. Using Eqs (6.6) and (6.7),

$$k_{11} = \frac{12EI}{3^3} + \frac{12EI}{3^3} = \frac{8EI}{9}$$

$$k_{21} = -\frac{12EI}{3^3} - \frac{12EI}{3^3} = -\frac{8EI}{9}$$

$$k_{31} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

$$k_{41} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

$$k_{51} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

$$k_{61} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at other coordinates as shown in Fig. 6.24(c) and compute the forces at all the coordinates. Using Eqs (6.6) and (6.7),

$$k_{12} = -\frac{12EI}{3^3} - \frac{12EI}{3^3} = -\frac{8EI}{9}$$

$$k_{22} = \frac{12EI}{3^3} + \frac{12EI}{3^3} + \frac{12EI}{3^3} + \frac{12EI}{6^3} = \frac{25EI}{18}$$

$$k_{32} = \frac{6EI}{3^2} = \frac{2EI}{3}$$

$$k_{42} = \frac{6EI}{3^2} = \frac{2EI}{3}$$

$$k_{52} = \frac{6EI}{3^2} - \frac{6EI}{3^2} = 0$$

$$k_{62} = \frac{6EI}{3^2} - \frac{6EI}{6^2} = \frac{EI}{2}$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at other coordinates as shown in Fig. 6.24(d) and compute the forces at all the coordinates. Using Eqs (6.8) and (6.9),

$$k_{13} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

$$k_{23} = \frac{6EI}{3^2} = \frac{2EI}{3}$$

$$k_{33} = \frac{4EI}{3} + \frac{4E(2I)}{6} = \frac{8EI}{3}$$

$$k_{43} = \frac{2E(2I)}{6} = \frac{2EI}{3}$$

$$k_{53} = \frac{2EI}{3}$$

$$k_{63} = 0$$

To generate the fourth column of the stiffness matrix give a unit displacement at coordinate 4 without any displacement at other coordinates as shown in Fig. 6.24(e) and compute the forces at all the coordinates. Using Eqs (6.8) and (6.9),

$$k_{14} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

$$k_{24} = \frac{6EI}{3^2} = \frac{2EI}{3}$$

$$k_{34} = \frac{2E(2I)}{6} = \frac{2EI}{3}$$

$$k_{44} = \frac{4E(2I)}{6} + \frac{4EI}{3} = \frac{8EI}{3}$$

$$k_{54} = 0$$

$$k_{64} = \frac{2EI}{3}$$

To generate the fifth column of the stiffness matrix, give a unit displacement at coordinates 5 without any displacement at other coordinates as shown in Fig. 6.24(f) and compute the forces at all the coordinates. Using Eqs (6.8) and (6.9),

$$k_{15} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

$$k_{25} = \frac{6EI}{3^2} - \frac{6EI}{3^2} = 0$$

$$k_{35} = \frac{2EI}{3}$$

$$k_{45} = 0$$

$$k_{55} = \frac{4EI}{3} + \frac{4EI}{3} + \frac{4E(2I)}{6} = 4EI$$

$$k_{65} = \frac{2E(2I)}{6} = \frac{2EI}{3}$$

To generate the sixth column of the stiffness matrix, give a unit displacement at coordinate 6 without any displacement at other coordinates as shown in Fig. 6.24(g) and compute the forces at all the coordinates. Using Eqs (6.8) and (6.9),

$$k_{16} = -\frac{6EI}{3^2} = -\frac{2EI}{3}$$

$$k_{26} = \frac{6EI}{3^2} - \frac{6EI}{6^2} = \frac{EI}{2}$$

$$k_{36} = 0$$

$$k_{46} = \frac{2EI}{3}$$

$$k_{56} = \frac{2E(2I)}{6} = \frac{2EI}{3}$$

$$k_{66} = \frac{4EI}{3} + \frac{4EI}{6} + \frac{4E(2I)}{6} = \frac{10EI}{3}$$

Thus all the elements of the stiffness matrix have been determined. It may be noted that resulting stiffness matrix is symmetrical.

Example 6.14

Determine the elements of the stiffness matrix for the portal frame with reference to the coordinates shown in Fig. 6.25(a).

Solution

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 to 8 without any displacement at other coordinates. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at other coordinates as shown in Fig. 6.25(b). Compute the forces at all the coordinates using Eqs (6.6) and (6.7).

$$k_{11} = \frac{12EI}{3^3} + \frac{12E(2I)}{3^3} + \frac{12EI}{3^3} = 1.78EI$$

$$k_{21} = -\frac{6EI}{3^2} = -0.67EI$$

$$k_{31} = -\frac{6E(2I)}{3^2} = -1.33EI$$

$$k_{41} = -\frac{6EI}{3^2} = -0.67EI$$

$$k_{51} = -\left[\frac{12EI}{3^3} + \frac{12E(2I)}{3^3} + \frac{12EI}{3^3}\right] = -1.78EI$$

$$k_{61} = -\frac{6EI}{3^2} = -0.67EI$$

$$k_{71} = -\frac{6E(2I)}{3^2} = -1.33EI$$

$$k_{81} = -\frac{6EI}{3^2} = -0.67EI$$

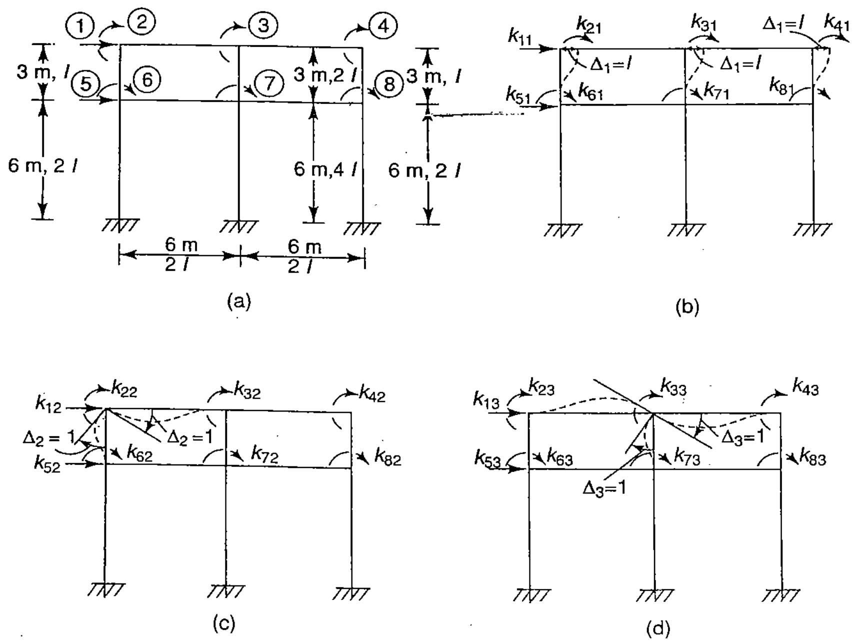


Fig. 6.25 (Contd)

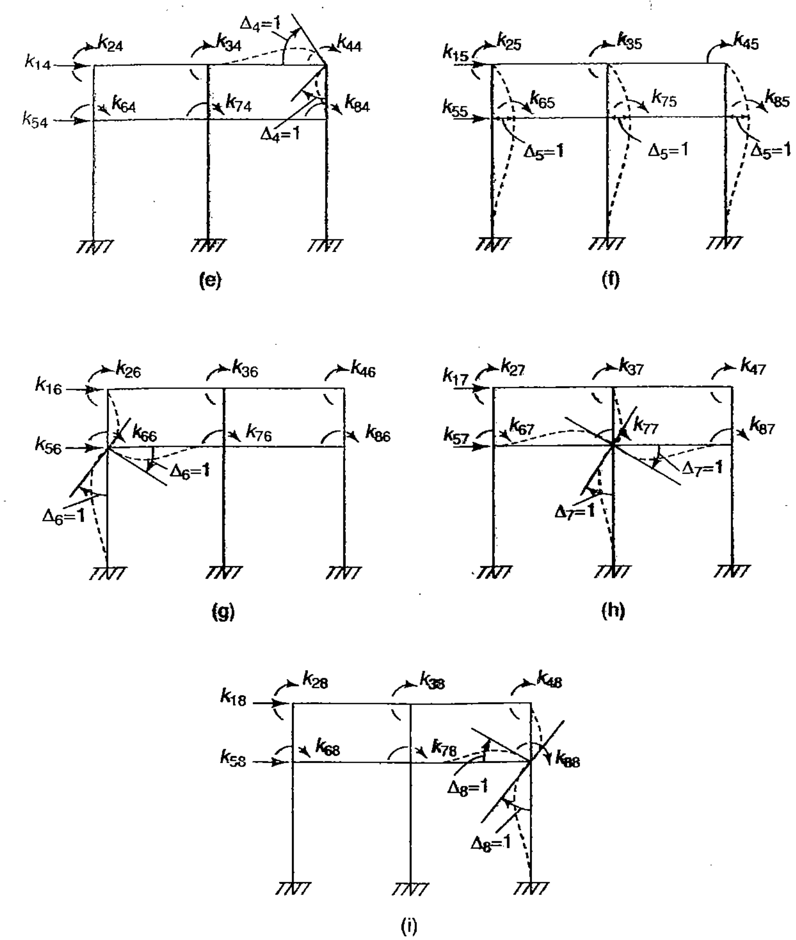


Fig. 6.25

To generate the second column of the stiffness matrix, give a unit displacement at coordiante 2 without any displacement at other coordinates as shown in Fig. 6.25(c). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{12} = -\frac{6EI}{3^2} = -0.67EI$$

$$k_{22} = \frac{4E(2I)}{6} + \frac{4EI}{3} = 2.67EI$$

$$k_{32} = \frac{2E(2I)}{6} = 0.67EI$$

$$k_{42} = 0$$

$$k_{52} = \frac{6EI}{3^2} = 0.67EI$$

$$k_{62} = \frac{2EI}{3} = 0.67EI$$

$$k_{72} = k_{82} = 0$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at other coordinates as shown in Fig. 6.25(d). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{13} = -\frac{6E(2I)}{3^2} = -1.33EI$$

$$k_{23} = \frac{2E(2I)}{6} = 0.67EI$$

$$k_{33} = \frac{4E(2I)}{6} + \frac{4E(2I)}{6} + \frac{4E(2I)}{3} = 5.33EI$$

$$k_{43} = \frac{2E(2I)}{6} = 0.67EI$$

$$k_{53} = \frac{6E(2I)}{3^2} = 1.33EI$$

$$k_{63} = 0$$

$$k_{73} = \frac{2E(2I)}{3} = 1.33EI$$

$$k_{83} = 0$$

To generate the fourth column of the stiffness matrix, give a unit displacement at coordinate 4 without any displacement at other coordinates as shown in Fig. 6.25(e). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{14} = -\frac{6EI}{3^2} = -0.67EI$$

$$k_{24} = 0$$

$$k_{34} = \frac{2E(2I)}{6} = 0.67EI$$

$$k_{44} = \frac{4E(2I)}{6} + \frac{4EI}{3} = 2.67EI$$

$$k_{54} = \frac{6EI}{3^2} = 0.67EI$$

$$k_{64} = k_{74} = 0$$

$$k_{84} = \frac{2EI}{3} = 0.67EI$$

To generate the fifth column of the stiffness matrix, gie a unit displacement at coordinate 5 without any displacement at other coordinates as shown in Fig. 6.25(f). Compute the forces at all the coordinates using Eqs (6.6) and (6.7).

$$k_{15} = -\left[\frac{12EI}{3^3} + \frac{12E(2I)}{3^3} + \frac{12EI}{3^3}\right] = -1.78EI$$

$$k_{25} = \frac{6EI}{3^2} = 0.67EI$$

$$k_{35} = \frac{6E(2I)}{3^2} = 1.33EI$$

$$k_{45} = \frac{6EI}{3^2} = 0.67EI$$

$$k_{55} = \frac{12EI}{3^3} + \frac{12E(2I)}{6^3} + \frac{12E(2I)}{3^3} + \frac{12E(4I)}{6^3}$$

$$+ \frac{12EI}{3^3} + \frac{12E(2I)}{6^3} = 2.22EI$$

$$k_{65} = \frac{6EI}{3^2} - \frac{6E(2I)}{6^2} = 0.33EI$$

$$k_{75} = \frac{6E(2I)}{3^2} - \frac{6E(4I)}{6^2} = 0.67EI$$

$$k_{85} = \frac{6EI}{3^2} - \frac{6E(2I)}{6^2} = 0.33EI$$

To generate the sixth column of the stiffness matrix, give a unit displacement at coordinate 6 without any displacement at other coordinates as shown in Fig. 6.25(g). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{16} = -\frac{6EI}{3^2} = -0.67EI$$

$$k_{26} = \frac{2EI}{3} = 0.67EI$$

$$k_{36} = k_{46} = 0$$

$$k_{56} = \frac{6EI}{3^2} - \frac{6E(2I)}{6^2} = 0.33EI$$

$$k_{66} = \frac{4EI}{3} + \frac{4E(2I)}{6} + \frac{4E(2I)}{6} = 4EI$$

$$k_{76} = \frac{2E(2I)}{6} = 0.67EI$$

$$k_{86} = 0$$

To generate the seventh column of the stiffness matrix, give a unit displacement at coordinate 7 without any displacement at other coordinates as shown in Fig. 6.25(h). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{17} = -\frac{6E(2I)}{3^2} = -1.33EI$$

$$k_{27} = 0$$

$$k_{37} = \frac{2E(2I)}{3} = 1.33EI$$

$$k_{47} = 0$$

$$k_{57} = \frac{6E(2I)}{3^2} - \frac{6E(4I)}{6^2} = 0.67EI$$

$$k_{67} = \frac{2E(2I)}{6} = 0.67EI$$

$$k_{77} = \frac{4E(2I)}{6} + \frac{4E(2I)}{3} + \frac{4E(2I)}{6} + \frac{4E(4I)}{6} = 8EI$$

$$k_{87} = \frac{2E(2I)}{6} = 0.67EI$$

To generate the eighth column of the stiffness matrix, give a unit displacement at coordinate 8 without any displacement at other coordinates as shown in Fig. 6.25(i). Compute the forces at all the coordinates using Eqs (6.8) and (6.9).

$$k_{18} = -\frac{6EI}{3^2} = -0.67EI$$

$$k_{28} = k_{38} = 0$$

$$k_{48} = \frac{2EI}{3} = 0.67EI$$

$$k_{58} = \frac{6EI}{3^2} - \frac{6E(2I)}{6^2} = 0.33EI$$

$$k_{68} = 0$$

$$k_{78} = \frac{2E(2I)}{6} = 0.67EI$$

$$k_{88} = \frac{4E(2I)}{6} + \frac{4EI}{3} + \frac{4E(2I)}{6} = 4EI$$

Thus all the elements of the stiffness matrix have been determined. It may be noted that the resulting stiffness matrix is symmetrical.

It has been shown in Sec. 4.3 that the stiffness matrix is symmetrical. Consequently, the elements belonging either to the upper triangle or the lower triangle only need be computed to save computational effort. Further saving in the computational effort is possible if the structure is symmetrical. Figure 6.26 shows a structure symmetrical about the vertical axis *YY*.

Coordinates p and p' are symmetrically located. Similarly, coordinates q and q' are symmetrical located. Coordinate i is any rotational coordinate located on the axis of symmetry in the case of even number of bays. If the number of bays is odd, coordinate i does not exist. Coordinate j is any sway coordinate.

From the concepts of symmetry, it is evident that

$$\begin{aligned} k_{pp} &= k_{p'p'} \\ k_{pq} &= k_{p'q'} \\ k_{pq'} &= k_{p'q} \\ k_{ip} &= k_{ip'} \\ k_{jp} &= k_{jp'} \end{aligned} \qquad (6.10)$$

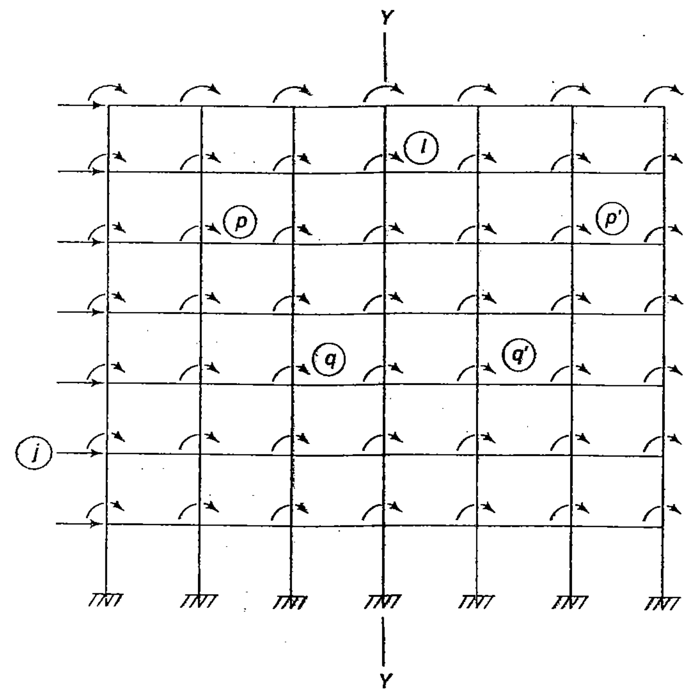


Fig. 6.26

Equation (6.10) can be used to reduce the number of elements of the stiffness matrix to be computed. Thus the computational effort can be reduced appreciably. Examples 6.12, 6.13 and 6.14 may be used to verify Eq. (6.10). For instance, in Example 6.14,

$$\begin{aligned} k_{22} &= k_{44} = 2.67EI \\ k_{62} &= k_{84} = 0.67EI \\ k_{28} &= k_{46} = 0 \\ k_{25} &= k_{45} = 0.67EI \\ k_{27} &= k_{47} = 0 \end{aligned}$$

6.6 DISPLACEMENT METHOD

In common practice the degree of static indeterminacy of large rigid-jointed frames is greater than the degree of kinematic indeterminacy. Hence, in comparison to the force method, the displacement method is generally found to be more suitable for the analysis of large rigid-jointed frames. The displacement method of analysis as applied to rigid-jointed plane frames may be described by the following steps:

(i) Determine the degree of freedom of the frame. The degree of freedom of rigid-jointed frames has been discussed in Sec. 1.7. In general, each joint has the freedom to rotate and each storey has the freedom to sway. Thus the degree of freedom of a typical rigid-jointed plane frame is equal to the sum of (a) the number of joints, (b) the number of storeys and (c) the degree of freedom of all supports.

(ii) Assign one coordinate to each independent displacement component. Thus a coordinate is assigned to each joint rotation, each storey sway and each displacement component at a support. Let the coordinates be numbered as 1, 2, ..., n.

(iii) Compute the forces due to the applied loads at all the coordinates without any displacement at the coordinates. Let these forces be designated as P_1', P_2', ..., P_n'. These are the foces at the coordinates in the restrained structure in which there is no displacement at any coordinate. Only local displacements at load points are permitted.

(iv) Let displacements Δ_1, Δ_2 ..., Δ_n be given at coordinates 1, 2, ..., n. Let $P_{1\Delta}$, $P_{2\Delta}$, ..., $P_{n\Delta}$ be the forces at coordinates 1, 2, ..., n on account of the displacements Δ_1, Δ_2 ..., Δ_n. From the force displacement relationships

$$\begin{bmatrix} P_{1\Delta} \\ P_{2\Delta} \\ \vdots \\ P_{n\Delta} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} \tag{a}$$

(v) The net forces at the coordinates are evidently the sum of the forces computed in steps (iii) and (iv)

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_n' \end{bmatrix} + \begin{bmatrix} P_{1\Delta} \\ P_{2\Delta} \\ \vdots \\ P_{n\Delta} \end{bmatrix} \tag{b}$$

Substituting from Eq. (a) into Eq. (b),

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_n' \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} \tag{c}$$

Solving Eq. (c) for the independent displacement components,

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} - \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_n' \end{bmatrix} \right\} \tag{6.11a}$$

Equation (6.11a) may be written in the compact form

$$[\Delta] = [k]^{-1} \{[P] - [P']\} \tag{6.11b}$$

This equation is the same as Eq. (5.4) derived earlier in Sec. 5.3. It may be noted that from the conditions of equilibrium, forces P_1, P_2, ..., P_n are equal to the external forces acting at coordinates. Hence these forces are known. Therefore, the displacements Δ_1, Δ_2, ..., Δ_n can be computed from the matrix Eq. (6.11). With all the displacement components known, the deformed shape of the frame is completely defined.

(vi) Compute the internal forces in the members of the frame by using the slope-deflection Eq. (2.47).

The above procedure is suitable when the supports are unyielding. In the case of yielding supports, coordinates should be assigned to those reactive forces along which settlements occur. However, this is not necessary. As an alternative solution, the additional restraining forces at the coordinates due to the settlement of supports should be computed and added algebraically to the restraining forces at the coordinates due to the applied loads. The procedure is illustrated by Ex. 6.18.

Example 6.15

Analyse the bent shown in Fig. 6.27(a).

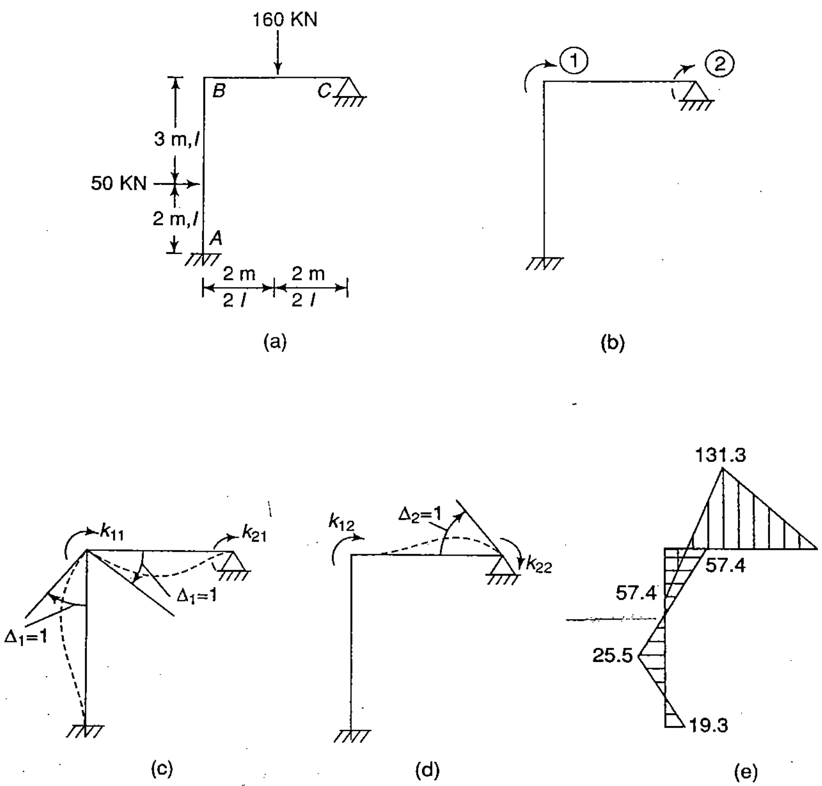


Fig. 6.27

Solution

As the frame cannot sway, rotations at B and C are the two independent displacement components. Hence coordinates 1 and 2 may be chosen as shown in Fig. 6.27(b). Forces P_1' and P_2' at coordinates 1 and 2 respectevely, due to the external loads other than those acting at the coordinates when no displacement is permitted at the coordinates, may be computed first. Considering member AB as fixed ended, the end moments are

$$M'_{AB} = -\frac{50 \times 2 \times 3^2}{5^2} = -36 \text{ kN}\cdot\text{m}$$

$$M'_{BA} = \frac{50 \times 3 \times 2^2}{5^2} = 24 \text{ kN}\cdot\text{m}$$

Similarly, considering member BC as fixed ended, the end moments are

$$M'_{BC} = -\frac{160 \times 2 \times 2^2}{4^2} = -80 \text{ kN}\cdot\text{m}$$

$$M'_{CB} = \frac{160 \times 2 \times 2^2}{4^2} = 80 \text{ kN}\cdot\text{m}$$

Hence,

$$P_1' = M'_{BA} + M'_{BC} = 24 - 80 = -56 \text{ kN}\cdot\text{m}$$

$$P_2' = M'_{CB} = 80 \text{ kN}\cdot\text{m}$$

As there at no external forces at coordinates 1 and 2,

$$P_1 = P_2 = 0$$

The stiffness matrix may now be developed. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at coordinate 2 as shown in Fig. 6.27(c) and compute the forces at coordinates 1 and 2.

$$k_{11} = \frac{4EI}{5} + \frac{4E(2I)}{4} = 2.8EI$$

$$k_{21} = \frac{2E(2I)}{4} = EI$$

Similarly, to generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at coordinate 1 as shown in Fig. 6.27(d) and compute the forces at coordinates 1 and 2.

$$k_{12} = \frac{2E(2I)}{4} = EI$$

$$k_{22} = \frac{4E(2I)}{4} = 2EI$$

Hence, the stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 2.8EI & EI \\ EI & 2EI \end{bmatrix}$$

Substituting into Eq. (6.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 2.8EI & EI \\ EI & 2EI \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -56 \\ 80 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \dfrac{41.74}{EI} \\ -\dfrac{60.87}{EI} \end{bmatrix}$$

Knowing the displacements, the end moments are obtained by using the slope-deflection Eq. (2.47).

$$M_{AB} = -36 + \frac{2EI}{5}\left(0 + \frac{41.74}{EI}\right) = -10.3 \text{ kN·m}$$

$$M_{BA} = 24 + \frac{2EI}{5}\left(\frac{2 \times 41.74}{EI} + 0\right) = 57.4 \text{ kN·m}$$

$$M_{BC} = -80 + \frac{2E(2I)}{4}\left(\frac{2 \times 41.74}{EI} - \frac{60.87}{EI}\right) = -57.4 \text{ kN·m}$$

$$M_{CB} = 80 + \frac{2E(2I)}{4}\left(-\frac{2 \times 60.87}{EI} + \frac{41.74}{EI}\right) = 0$$

The bending-moment diagram drawn on the compression side is shown in Fig. 6.27(e).

Example 6.16

Analyse the structure shown in Fig. 6.28(a).

Solution

The frame cannot sway because the translation of the joint in any direction is not possible. Hence, rotations at O and A are the two indepentent displacement components. Coordinates 1 and 2 may, therefore, be chosen as shown in Fig. 6.28(b). Assuming member OA to be fixed-ended, the end moments are

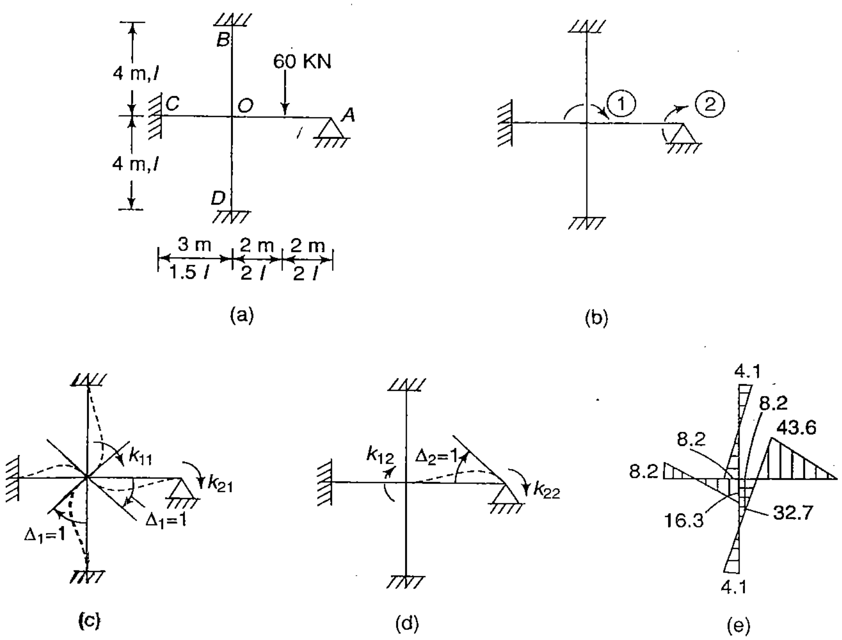


Fig. 6.28

$$M'_{OA} = -\frac{60 \times 2 \times 2^2}{4^2} = -30 \text{ kN·m}$$

$$M'_{AO} = \frac{60 \times 2 \times 2^2}{4^2} = 30 \text{ kN·m}$$

Hence,

$$P'_1 = M'_{OA} = -30 \text{ kN·m}$$

$$P'_2 = M'_{AO} = 30 \text{ kN·m}$$

As there are no external forces at coordinates 1 and 2,

$$P_1 = P_2 = 0$$

The stiffness matrix may now be developed. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at coordinate 2 as shown in Fig. 6.28(c) and compute the forces at coordinates 1 and 2.

$$k_{11} = \frac{4E(2I)}{4} + \frac{4EI}{4} + \frac{4E(1.5I)}{3} + \frac{4EI}{4} = 6EI$$

$$k_{21} = \frac{2E(2I)}{4} = EI$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at coordinate 1 as shown in Fig. 6.28(d) and compute the forces at coordinates 1 and 2.

$$k_{12} = \frac{2E(2I)}{4} = EI$$

$$k_{22} = \frac{4E(2I)}{4} = 2EI$$

Substituting into Eq. (6.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 6EI & EI \\ EI & 2EI \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -30 \\ 30 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \dfrac{8.18}{EI} \\ -\dfrac{19.09}{EI} \end{bmatrix}$$

The end moments in the members are obtained by using the slope-deflection Eq. (2.47).

$$M_{OA} = -30 + \frac{2E(2I)}{4}\left[\frac{2 \times 8.18}{EI} - \frac{19.09}{EI}\right] = -32.72 \text{ kN·m}$$

$$M_{AO} = 30 + \frac{2E(2I)}{4}\left[\frac{-2 \times 19.09}{EI} + \frac{8.18}{EI}\right] = 0$$

$$M_{OB} = \frac{2EI}{4}\left[\frac{2 \times 8.18}{EI} + 0\right] = 8.18 \text{ kN.m}$$

$$M_{BO} = \frac{2EI}{4}\left[\frac{8.18}{EI} + 0\right] = 4.09 \text{ kN·m}$$

$$M_{OC} = \frac{2E(1.5I)}{3}\left[\frac{2 \times 8.18}{EI} + 0\right] = 16.36 \text{ kN·m}$$

$$M_{CO} = \frac{2E(1.5I)}{3}\left[\frac{8.18}{EI} + 0\right] = 8.18 \text{ kN·m}$$

$$M_{OD} = \frac{2EI}{4}\left[\frac{2 \times 8.18}{EI} + 0\right] = 8.18 \text{ kN·m}$$

$$M_{DO} = \frac{2EI}{4}\left[\frac{8.18}{EI} + 0\right] = 4.09 \text{ kN·m}$$

The bending-moment diagram drawn on compression side is shown in Fig. 6.28(e).

Example 6.17

Analyse the portal frame shown in Fig. 6.29(a).

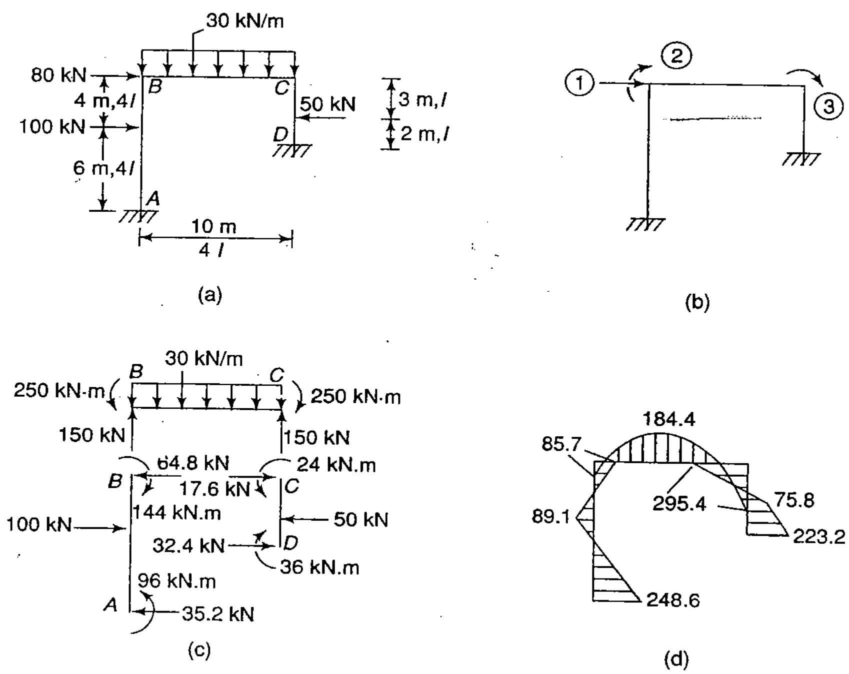


Fig. 6.29

Solution

The degree of freedom of the frame is three because the horizontal displacement of joint B and the rotations of joints B and C are the three independent displacement components. It may be noted that the horizontal displacement of joint C is the same as that of joint B. Hence the coordinates 1, 2 and 3 may be chosen as shown in Fig. 6.29(b). The stiffness matrix for the frame with reference to these coordinates has been developed in Ex. 4.10.

$$[k] = \begin{bmatrix} 0.144EI & -0.240EI & -0.240EI \\ -0.240EI & 3.200EI & 0.800EI \\ -0.240EI & 0.800EI & 2.400EI \end{bmatrix}$$

The forces P_1', P_2' and P_3' at coordinates 1, 2 and 3 due to the external loads other than those acting at the coordinates when no displacement is permitted at the coordinates may be computed by considering members AB, BC and CD as fixed-ended members. The free-body diagrams of members AB, BC and CD considered as fixed-ended members are shown in Fig. 6.29(c). The fixed-end forces shown in the free-body diagram may be calculated as follows:

Considering member AB,

$$M'_{AB} = -\frac{100 \times 6 \times 4^2}{10^2} = -96 \text{ kN·m}$$

$$M'_{AB} = \frac{100 \times 4 \times 6^2}{10^2} = 144 \text{ kN·m}$$

The end reactions of 35.2 kN and 64.8 kN acting at A and B respectively as shown in Fig. 6.29(c) may be computed by using the equations of statics.

Next considering member BC,

$$M'_{BC} = -\frac{30 \times 10^2}{12} = -2.50 \text{ kN·m}$$

$$M'_{CB} = \frac{30 \times 10^2}{12} = 250 \text{ kN·m}$$

The end reactions at B and C are 150 kN each as shown in Fig. 6.29(c).

Finally considering member CD,

$$M'_{CD} = -\frac{50 \times 3 \times 2^2}{5^2} = -24 \text{ kN·m}$$

$$M'_{DC} = \frac{50 \times 2 \times 3^2}{5^2} = 36 \text{ kN·m}$$

The end reactions of 21.2 kN and 28.8 kN acting at C and D respectively as shown in Fig. 6.29(c) may be computed by using the equations of statics.

The net forces at coordinates 1, 2 and 3 may be computed as follows:

$$P_1' = -64.8 + 17.6 = -47.2 \text{ kN}$$
$$P_2' = 144 - 250 = -106 \text{ kN}\cdot\text{m}$$
$$P_3' = 250 - 24 = 226 \text{ kN}\cdot\text{m}$$

The external loads acting at the coordinates are

$$P_1 = 80 \text{ kN}$$
$$P_2 = P_3 = 0$$

Substituting into Eq. (6.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 0.144EI & -0.240EI & -0.240EI \\ -0.240EI & 3.200EI & 0.800EI \\ -0.240EI & 0.800EI & 2.400EI \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 80 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -47.2 \\ -106 \\ 226 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \dfrac{1028.91}{EI} \\ \dfrac{117.93}{EI} \\ \dfrac{-30.58}{EI} \end{bmatrix}$$

Knowing the displacements, the end moments in the members are obtained by using the slope-deflection Eq. (2.47).

$$M_{AB} = -96 + \frac{2E(4I)}{10}\left(0 + \frac{117.93}{EI} - \frac{3 \times 1028.91}{10EI}\right)$$
$$= -248.6 \text{ kN}\cdot\text{m}$$

$$M_{BA} = 144 + \frac{2E(4I)}{10}\left(\frac{2 \times 117.93}{EI} + 0 - \frac{3 \times 1028.9}{10EI}\right)$$
$$= 85.7 \text{ kN}\cdot\text{m}$$

$$M_{BC} = -250 + \frac{2E(4I)}{10}\left(\frac{2 \times 117.93}{EI} - \frac{30.58}{EI}\right)$$
$$= -85.7 \text{ kN}\cdot\text{m}$$

$$M_{CB} = 250 + \frac{2E(4I)}{10}\left(-\frac{2 \times 30.58}{EI} + \frac{117.93}{EI}\right)$$
$$= 295.4 \text{ kN}\cdot\text{m}$$

$$M_{CD} = -24 + \frac{2EI}{5}\left(\frac{-2 \times 30.58}{EI} + 0 - \frac{3 \times 1028.91}{5EI}\right)$$
$$= -295.4 \text{ kN}\cdot\text{M}$$

$$M_{DC} = 36 + \frac{2EI}{5}\left(0 - \frac{30.58}{EI} - \frac{3 \times 1028.91}{5EI}\right)$$
$$= -223.2 \text{ kN}\cdot\text{m}$$

The bending-moment diagram drawn on the compression side is shown in Fig. 6.29(d).

As a check, the shear Eq. (6.3) may be used. Sum of the internal shear forces in the columns immediately below the top of the frame,

$$\Sigma Q = -\left[\frac{-248.6 + 85.7 + 100 \times 6}{10} + \frac{-295.4 - 223.2 - 50 \times 2}{5}\right] = 80 \text{ kN}$$

which is equal to the applied horizontal force at point *B*.

Example 6.18

Analyse the portal frame of Fig. 6.30(a) if the yielding of support D to the right and downwards in kN-m units are 200/EI and 500/EI respectively.

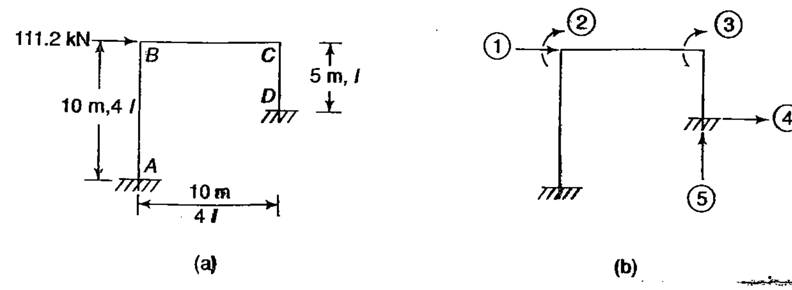


Fig. 6.30

Solution

Herein, two alternative solutions of the problem are given. In the first solution, a coordinate has also been assigned to each of the specified settlements at the supports. However, this is not necessary as illustrated by the second solution.

(i) Assign coordinates 1 to 5 as indicated in Fig. 6.30(b). Proceeding in the usual manner, the stiffness matrix with reference to the chosen coordinates is found to be

$$[k] = EI \begin{bmatrix} 0.144 & -0.24 & -0.24 & -0.096 & 0 \\ -0.24 & 3.2 & 0.8 & 0 & 0.24 \\ -0.24 & 0.8 & 2.4 & 0.24 & 0.24 \\ -0.096 & 0 & 0.24 & 0.096 & 0 \\ 0 & 0.24 & 0.24 & 0 & 0.048 \end{bmatrix} \qquad \text{(a)}$$

As there are no external loads other than those acting at the coordinates,

$$P_1' = P_2' = P_3' = P_4' = P_5' = 0$$

Also from the given data,

$$P_1 = 111.2 \text{ kN} \qquad P_2 = P_3 = 0 \qquad \Delta_4 = \frac{200}{EI}$$

$$\Delta_5 = -\frac{500}{EI}$$

Substituting into Eq. (6.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \dfrac{200}{EI} \\ -\dfrac{500}{EI} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.144 & -0.24 & -0.24 & -0.096 & 0 \\ -0.24 & 3.2 & 0.8 & 0 & 0.24 \\ -0.24 & 0.8 & 2.4 & 0.24 & 0.24 \\ -0.096 & 0 & 0.24 & 0.096 & 0 \\ 0 & 0.24 & 0.24 & 0 & 0.048 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 111.2 \\ 0 \\ 0 \\ P_4 \\ P_5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Rearranging the terms,

$$\left[\begin{array}{c} 111.2 \\ 0 \\ 0 \\ \hline P_4 \\ P_5 \end{array}\right] = EI \left[\begin{array}{ccc|cc} 0.144 & -0.24 & -0.24 & -0.096 & 0 \\ -0.24 & 3.2 & 0.8 & 0 & 0.24 \\ -0.24 & 0.8 & 2.4 & 0.24 & 0.24 \\ \hline -0.096 & 0 & 0.24 & 0.096 & 0 \\ 0 & 0.24 & 0.24 & 0 & 0.048 \end{array}\right] \left[\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \hline \dfrac{200}{EI} \\ -\dfrac{500}{EI} \end{array}\right]$$

Partitioning the matrices as indicated by dotted lines, the above equation may be split up into the following two equations:

$$\begin{bmatrix} 111.2 \\ 0 \\ 0 \end{bmatrix} = EI \begin{bmatrix} 0.144 & -0.24 & -0.24 \\ -0.24 & 3.2 & 0.8 \\ -0.24 & 0.8 & 2.4 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} + \begin{bmatrix} -0.096 & 0 \\ 0 & 0.24 \\ 0.24 & 0.24 \end{bmatrix} \begin{bmatrix} 200 \\ -500 \end{bmatrix} \qquad \text{(b)}$$

$$\begin{bmatrix} P_4 \\ P_5 \end{bmatrix} = EI \begin{bmatrix} -0.096 & 0 & 0.24 \\ 0 & 0.24 & 0.24 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} + \begin{bmatrix} 0.096 & 0 \\ 0 & 0.048 \end{bmatrix} \begin{bmatrix} 200 \\ -500 \end{bmatrix} \qquad \text{(c)}$$

Solving Eq. (b),

$$\Delta_1 = \frac{1283.69}{EI} \qquad \Delta_2 = \frac{102.76}{EI} \qquad \Delta_3 = \frac{124.08}{EI}$$

Substituting these values into Eq. (c),

$$P_4 = -74.25 \text{ kN} \qquad P_5 = 30.44 \text{ kN}$$

Using the slope-deflection Eq. (2.47), the moment at support D may be computed.

$$M_{DC} = \frac{2EI}{5}\left[\frac{124.08}{EI} - \frac{3\left(\dfrac{1283.69}{EI} - \dfrac{200}{EI}\right)}{5}\right]$$

$$= -210.46 \text{ kN·m}$$

Hence, the horizontal and vertical reactions and the bending moment at support D are 74.25 kN (towards left), 30.44 kN (upward) and 210.46 kN·m counter-clockwise respectively. It may be noted that these reactive forces are the same as those computed in Ex. 6.2.

(ii) Assign coordinates 1, 2 and 3 as shown in Fig. 6.29(b). The stiffness matrix with reference to these coordinates as developed in Ex. 6.17 is

$$[k] = EI\begin{bmatrix} 0.144 & -0.240 & -0.240 \\ -0.240 & 3.200 & 0.800 \\ -0.240 & 0.800 & 2.400 \end{bmatrix}$$

From the given data,

$$P_1 = 111.2 \text{ kN} \qquad P_2 = P_3 = 0$$

In this alternative solution, coordinates have not been assigned to the displacements due to the settlement of support at D. Hence the forces produced on account of the displacements at support D should be included in the elements of matrix$[P']$. As there are no external loads other than those acting at the coordinates, the elements of the matrix $[P']$ are the forces at the coordinates, only due to the settlement of the support at D.

Considering first, the restraining forces due to the horizontal displacement of $\dfrac{200}{EI}$ towards right at D.

$$P_1' = -\frac{12EI}{5^3} \times \frac{200}{EI} = -19.2 \text{ kN}$$

$$P_2' = 0$$

$$P_3' = \frac{6EI}{5^2} \times \frac{200}{EI} = 48 \text{ kN·m}$$

Considering next, the restaining forces due to the vertical displacement of $\dfrac{500}{EI}$ downward at D,

$$P_1' = 0$$

$$P_2' = P_3' = \frac{6E(4I)}{10^2} \times \frac{500}{EI} = -120 \text{ kN·m}$$

Hence, the toral restraining forces due to the settlement of support are

$$P_1' = -19.2 \text{ kN}$$
$$P_2' = -120 \text{ kN·m}$$
$$P_3' = -72 \text{ kN}$$

Substituting into Eq. (6.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.144 & -0.240 & -0.240 \\ -0.240 & 3.200 & 0.800 \\ -0.240 & 0.800 & 2.400 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 111.2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -19.2 \\ -120 \\ -72 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1283.69/EI \\ 102.76/EI \\ 124.08/EI \end{bmatrix}$$

These displacements are the same as obtained in solution (i). Knowing the displacements, the bending moments may be computed by using the slope-deflection Eq. 2.47.

Example 6.19

Analyse the frame shown in Fig. 6.31(a).

Solution

In this problem the horizontal displacement of joint B and the rotations of joints B, D and F are the independent displacement components. It may be noted that the horizontal displacements of joints D and F are the same as that of joint B. Hence, the degree of freedom of the frame is four. Consequently, coordinates 1 to 4 as shown in Fig. 6.31(b) may be chosen.

Forces P_1', P_2', P_3' and P_4' at coordinates 1 to 4 due to external loads other than those acting at the coordinates when no displacement is permitted at the coordinates may be computed by considering all the members as fixed-ended members. The free-body diagrams of all the members considered as fixed-ended members are shown in Fig. 6.31(c). The end reactions shown in the free-body diagrams may be computed in the same manner as in Ex. 6.17. The net forces at coordinates 1 to 4 may be computed as follows:

$$P_1' = -150 - 20 + 90 = -80 \text{ kN}$$
$$P_2' = 125 - 120 = 5 \text{ kN·m}$$
$$P_3' = 80 + 20 - 40 = 60 \text{ kN·m}$$
$$P_4' = 80 - 45 = 35 \text{ kN·m}$$

From the given data,

$$P_1 = 100 \text{ kN}$$
$$P_2 = P_3 = P_4 = 0$$

Next, the stiffness matrix of the frame with reference to the chosen coordinates may be developed. To generate the first column of the stiffness matrix, give a unit displacement

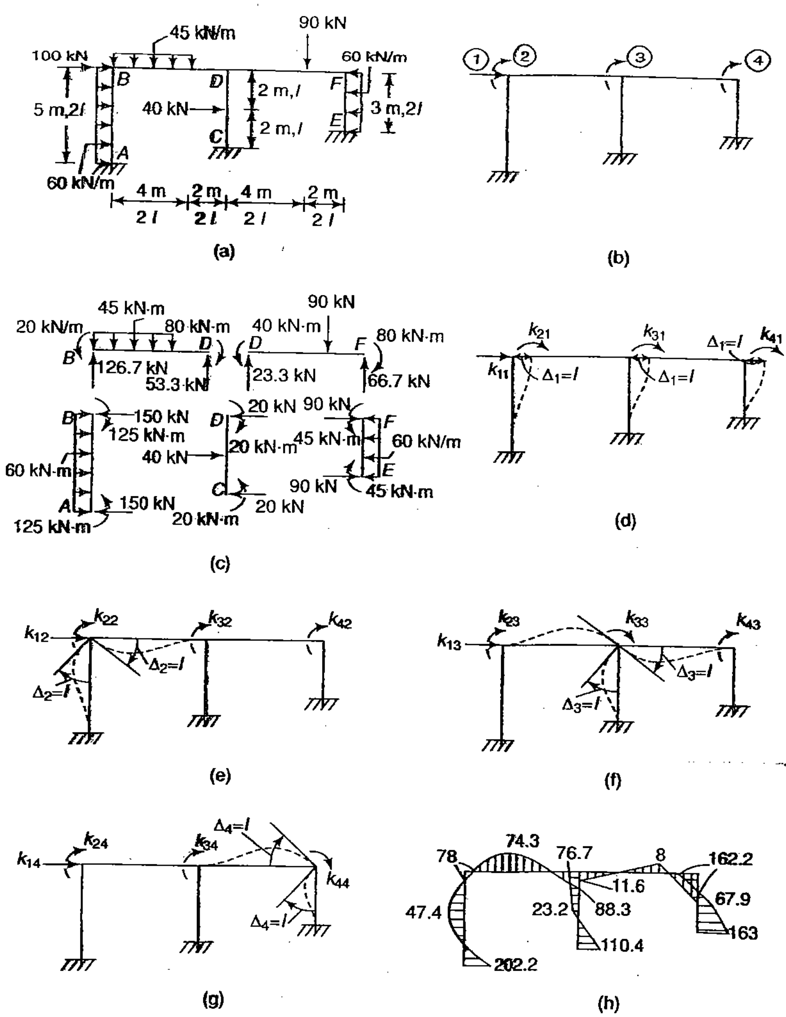


Fig. 6.31

at coordinate 1 without any displacement at coordinates 2, 3 and 4 as shown in Fig. 6.31(d) and compute the forces at all the coordinates. Using Eqs (6.6) and (6.7),

$$k_{11} = \frac{12E(2I)}{5^3} + \frac{12EI}{4^3} + \frac{12E(2I)}{3^3} = 1.2683EI$$

$$k_{21} = -\frac{6E(2I)}{5^2} = -0.4800EI$$

$$k_{31} = -\frac{6EI}{4^2} = -0.3750EI$$

$$k_{41} = -\frac{6E(2I)}{3^2} = -1.3333EI$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at coordinates 1, 3 and 4 as shown in Fig. 6.31(e) and compute the forces at all the coordinates. Using Eqs (6.8) and (6.9),

$$k_{12} = -\frac{6E(2I)}{5^2} = -0.4800EI$$

$$k_{22} = \frac{4E(2I)}{5} + \frac{4E(2I)}{6} = 2.9333EI$$

$$k_{32} = \frac{2E(2I)}{6} = 0.6667EI$$

$$k_{42} = 0$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at coordinates 1, 2 and 4 as shown in Fig. 6.31(f) and compute the forces at all the coordinates. Using Eqs (6.8) and (6.9),

$$k_{13} = -\frac{6EI}{4^2} = -0.3750EI$$

$$k_{23} = \frac{2E(2I)}{6} = 0.6667EI$$

$$k_{33} = \frac{4E(2I)}{6} + \frac{4EI}{4} + \frac{4EI(2I)}{6} = 3.6667EI$$

$$k_{43} = \frac{2E(2I)}{6} = 0.6667EI$$

To generate the fourth column of the stiffness matrix, give a unit displacement at coordinate 4 without any displacement at coordinates 1, 2 and 3 as shown in Fig. 6.31(g) and compute the forces at all the coordinates. Using Eqs (6.8) and (6.9),

$$k_{14} = -\frac{6E(2I)}{3^2} = -1.3333EI$$

$$k_{24} = 0$$

$$k_{34} = \frac{2E(2I)}{6} = 0.6667EI$$

$$k_{44} = \frac{4E(2I)}{6} + \frac{4E(2I)}{3} = 4EI$$

Substituting into Eq. (6.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 1.2683 & -0.4800 & -0.3750 & -1.3333 \\ -0.4800 & 2.9333 & 0.6667 & 0 \\ -0.3750 & 0.6667 & 3.6667 & 0.6667 \\ -1.3333 & 0 & 0.6667 & 4.0000 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -80 \\ 5 \\ 60 \\ 35 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \dfrac{224.00}{EI} \\ \dfrac{37.86}{EI} \\ -\dfrac{12.71}{EI} \\ \dfrac{68.02}{EI} \end{bmatrix}$$

Knowing the displacements, the end moments in the members are obtained by using the slope-deflection Eq. (2.47).

$$M_{AB} = -125 + \frac{2E(2I)}{5}\left(0 + \frac{37.86}{EI} - \frac{3 \times 224.00}{5EI}\right)$$

$$= -202.2 \text{ kN·m}$$

$$M_{BA} = 125 + \frac{2E(2I)}{5}\left(\frac{2 \times 37.86}{EI} + 0 - \frac{3 \times 224.00}{5EI}\right)$$

$$= 78 \text{ kN·m}$$

$$M_{BD} = -120 + \frac{2E(2I)}{6}\left(\frac{2 \times 37.86}{EI} - \frac{12.71}{EI}\right)$$

$$= -78 \text{ kN·m}$$

$$M_{DB} = 80 + \frac{2E(2I)}{6}\left(-\frac{2 \times 12.71}{EI} + \frac{37.86}{EI}\right)$$

$$= 88.3 \text{ kN·m}$$

$$M_{DC} = 20 + \frac{2EI}{4}\left(-\frac{2 \times 12.71}{EI} + 0 - \frac{3 \times 224.00}{4EI}\right)$$

$$= -76.7 \text{ kN·m}$$

$$M_{DF} = -40 + \frac{2E(2I)}{6}\left(-\frac{2 \times 12.71}{EI} + \frac{68.02}{EI}\right)$$

$$= -11.6 \text{ kN·m}$$

$$M_{CD} = -20 + \frac{2EI}{4}\left(0 - \frac{12.71}{EI} - \frac{3 \times 224.00}{4EI}\right)$$
$$= -110.4 \text{ kN·m}$$

$$M_{FD} = 80 + \frac{2E(2I)}{6}\left(\frac{2 \times 68.02}{EI} - \frac{12.71}{EI}\right)$$
$$= 162.2 \text{ kN·m}$$

$$M_{FE} = -45 + \frac{2E(2I)}{3}\left(\frac{2 \times 68.02}{EI} + 0 - \frac{3 \times 224.00}{3EI}\right)$$
$$= 162.2 \text{ kN·m}$$

$$M_{EF} = 45 + \frac{2E(2I)}{3}\left(0 + \frac{68.02}{EI} - \frac{3 \times 224.00}{3EI}\right)$$
$$= -163 \text{ kN·m}$$

The bending-moment diagram drawn on the compression side is shown in Fig. 6.31(h).

As a check, the shear Eq. (6.3) may be used. The sum of the internal shear forces in the columns immediately below the top of the frame,

$$\Sigma Q = -\left[\frac{-202.2 + 78 + 300 \times 2.5}{5} + \frac{-110.4 - 76.7 + 40 \times 2}{4} + \frac{-163 - 162.2 - 180 \times 1.5}{3}\right]$$
$$= 100 \text{ kN}$$

which is equal to the applied horizontal force at point B.

Example 6.20

Analyse the rigid-jointed frame shown in Fig. 6.32(a).

Solution

In this problem the horizontal displacements of joints A and C and the rotations of joints of A, B, C and D are the independent displacement components. It may be noted that the horizontal displacement of joint B is the same as that of joint A and the horizontal displacement of joint D is the same as that of joint C. Hence the degree of freedom of the frame is six. Consequently, coordinates 1 to 6 as shown in Fig. 6.32(b) may be chosen.

Forces P_1' to P_6' at coordinates 1 to 6 due to the external loads other than those acting at the coordinates may be computed by considering all the members as fixed-ended members.

$$P_1' = P_2' = 0$$

$$-P_3' = P_4' = \frac{30 \times 6^2}{12} = 90 \text{ kN·m}$$

$$-P_5' = P_6' = \frac{60 \times 6^2}{12} = 180 \text{ kN·m}$$

From the given data,

$$P_1 = 20 \text{ kN} \qquad P_2 = -36.45 \text{ kN} \qquad P_3 = P_4 = P_5 = P_6 = 0$$

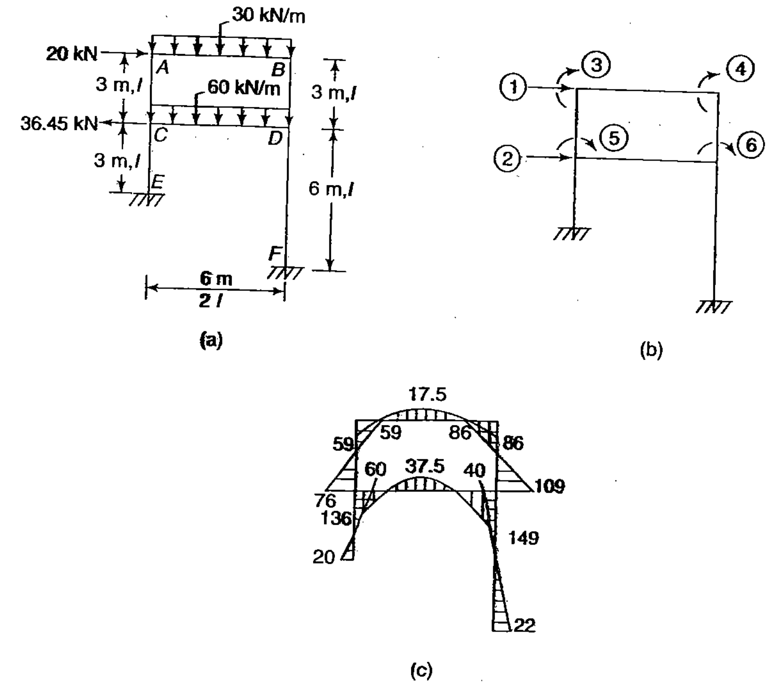


Fig. 6.32

The elements of the stiffness matrix have already been determined in Ex. 6.13. Stiffness matrix $[k]$ is found to be

$$[k] = \frac{EI}{18}\begin{bmatrix} 16 & -16 & -12 & -12 & -12 & -12 \\ -16 & 25 & 12 & 12 & 0 & 9 \\ -12 & 12 & 48 & 12 & 12 & 0 \\ -12 & 12 & 12 & 48 & 0 & 12 \\ -12 & 0 & 12 & 0 & 72 & 12 \\ -12 & 9 & 0 & 12 & 12 & 60 \end{bmatrix}$$

Substituting into Eq. (6.11) and solving for the displacements,

$$\Delta_1 = \frac{6.6186}{EI} \qquad \Delta_2 = \frac{2.8619}{EI} \qquad \Delta_3 = \frac{3.3155}{EI}$$

$$\Delta_4 = -\frac{1.9350}{EI} \qquad \Delta_5 = \frac{5.9258}{EI} \qquad \Delta_6 = -\frac{5.3151}{EI}$$

Knowing the displacements, the end moments may be computed by using the slope-deflection Eq. (2.47). The bending-moment diagram drawn the compresion side is shown in Fig. 6.23(c).

Example 6.21

Analyse the two-bay double storey portal frame shown in Fig. 6.33(a).

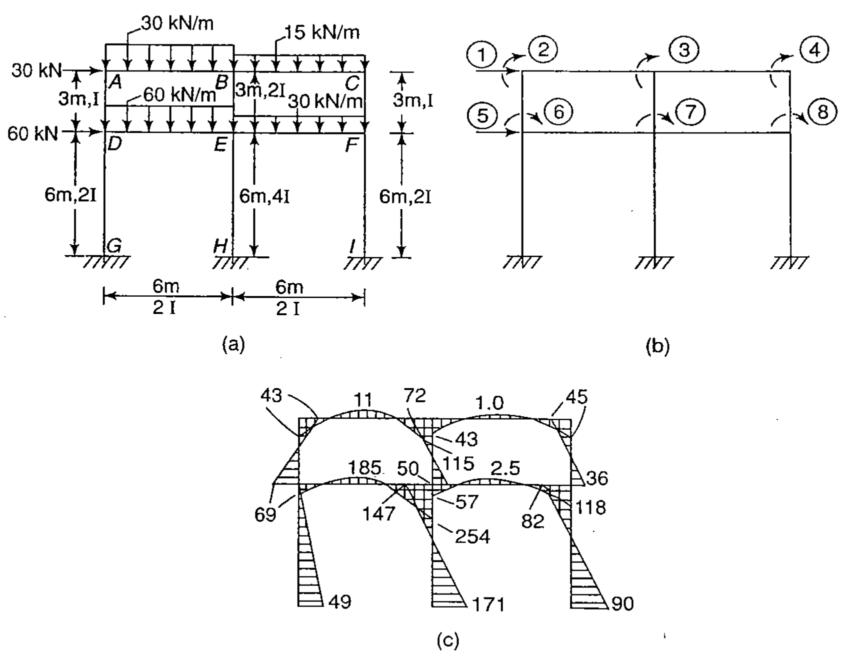


Fig. 6.33

Solution

In this problem the horizontal displacement of joints A and D and the rotations of joints A, B, C, D, E and F are the independent displacement components. It may be noted that the horizontal displacements of joints B and C are the same as that of joint A and the horizontal displacements of joints E and F are the same as that of joint D. Hence the

degree of freedom of the frame is eight. Consequently, the coordinates 1 to 8 as shown in Fig. 6.33(b) may be chosen.

Forces P_1' to P_8' at coordinates 1 to 8 due to the external loads other than those acting at the coordinates may be computed by considering all the members as fixed-ended members.

$$P_1' = 0$$

$$P_2' = -\frac{30 \times 6^2}{12} = -90 \text{ kN·m}$$

$$P_3' = \frac{30 \times 6^2}{12} - \frac{15 \times 6^2}{12} = 45 \text{ kN·m}$$

$$P_4' = \frac{15 \times 6^2}{12} = 45 \text{ kN·m}$$

$$P_5' = 0$$

$$P_6' = -\frac{60 \times 6^2}{12} = -180 \text{ kN·m}$$

$$P_7' = \frac{60 \times 6^2}{12} - \frac{30 \times 6^2}{12} = -90 \text{ kN·m}$$

$$P_8' = \frac{30 \times 6^2}{12} = 90 \text{ kN·m}$$

From the given data,

$$P_1 = 30 \text{ kN} \qquad P_5 = 60 \text{ kN} \qquad P_2 = P_3 = P_4 = P_6 = P_7 = P_8 = 0$$

The elements of the stiffness matrix have already been determined in Ex. 6.14. Stiffness matrix $[k]$ is found to be

$$[k] = \frac{EI}{9}\begin{bmatrix} 16 & -6 & -12 & -6 & -16 & -6 & -12 & -6 \\ -6 & 24 & 6 & 0 & 6 & 6 & 0 & 0 \\ -12 & 6 & 48 & 6 & 12 & 0 & 12 & 0 \\ -6 & 0 & 6 & 24 & 6 & 0 & 0 & 6 \\ -16 & 6 & 12 & 6 & 20 & 3 & 6 & 3 \\ -6 & 6 & 0 & 0 & 3 & 36 & 6 & 0 \\ -12 & 0 & 12 & 0 & 6 & 6 & 72 & 6 \\ -6 & 0 & 0 & 6 & 3 & 0 & 6 & 36 \end{bmatrix}$$

Substituting into Eq. (6.11) and solving for the displacements,

$$\Delta_1 = \frac{371.250}{EI} \qquad \Delta_2 = \frac{33.926}{EI} \qquad \Delta_3 = \frac{1.971}{EI} \qquad \Delta_4 = -\frac{1.607}{EI}$$

$$\Delta_5 = \frac{294.570}{EI} \qquad \Delta_6 = \frac{73.481}{EI} \qquad \Delta_7 = \frac{18.684}{EI} \qquad \Delta_8 = \frac{11.880}{EI}$$

Knowing the displacement components, the end moments may be computed by using the slope-deflection Eq. (2.47). The bending-moment diagram drawn on the compression side is shown in Fig. 6.33(c).

6.7 DISPLACEMENT METHOD FOR NON-RECTANGULAR FRAMES

From the preceding discussion of rectangular frames, it is evident that when a rectangular frame sways horizontally, only the columns undergo rotation. The beams do not rotate as they move parallel to themselves. This is not true in the case of non-rectangular frames. All members of these frames undergo rotation as these frames sway horizontally. As a result the expressions for the stiffness elements, which depend on the geometry of the frame, become complicated.

Consider the non-rectangular frame *ABCD* shown in Fig. 6.34(a). The inclinations, lengths and moments of inertia of the members of the frame are indicated on the figure. In order to develop the stiffness matrix for the frame, it is necessary to derive expressions for the rotation of the members of the frame due to a unit horizontal displacement (sway) at joint *B*. In Fig. 6.34(a), a unit horizontal displacement has been given to joint *B* so that joints *B* and *C* move to the new locations B_1 and C_1 respectively. As the length of member *AB* cannot change, point B_1 is located on the normal to member *AB* at *B*. Similarly, as the length of number *CD* cannot change, point C_1 is located on the normal to the member *CD* at *C*. BB_2 is the horizontal projection of line BB_1 and is evidently equal to 1 because a unit horizontal displacement has been given to joint *B*.

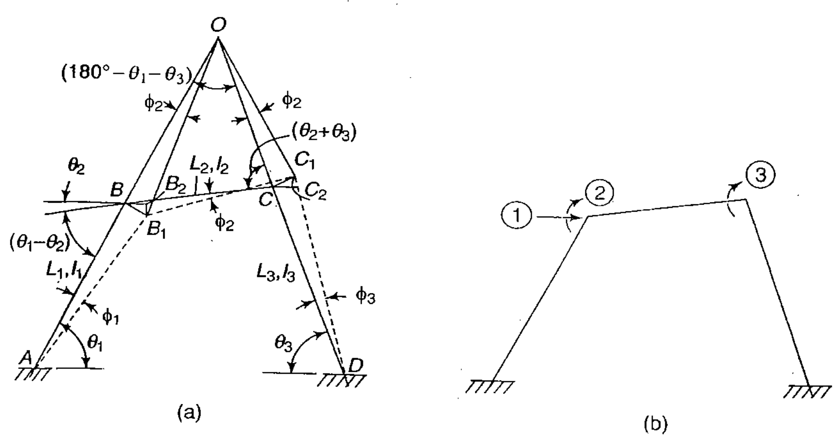


Fig. 6.34 (Contd)

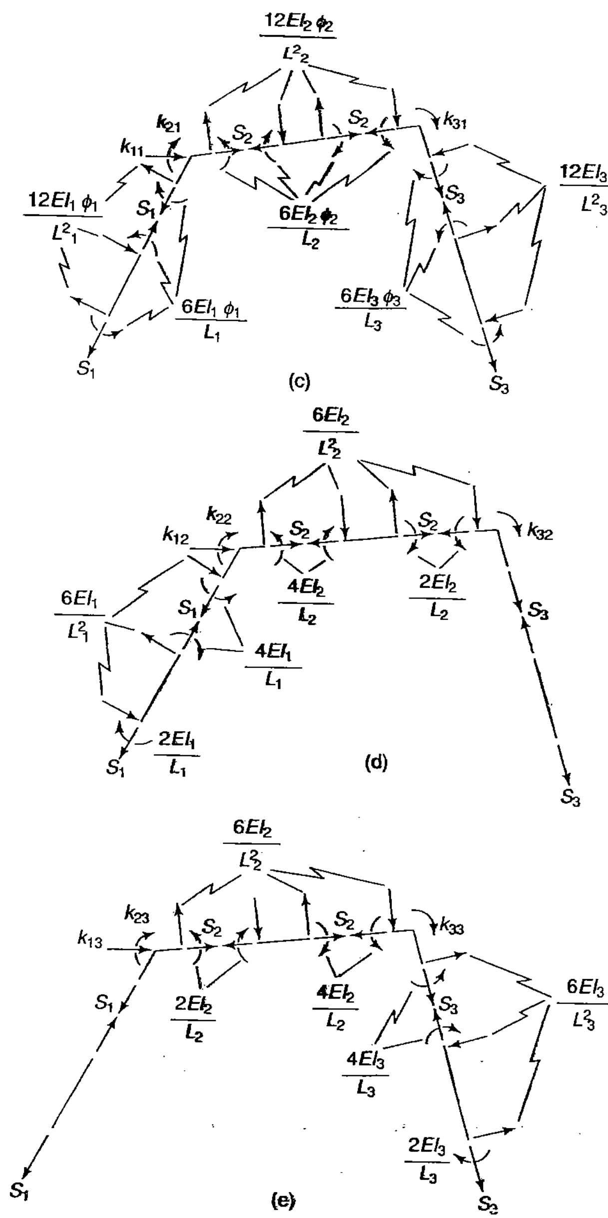


Fig. 6.34

To compute the rotations of members AB, BC and CD due to a unit horizontal displacement at B, the concept of instantaneous centre of rotation is useful. The instantaneous centre of rotation of a rigid-body is defined as the point about which the entire rigid body rotates. It is evident that all points on the rigid-body sweep the same angle about the centre of rotation. As point A is fixed in position, member AB rotates with point A as the centre of rotation. Hence BB_1 is normal to AB. Similarly, point D is the centre of rotation for member CD and hence CC_1 is normal to CD. If O is the point of intersection of lines AB and DC, it follows that O is the centre of rotation for member BC. From the geometry of the frame,

$$\frac{OB}{\sin(\theta_2+\theta_3)} = \frac{OC}{\sin(\theta_1-\theta_2)} = \frac{L_2}{\sin(180^\circ-\theta_1-\theta_3)}$$

$$OB = \frac{L_2 \sin(\theta_2+\theta_3)}{\sin(\theta_1+\theta_3)}$$

$$OC = \frac{L_2 \sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_3)}$$

As, $BB_2 = 1,$

$$BB_1 = \frac{1}{\sin\theta_1}$$

$$\phi_1 = \frac{BB_1}{L_1} = \frac{1}{L_1 \sin\theta_1} \text{ (clockwise)} \tag{6.12a}$$

$$\phi_2 = \frac{BB_1}{OB} = \frac{\sin(\theta_1+\theta_3)}{L_2 \sin\theta_1 \sin(\theta_2+\theta_3)} \text{ (counter-clockwise)} \tag{6.12b}$$

$$CC_1 = OC \times \phi_2$$

$$= \frac{\sin(\theta_1-\theta_2)}{\sin\theta_1 \sin(\theta_2+\theta_3)}$$

$$\phi_3 = \frac{CC_1}{L_3} = \frac{\sin(\theta_1-\theta_2)}{L_3 \sin\theta_1 \sin(\theta_2+\theta_3)} \text{ (clockwise)} \tag{6.12c}$$

It may be noted that the horizontal displacement of joint C,

$$CC_2 = CC_1 \sin\theta_3$$

$$= \frac{\sin(\theta_1-\theta_2)\sin\theta_3}{\sin\theta_1 \sin(\theta_2+\theta_3)}$$

If follows that if a horizontal displacement, Δ_B is given to joint B, the horizontal displacement of joint C, Δ_C is given by the equation,

$$\Delta_C = \frac{\sin(\theta_1-\theta_2)\sin\theta_3}{\sin(\theta_2+\theta_3)\sin\theta_1}\Delta_B \tag{a}$$

Equation (a) shows that the horizontal displacement of joint C is related to the horizontal displacement of joint B and is, consequently, not an independent displacement component.

The degree of freedom of the frame is three, because the independent displacement components are the horizontal displacement of joint B and the rotations of joints B and C. It has already been seen that the horizontal displacement of joint C is not an independent displacement component because it is related to the horizontal displacement of joint B through Eq. (a). Consequently, coordinates 1, 2 and 3 may be chosen as shown in Fig. 6.34(b).

In order to develop the stiffness matrix, a unit displacement may be given sucessively at coordinates 1, 2 and 3. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at coordinates 2 and 3. This displacement gives rise to rotations ϕ_1, ϕ_2 and ϕ_3 of members AB, BC and CD as explained above. The free-body diagrams of the three members and joints B and C are shown in Fig. 6.34(c). In these diagrams S_1, S_2 and S_3 are the axial forces in members AB, BC and CD respectively. These axial forces have been assumed to be tensile.

Member AB undergoes the clockwise rotation ϕ_1. Hence, using Table 2.16, the counter-clockwise couple at each end of the member is $6EI_1\,\phi_1/L_1$ and the transverse force at each end is $12EI_1\phi_1/L_1^2$. The directions of the transverse forces must, evidently, be such as to produce a clockwise couple so that the counter-clockwise end couples are balanced.

Member BC undergoes a counter-clockwise rotation ϕ_2. Consequently, the end couples in this member are clockwise and the transverse forces at the ends are directed so as to produce a counter-clockwise couple.

Member CD undergoes a clockwise rotation ϕ_3. Consequently, the end couples in this member are counter-clockwise and the transverse forces at the ends are directed so as to produce a clockwise couple.

In the free-body diagram of joint B, k_{11} is the horizontal force required to produce a unit horizontal displacement of joint B. Couple k_{21} is required at joint B to prevent its rotation. The other forces acting at the joint are equal in magnitude and opposite in sense to those acting at end B of members AB and BC.

In the free-body diagram of joint C, k_{31} is the couple required at joint C to prevent its rotation. The other forces acting at the joint are equal in magnitude and opposite in sense to those acting at end C of members BC and CD.

For the equilibrium of joint B,

$$k_{11} - S_1 \cos\theta_1 + S_2 \cos\theta_2 - \frac{12EI_1\phi_1}{L_1^2}\sin\theta_1 - \frac{12EI_2\phi_2}{L_2^2}\sin\theta_2 = 0 \quad \text{(b)}$$

$$-S_1 \sin\theta_1 + S_2 \sin\theta_2 + \frac{12EI_1\phi_1}{L_1^2}\cos\theta_1 + \frac{12EI_2\phi_2}{L_2^2}\cos\theta_2 = 0$$

$$k_{21} + \frac{6EI_1\phi_1}{L_1} - \frac{6EI_2\phi_2}{L_2} = 0$$

For the equilibrium of joint C,

$$S_2 \cos\theta_2 - S_3 \sin\theta_3 + \frac{12EI_2\phi_2}{L_2^2}\sin\theta_2 + \frac{12EI_3\phi_3}{L_3^2}\sin\theta_3 = 0$$

$$S_2 \sin\theta_2 + S_3 \sin\theta_3 + \frac{12EI_2\phi_2}{L_2^2}\cos\theta_2 + \frac{12EI_3\phi_3}{L_3^2}\cos\theta_3 = 0 \quad \text{(c)}$$

$$k_{31} - \frac{6EI_2\phi_2}{L_2} + \frac{6EI_3\phi_3}{L_3} = 0$$

Solving Eqs (b) and (c),

$$k_{11} = \frac{12EI_1\phi_1}{L_1^2\sin\theta_1} + \frac{12EI_2\phi_2}{L_2^2}\frac{\sin(\theta_1+\theta_3)}{\sin\theta_1\sin(\theta_2+\theta_3)}$$

$$-\frac{12EI_3\phi_3}{L_3^2}\frac{\sin(\theta_1-\theta_2)}{\sin\theta_1\sin(\theta_2+\theta_3)} \quad \text{(d)}$$

$$k_{21} = -\frac{6EI_1\phi_1}{L_1} + \frac{6EI_2\phi_2}{L_2}$$

$$k_{31} = \frac{6EI_2\phi_2}{L_2} - \frac{6EI_3\phi_3}{L_3}$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at coordinates 1 and 3. In this case the members do not rotate because the horizontal displacement of the frame is prevented. The free-body diagrams of the three members and joints B and C are shown in Fig. 6.34(d). Table 2.16 has been used in obtaining the end couples and the transverse forces at the ends of members AB and BC. Member CD does not bend, it carries only an axial force.

For the equilibrium of joint B,

$$k_{12} - S_1 \cos\theta_1 + S_2 \cos\theta_2 + \frac{6EI_1}{L_1^2}\sin\theta_1 - \frac{6EI_2}{L_2^2}\sin\theta_2 = 0$$

$$-S_1 \sin\theta_1 + S_2 \sin\theta_2 - \frac{6EI_1}{L_1^2}\cos\theta_1 + \frac{6EI_2}{L_2^2}\cos\theta_2 = 0 \quad \text{(e)}$$

$$k_{22} - \frac{4EI_1}{L_1} - \frac{4EI_2}{L_2} = 0$$

For the equilibrium of joint C,

$$S_2 \cos\theta_2 - S_3 \cos\theta_3 - \frac{6EI_2}{L_2^2}\sin\theta_2 = 0$$

$$S_2 \sin\theta_2 + S_3 \sin\theta_3 + \frac{6EI_2}{L_2^2}\cos\theta_2 = 0 \quad \text{(f)}$$

$$k_{32} - \frac{2EI_2}{L_2} = 0$$

Solving Eqs (e) and (f),

$$k_{12} = -\frac{6EI_1}{L_1^2\sin\theta_1} + \frac{6EI_2}{L_2^2}\frac{\sin(\theta_1+\theta_3)}{\sin\theta_1\sin(\theta_2+\theta_3)}$$

$$k_{22} = \frac{4EI_1}{L_1} + \frac{4EI_2}{L_2} \quad \text{(g)}$$

$$k_{32} = \frac{2EI_2}{L_2}$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at coordinates 1 and 2. In this case the members do not rotate because the horizontal displacement of the frame is prevented. The free-body diagrams of the three members and joints B and C are shown in Fig. 6.34(e). Table 2.16 has been used in obtaining the end couples and the transverse forces at the ends of members BC and CD. The member AB does not bend, it carries only an axial force.

For the equilibrium of joint B,

$$k_{13} - S_1 \cos\theta_1 + S_2 \cos\theta_2 - \frac{6EI_2}{L_2^2}\sin\theta_2 = 0$$

$$-S_1 \sin\theta_1 + S_2 \sin\theta_2 + \frac{6EI_2}{L_2^2}\cos\theta_2 = 0 \quad \text{(h)}$$

$$k_{23} - \frac{2EI_2}{L_2} = 0$$

For the equilibrium of joint C,

$$S_2 \cos \theta_2 - S_2 \cos \theta_3 - \frac{6EI_2}{L_2^2} \sin \theta_2 - \frac{6EI_3}{L_3^2} \sin \theta_3 = 0$$

$$S_2 \sin \theta_2 + S_3 \sin \theta_3 + \frac{6EI_2}{L_2^2} \cos \theta_2 - \frac{6EI_3}{L_3^2} \cos \theta_3 = 0 \qquad \text{(i)}$$

$$k_{33} - \frac{4EI_2}{L_2} - \frac{4EI_3}{L_3} = 0$$

Solving Eqs (h) and (i),

$$k_{13} = \frac{6EI_2}{L_2^2} \frac{\sin(\theta_1 + \theta_3)}{\sin \theta_1 \sin(\theta_2 + \theta_3)} - \frac{6EI_3}{L_3^2} \frac{\sin(\theta_1 - \theta_2)}{\sin \theta_1 \sin(\theta_2 + \theta_3)} \qquad \text{(j)}$$

$$k_{23} = \frac{2EI_2}{L_2}$$

$$k_{33} = \frac{4EI_2}{L_2} + \frac{4EI_3}{L_3}$$

The final expression for the elements of the stiffness matrix are obtained by substituting the values of ϕ_1, ϕ_2 and ϕ_3 from Eq. (6.12).

$$k_{11} = \frac{12EI_1}{L_1^3 \sin^2 \theta_1} + \frac{12EI_2 \sin^2(\theta_1 + \theta_3)}{L_2^3 \sin^2 \theta_1 \sin^2(\theta_2 + \theta_3)} + \frac{12EI_3 \sin^2(\theta_1 - \theta_2)}{L_3^3 \sin^2 \theta_1 \sin^2(\theta_2 + \theta_3)}$$

$$k_{22} = \frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}$$

$$k_{33} = \frac{4EI_2}{L_2} + \frac{4EI_3}{L_3} \qquad (6.13)$$

$$k_{21} = k_{12} = -\frac{6EI_1}{L_2^2 \sin \theta_1} + \frac{6EI_2 \sin(\theta_1 + \theta_3)}{L_2^2 \sin \theta_1 \sin(\theta_2 + \theta_3)}$$

$$k_{23} = k_{32} = \frac{2EI_2}{L_2}$$

$$k_{31} = k_{13} = \frac{6EI_2 \sin(\theta_1 + \theta_3)}{L_2^2 \sin \theta_1 \sin(\theta_2 + \theta_3)} - \frac{6EI_3 \sin(\theta_1 - \theta_2)}{L_3^2 \sin \theta_1 \sin(\theta_2 + \theta_3)}$$

Forces P_1', P_2' and P_3' at coordinates 1, 2 and 3 respectively due to the external loads other than those acting at the coordinates may be computed by considering the free-body diagrams of members AB, BC and CD and joints B and C. Forces P_1', P_2' and P_3' are the net forces at the coordinates required to maintain the equilibrium of joints B and C. Displacements Δ_1, Δ_2 and Δ_3 may be computed next by substituting into Eq. (6.11). Finally, the bending moments at the ends of the members may be calculated by using the slope-deflection Eq. (2.47)

Example 6.22

Analyse the portal frame with inclined legs shown in Fig. 6.35(a).

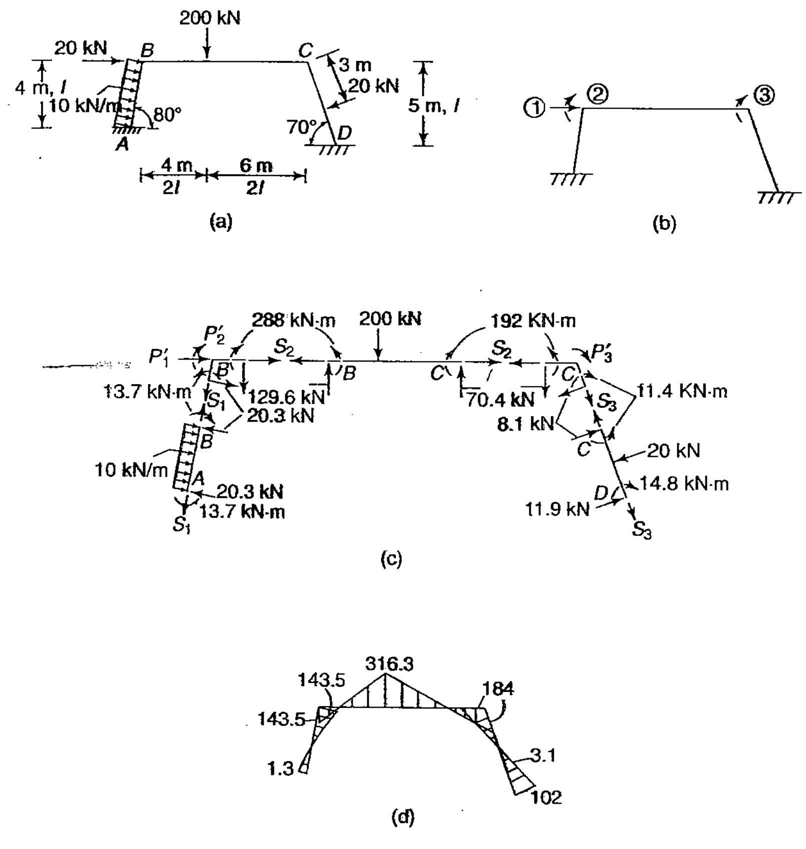


Fig. 6.35

Solution

In this problem the independent displacement components are the horizontal displacement of joint B and the rotations of joints B and C as discussed in Sec. 6.7. Consequently, coordinates 1, 2 and 3 may be chosen as shown in Fig. 6.35(b).

Using the notations of Sec. 6.7.,

$$\theta_1 = 80^\circ \qquad L_1 = 4 \text{ cosec } 80^\circ = 4.06 \text{ m} \qquad I_1 = I$$
$$\theta_2 = 0 \qquad L_2 = 10 \text{ m} \qquad I_2 = 2I$$
$$\theta_3 = 70^\circ \qquad L_3 = 5 \text{ cosec } 70^\circ = 5.32 \text{ m} \qquad I_3 = I$$

Rotations ϕ_1, ϕ_2 and ϕ_3 of members AB, BC and CD due to a unit displacement at coordinate 1 are obtained by using Eq. (6.12).

$$\phi_1 = \frac{1}{4.06 \sin 80^\circ} = 0.25 \text{ radian (clockwise)}$$

$$\phi_2 = \frac{\sin (80^\circ + 70^\circ)}{10 \sin 80^\circ \sin 70^\circ} = 0.05403 \text{ radian (counter-clockwise)}$$

$$\phi_3 = \frac{\sin 80^\circ}{5.32 \sin 80^\circ \sin 70^\circ} = 0.20 \text{ radian (clockwise)}$$

If Δ_1 is the displacement at coordinate 1, the rotations of the members are

$$\begin{aligned} \phi_{AB} &= 0.25\Delta_1 \text{ (clockwise)} \\ \phi_{BC} &= 0.05403\,\Delta_1 \text{ (counter-clockwise)} \\ \phi_{CD} &= 0.20\Delta_1 \text{ (clockwise)} \end{aligned} \tag{a}$$

Substituting these values in Eq. (6.13), the stiffness elements are

$$k_{11} = \frac{12EI}{4.06^3 \sin^2 80^\circ} + \frac{12E(2I) \sin^2 (80^\circ + 70^\circ)}{10^3 \sin^2 80^\circ \sin^2 70^\circ}$$
$$+ \frac{12EI \sin^2 80^\circ}{5.32^3 \sin^2 80^\circ \sin^2 70^\circ} = 0.283EI$$

$$k_{22} = \frac{4EI}{4.06} + \frac{4E(2I)}{10} = 1.785EI$$

$$k_{33} = \frac{4E(2I)}{10} + \frac{4EI}{5.32} = 1.550EI$$

$$k_{21} = k_{12} = -\frac{6EI}{4.06^2 \sin 80^\circ} + \frac{6E(2I) \sin (80^\circ + 70^\circ)}{10^2 \sin 80^\circ \sin 70^\circ}$$
$$= -0.305\, EI$$

$$k_{23} = k_{32} = \frac{2E(2I)}{10} = 0.4EI$$

$$k_{31} = k_{13} = \frac{6E(2I) \sin (80^\circ + 70^\circ)}{10^2 \sin 80^\circ \sin 70^\circ} - \frac{6EI \sin 80^\circ}{5.32^2 \sin 80^\circ \sin 70^\circ}$$
$$= -0.161EI$$

Thus stiffness matrix $[k]$ with reference to the chosen coordinates is given by the equation

$$[k] = EI \begin{bmatrix} 0.283 & -0.305 & -0.161 \\ -0.305 & 1.785 & 0.400 \\ -0.161 & 0.400 & 1.550 \end{bmatrix}$$

Forces P_1', P_2' and P_3' at coordinates 1, 2 and 3 due to the external loads other than those acting at the coordinates may be computed by considering all the members as fixed-ended members. The free-body diagrams of all the members and joints B and C are shown in Fig. 6.35(c).

For the equilibrium of joint B,

$$\begin{aligned} &P_1' - S_1 \cos 80^\circ + S_2 + 20.3 \sin 80^\circ = 0 \\ &S_1 \sin 80^\circ + 20.3 \cos 80^\circ + 129.6 = 0 \\ &P_2' + 288 - 13.7 = 0 \end{aligned} \tag{b}$$

For the equilibrium of joint C,

$$\begin{aligned} &S_2 - S_3 \cos 70^\circ + 8.1 \sin 70^\circ = 0 \\ &S_3 \sin 70^\circ + 70.4 + 8.1 \cos 70^\circ = 0 \\ &P_3' + 11.4 - 192 = 0 \end{aligned} \tag{c}$$

Solving Eqs (b) and (c),

$$P_1' = -9.3 \text{ kN}$$
$$P_2' = -274.3 \text{ kN·m}$$
$$P_3' = 180.6 \text{ kN·m}$$

The external loads at the coordinates are

$$P_1 = 20 \text{ kN}$$
$$P_2 = 0$$
$$P_3 = 0$$

Substituting into Eq. (6.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 0.283 & -0.305 & -0.161 \\ -0.305 & 1.785 & 0.400 \\ -0.161 & 0.400 & 1.550 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -9.3 \\ -274.3 \\ 180.6 \end{bmatrix} \right\}$$

$$= \frac{1}{EI} \begin{bmatrix} 270.02 \\ 233.05 \\ 148.45 \end{bmatrix}$$

Knowing the displacements, the end moments in the members are obtained by using Eq. (a) and the slope-deflection Eq. (2.48).

$$M_{AB} = -13.7 + \frac{2EI}{4.06}\left(0 + \frac{233.05}{EI} - \frac{3 \times 270.02 \times 0.25}{EI}\right)$$

$$= 1.3 \text{ kN·m}$$

$$M_{BA} = 13.7 + \frac{2EI}{4.06}\left(\frac{2 \times 233.05}{EI} - \frac{3 \times 270.02 \times 0.25}{EI}\right)$$

$$= 143.5 \text{ kN·m}$$

$$M_{BC} = -288 + \frac{2E(2I)}{10}\left(\frac{2 \times 233.05}{EI} - \frac{148.45}{EI} + \frac{3 \times 270.02 \times 0.05403}{EI}\right)$$

$$= -143.5 \text{ kN·m}$$

$$M_{CB} = 192 + \frac{2E(2I)}{10}\left(\frac{-2 \times 148.45}{EI} + \frac{233.05}{EI} + \frac{3 \times 270.02 \times 0.05403}{EI}\right)$$

$$= 184 \text{ kN·m}$$

$$M_{CD} = -11.4 + \frac{2EI}{5.32}\left(\frac{-2 \times 148.45}{EI} - \frac{3 \times 270.02 \times 0.20}{EI}\right)$$

$$= -184 \text{ kN·m}$$

$$M_{DC} = 14.8 + \frac{2EI}{5.32}\left(\frac{-148.45}{EI} - \frac{3 \times 270.02 \times 0.20}{EI}\right)$$

$$= -102 \text{ kN·m}$$

The bending-moment diagram drawn on the compression side is shown in Fig. 6.35(d).

6.8 COMPARISON OF METHODS

In the preceding sections the force and the displacement methods for the analysis of rigid-jointed plane frames have been discussed. A large number of problems have also been solved by both the methods. It is quite evident that in the case of rigid-jointed plane frames, the development of the stiffness matrix is much simpler and quicker as compared to the development of the flexibility matrix. Besides, the degree of kinematic indeterminacy of these frames is generally smaller than the degree of static indeterminacy except in the case of small frames. For example, the degrees of static and kinematic indeterminacies of the frame shown in Fig. 6.16 are 30 and 20 respectively. It, therefore, follows that for the analysis of large rigid-jointed plane frames, the displacement method is generally preferable as compared to the force method.

PROBLEMS

6.1 Analyse the right angled bent of Fig. 6.36 by the force method. Hence calculate the horizontal reaction at A. Verify the result by the displacement method.

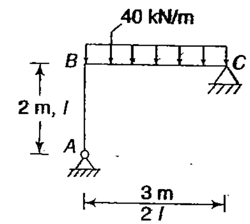


Fig. 6.36

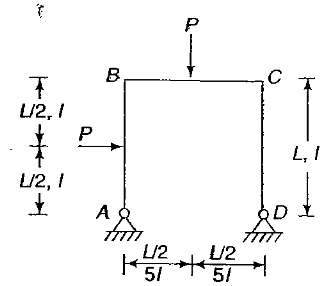


Fig. 6.37

6.2 Analyse the portal frame of Fig. 6.37 by the force method. Hence calculate the bending moments at B and C.

6.3 Using the force method, analyse the frame shown in Fig. 6.38. Hence determine the vertical reactions at supports A and D.

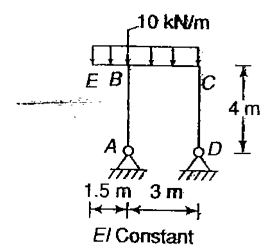


Fig. 6.38

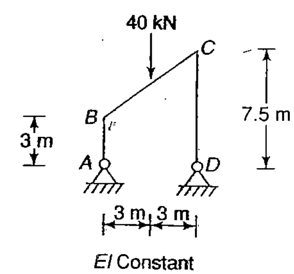


Fig. 6.39

6.4 Analyse the portal frame of Fig. 6.39 by the force method treating the horizontal reaction at D as the redundant. Hence calculate the bending moments at B and C.

6.5 Using the force method, analyse the portal frame of Fig. 6.40 treating the horizontal reaction at D as the redundant. Hence calculate the bending moments at B and C.

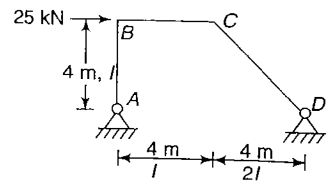


Fig. 6.40

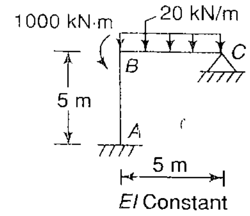


Fig. 6.41

6.6 Using the force method, analyse the right angled bent of Fig. 6.41 treating the two reactions at support C as the redundants. Hence calculate the support reactions at C.

6.7 Analyse the frame of Fig. 6.41 by the force method in which the bending moments at A and B are treated as the redundants. Hence calculate the support reactions at A.

6.8 Using the displacement method, analyse the frame of Fig. 6.41. Hence calculate the bending moments at A and B.

6.9 Adopting the following three alternatives regarding the choice of the redundants, analyse the portal frame of Fig. 6.42 by the force method:
(i) the two reaction components at support A as the redundants
(ii) the horizontal reaction and the bending moment at D as the redundants
(iii) the bending moments at B and D as the redundants.

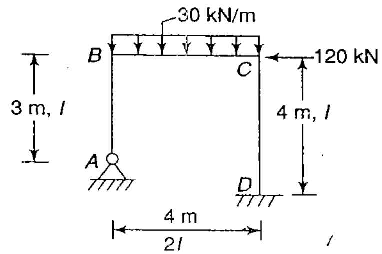


Fig. 6.42

Hence calculate the bending moments at B, C and D. Verify the result by the displacement method.

6.10 Using the force method, analyse the portal frames shown in Fig. 6.43. Hence determine the bending moment at C.

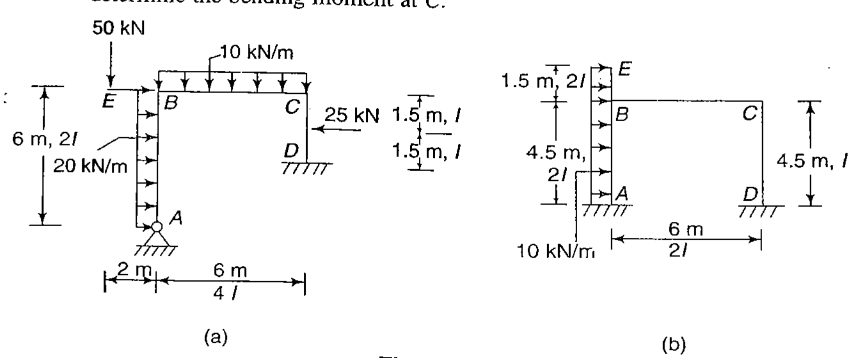


Fig. 6.43

6.11 Analyse the frame of Fig. 6.44 using the force method. Hence calculate the bending moment at B.

6.12 Analyse the frame of Fig. 6.44 using the displacement method. Hence calculate the support reactions at D.

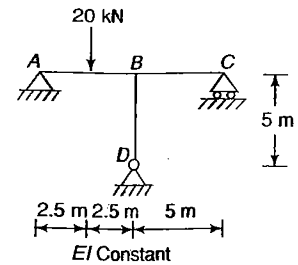


Fig. 6.44

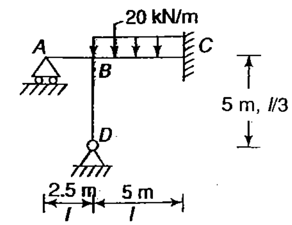


Fig. 6.45

6.13 Using the displacement method, analyse the frame shown in Fig. 6.45. Hence determine the bending moment at B.

6.14 Which of the two methods is better for the analysis of the frame shown in Fig. 6.46? Use it to analyse the frame and compute the support reactions at C.

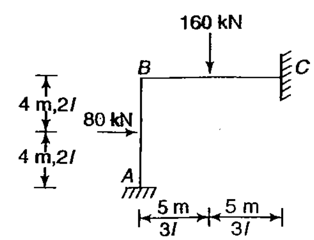


Fig. 6.46

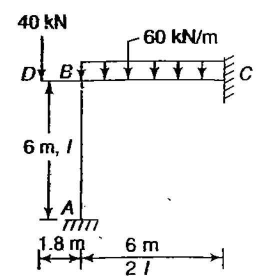


Fig. 6.47

6.15 Analyse the frame of Fig. 6.47 by the displacement method. Hence calculate the support reactions at C. Verify the result by the force method.

6.16 Analyse the portal frames of Fig. 6.48 using the displacement method. Hence determine the fixed-end moments at A and D. Verify the results by the force method.

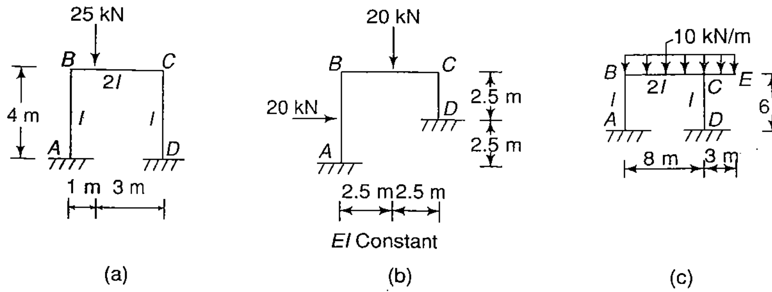


Fig. 6.48

6.17 Analyse the portal frames with inclined legs shown in Fig. 6.49 by the displacement method. Hence determine the fixed-end moments at *A* and *D*. Verify the result by the force method.

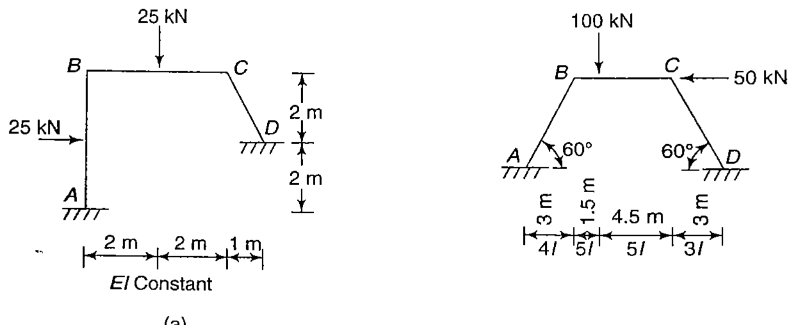


Fig. 6.49

6.18 Using the force method, analyse the gable frame shown in Fig. 6.50. Hence determine the end moments in member *BC*.

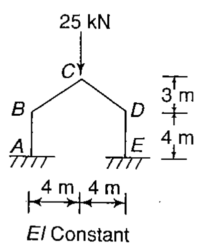


Fig. 6.50

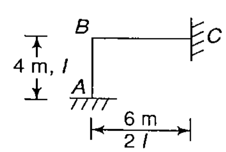


Fig. 6.51

6.19 Analyse the right angled bent shown in Fig. 6.51 by both the methods if the support at *A* permits a clockwise rotation of 0.001 radian. Hence determine the fixed-end moments at *A* and *C*.

6.20 If the support at *A* in the frame of Fig. 6.52 settles downwards by 20 mm, calculate the bending moments at *A* and *D* by the force method.

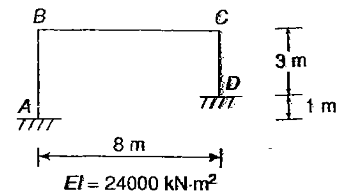


Fig. 6.52

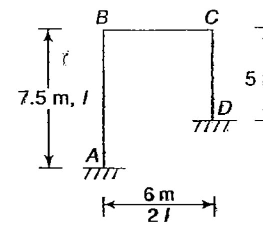


Fig. 6.53

6.21 In the portal frame of Fig. 6.53, the support at *D* permits a clockwise rotation of 0.002 radian and a vertical downward settlement of 4.17 mm. Determine the fixed-end moments at *A* and *D*. $EI = 6 \times 10^4$ kN·m^2.

6.22 What are the degrees of static and kinematic indeterminacies of the frame shown in Fig. 6.54? Analyse the frame using the force method and determine the support reactions at *A* and *F*.

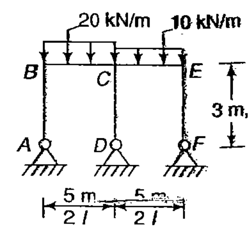


Fig. 6.54

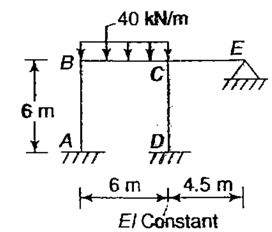


Fig. 6.55

6.23 Determine the degrees of static and kinematic indeterminacies of the frame shown in Fig. 6.55. Using the displacement method analyse the frame and compute the three reaction components at *A*.

6.24 Which method would you prefer for analysing the frame shown in Fig. 6.56? Use it to calculate the support reactions at *E* and *F*.

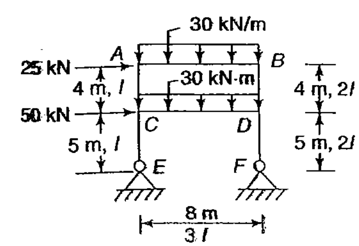


Fig. 6.56

6.25 Explain why the displacement method is preferable for the analysis of the frames shown in Fig. 6.57. Using the displacement method, determine the bending moment at B.

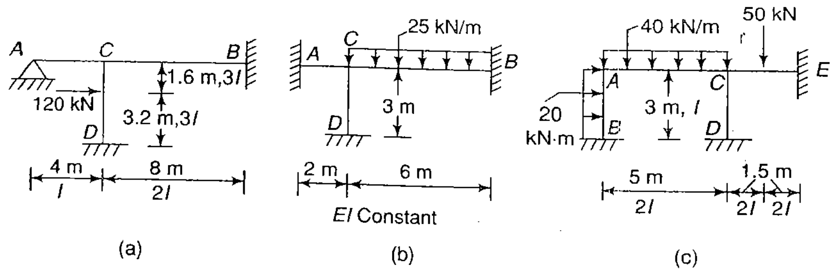


Fig. 6.57

6.26 Using the displacement method, analyse the frame of Fig. 6.58. Hence determine the fixed-end moments at B and F. Verify the result by the force method.

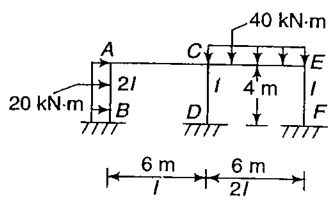


Fig. 6.58

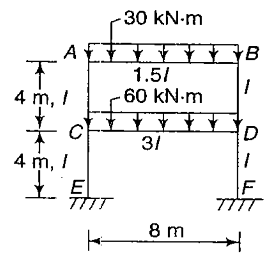


Fig. 6.59

6.27 Using the displacement method, analyse the frame of Fig. 6.59. Hence determine the support reactions at F. Verify the result by the force method.

6.28 Using the force method, analyse the rigid-jointed frame of Fig. 6.60 in which an internal hinge has been provided at the centre of member BC. Hence determine the reaction components at the hinged support D.

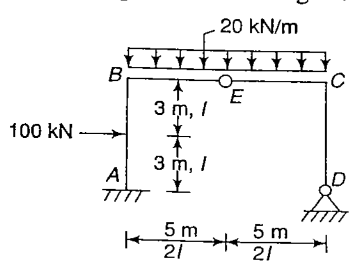


Fig. 6.60

6.29 What are the degrees of static and kinematic indeterminacies of the rigid-jointed frame shown in Fig. 6.61? The frame has an internal hinge in member BC as shown in the figure. Analyse the frame by the force method. Hence calculate the bending moments at supports A and D.

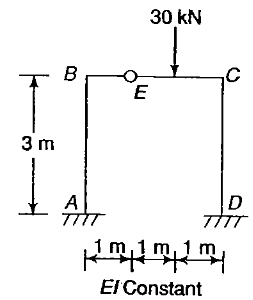


Fig. 6.61

7 PIN-JOINTED PLANE FRAMES

7.1 INTRODUCTION

A pin-jointed frame is a skeletal structure formed by the assembly of members which meet at pin joints. The pin joints do not offer any resistance to the change in the angles between the members meeting at a joint. Hence, in order to maintain internal stability, a pin-jointed frame should have enough members so that small changes in the angles between the members cannot occur without axial straining of the members of the frame. It has been seen in Sec. 1.8 that a necessary but not sufficient condition for internal stability and statical determinacy of a pin-jointed plane frame is that it must have $2j - 3$ members where j is the number of joints. In the analysis of pin-jointed frames it is commonly assumed that the members are straight and the loads are applied only at the joints. Under these conditions the members of a pin-jointed frame carry only axial forces. It follows that the members of a pin-jointed frame remain straight even after the deformation of the frame. A member of a pin-jointed frame undergoes either elongation or contraction during the deformation of the frame resulting in small displacements of the joints. The geometry of the displaced frame is completely defined if the displacements of the joints are known. These basic assumptions will be utilised in the development of force and displacement methods of matrix analysis of pin-jointed plane frames discused in subsequent sections. Member forces will be taken as positive if tensile and negative if compressive.

7.2 DISPLACEMENT OF A PIN-JOINTED PLANE FRAME

It has been shown in Sec. 2.12 that the displacement at coordinate j in a pin-jointed plane frame is given by the equation

$$\Delta_j = \sum \frac{Ss_jL}{AE} \tag{7.1}$$

where S = force in a member due to applied loads

s_j = force in the member due to a unit force at coordinate j

$\frac{L}{AE}$ = flexibility of the member.

The summation has to be carried out to include all members of the frame.

It has also been shown in Sec. 2.12 that δ_{ij}, the displacement at coordinate i due to a unit force at coordinate j, is given by the equation

$$\delta_{ij} = \sum \frac{s_i s_j L}{AE} \tag{7.2}$$

where s_i and s_j are the forces in a member due to a unit force at coordinates i and j respectively.

Equation (7.1) may be used to compute the displacement of a joint in any chosen direction on account of the applied loads. The elements of the flexibility matrix of a pin-jointed frame may be calculated with the help of Eq. (7.2). It should be noted that when a coordinate j corresponds to an axial tensile force in a member, it will be represented by a pair of straight arrows pointing towards each other as discussed in Sec. 2.4.

Example 7.1

Develop the flexibility matrix for the pin-jointed plane frame with reference to coordinates 1 and 2 shown in Fig. 7.1(a). The numbers in parentheses are the cross-sectional areas of the members in mm².

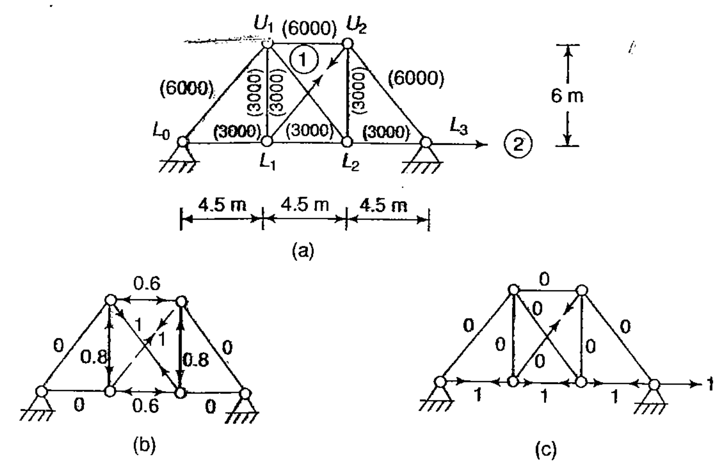


Fig. 7.1

Solution

To develop the flexibility matrix, the forces s_1 and s_2 in all the members of the frame due to a unit force at coordinates 1 and 2 respectively, have to be computed. Figure 7.1(b) shows the forces s_1 in all the members due to a unit force at coordinate 1. These forces are listed in column 3 of Table 7.1. Similarly, Fig. 7.1(c) shows the forces s_2 in all the members of the frame due to a unit force at coordinate 2. These forces are listed in column 4 of Table 7.1.

Table 7.1

Member	$\frac{L}{A}$ mm/mm²	s_1	s_2	$\frac{s_1^2L}{A}$ mm/mm²	$\frac{s_2^2L}{A}$ mm/mm²	$\frac{s_1s_2L}{A}$ mm/mm²
1	2	3	4	5	6	7
U_1U_2	0.75	–0.6	0	0.27	0	0
L_0L_1	1.5	0	1	0	1.50	0
L_1L_2	1.5	–0.6	1	0.54	1.50	–0.90
L_2L_3	1.5	0	1	0	1.50	0
U_1L_0	1.25	0	0	0	0	0
U_1L_1	2.0	–0.8	0	1.28	0	0
U_1L_2	2.5	1	0	2.5	0	0
U_2L_2	2.0	–0.8	0	1.28	0	0
U_2L_3	1.25	0	0	0	0	0
U_2L_1	2.5	1	0	2.5	0	0
			Σ	8.37	4.50	–0.90
			Tension	+		
			Compression	–		

Substituting from Table 7.1 into Eq. 7.2,

$$\delta_{11} = \sum \frac{s_1^2 L}{AE} = \frac{1}{E}\sum \frac{s_1^2 L}{A} = \frac{8.37}{E}$$

$$\delta_{21} = \delta_{12} = \sum \frac{s_1 s_2 L}{AE} = \frac{1}{E}\sum \frac{s_1 s_2 L}{A} = -\frac{0.90}{E}$$

$$\delta_{22} = \sum \frac{s_2^2 L}{AE} = \frac{1}{E}\sum \frac{s_2^2 L}{A} = \frac{4.50}{E}$$

Hence, the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{E}\begin{bmatrix} 8.37 & -0.9 \\ -0.9 & 4.5 \end{bmatrix}$$

Example 7.2

Develop the flexibility matrix for the pin-jointed plane frame with reference to coordinates 1, 2 and 3 shown in Fig. 7.2(a). Axial flexibility of each member of the frame is 0.02 mm/kN.

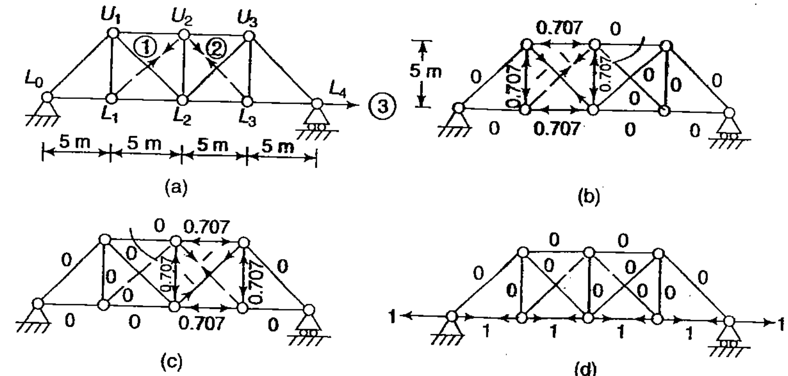


Fig. 7.2

Solution

To develop the flexibility matrix, forces s_1, s_2 and s_3 in all the members of the frame due to a unit force at coordinates 1, 2 and 3 respectively have to be computed. Fig. 7.2(b) shows the forces s_1 in all the members due to a unit force at coordinate 1. These forces are also listed in column 2 of Table 7.2. Similarly, Fig. 7.2(c) and (d) show forces s_2 and s_3 in all the members due to a unit force at coordinates 2 and 3 respectively. These forces are listed in columns 3 and 4 of Table 7.2.

Table 7.2

Member	s_1	s_2	s_3	s_1^2	s_2^2	s_3^2	s_1s_2	s_2s_3	s_3s_1
1	2	3	4	5	6	7	8	9	10
U_1U_2	–0.707	0	0	0.50	0	0	0	0	0
U_2U_3	0	–0.707	0	0	0.50	0	0	0	0
L_0L_1	0	0	1.00	0	0	1.00	0	0	0
L_1L_2	–0.707	0	1.00	0.50	0	1.00	0	0	–0.707
L_2L_3	0	–0.707	1.00	0	0.50	1.00	0	–0.707	0
L_3L_4	0	0	1.00	0	0	1.00	0	0	0
U_1L_0	0	0	0	0	0	0	0	0	0
U_1L_1	–0.707	0	0	0.50	0	0	0	0	0
U_1L_2	1.00	0	0	1.00	0	0	0	0	0
U_2L_2	–0.707	–0.707	0	0.50	0.50	0	0.50	0	0
U_3L_2	0	1.00	0	0	1.00	0	0	0	0
U_3L_3	0	–0.707	0	0	0.50	0	0	0	0
U_3L_4	0	0	0	0	0	0	0	0	0
L_1U_2	1.00	0	0	1.00	0	0	0	0	0
U_2L_3	0	1.00	0	0	1.00	0	0	0	0
			Σ	4.00	4.00	4.00	0.50	–0.707	–0.707
			Tension	+					
			Compression	–					

Substituting from Table 7.2 into Eq. 7.2,

$$\delta_{11} = \sum \frac{s_1^2 L}{AE} = \frac{L}{AE}\sum s_1^2 = \frac{4.0}{50}$$

$$\delta_{21} = \delta_{12} = \sum \frac{s_1 s_2 L}{AE} = \frac{L}{AE}\sum s_1 s_2 = \frac{0.5}{50}$$

$$\delta_{31} = \delta_{13} = \sum \frac{s_1 s_3 L}{AE} = \frac{L}{AE}\sum s_1 s_3 = -\frac{0.707}{50}$$

$$\delta_{22} = \sum \frac{s_2^2 L}{AE} = \frac{L}{AE}\sum s_2^2 = \frac{4.0}{50}$$

$$\delta_{32} = \delta_{23} = \sum \frac{s_2 s_3 L}{AE} = \frac{L}{AE}\sum s_2 s_3 = -\frac{0.707}{50}$$

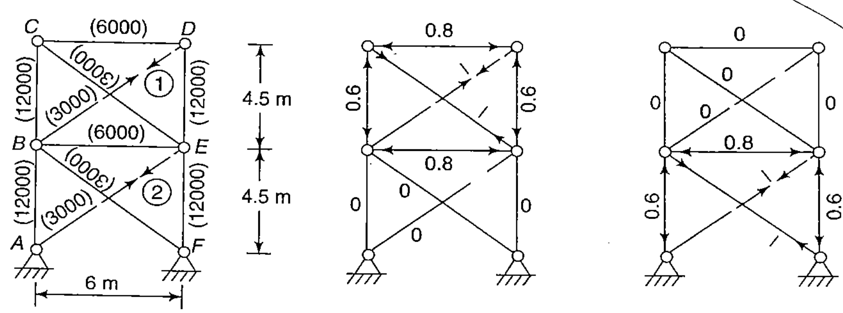
$$\delta_{33} = \sum \frac{s_3^2 L}{AE} = \frac{L}{AE}\sum s_3^2 = \frac{4.0}{50}$$

Hence, the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{50}\begin{bmatrix} 4.0 & 0.5 & -0.707 \\ 0.5 & 4.0 & -0.707 \\ -0.707 & -0.707 & 4.0 \end{bmatrix}$$

Example 7.3

Develop the flexibility matrix for the pin-jointed plane frame with reference to coordinates 1 and 2 shown in Fig. 7.3(a). The numbers in parentheses are the cross-sectional areas of the members in mm².



(a) (b) (c)

Fig. 7.3

Solution

To develop the flexibility matrix, forces s_1 and s_2 in all the members of the frame due to a unit force at coordinates 1 and 2 respectively have to be computed. Figure 7.3(b) shows forces s_1 in all the members due to a unit force at coordinate 1. These forces are listed in column 3 of Table 7.3. Similarly, Fig. 7.3(c) shows forces s_2 in all the members due to a unit force at coordinate 2. These forces are listed in column 4 of Table 7.3.

Table 7.3

Member	$\frac{L}{A}$ mm/mm²	s_1	s_2	$\frac{s_1^2 L}{A}$ mm/mm²	$\frac{s_2^2 L}{A}$ mm/mm²	$\frac{s_1 s_2 L}{A}$ mm/mm²
1	2	3	4	5	6	7
AB	0.375	0	−0.6	0	0.135	0
BC	0.375	−0.6	0	0.135	0	0
DE	0.375	−0.6	0	0.135	0	0
EF	0.375	0	−0.6	0	0.135	0
CD	1.0	−0.8	0	0.64	0	0
BE	1.0	−0.8	−0.8	0.64	0.64	0.64
CE	2.5	1.0	0	2.5	0	0
BF	2.5	0	1.0	0	2.5	0
BD	2.5	1.0	0	2.5	0	0
AE	2.5	0	1.0	0	2.5	0
			Σ	6.55	5.91	0.64

Tension +
Compression −

Substituting from Table 7.3 into Eq. (7.2),

$$\delta_{11} = \sum \frac{s_1^2 L}{AE} = \frac{1}{E}\sum \frac{s_1^2 L}{A} = \frac{6.55}{E}$$

$$\delta_{21} = \delta_{12} = \sum \frac{s_1 s_2 L}{AE} = \frac{1}{E}\sum \frac{s_1 s_2 L}{A} = \frac{0.64}{E}$$

$$\delta_{22} = \sum \frac{s_2^2 L}{AE} = \frac{1}{E}\sum \frac{s_2^2 L}{A} = \frac{5.91}{E}$$

Hence the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{E}\begin{bmatrix} 6.55 & 0.64 \\ 0.64 & 5.91 \end{bmatrix}$$

7.3 FORCE METHOD

The force method for the analysis of pin-jointed plane frames begins with the determination of the degree of static indeterminacy and identification of the redundants. The degree of static indeterminacy of pin-jointed plane frames has been discussed in Sec. 1.6. The frame may be statically indeterminate internally as well as externally. To make the frame statically determinate internally, a sufficient number of cuts should be introduced. Similarly, to make the frame statically determinate externally, a sufficient number of external reaction components may be released. A coordinate should be assigned to each of the internal forces released at a cut and to each of the external reaction components released.

To illustrate the manner in which coordinates are assigned in the force method of analysis of pin-jointed plane frames, consider the structure shown in Fig. 7.4(a). The frame has 10 joints. Hence, $2j - 3 = 2 \times 10 - 3 = 17$ members are necessary to make the frame internally stable and determinate.

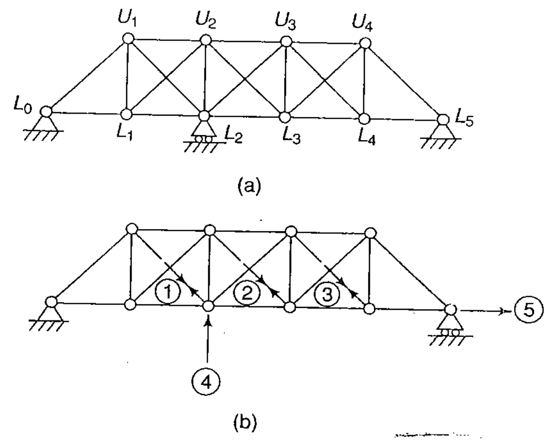


Fig. 7.4

The actual number of members is 20. Hence the frame is indeterminate internally to the third degree. Thus to make the frame statically determinate internally, three cuts have to be introduced. Let the cuts be provided in the members U_1L_2, U_2L_3 and U_3L_4. Hence, coordinates 1, 2, and 3 should be chosen as shown in Fig. 7.4(b). These coordinates correspond to the redundant forces P_1, P_2 and P_3 in members U_1L_2, U_2L_3 and U_3L_4 respectively. Each of the three coordinates comprises a pair of straight arrows pointing towards each other indicating that the released member forces are assumed to be tensile. The total number of external reaction components is five. Hence the degree of external indeterminacy is $5 - 3 = 2$. To make the frame statically determinate externally, two reaction components should be treated as redundant and released. The vertical reaction P_4 of the roller support at L_2 and the horizontal reaction P_5 of the hinge support at L_5 may be chosen for this purpose. Hence coordinates 4 and 5 should be assigned as shown in Fig. 7.4(b). These coordinates correspond to external redundants P_4 and P_5.

After the selection of the released structure and the system of coordinates, the force method for the analysis of pin-jointed plane frames is the same as for continuous beams discussed in Sec. 5.2. The chosen redundants may be determined from the compatibility conditions which lead to the equation

$$[P] = [\delta]^{-1}\{[\Delta] - [\Delta_L]\} \tag{7.3}$$

In the case of unyielding supports, Eq. (7.3) takes the form

$$[P] = -[\delta]^{-1}[\Delta_L] \tag{7.4}$$

In the case of yielding supports with prespecified settlements, Eq. (7.3) may be used in which the appropriate values of the prespecified settlements may be substituted into matrix $[\Delta]$.

Example 7.4

(i) Analyse the pin-jointed plane frame shown in Fig. 7.5(a). The flexibility for each member is 0.025 mm/kN.

(ii) If member L_1U_2 of the pin-jointed plane frame shown in Fig. 7.5(a) is too long by 2 mm, determine the forces in the members of the frame due to self-straining only.

(iii) If member L_1U_2 of the pin-jointed plane frame shown in Fig. 7.5(a) undergoes a rise of temperature of 32°C, determine the forces in the members of the frame due to rise of temperature only. Take coefficeint of expansion, $\alpha = 11.0 \times 10^{-6}$ per °C.

(iv) If the pin-jointed plane frame shown in Fig. 7.5(a) undergoes a uniform rise of temperature of 32°C, calculate the forces in the members of the frame due to rise of temperature only.

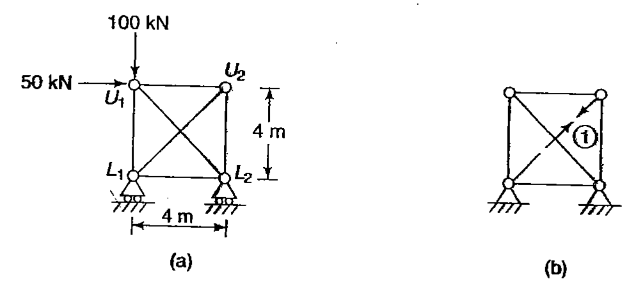


Fig. 7.5

Solution

(i) The number of joints is four. Hence the number of members required to make the frame stable and statically determinate internally is $2 \times 4 - 3 = 5$. The frame has six members. Hence the degree of internal indeterminacy is $6 - 5 = 1$. A cut may be introduced in diagonal member L_1U_2 to obtain the released structure as shown in Fig. 7.5(b). Coordinate 1 may be assigned to the member force thus released as shown in the figure. Forces S in the members of the released structure due to the applied loads are listed in column 2 of Table 7.4. Forces s_1 in the members of the released structure due to a unit force at coordinate 1 are listed in column 3 of Table 7.4.

Table 7.4

Member	S (kN)	s_l	s_l^2	Ss_l (kN)	Net force case (i) (kN)	Net force case (ii) (kN)
1	2	3	4	5	6	7
L_1L_2	50	−0.707	0.5	−35.35	37.5	14.14
L_2U_2	0	−0.707	0.5	0	−12.5	14.14
U_1U_2	0	−0.707	0.5	0	−12.5	14.14
L_1U_1	−50	−0.707	0.5	35.35	−62.5	14.14
L_2U_1	−70.7	1.0	1.0	−70.7	−53.0	−20.0
L_1U_2	0	1.0	1.0	0	17.7	−20.0
		Σ	4.0	−70.7		
				Tension	+	
				Compression	−	

Substituting from Table 7.4 into Eqs (7.1) and (7.2),

$$\Delta_{1L} = \sum \frac{Ss_1L}{AE} = \frac{L}{AE}\sum Ss_1$$

$$= -0.025 \times 70.7 = -1.77 \text{ mm}$$

$$\delta_{11} = \sum \frac{s_1^2 L}{AE} = \frac{L}{AE}\sum s_1^2 = 0.025 \times 4.0 = 0.1$$

Force P_1 in redundant member L_1U_2 may be calculated by using Eq. (7.4). It may be noted that in the present problem, all the matrices in the equation have only one element because there is only one coordinate.

$$[P_1] = -[\delta_{11}]^{-1}[\Delta_{1L}] = -[0.1]^{-1}[-1.77] = [17.7]$$

Knowing force P_1 in redundant member L_1U_2, the forces in other members may be calculated by adding the forces caused by the applied loads and the redundant as indicated by the equation

$$\text{Net force} = S + P_1s_1$$

These forces are listed in column 6 of Table 7.4.

(ii) As the effect of self-straining alone has to be considered, the external loads may be taken to be zero. Hence, $\Delta_{1L} = 0$. As redundant member L_1U_2, which is responsible for the initial lack of fit, is 2 mm too long, it would be under compression. Hence, $\Delta_1 = -2$ mm.
Substituting into Eq. (7.3),

$$[P_1] = [0.1]^{-1}\{[-2] - [0]\} = [-20.0]$$

Knowing force P_1 in redundant member L_1U_2, the forces in other members may be calculated. These forces are listed in column 7 of Table 7.4.

(iii) A free thermal expansion of redundant member $L_1U_2 = 5656 \times 11.0 \times 10^{-6} \times 32 = 2$ mm. Consequently, redundant member L_1U_2 is too long by 2 mm causing the same initial lack of fit as in case (ii). Hence the forces induced in the members of the frame are the same as in case (ii) and are listed in column 7 of Table 7.4.

(iv) It may be noted that a structure whether statically determinate or indeterminate internally, does not develop internal forces due to a uniform change of temperature unless the structure is externally indeterminate. As in the present case the structure is externally determinate, no forces would be set up due to the uniform rise of temperature.

Example 7.5

(i) Analyse the pin-jointed plane frame shown in Fig. 7.6(a). The numbers in parentheses are the cross-sectional areas of the members in mm².

(ii) If member L_1U_2 of the pin-jointed plane frame shown in Fig. 7.6(a) is too long by 3 mm, determine the forces in the members of frame due to self-straining only. Take E = 200 kN/mm².

(iii) If member L_1U_2 of the pin-jointed plane frame shown in Fig. 7.6(a) undergoes a rise of temperature of 36.4°C, determine the forces in the members of the frame due to rise of temperature only. Take coefficient of expansion, $\alpha = 11.0 \times 10^{-6}$ per °C.

(iv) If the pin-jointed plane frame shown in Fig. 7.6(a) undergoes a rise of temperature of 40°C, determine the forces in the members of the frame due to change of temperature only.

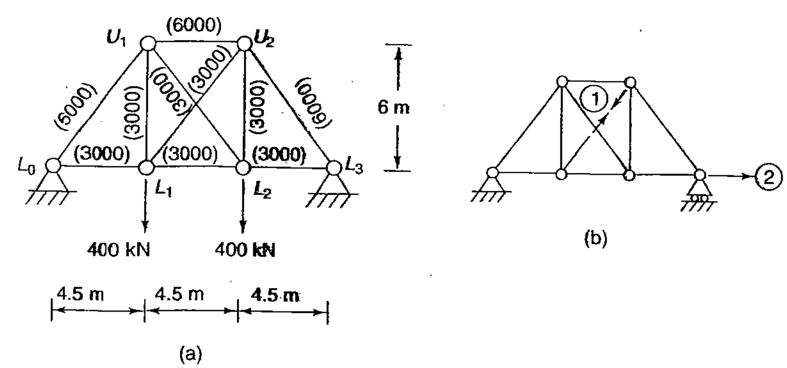


Fig. 7.6

Solution

(i) The number of joints is six. Hence the number of members required to make the frame stable and statically determinate internally is 2 × 6 − 3 = 9. The frame has ten members. Hence the degree of internal indeterminacy is $10 - 9 = 1$. A cut may be provided in diagonal L_1U_2 to make the frame statically determinate internally. The total number of reaction components is four. Hence the degree of external indeterminacy is $4 - 3 = 1$. The frame may be made determinate externally by replacing the hinge support at L_3 by a roller support, thereby releasing the horizontal reaction component at L_3. Coordinate 1 may be assigned to the redundant force in member L_1U_2 and coordinate 2 to the horizontal reaction component at L_3. The released structure and the chosen coordinates are shown in Fig. 7.6(b). Forces S in the members of the released structure due to the applied loads are listed in column 3 of Table 7.5(a). Forces s_1 and s_2 in the members of the released structure due to a unit force at coordinates 1 and 2 respectively have already been computed in Ex. 7.1. These forces are listed in columns 4 and 5 of Table 7.5(a).

Table 7.5(a)

Member	$\frac{L}{A}$ (mm/mm²)	S (kN)	s_1	s_2	$\frac{Ss_1L}{A}$ (kN/mm)	$\frac{Ss_2L}{A}$ (kN/mm)
1	2	3	4	5	6	7
U_1U_2	0.75	−300	−0.6	0	135	0
L_0L_1	1.5	300	0	1	0	450
L_1L_2	1.5	300	−0.6	1	−270	450
L_2L_3	1.5	300	0	1	0	450
U_1L_0	1.25	−500	0	0	0	0
U_1L_1	2.0	400	−0.8	0	−640	0
U_1L_2	2.5	0	1.0	0	0	0
U_2L_2	2.0	400	−0.8	0	−640	0
U_2L_3	1.25	−500	0	0	0	0
U_2L_1	2.5	0	1.0	0	0	0
				Σ	−1415	1350

Tension +
Compression −

Substituting from Table 7.5(a) into Eq. (7.1),

$$\Delta_{1L} = \sum \frac{Ss_1L}{AE} = \frac{1}{E}\sum \frac{Ss_1L}{A} = -\frac{1415}{E}$$

$$\Delta_{2L} = \sum \frac{Ss_2L}{AE} = \frac{1}{E}\sum \frac{Ss_2L}{A} = \frac{1350}{E}$$

The flexibility matrix for the released structure has already been developed in Ex. 7.1.

As the supports are unyielding, substituting into Eq. (7.4),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -E\begin{bmatrix} 8.37 & -0.90 \\ -0.90 & 4.50 \end{bmatrix}\begin{bmatrix} -\frac{1415}{E} \\ \frac{1350}{E} \end{bmatrix} = \begin{bmatrix} 139.8 \\ -272 \end{bmatrix}$$

Knowing forces P_1 and P_2, the forces in other members may be calculated by adding the forces caused by the applied loads and the redundants as indicated by the equation

$$\text{Net force} = S + P_1s_1 + P_2s_2$$

These forces are listed in column 2 of Table 7.5(b).

Table 7.5(b)

Member (kN)	Force, case (i) (kN)	Force, case (ii) (kN)	Force, case (iv)
1	2	3	4
U_1U_2	−383.9	44.0	17.4
L_0L_1	28.0	−14.7	−269.8
L_1L_2	−55.9	29.3	−252.4
L_2L_3	28.0	−14.7	269.8
U_1L_0	−500.0	0	0
U_1L_1	288.2	58.6	23.2
U_1L_2	139.8	−73.3	−29.0
U_2L_2	288.2	58.6	23.2
U_2L_3	−500.0	0	0
U_2L_1	139.8	−73.3	−29
		Tension	+
		Compression	−

(ii) As the effect of self-straining alone has to be considered, the external loads may be taken to be zero. Hence, $[\Delta_L]$ is a null matrix. As redundant member L_1U_2, which is responsible for the initial lack of fit, is 3 mm too long, it would be under compression. Hence, $\Delta_1 = -3$ mm and $\Delta_2 = 0$. Substituting into Eq. (7.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 200\begin{bmatrix} 8.37 & -0.90 \\ -0.90 & 4.50 \end{bmatrix}^{-1}\left\{\begin{bmatrix} -3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} -73.3 \\ -14.7 \end{bmatrix}$$

Knowing forces P_1 and P_2, the forces in other members of the frame may be calculated. These forces are listed in column 3 of Table 7.5(b).

(iii) A free thermal expansion of redundant member $L_1U_2 = 7500 \times 11.0 \times 10^{-6} \times 36.4 = 3$ mm. Consequently, redundant member L_1U_2 is too long by 3 mm causing the same initial lack of fit as in case (ii). Hence the forces induced in the members of the frame are the same as in case (ii) and are listed in column 3 of Table 7.5(b).

(iv) As the structure is statically indeterminate externally, even a uniform change of temperature will induce internal forces in the structure. The free thermal expansion of the bottom chord $L_0L_3 = 13500 \times 11.0 \times 10^{-6} \times 40 = 5.94$ mm. Hence, $\Delta_1 = 0$ and $\Delta_2 = -5.94$ mm. For no load condition, $\Delta_{1L} = 0$ and $\Delta_{2L} = 0$. Substituting into Eq. (7.3),

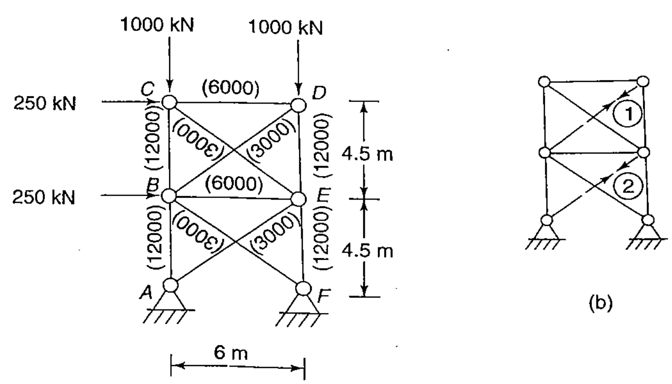
$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 200\begin{bmatrix} 8.37 & -0.90 \\ -0.90 & 4.50 \end{bmatrix}^{-1}\left\{\begin{bmatrix} 0 \\ -5.94 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} -29 \\ -269.8 \end{bmatrix}$$

Knowing forces P_1 and P_2, the forces in other members of the frame may be calculated. These forces are listed in column 4 of Table 7.5(b).

Example 7.6

Analyse the pin-jointed plane frame shown in Fig. 7.7(a). The numbers in parentheses are the cross-sectional areas of the members in mm².



(a) (b)

Fig. 7.7

Solution

Total number of joints, $j = 6$. Hence the number of independent equations is $2j = 2 \times 6 = 12$. As the frame has 10 members, the number of unknown member forces, $m = 10$. As there are two hinged supports, the number of reaction components, $r = 2 \times 2 = 4$. Thus the total number of unknowns is $(m + r) = 10 + 4 = 14$. Degree of static indeterminacy, $D_s = 14 - 12 = 2$. In order to make the frame statically determinate, cuts

may be provided in members BD and AE. Coordinates 1 and 2 may be assigned to the redundant forces in members BD and AE. The released structure and the chosen coordinates are shown in Fig. 7.7(b). Forces S in the members of the released structure due to the applied loads are listed in column 3 of Table 7.6. Forces s_1 and s_2 in the members of the released structure due to a unit force at coordinates 1 and 2 respectively, have already been computed in Ex. 7.3. These forces are listed in columns 4 and 5 of Table 7.6.

Table 7.6

Member	$\frac{L}{A}$ (mm/mm²)	S (kN)	s_1	s_2	$\frac{Ss_1L}{A}$ (kN/mm)	$\frac{Ss_2L}{A}$ (kN/mm)	Net force = $S + \Sigma Ps$ (kN)
1	2	3	4	5	6	7	8
AB	0.375	−437.5	0	−0.6	0	98.44	−575.1
BC	0.375	−812.5	−0.6	0	182.81	0	−851.6
DE	0.375	−1000.0	−0.6	0	225.00	0	−1039.1
EF	0.375	−1187.5	0	−0.6	0	267.19	−1325.1
CD	1.00	0	−0.8	0	0	0	−52.1
BE	1.00	250.0	−0.8	−0.8	−200.00	−200.00	14.5
CE	2.50	−312.5	1.0	0	−781.25	0	−247.4
BF	2.50	−625.0	0	1.0	0	−1562.50	−395.7
BD	2.50	0	1.0	0	0	0	65.1
AE	2.50	0	0	1.0	0	0	229.3
				Σ	−573.44	−1396.87	

Tension +
Compression −

Substituting from Table 7.6 into Eq. (7.1),

$$\Delta_{1L} = \sum \frac{Ss_1L}{AE} = \frac{1}{E}\sum \frac{Ss_1L}{A} = -\frac{573.44}{E}$$

$$\Delta_{2L} = \sum \frac{Ss_2L}{AE} = \frac{1}{E}\sum \frac{Ss_2L}{A} = -\frac{1396.87}{E}$$

The flexibility matrix $[\delta]$ for the released structure has been developed in Ex. 7.3.

Substituting into Eq. (7.4),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -E\begin{bmatrix} 6.55 & 0.64 \\ 0.64 & 5.91 \end{bmatrix}^{-1}\begin{bmatrix} -573.44/E \\ -1396.87/E \end{bmatrix} = \begin{bmatrix} 65.1 \\ 229.3 \end{bmatrix}$$

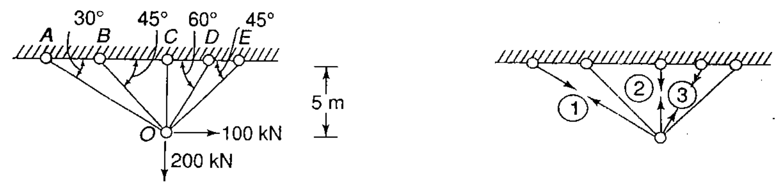
Knowing forces P_1 and P_2, the forces in other members of the frame may be calculated by adding the forces caused by the applied loads and the redundants as indicated by the equation

$$\text{Net force} = S + P_1 s_1 + P_2 s_2$$

These forces are listed in column 8 of Table 7.6.

Example 7.7

Analyse the pin-jointed structure shown in Fig. 7.8(a). The cross-sectional area of each member is 2000 mm². Take E = 200 kN/mm².



(a) (b)

Fig. 7.8

Solution

At joint O there are five unknown member forces and only two independent equations are available there. Hence, the degree of static indeterminacy, $D_s = 5 - 2 = 3$. In order to make the frame statically determinate, cuts may be provided in members OA, OC and OD. Coordinates 1, 2 and 3 may be assigned to the redundant forces in members OA, OC and OD respectively. The released structure and the chosen coordinates are shown in Fig. 7.8(b). Forces S in the members of the released structure due to the applied loads are listed in column 3 of Table 7.7(a). Forces s_1, s_2 and s_3 in the members of the released structure due to a unit force at coordinates 1, 2 and 3 respectively are listed in columns 4, 5 and 6 of Table 7.7(a). Table 7.7(b) shows the necessary computations for the evaluation of the elements of the flexibility matrix.

Table 7.7(a)

Member	L (m)	S (kN)	s_1	s_2	s_3	Ss_1L (kN-m)	Ss_2L (kN-m)	Ss_3L (kN-m)
1	2	3	4	5	6	7	8	9
OB	7.071	212.1	−0.966	−0.707	−0.259	−1448.8	−1060.3	−388.4
OE	7.071	70.7	0.259	−0.707	−0.966	129.5	−353.4	−482.9
OA	10.0	0	1.0	0	0	0	0	0
OC	5.0	0	0	1.0	0	0	0	0
OD	5.774	0	0	0	1.0	0	0	0
					Σ	−1319.3	−1413.7	−871.3

Tension +
Compression −

Table 7.7(b)

Member	s_1^2L (m)	s_2^2L (m)	s_3^2L (m)	s_1s_2L (m)	s_2s_3L (m)	s_3s_1L (m)	Not force = $S + \Sigma Ps$ (kN)
1	2	3	4	5	6	7	8
OB	6.6	3.54	0.47	4.83	1.30	1.77	87
OE	0.47	3.54	6.6	−1.30	4.83	−1.77	−1
OA	10.0	0	0	0	0	0	60
OC	0	5.0	0	0	0	0	86
OD	0	0	5.774	0	0	0	27
Σ	17.07	12.08	12.844	3.53	6.13	0	

Tension +
Compression −

Substituting from Table 7.7(a) into Eq. (7.1),

$$\Delta_{1L} = \sum \frac{Ss_1L}{AE} = \frac{1}{AE}\sum Ss_1L$$

$$= \frac{1}{2000 \times 200} \times (-1319.3) = -\frac{1319.3}{400000}\ \text{m}$$

$$\Delta_{2L} = \sum \frac{Ss_2L}{AE} = \frac{1}{AE}\sum Ss_2L = -\frac{1413.7}{400000}\ \text{m}$$

$$\Delta_{3L} = \sum \frac{Ss_3L}{AE} = \frac{1}{AE}\sum Ss_3L = -\frac{871.3}{400000}\ \text{m}$$

Substituting from Table 7.7(b) into Eq. (7.2),

$$\delta_{11} = \sum \frac{s_1^2L}{AE} = \frac{1}{AE}\sum s_1^2L = \frac{17.07}{400000}$$

$$\delta_{21} = \delta_{12} = \sum \frac{s_1s_2L}{AE} = \frac{1}{AE}\sum s_1s_2L = \frac{3.53}{400000}$$

$$\delta_{31} = \delta_{13} = \sum \frac{s_1s_3L}{AE} = \frac{1}{AE}\sum s_1s_3L = 0$$

$$\delta_{22} = \sum \frac{s_2^2L}{AE} = \frac{1}{Ae}\sum s_2^2L = \frac{12.08}{400000}$$

$$\delta_{32} = \delta_{23} = \sum \frac{s_2s_3L}{AE} = \frac{1}{AE}\sum s_2s_3L = \frac{6.13}{400000}$$

$$\delta_{33} = \sum \frac{s_3^2L}{AE} = \frac{1}{AE}\sum s_3^2L = \frac{12.844}{400000}$$

Substituting into Eq. (7.4)

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -400000 \begin{bmatrix} 17.07 & 3.53 & 0 \\ 3.53 & 12.08 & 6.13 \\ 0 & 6.13 & 12.844 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1319.3}{400000} \\ -\frac{1413.7}{400000} \\ -\frac{871.3}{400000} \end{bmatrix} = \begin{bmatrix} 60 \\ 86 \\ 27 \end{bmatrix}$$

Knowing forces P_1, P_2 and P_3, the forces in the other members of the structure may be calculated by adding the forces caused by the applied loads and the redundants as indicated by the equation

$$\text{Net force} = S + P_1 s_1 + P_2 s_2 + P_3 s_3$$

These forces are listed in column 8 of Table 7.7(b).

Example 7.8

Analyse the pin-jointed plane frame shown in Fig. 7.9(a) for the following support conditions:

(i) There is no displacement at support L_4.

(ii) The horizontal displacement at support L_4 is 6 mm towards right.

Axial flexibility of each member of the frame is 0.02 mm/kN.

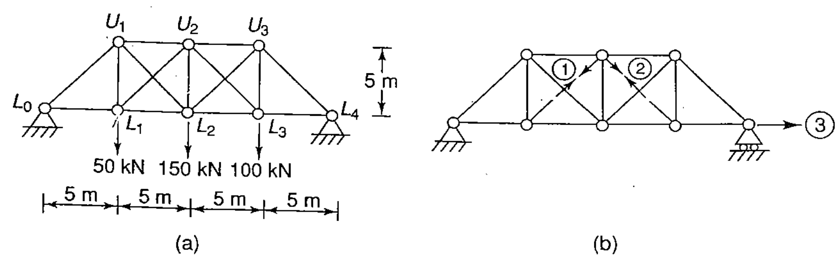


Fig. 7.9

Solution

The number of joints is 8. Hence the number of members required to make the frame stable and statically determinate internally is $2 \times 8 - 3 = 13$. The frame has 15 members. Hence the degree of internal indeterminacy is $15 - 13 = 2$. Cuts may be provided in diagonals L_1U_2 and U_2L_3 to make the frame statically determinate internally. The total number of reaction components is four. Hence the degree of external indeterminacy is $4 - 3 = 1$. The frame may be made statically determinate externally by replacing the hinge support at L_4 by a roller support, thereby releasing the horizontal reaction component at L_4. Coordinates 1 and 2 may be assigned to the redundant forces in members L_1U_2 and U_2L_3 and coordinate 3 to the horizontal reaction component at L_4.

The released structure and the chosen coordinates are shown in Fig. 7.9(b). Forces S in the members of the released structure due to the applied loads are listed in column 2 of Table 7.8(a). Forces s_1, s_2 and s_3 in the members of the released structure due to a unit force at coordinates 1, 2 and 3 respectively have already been computed in Ex. 7.2. These are listed in columns 3, 4 and 5 of Table 7.8(a).

Table 7.8(a)

Member	S *(kN)*	s_1	s_2	s_3	Ss_1 *(kN)*	Ss_2 *(kN)*	Ss_3 *(kN)*
1	2	3	4	5	6	7	8
U_1U_2	−225	−0.707	0	0	159.1	0	0
U_2U_3	−225	0	−0.707	0	0	159.1	0
L_0L_1	137.5	0	0	1.0	0	0	137.5
L_1L_2	137.5	−0.707	0	1.0	−97.2	0	137.5
L_2L_3	162.5	0	−0.707	1.0	0	−114.9	162.5
L_3L_4	162.5	0	0	1.0	0	0	162.5
U_1L_0	−194.5	0	0	0	0	0	0
U_1L_1	50.0	−0.707	0	0	−35.4	0	0
U_1L_2	123.7	1.0	0	0	123.7	0	0
U_2L_2	0	−0.707	−0.707	0	0	0	0
U_3L_2	88.4	0	1.0	0	0	88.4	0
U_3L_3	100.0	0	−0.707	0	0	−70.7	0
U_3L_4	−229.8	0	0	0	0	0	0
U_2L_3	0	0	1.0	0	0	0	0
U_2L_1	0	1.0	0	0	0	0	0
				Σ	150.2	61.9	600.0
					Tension	+	
					Compression	−	

Substituting from Table 7.8(a) into Eq. (7.1),

$$\Delta_{1L} = \sum \frac{Ss_1 L}{AE} = \frac{L}{AE} \sum Ss_1 = 0.02 \times 150.2 = 3.004 \text{ mm}$$

$$\Delta_{2L} = \sum \frac{Ss_2 L}{AE} = \frac{L}{AE} \sum Ss_2 = 0.02 \times 61.9 = 1.238 \text{ mm} \qquad \text{(a)}$$

$$\Delta_{3L} = \sum \frac{Ss_3 L}{AE} = \frac{L}{AE} \sum Ss_3 = 0.02 \times 600 = 12 \text{ mm}$$

The flexibility matrix for the released structure has already been obtained in Ex. 7.2. The values of redundant forces P_1, P_2 and P_3 depend upon the support conditions.

(i) As in this case supports are unyilding, substituting into Eq. (7.4)

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -50 \begin{bmatrix} 4.0 & 0.5 & -0.707 \\ 0.5 & 4.0 & -0.707 \\ -0.707 & -0.707 & 4.0 \end{bmatrix} \begin{bmatrix} 3.004 \\ 1.238 \\ 12.0 \end{bmatrix}^{-1} = \begin{bmatrix} -62.5 \\ -37.3 \\ -167.6 \end{bmatrix}$$

Knowing forces P_1, P_2 and P_3, the forces in other members of the frame may be calculated by adding the forces caused by the applied loads and the redundants as indicated by the equation

$$\text{Net force} = S + P_1 s_1 + P_2 s_2 + P_3 s_3$$

These forces are listed in column 2 of Table 7.8(b).

Table 7.8(b)

Member	*Net Force in kN*	
	Case (i)	*Case (ii)*
1	2	3
U_1U_2	−180.8	−189.6
U_2U_3	−198.6	−207.5
L_0L_1	−30.1	49.3
L_1L_2	14.2	84.7
L_2L_3	21.3	91.8
L_3L_4	−5.1	74.3
U_1L_0	−194.5	−194.5
U_1L_1	94.2	85.4
U_1L_2	61.2	73.7
U_2L_2	70.6	52.9
U_3L_2	51.1	63.6
U_3L_3	126.4	117.5
U_3L_4	−229.8	−229.8
L_1U_2	−6.25	−50.0
U_2L_3	−37.3	−24.8
	Tension	+
	Compression	−

(ii) In this case the horizontal displacement at support L_4 is 6 mm towards right. Hence, Δ_3 = 6 mm. Also, for the continuity of the structure $\Delta_1 = \Delta_2 = 0$. Substituting into Eq. (7.3),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 50 \begin{bmatrix} 4.0 & 0.5 & -0.707 \\ 0.5 & 4.0 & -0.707 \\ -0.707 & -0.707 & 4.0 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 3.004 \\ 1.238 \\ 12.0 \end{bmatrix} \right\} = \begin{bmatrix} -50.0 \\ -24.8 \\ -88.2 \end{bmatrix}$$

Knowing forces P_1, P_2 and P_3, the forces in the other members of the frame may be calculated. These forces are listed in column 3 of Table 7.8(b).

7.4 STIFFNESS OF A PIN JOINT

A pin joint offers resistance to translation because it entails elongations or contractions of the members meeting at the joint. The translational stiffness of a pin joint in any chosen direction is defined as the force required to produce unit displacement in the chosen direction. It follows that the stiffness of a pin joint depends upon the axial stiffness of the members meeting at the joint.

Figure 7.10 shows a typical pin-joint O of a pin-jointed plane frame. At joint O under consideration, coordinates i and j have been chosen along the positive directions of x and y axes respectively.

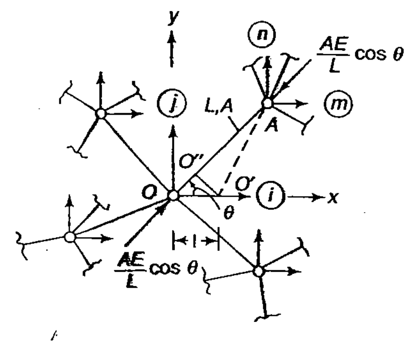


Fig. 7.10

7.4.1. Displacement Along Coordinate *i*

When a unit displacement is given at coordinate i without any displacement at other coordinates so that joint O moves to the new position O', the members meeting at joint O undergo changes in lengths. The contraction of member OA is $OO'' = \cos\theta$. The compressive force required to produce this contraction of member OA is evidently $\frac{AE}{L}\cos\theta$. The component of this force along coordinate i is $\frac{AE}{L}\cos^2\theta$. Similarly, considering other members meeting at joint O, it is evident that net force k_{ii} required to displace joint O by unit distance along coordinate i is given by the equation

$$k_{ii} = \sum \frac{AE}{L}\cos^2\theta \qquad (7.5)$$

where $\frac{AE}{L}$ = axial stiffness of a member meeting at joint O

θ = inclination of the member measured counter-clockwise from the positive direction of x-axis.

Summation should be carried out to include all the members meeting at joint O.

The force at coordinate j due to the compressive force $\frac{AE}{L}\cos\theta$ in member OA is equal to $\frac{AE}{L}\cos\theta\sin\theta$. Hence considering all the members meeting at joint O, the net force k_{ji} at coordinate j due to a unit displacement at coordinate i is given by the equation

$$k_{ji} = \sum \frac{AE}{L}\cos\theta\sin\theta \tag{7.6}$$

If m and n are the coordinates at joint A along the positive directions of x and y axes respectively, the force k_{mi} at coordinate m due to a unit displacement at coordinate i is equal to the component of the compressive force $\frac{AE}{L}\cos\theta$ in member OA in the direction of coordinate m. Hence,

$$k_{mi} = -\frac{AE}{L}\cos^2\theta \tag{7.7}$$

Similarly, force k_{ni} at coordinate n due to a unit displacement at coordinate i is given by the equation

$$k_{ni} = -\frac{AE}{L}\cos\theta\sin\theta \tag{7.8}$$

7.4.2 Displacement Along Coordinate j

When a unit displacement is given to joint O along coordinate j, the contraction of member OA is equal to sin θ. The compressive force required to produce this contraction to member OA is $\frac{AE}{L}\sin\theta$. The component of this force along coordinate j is $\frac{AE}{L}\sin^2\theta$. Considering all the members meeting at joint O, the net force k_{jj} required to displace joint O by unit distance along coordinate j is given by the equation

$$k_{jj} = \sum \frac{AE}{L}\sin^2\theta \tag{7.9}$$

Similarly, forces k_{ij}, k_{mj} and k_{nj} at coordinates i, m and n due to a unit displacement along coordinate j may be computed. These are given by the following equations:

$$k_{ij} = \sum \frac{AE}{L}\sin\theta\cos\theta \tag{7.10a}$$

$$k_{mj} = -\frac{AE}{L}\sin\theta\cos\theta \tag{7.10b}$$

$$k_{nj} = -\frac{AE}{L}\sin^2\theta \tag{7.10c}$$

It may be noted that the algebraic sum of the forces at the coordinates due to a unit displacement at coordinate i vanishes, thereby satisfying the basic condition of static equilibrium, $\Sigma X = 0$. Similarly, the algebraic sum of the forces at the coordinates due to a unit displacement at coordinate j vanishes, thus satisfying the condition, $\Sigma Y = 0$.

From Eqs (7.5) to (7.10) the following inferences may be drawn. These are useful in the computation of the stiffness elements.

(i) The stiffness element k_{pq} is given by the equation

$$k_{pq} = \sum \frac{AE}{L}\cos^2\theta \tag{7.11a}$$

if $p = q$ and the coordinates p and q are directed along the x-axis.

(ii) The stiffness element k_{pq} is given by the equation

$$k_{pq} = \sum \frac{AE}{L}\sin^2\theta \tag{7.11b}$$

if $p = q$ and coordinates p and q are directed along the y-axis.

(iii) The stiffness element k_{pq} is given by the equation

$$k_{pq} = \sum \frac{AE}{L}\sin\theta\cos\theta \tag{7.11c}$$

if coordinates p and q are located at the same joint and are orthogonal.

(iv) The stiffness element k_{pq} is given by the equation

$$k_{pq} = -\frac{AE}{L}\cos^2\theta \tag{7.11d}$$

if coordinates p and q are located at the two ends of the member and are directed along the x-axis.

(v) The stiffness element k_{pq} is given by the equation

$$k_{pq} = -\frac{AE}{L}\sin^2\theta \tag{7.11e}$$

if coordinates p and q are located at the two ends of the member and are directed along the y-axis.

(vi) The stiffness element k_{pq} is given by the equation

$$k_{pq} = -\frac{AE}{L}\sin\theta\cos\theta \tag{7.11f}$$

if coordinates p and q are located at the two ends of the member and are orthogonal.

It may be noted that the above expressions are based on the assumptions that coordinates p and q are taken along the positive directions of the x- and y-axes and the angle θ is always measured counter clockwise from the positive direction of x-axis.

It may sometimes become necessary to choose coordinates at a joint along directions other than horizontal and vertical directions (Ex. 7.17). Referring to Fig. 7.10, if coordinates m and n at joint A are rotated counter-clockwise through an angle α as indicated in Fig. 7.11, the stiffness elements can be expressed as follows:

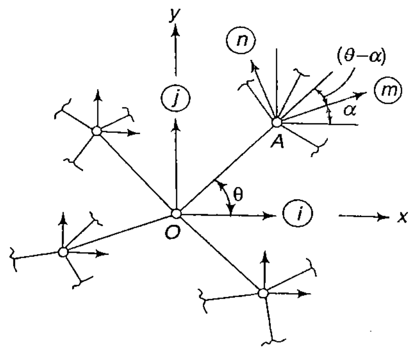


Fig. 7.11

$$
\begin{aligned}
k_{ii} &= \sum \frac{AE}{L} \cos^2 \theta \\
k_{jj} &= \sum \frac{AE}{L} \sin^2 \theta \\
k_{ij} = k_{ji} &= \sum \frac{AE}{L} \sin \theta \cos \theta \\
k_{mi} = k_{im} &= -\frac{AE}{L} \cos \theta \cos(\theta - \alpha) \\
k_{ni} = k_{in} &= -\frac{AE}{L} \cos \theta \sin(\theta - \alpha) \\
k_{mj} = k_{jm} &= -\frac{AE}{L} \sin \theta \cos(\theta - \alpha) \\
k_{nj} = k_{jn} &= -\frac{AE}{L} \sin \theta \sin(\theta - \alpha)
\end{aligned}
\qquad (7.12)
$$

$$
\begin{aligned}
k_{mm} &= \sum \frac{AE}{L} \cos^2 (\theta - \alpha) \\
k_{nn} &= \sum \frac{AE}{L} \sin^2 (\theta - \alpha) \\
k_{mn} = k_{nm} &= \sum \frac{AE}{L} \cos (\theta - \alpha) \sin(\theta - \alpha)
\end{aligned}
$$

Example 7.9

Calculate the stiffness elements k_{11}, k_{21}, k_{31}, k_{41}, k_{12}, k_{22}, k_{32} and k_{42} with reference to the coordinates shown in Fig. 7.12. The lengths of the members are shown in the figure. The numbers in parentheses are the cross-sectional areas of the members in mm². Take $E = 200$ kN/mm².

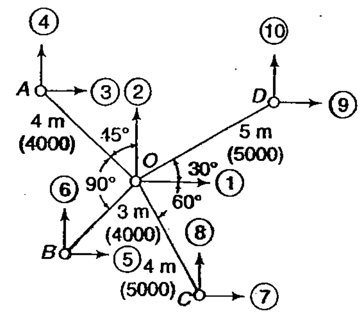


Fig. 7.12

Solution

The computations required for the evaluation of the stiffness elements may be carried out as indicated in Table 7.9. The axial stiffness $\frac{AE}{L}$ and the inclination θ of each member are listed in columns 2 and 3 respectively

Table 7.9

Member	$\frac{AE}{L}$ (kN/mm)	θ	$\cos\theta$	$\sin\theta$	$\frac{AE}{L}\cos^2\theta$	$\frac{AE}{L}\sin^2\theta$	$\frac{AE}{L}\times$ $\cos\theta \sin\theta$
1	2	3	4	5	6	7	8
OA	200.00	135°	−0.7071	0.7071	100.00	100.00	−100.00
OB	266.67	225°	−0.7071	−0.7071	133.33	133.33	133.33
OC	250.00	300°	0.5000	−0.8660	62.50	187.50	−108.25
OD	200.00	30°	0.8660	0.5000	150.00	50.00	86.60
				Σ	445.83	470.83	11.68

Using Table 7.9 and Eq. (7.11), the stiffness elements in kN/mm are

$$k_{11} = \sum \frac{AE}{L} \cos^2 \theta = 445.83$$

$$k_{21} = k_{12} = \sum \frac{AE}{L} \sin \theta \cos \theta = 11.68$$

$$k_{31} = -\frac{AE}{L} \cos^2 \theta \text{ (for member } OA) = -100.00$$

$$k_{41} = -\frac{AE}{L} \sin \theta \cos \theta \text{ (for member } OA) = 100.00$$

$$k_{22} = \sum \frac{AE}{L} \sin^2 \theta = 470.83$$

$$k_{32} = -\frac{AE}{L} \sin \theta \cos \theta \text{ (for member } OA) = 100.00$$

$$k_{42} = -\frac{AE}{L} \sin^2 \theta \text{ (for member } OA) = -100.00$$

7.5 MEMBER FORCES

Figure 7.13 shows a typical member AB connecting joints A and B of a pin-jointed plane frame. The force in member AB can be calculated if the displacements at the two ends of the member are known. Let A' and B' be the displaced positions of joints A and B. The components of the displacement at joint A along x- and y-axes are Δ_{Ax} and Δ_{Ay} respectively. Similarly Δ_{Bx} and Δ_{By} are the components of the displacement of joint B along x- and y-axes respectively. From the figure it is clear that the shortening of the member due to the displacement of joint A is $(\Delta_{Ax} \cos \theta_{AB} + \Delta_{Ay} \sin \theta_{AB})$. Also, the elongation of the member due to the displacement of joint B is $(\Delta_{Bx} \cos \theta_{AB} + \Delta_{By} \sin \theta_{AB})$. Hence the net shortening of the member is $[(\Delta_{Ax} - \Delta_{Bx}) \cos \theta_{AB} + (\Delta_{Ay} - \Delta_{By}) \sin \theta_{AB}]$. Consequently, the force in member AB is given by the equation

$$S_{AB} = -\frac{AE}{L}\left[\left(\Delta_{Ax} - \Delta_{Bx}\right)\cos \theta_{AB} + \left(\Delta_{Ay} - \Delta_{By}\right)\sin \theta_{AB}\right] \quad (7.13a)$$

Minus sign on the right hand side of the equation indicates that the force S_{AB} is compressive.

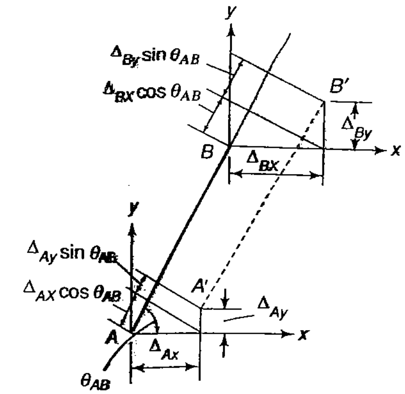


Fig. 7.13

If the member is reckoned as BA instead of AB, then the inclination of the member is measured at end B and the equation for the force in the member may be written as

$$S_{BA} = -\frac{AE}{L}\left[\left(\Delta_{Bx} - \Delta_{Ax}\right)\cos \theta_{BA} + \left(\Delta_{By} - \Delta_{Ay}\right)\sin \theta_{BA}\right] \quad (7.13b)$$

It may be noted that the inclination of a member at any joint is always measured counter-clockwise from the positive direction of x-axis. Also, S_{AB} and S_{BA} will be found to be equal because the axial force in a member of a pin-jointed frame is constant throughout its length. Forces in other members of the pin-jointed frame may be computed in a similar manner.

Example 7.10

Figure 7.14 shows member AB of a pin-jointed plane frame. Determine the force in the member if the displacements of joints A and B are

$$\Delta_{Ax} = 1.5 \text{ mm}$$
$$\Delta_{Ay} = -2.5 \text{ mm}$$
$$\Delta_{Bx} = -0.5 \text{ mm}$$
$$\Delta_{By} = -2 \text{ mm}$$

The length and cross-sectional area of the member are 5000 mm and 4000 mm² respectively. Take E = 200 kN/mm².

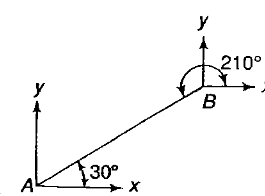


Fig. 7.14

Solution

The axial stiffness of the member,

$$\frac{AE}{L} = \frac{4000 \times 200}{5000} = 160 \text{ kN/mm}$$

Inclination of member AB at joint A,

$$\theta_{AB} = 30°$$

Substituting into Eq. 7.13(a), the force in member AB,

$$S_{AB} = -160[1.5 + 0.5](0.866) + (-2.5 + 2.0)(0.5)]$$
$$= -237.12 \text{ kN}$$

The minus sign indicates that the force in the member is compressive.

Alternatively, if the member is taken as BA, the inclination of BA at joint B,

$$\theta_{BA} = 210°$$

Substituting into Eq. 7.13(b), the force in member BA,

$$S_{BA} = -160[(-0.5 - 1.5)(-0.866) + (-2.0 + 2.5)(-0.5)]$$
$$= -237.12 \text{ kN}$$

It may be noted that S_{AB} and S_{BA} are equal.

Example 7.11

Figure 7.15 shows a pin-jointed triangular frame ABC. Calculate the forces in the members of the frame if the horizontal displacements of joints A and C are 0.568 mm towards right and 1.2 mm towards left respectively and the upward displacement of joint A is 2.576 mm. The numbers in parentheses are the cross-sectional areas of the members in mm². Take E = 200 kN/mm².

Solution

From the given data,

$$\Delta_{Ax} = 0.568 \text{ mm}$$
$$\Delta_{Ay} = 2.576 \text{ mm}$$
$$\Delta_{Cx} = -1.2 \text{ mm}$$
$$\Delta_{Bx} = \Delta_{By} = \Delta_{Cy} = 0$$

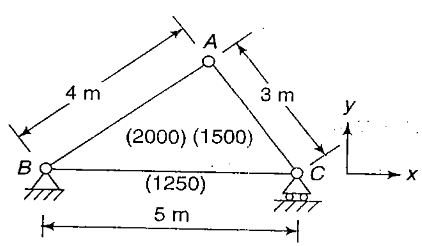


Fig. 7.15

For member BA,

$$\frac{AE}{L} = \frac{2000 \times 200}{4000} = 100 \text{ kN/mm}$$

$$\theta_{BA} = \cos^{-1} 0.8 = 36°52'$$

$$S_{BA} = -100[(0 - 0.568) \cos 36°52' + (0 - 2.576) \sin 36°52']$$
$$= -100[(-0.568)(0.8) + (-2.576)(0.6)]$$
$$= 200 \text{ kN (tensile)}$$

For member BC

$$\frac{AE}{L} = \frac{1250 \times 200}{5000} = 50 \text{ kN/mm}$$

$$\theta_{BC} = 0°$$

$$S_{BC} = -50[(0 + 1.2) \cos 0° + (0 - 0) \sin 0°]$$
$$= -60 \text{ kN (compressive)}$$

For member CA,

$$\frac{AE}{L} = \frac{1500 \times 200}{3000} = 100 \text{ kN/mm}$$

$$\theta_{CA} = 180° - \cos^{-1} 0.6 = 180° - 53°8' = 126°52'$$

$$S_{CA} = -100[(-1.2 - 0.568) \cos 126°52' + (0 - 2.576) \sin 126°52']$$
$$= 100 \text{ kN (tensile)}$$

7.6 DISPLACEMENT METHOD

The displacement method of analysis of pin-jointed frames begins with the determination of degree of freedom of the structure which is equal to the number of independent displacement components. The degree of kinematic indeterminacy has been discussed in Sec. 1.7. It may be noted that as per Eq. (1.20c), the degree of freedom of a pin-jointed plane frame is $(2j - r)$ where j is the number of joints and r is the number of external reaction components. After the independent displacement components have been identified, a coordinate is assigned to each of them. The stiffness matrix with reference to the chosen coordinates is then developed. The elements of the stiffness matrix can be computed by using Eq. (7.11). As the forces acting at the coordinates are known, the force-displacement relationship can be used to determine the displacements at the coordinates. After the displacements are known, the member forces may be computed by using Eq. (7.13). The method may be described systematically by the following steps:

(i) Determine the degree of freedom of the structure.

(ii) Assign one coordinate to each of the independent displacement components. Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be the independent displacement components at coordinates 1, 2, ..., n.

(iii) Develop the stiffness matrix with reference to the chosen coordinates. To generate the jth column of the stiffness matrix, give a unit displacement at coordinate j without any displacement at other coordinates. Using Eq. (7.11), compute the forces $k_{1j}, k_{2j}, \ldots, k_{nj}$ at coordinates 1, 2, ..., n. These forces constitute the elements of the jth column of the stiffness matrix.

(iv) Use the force-displacement relationship to determine the displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$.

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} \tag{a}$$

Rearranging the terms of Eq. (a),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}^{-1} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \tag{7.14a}$$

Equation (7.14a) may be written in the compact form

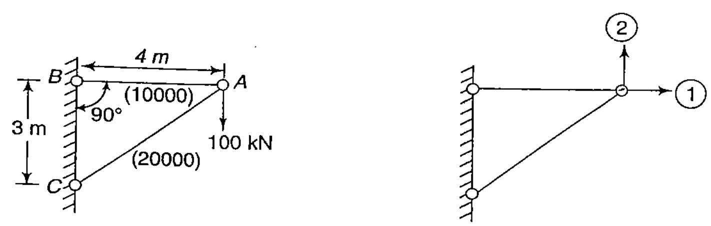
$$[\Delta] = [k]^{-1}[P] \tag{7.14b}$$

(v) Compute the forces in the members of the frame by using Eq. (7.13).

The above procedure is suitable when the supports are unyielding. In the case of yielding supports, coordinates should also be assigned to those reactive forces along which settlements occur. However, this is not necessary. As an alternative solution, the net forces at the coordinates due to the external loads and the settlements of the supports should be taken as the elements of the matrix $[P]$. The procedure is illustrated by Ex. 7.13(iii).

Example 7.12

Figure 7.16(a) shows a jib-crane carrying vertical load of 10 kN at A. Determine the displacement of joint A. Hence calculate the forces in members AB and AC. The numbers in parentheses are the cross-sectional areas of the members in mm². Take E = 200 kN/mm².



(a) (b)

Fig. 7.16

Solution

Joint A can move in the vertical and horizontal directions. No movement is possible at joints B and C. Hence the degree of freedom of the structure is two. Let coordinates 1 and 2 be chosen as shown in Fig. 7.16(b) and Δ_1 and Δ_2 be the displacements at coordinates 1 and 2 due to the applied loads, viz., $P_1 = 0$ and $P_2 = -100$ kN. The stiffness matrix with reference to the chosen coordinates may be developed by giving a unit displacement successively at coordinates 1 and 2. The necessary computations for the evaluation of the stiffness elements have been listed in Table 7.10.

Table 7.10

Member	$\frac{AE}{L}$ (kN/mm)	θ	$\cos\theta$	$\sin\theta$	$\frac{AE}{L}\cos^2\theta$ (kN/mm)	$\frac{AE}{L}\sin^2\theta$ (kN/mm)	$\frac{AE}{L}\times\sin\theta\cos\theta$ (kN/mm)
1	2	3	4	5	6	7	8
AB	500	180°	−1.00	0	500	0	0
AC	800	216°52′	−0.80	−0.60	512	288	384
				Σ	1012	288	384

Using Table 7.10 and Eq. (7.11), the stiffness elements in kN/mm are

$$k_{11} = \sum \frac{AE}{L}\cos^2\theta = 1012$$

$$k_{21} = k_{12} = \sum \frac{AE}{L}\sin\theta\cos\theta = 384$$

$$k_{22} = \sum \frac{AE}{L}\sin^2\theta = 288$$

Substituting into Eq. (7.14),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 1012 & 384 \\ 384 & 288 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -100 \end{bmatrix} = \begin{bmatrix} 0.267 \\ -0.703 \end{bmatrix}$$

Knowing the displacements Δ_1 and Δ_2, the member forces may be computed by using Eq. (7.13).

For member AB,

$$\frac{AE}{L} = 500 \text{ kN/mm}$$

$$\theta_{AB} = 180°$$

$$S_{AB} = -500[(0.267 - 0)\cos 180° + (-0.703 - 0)\sin 180°]$$

$$= 133.5 \text{ kN (tensile)}$$

For member AC,

$$\frac{AE}{L} = 800 \text{ kN/mm}$$

$$\theta_{AC} = 216°52'$$

$$S_{AC} = -800[(0.267 - 0)\cos 216°52' + (-0.703 - 0)\sin 216°52']$$
$$= -166.6 \text{ kN (compressive)}$$

Example 7.13

(i) Figure 7.17(a) shows a three wire system supporting a load W at joint O. The axial stiffness of members OA, OB and OC are K_1, K_2 and K_3 and their inclinations are θ_1, θ_2 and θ_3 as shown in the figure. Determine the horizontal and vertical displacements of joint O and the forces in wires OA, OB and OC.

(ii) Analyse the three wire system shown in Fig. 7.17(b). The numbers in parentheses are the cross-sectional areas of the wires in mm². Take E = 200 kN/mm².

(iii) Referring to Fig. 7.17(b), if the hinge-support at B settles downwards by 0.05 mm, calculate the forces in the wires OA, OB and OC.

Solution

(i) The degree of freedom is two because joint O can move in the horizontal and vertical directions. Let coordinates 1 and 2 be chosen as shown in Fig. 7.17(c) and Δ_1 and Δ_2 be the displacements at coordinates 1 and 2 due to the applied loads, viz., $P_1 = 0$ and $P_2 = -W$. The stiffness matrix with reference to the chosen coordinates may be developed by giving a unit displacement successively at coordinates 1 and 2. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1.

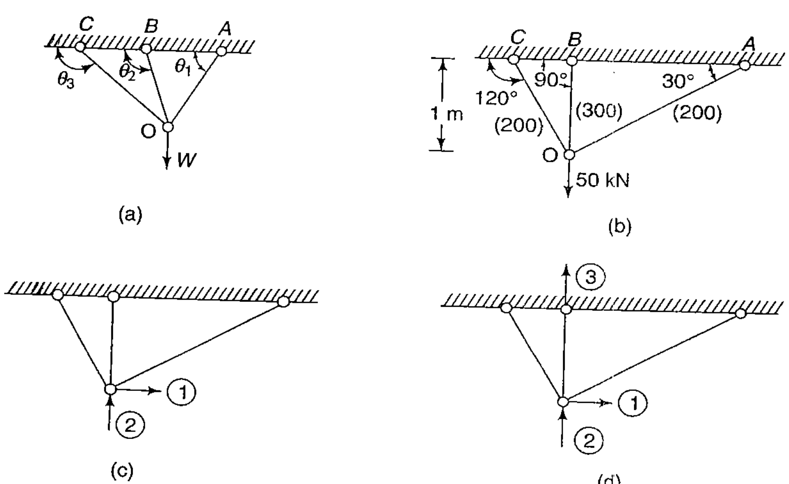


Fig. 7.17

$$k_{11} = \sum \frac{AE}{L}\cos^2\theta = K_1\cos^2\theta_1 + K_1\cos^2\theta_2 + K_3\cos^2\theta_3 \quad \text{(a)}$$

$$k_{21} = \sum \frac{AE}{L}\sin\theta\cos\theta = K_1\sin\theta_1\cos\theta_1 + K_2\sin\theta_2\cos\theta_2 + K_3\sin\theta_3\cos\theta_3 \quad \text{(b)}$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2.

$$k_{12} = \sum \frac{AE}{L}\sin\theta\cos\theta = K_1\sin\theta_1\cos\theta_1 + K_2\sin\theta_2\cos\theta_2 + K_3\sin\theta_3\cos\theta_3 \quad \text{(c)}$$

$$k_{22} = \sum \frac{AE}{L}\sin^2\theta = K_1\sin^2\theta_1 + K_2\sin^2\theta_2 + K_3\sin^2\theta_3 \quad \text{(d)}$$

Substituting into Eq. (7.14),

$$\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix} = \begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}^{-1}\begin{bmatrix}0\\ -W\end{bmatrix}$$

$$= \frac{1}{k_{11}k_{22} - k_{12}k_{21}}\begin{bmatrix}k_{22} & -k_{21}\\ -k_{12} & k_{11}\end{bmatrix}\begin{bmatrix}0\\ -W\end{bmatrix} \quad \text{(e)}$$

Solving Eq. (c),

$$\Delta_1 = \frac{k_{21}}{k_{11}k_{22} - k_{12}k_{21}}W \quad \text{(f)}$$

$$\Delta_2 = \frac{-k_{11}}{k_{11}k_{22} - k_{12}k_{21}}W \quad \text{(g)}$$

where the stiffness elements k_{11}, k_{21}, k_{12} and k_{22} are given by Eq. (a) to (d)
The member forces may now be determined by using Eq. (7.13):

$$S_{OA} = -K_1[\Delta_1\cos\theta_1 + \Delta_2\sin\theta_1] \quad \text{(h)}$$
$$S_{OB} = -K_2[\Delta_1\cos\theta_2 + \Delta_2\sin\theta_2] \quad \text{(i)}$$
$$S_{OC} = -K_3[\Delta_1\cos\theta_3 + \Delta_2\sin\theta_3] \quad \text{(j)}$$

where the displacements Δ_1 and Δ_2 are given by Eq. (f) and (g).

(ii) For the three wire system shown in Fig. 7.17(b),

$P_1 = 0$
$P_2 = -50$ kN
$\theta_1 = 30°$
$\theta_2 = 90°$
$\theta_3 = 120°$

The member stiffnesses are

$$K_1 = \frac{200 \times 200}{2000} = 20 \text{ kN/mm}$$

$$K_2 = \frac{300 \times 200}{1000} = 60 \text{ kN/mm}$$

$$K_3 = \frac{200 \times 200\sqrt{3}}{2000} = 20\sqrt{3} \text{ kN/mm}$$

The elements of the stiffness matrix are

$$k_{11} = 20 \times \frac{3}{4} + 60 \times 0 + 20\sqrt{3} \times \frac{1}{4} = 23.66$$

$$k_{21} = k_{12} = 20 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + 60 \times 1 \times 0 + 20\sqrt{3}\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= -6.34$$

$$k_{22} = 20 \times \frac{1}{4} + 60 \times 1 + 20\sqrt{3} \times \frac{3}{4} = 90.98$$

Substituting into Eq. (f) to (j),

$$\Delta_1 = -0.15 \text{ mm}$$
$$\Delta_2 = -0.56 \text{ mm}$$
$$S_{OA} = 8.2 \text{ kN (tensile)}$$
$$S_{OB} = 33.6 \text{ kN (tensile)}$$
$$S_{OC} = 14.2 \text{ kN (tensile)}$$

(iii) Herein two alternative solutions are given for this case. In solution (A), a coordinate has also been assigned to the displacement at support B. However, this is not necessary as illustrated in solution (B).

(A) In this solution the coordinates are chosen as shown in Fig. 7.17(d). The stiffness matrix with reference to coordinates 1, 2 and 3 may be developed by giving a unit displacement successively at coordinates 1, 2 and 3 and is found to be

$$[k] = \begin{bmatrix} 23.66 & -6.34 & 0 \\ -6.34 & 90.98 & -60 \\ 0 & -60 & 60 \end{bmatrix}$$

From the given data,

$$P_1 = 0 \qquad P_2 = -50 \text{ kN} \qquad \Delta_3 = -0.05 \text{ mm}$$

Substituting into Eq. (7.14),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ -0.05 \end{bmatrix} = \begin{bmatrix} 23.66 & -6.34 & 0 \\ -6.34 & 90.98 & -60 \\ 0 & -60 & 60 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -50 \\ P_3 \end{bmatrix}$$

Rearranging the terms,

$$\left[\begin{array}{c} 0 \\ -50 \\ \hline P_3 \end{array}\right] = \left[\begin{array}{cc:c} 23.66 & -6.34 & 0 \\ -6.34 & 90.98 & -60 \\ \hline 0 & -60 & 60 \end{array}\right] \left[\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \hline -0.05 \end{array}\right]$$

Partitioning the matrices as indicated by dotted lines, the above equation may be split up into the following two equations:

$$\begin{bmatrix} 0 \\ -50 \end{bmatrix} = \begin{bmatrix} 23.66 & -6.34 \\ -6.34 & 90.98 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -60 \end{bmatrix} [-0.05] \qquad \text{(k)}$$

$$[P_3] = [0 - 60] \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} + [60][-0.05] \qquad \text{(l)}$$

Solving Eq. (k),

$$\Delta_1 = -0.159 \text{ mm} \qquad \Delta_2 = 0.594 \text{ mm}$$

Substituting these values into Eq. (1),

$$P_3 = 32.6 \text{ kN}$$

Member forces may now be determined by using Eq. (7.13).

For member OA,

$$\frac{AE}{L} = 20 \text{ kN/mm}$$
$$\theta_{OA} = 30°$$
$$S_{OA} = -20\,[-0.159 \cos 30° - 0.594 \sin 30°]$$
$$= 8.7 \text{ kN (tensile)}$$

For member OC,

$$\frac{AE}{L} = 20\sqrt{3} \text{ kN/mm}$$
$$\theta_{OC} = 120°$$
$$S_{OC} = -20\sqrt{3}\,[-0.159 \cos 120° - 0.594 \sin 120°]$$
$$= 15.1 \text{ kN (tensile)}$$

For member OB,

$$S_{OB} = P_3 = 32.6 \text{ kN (tensile)}$$

(B) In this solution the same coordinates as in case (ii) may be adopted. Thus the chosen coordinates are shown in Fig. 7.17(c). The elements of the stiffness matrix with reference to coordinates 1 and 2 as computed in case (ii) are

$$k_{11} = 23.66 \qquad k_{12} = k_{21} = -6.34 \qquad k_{22} = 90.98$$

In this alternative solution, coordinate has not been assignd to the displacement due to the settlement of the support at B. Hence it is necessary to use Eq. (4.48) in which the elements of the matrix $[P']$ are due to the settlement of the support at B.

$$P_1' = 0$$

$$P_2' = \frac{300 \times 200}{1000} \times 0.05 = 3 \text{ kN}$$

From the given data,

$$P_1 = 0$$
$$P_2 = -50 \text{ kN}$$

Substituting into Eq. (7.15),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 23.66 & -6.34 \\ -6.34 & 90.98 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ -50 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\} = \begin{bmatrix} -0.159 \\ -0.594 \end{bmatrix}$$

These displacements are same as obtained in case (A).

Example 7.14

Analyse the pin-jointed structure shown in Fig. 7.18(a). The cross-sectional area of each member is 2000 mm². Take E = 200 kN/mm².

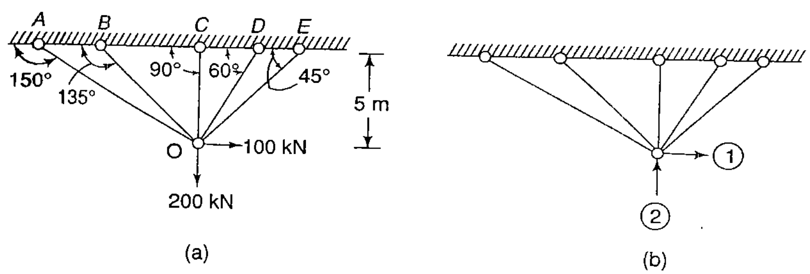


Fig. 7.18

Solution

The degree of freedom is two because joint O can move in the horizontal and vertical directions and no movement is possible at joints A, B, C, D and E. Let coordinates 1 and 2 be chosen as shown in Fig. 7.18(b). The necessary computations for the evaluation of stiffness elements have been listed in Table 7.11.

Table 7.11

Member	$\frac{AE}{L}$ (kN/mm)	θ	$\cos\theta$	$\sin\theta$	$\frac{AE}{L}\cos^2\theta$ (kN/mm)	$\frac{AE}{L}\sin^2\theta$ (kN/mm)	$\frac{AE}{L} \times \sin\theta\cos\theta$ (kN/mm)
1	2	3	4	5	6	7	8
OA	400.00	150°	−0.866	0.500	300.00	100.00	−173.20
OB	565.69	135°	−0.707	0.707	282.85	282.85	−282.85
OC	800.00	90°	0	1.000	0	800.00	0
OD	692.76	60°	0.500	0.866	173.19	519.57	300.00
OE	565.69	45°	0.707	0.707	282.85	282.85	282.85
				Σ	1038.89	1985.27	126.80

Using Eq. (7.11),

$$k_{11} = \sum \frac{AE}{L}\cos^2\theta = 1038.89$$

$$k_{21} = k_{12} = \sum \frac{AE}{L}\sin\theta\cos\theta = 126.80$$

$$k_{22} = \sum \frac{AE}{L}\sin^2\theta = 1985.27$$

From the given data,

$$P_1 = 100 \text{ kN} \qquad P_2 = -200 \text{ kN}$$

Substituting into Eq. (7.14),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 1038.89 & 126.80 \\ 126.80 & 1985.27 \end{bmatrix} \begin{bmatrix} 100 \\ -200 \end{bmatrix} = \begin{bmatrix} 0.109 \\ -0.108 \end{bmatrix}$$

Knowing the displacements Δ_1 and Δ_2, the member forces may be computed by using Eq. (7.13).

For member OA,

$$\frac{AE}{L} = 400.0 \text{ kN/mm}$$
$$\theta_{OA} = 150°$$
$$S_{OA} = -400.0[(0.109 - 0)\cos 150° + (-0.108 - 0)\sin 150°]$$
$$= 60 \text{ kN (tensile)}$$

For member OB,

$$\frac{AE}{L} = 565.69 \text{ kN/mm}$$
$$\theta_{OB} = 135°$$
$$S_{OB} = -565.69[(0.109 - 0)\cos 135° + (-0.108 - 0)\sin 135°]$$
$$= 87 \text{ kN (tensile)}$$

For member OC,

$$\frac{AE}{L} = 800.0 \text{ kN/mm}$$
$$\theta_{OC} = 90°$$
$$S_{OC} = -800.0[(0.109 - 0)\cos 90° + (-0.108 - 0)\sin 90°]$$
$$= 86 \text{ kN (tensile)}$$

For number OD,

$$\frac{AE}{L} = 692.76 \text{ kN/mm}$$
$$\theta_{OD} = 60°$$
$$S_{OD} = -692.76[(0.109 - 0)\cos 60° + (-0.108 - 0)\sin 60°]$$
$$= 27 \text{ kN (tensile)}$$

For member OE,

$$\frac{AE}{L} = 565.69 \text{ kN/mm}$$
$$\theta_{OE} = 45^\circ$$
$$S_{OE} = -565.69[(0.109 - 0) \cos 45^\circ + (-0.108 - 0) \sin 45^\circ]$$
$$= -1 \text{ kN (compressive)}$$

Example 7.15

Analyse the pin-jointed frame shown in Fig. 7.19(a). The lengths of the members are shown in the figure. The numbers in parentheses are the cross-sectional areas of the members in mm². Take E = 200 kN/mm².

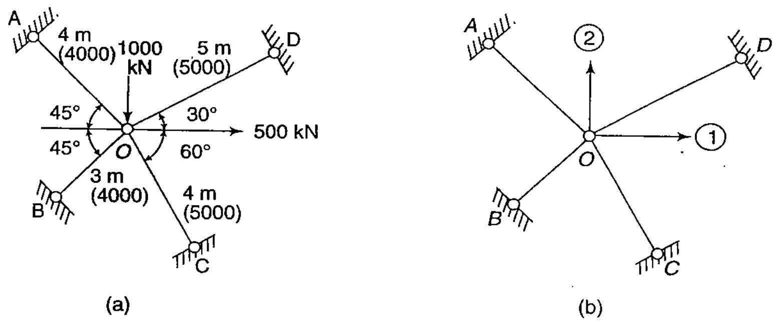


Fig. 7.19

Solution

The degree of freedom is two because O can move in the horizontal and vertical directions and no movement is possible of joints A, B, C and D. Let coordinates 1 and 2 be chosen as shown in Fig. 7.19(b) and Δ_1 and Δ_2 be the displacements at coordinates 1 and 2 due to the applied loads, viz., $P_1 = 500$ kN and $P_2 = -1000$ kN. The stiffness matrix with reference to the chosen coordinates may be developed by giving a unit displacement sucessively at coordinates 1 and 2. The stiffness elements as already obtained in Ex. 7.9 are

$$k_{11} = 445.83$$
$$k_{21} = k_{12} = 11.68$$
$$k_{22} = 470.83$$

Substituting into Eq. (7.14),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 445.83 & 11.68 \\ 11.68 & 470.83 \end{bmatrix}^{-1} \begin{bmatrix} 500 \\ -1000 \end{bmatrix} = \begin{bmatrix} 1.178 \\ -2.153 \end{bmatrix}$$

Knowing the displacement Δ_1 and Δ_2, the member forces may be computed by using Eq. (7.13).

For member OA,

$$\frac{AE}{L} = 200.0 \text{ kN/mm}$$
$$\theta_{OA} = 135^\circ$$
$$S_{OA} = -200.0\,[(1.178 - 0) \cos 135^\circ + (-2.153 - 0) \sin 135^\circ]$$
$$= 471 \text{ kN (tensile)}$$

For member OB,

$$\frac{AE}{L} = 266.67 \text{ kN/mm}$$
$$\theta_{OB} = 225^\circ$$
$$S_{OB} = -266.67[(1.178 - 0] \cos 225^\circ + (-2.153 - 0) \sin 225^\circ]$$
$$= -184 \text{ kN (compressive)}$$

For member OC,

$$\frac{AE}{L} = 250.0 \text{ kN/mm}$$
$$\theta_{OC} = 300^\circ$$
$$S_{OC} = -250.0\,[(1.178 - 0) \cos 300^\circ + (-2.153 - 0) \sin 300^\circ]$$
$$= -613 \text{ kN (compressive)}$$

For member OD,

$$\frac{AE}{L} = 200.0 \text{ kN/mm}$$
$$\theta_{OD} = 30^\circ$$
$$S_{OD} = -200.0[(1.178 - 0) \cos 30^\circ + (-2.153 - 0) \sin 30^\circ]$$
$$= 11 \text{ kN (tensile)}$$

Example 7.16

Analyse the pin-jointed plane frame shown in Fig. 7.20(a). The axial stiffness for each number is 40 kN/mm

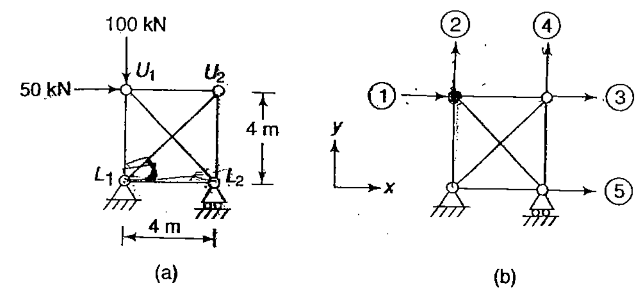


Fig. 7.20

Solution

Each of joints U_1 and U_2 has two degree of freedom because they can move in the horizontal and vertical directions. Joint L_2, supported by a roller, has a single degree of freedom because it can move in the horizontal direction only. Joint L_1, supported by a hinge, has zero degree of freedom because it cannot move in any direction. Hence the degree of freedom of the structure is five. Consequently, coordinates 1 to 5 may be chosen as shown in Fig. 7.20(b). Let Δ_1 to Δ_5 be the displacements at coordinates 1 to 5 due to the applied loads, viz., $P_1 = 50$ kN, $P_2 = -100$ kN, $P_3 = P_4 = P_5 = 0$. The stiffness matrix with reference to the chosen coordinates may be developed by giving a unit displacement successively at coordinates 1 to 5. The necessary computations for the evaluation of the stiffness elements have been listed in Table 7.12.

Table 7.12

Member	θ	$\cos\theta$	$\sin\theta$	$\cos^2\theta$	$\sin^2\theta$	$\sin\theta\cos\theta$
1	2	3	4	5	6	7
L_1L_2	0	1.000	0	1.0	0	0
L_2U_2	90°	0	1.000	0	1.0	0
U_2U_1	180°	– 1.000	0	1.0	0	0
U_1L_1	270°	0	– 1.000	0	1.0	0
L_1U_2	45°	0.707	0.707	0.5	0.5	0.5
L_2U_1	135°	– 0.707	0.707	0.5	0.5	– 0.5

Using Eq. (7.11) and Table 7.12,

$$k_{11} = \sum\frac{AE}{L}\cos^2\theta \text{ the summation to include } U_1U_2,\ U_1L_1 \text{ and } U_1L_2$$
$$= 40(1.0 + 0 + 0.5) = 60$$

$$k_{21} = k_{12} = \sum\frac{AE}{L}\sin\theta\cos\theta \text{ the summation to include } U_1U_2,\ U_1L_1 \text{ and } U_1L_2$$
$$= 40\,(0 + 0 - 0.5) = -20$$

$$k_{31} = k_{13} = -\frac{AE}{L}\cos^2\theta \text{ for } U_2U_1$$
$$= -40 \times 1.0 = -40$$

$$k_{41} = k_{14} = -\frac{AE}{L}\sin\theta\cos\theta \text{ for } U_2\,U_1$$
$$= -40\,(0) = 20$$

$$k_{51} = k_{15} = -\frac{AE}{L}\cos^2\theta \text{ for } L_2U_1$$
$$= -40 \times 0.5 = -20$$

$$k_{22} = \sum\frac{AE}{L}\sin^2\theta \text{ the summation to include } U_1U_2,\ U_1L_1 \text{ and } U_1L_2$$
$$= 40(0 + 1.0 + 0.5) = 60$$

$$k_{32} = k_{23} = -\frac{AE}{L}\sin\theta\cos\theta \text{ for } U_2U_1$$
$$= -40(0) = 0$$

$$k_{42} = k_{24} = -\frac{AE}{L}\sin^2\theta \text{ for } U_2U_1$$
$$= -40(0) = 0$$

$$k_{52} = k_{25} = -\frac{AE}{L}\sin\theta\cos\theta \text{ for } L_2U_1$$
$$= -40(-0.5) = 20$$

$$k_{33} = \sum\frac{AE}{L}\cos^2\theta \text{ the summation to include } U_2L_2,\ U_2U_1 \text{ and } U_2L_1$$
$$= 40(0 + 1.0 + 0.5) = 60$$

$$k_{43} = k_{34} = \sum\frac{AE}{L}\sin\theta\cos\theta \text{ the summation to include } U_2L_2,\ U_2U_1 \text{ and } U_2L_1$$
$$= 40(0 + 0 + 0.5)$$
$$= 20$$

$$k_{53} = k_{35} = -\frac{AE}{L}\cos^2\theta\ \ L_2U_2$$
$$= -40(0) = 0$$

$$\begin{bmatrix}0\\0\end{bmatrix} = [P] + [K][\Delta]$$
$$0 = 50 + -$$

$$k_{44} = \sum\frac{AE}{L}\sin^2\theta \text{ the summation to include } U_2L_2,\ U_2L_1 \text{ and } U_2U_1$$
$$= 40(1.0 + 0 + 0.5) = 60$$

$$k_{54} = k_{45} = -\frac{AE}{L}\sin\theta\cos\theta \text{ for } L_2U_2$$
$$= -40(0) = 0$$

$$k_{55} = \sum\frac{AE}{L}\cos^2\theta \text{ the summation to include } L_2L_1,\ L_2U_2 \text{ and } L_2U_1$$
$$= 40(1.0 + 0 + 0.5) = 60$$

Hence, the stiffness matrix $[k]$ is given by the equation

$$P \cdot K^{-1} = \Delta$$

$$[k] = 20\begin{bmatrix} 3 & -1 & -2 & 0 & -1 \\ -1 & 3 & 0 & 0 & 1 \\ -2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 & 3 \end{bmatrix}$$

Substituing into Eq. (7.14) and solving for the displacements,

$\Delta_1 = 1.25$ mm $\Delta_2 = -1.5625$ $\Delta_3 = 0.9375$ mm

$\Delta_4 = -0.3125$ mm $\Delta_5 = 0.9375$ mm

Knowing displacements Δ_1 to Δ_5, the member forces may be computed by using Eq. (7.13).

Example 7.17

Figure 7.21(a) shows a triangular frame carrying a load of 200 kN at A. Determine the displacements of joints A and C. Hence calculate the forces in the members of the frame. The axial stiffness for each member is 100 kN/mm.

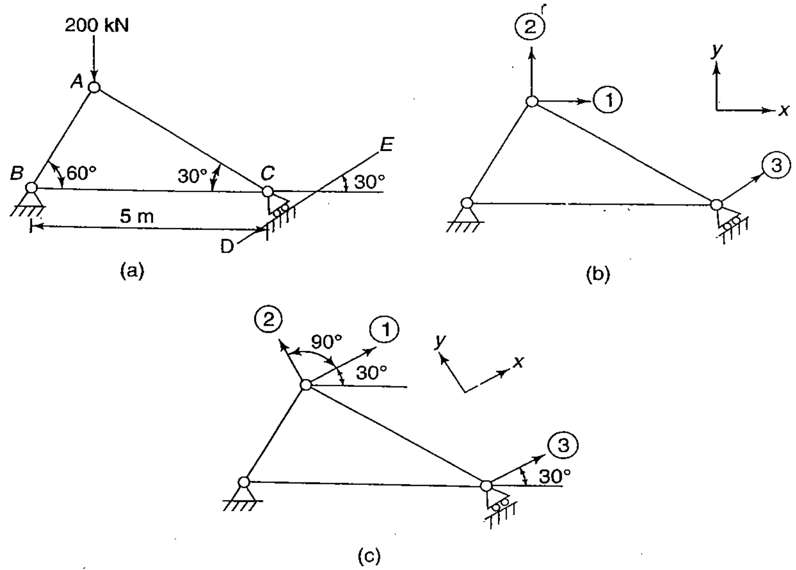


Fig. 7.21

Solution

While joint A can move along horizontal and vertical directions, jonit C can move along the inclined plane DE. Hence, coordinates 1, 2 and 3 may be chosen as shown in Fig. 7.21(b). Let Δ_1, Δ_2 and Δ_3 be the displacements at coordinates 1, 2 and 3 due to the applied loads, viz., $P_1 = 0$, $P_2 = -200$ kN and $P_3 = 0$. The stiffness matrix with reference to the chosen coordinates may be developed by giving a unit displacement successively at coordinates 1, 2 and 3.

Using Eq. (7.12),

$$k_{11} = \sum \frac{AE}{L} \cos^2 \theta = 100 [\cos^2 \theta_{AB} + \cos^2 \theta_{AC}]$$
$$= 100 [\cos^2 240° + \cos^2 330°]$$
$$= 100$$

$$k_{22} = \sum \frac{AE}{L} \sin^2 \theta = 100 [\sin^2 \theta_{AB} + \sin^2 \theta_{AC}]$$
$$= 100 [\sin^2 240° + \sin^2 330°]$$
$$= 100$$

$$k_{33} = \sum \frac{AE}{L} \cos^2(\theta - \alpha) = 100 [\cos^2(\theta_{CA} - 30°) + \cos^2(\theta_{CB} - 30°)]$$
$$= 100 [\cos^2 120° + \cos^2 150°]$$
$$= 100$$

$$k_{12} = k_{21} = \sum \frac{AE}{L} \sin \theta \cos \theta$$
$$= 100 [\sin \theta_{AB} \cos \theta_{AB} + \sin \theta_{AC} \cos \theta_{AC}]$$
$$= 100 [\sin 240° \cos 240° + \sin 330° \cos 330°]$$
$$= 0$$

$$k_{13} = k_{31} = -\frac{AE}{L} \cos \theta \cos (\theta - \alpha) \text{ for member } AC$$
$$= -100 \cos 330° \cos (330° - 30°)$$
$$= -43.3$$

$$k_{23} = k_{32} = -\frac{AE}{L} \sin \theta \cos (\theta - \alpha) \text{ for member } AC$$
$$= -100 \sin 330° \cos (330° - 30°)$$
$$= 25$$

Hence, stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 100 & 0 & -43.3 \\ 0 & 100 & 25 \\ -43.3 & 25 & 100 \end{bmatrix}$$

Substituting into Eq. (7.14),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 100 & 0 & -43.3 \\ 0 & 100 & 25 \\ -43.3 & 25 & 100 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -200 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.29 \\ -2.17 \\ 0.67 \end{bmatrix}$$

Knowing displacements Δ_1, Δ_2 and Δ_3, the member forces may be computed by using Eq. (7.13). While using Eq. (7.13), the components of joint displacements along x- and y-axes are required.

$$\Delta_{Ax} = 0.29 \text{ mm}$$
$$\Delta_{Ay} = -2.17 \text{ mm}$$
$$\Delta_{Bx} = \Delta_{By} = 0$$
$$\Delta_{Cx} = 0.67 \cos 30° = 0.58 \text{ mm}$$
$$\Delta_{Cy} = 0.67 \sin 30° = 0.33 \text{ mm}$$

For member AB,

$$\frac{AE}{L} = 100 \text{ kN/mm}$$

$$\theta_{AB} = 240°$$
$$S_{AB} = -100\,[(0.29-0)\cos 240° + (-2.17-0)\sin 240°]$$
$$= -173 \text{ kN (compressive)}$$

For member AC,

$$\frac{AE}{L} = 100 \text{ kN/mm}$$
$$\theta_{AC} = 330°$$
$$S_{AC} = -100\,[(0.29-0.58)\cos 330° + (-2.17-0.33)\sin 330°]$$
$$= -100 \text{ kN (compressive)}$$

For member BC,

$$\frac{AE}{L} = 100 \text{ kN/mm}$$
$$\theta_{BC} = 0$$
$$S_{BC} = -100\,[(0-0.58)\cos 0° + (0-0.33)\sin 0°]$$
$$= 58 \text{ kN (tensile)}.$$

An alternative solution of the problem may be obtained by choosing coordinates 1, 2 and 3 as shown in Fig. 7.21(c). In this case the x-axis has been chosen parallel to the plane of rollers, i.e., inclined at an angle of 30° to the horizontal. The external forces acting at the coordinates 1, 2 and 3 are

$$P_1 = -200 \cos 60° = -100 \text{ kN}$$
$$P_2 = -200 \sin 60° = -173.2 \text{ kN}$$
$$P_3 = 0$$

The stiffness matrix with reference to the chosen coordinates may be developed by giving a unit displacement successively at coordinates 1, 2 and 3. The necessary computations for the evaluation of elements of the stiffness matrix have been listed in Table 7.13.

Table 7.13

Member	θ	$\cos\theta$	$\sin\theta$	$\cos^2\theta$	$\sin^2\theta$	$\sin\theta\cos\theta$
1	2	3	4	5	6	7
AB	210°	– 0.866	– 0.500	0.750	0.250	0.433
AC	300°	0.500	– 0.866	0.250	0.750	– 0.433
BC	330°	0.866	– 0.500	0.750	0.250	– 0.433

Using Table 7.13 and Eq. (7.11),

$$k_{11} = \sum \frac{AE}{L}\cos^2\theta \text{ the summation to include } AB \text{ and } AC$$
$$= 100\,(0.750 + 0.250) = 100$$
$$k_{22} = \sum \frac{AE}{L}\sin^2\theta \text{ the summation to include } AB \text{ and } AC$$

$$= 100\,(0.250 + 0.750) = 100$$
$$k_{21} = k_{12} = \sum \frac{AE}{L}\sin\theta\cos\theta \text{ the summation to include } AB \text{ and } AC$$
$$= 100\,(0.433 - 0.433) = 0$$
$$k_{13} = k_{31} = -\frac{AE}{L}\cos^2\theta \text{ for } AC$$
$$= -100\,(0.25) = -25$$
$$k_{23} = k_{32} = -\frac{AE}{L}\sin\theta\cos\theta \text{ for } AC$$
$$= -100\,(-0.433) = 43.3$$
$$k_{33} = \sum \frac{AE}{L}\cos^2\theta \text{ the summation to include } AC \text{ and } BC$$
$$= 100\,(0.250 + 0.750) = 100$$

Hence, stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 100 & 0 & -25 \\ 0 & 100 & 43.3 \\ -25 & 43.3 & 100 \end{bmatrix}$$

Substituting into Eq. (7.14),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 100 & 0 & -25 \\ 0 & 100 & 43.3 \\ -25 & 43.3 & 100 \end{bmatrix}^{-1} \begin{bmatrix} -100 \\ -173.2 \\ 0 \end{bmatrix} = -\begin{bmatrix} -0.83 \\ 2.02 \\ 0.67 \end{bmatrix}$$

Knowing the displacements Δ_1, Δ_2 and Δ_3, the member forces may be computed by using Eq. (7.13).

For member AB,

$$\frac{AE}{L} = 100 \text{ kN/mm}$$
$$\theta_{AB} = 210°$$
$$S_{AB} = -100\,[(-0.83-0)\cos 210° + (-2.02-0)\sin 210°]$$
$$= -173 \text{ kN (compressive)}$$

For member AC,

$$\frac{AE}{L} = 100 \text{ kN/mm}$$
$$\theta_{AC} = 300°$$
$$S_{AC} = -100\,[(-0.83-0.67)\cos 300° + (-2.02-0)\sin 300°]$$
$$= -100 \text{ kN (compressive)}$$

For member BC,

$$\frac{AE}{L} = 100 \text{ kN/mm}$$

$$\theta_{BC} = 330°$$

$$S_{BC} = -100\,[(0-0.67)\cos 330° + (0-0)\sin 330°]$$

$$= 58 \text{ kN (tensile)}$$

7.7 COMPARISON OF METHODS

In the preceding sections the force and displacement methods for the analysis of pin-jointed plane frames have been discussed. A large number of problems have also been solved by both the methods. It is evident that in the case of pin-jointed plane frames the development of the stiffness matrix is simpler as compared to the development of the flexibility matrix, but the degree of kinematic indeterminacy of these frames is generally much larger than the degree of static indeterminacy. For example, the degrees of static and kinematic indeterminacies of the truss of Fig. 7.9 are 3 and 12 respectively. There can, however, be exceptions to this in certain cases. For instance, the degrees of static and kinematic indeterminacies of the frame of Fig. 7.18 are 3 and 2 respectively. In general, it may be stated that for the analysis of pin-jointed plane frames the force method is preferable as compared to the displacement method because in a majority of cases the degree of kinematic indeterminacy is much larger than the degree of static indeterminacy.

PROBLEMS

7.1 Determine the degrees of static and kinematic indeterminacies of the pin-jointed frame shown in Fig. 7.22. Analyse the frame by the force and displacement methods. Hence determine the force in member AB. In the figure, the numbers in parentheses are the cross-sectional areas of the members in mm^2. E is constant.

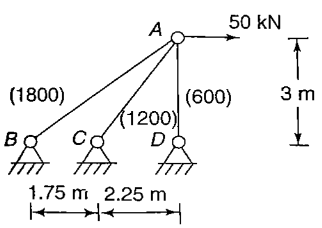


Fig. 7.22

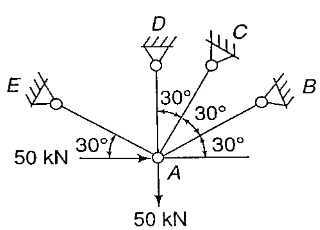


Fig. 7.23

7.2 Analyse the pin-jointed frame of Fig. 7.23 by the force method. Hence determine the force in member AE. The axial flexibility, L/AE, is the same for all the members. Verify the result by the displacement method.

7.3 Analyse the pin-jointed frame of Fig. 7.24 by the force method. Hence determine the force in member AC. Verify the result by the displacement method. All members have the same cross-sectional area.

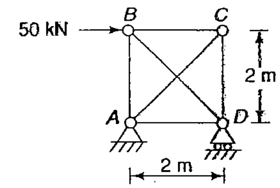


Fig. 7.24

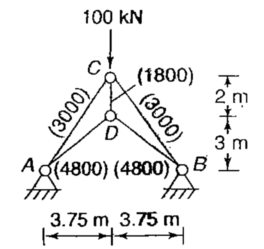


Fig. 7.25

7.4 Analyse the pin-jointed frame of Fig. 7.25 by the force method. Hence calculate the force in member CD. In the figure, the numbers in parentheses are the cross-sectional areas of the members in mm^2.

7.5 Analyse the pin-jointed frame of Fig. 7.26 by the force method. Hence determine the force in member CD. The numbers in parentheses are the cross-sectional areas of the members in mm^2. Verify the result by the displacement method.

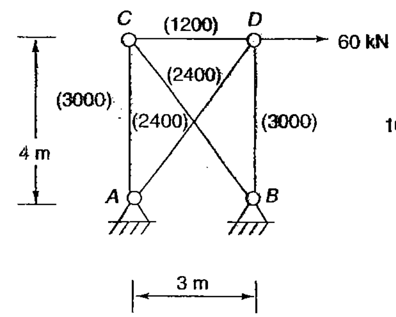


Fig. 7.26

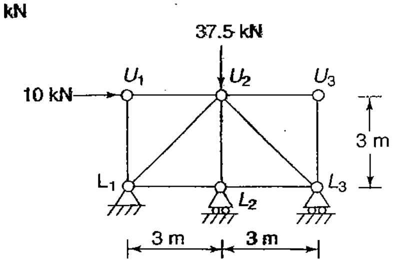


Fig. 7.27

7.6 Analyse the pin-jointed frame of Fig. 7.27 by the force method treating the support reaction at L_3 as the redundant. Hence determine the force in members U_1U_2 and U_2U_3. Verify the result by treating the support reaction at L_2 as the redundant. Areas of cross-section are the same for all members.

7.7 Using the force method, analyse the pin-jointed frame of Fig. 7.28. Hence determine the forces in members AD and BC. All members have the same value of AE.

7.8 Using the force method, analyse the pin-jointed frame of Fig. 7.29. Hence determine the force in member CD. AE is the same for all the members.

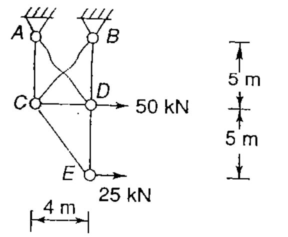


Fig. 7.28

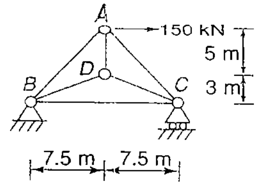


Fig. 7.29

7.9 Treating the reaction at support G as the redundant, analyse the pint-jointed frame of Fig. 7.30 by the force method. Hence determine the force in members BG and GH. Verify the result with the help of an alternative solution in which the reaction at support E is treated as the redundant. The numbers in parentheses are the cross sectional areas of the members in mm^2.

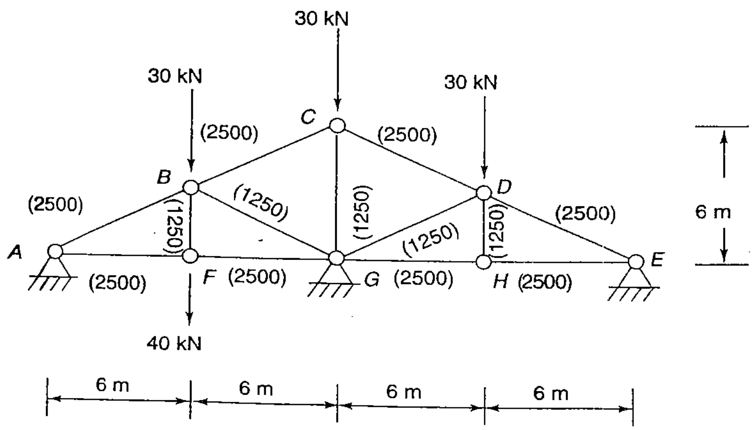


Fig. 7.30

7.10 Using the force mehod, analyse the pin-jointed frame of Fig. 7.31 treating the force in diagonal U_2L_1 as the redundant. Hence determine the force in member U_1L_2. Check the result by teating the force in member U_1L_2 as the redundant. The numbers in parentheses are the cross-sectional areas of the members in mm^2.

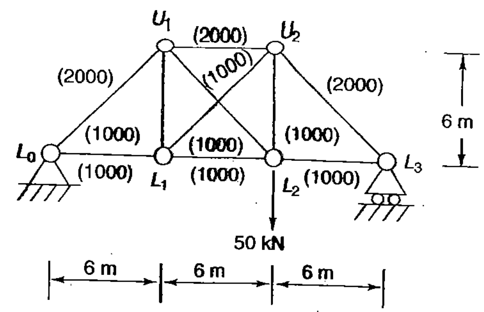


Fig. 7.31

7.11 Using the force method, determine the forces in members AC, BD and CD of the pin-jointed frame shown in Fig. 7.32. All members have the same value of AE.

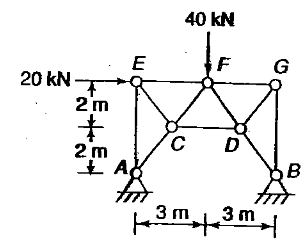


Fig. 7.32

7.12 Using the force method, analyse the pin-jointed frames of Fig. 7.33 treating the forces in diagonals U_1L_2 and U_2L_3 as the redundants. Hence determine the forces in members U_1L_1 and U_2U_3. Check the result by adopting the forces in diagonals L_1U_2 and L_2U_3 as the redundants. The numbers in parentheses are the cross-sectional areas of the members in mm^2.

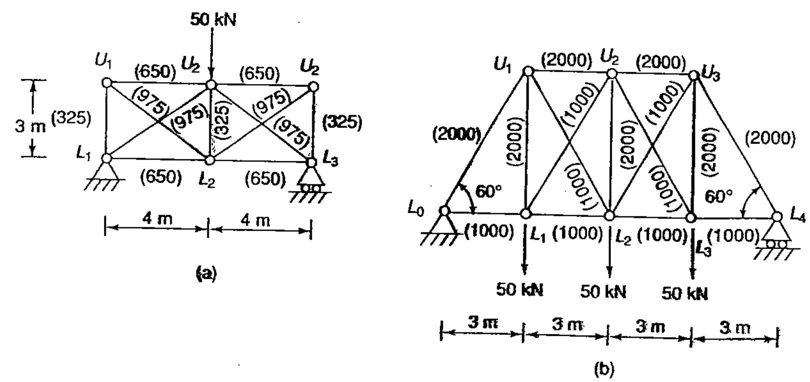


Fig. 7.33

7.13 Using the force method, analyse the pin-jointed frame of Fig. 7.34 treating the force in member U_1U_2 as the redundant. Hence determine the force in number U_1L_1. Verify the result by adopting the force in diagonal U_1L_2 as the redundant. The numbers in parentheses are the cross-sectional areas of the members in mm^2.

7.14 Analyse the pin-jointed frame of Fig. 7.35 by the force method in which the forces in diagonals *AD* and *DE* are chosen as the redundants. Hence calculate

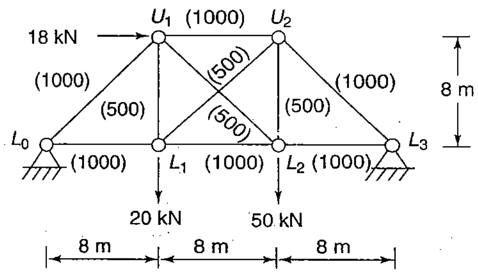


Fig. 7.34

the forces in members *BC* and *CF*. Verify the result by treating the forces in member *BC* and *CF* as the redundants. The axial flexibility, *L*/*AE*, is the same for all members.

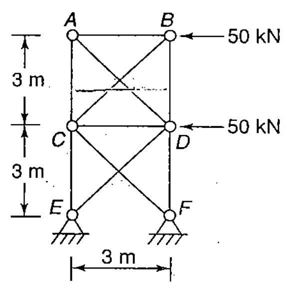


Fig. 7.35

7.15 Analyse the pin-jointed frame of Fig. 7.36 by the force method adopting the force in diagonal U_2L_3 and the vertical reaction at support L_2 as the redundants. Hence determine the force in member U_2U_3. Verify the result using an alternative choice of redundants. The numbers in parentheses are the cross-sectional areas of the members in mm^2.

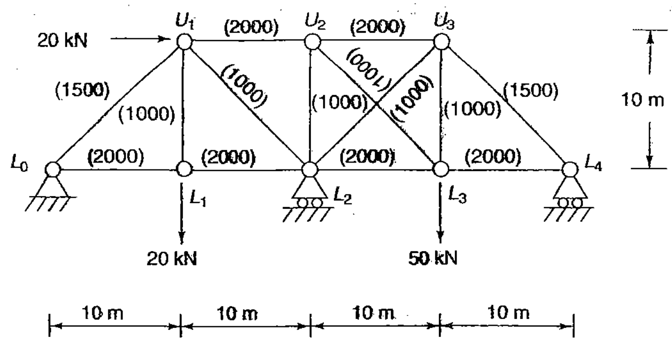


Fig. 7.36

7.16 What are the degrees of static and kinematic indeterminacies of the pin-jointed frame of Fig. 7.37? Using the force method, determine the forces in members *AE* and *DG*. All members are of equal length and have the same cross-sectional area.

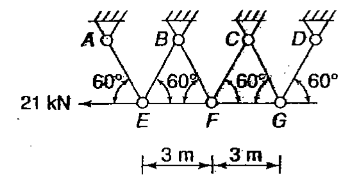


Fig. 7.37

7.17 Using the force method, analyse the pin-jointed frame of Fig. 7.38. Hence determine the force in member U_1L_2. All members have the same area of cross-section.

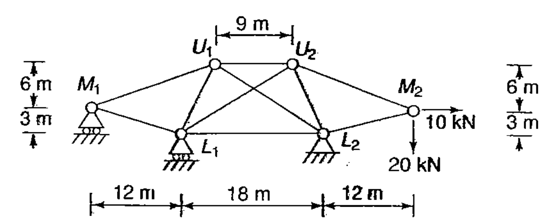


Fig. 7.38

7.18 Develop the stiffness matrix with reference to coordinates 1 to 4 for the pin-jointed frame of Fig. 7.39. All members of the frame have the same cross-sectional area, $A = 2000$ mm^2. $E = 200$ kN/mm^2.

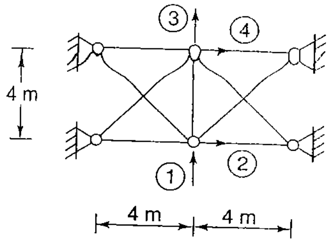


Fig. 7.39

7.19 Analyse the pin-jointed frame of Fig. 7.40 by the displacement method. Hence determine the force in member BC. Verify the result by the force method. The numbers in parentheses are the cross-sectional areas of the members in mm^2.

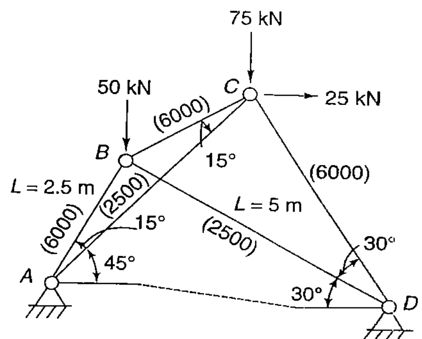


Fig. 7.40

7.20 Using the force method, analyse the pin-jointed frame of Fig. 7.41 treating the forces in numbers U_1U_2, U_2U_3 and U_3U_4 as the redundants. Hence determine the forces in diagonals U_2L_1, U_3L_2 and U_3L_4. Check the result by treating these forces as the redundants. The numbers in parentheses are cross-sectional areas of the members in mm^2.

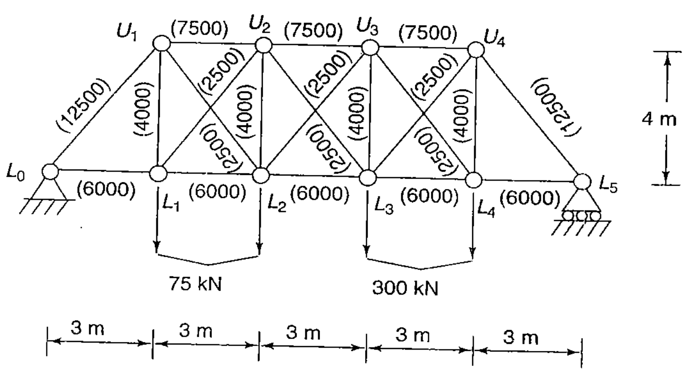


Fig. 7.41

7.21 Determine the degrees of internal and external static indeterminacies of the pin-jointed frame shown in Fig. 7.42. Analyse the frame using the force method. Hence calculate the forces in members U_2L_3 and U_4L_3. Check the result by an alternative choice of redundants. The numbers in parentheses are the cross-sectional areas of the members in mm^2.

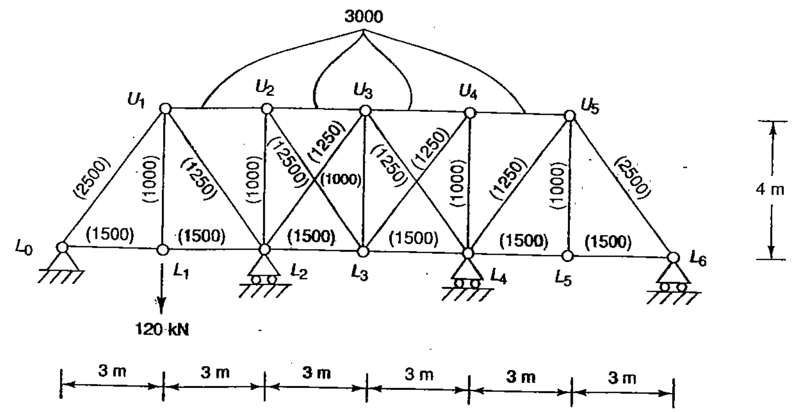


Fig. 7.42

8 RIGID-JOINTED SPACE FRAMES

8.1 INTRODUCTION

In practice the rigid-jointed frames such as building frames are usually three-dimensional space structures. However, for simplicity in design, the space structure is considered as a number of independent plane frames. Actually the frames interact with one another leading to a redistribution of internal forces and the development of torsional moments in the members of the space frame. Although the neglect of the torsional moments results in considerable simplification, it is neither economical nor safe in all cases. Hence, it may become necessary in certain instances, particularly in the case of important structures, to consider the three-dimensional effects. A grid structure is another example in which the neglect of twisting moments is neither safe nor economical. Although the members of a grid structure, generally, lie in the plane, the twisting moments are present because the external loads are normal to the plane of the grid. Just as in the case of two-dimensional frames, the two main methods, viz., the force method and the displacement method, may be used for the analysis of rigid-jointed space frames.

In dealing with rigid-jointed space frames, the rotational coordinates corresponding to rotations about the three Cartesian axes and couples about them will be represented by double-headed arrows in accordance with the vector notation and the right-handed screw system (Sec. 1.4). For instance, referring to Fig. 8.1, coordinates 1, 2 and 3 represent clockwise rotations about x-, y- and z-axes respectively while looking towards the positive directions of the axes. Coordinates corresponding to linear displacements will continue to be represented by single-headed arrows as in the case of plane frames.

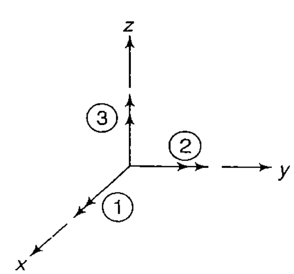


Fig. 8.1

8.2 FORCE METHOD

As in the case of rigid-jointed plane frames, the force method for the analysis of rigid-jointed space frames begins with the determination of the degree of static indeterminacy and the identification of the redundants. The degree of static indeterminacy of rigid-jointed space frames has been discussed in Sec. 1.6. The released structure is obtained by releasing all the redundants. Care should be exercised in selecting the redundants so that the released structure is statically determinate, stable and as simple as possible. A coordinate is assigned to each redundant thus released. As usual the coordinates assigned to internal redundants may be represented by a pair of arrows and those assigned to external redundants by single arrows as discussed in Sec. 2.4. For the continuity of the structure, the net displacements at the coordinates assigned to internal redundants must be zero. Similarly, in the case of unyielding supports, the net displacements at the coordinates assigned to external redundants must vanish. These conditions of compatibility of displacements give a sufficient number of equations for the determination of all the redundants. Thus, depending upon the support conditions, the redundants may be computed by using Eq. (6.1) or (6.2).

The unit load method may be used for determining the elements of the flexibility matrix and the displacements due to applied loads. The computation of displacements by the unit load method is based on strain energy expressions. In the case of rigid-jointed space frames, the members are subjected to torsion in addition to the bending moments about the major and minor axes. They are also subjected to axial and shear forces but the strain energy due to these forces is small and is commonly ignored. Hence, the strain energy u of a member of a rigid-jointed space structure may be expressed by the equation.

$$u = \int \frac{M_u^2 ds}{2EI_u} + \int \frac{M_v^2 ds}{2EI_v} + \int \frac{T^2 ds}{2GK} \tag{8.1}$$

where M_u, M_v = bending moments about the major and minor axes of the member respectively

I_u, I_v = moments of inertia about the major and minor axes of the member respectively

G = shear modulus of elasticity

K = torsion constant

Total strain energy U of the entire structure is obtained by adding the strain energies of all the constituent members.

$$U = \sum \left[\int \frac{M_u^2 ds}{2EI_u} + \int \frac{M_v^2 ds}{2EI_v} + \int \frac{T^2 ds}{2GK} \right] \tag{8.2}$$

Using Castigliano's theorem Part II, Sec. 2.10, the displacement at any coordinate j is given by the equation

$$\Delta_j = \frac{\partial U}{\partial P_j} \tag{8.3}$$

Substituting from Eq. (8.2) into Eq. (8.3), the displacement Δ_{jL} at coordinate j due to applied loads may be written as

$$\Delta_{jL} = \sum\left[\int M_u \frac{\partial M_u}{\partial P_j}\frac{ds}{EI_u} + \int M_v \frac{\partial M_v}{\partial P_j}\frac{ds}{EI_v} + \int T\frac{\partial T}{\partial P_j}\frac{ds}{GK}\right]$$

or

$$\Delta_{jL} = \sum\left[\int \frac{M_u m_u ds}{EI_u} + \int \frac{M_v m_v ds}{EI_v} \int \frac{Tt ds}{GK}\right] \tag{8.4}$$

where

$$m_u = \frac{\partial M_u}{\partial P_j}$$

$$m_v = \frac{\partial M_v}{\partial P_j}$$

$$t = \frac{\partial T}{\partial P_j}$$

It may be noted that m_u, m_v and t are the rates of change of M_u, M_v and T with respect to the force P_j at coordinate j. Hence, m_u, m_v and t are also equal to the bending moment about the major axis, the bending moment about the minor axis and the torque respectively due to a unit force at coordinate j. Thus m_u, m_v and t may be computed by applying a unit force at coordinate j at which the displacement has to be determined. Equation (8.4), therefore, represents the unit load method for the determination of displacements of a rigid-jointed space structure. Similarly, the displacements at other coordinates due to the applied loads, which constitute the elements of the second matrix on the right-hand side of Eq. (6.2), may be computed.

The unit load method may also be used conveniently to calculate the elements of the flexibility matrix. To generate the jth column of the flexibility matrix, a unit load may be applied at coordinate j and the displacements at all the coordinates computed. These displacements constitute the elements of the jth column of the flexibility matrix. Thus the element δ_{ij}, which is the displacement at coordinate i due to a unit force at coordinate j, may be computed by using the equation

$$\delta_{ij} = \sum\left[\int \frac{m_{ui} m_{uj} ds}{EI_u} + \int \frac{m_{vi} m_{vj} ds}{EI_v} + \int \frac{t_i t_j ds}{GK}\right] \tag{8.5}$$

where m_{ui}, m_{uj} = bending moments about the major axis due a unit force at coordinates i and j respectively

m_{vi}, m_{vj} = bending moments about the minor axis due to a unit force at coordinates i and j respectively

t_i, t_j = twisting moments due to a unit force at coordinates i and j respectively

The other columns of the flexibility matrix may be generated in a similar manner.

The foregoing analysis is based on the energy expression which includes the strain energies due to bending moments about the major and minor axes and twisting moments. In many practical problems the bending moments about the minor axis and the twisting moments are small as compared to the bending moments about the major axis. The relative magnitudes of the three types of moments depend upon the configuration of the structure and the type of loading. If the minor axis bending moments and twisting moments are small in comparison to the major axis bending moments, an approximate analysis which results in considerable simplification, may be obtained by ignoring the last two terms on the right-hand sides of Eqs (8.2), (8.4) and (8.5).

Example 8.1

A cranked bar ABCD of rectangular cross-section is free at A and fixed at D as shown in Fig. 8.2. Develop the flexibility matrix with reference to coordinates 1, 2 and 3 shown in the figure. Also calculate the displacements Δ_1, Δ_2 and Δ_3 at end A due to loads $P_1 = 1.5$ kN, $P_2 = 1.2$ kN and $P_3 = -2$ kN at coordinates 1, 2 and 3 respectively. Take $E = 200$ kN/mm^2 and $G = 80$ kN/mm^2.

Solution

To develop the flexibility matrix, a unit force may be applied successively at coordinates 1, 2 and 3. Thus to generate the first column of the flexibility matrix, apply a unit force at coordinate 1 and compute the displacements δ_{11}, δ_{21} and δ_{31} at coordinates 1, 2 and 3 respectively. In a similar manner the other columns of the flexibility matrix may be generated. The computations necessary for this purpose are shown in Table 8.1. The expressions for bending moments and twisting moments in the members can be derived readily by considering the free bodies of members AB, BC and CD in that order. The cross-sectional properties of members I_u, I_v and K may be calculated from the given cross-sectional dimensions.

Thus for member AB,

$$I_u = \frac{1}{12}\times 40\times 120^3 = 5.76\times 10^6 \text{ mm}^4$$

$$I_v = \frac{1}{12}\times 120\times 40 = 0.64\times 10^6 \text{ mm}^4$$

Torsion constant K may be computed by using Table 2.2.

$$K = 120\times 40^3\left[\frac{1}{3} - 0.21\times\frac{40}{120}\left(1 - \frac{40^4}{12\times 120^4}\right)\right]$$

$$= 2.023\times 10^6 \text{ mm}^4$$

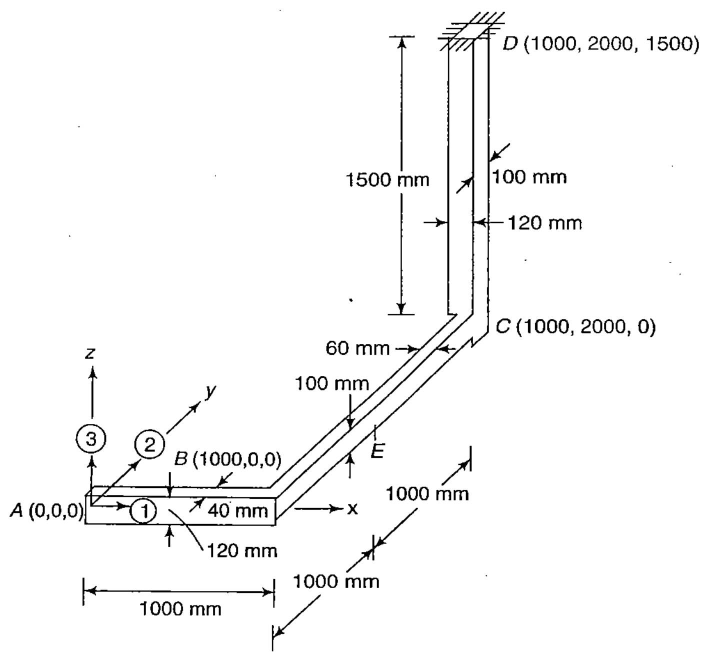


Fig. 8.2

The cross-sectional properties for members *BC* and *CD* may be computed in a similar manner. These are listed in Table 8.1.

The elements of the flexibility may be computed by using Table 8.1 and Eq. (8.5).

$$\delta_{11} = \int_0^{2000} \frac{x^2 dx}{200 \times 1.8 \times 10^6} + \int_0^{1500} \frac{(-x^2)dx}{200 \times 14.4 \times 10^6}$$

$$+ \int_0^{1500} \frac{2000^2 dx}{80 \times 19.844 \times 10^6}$$

$$= 11577.5 \times 10^{-3}$$

$$\delta_{22} = \int_0^{1000} \frac{(-x)^2 dx}{200 \times 0.64 \times 10^6} + \int_0^{2000} \frac{(-1000)^2 dx}{200 \times 1.8 \times 10^6}$$

$$+ \int_0^{1500} \frac{x^2 dx}{200 \times 10 \times 10^6} + \int_0^{1500} \frac{(-1000)^2 dx}{80 \times 19.844 \times 10^6}$$

$$= 9667.0 \times 10^{-3}$$

Table 8.1

Member	*AB*	*BC*	*CD*
1	2	3	4
I_u, mm^4	5.76×10^6	5×10^6	14.4×10^6
I_v, mm^4	0.64×10^6	1.8×10^6	10×10^6
K, mm^4	2.023×10^6	4.508×10^6	19.844×10^6
Origin	*A*	*B*	*C*
Limits, mm	0 to 1000	0 to 2000	0 to 1500
M_u, kN·mm	$-2x$	$2x$	$-1.5x-2000$
M_v, kN·mm	$-1.2x$	$1.5x-1200$	$1.2x+4000$
T, kN·mm	0	-2000	1800
m_{u1}	0	0	$-x$
m_{u2}	0	0	0
m_{u3}	x	$-x$	1000
m_{v1}	0	x	0
m_{v2}	$-x$	-1000	x
m_{v3}	0	0	-2000
t_1	0	0	2000
t_2	0	0	-1000
t_3	0	1000	0

$$\delta_{33} = \int_0^{1000} \frac{x^2 dx}{200 \times 5.76 \times 10^6} + \int_0^{2000} \frac{(-x)^2 dx}{200 \times 5 \times 10^6}$$

$$+ \int_0^{2000} \frac{1000^2 dx}{80 \times 4.508 \times 10^6} + \int_0^{1500} \frac{100^2 dx}{200 \times 14.4 \times 10^6}$$

$$+ \int_0^{1500} \frac{(-2000)^2 dx}{200 \times 10 \times 10^6}$$

$$= 12022.5 \times 10^{-3}$$

$$\delta_{12} = \delta_{21} = \int_0^{2000} \frac{-1000x dx}{200 \times 1.8 \times 10^6} + \int_0^{1500} \frac{(2000)(-1000)dx}{80 \times 19.844 \times 10^6}$$

$$= -7445.0 \times 10^{-3}$$

$$\delta_{23} = \delta_{32} = \int_0^{1500} \frac{-2000x dx}{200 \times 10 \times 10^6} = -1125.0 \times 10^{-3}$$

$$\delta_{31} = \delta_{13} = \int_0^{1500} \frac{-1000x dx}{200 \times 14.4 \times 10^6} = -390.6 \times 10^{-3}$$

Hence, the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = 10^{-3}\begin{bmatrix} 11577.5 & -7445.0 & -390.6 \\ -7445.0 & 9667.0 & -1125.0 \\ -390.6 & -1125.0 & 12022.5 \end{bmatrix}$$

The displacements at the coordinates due to the given loads may be calculated by multiplying the flexibility matrix and the load matrix.

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = 10^{-3}\begin{bmatrix} 11577.5 & -7445.0 & -390.6 \\ -7445.0 & 9667.0 & -1125.0 \\ -390.6 & -1125.0 & 12022.5 \end{bmatrix}\begin{bmatrix} 1.5 \\ 1.2 \\ -2.0 \end{bmatrix}$$

$$= \begin{bmatrix} 9.2135 \\ 2.6829 \\ -25.9809 \end{bmatrix}$$

Hence,

$$\Delta_1 = 9.2135 \text{ mm}$$

$$\Delta_2 = 2.6829 \text{ mm}$$

and

$$\Delta_3 = -25.9809 \text{ mm}$$

As an alternative procedure, the displacements due to the given loads may be computed by using Table 8.1 and Eq. (8.4).

$$\Delta_1 = \int_0^{2000} \frac{(1.5x - 1200)x dx}{200 \times 1.8 \times 10^6} + \int_0^{1500} \frac{(-1.5x - 2000)(-x)dx}{200 \times 14.4 \times 10^6}$$

$$+ \int_0^{1500} \frac{1800 \times 2000 dx}{80 \times 19.844 \times 10^6} = 9.2135 \text{ mm}$$

$$\Delta_2 = \int_0^{1000} \frac{(-1.2x)x dx}{200 \times 0.64 \times 10^6} + \int_0^{2000} \frac{(1.5x - 1200)(-1000)dx}{200 \times 1.8 \times 10^6}$$

$$+ \int_0^{1500} \frac{(1.2x + 4000)x dx}{200 \times 10 \times 10^6} + \int_0^{1500} \frac{1800(-1000)dx}{80 \times 19.844 \times 10^6}$$

$$= 2.6829 \text{ mm}$$

$$\Delta_3 = \int_0^{1000} \frac{(-2x)x dx}{200 \times 5.76 \times 10^6} + \int_0^{2000} \frac{2x(-x)\, dx}{200 \times 5 \times 10^6}$$

$$+ \int_0^{2000} \frac{(-2000)(1000)dx}{80 \times 4.508 \times 10^6} + \int_0^{1500} \frac{(-1.5x - 2000)(1000)dx}{200 \times 14.4 \times 10^6}$$

$$+ \int_0^{1500} \frac{(1.2x + 4000)(-2000)dx}{200 \times 10 \times 10^6}$$

$$= -25.9809 \text{ mm}$$

Example 8.2

The cranked bar of Ex. 8.1 is hinged at A and fixed at D. It carries a uniformly distributed vertical load of 2 kN/m on member AB and a vertical load of 1 kN on member BC at its centre E. Compute the reactions at hinge-support A.

Solution

The degree of static indeterminacy is three because if the three reaction components at hinge support A are released, a statically determinate cantilever bar is obtained. Hence, coordinates 1, 2 and 3 corresponding to the three reaction components P_1, P_2 and P_3 may be chosen as shown in Fig. 8.2. The displacements at the coordinates due to the applied loads may be computed by using the unit load method. The computations necessary for this purpose are shown in Table 8.2.

Table 8.2

Member	*AB*	*BE*	*EC*	*CD*
1	2	3	4	5
I_u, mm^4	5.76×10^6	5×10^6	5×10^6	14.4×10^6
I_v, mm^4	0.64×10^6	1.8×10^6	1.8×10^6	10×10^6
K, mm^4	2.023×10^6	4.508×10^6	4.508×10^6	19.844×10^6
Origin	A	B	B	C
Limits, mm	0 to 1000	0 to 1000	1000 to 2000	0 to 1500
M_u, kN·mm	$-0.001x^2$	$2x$	$2x + (x - 1000)$	-1000
M_v kN·mm	0	0	0	5000
T, kN·mm	0	-1000	-1000	0
m_{u1}	0	0	0	$-x$
m_{u2}	0	0	0	0
m_{u3}	x	$-x$	$-x$	1000
m_{v1}	0	x	x	0
m_{v2}	$-x$	-1000	-1000	x
m_{v3}	0	0	0	-2000
t_1	0	0	0	2000
t_2	0	0	0	-1000
t_3	0	1000	1000	0

Using Table 8.2 and Eq. (8.4),

$$\Delta_{1L} = \int_0^{1500} \frac{(-1000)(-x)dx}{200 \times 14.4 \times 10^6} = 0.3906 \text{ mm}$$

$$\Delta_{2L} = \int_0^{1500} \frac{5000xdx}{200 \times 10 \times 10^6} = 2.8125 \text{ mm}$$

$$\Delta_{3L} = \int_0^{1000} \frac{-0.001x^2(x)dx}{200 \times 5.76 \times 10^6} + \int_0^{1000} \frac{2x(-x)dx}{200 \times 5 \times 10^6}$$

$$+ \int_0^{1000} \frac{(-1000)(1000)dx}{80 \times 4.508 \times 10^6} + \int_{1000}^{2000} \frac{\{2x + (x - 1000)\}(-x)dx}{200 \times 5 \times 10^6}$$

$$+ \int_{1000}^{2000} \frac{(-1000)(1000)dx}{80 \times 4.508 \times 10^6} + \int_0^{1500} \frac{(-1000)(1000)dx}{200 \times 14.4 \times 10^6}$$

$$+ \int_0^{1500} \frac{5000(-2000)dx}{200 \times 10 \times 10^6}$$

$$= -17.6774 \text{ mm}$$

The flexibility matrix for the bar with reference to coordinates 1, 2 and 3 has already been developed in Ex. 8.1. Substituting into Eq. (6.2),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -10^3 \begin{bmatrix} 11577.5 & -7445.0 & -390.6 \\ -7445.0 & 9667.0 & -1125.0 \\ -390.6 & -1125.0 & 12022.5 \end{bmatrix}^{-1} \begin{bmatrix} 0.3906 \\ 2.8125 \\ -17.6744 \end{bmatrix}$$

$$= \begin{bmatrix} -0.1267 \\ -0.2203 \\ 1.4456 \end{bmatrix}$$

Hence, $P_1 = -0.1267$ kN
$P_2 = -0.2203$ kN
$P_3 = 1.4456$ kN

8.3 STIFFNESS OF RECTANGULAR FRAMES

A building frame is a common example of a rigid-jointed space frame. It usually comprises several rows of columns connected by series of beams at each floor level along two orthogonal directions. Choosing x- and y-axes in the horizontal plane along the two orthogonal directions and z-axis in the vertical direction, a typical building frame can be visualised as a series of plane frames in the x-z plane connected by series of beams parallel to the y-axis at each floor level. It can also be visualised as a series of plane frames in the y-z plane

connected by series of beams parallel to the x-axis at each floor level. Whenever, one of the plane frames deforms due to applied loads, it tends to deform the plane frames parallel to it through the interconnecting beams. This interaction between parallel plane frames represents the three dimensional action in a space frame.

In using the displacement method for the analysis of a rigid-jointed space frame, it is necessary to develop the expressions for the stiffness of a typical frame, it is necessary to develop the expressions for the stiffness of a typical joint of the frame. Figure 8.3 shows a typical joint O of a multistoreyed building frame having several bays in both directions. Joint O is located at the ith floor level. It belongs to the jth frame among the series of frames parallel to the x-z plane. Similarly, it belongs to the kth frame among the series of frames parallel to the y-z plane. It has been shown in Sec. 1.7 that, in general, a rigid-joint has six degrees of freedom corresponding to three linear displacements along the cartesian axes and three rotations about these axes. In the present case, the displacement of joint O along the z (vertical) direction is not possible because

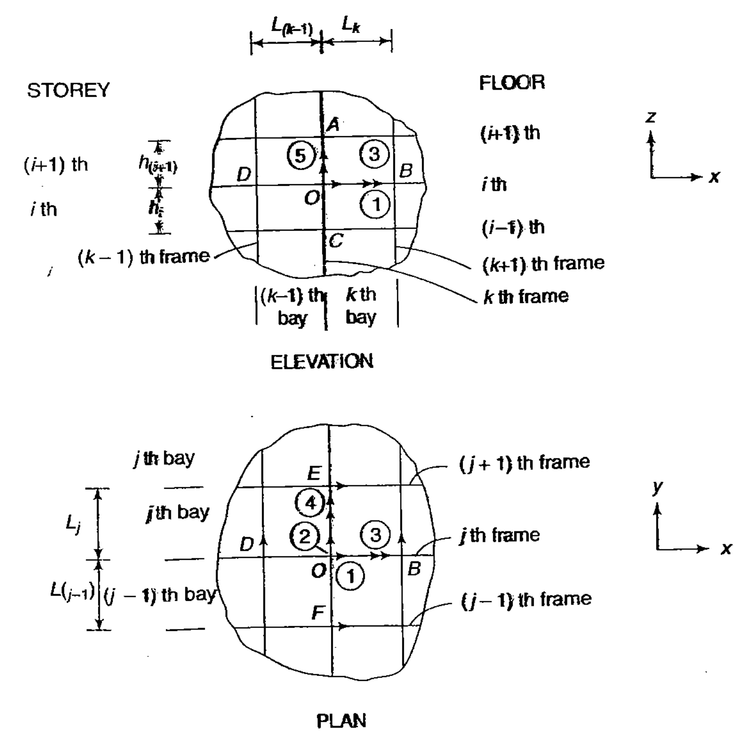


Fig. 8.3

the members of a rigid-jointed frame are assumed to be inextensible. Hence, there are five independent displacement components at joint O. Coordinates 1 and 2 have been assigned to the horizontal displacement (sway) along x and y directions respectively. Coordinates 3, 4 and 5 have been assigned to the rotations about x-, y- and z-axes respectively. The three rotational coordinates are indicated by double-headed arrows. Coordinates are assigned to all other joints of the frame in a similar manner. It should, however, be noted that in any particular plane frame (from amongst the series of plane frames constituting the space frame under consideration) only one horizontal (sway) coordinate is required at each floor level either in the x or y direction because the beams are assumed to be inextensible. Thus, while three rotational coordinates are required at each joint, the sway coordinates have to be assigned keeping in view the inextensibility of the beams. For example, only one sway coordinate is required at joints B, D, E and F as shown in the figure. The rotational coordinates at these joints have not been shown in the figure. At each of the joints A and C, two sway coordinates (not shown in the figure) are required in addition to three rotational coordinates.

The stiffness of rigid-jointed rectangular plane frames has been discussed in Sec. 6.5. The expressions for the stiffness of rigid-jointed space frames can be developed on similar lines by including the three dimensional effect. To derive the expression for stiffness element k_{11}, note that when a unit displacement is given at coordinate 1 without any displacement at the remaining coordinates as shown in Fig. 8.4, all columns of the jth frame belonging to the ith and $(i + 1)$th storeys are bent. In addition, all beams of the ith floor belonging to the $(j-1)$th and jth bays are subjected to bending action. Consequently, using Table 2.16, the expression for k_{11} may be written as

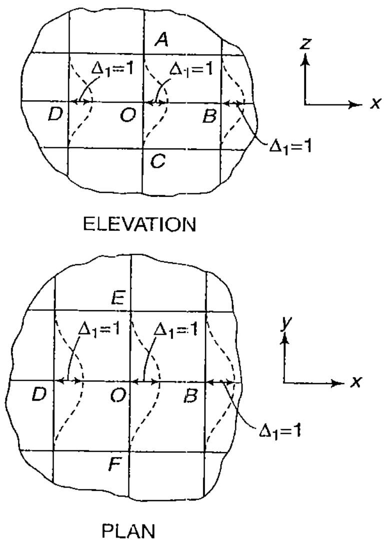


Fig. 8.4

$$k_{11} = \sum^{j\text{th frame, } i\text{th storey}} \frac{12EI_y}{h_i^3} + \sum^{j\text{th frame, } (i+1)\text{th storey}} \frac{12EI_y}{h^3_{(i+1)}}$$

$$+ \sum^{(j-1)\text{th bay, } i\text{th floor}} \frac{12EI_z}{L^3_{(j-1)}} + \sum^{j\text{th bay, } i\text{th floor}} \frac{12EI_z}{L_j^3} \qquad (8.6a)$$

Similarly, the expression for k_{22} may be written as

$$k_{22} = \sum^{k\text{th frame, } i\text{th storey}} \frac{12EI_x}{h_i^3} + \sum^{k\text{th frame, } (i+1)\text{th storey}} \frac{12EI_x}{h^3_{(i+1)}}$$

$$+ \sum^{(k-1)\text{th bay, } i\text{th floor}} \frac{12EI_z}{L^3_{(k-1)}} + \sum^{k\text{th bay, } i\text{th floor}} \frac{12EI_z}{L_k^3} \qquad (8.6b)$$

where I_x, I_y and I_z = moments of inertia of the cross-sections of the members about centroidal axes parallel to x-, y- and z-axes respectively.

The first two terms on the right hand sides of Eq. (8.6) represent the resistance offered by the columns and the last two terms represent the resistance offered by the beams. In Eq. (8.6a) the last two terms on the right hand side represent the resistance offered by the beams connecting the jth frame to the $(j-1)$th and $(j+1)$th frames. Similarly, the last two terms on the right hand side of Eq. (8.6b) represent the resistance offered by the beams connecting the kth frame to the $(k-1)$th and $(k+1)$th frames. These terms, therefore, represent the interaction between adjacent frames in a three dimensional structure. It may be noted that if these terms are ignored, i.e., the interaction between the adjacent frames is neglected, Eq. (8.6) reverts to Eq. (6.6) derived for rigid-jointed plane frames.

When a unit displacement is given at coordinate 1 as shown in Fig. 8.4, each column of the jth frame belonging to ith and $(i+1)$th storeys carries at its ends a transverse force equal to $12EI_y/h^3$ and a bending couple equal to $6EI_y/h^2$ in the x-z plane. The remaining four force components at each end of these columns are zero. Hence, the values of the elements of the stiffness matrix corresponding to the coordinates located at the ends of these columns can be readily determined. In a similar manner, each beam of the ith floor belonging to the $(j-1)$th and jth bays carries a transverse force equal to $12EI_z/L^3$ and a bending couple equal to $6EI_z/L^2$ in the x-y plane. The remaining four force components at each end of these beams are zero. Hence, the values of the elements of the stiffness matrix corresponding to the coordinates located at the ends of these beams can be readily determined.

When a unit displacement is given at coordinate 3, i.e., a unit rotation about the x-axis as shown in Fig. 8.5, columns OA and OC and beams OE and OF undergo flexural rotation whereas beams OB and OD parallel to the x-axis undergo torsional rotation. Hence, using the stiffness expressions from Table 4.1, the expression for k_{33} may be written as

$$k_{33} = \left(\frac{4EI_x}{h}\right)_{OA} + \left(\frac{4EI_x}{h}\right)_{OC} + \left(\frac{4EI_x}{L}\right)_{OE}$$

$$\left(\frac{4EI_x}{L}\right)_{OF} + \left(\frac{GK}{L}\right)_{OB} + \left(\frac{GK}{L}\right)_{OD} \qquad (8.7a)$$

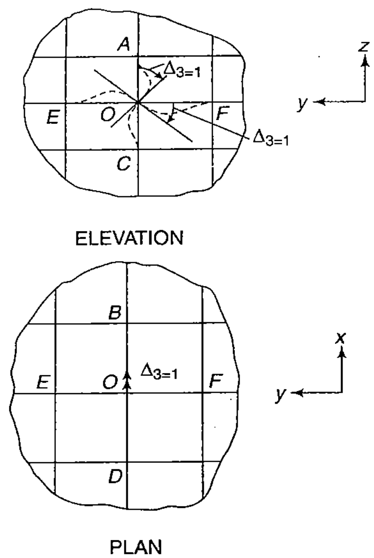


Fig. 8.5

When a unit displacement is given at coordinate 4, i.e., a unit rotation about the y-axis, members OE and OF parallel to the y-axis undergo torsional rotation. The remaining four members meeting at joint O undergo flexural rotation. In a similar manner, when a unit displacement is given at coordinate 5, i.e., a unit rotation about the z-axis, members OA and OC parallel to the z-axis undergo torsional rotation. The remaining four members meeting at joint O undergo flexural rotation. Consequently, the expressions for k_{44} and k_{55} may be written as

$$k_{44} = \left(\frac{4EI_y}{h}\right)_{OA} + \left(\frac{4EI_y}{h}\right)_{OC} + \left(\frac{4EI_y}{L}\right)_{OB} + \left(\frac{4EI_y}{L}\right)_{OD} + \left(\frac{GK}{L}\right)_{OE} + \left(\frac{GK}{L}\right)_{OF} \tag{8.7b}$$

$$k_{55} = \left(\frac{4EI_z}{L}\right)_{OB} + \left(\frac{4EI_z}{L}\right)_{OD} + \left(\frac{4EI_z}{L}\right)_{OE} + \left(\frac{4EI_z}{L}\right)_{OF} + \left(\frac{GK}{h}\right)_{OA} + \left(\frac{GK}{h}\right)_{OC} \tag{8.7c}$$

In evaluating the elements of the stiffness matrix corresponding to the coordinates located at joints A to F, it may be noted that carry-over factor is $\frac{1}{2}$

in the case of bending couples and (– 1) in the case of twisting couples. Hence, the values of these elements of the stiffness matrix may be written readily. Referring to Figs 8.4 and 8.5 and using Table 2.16 it may verified that

$$\begin{aligned}
k_{21} &= k_{31} = 0\\
k_{41} &= \left(\frac{6EI_y}{h^2}\right)_{OA} - \left(\frac{6EI_y}{h^2}\right)_{OC}\\
k_{51} &= \left(\frac{6EI_z}{L^2}\right)_{OF} - \left(\frac{6EI_z}{L^2}\right)_{OE}\\
k_{12} &= k_{42} = 0\\
k_{32} &= \left(\frac{6EI_x}{L^2}\right)_{OC} - \left(\frac{6EI_x}{L^2}\right)_{OA}\\
k_{52} &= \left(\frac{6EI_z}{L^2}\right)_{OB} - \left(\frac{6EI_z}{L^2}\right)_{OD}\\
k_{13} &= k_{43} = k_{53} = 0\\
k_{23} &= \left(\frac{6EI_x}{h^2}\right)_{OC} - \left(\frac{6EI_x}{h^2}\right)_{OA}\\
k_{14} &= \left(\frac{6EI_y}{h^2}\right)_{OA} - \left(\frac{6EI_y}{h^2}\right)_{OC}\\
k_{24} &= k_{34} = k_{54} = 0\\
k_{15} &= \left(\frac{6EI_z}{L^2}\right)_{OF} - \left(\frac{6EI_z}{L^2}\right)_{OE}\\
k_{25} &= \left(\frac{6EI_z}{L^2}\right)_{OB} - \left(\frac{6EI_z}{L^2}\right)_{OD}\\
k_{35} &= k_{45} = 0
\end{aligned} \tag{8.8}$$

Using the expressions derived above, it is possible to develop the stiffness matrix for a three-dimensional building frame.

Example 8.3

Determine the degree of freedom of the three-storeyed building frame shown in Fig. 8.6. Hence select a suitable system of coordinates for the frame. Compute the elements of the stiffness matrix with reference to the coordinates located at joint O. The cross-

sectional dimensions of all the columns of the frame are 300 mm × 300 mm. All beams parallel to the x-axis are 300 mm in width and 500 mm in depth. *All beams parallel to the x-axis are 300 mm in width and 500 mm in depth.* *All beams parallel to the y-axis are 300 mm in width and 600 mm in depth. Take E = 10 kN/mm² and G = 4 kN/mm².*

Solution

Each floor has nine joints. Therefore, the total number of joints, excluding the fixed column bases, is 27. As each joint can rotate about the three cartesian axes, the total

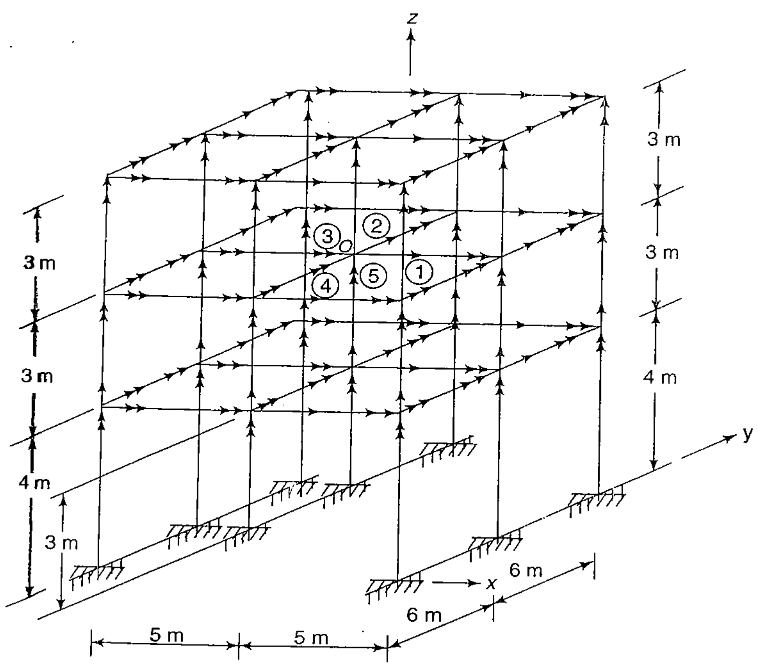


Fig. 8.6

number of independent joint rotations are $3 \times 27 = 81$. All the plane frames parallel to the x-z and y-z planes can sway in the horizontal direction and each frame has an independent sway displacement at each floor level. Consequently, there are six sway displacements at each floor level as shown in Fig. 8.6. Hence, the total number of sway displacements is $3 \times 6 = 18$. Consequently, the degree of freedom of the structure, which corresponds to the total number of independent displacement components, is $81 + 18 = 99$. A coordinate may be assigned to each one of these 99 independent displacement components as shown in Fig. 8.6. In the figure, numbers have been given only to those coordinates which are located at joint O.

For beams parallel to the x-axis,

$$I_y = \frac{1}{12} \times 300 \times 500^3 = 3.125 \times 10^9 \text{ mm}^4$$

$$I_z = \frac{1}{12} \times 500 \times 300^2 = 1.125 \times 10^9 \text{ mm}^4$$

$$K = 500 \times 300^3 \left[\frac{1}{3} - 0.21 \times \frac{300}{500}\left(1 - \frac{300^4}{12 \times 500^4}\right)\right]$$

$$= 2.81733 \times 10^9 \text{ mm}^4$$

For beams parallel to the y-axis,

$$I_x = \frac{1}{12} \times 300 \times 600^3 = 5.4 \times 10^9 \text{ mm}^4$$

$$I_z = \frac{1}{12} \times 600 \times 300^3 = 1.35 \times 10^9 \text{mm}^4$$

$$K = 600 \times 300^3 \left[\frac{1}{3} - 0.21 \times \frac{300}{600}\left(1 - \frac{300^3}{12 \times 600^4}\right)\right]$$

$$= 3.70781 \times 10^9 \text{ mm}^4$$

For columns,

$$I_x = \frac{1}{12} \times 300 \times 300^3 = 0.675 \times 10^9 \text{ mm}^4$$

$$I_y = \frac{1}{12} \times 300 \times 300^3 = 0.675 \times 10^9 \text{ mm}^4$$

$$K = 300 \times 300^3 \left[\frac{1}{3} - 0.21 \times \frac{300}{300}\left(1 - \frac{300^4}{12 \times 300^4}\right)\right]$$

$$= 1.14072 \times 10^9 \text{ mm}^4$$

The required stiffness elements can be computed by using Eqs (8.6), (8.7) and (8.8). Their values in kN-mm are

$$k_{11} = 3\left(\frac{12 \times 10 \times 0.675 \times 10^9}{3000^3}\right) + 3\left(\frac{12 \times 10 \times 0.675 \times 10^9}{3000^3}\right)$$

$$+ 3\left(\frac{12 \times 10 \times 1.35 \times 10^9}{6000^3}\right) + 3\left(\frac{12 \times 10 \times 1.35 \times 10^9}{6000^3}\right)$$

$$= 10.5$$

$$k_{22} = 3\left(\frac{12 \times 10 \times 0.675 \times 10^9}{3000^3}\right) + 3\left(\frac{12 \times 10 \times 0.675 \times 10^9}{3000^3}\right)$$

$$+ 3\left(\frac{12 \times 10 \times 1.125 \times 10^9}{5000^3}\right) + 3\left(\frac{12 \times 10 \times 1.125 \times 10^9}{5000^3}\right)$$

$$= 12.48$$

$$k_{33} = \frac{4 \times 10 \times 0.675 \times 10^9}{3000} + \frac{4 \times 10 \times 0.675 \times 10^9}{3000}$$

$$+ \frac{4 \times 10 \times 5.4 \times 10^9}{6000} + \frac{4 \times 10 \times 5.4 \times 10^9}{6000}$$

$$+ \frac{4 \times 2.81733 \times 10^9}{5000} + \frac{4 \times 2.81733 \times 10^9}{5000} = 94.5077 \times 10^6$$

$$k_{44} = \frac{4 \times 10 \times 0.675 \times 10^9}{3000} + \frac{4 \times 100 \times 0.675 \times 10^9}{3000}$$

$$+ \frac{4 \times 10 \times 3.125 \times 10^9}{5000} + \frac{4 \times 10 \times 3.125 \times 10^9}{5000}$$

$$+ \frac{4 \times 3.70781 \times 10^9}{6000} \times \frac{4 \times 3.70781 \times 10^9}{6000} = 72.9437 \times 10^6$$

$$k_{55} = \frac{4 \times 10 \times 1.125 \times 10^9}{5000} + \frac{4 \times 10 \times 1.125 \times 10^9}{5000}$$

$$+ \frac{4 \times 10 \times 1.35 \times 10^9}{6000} + \frac{4 \times 10 \times 1.35 \times 10^9}{6000}$$

$$+ \frac{4 \times 1.14072 \times 10^9}{3000} + \frac{4 \times 1.14072 \times 10^9}{3000} = 39.0419 \times 10^6$$

$$k_{12} = k_{21} = k_{13} = k_{31} = k_{24} = k_{42} = k_{34}$$

$$= k_{43} = k_{35} = k_{53} = k_{54} = k_{45} = 0$$

$$k_{14} = k_{41} = \frac{6 \times 10 \times 0.675 \times 10^9}{3000^2} - \frac{6 \times 10 \times 0.675 \times 10^9}{3000^2} = 0$$

$$k_{15} = k_{51} = \frac{6 \times 10 \times 1.35 \times 10^9}{6000^2} - \frac{6 \times 10 \times 1.35 \times 10^9}{6000^2} = 0$$

$$k_{23} = k_{32} = \frac{6 \times 10 \times 0.675 \times 10^9}{3000^2} - \frac{6 \times 10 \times 0.675 \times 10^9}{3000^2} = 0$$

$$k_{25} = k_{52} = \frac{6 \times 10 \times 1.35 \times 10^9}{5000^2} - \frac{6 \times 10 \times 1.35 \times 10^9}{5000^2} = 0$$

Example 8.4

Determine the degree of freedom of the rigid-jointed space frame shown in Fig. 8.7. Hence, select a suitable system of coordinates and develop the stiffness matrix with reference to the chosen coordinates. The cross-sectional dimensions of columns AB and CD are 300 × 300 mm. Beams BC, BE and CF are 300 mm in width and 600 mm in depth. Take E = 10 kN/mm² and G = 4 kN/mm².

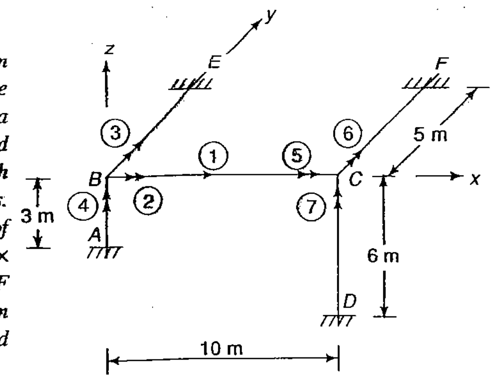


Fig. 8.7

Solution

At fixed supports A, D, E and F, no movement, whatsoever, is possible. Hence the degree of freedom at the supports is zero. The degree of freedom of joint B is four because it can move along the x-axis and rotate about the three cartesian axes. The degree of freedom of joint C is three because it can rotate about the three cartesian axes. The displacement of joint C along the x-axis is the same as that of joint B. Hence it is not an independent displacement component. It may also be noted that the displacements of joints B and C along y- and z-axes are not possible due to the assumed inextensibility of the members. Consequently, the degree of freedom of the structure is seven. Coordinates 1 to 7 may, therefore, be selected as shown in Fig. 8.7. Coordinate 1 has been assigned to the horizontal (sway) displacement of the frame along the x-axis. Coordinates 2, 3 and 4 have been assigned to the rotations of joint B about the x-, y- and z-axes respectively. Coordinates 5, 6 and 7 correspond to the rotations of joint C about x-, y- and z-axes respectively.

For beam BC,

$$I_y = \frac{1}{12} \times 300 \times 600^3 = 5.4 \times 10^9 \text{ mm}^4$$

$$I_z = \frac{1}{12} \times 600 \times 300^3 = 1.35 \times 10^9 \text{ mm}^4$$

$$K = 600 \times 300^3 \left[\frac{1}{3} - 0.21 \times \frac{300}{600}\left(1 - \frac{300}{12 \times 600^4}\right)\right]$$

$$= 3.699 \times 10^9 \text{ mm}^4$$

For beams BE and CF,

$$I_x = \frac{1}{12} \times 300 \times 600^3 = 5.4 \times 10^9 \text{ mm}^4$$

$$I_z = \frac{1}{12} \times 600 \times 300^3 = 1.35 \times 10^9 \text{ mm}^4$$

$= 3.699 \times 10^9$ mm

For columns *AB* and *CD*,

$$I_x = I_y = \frac{1}{12} \times 300 \times 300^3 = 0.675 \times 10^9 \text{ mm}^4$$

$$K = 300 \times 300^3 \left[\frac{1}{3} - 0.21 \times \frac{300}{300}\left(1 - \frac{300^4}{12 \times 300^4}\right)\right]$$

$$= 1.141 \times 10^9 \text{ mm}^4$$

The stiffness matrix can be developed by giving a unit displacement successively at coordinates 1 to 7. To generate the first column of the stiffness matrix, give a unit displacement at coordinate 1 without any displacement at the remaining coordinates. In this case columns *AB* and *CD* bend about the *y*-axis. Beams *BE* and *CF* bend about the *z*-axis and beam *BC* moves parallel to itself without bending. Hence, using the stiffness expressions from Table 4.1, the elements of the first column of the stiffness matrix may be computed as follows:

$$k_{11} = \frac{12 \times 10 \times 0.675 \times 10^9}{3000^3} + \frac{12 \times 10 \times 1.35 \times 10^9}{5000^3} + \frac{12 \times 10 \times 0.675 \times 10^9}{6000^3} + \frac{12 \times 10 \times 1.35 \times 10^9}{5000^3}$$

$$= 5.967$$

$$k_{21} = 0$$

$$k_{31} = -\frac{6 \times 10 \times 0.675 \times 10^9}{3000^2} = -4500$$

$$k_{41} = -\frac{6 \times 10 \times 1.35 \times 10^9}{5000^2} = -3240$$

$$k_{51} = 0$$

$$k_{61} = -\frac{6 \times 10 \times 0.675 \times 10^9}{6000^2} = -1125$$

$$k_{71} = -\frac{6 \times 10 \times 1.35 \times 10^9}{5000^2} = -3240$$

To generate the second column of the stiffness matrix, give a unit displacement at coordinate 2 without any displacement at the remaining coordinates. In this case memebrs *BA* and *BE* bend and member *BC* twists about the *x*-axis. Member *CD* and *CF* remain undeformed. Hence, using the stiffness expressions from Table 4.1, the elements of the second column of the stiffness matrix may be computed as follows:

$$k_{12} = 0$$

$$k_{22} = \frac{4 \times 10 \times 0.675 \times 10^9}{3000} + \frac{4 \times 10 \times 5.4 \times 10^9}{5000}$$

$$+ \frac{4 \times 3.699 \times 10^9}{10000} = 53.68 \times 10^6$$

$$k_{32} = 0$$

$$k_{42} = 0$$

$$k_{52} = -\frac{4 \times 3.699 \times 10^9}{10000} = -1.48 \times 10^6$$

$$k_{62} = 0$$

$$k_{72} = 0$$

To generate the third column of the stiffness matrix, give a unit displacement at coordinate 3 without any displacement at the remaining coordinates. In this case members *BA* and *BC* bend and member *BE* twists about *y*-axis. Members CD and CF remain undeformed. Hence, using the stiffness expressions from Table 4.1, the elements of the third column of the stiffness matrix may be computed as follows:

$$k_{13} = -\frac{6 \times 10 \times 0.675 \times 10^9}{3000^2} = -4500$$

$$k_{23} = 0$$

$$k_{33} = \frac{4 \times 10 \times 0.675 \times 10^9}{3000} + \frac{4 \times 10 \times 5.4 \times 10^9}{10000} + \frac{4 \times 3.699 \times 10^9}{5000} = 33.56 \times 10^6$$

$$k_{43} = 0$$

$$k_{53} = 0$$

$$k_{63} = \frac{2 \times 10 \times 5.4 \times 10^9}{10000} = 10.8 \times 10^6$$

$$k_{73} = 0$$

To generate the fourth column of the stiffness matrix, give a unit displacement at coordinate 4 without any displacement at the remaining coordinates. In this case members *BC* and *BE* bend and member *BA* twists about the *z*-axis. Hence, using the stiffness expressions from Table 4.1, the elements of the fourth column of the stiffness matrix may be computed as follows:

$$k_{14} = -\frac{6 \times 10 \times 1.35 \times 10^9}{5000^2} = -3240$$

$$k_{24} = 0$$

$$k_{34} = 0$$

$$k_{44} = \frac{4 \times 10 \times 1.35 \times 10^9}{10000} + \frac{4 \times 10 \times 1.35 \times 10^9}{5000}$$

$$+\frac{4 \times 1.141 \times 10^9}{3000} = 17.72 \times 10^6$$

$$k_{54} = 0$$

$$k_{64} = 0$$

$$k_{74} = \frac{2 \times 10 \times 1.35 \times 10^9}{10000} = 2.7 \times 10^6$$

In a similar manner, the elements of the fifth, sixth and seventh columns may be generated. Hence, the stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 5.967 & 0 & -4500 & -3240 & 0 & -1125 & -3240 \\ 0 & 53680000 & 0 & 0 & -1480000 & 0 & 0 \\ -4500 & 0 & 33560000 & 0 & 0 & 10800000 & 0 \\ -3240 & 0 & 0 & 17720000 & 0 & 0 & 2700000 \\ 0 & -1480000 & 0 & 0 & 49180000 & 0 & 0 \\ -1125 & 0 & 10800000 & 0 & 0 & 29060000 & 0 \\ -3240 & 0 & 0 & 2700000 & 0 & 0 & 16960000 \end{bmatrix}$$

8.4 STIFFNESS OF GRID STRUCTURES

Grid structures are frequently used in buildings and bridges. Although all members of a grid generally lie in one plane, they are subjected to torsional moments because the external loads act normal to the plane of the grid. Figure 8.8 shows a typical joint of an irregular grid. If the plane of the grid coincides with the x-y (horizontal) plane, joint A can rotate in the y-z and x–z planes about x- and y-axes respectively. It can have a linear displacement in the z

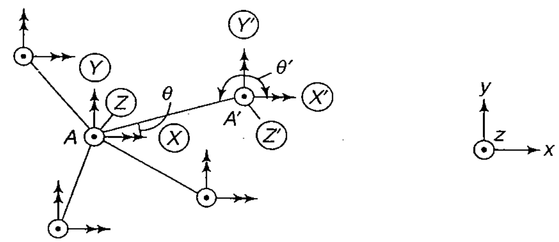


Fig. 8.8

(vertical) direction. The linear displacements in x and y directions and rotation in the x-y plane are not possible if the loading is vertical. Consequently, the degree of freedom of joint A is 3, the independent displacement components being the rotations about x- and y-axes and linear displacement along the z-axis. Coordinates X, Y and Z have been assigned to the independent

displacement components at joint A. Similarly, coordinates may be assigned to the three independent displacement components at each of the remaining joints. For instance, at joint A', coordinates X', Y' and Z' have been assigned to the rotations about the x- and y-axes and linear displacement in the z direction respectively. Expressions for the elements of the stiffness matrix may be derived by giving a unit displacement at each coordinate successively.

To generate the Xth column of the stiffness matrix, give a unit displacement at coordinate X, i.e., a unit rotation about the x-axis. The torsional and flexural rotations of member $AA¢$ are obtained by resolving the rotational vector parallel and perpendicular to the axis of the member. Hence, the torsional and flexural rotations of member $AA¢$ are cos q and sin q respectively. Using the expressions for torsional and flexural stiffnesses given in Table 4.1, the twisting moment and bending moment at end A of the member $AA¢$ are $\frac{GK}{L}$ cos q and $\frac{4EI}{L}\sin\theta$ respectively as shown in Fig. 8.9. This figure shows the free-body diagram of member $AA¢$ due to a unit displacement at coordinate X. The components of these torsional and flexural couples along coordinate X are $\frac{GK}{L}\cos^2\theta$ and $\frac{4EI}{L}\sin^2\theta$. Hence, the resisting couple about the x-axis offered by member AA' for a unit displacement at coordinate X is $\left(\frac{GK}{L}\cos^2\theta + \frac{4EI}{L}\sin^2\theta\right)$. Similar resistances are offered by other members meeting at joint A. Hence the expression for k_{XX} may be written as

$$k_{XX} = \sum\left(\frac{GK}{L}\cos^2\theta + \frac{4EI}{L}\sin^2\theta\right) \tag{8.9a}$$

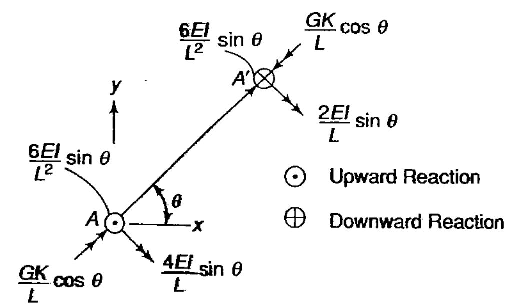


Fig. 8.9

The summation should include all members meeting at joint A. The angle θ is measured counter-clockwise from the positive direction of the x-axis at A.

The components of the torsional and flexural couples in member AA' along coordinate Y are $\frac{GK}{L}\cos\theta\sin\theta$ and $-\frac{4EI}{L}\sin\theta\cos\theta$. Similar components are offered by the torsional and flexural moments in the other members meeting at joint A. Hence, the expression for k_{YX} may be written as

$$k_{YX} = \sum\left(\frac{GK}{L} - \frac{4EI}{L}\right)\sin\theta\cos\theta \tag{8.9b}$$

The expression for k_{ZX} can be derived by computing the force required at coordinate Z when a unit displacement is given at coordinate X. Considering again member AA', the torsional moment does not produce a vertical reaction. The flexural moment at end A in member AA' is $\frac{4EI}{L}\sin\theta$. As the carry-over factor is $\frac{1}{2}$, the flexural moment at A' is $\frac{2EI}{L}\sin\theta$. Consequently, the vertical reaction at joint A, offered by member AA' is $\frac{6EI}{L^2}\sin\theta$. Similar vertical reactions are offered by other members meeting at joint A. Hence,

$$k_{ZX} = \sum\frac{6EI}{L^2}\sin\theta \tag{8.9c}$$

The torsional and flexural moments at end A' of member AA' are $\frac{GK}{L}\cos\theta$ and $\frac{2EI}{L}\sin\theta$ as shown in Fig. 8.9. The forces required at coordinates X', Y' and Z' to sustain these torsional and flexural couples in member AA' are evidently the values of $k_{X'X}$, $K_{Y'X}$ and $K_{Z'X}$ respectively. Hence,

$$k_{X'X} = \left(\frac{2EI}{L}\sin^2\theta - \frac{GK}{L}\cos^2\theta\right) \tag{8.9d}$$

$$k_{Y'X} = -\left(\frac{2EI}{L} + \frac{GK}{L}\right)\sin\theta\cos\theta \tag{8.9e}$$

$$k_{Z'X} = -\frac{6EI}{L^2}\sin\theta \tag{8.9f}$$

To generate the Yth column of the stiffness matrix, give a unit displacement at coordinate Y, i.e., a unit rotation about the y-axis. The torsional and flexural

rotations of member AA' are $\sin\theta$ and $\cos\theta$ respectively. Consequently, the torsional and flexural moments at end A of member AA' are $\frac{GK}{L}\sin\theta$ and $\frac{4EI}{L}\cos\theta$ respectively as shown in the free-body diagram of the member AA' in Fig. 8.10. The free-body diagrams of other members of the frame meeting at joint A due to a unit displacement at coordinate Y may be drawn in a similar manner. Proceeding in a manner similar to that employed for developing the stiffness expressions due to a unit displacement at coordinate X, the stiffness expressions due to a unit displacement at coordinate Y may be developed.

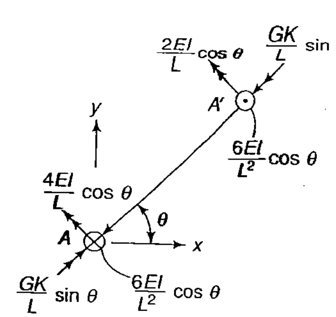


Fig. 8.10

$$k_{XY} = \sum\left(\frac{GK}{L} - \frac{4EI}{L}\right)\sin\theta\cos\theta \tag{8.10a}$$

$$k_{YY} = \sum\left(\frac{GK}{L}\sin^2\theta + \frac{4EI}{L}\cos^2\theta\right) \tag{8.10b}$$

$$k_{ZY} = -\sum\frac{6EI}{L^2}\cos\theta \tag{8.10c}$$

$$k_{X'Y} = -\left(\frac{GK}{L} + \frac{2EI}{L}\right)\sin\theta\cos\theta \tag{8.10d}$$

$$k_{Y'Y} = \left(\frac{2EI}{L}\cos^2\theta - \frac{GK}{L}\sin^2\theta\right) \tag{8.10e}$$

$$k_{Z'Y} = \frac{6EI}{L^2}\cos\theta \tag{8.10f}$$

To generate the Zth column of the stiffness matrix, give a unit displacement at coordinate Z, i.e., a unit upward deflection at joint A. The displacements at coordinates X and Y are prevented. The members do not undergo torsional and flexural rotations at joint A. Consequently, the members bend without torsion. The free-body diagram of member AA' is shown in Fig. 8.11. Using Table 2.16, the flexural couple and the vertical reaction

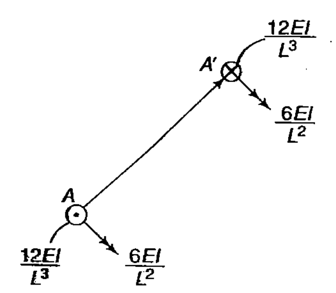


Fig. 8.11

at each end of member AA' are $6EI/L^2$ and $12EI/L^3$ respectively. Similarly, considering other members meeting at joint A, the total vertical force required at joint A for a unit vertical displacement is given by the equation

$$k_{ZZ} = \sum \frac{12EI}{L^3} \tag{8.11a}$$

Considering the free-body diagram of member AA' shown in Fig. 8.11 and resolving the forces at end A along coordinates X and Y, expressions for k_{XZ} and k_{YZ} may be written as

$$k_{XZ} = \sum \frac{6EI}{L^2} \sin\theta \tag{8.11b}$$

$$k_{YZ} = -\sum \frac{6EI}{L^2} \cos\theta \tag{8.11c}$$

Similarly, considering the free-body diagram of member AA' and resolving the forces acting at end A' along coordinates X', Y' and Z', the expressions for $k_{X'Z}$, $k_{Y'Z}$ and $k_{Z'Z}$ may be written as

$$k_{X'Z} = \frac{6EI}{L^2} \sin\theta \tag{8.11d}$$

$$k_{Y'Z} = -\frac{6EI}{L^2} \cos\theta \tag{8.11e}$$

$$k_{Z'Z} = -\frac{12EI}{L^3} \tag{8.11f}$$

Equations (8.9), (8.10) and (8.11) are sufficient to develop the stiffness matrix for any irregular grid. In the derivation of the above equations, it is assumed that all members of the grid lie in one plane and that external loads are normal to the plane of the grid.

Examples 8.5

Determine the degree of freedom of the grid shown in Fig. 8.12. Hence select a suitable system of coordintes and develop the stiffness matrix. The members are 300 mm in width and 600 mm in depth. Take E = 12 kN/mm² and G = 5 kN/mm².

Solution

At fixed supports A, B, C and D, no displacement is possible. At joint O, the degree of freedom is three. Coordinates 1 and 2 may be assigned to the rotations of joint O about x-and y-axes and coordinate 3 to the displacement along the z-axis as shown in Fig. 8.12.

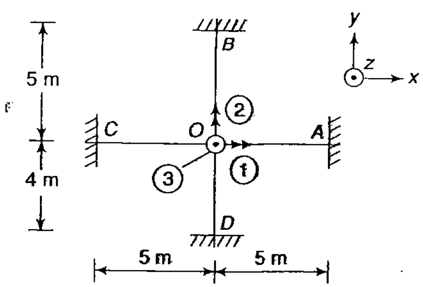


Fig. 8.12

As all the members of the grid have the same cross-section, their moment of inertia and torsion constant are also the same.

$$I = \frac{1}{12} \times 300 \times 600^3 = 5.4 \times 10^9 \text{ mm}^4$$

$$K = 600 \times 300^3 \left[\frac{1}{3} - 0.21 \times \frac{300}{600}\left(1 - \frac{300^4}{12 \times 600^4}\right)\right]$$

$$= 3.699 \times 10^9 \text{ mm}^4$$

The inclinations of members OA, OB, OC and OD measured counter-clockwise from the positive direction of the x-axis are 0°, 90°, 180° and 270° respectively. The elements of the stiffness matrix with reference to the chosen coordinates may be computed by using Eqs (8.9), (8.10) and (8.11).

$$k_{11} = \frac{5 \times 3.699 \times 10^9}{5000} \cos^2 0^0 + \frac{5 \times 3.699 \times 10^9}{5000} \cos^2 180^\circ$$

$$+ \frac{4 \times 12 \times 5.4 \times 10^9}{5000} \sin^2 90^\circ + \frac{4 \times 12 \times 5.4 \times 10^9}{4000} \sin^2 270^\circ$$

$$= 124.04 \times 10^6$$

$$k_{22} = \frac{5 \times 3.699 \times 10^9}{5000} \sin^2 90^\circ + \frac{5 \times 3.699 \times 10^9}{4000} \sin^2 270^\circ$$

$$+ \frac{4 \times 12 \times 5.4 \times 10^9}{5000} \cos^2 0^\circ + \frac{4 \times 12 \times 5.4 \times 10^9}{5000} \cos^2 180^\circ$$

$$= 112 \times 10^6$$

$$k_{33} = \frac{12 \times 12 \times 5.4 \times 10^9}{5000^3} + \frac{12 \times 12 \times 5.4 \times 10^9}{5000^3}$$

$$+ \frac{12 \times 12 \times 5.4 \times 10^9}{5000^3} + \frac{12 \times 12 \times 5.4 \times 10^9}{4000^3} = 30.8$$

$$k_{12} = k_{21} = 0$$

$$k_{23} = k_{32} = -\frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\cos 0^\circ - \frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\cos 90^\circ$$

$$-\frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\cos 180^\circ - \frac{6 \times 12 \times 5.4 \times 10^9}{4000^2}\cos 270^\circ$$

$$= 0$$

$$k_{31} = k_{13} = \frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\sin 0^\circ + \frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\sin 90^\circ$$

$$+\frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\sin 180^\circ + \frac{6 \times 12 \times 5.4 \times 10^9}{4000^2}\sin 270^\circ$$

$$= -8748$$

Hence, stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 124.04 \times 10^6 & 0 & -8748 \\ 0 & 112 \times 10^6 & 0 \\ -8748 & 0 & 30.8 \end{bmatrix}$$

Example 8.6

Determine the degree of freedom of grid shown in Fig. 8.13. Hence select a suitable system of coordinates and develop the stiffness matrix. The members are 300 mm in width and 600 mm in depth. Take $E = 12$ kN/mm² and $G = 5$ kN/mm².

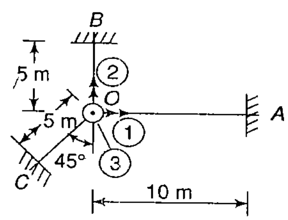


Fig. 8.13

Solution

The degree of freedom of the structure is evidently three because joint O can rotate about the x- and y-axes and move along z-axis. Coordinates 1 and 2 may be assigned to the rotations of joint O about x-and y-axes and coodinate 3 to the displacement along the z-axis as shown in Fig. 8.13. As all the members of the grid have the same cross-section, their moment of inertia and torsion constant are also the same.

$$I = \frac{1}{12} \times 300 \times 600^3 = 5.4 \times 10^9 \text{ mm}^4$$

$$K = 600 \times 300^3\left[\frac{1}{3} - 0.21 \times \frac{300}{600}\left(1 - \frac{300^4}{12 \times 600^4}\right)\right]$$

$$= 3.699 \times 10^9 \text{ mm}^4$$

Also, the inclinations of members OA, OB and OC measured counter-clockwise from the positive direction of the x-axis are 0°, 90° and 225° respectively. The elements of the stiffness matrix may be computed by using Eqs (8.9), (8.10) and (8.11).

$$k_{11} = \frac{5 \times 3.699 \times 10^9}{10000} + \frac{4 \times 12 \times 5.4 \times 10^9}{5000}$$

$$+\frac{5 \times 3.699 \times 10^9}{5000}\cos^2 225^\circ + \frac{4 \times 12 \times 5.4 \times 10^9}{5000}\sin^2 225^\circ$$

$$= 81.46 \times 10^6$$

$$k_{22} = \frac{4 \times 12 \times 5.4 \times 10^9}{10000} + \frac{5 \times 3.699 \times 10^9}{5000} + \frac{5 \times 3.699 \times 10^9}{5000}\sin^2 225^\circ$$

$$+\frac{4 \times 12 \times 5.4 \times 10^9}{5000}\cos^2 225^\circ = 57.39 \times 10^6$$

$$k_{33} = \frac{12 \times 12 \times 5.4 \times 10^9}{10000^3} + \frac{12 \times 12 \times 5.4 \times 10^9}{5000^3} + \frac{12 \times 12 \times 5.4 \times 10^9}{5000^3}$$

$$= 13.22$$

$$k_{12} = k_{21} = \left(\frac{5 \times 3.699 \times 10^9}{5000} - \frac{4 \times 12 \times 5.4 \times 10^9}{5000}\right)\sin 225^\circ \cos 225^\circ$$

$$= -24.07 \times 10^6$$

$$k_{23} = k_{32} = -\frac{6 \times 12 \times 5.4 \times 10^9}{10000^2} - \frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\cos 225^\circ$$

$$= 7108.9$$

$$k_{31} = k_{13} = \frac{6 \times 12 \times 5.4 \times 10^9}{5000^2} + \frac{6 \times 12 \times 5.4 \times 10^9}{5000^2}\sin 225^\circ$$

$$= 4555.1$$

Hence, stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 81.46 \times 10^6 & -24.07 \times 10^6 & 4555.1 \\ -24.07 \times 10^6 & 57.39 \times 10^6 & 7108.9 \\ 4555.1 & 7108.9 & 13.22 \end{bmatrix}$$

Example 8.7

Determine the degree of freedom of the grid shown in Fig. 8.14. Hence select a suitable system of coordinates and develop the stiffness matrix. Members AB, BC, DE and EF are 450 mm in width and 900 mm in depth. Member BE is 300 mm in width and 900 mm in depth. Take $E = 12$ kN/mm² and $G = 5$ kN/mm².

Solution

The degree of freedom at fixed supports A, C, D and F is zero. At each of the joints B and E, the degree of freedom is three. Hence the degree of freedom of the structure is six. At joint B, coordinates 1 and 2 may be assigned to the rotations about the x- and y-axes respectively and coordinate 3 to the displacement along the z-axis. Similarly, at joint E, coordinates 4 and 5 may be assigned to the rotations about the x- and y-axes and coordinate 6 to the displacement along the z-axis.

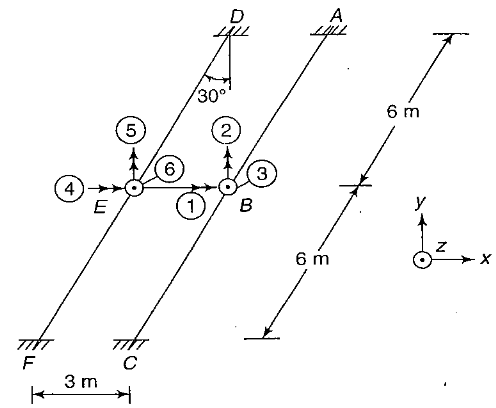


Fig. 8.14

For members AB, BC, DE and EF,

$$I = \frac{1}{12} \times 450 \times 900^3 = 27.34 \times 10^9 \text{ mm}^4$$

$$K = 900 \times 450^3 \left[\frac{1}{3} - 0.21 \times \frac{450}{900}\left(1 - \frac{450^4}{12 \times 900^4}\right)\right] = 18.77 \times 10^9 \text{ mm}^4$$

For member BE,

$$I = \frac{1}{12} \times 300 \times 900^3 = 18.23 \times 10^9 \text{ mm}^4$$

$$K = 900 \times 300^3 \left[\frac{1}{3} - 0.21 \times \frac{300}{900}\left(1 - \frac{300^4}{12 \times 900^4}\right)\right] = 6.401 \times 10^9 \text{ mm}^4$$

At joint B, the inclinations of members BA, BE and BC measured counter-clockwise from the positive direction of the x-axis are 60°, 180° and 240° respectively. Similarly, at joint E, the inclinations of members EB, ED and EF are 0°, 60° and 240° respectively. Hence, the elements of the stiffness matrix may be computed by using Eqs (8.9), (8.10) and (8.11).

$$\begin{aligned} k_{11} &= \frac{5 \times 18.77 \times 10^9}{6000}\cos^2 60° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\sin^2 60° \\ &\quad + \frac{5 \times 6.401 \times 10^9}{3000}\cos^2 180° + \frac{4 \times 12 \times 18.23 \times 10^9}{6000}\sin^2 180° \\ &\quad + \frac{5 \times 18.77 \times 10^9}{6000}\cos^2 240° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\sin^2 240° \\ &= 346.57 \times 10^6 \end{aligned}$$

$$\begin{aligned} k_{22} &= \frac{5 \times 18.77 \times 10^9}{6000}\sin^2 60° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\cos^2 60° \\ &\quad + \frac{5 \times 6.401 \times 10^9}{3000}\sin^2 180° + \frac{4 \times 12 \times 18.23 \times 10^9}{3000}\cos^2 180° \\ &\quad + \frac{5 \times 18.77 \times 10^9}{6000}\sin^2 240° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\cos^2 240° \\ &= 424.5 \times 10^6 \end{aligned}$$

$$\begin{aligned} k_{33} &= \frac{12 \times 12 \times 27.34 \times 10^9}{6000^3} + \frac{12 \times 12 \times 18.23 \times 10^9}{3000^3} \\ &\quad + \frac{12 \times 12 \times 27.34 \times 10^9}{6000^3} = 133.68 \end{aligned}$$

$$\begin{aligned} k_{44} &= \frac{5 \times 6.401 \times 10^9}{3000}\cos^2 0° + \frac{4 \times 12 \times 18.23 \times 10^9}{3000}\sin^2 0° \\ &\quad + \frac{5 \times 18.77 \times 10^9}{6000}\cos^2 60° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\sin^2 60° \\ &\quad + \frac{5 \times 18.77 \times 10^9}{6000}\cos^2 240° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\sin^2 240° \\ &= 346.57 \times 10^6 \end{aligned}$$

$$\begin{aligned} k_{55} &= \frac{5 \times 6.401 \times 10^9}{3000}\sin^2 0° + \frac{4 \times 12 \times 18.23 \times 10^9}{3000}\cos^2 0° \\ &\quad + \frac{5 \times 18.77 \times 10^9}{6000}\sin^2 60° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\cos^2 60° \\ &\quad + \frac{5 \times 18.77 \times 10^9}{6000}\sin^2 240° + \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\cos^2 240° \\ &= 424.5 \times 10^6 \end{aligned}$$

$$\begin{aligned} k_{66} &= \frac{12 \times 12 \times 18.23 \times 10^9}{3000^3} + \frac{12 \times 12 \times 27.34 \times 10^9}{6000^3} \\ &\quad + \frac{12 \times 12 \times 27.34 \times 10^9}{6000^3} = 133.38 \end{aligned}$$

$$k_{12} = k_{21} = \left(\frac{5 \times 18.77 \times 10^9}{6000} - \frac{4 \times 12 \times 27.34 \times 10^9}{6000}\right)\sin 60° \cos 60°$$

$$+\left(\frac{5\times 6.401\times 10^9}{3000}-\frac{4\times 12\times 18.23\times 10^9}{3000}\right)\sin 180^\circ\cos 180^\circ$$

$$+\left(\frac{5\times 18.77\times 10^9}{6000}-\frac{4\times 12\times 27.34\times 10^9}{6000}\right)\sin 240^\circ\cos 240^\circ$$

$$=-175.871\times 10^6$$

$$k_{13}=k_{31}=\frac{6\times 12\times 27.34\times 10^9}{6000^2}\sin 60^\circ+\frac{6\times 12\times 18.23\times 10^9}{3000^2}\sin 180^\circ$$

$$+\frac{6\times 12\times 27.34\times 10^9}{6000^2}\sin 240^\circ=0$$

$$k_{23}=k_{32}=-\frac{6\times 12\times 27.34\times 10^9}{6000^2}\cos 60^\circ-\frac{6\times 12\times 18.23\times 10^9}{3000^2}\cos 180^\circ$$

$$-\frac{6\times 12\times 27.34\times 10^9}{6000^2}\cos 240^\circ=145840$$

$$k_{45}=k_{54}=\left(\frac{5\times 6.401\times 10^9}{3000}-\frac{4\times 12\times 18.23\times 10^9}{3000}\right)\sin 0^\circ\cos 0^\circ$$

$$+\left(\frac{5\times 18.77\times 10^9}{6000}-\frac{4\times 12\times 27.34\times 10^9}{6000}\right)\sin 60^\circ\cos 60^\circ$$

$$+\left(\frac{5\times 18.77\times 10^9}{6000}-\frac{4\times 12\times 27.34\times 10^9}{6000}\right)\sin 240^\circ\cos 240^\circ$$

$$=-175.871\times 10^6$$

$$k_{46}=k_{64}=\frac{6\times 12\times 18.23\times 10^9}{3000^2}\sin 0^\circ+\frac{6\times 12\times 27.34\times 10^9}{6000}\sin 60^\circ$$

$$+\frac{6\times 12\times 27.34\times 10^9}{6000^2}\sin 240^\circ=0$$

$$k_{56}=k_{65}=-\frac{6\times 12\times 18.23\times 10^9}{3000^2}\cos 0^\circ-\frac{6\times 12\times 27.34\times 10^9}{6000^2}\cos 60^\circ$$

$$-\frac{6\times 12\times 27.34\times 10^9}{6000}\cos 240^\circ=-145840$$

$$k_{41}=k_{14}=\frac{2\times 12\times 18.23\times 10^9}{3000}\sin^2 180^\circ-\frac{5\times 6.401\times 10^9}{3000}\cos^2 180^\circ$$

$$=-10.668\times 10^6$$

$$k_{51}=k_{15}=-\left(\frac{2\times 12\times 18.23\times 10^9}{3000}+\frac{5\times 6.401\times 10^9}{3000}\right)\sin 180^\circ\cos 180^\circ$$

$$=0$$

$$k_{61}=k_{16}=-\frac{6\times 12\times 18.23\times 10^9}{3000^2}\sin 180^\circ=0$$

$$k_{42}=k_{24}=-\left(\frac{5\times 6.401\times 10^9}{3000}+\frac{2\times 12\times 18.23\times 10^9}{3000}\right)\sin 180^\circ\cos 180^\circ$$

$$=0$$

$$k_{52}=k_{25}=\frac{2\times 12\times 18.23\times 10^9}{3000}\cos^2 180^\circ-\frac{5\times 6.401\times 10^9}{3000}\sin^2 180^\circ$$

$$=145.84\times 10^6$$

$$k_{62}=k_{26}=\frac{6\times 12\times 18.23\times 10^9}{3000^2}\cos 180^\circ=-145840$$

$$k_{43}=k_{34}=\frac{6\times 12\times 18.23\times 10^9}{3000^2}\sin 180^\circ=0$$

$$k_{53}=k_{35}=-\frac{6\times 12\times 18.23\times 10^9}{3000^2}\cos 180^\circ=145840$$

$$k_{63}=k_{36}=-\frac{12\times 12\times 18.23\times 10^9}{3000^2}=-97.23$$

Hence, stiffness matrix $[k]$ is given by the equation

$$[k]=\begin{bmatrix} 346.57\times 10^6 & -175.871\times 10^6 & 0 & -10.668\times 10^6 & 0 & 0\\ -175.871\times 10^6 & 424.5\times 10^6 & 145840 & 0 & 145.84\times 10^6 & -145840\\ 0 & 145840 & 133.68 & 0 & 145840 & -97.23\\ -10.668\times 10^6 & 0 & 0 & 346.57\times 10^6 & -175.871\times 10^6 & 0\\ 0 & 145.84\times 10^6 & 145840 & -175.871\times 10^6 & 424.5\times 10^6 & -145840\\ 0 & -145840 & -97.23 & 0 & -145840 & 133.68 \end{bmatrix}$$

8.5 DISPLACEMENT METHOD

As in the case of rigid-jointed plane frames, the method begins with the determination of the degree of freedom of the structure and identification of the independent displacement components. The degree of freedom has been discussed in Sec. 1.7 and also in Secs. 8.3 and 8.4. A coordinate is then assigned to each independent displacement component and the stiffness matrix with reference to these coordinates developed. Thereafter, as in the case of rigid-jointed plane frames, the displacement components may be computed by using the equation

$$[\Delta] = [k]^{-1}\ \{[P] - [P']\} \tag{8.12}$$

After the displacement components are known, the bending moments in the members may be computed by using the slope-deflection Eq. (2.47). The twisting moments in the members may be computed by multiplying the torsional rotations by the respective torsional stiffness, *GK/L*.

Example 8.8

Analyse the rigid-jointed space frame of Ex. 8.4. The external loads acting on the frame are shown in Fig. 8.15. Compute the bending and twisting moments at the ends of member BC.

Solution

In Ex. 8.4, it has been shown that the degree of freedom of the frame is seven. Coordinates 1 to 7 have been assigned to the seven independent displacement components of the frame. The displacement components Δ_1 to Δ_7 at the chosen coordinates may be computed by using Eq. (8.12).

Forces P'_1 to P'_7 at coordinates 1 to 7, due to the external loads other than those acting at the coordinates when no displacement is permitted at the coordinates, may be computed by considering all the members as fixed-ended members.

$$P'_1 = P'_2 = 0 \quad P'_3 = -\frac{60 \times 4000 \times 6000^2}{10000^2} = -86400 \text{ kN·mm}$$

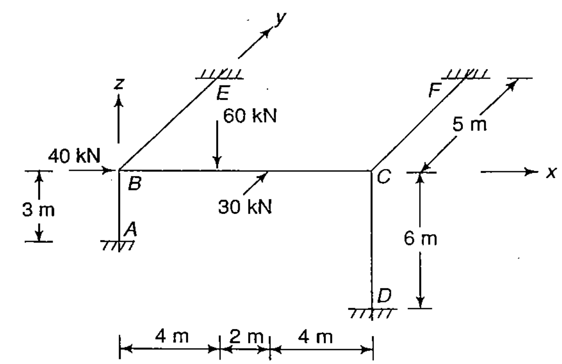


Fig. 8.15

$$P'_4 = -\frac{30 \times 6000 \times 4000^2}{10000^2} = -28800 \text{ kN·mm} \quad P'_5 = 0$$

$$P'_6 = \frac{6 \times 6000 \times 4000^2}{10000^2} = 57600 \text{ kN·mm}$$

$$P'_7 = \frac{30 \times 4000 \times 6000^2}{10000^2} = 43200 \text{ kN·mm}$$

The external loads acting at the coordinates are

$$P_1 = 40 \text{ kN} \quad P_2 = P_3 = P_4 = P_5 = P_6 = P_7 = 0$$

Stiffness matrix with reference to the chosen coordinates has already been developed in Ex. 8.4.

Substituting into Eq. (8.12),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \end{bmatrix} = \begin{bmatrix} 5.967 & 0 & -4500 & -3240 & 0 & -1125 & -3240 \\ 0 & 53.68\times10^6 & 0 & 0 & -1.48\times10^6 & 0 & 0 \\ -4500 & 0 & 33.56\times10^6 & 0 & 0 & 10.8\times10^6 & 0 \\ -3240 & 0 & 0 & 17.72\times10^6 & 0 & 0 & 2.7\times10^6 \\ 0 & -1.48\times10^6 & 0 & 0 & 49.18\times10^6 & 0 & 0 \\ -1125 & 0 & 10.8\times10^6 & 0 & 0 & 29.06\times10^6 & 0 \\ -3240 & 0 & 0 & 2.7\times10^6 & 0 & 0 & 16.96\times10^6 \end{bmatrix}^{-1}$$

$$\times \left\{ \begin{bmatrix} 40 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -86400 \\ -28800 \\ 0 \\ 57600 \\ 43200 \end{bmatrix} \right\}$$

Solving for the displacement components,

$\Delta_1 = 14$ mm $\quad \Delta_2 = 0 \quad \Delta_3 = 0.00525$ radian $\quad \Delta_4 = 0.00362$ radian

$\Delta_5 = 0 \quad \Delta_6 = -0.00348 \quad \Delta_7 = -0.00095$ radian

Knowing all the displacement components at joints *B* and *C*, the twisting and bending moments at the ends of member *BC* can be calculated. The twisting moment in member *BC*

$$T = \frac{GK}{L}(\Delta_2 \sim \Delta_5)$$

$$= \frac{4 \times 3.699 \times 10^9}{10000}(0 - 0) = 0$$

The bending moments at the ends of member BC may be calculated by using the slope-deflection Eq. (2.47). At end B,

$$M_y = M_y^F + \frac{2EI_y}{L}(2\Delta_3 + \Delta_6)$$

$$= -86400 + \frac{2 \times 10 \times 5.4 \times 10^9}{10000}[2(0.00525) - 0.00348]$$

$$= -10584 \text{ kN·mm}$$

$$M_z = M_z^F + \frac{2EI_z}{L}(2\Delta_4 + \Delta_7)$$

$$= -28800 + \frac{2 \times 10 \times 1.35 \times 10^9}{10000}[2(0.00362) - 0.00095]$$

$$= -11817 \text{ kN·mm}$$

Similarly at end C,

$$M_y = 57600 + \frac{2 \times 10 \times 5.4 \times 10^9}{10000}[2(-0.00348) + 0.00525]$$

$$= 39132 \text{ kN·mm}$$

$$M_z = 43200 + \frac{2 \times 10 \times 1.35 \times 10^9}{10000}[2(-0.00095) + 0.00362]$$

$$= 47844 \text{ kN·mm}$$

Example 8.9

If the grid of Ex. 8.5 carries a vertical downward load of 200 kN at joint O, calculate the displacements of joint O. Hence determine the bending and twisting moments at the ends of members OA, OB, OC and OD.

Solution

In Ex. 8.5 it has been shown that the degree of freedom of the frame is three. Coordinates 1, 2 and 3 have been assigned to the three independent displacement components of the frame. The displacement components Δ_1, Δ_2 and Δ_3 at the chosen coordinates may be computed by using Eq. (8.12).

In the present example, there are no loads other than those acting at the coordinates. Hence,

$$P_1' = P_2' = P_3' = 0$$

The external loads acting at the coordinates are

$$P_1 = P_2 = 0 \quad P_3 = -200 \text{ kN}$$

The stiffness matrix has already been developed in Ex. 8.5. Substituting into Eq. (8.12),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 124.04 \times 10^6 & 0 & -8748 \\ 0 & 112 \times 10^6 & 0 \\ -8748 & 0 & 30.8 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \\ -200 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Solving for the displacement components,

$$\Delta_1 = -0.000467 \text{ radian} \quad \Delta_2 = 0$$

$$\Delta_3 = -6.62624 \text{ mm}$$

From Fig. 8.12 it is clear that the torsional rotations of members OA, OB, OC and OD are 0.000467 radian, 0, 0.000467 radian and 0; and the flexural rotations are 0, 0.000467 radian, 0 and 0.000467 radian respectively. Hence the twisting moments in the members are

$$T_{OA} = \frac{5 \times 3.699 \times 10^9}{5000} \times 0.000467 = 1727.4 \text{ kN·mm}$$

$$T_{OB} = 0$$

$$T_{OC} = \frac{5 \times 3.699 \times 10^9}{5000} \times 0.000467 = 1727.4 \text{ kN·mm}$$

$$T_{OD} = 0$$

The bending moments at the ends of members OA, OB, OC and OD may be computed by using the slope-deflection Eq. (2.47).

$$M_{OA} = \frac{2 \times 12 \times 5.4 \times 10^9}{5000}\left[\frac{-3(-6.62624)}{5000}\right]$$

$$= 103100 \text{ kN·mm (sagging)}$$

$$M_{AO} = 103100 \text{ kN·mm (hogging)}$$

$$M_{OB} = \frac{2 \times 12 \times 5.4 \times 10^9}{5000}\left[2(0.000467) - \frac{3(-6.62624)}{5000}\right]$$

$$= 127300 \text{ kN·mm (sagging)}$$

$$M_{BO} = \frac{2 \times 12 \times 5.4 \times 10^9}{5000}\left[0.000467 - \frac{3(-6.62624)}{5000}\right]$$

$$= 115200 \text{ kN·mm (hogging)}$$

$$M_{OC} = \frac{2 \times 12 \times 5.4 \times 10^9}{5000}\left[\frac{-3(6.62624)}{5000}\right]$$

$$= 103100 \text{ kN·mm (sagging)}$$

$$M_{CO} = 103100 \text{ kN·mm (hogging)}$$

$$M_{OD} = \frac{2 \times 12 \times 5.4 \times 10^9}{4000}\left[2(0.000467) - \frac{3(6.62624)}{4000}\right]$$

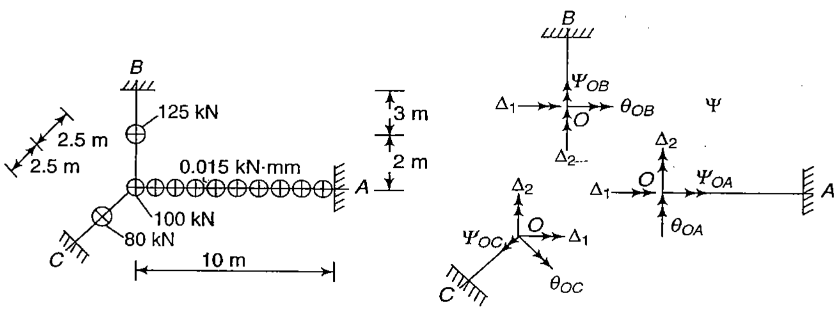
$$= 130800 \text{ kN}\cdot\text{mm (sagging)}$$

$$M_{DO} = \frac{2 \times 12 \times 5.4 \times 10^9}{4000}\left[0.000467 - \frac{3(6.62624)}{4000}\right]$$

$$= 145900 \text{ kN}\cdot\text{mm (hogging)}$$

Example 8.10

Calculate the twisting and bending moments at the ends of members OA, OB and OC for the grid of Ex. 8.6. The external loads acting on the grid are shown in Fig. 8.16(a). Member OA carries a vertical downward load of 0.015 kN/mm. Member OB carries a vertical downward load of 125 kN at 2 m from O. Member OC carries a vertical downward load of 80 kN at its centre. A vertical downward load of 100 kN acts at joint O.



(a) (b)

Fig. 8.16

Solution

In Ex. 8.6 it has been shown that the degree of freedom of the frame is three. Coordinates 1, 2 and 3 have been assigned to three independent displacement components of the frame. The displacement components Δ_1, Δ_2 and Δ_3 at the chosen coordinates may be computed by using Eq. (8.12).

Net forces P_1', P_2' and P_3' at coordinates 1, 2 and 3 due to the external loads other than those actinng at the coordinates, when no displacement is permitted at the coordinates, may be computed by considering all the members as fixed-ended members. The fixed-end moments are

$$M_{OA}^F = M_{AO}^F \frac{0.015 \times 10000^2}{12} = 125000 \text{ kN}\cdot\text{mm (hogging)}$$

$$M_{OB}^F = \frac{125 \times 2000 \times 3000^2}{5000^2} = 90000 \text{ kN}\cdot\text{mm (hogging)}$$

$$M_{BO}^F = \frac{125 \times 3000 \times 2000^2}{5000^2} = 60000 \text{ kN}\cdot\text{mm (hogging)}$$

$$M_{OC}^F = M_{CO}^F = \frac{80 \times 5000}{8} = 50000 \text{ kN}\cdot\text{mm (hogging)}$$

Forces P_1' and P_2' at coordinates 1 and 2 are obtained by resolving these couples along coordinates 1 and 2 respectively.

$$P_1' = 90000 - 50000 \sin 45° = 54645 \text{ kN}\cdot\text{mm}$$

$$P_2' = -125000 + 50000 \cos 45° = -89645 \text{ kN}\cdot\text{mm}$$

Force P_3' at coordinate 3 is obtained by adding algebraically the vertical reactions at end O of member OA, OB and OC treated as fixed-ended members.

$$\text{Vertical reaction at } O \text{ in member } OA = \frac{0.015 \times 10000}{2} = 75 \text{ kN}$$

$$\text{Vertical reaction at } O \text{ in member } OB = \frac{125 \times 3000}{5000} + \frac{90000 - 60000}{5000} = 81 \text{ kN}$$

$$\text{Vertical reaction at } O \text{ in member } OC = \frac{80}{2} = 40 \text{ kN}$$

Hence, $\quad P_3' = 75 + 81 + 40 = 196$ kN

The external loads acting at the coordinates are

$$P_1 = P_2 = 0 \qquad P_3 = -100 \text{ kN}$$

The stiffness matrix has already been developed in Ex. 8.6. Substituting in to Eq. (8.12),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 81.46 \times 10^6 & -24.07 \times 10^6 & 4555.1 \\ -24.07 \times 10^6 & 57.39 \times 10^6 & 7108.9 \\ 4555.1 & 7108.9 & 13.22 \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} 0 \\ 0 \\ -100 \end{bmatrix} - \begin{bmatrix} 54645 \\ -89645 \\ 196 \end{bmatrix} \right\}$$

Solving for the displacement components,

$$\Delta_1 = 0.002354 \text{ radian} \qquad \Delta_2 = 0.005793 \text{ radian} \qquad \Delta_3 = 25.604 \text{ mm}$$

Rotations Δ_1 and Δ_2 occurring at end O of members OA, OB and OC are shown in Fig. 8.16(b). The torsional and flexural rotations of members OA, OB and OC at joint O can be determined by resolving rotations Δ_1 and Δ_2 in the directions parallel and normal to the axes of the members. The torsional rotations of the members are

$$\psi_{OA} = \Delta_1 = 0.002354 \text{ radian}$$

$$\psi_{OB} = \Delta_2 = 0.005793 \text{ radian}$$

$$\psi_{OC} = -(\Delta_1 \sin 45° + \Delta_2 \cos 45°) = -0.005761 \text{ radian}$$

Hence, the twisting moments in members OA, OB and OC, which have the same sense as the torsional rotations, are

$$T_{OA} = \frac{5 \times 3.699 \times 10^9}{10000}(0.002354) = 8700 \text{ kN}\cdot\text{mm}$$

$$T_{OB} = \frac{5 \times 3.699 \times 10^9}{10000}(0.005793) = 21400 \text{ kN}\cdot\text{mm}$$

$$T_{OC} = \frac{5 \times 3.699 \times 10^9}{5000}(-0.005761) = -21300 \text{ kN}\cdot\text{mm}$$

The flexural rotations of members OA, OB and OC as shown Fig. 8.16(b) are

$$\theta_{OA} = \Delta_2 = 0.005793 \text{ radian}$$

$$\theta_{OB} = -\Delta_1 = -0.002354 \text{ radian}$$

$$\theta_{OC} = \Delta_1 \cos 45° - \Delta_2 \sin 45° = -0.002432 \text{ radian}$$

The bending moments at the ends of members OA, OB and OC can be calculated by usign the slope-deflection Eq. (2.47).

$$M_{OA} = 124700 \text{ kN}\cdot\text{mm (sagging)}$$

$$M_{AO} = 299600 \text{ kN}\cdot\text{mm (hogging)}$$

$$M_{OB} = 186200 \text{ kN}\cdot\text{mm (sagging)}$$

$$M_{BO} = 397200 \text{ kN}\cdot\text{mm (hogging)}$$

$$M_{OC} = 222100 \text{ kN}\cdot\text{mm (sagging)}$$

$$M_{CO} = 385100 \text{ kN}\cdot\text{mm (hogging)}$$

Example 8.11

Calculate the twisting and bending moments at the ends of the members of the grid of Ex. 8.7. The external loads acting on the grid are shown in Fig. 8.17(a). Members BA, BC, ED and EF carry vertical downward loads of 0.02 kN/mm. A vertical downward load of100 kN acts at joint B.

Solution

In Ex. 8.7 it has been shown that the degree of freedom of the frame is six. Coordinates 1, 2 and 3 have been assigned to the three independent displacement components at joint B and coordinates 4, 5 and 6 have been assigned to the three independent displacement components at joint E. The displacement components Δ_1 to Δ_6 at the chosen coordinates may be computed by using Eq. (8.12).

Net forces P_1' to P_6' at coordinates 1 to 6 due to external loads other than those acting at the coordinates, when no displacement is permitted at the coordinates, may be computed by considering all the members as fixed-ended members. The fixed-end moments are

$$M_{AB}^F = M_{BA}^F = M_{BC}^F = M_{DE}^F = M_{ED}^F = M_{EF}^F = M_{FE}^F$$

$$= \frac{0.02 \times 6000^2}{12} = 60000 \text{ kN}\cdot\text{mm}$$

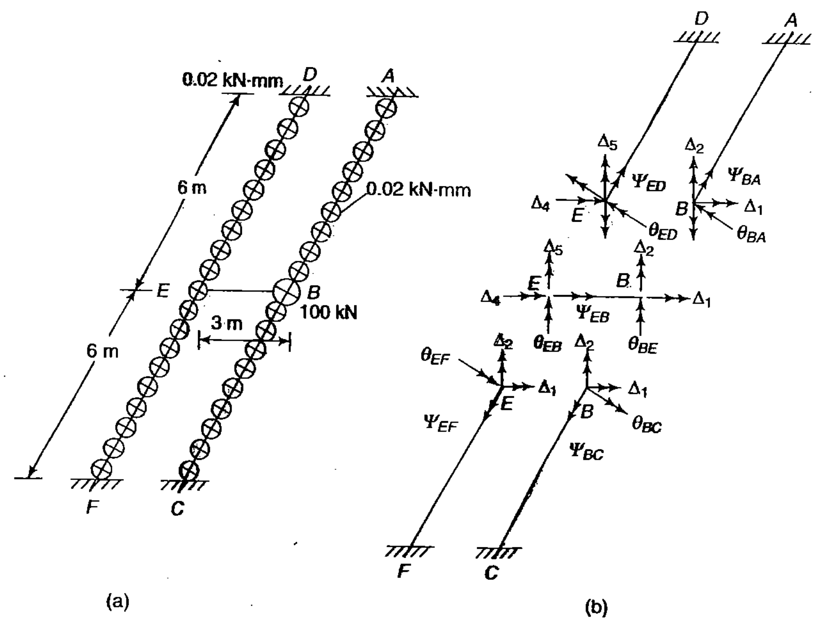


Fig. 8.17

Forces P_1' and P_2' at coordinates 1 and 2 are obtained by resolving M_{BA}^F and M_{BC}^F along coordinates 1 and 2 respectively. Similarly, forces P_4' and P_5' at coordinate 4 and 5 are obtained by resolving M_{ED}^F and M_{EF}^F along coordinates 4 and 5 respectively.

$$P_1' = P_2' = P_4' = P_5' = 0$$

Forces P_3' at coordinate 3 is obtained by adding algebraically the vertical reactions at ends B of members BA, BC and BE. Similarly, force P_6' at coordinate 6 is obtained by adding algebraically the vertical reactions at ends E of members ED, EF and EB.

$$P_3' = \frac{0.02 \times 6000}{2} + \frac{0.02 \times 6000}{2} = 120 \text{ kN}$$

$$P_6' = \frac{0.02 \times 6000}{2} + \frac{0.02 \times 6000}{2} = 120 \text{ kN}$$

The external loads acting at the coordinates are

$$P_1 = P_2 = P_4 = P_5 = P_6 = 0 \qquad P_3 = -100 \text{ kN}$$

The stiffness matrix has already been developed in Ex. 8.7. Substituting into Eq. (8.12),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{bmatrix} = \begin{bmatrix} 346.57\times 10^6 & -175.871\times 10^6 & 0 & -10.688\times 10^6 & 0 & 0 \\ -175.871\times 10^6 & 424.5\times 10^6 & 145840 & 0 & 145.84\times 10^6 & -145840 \\ 0 & 145840 & 133.68 & 0 & 145840 & -97.23 \\ -10.668\times 10^6 & 0 & 0 & 346.57\times 10^6 & -175.871\times 10^6 & 0 \\ 0 & 145.84\times 10^6 & 145840 & -175.871\times 10^6 & 424.5\times 10^6 & -145840 \\ 0 & -145.840 & -97.23 & 0 & -145840 & 133.68 \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 120 \\ 0 \\ 0 \\ 120 \end{bmatrix} \right\}$$

Solving for the independent displacement components,

$$\Delta_1 = \Delta_4 = 301.2 \times 10^{-6} \text{ radian} \quad \Delta_3 = -5.6066 \text{ mm}$$

$$\Delta_2 = \Delta_5 = 575.2 \times 10^{-6} \text{ radian} \quad \Delta_5 = -3.7204 \text{ mm}$$

Rotations Δ_1 and Δ_2 occurring at ends *B* of members *BA*, *BC*, and *BE* and rotations Δ_4 and Δ_5 occurring at ends *E* of members *ED*, *EF* and *EB* are shown in Fig. 8.17(b). The torsional and flexural rotaitons of members *BA*, *BC* and *BE* at joint *B* can be determined by resolving rotations Δ_1 and Δ_2 in the directions parallel and normal to the axes of the members. Similarly, the torsional and flexural rotations of members *ED*, *EF* and *EB* at joint *E* can be determined by resolving rotations Δ_4 and Δ_5 in the directions parallel and normal to the axes of the members.

The torsional rotations of the members as shown in Fig. 8.17(b) are

$$\psi_{BA} = \Delta_1 \cos 60° + \Delta_2 \cos 30° = 648.7 \times 10^{-6} \text{ radian}$$

$$\psi_{BC} = -(\Delta_1 \cos 60° + \Delta_2 \cos 30°) = -648.7 \times 10^{-6} \text{ radian}$$

$$\psi_{BE} = \psi_{EB} = \Delta_1 \sim \Delta_4 = 0$$

$$\psi_{ED} = \Delta_4 \cos 60° + \Delta_5 \cos 30° = 648.7 \times 10^{-6} \text{ radian}$$

$$\psi_{EF} = -(\Delta_4 \cos 60° + \Delta_5 \cos 30°) = -648.7 \times 10^{-6} \text{ radian}$$

Hence, the twisting moments (having the same sense as torsional rotations) in the members are

$$T_{BA} = \frac{5 \times 18.77 \times 10^9}{6000} (648.7 \times 10^{-6}) = 10100 \text{ kN·mm}$$

$$T_{BC} = \frac{5 \times 18.77 \times 10^9}{6000} \left(-648.7 \times 10^{-6}\right) = -10100 \text{ kN·mm}$$

$$T_{BE} = 0$$

$$T_{ED} = \frac{5 \times 18.77 \times 10^9}{6000} (648.7 \times 10^{-6}) = 10100 \text{ kN·mm}$$

$$T_{EF} = \frac{5 \times 18.77 \times 10^9}{6000} (-648.7 \times 10^{-6}) = -10100 \text{ kN·mm}$$

The flexural rotations of the members as shown in Fig. 8.17(b) are

$$\theta_{BA} = \Delta_2 \sin 30° - \Delta_1 \sin 60° = 26.8 \times 10^{-6} \text{ radian}$$

$$\theta_{BC} = -(\Delta_2 \sin 30° - \Delta_1 \sin 60°) = -26.8 \times 10^{-6} \text{ radian}$$

$$\theta_{BE} = \Delta_2 = 575.2 \times 10^{-6} \text{ radian}$$

$$\theta_{ED} = \Delta_5 \sin 30° - \Delta_4 \sin 60° = 26.8 \times 10^{-6} \text{ radian}$$

$$\theta_{EF} = -(\Delta_5 \sin 30° - \Delta_4 \sin 60°) = -26.8 \times 10^{-6} \text{ radian}$$

$$\theta_{EB} = \Delta_5 = 575.2 \times 10^{-6} \text{ radian}$$

The bending moments at the ends of the members can be calculated by using the slope-deflection Eq. (2.47).

M_{BA} = 252400 kN·mm (sagging)
M_{AB} = 369500 kN·mm (hogging)
M_{BC} = 240700 kN·mm (sagging)
M_{CB} = 363600 kN·mm (hogging)
M_{ED} = 149300 kN·mm (sagging)
M_{DE} = 266400 kN·mm (hogging)
M_{EF} = 137600 kN·mm (sagging)
M_{FE} = 260500 kN·mm (hogging)
M_{BE} = 23400 kN·mm (sagging)
M_{EB} = 23400 kN·mm (hogging)

8.6 COMPARISON OF METHODS

In the preceding sections the force and displacement methods for the analysis of rigid-jointed space frames have been discussed. Examples to illustrate the two main methods have been given. It may be noted that in the case of rigid-jointed space frames, the development of the stiffness matrix is simpler as compared to the development of the flexibility matrix. Also, the degree of kinematic indeterminacy of these frames is generally smaller than the degree of static indeterminacy. For example, the degrees of kinematic and static indeterminacies of the frame of Ex. 8.4 are 7 and 18 respectively. Hence, in general it may be stated that for the analysis of rigid-jointed space frames, the displacement method is preferable as compared to the force method.

PROBLEMS

8.1 Analyse the space frame shown in Fig. 8.18 by the displacement method. The frame carries a vertical downward load of 200 kN at the centre of member *BC*. Hence determine the bending moments at *A* and *C* and the twisting moment at *D*. *EI* is the same for all the members and $GK = 0.5\ EI$.

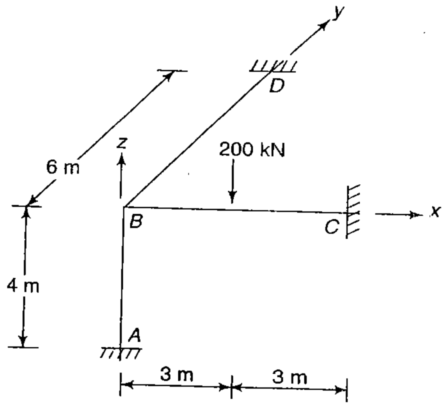


Fig. 8.18

8.2 Analyse the frame of Fig. 8.18 if the frame has hinge supports at *A* and *C* and member *BD* is fixed at *D*. Hence determine the twisting moment at *D*.

8.3 Analyse the frame of Fig. 8.18 if in addition to the load of 200 kN shown in the figure, a horizontal load of 120 kN acts at the centre of member *BD* towards the positive direction of the *x*-axis. Calculate the bending and twisting moments at the fixed supports *A*, *C* and *D*.

8.4 The members of the rigid-jointed frame shown in Fig. 8.19 lie in the horizontal plane. End *A* is fixed and a spherical seating is provided at end *C*. A vertical downward load of 23 kN acts at the centre of member *BC*. Analyse the frame by the force method. Hence determine the vertical reaction at support *C*. $EI/GK = 1.25$ and $L_1 = L_2 = 4$ m.

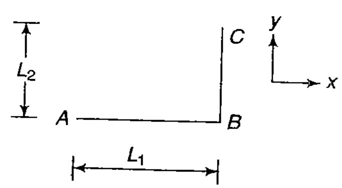


Fig. 8.19

8.5 Calculate the vertical reaction at support *C* for the frame of Fig. 8.19 if end *A* is fixed and a spherical seating is provided at *C*. It carries a downward vertical load of 10 kN/m uniformly spread over the entire frame. $EI/GK = 1.25$ and $L_1 = L_2 = 5$ m.

8.6 Ends *A* and *C* of the frame shown in Fig. 8.19 are fixed and a vertical downward load of 50 kN acts at the centre of *AB*. Determine the displacements at joint *B* if $L_1 = L_2 = 4$ m and $EI/GK = 1.25$.

8.7 Ends *A* and *C* of the frame shown in Fig. 8.19 are fixed. $L_1 = 10$ m and $L_2 = 8$ m. Calculate the support reactions at *C*, if member *AB* carries a vertical downward load of 40 kN at a distance of 8 m from *A*. $EI/GK = 1$.

8.8 Analyse the rigid-jointed frame shown in Fig. 8.20. The frame lies in the horizontal plane. Hence determine the bending moment at *E* if a vertical downward load of 20 kN acts at *E*. $EI/GK = 4$.

8.9 Determine the downward deflection at *E* if the frame of Fig. 8.20 carries a downward vertical load of 10 kN/m uniformly spread over the entire frame. $EI/GK = 4$.

8.10 Analyse the grid shown in Fig. 8.21 by the displacement method. A vertical downward load of 90 kN acts at *D*. Calculate the twisting moments in *DA* and *DB*. $EI/GK = 4/3$.

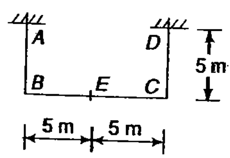


Fig. 8.20

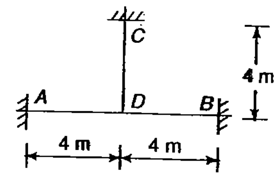


Fig. 8.21

8.11 The frame shown in Fig. 8.22 lies in the horizontal plane and carries a vertical downward load of 20 kN at joint *B*. Determine the twisting moment in member *AB*. Take $EI/GK = 1.25$.

8.12 Using the displacement method, analyse the frame *ABCD* shown in Fig. 8.23. The frame lies in the horizontal plane. Member *BC* carries a vertical downward load of 50 kN at its centre. Hence determine the twisting moment at *A*. Verify the result by the force method. Take $EI/GK = 2.0$.

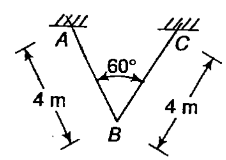


Fig. 8.22

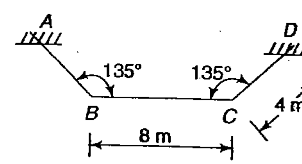


Fig. 8.23

8.13 The rigid-jointed frame *ABCD* of Fig. 8.24 lies in the horizontal plane. Analyse the frame using the force method. The frame carries a vertical downward load of 20 kN at joint *B*. Determine the support reactions at *A*. Take $EI/GK = 1.25$. Verify the result by the displacement method.

8.14 The grid frame shown in Fig. 8.25 lies in the horizontal plane. Members *AO* and *OC* carry a downward vertical load of 10 kN-m uniformly distributed over the whole length. Analyse the frame by the displacement method. Hence determine the support reactions at *A*. Take $EI/GK = 2.0$

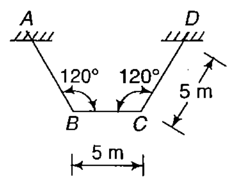


Fig. 8.24

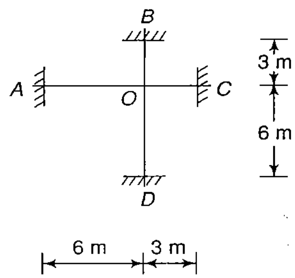


Fig. 8.25

8.15 Determine the elements of the first two rows of the stiffness matrix with reference to coordinates 1, 2, ... 12 for the grid shown in Fig. 8.26. Take EI = 12500 kN·m^2 and GK = 6250 kN·m^2.

8.16 The girder system shown in Fig. 8.27 has to support the load of a typical grid floor of a building. Longitudinal girders AB and CD carry a uniformly distributed load of 20.64 kN/m inclusive of self-weight. The uniformly distributed load on cross girders EF and GH, inclusive of self-weight, is 18.48 kN/m. Take EL = 364900 kN·m^2 and GK = 113900 kN·m^2 for all the girders. Analyse the grid by the displacement method. Use the double symmetry of the structure to reduce the number of coordinates to 3 as shown in the figure. Develop the stiffness matrix with reference to the coordinates 1, 2, and 3. Hence determine the bending and twisting moments at A.

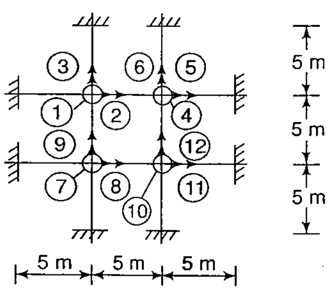


Fig. 8.26

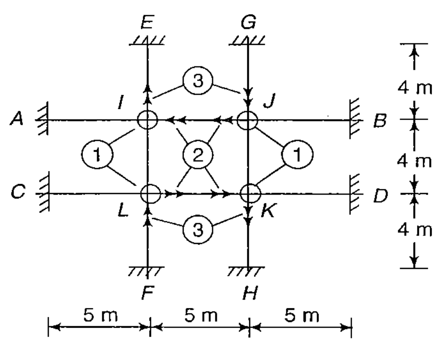


Fig. 8.27

8.17 If the grid of Prob. 8.16 is simply supported in bending and fixed in torsion at all the supports, develop the stiffness matrix with reference to the coordinates 1, 2 and 3. Hence analyse the grid by the displacement method and determine the twisting moments at A and E.

8.18 Analyse the grid frame of Fig. 8.17 if it carries only a load of 100 kN at joint B. Hence determine the bending and twisting moments at D.

9 PIN-JOINTED SPACE FRAMES

9.1 INTRODUCTION

A large number of pin-jointed frames commonly encountered in practice, such as radio and transmission towers, are three-dimensional space frames. The members of the pin-jointed space frames carry only axial forces, provided the loads are applied at the joints and the members are straight. Hence, the nature of stress in the members of a pin-jointed frame is the same whether it is a plane frame or a space frame. Just as in the case of pin-jointed plane frames, the two main methods, viz., the force method and the displacement method, may be used for the analysis of pin-jointed space frames.

9.2 TENSION-COEFFICIENT METHOD

The tension-coefficient method provides a neat approach for the determination of member forces in a pin-jointed space frame. The method is based on the fundamental equations of static equilibrium and may be readily applied for the solution of statically determinate pin-jointed space frames. In this method the three equations of static equilibrium, one along each of the three cartesian coordinate axes, are written at each joint. This procedure provides the sufficient number of equations for the determination of the member forces and the external reactive forces, provided that the frame is statically determinate. The conditions of static determinacy for pin-jointed space frames have been discussed in Sec. 1.6.

Consider a member AB, connecting joints A and B of a pin-jointed space frame. The length of member AB is given by the equation

$$L_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \tag{9.1}$$

where (x_A, y_A, z_A) and (x_B, y_B, z_B) are the coordinates of joints A and B respectively with reference to a chosen system of cartesian coordinates.

The axial force in member AB may be expressed as

$$S_{AB} = t_{AB}L_{AB} \tag{9.2}$$

where t_{AB} is the tension-coefficient of member AB.

To start the analysis, it is presumed that all members of the frame are in tension. If a tension coefficient of a member is found to be negative it means that the member is in compression. Consider the equilibrium of joint A. Member AB exerts a force S_{AB} at joint A directed towards joint B. The component of this force along the x-axis is given by the equation

$$X_{AB} = S_{AB} \frac{(x_B - x_A)}{L_{AB}}$$

where $\frac{(x_B - x_A)}{L_{AB}}$ is the direction cosine of line AB with the x-axis.

Using Eq. (9.2), the preceding equation may be rewritten as

$$X_{AB} = t_{AB}(x_B - x_A)$$

Similar equations can be derived for the components along the y- and z-axes giving the following equations:

$$\begin{aligned} X_{AB} &= t_{AB}(x_B - x_A) \\ Y_{AB} &= t_{AB}(y_B - y_A) \\ Z_{AB} &= t_{AB}(z_B - z_A) \end{aligned} \tag{9.3}$$

For the equilibrium of joint A along the three coordinate axes, the following equations must be satisfied:

$$\begin{aligned} X_A + \Sigma\, t_{AB}(x_B - x_A) &= 0 \\ Y_A + \Sigma\, t_{AB}(y_B - y_A) &= 0 \\ Z_A + \Sigma\, t_{Ab}(z_B - z_A) &= 0 \end{aligned} \tag{9.4}$$

where X_A, Y_A and Z_A are the components of the external force acting at joint A along x-, y- and z-axes respectively. The summation should be carried out so as to include all the members meeting at joint A.

Similar equations can be written for all the joints of the frame. If j is the number of joints, the total number of equations thus obtained are $3j$. For an internally determinate and stable pin-jointed space frame, the total number of unknown member forces is $(3j - 6)$. The number of unknown reaction components in an externally determinate and stable pin-jointed space frame is six. Hence, the number of unknowns in a statically determinate and stable pin-jointed space frame is $3j$. Thus Eq. (9.4) is sufficient for the determination of all the tension coefficients and reaction components. After knowing the tension coefficients, the member forces can be determined by using Eq. (9.2).

The tension-coefficient method may be summarised by the following steps:

(i) Choose a cartesian system of coordinates and determine the coordinates of all the joints of the pin-jointed space frame.

(ii) Determine the components of the external forces acting at the joints along the coordinate axes.

(iii) Write down the equations of equilibrium (Eq. 9.4), for all the joints. For a pin-jointed space frame with j joints, the total number of equations of equilibrium are $3j$, as there are three equations of equilibrium at each joint. These equations may be expressed in the following matrix form:

$$[a]\,[x] = [c] \tag{9.5}$$

where $[a]$ = coefficient matrix of order $3j \times 3j$
$[x]$ = column matrix of order $3j \times 1$ whose elements are the unknown tension coefficients and the reaction components
$[c]$ = column matrix of order $3j \times 1$ whose elements are constants.

The unknown tension coefficients and reaction components can be determined by premultiplying both sides of Eq. (9.5) by $[a]^{-1}$

$$[x] = [a]^{-1}[c] \tag{9.6}$$

(iv) Calculate the lengths of all the members using Eq. (9.1).

(v) Determine the member forces using Eq. (9.2).

It may be noted that although the tension coefficient method is eminently suitable for the analysis of pin-jointed space frames, it can also be used for the analysis of pin-jointed plane frames. If the plane frame is assumed to lie in the x-y plane, the z coordinate of all the joints is zero. The tension coefficient method is essentially the same as the method of joints commonly used for the analysis of pin-jointed plane frames. 62·48

Example 9.1

Using the tension-coefficient method, calculate the forces in the members of the pin-jointed space frame shown in Fig. 9.1. The numbers in parentheses are the cartesian coordinates of the joints of the frame.

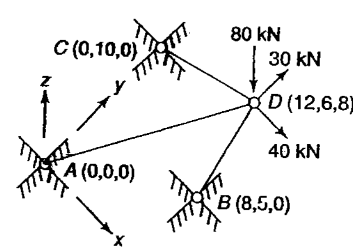


Fig. 9.1

Solution

The lengths of members DA, DB, and DC may be computed by using Eq. (9.1).

$$L_{DA} = \sqrt{(0-12)^2 + (0-6)^2 + (0-8)^2} = 15.620 \text{ m}$$

$$L_{DB} = \sqrt{(8-12)^2 + (5-6)^2 + (0-8)^2} = 9.000 \text{ m} \tag{a}$$

$$L_{DC} = \sqrt{(0-12)^2 + (10-6)^2 + (0-8)^2} = 14.967 \text{ m}$$

Components of the external forces acting at joint D are

$$X_D = 40 \text{ kN} \qquad Y_D = 30 \text{ kN} \qquad Z_D = -80 \text{ kN}$$

Considering the equilibrium of joint D and using Eq. (9.4),

$$\begin{aligned} t_{DA}(0-12) + t_{DB}(8-12) + t_{DC}(0-12) + 40 &= 0 \\ t_{DA}(0-6) + t_{DB}(5-6) + t_{DC}(10-6) + 30 &= 0 \\ t_{DA}(0-8) + t_{DB}(0-8) + t_{DC}(0-8) - 80 &= 0 \end{aligned} \qquad \text{(b)}$$

Solving Eq. (b) for the tension coefficients,

$$t_{DA} = 9 \text{ kN/m} \qquad t_{DB} = -20 \text{ kN/m} \qquad t_{DC} = 1 \text{ kN/m} \qquad \text{(c)}$$

Substituting from Eqs (a) and (c) into Eq. (9.2), the member forces obtained are

$$S_{DA} = 140.58 \text{ kN} \qquad S_{DB} = -180 \text{ kN} \qquad S_{DC} = 14.97 \text{ kN}$$

The minus sign shows that the force in member DB is compressive.

The external reaction components at the supports may be computed by considering the equilibrium of joints A, B and C using Eq. (9.4). For the equilibrium of joint A,

$$\begin{aligned} 9(12-0) + X_A &= 0 \\ 9(6-0) + Y_A &= 0 \\ 9(8-0) + Z_A &= 0 \end{aligned} \qquad \text{(d)}$$

Solving Eq. (d) for the external reaction components at joint A,

$$X_A = -108 \text{ kN} \qquad Y_A = -54 \text{ kN} \qquad Z_A = -72 \text{ kN}$$

The external reaction components at joints B and C can be calculated in a similar manner. Considering the free body of the entire structure, it may be verified that the equations of static equilibrium, Eqs (1.1) and (1.2), are satisfied.

Example 9.2

Calculate the member forces in the pin-jointed space frame shown in Fig. 9.2. The frame is supported by hinges at A, B and C. The numbers in parentheses are the cartesian coordinates of joints of the frame.

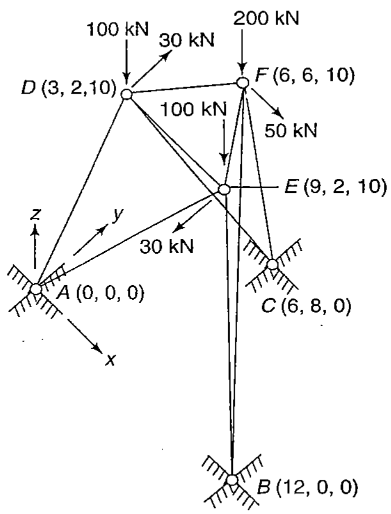


Fig. 9.2

Solution

The frame has nine members and therefore there are 9 tension coefficients. At each support there are three reaction components. Thus there are nine reaction components. Hence, the total number of unknowns is 18. These unknowns can be determined by considering the equilibrium of the six joints.

Considering the equilibrium of joint D and using Eq. (9.4),

$$\begin{aligned} t_{DA}(0-3) + t_{DC}(6-3) + t_{DE}(9-3) + t_{DF}(6-3) + 0 &= 0 \\ t_{DA}(0-2) + t_{DC}(8-2) + t_{DE}(2-2) + t_{DF}(6-2) + 30 &= 0 \\ t_{DA}(0-10) + t_{DC}(0-10) + t_{DE}(10-10) + t_{DF}(10-10) - 100 &= 0 \end{aligned} \qquad \text{(a)}$$

Equation (a) can be simplified and rewritten as

$$\begin{aligned} -3t_{DA} + 3t_{DC} + 6t_{DE} + 3t_{DF} &= 0 \\ -2t_{DA} + 6t_{DC} + 4t_{DF} &= -30 \\ -10t_{DA} - 10t_{DC} &= 100 \end{aligned} \qquad \text{(b)}$$

Similar equations can be written by considering the equilibrium of joints E and F. Thus the equilibrium of joints D, E and F lead to the following set of nine equations:

$$\begin{aligned} -3t_{DA} + 3t_{DC} + 6t_{DE} + 3t_{DF} &= 0 \\ -2t_{DA} + 6t_{DC} + 4t_{DF} &= -30 \\ -10t_{DA} - 10t_{DC} &= 100 \\ -9t_{EA} + 3t_{EB} - 6t_{ED} - 3t_{EF} &= 0 \\ -2t_{EA} - 2t_{EB} + 4t_{EF} &= 30 \\ -10t_{EA} - 10t_{EB} &= 100 \\ -3t_{FD} + 6t_{FB} + 3t_{FE} &= -50 \\ -4t_{FD} - 6t_{FB} - 2t_{FC} - 4t_{FE} &= 0 \\ -10t_{FB} - 10t_{FC} &= 200 \end{aligned} \qquad \text{(c)}$$

Similarly, considering the equilibrium of the joints A, B and C, the following set of nine equations is obtained:

$$\begin{aligned} -3t_{AD} + 9t_{AE} &= -X_A \\ 2t_{AD} + 2t_{AE} &= -Y_A \\ 10t_{AD} + 10t_{AE} &= -Z_A \\ -3t_{BE} - 6t_{BF} &= -X_B \\ 2t_{BE} + 6t_{BF} &= -Y_B \\ 10t_{BE} + 10t_{BF} &= -Z_B \\ -3t_{CD} &= -X_C \\ -6t_{CD} - 2t_{CF} &= -Y_C \\ 10t_{CD} + 10t_{CF} &= -Z_C \end{aligned} \qquad \text{(d)}$$

All the unknown tension-coefficients and the reaction components can be obtained by expressing Eqs (c) and (d) in the matrix form, Eq. (9.5), and carrying out the matrix inversion indicated by Eq. (9.6). This procedure requires the inversion of a matrix of order 18×18. In the present case it may be noted that Eqs (c) involve only the tension-coefficients. The unknown reaction components do not appear in these equations. Hence, Eqs (c) are sufficient for the determination of all the tension coefficients. The solution of the set of simultaneous Eq. (c) is equivalent to the inversion of a matrix of order 9×9. Consequently, this approach would appear to be preferable. Solving Eqs (c) for the tension coefficients,

$t_{BF} = -7.92$ kN/m $\quad t_{CF} = -12.08$ kN/m

$t_{DF} = 3.33$ kN/m $\quad t_{EF} = 2.50$ kN/m

$t_{AE} = -3.75$ kN/m $\quad t_{BE} = -6.25$ kN/m

$t_{DE} = 1.25$ kN/m $\quad t_{AD} = -2.08$ kN/m

$t_{CD} = -7.92$ kN/m

The lengths of the members of the frame may be computed by using Eq. (9.1).

$L_{BF} = 13.115$ m $\quad L_{CF} = 10.198$ m

$L_{DF} = 5.000$ m $\quad L_{EF} = 5.000$ m

$L_{AE} = 13.601$ m $\quad L_{BE} = 10.630$ m

$L_{DE} = 6.000$ m $\quad L_{AD} = -10.630$ m

$L_{CD} = 12.042$ m

Knowing the tension-coefficients and the lengths of the members of the frame, the axial forces in the members may be computed by using Eq. (9.2).

$S_{BF} = -103.87$ kN $\quad S_{CF} = -123.19$ kN

$S_{DF} = 16.67$ kN $\quad S_{EF} = 12.50$ kN

$S_{AE} = -51.00$ kN $\quad S_{BE} = -66.44$ kN

$S_{DE} = 7.50$ kN $\quad S_{AD} = -22.11$ kN

$S_{CD} = -95.37$ kN

The reaction components at the supports may be evaluated by substituting the values of tension coefficients in Eq. (d). The reaction components are found to be

$X_A = 40.00$ kN $\quad Y_A = 11.67$ kN $\quad Z_A = 58.33$ kN

$X_B = -66.25$ kN $\quad Y_B = 60.00$ kN $\quad Z_B = 141.67$ kN

$X_C = -23.75$ kN $\quad Y_C = -71.67$ kN $\quad Z_C = 200.00$ kN

It may be checked that all the six equations of static equilibrium, Eqs (1.1) and (1.2), are identically satisfied.

$$40.00 - 66.25 - 23.75 + 50.00 = 0$$

$$11.67 + 60.00 - 71.67 - 30.00 + 30.00 = 0$$

$$58.33 + 141.67 + 200.00 - 100.00 - 100.00 - 200.00 = 0$$

$$200.00 \times 8 - 100.00 \times 2 - 100.00 \times 2 - 200.00 \times 6 - 30.00 \times 10 - 30.00 \times 10 = 0$$

$$200.00 \times 6 + 141.67 \times 12 - 100.00 \times 3 - 100.00 \times 9 - 200.00 \times 6 - 50.00 \times 10 = 0$$

$$60.00 \times 12 + 23.75 \times 8 - 71.67 \times 6 + 30.00 \times 3 - 30.00 \times 9 - 50.00 \times 6 = 0$$

Example 9.3

Calculate the member forces in the pin-jointed space frame shown in Fig. 9.3. The frame is supported by a hinge at A and by a roller at B which permits movement freely along the x-axis only. The support at C permits movement freely along x- and y-axes. The numbers in parentheses are the cartesian coordinates of the joints of the frame.

Solution

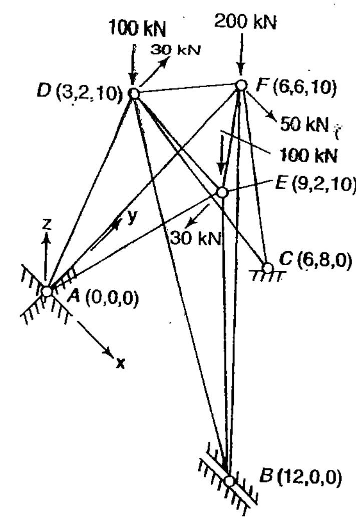


Fig. 9.3

The frame has 12 members and therefore there are twelve tension coefficients. There are three reaction components at A, two at B and one at C. Thus there are six reaction components. Hence, the total number of unknowns is 18. These unknowns can be determined by considering the equilibrium of the six joints. This procedure leads to the inversion of a matrix of order 18×18 as in Ex. 9.2.

An alternative approach in the present case is to compute the reaction components at the supports first. As the structure is statically determinate externally, all the external reaction components can be determined by considering the free-body diagram of the entire frame. Using equations of static equilibrium, Eqs (1.1) and (1.2), the following set of six equations can be written as:

$$\begin{aligned}
\Sigma F_x &= X_A + 50 = 0 \\
\Sigma F_y &= Y_A + Y_B + 30 - 30 = 0 \\
\Sigma F_z &= Z_A + Z_B + Z_C - 100 - 100 - 200 = 0 \\
\Sigma M_x &= 100 \times 2 + 100 \times 2 + 200 \times 6 - Z_C \times 8 = 0 \\
\Sigma M_y &= Z_B \times 12 + Z_C \times 6 - 100 \times 3 - 100 \times 9 - 200 \times 6 \\
&\quad - 50 \times 10 = 0 \\
\Sigma M_z &= Y_B \times 12 + 30 \times 3 - 30 \times 9 - 50 \times 6 = 0
\end{aligned} \qquad \text{(a)}$$

Solving Eq. (a) for the external reaction components,

$$\begin{matrix} X_A = -50.00 \text{ kN} & Y_A = -40.00 \text{ kN} & Z_A = 58.33 \text{ kN} \\ Y_B = 40.00 \text{ kN} & Z_B = 141.67 \text{ kN} & Z_C = 200.00 \text{ kN} \end{matrix} \qquad \text{(b)}$$

Considering the equilibrium of joints A, B and C and using Eq. (9.4), the following set of 9 equations can be written as

$$\begin{aligned}
3t_{AD} + 9t_{AE} + 6t_{AF} &= -X_A \\
2t_{AD} + 2t_{AE} + 6t_{AF} &= -Y_A \\
10t_{AD} + 10t_{AE} + 10t_{AF} &= -Z_A \\
9t_{BD} + 3t_{BE} + 6t_{BF} &= 0 \\
2t_{BD} + 2t_{BE} + 6t_{BF} &= -Y_B \\
10t_{BD} + 10t_{BE} + 10t_{BF} &= -Z_B \\
3t_{CD} - 3t_{CE} &= 0 \\
6t_{CD} + 6t_{CE} + 2t_{CF} &= 0 \\
10t_{CD} + 10t_{CE} + 10t_{CF} &= -Z_C
\end{aligned} \qquad \text{(c)}$$

Substituting from Eq. (b) into Eq. (c) and solving Eq. (c), the values of tension coefficients can be calculated. These are found to be

$$\begin{aligned} t_{AD} &= -23.54 \text{ kN/m} & t_{AE} &= 4.79 \text{ kN/m} \\ t_{AF} &= 12.92 \text{ kN/m} & t_{BD} &= 8.54 \text{ kN/m} \\ t_{BE} &= -19.79 \text{ kN/m} & t_{BF} &= -2.92 \text{ kN/m} \\ t_{CD} &= 5.00 \text{ kN/m} & t_{CE} &= 5.00 \text{ kN/m} \\ t_{CF} &= -30.00 \text{ kN/m} \end{aligned} \tag{d}$$

Next, considering the equilibrium of joints D, E and F and using Eq. (9.4), the following set of 9 equations may be written as

$$\begin{aligned} -3t_{DA} + 9t_{DB} + 3t_{DC} + 6t_{DE} + 3t_{DF} &= 0 \\ -2t_{DA} - 2t_{DB} + 6t_{DC} \qquad + 4t_{DF} &= -30 \\ -10t_{DA} - 10t_{DB} - 10t_{DC} &= 100 \\ -9t_{EA} + 3t_{EB} - 3t_{EC} - 6t_{ED} \quad -3t_{EF} &= 0 \\ -2t_{EA} - 2t_{EB} + 6t_{EC} \qquad + 4t_{EF} &= 30 \\ -10t_{EA} - 10t_{EB} - 10t_{EC} &= 100 \\ -6t_{FA} + 6t_{FB} - 3t_{FD} \qquad + 3t_{FE} &= -50 \\ -6t_{FA} - 6t_{FB} - 4t_{FD} + 2t_{FC} - 4t_{FE} &= 0 \\ -10t_{FA} - 10t_{FB} - 10t_{FC} &= 200 \end{aligned} \tag{e}$$

The remaining three tension-coefficients t_{DE}, t_{DF} and t_{EF} can be determined by substituting from Eq. (d) into first, second and fifth equations of set (e) and solving them. These tension-coefficients are found to be

$$\begin{aligned} t_{DE} &= -15.83 \text{ kN/m} & t_{DF} &= -22.50 \text{ kN/m} \\ t_{EF} &= -7.50 \text{ kN/m} \end{aligned}$$

The remaining equations of set (e) can be used as a check. They must be satisfied identically.

The lengths of the members of the frame may be computed by using Eq. (9.1).

$$\begin{aligned} L_{AD} &= 10.630 \text{ m} & L_{AE} &= 13.601 \text{ m} \\ L_{AF} &= 13.115 \text{ m} & L_{BD} &= 13.601 \text{ m} \\ L_{BE} &= 10.630 \text{ m} & L_{BF} &= 13.115 \text{ m} \\ L_{CD} &= 12.042 \text{ m} & L_{CE} &= 12.042 \text{ m} \\ L_{CF} &= 10.198 \text{ m} & L_{FD} &= 5.000 \text{ m} \\ L_{DE} &= 6.000 \text{ m} & L_{EF} &= 5.000 \text{ m} \end{aligned}$$

Knowing the tension-coefficients and lengths of the members of the frame, the axial force in the members may be computed by using Eq. (9.2).

$$\begin{aligned} S_{AD} &= -250.25 \text{ kN} & S_{AE} &= 65.17 \text{ kN} \\ S_{AF} &= 169.40 \text{ kN} & S_{BD} &= 116.18 \text{ kN} \\ S_{BE} &= -210.39 \text{ kN} & S_{BF} &= -38.25 \text{ kN} \\ S_{CD} &= 60.21 \text{ kN} & S_{CE} &= 60.21 \text{ kN} \\ S_{CF} &= -305.94 \text{ kN} & S_{FD} &= -112.50 \text{ kN} \\ S_{DE} &= -95.00 \text{ kN} & S_{EF} &= -37.50 \text{ kN} \end{aligned}$$

9.3 DISPLACEMENT OF PIN-JOINTED SPACE FRAMES

As in the case of pin-jointed plane frames, the displacements of pin-jointed space frames may be computed by using the unit-load method given in Sec. 2.12. Consider the displacements of a statically determinate pin-jointed space frame due to a given system of external loads. Forces S in the members of the space frame may be calculated by using the tension-coefficient method given in the preceding section. In order to determine the displacement Δ_j at any coordinate j, apply a unit force at coordinate j and calculate the member forces s_j due to the unit force. According to the unit-load method, the displacement Δ_j is given by the equation

$$\Delta_j = \sum \frac{S s_j L}{AE} \tag{9.7}$$

where L/AE is the flexibility of a member. The summation should be carried out to include all the members of the frame.

The displacement δ_{ij} at coordinate i due to a unit force at coordinate j may be computed in a similar manner. It is given by the equation

$$\delta_{ij} = \sum \frac{s_i s_j L}{AE} \tag{9.8}$$

where s_i and s_j are the forces in a member due to a unit force at coordinates i and j respectively.

Equation (9.7) may be used to compute the displacement of a joint in any chosen direction on account of the applied loads. The elements of the flexibility matrix for a pin-jointed space frame may be calculated with the help of Eq. (9.8).

Example 9.4

For the pin-jointed space frame shown in Fig. 9.4, calculate the displacements at coordinates 1, 2 and 3 due to loads $P_1 = 40$ kN, $P_2 = 30$ kN and $P_3 = -80$ kN acting at coordinates 1, 2 and 3 respectively. Also, develop the flexibility matrix for the frame. The numbers in parentheses by the sides of members are the cross-sectional areas of the members of the frame in mm². Take $E = 200$ kN/mm².

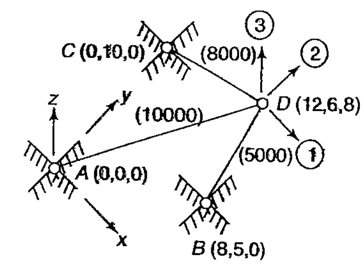


Fig. 9.4

Solution

The frame in this example is the same as the one analysed earlier in Ex. 9.1. The lengths of members DA, DB and DC have been computed and are given in Eq. (a) of Ex. 9.1. The term L/A for each member has been listed in column 2 of Table 9.1. The displacements at coordinates 1, 2 and 3 due to the applied loads may be calculated by using the unit-load method. For this purpose it is necessary to calculate member forces S, s_1, s_2 and s_3. Member forces S have already been calculated in Ex. 9.1 and are listed in column 3 of Table 9.1.

Table 9.1

Member	L/A (mm^{-1})	S (kN)	s_1	s_2	s_3	Ss_1L/A (kN/mm)	Ss_2L/A (kN/mm)	Ss_3L/A (kN/mm)
1	2	3	4	5	6	7	8	9
DA	1.5620	140.58	0.976	1.562	–0.683	214.316	342.993	–149.977
DB	1.8000	–180.00	–1.125	0	1.688	364.500	0	–546.912
DC	1.8708	14.97	0.935	–1.497	–0.281	26.185	–41.925	–7.870
					Σ	605.001	301.068	–704.759
						Tension	+	
						Compression	–	

To calculate member forces s_1 apply a unit force at coordinate 1. Considering the equilibrium of joint D and using Eq. (9.4),

$$t_{DA}(0-12)+t_{DB}(8-12)+t_{DC}(0-12)+1=0$$
$$t_{DA}(0-6)\ +t_{DB}(5-6)\ +t_{DC}(10-6)=0$$
$$t_{DA}(0-8)\ +t_{DB}(0-8)\ +t_{DC}(0-8)=0$$

Solving these equations for the tension coefficients and using Eq. (9.2), member forces s_1 are found to be

$$S_{DA}=0.976 \qquad S_{DB}=-1.125 \qquad S_{DC}=0.935 \tag{a}$$

To calculate member forces s_2, apply a unit force at coordinate 2. Considering the equilibrium of joint D and using Eq. (9.4),

$$t_{DA}(0-12)+t_{DB}(8-12)+t_{DC}(0-12)=0$$
$$t_{DA}(0-6)\ +t_{DB}(5-6)\ +t_{DC}(10-6)+1=0$$
$$t_{DA}(0-8)\ +t_{DB}(0-8)\ +t_{DC}(0-8)=0$$

Solving these equations for the tension-coefficients and using Eq. (9.2), member forces s_2 are found to be

$$S_{DA}=1.562 \qquad S_{DB}=0 \qquad S_{DC}=-1.497 \tag{b}$$

To calculate member forces s_3, apply a unit force at coordinate 3. Considering the equilibrium of joint D and using Eq. (9.4),

$$t_{DA}(0-12)+t_{DB}(8-12)+t_{DC}(0-12)=0$$
$$t_{DA}(0-6)\ +t_{DB}(5-6)\ +t_{DC}(10-6)=0$$
$$t_{DA}(0-8)\ +t_{DB}(0-8)\ +t_{DC}(0-8)+1=0$$

Solving these equations for the tension coefficients and using Eq. (9.2), member forces s_3 are found to be

$$S_{DA}=-0.683 \qquad S_{DB}=1.688 \qquad S_{DC}=-0.281 \tag{c}$$

Member forces s_1, s_2 and s_3 obtained in Eqs (a), (b) and (c) are listed in columns 4, 5 and 6 of Table 9.1. Displacements Δ_1, Δ_2 and Δ_3 at coordinates 1, 2 and 3 due to the applied loads may be computed by using Eq. (9.7).

$$\Delta_1=\sum\frac{Ss_1L}{AE}=\frac{1}{E}\sum\frac{Ss_1L}{A}=\frac{1}{200}\times 605.001=3.03\ \text{mm}$$

$$\Delta_2=\sum\frac{Ss_2L}{AE}=\frac{1}{E}\sum\frac{Ss_2L}{A}=\frac{1}{200}\times 301.068=1.51\ \text{mm}$$

$$\Delta_3=\sum\frac{Ss_3L}{AE}=\frac{1}{E}\sum\frac{Ss_3L}{A}=\frac{1}{200}(-704.759)=-3.52\ \text{mm}$$

The elements of the flexibility matrix can be calculated by using Eq. (9.8). The necessary computations have been carried out in Table 9.2

Table 9.2

Member	s_1^2L/A (mm^{-1})	s_2^2L/A (mm^{-1})	s_3^2L/A (mm^{-1})	s_1s_2L/A (mm^{-1})	s_2s_3L/A (mm^{-1})	s_3s_1L/A (mm^{-1})
1	2	3	4	5	6	7
DA	1.4879	3.8110	0.7287	2.3813	–1.6664	–1.0412
DB	2.2781	0	5.1288	0	0	–3.4182
DC	1.6355	4.1925	0.1477	–2.6185	0.7870	–0.4915
Σ	5.4015	8.0035	6.0052	–0.2372	–0.8794	–4.9509

$$\delta_{11}=\sum\frac{s_1^2L}{AE}=\frac{1}{E}\sum\frac{s_1^2L}{A}=\frac{1}{200}\times 5.4015=0.0270$$

$$\delta_{22}=\sum\frac{s_2^2L}{AE}=\frac{1}{E}\sum\frac{s_2^2L}{A}=\frac{1}{200}\times 8.0035=0.0400$$

$$\delta_{33}=\sum\frac{s_3^2L}{AE}=\frac{1}{E}\sum\frac{s_3^2L}{A}=\frac{1}{200}\times 6.0052=0.0300$$

$$\delta_{12}=\delta_{21}=\sum\frac{s_1s_2L}{AE}=\frac{1}{E}\sum\frac{s_1s_2L}{A}=\frac{1}{200}(-0.2372)=-0.0012$$

$$\delta_{23}=\delta_{32}=\sum\frac{s_2s_3L}{AE}=\frac{1}{E}\sum\frac{s_2s_3L}{A}=\frac{1}{200}(-0.8794)=-0.0044$$

$$\delta_{31}=\delta_{13}=\sum\frac{s_3s_1L}{AE}=\frac{1}{E}\sum\frac{s_3s_1L}{A}=\frac{1}{200}(-4.9509)=-0.0248$$

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta]=\begin{bmatrix}0.0270 & -0.0012 & -0.0248\\ -0.0012 & 0.0400 & -0.0044\\ -0.0248 & -0.0044 & 0.0300\end{bmatrix}$$

It may be noted that the displacements Δ_1, Δ_2 and Δ_3 may also be computed by using the load-displacement relationship, Eq. (4.23).

$$[\Delta] = [\delta][P]$$

or

$$\begin{bmatrix}\Delta_1\\\Delta_2\\\Delta_3\end{bmatrix} = \begin{bmatrix}0.0270 & -0.0012 & -0.0248\\ -0.0012 & 0.0400 & -0.0044\\ -0.0248 & -0.0044 & 0.0300\end{bmatrix}\begin{bmatrix}40\\30\\-80\end{bmatrix} = \begin{bmatrix}3.03\\1.50\\-3.52\end{bmatrix}$$

Example 9.5

Figure 9.5 shows a pin-jointed space frame supported by hinges at A, B and C. Cuts have been introduced in members BD, CE and AF. Coordinates 1, 2 and 3 are located at these cuts as shown in the figure. Calculate the displacements at coordinates 1, 2 and 3 due to the applied loads. Also, develop the flexibility matrix for the frame. The flexibility L/AE of each member of the frame is 0.02 mm/kN.

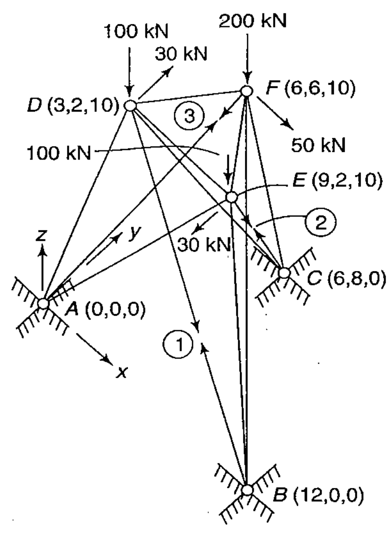


Fig. 9.5

Solution

The displacements at coordinates 1, 2 and 3 due to the applied loads may be calculated by using the unit-load method. For this purpose it is necessary to calculate member forces S, s_1, s_2 and s_3. Forces S in the members, except in those in which cuts have been provided, have already been calculated in Ex. 9.2. The forces in the members in which cuts have been provided are evidently zero. These forces are listed in column 2 of Table 9.3.

To calculate member forces s_1, apply a unit force at coordinate 1. The components of the unit force along the three cartesian axes at joint D are

$$X_D = \text{direction cosine of } DB \text{ with the } x\text{-axis} = \frac{x_B - x_D}{L_{DB}} = 0.662$$

$$Y_D = \text{direction cosine of } DB \text{ with the } y\text{-axis} = \frac{y_B - y_D}{L_{DB}} = -0.147$$

$$Z_D = \text{direction cosine of } DB \text{ with the } z\text{-axis} = \frac{z_B - z_D}{L_{DB}} = -0.735$$

It may be noted that the cartesian components of the unit force at joint B are equal in magnitude and opposite in sign to those at joint D. Considering the equilibrium of joints D, E and F and using Eq. (9.4),

$$\begin{aligned}
-3t_{DA} + 3t_{DC} + 6t_{DE} + 3t_{DF} &= -0.662\\
-2t_{DA} + 6t_{DC} \qquad + 4t_{DF} &= 0.147\\
-10t_{DA} - 10t_{DC} &= 0.735\\
-9t_{EA} + 3t_{EB} - 6t_{ED} - 3t_{EF} &= 0\\
-2t_{EA} - 2t_{EB} \qquad + 4t_{EF} &= 0\\
-10t_{EA} - 10t_{EB} &= 0\\
6t_{FB} - 3t_{FD} \qquad + 3t_{FE} &= 0\\
-6t_{FB} - 4t_{FD} + 2t_{FC} - 4t_{FE} &= 0\\
-10t_{FB} \qquad - 10t_{FC} &= 0
\end{aligned} \qquad \text{(a)}$$

Solving Eq. (a) for the tension-coefficients and using Eq. (9.2), forces s_1 are calculated. These forces are listed in column 3 of Table 9.3.

Table 9.3

Member	S (kN)	s_1	s_2	s_3	Ss_1 (kN)	Ss_2 (kN)	Ss_3 (kN)
1	2	3	4	5	6	7	8
BF	103.87	0	–1.089	0	0	–113.07	0
CF	123.19	0	0.846	0.775	0	104.24	95.48
DF	16.67	0	0	0.760	0	0	–12.65
EF	12.50	0	0.833	0	0	–10.38	0
AE	51.00	1.006	0	0	–51.34	0	0
BE	66.44	0.787	0.882	0	52.26	58.62	0
DE	7.50	0.882	0	0	–6.62	0	0
AD	22.11	0.787	0	0.808	17.39	0	17.86
CD	95.37	0	0	0.915	0	0	–87.28
AF	0	0	0	1.000	0	0	0
BD	0	1.000	0	0	0	0	0
CE	0	0	1.000	0	0	0	0
				Σ	11.69	39.44	13.41

Tension +
Compression –

To calculate member forces s_2, apply a unit force at coordinate 2. The components of the unit force along the three cartesian axes at joint E are

$$X_E = \text{direction cosine of } EC \text{ with the } x\text{-axis} = \frac{x_C - x_E}{L_{EC}} = -0.249$$

$$Y_E = \text{direction cosine of } EC \text{ with the } y\text{-axis} = \frac{y_C - y_E}{L_{EC}} = 0.498$$

$$Z_E = \text{direction cosine of } EC \text{ with the } z\text{-axis} = \frac{z_C - z_E}{L_{EC}} = -0.830$$

Considering the equilibrium of joints D, E and F and using Eq. (9.4), member forces s_2 may be obtained. These forces are listed in column 4 of Table 9.3.

Similarly, to calculate member forces s_3, apply a unit force at coordinate 3. The components of the unit force along the three cartesian axes at joint F are

$$X_F = \text{direction cosine of } FA \text{ with the } x\text{-axis} = \frac{x_A - x_F}{L_{FA}} = -0.457$$

$$Y_F = \text{direction cosine of } FA \text{ with the } y\text{-axis} = \frac{y_A - y_F}{L_{FA}} = -0.457$$

$$Z_F = \text{direction cosine of } FA \text{ with the } z\text{-axis} = \frac{z_A - z_F}{L_{FA}} = -0.762$$

Considering the equilibrium of joints D, E and F and using Eq. (9.4), member forces s_3 may be obtained. These forces are listed in column 5 of Table 9.3.

Displacements Δ_1, Δ_2 and Δ_3 at coordinates 1, 2 and 3 due to the applied loads may be computed by using Eq. (9.7).

$$\Delta_1 = \sum \frac{Ss_1L}{AE} = \frac{L}{AE}\sum Ss_1 = 0.02 \times 11.69 = 0.234 \text{ mm}$$

$$\Delta_2 = \sum \frac{Ss_2L}{AE} = \frac{L}{AE}\sum Ss_2 = 0.02 \times 39.44 = 0.789 \text{ mm}$$

$$\Delta_3 = \sum \frac{Ss_3L}{AE} = \frac{L}{AE}\sum Ss_3 = 0.02 \times 13.41 = 0.268 \text{ mm}$$

The elements of the flexibility matrix can be calculated by using Eq. (9.8). The necessary computations have been carried out in Table 9.4

Table 9.4

Member	s_1^2	s_2^2	s_3^2	s_1s_2	s_2s_3	s_3s_1
1	2	3	4	5	6	7
BF	0	1.185	0	0	0	0
CF	0	0.716	0.601	0	0.656	0
DF	0	0	0.578	0	0	0
EF	0	0.694	0	0	0	0
AE	1.013	0	0	0	0	0
BE	0.619	0.778	0	0.694	0	0
DE	0.778	0	0	0	0	0
AD	0.619	0	0.653	0	0	0.636
CD	0	0	0.838	0	0	0
AF	0	0	1.000	0	0	0
BD	1.000	0	0	0	0	0
CE	0	1.000	0	0	0	0
Σ	4.029	4.373	3.670	0.694	0.656	0.636

$$\delta_{11} = \sum \frac{s_1^2L}{AE} = \frac{L}{AE}\sum s_1^2 = 0.02 \times 4.029 = 0.08058$$

$$\delta_{22} = \sum \frac{s_2^2L}{AE} = \frac{L}{AE}\sum s_2^2 = 0.02 \times 4.373 = 0.08746$$

$$\delta_{33} = \sum \frac{s_3^2L}{AE} = \frac{L}{AE}\sum s_3^2 = 0.02 \times 3.670 = 0.07340$$

$$\delta_{12} = \delta_{21} = \sum \frac{s_1s_2L}{AE} = \frac{L}{AE}\sum s_1s_2 = 0.02 \times 0.694 = 0.01388$$

$$\delta_{23} = \delta_{32} = \sum \frac{s_2s_3L}{AE} = \frac{L}{AE}\sum s_2s_3 = 0.02 \times 0.656 = 0.01312$$

$$\delta_{31} = \delta_{13} = \sum \frac{s_3s_1L}{AE} = \frac{L}{AE}\sum s_3s_1 = 0.02 \times 0.636 = 0.01272$$

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \begin{bmatrix} 0.08058 & 0.01388 & 0.01272 \\ 0.01388 & 0.08746 & 0.01312 \\ 0.01272 & 0.01312 & 0.07340 \end{bmatrix}$$

Example 9.6

Figure 9.6 shows a pin-jointed space frame. The frame is supported by a hinge at A and by a roller at B which permits movement freely along the x-axis only. The support at C permits movement freely along x- and y-axes. Calculate the displacements at co-ordinates 1, 2 and 3 due to the applied loads. Also, develop the flexibility matrix for the frame. The flexibility, L/AE of each member of the frame is 0.02 mm/kN.

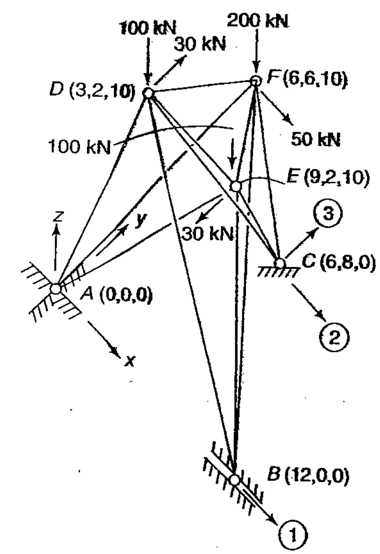


Fig. 9.6

Solution

The displacements at coordinates 1, 2 and 3 due to the applied loads may be calculated by using the unit-load method. For this purpose it is necessary to calculate member forces S, s_1, s_2 and s_3. Member forces S have already been calculated in Ex. 9.3 and are listed in column 2 of Table 9.5.

To calculate member forces s_1, apply a unit force at coordinate 1. Member forces s_1

may be calculated in a manner similar to that for member forces S. Likewise, member forces s_2 and s_3 due to a unit force at coordinates 2 and 3 successively may be computed. Member forces s_1, s_2 and s_3 are listed in columns 3, 4 and 5 of Table 9.5.

Displacements Δ_1, Δ_2 and Δ_3 at coordinates 1, 2 and 3 due to the applied loads may be computed by using Eq. (9.7).

Table 9.5

Member	S (kN)	s_1	s_2	s_3	Ss_1 (kN)	Ss_2 (kN)	Ss_3 (kN)
1	2	3	4	5	6	7	8
AF	169.40	0	2.192	1.640	0	371.11	277.78
BF	–38.25	0	–2.192	1.640	0	83.87	–62.78
CF	–305.94	0	0	–2.550	0	0	780.00
DF	112.50	0	–1.665	–1.250	0	187.31	140.63
EF	–37.50	0	1.665	–1.250	0	–62.44	46.88
AE	65.17	2.271	1.129	–0.857	147.99	73.55	–55.83
BE	–210.39	–1.775	0.882	–0.670	373.46	–185.61	140.89
CE	60.21	0	–2.011	1.505	0	–121.08	90.63
DE	–95.00	–1.998	–1.002	0.750	189.77	95.17	–71.24
AD	–250.25	–1.775	–2.657	–0.670	444.22	665.01	167.58
BD	116.18	2.271	1.129	–0.857	263.84	131.13	–99.53
CD	60.21	0	2.011	1.505	0	121.08	90.63
				Σ	1419.28	1359.10	1445.64

Tension +
Compression –

$$\Delta_1 = \sum \frac{Ss_1L}{AE} = \frac{L}{AE}\sum Ss_1 = 0.02 \times 1419.28 = 28.39 \text{ mm}$$

$$\Delta_2 = \sum \frac{Ss_2L}{AE} = \frac{L}{AE}\sum Ss_2 = 0.02 \times 1359.10 = 27.18 \text{ mm}$$

$$\Delta_3 = \sum \frac{Ss_3L}{AE} = \frac{L}{AE}\sum Ss_3 = 0.02 \times 1445.64 = 28.91 \text{ mm}$$

The elements of the flexibility matrix can be calculated by using Eq. (9.8). The necessary computations have been carried out in Table 9.6.

$$\delta_{11} = \sum \frac{s_1^2L}{AE} = \frac{L}{AE}\sum s_1^2 = 0.02 \times 20.620 = 0.41240$$

$$\delta_{22} = \sum \frac{s_2^2L}{AE} = \frac{L}{AE}\sum s_2^2 = 0.02 \times 34.627 = 0.69254$$

$$\delta_{33} = \sum \frac{s_3^2L}{AE} = \frac{L}{AE}\sum s_3^2 = 0.02 \times 22.461 = 0.44922$$

$$\delta_{12} = \delta_{21} = \sum \frac{s_1s_2L}{AE} = \frac{L}{AE}\sum s_1s_2 = 0.02 \times 10.270 = 0.20540$$

$$\delta_{23} = \delta_{32} = \sum \frac{s_2s_3L}{AE} = \frac{L}{AE}\sum s_2s_3 = 0.02\,(-1.497) = -0.02994$$

$$\delta_{31} = \delta_{13} = \sum \frac{s_3s_1L}{AE} = \frac{L}{AE}\sum s_3s_1 = 0.02\,(-3.014) = -0.06028$$

Table 9.6

Member	s_1^2	s_2^2	s_3^2	s_1s_2	s_2s_3	s_3s_1
1	2	3	4	5	6	7
AF	0	4.797	2.688	0	3.591	0
BF	0	4.797	2.688	0	–3.591	0
CF	0	0	6.500	0	0	0
DF	0	2.778	1.563	0	2.083	0
EF	0	2.778	1.563	0	–2.083	0
AE	5.159	1.274	0.734	2.564	–0.967	–1.946
BE	3.151	0.778	0.448	–1.566	–0.591	1.189
CE	0	4.044	2.266	0	–3.027	0
DE	4.000	1.000	0.563	2.000	–0.752	–1.500
AD	3.151	7.063	0.448	4.708	–1.780	1.189
BD	5.159	1.274	0.734	2.564	–0.967	–1.946
CD	0	4.044	2.266	0	3.027	0
Σ	20.620	34.627	22.461	10.270	–1.497	–3.014

Hence, the required flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \begin{bmatrix} 0.41240 & 0.20540 & -0.06028 \\ 0.20540 & 0.69254 & -0.02994 \\ -0.06028 & -0.02994 & 0.44922 \end{bmatrix}$$

9.4 FORCE METHOD

As in the case of pin-jointed plane frames, the method begins with the determination of degree of static indeterminacy, n. The basic determinate structure is obtained by removing n redundants. Of the numerous alternatives available for the choice of redundants, the one which leads to a simple basic determinate structure may be selected for the solution of the problem. Care should also be taken to see that the basic determinate structure is stable. The n

redundants thus removed may comprise internal member forces or external reaction components or a combination of both. A coordinate is assigned to each of the redundants thus removed. The displacements at all the coordinates are computed by using the unit-load method. The displacement at a coordinate corresponding to an internal redundant is zero for the continuity of the structure. The displacement at a coordinate corresponding to an external redundant is zero for an unyielding support or has a prespecified value in the case of a yielding support. Thus the net displacements at all coordinates are known. Hence, depending upon the support conditions, the redundants may be computed by using Eqs (7.3) or (7.4).

EXAMPLE 9.7

Figure 9.7 shows a pin-jointed space frame resting on hinge supports at A, B and C. Determine the forces in all the members of the frame. The numbers in parentheses are the cartesian coordinates of the joints. The flexibility, L/AE, of each member of the frame is 0.02 mm/kN.

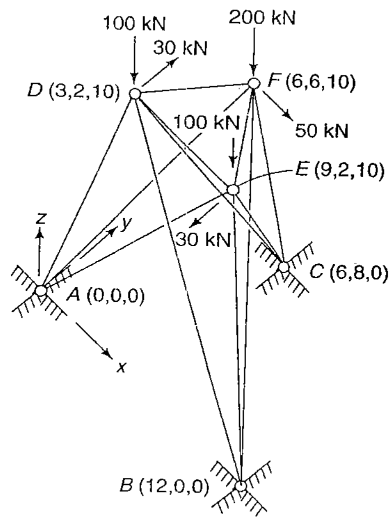


Fig. 9.7

Solution

The frame has 12 members and 9 external reaction components. Hence, the total number of unknowns is 21. The total number of equations of equilibrium is 18, because 3 equations of equilibrium can be written for each of the six joints of the frame. Consequently, the degree of static indeterminacy of the structure is 21 – 18 = 3. There are numerous ways in which the redundants can be chosen. Herein, two alternative approaches are presented.

(i) To obtain the released structure, the forces in diagonal members *DB*, *EC* and *FA* may be chosen as redundants and released by introducing cuts in these members. Coordinates 1, 2 and 3 may be assigned to the internal forces thus released as shown in Fig. 9.5. As the supports are unyielding, forces P_1, P_2 and P_3 in the redundant members may be computed by using Eq. (7.4). The flexibility matrix has already been developed in Ex. 9.5. The displacements at coordinates 1, 2 and 3 due to the applied loads have also been computed in the same example. Hence, matrix $[\Delta_L]$ is given by the equation

$$[\Delta_L] = \begin{bmatrix} 0.234 \\ 0.789 \\ 0.268 \end{bmatrix}$$

Substituting into Eq. (7.4)

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -\begin{bmatrix} 0.08058 & 0.01388 & 0.01272 \\ 0.01388 & 0.08746 & 0.01312 \\ 0.01272 & 0.01312 & 0.0734 \end{bmatrix}^{-1} \begin{bmatrix} 0.234 \\ 0.789 \\ 0.268 \end{bmatrix}$$

$$= -\begin{bmatrix} 1.12 \\ 8.55 \\ 1.93 \end{bmatrix}$$

Hence,

$$P_1 = -1.12 \text{ kN} \qquad P_2 = -8.55 \text{ kN} \qquad P_3 = -1.93 \text{ kN}$$

Knowing forces P_1, P_2 and P_3, the forces in the other members of the frame may be calculated by adding the forces caused by the applied loads and the redundants as indicated by the equation

$$\text{Net force} = S + P_1 s_1 + P_2 s_2 + P_3 s_3 \tag{a}$$

These forces are found to be

$S_{AF} = -1.93$ kN $\quad S_{BF} = -113.18$ kN $\quad S_{CF} = -114.46$ kN

$S_{DF} = 18.14$ kN $\quad S_{EF} = 19.62$ kN $\quad S_{AE} = -52.13$ kN

$S_{BE} = -58.02$ kN $\quad S_{CE} = -8.55$ kN $\quad S_{DE} = 8.49$ kN

$S_{AD} = 19.67$ kN $\quad S_{BD} = -1.12$ kN $\quad S_{CD} = -97.14$ kN

(ii) As an alternative, the external reaction components X_B at *B* and X_C and Y_C at *C* may be chosen as redundants. Coordinates 1, 2 and 3 may be assigned to these redundant reaction components as shown in Fig. 9.6. As the supports are unyielding, the net displacements Δ_1, Δ_2 and Δ_3 at coordinates 1, 2 and 3 are evidently zero. Hence, the redundant reaction components P_1, P_2 and P_3 at coordinates 1, 2 and 3 may be computed by using Eq. (7.4). The flexibility matrix has already been developed in Ex. 9.6. The displacements at coordinates 1, 2 and 3 due to the applied loads have also been computed in the same example. Hence, matrix $[\Delta_L]$ is given by the equation

$$[\Delta_L] = \begin{bmatrix} 28.39 \\ 27.18 \\ 28.91 \end{bmatrix}$$

Substituting into Eq. (7.4),

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -\begin{bmatrix} 0.41240 & 0.20540 & -0.06028 \\ 0.20540 & 0.69254 & -0.02994 \\ -0.06028 & -0.02994 & 0.44922 \end{bmatrix}^{-1} \begin{bmatrix} 28.39 \\ 27.18 \\ 28.19 \end{bmatrix}$$

$$= -\begin{bmatrix} 68.80 \\ 22.09 \\ 75.07 \end{bmatrix}$$

Knowing the redundant reaction components P_1, P_2 and P_3, the forces in the members of the frame may be calculated by using Eq. (a). It may be checked that the member forces are the same as in (i).

9.5 STIFFNESS OF A PIN-JOINT

The translational stiffness of a pin-joint of a plane frame has been discussed in Sec. 7.4. As in the case of plane frames, a pin-joint of a space-frame offers resistance to translation because it entails changes in the lengths of the members meeting at the joint. The translational stiffness of a pin-joint of a space frame in any chosen direction is defined as the force required to produce unit translation in the chosen direction.

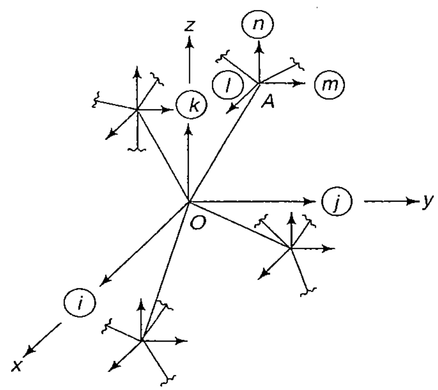


Fig. 9.8

Consider the translational stiffness of the typical pin-joint O shown in Fig. 9.8. At joint O under consideration, coordinates i, j and k have been chosen along the positive directions of the x- y-, and z-axes respectively.

9.5.1 Translation along Coordinate *i*

When a unit translation is given to joint O along coordinate i, the members meeting at joint O will undergo changes in lengths. The contraction of member OA is equal to C_x where C_x is the direction cosine of member OA with the x-axis. The compressive force required to produce this contraction of the member is evidently $\frac{AE}{L}C_x$. The component of this force along coordinate i is $\frac{AE}{L}C_x^2$. Similarly, considering other members meeting at joint O, it is evident that the net force k_{ii} required to translate joint O by unit distance along coordinate i is given by the equation

$$k_{ii} = \sum \frac{AE}{L} C_x^2 \tag{9.9a}$$

where $\frac{AE}{L}$ = axial stiffness of a member meeting at joint O.

The summation should be carried out so as to include all the members meeting at joint O.

The force at coordinate j due to the compressive force in member OA is equal to $\frac{AE}{L}C_xC_y$ where C_y is the direction cosine of member OA with y-axis. Hence, considering all the members meeting at joint O, net force k_{ji} at coordinate j due to a unit displacement at coordinate i is given by the equation

$$k_{ji} = \sum \frac{AE}{L} C_x C_y \tag{9.9b}$$

Similarly, net force k_{ki} at coordinate k due to a unit displacement at coordinate i is given by the equation

$$k_{ki} = \frac{AE}{L} C_x C_z \tag{9.9c}$$

where C_z is the direction cosine of the member OA with the z-axis.

If l, m and n are the coordinates at joint A along the positive directions of x-, y-, and z-axes respectively, forces k_{li}, k_{mi} and k_{ni} at coordinates l, m and n due to a unit displacement at coordinate l are equal to the components of the force in member OA along coordinates l, m and n. Hence,

$$k_{li} = -\frac{AE}{L} C_x^2 \tag{9.9d}$$

$$k_{mi} = -\frac{AE}{L} C_x C_y \tag{9.9e}$$

$$k_{ni} = -\frac{AE}{L} C_x C_z \tag{9.9f}$$

9.5.2 Translation along Coordinate *j*

When a unit translation is given to joint O along coordinate j, the contraction of member OA is equal to C_y. The compressive force required to produce this contraction of member OA is $\frac{AE}{L}C_y$. Proceeding as in Sec. 9.5.1, the forces at coordinates i, j, k, l, m and n due to a unit displacement at coordinate j may be computed. These forces are

$$k_{ij} = \sum \frac{AE}{L} C_y C_x \tag{9.9g}$$

$$k_{jj} = \sum \frac{AE}{L} C_y^2 \tag{9.9h}$$

$$k_{kj} = \sum \frac{AE}{L} C_y C_z \tag{9.9i}$$

$$k_{lj} = -\frac{AE}{L} C_y C_x \tag{9.9j}$$

$$k_{mj} = -\frac{AE}{L} C_y^2 \quad (9.9k)$$

$$k_{nj} = -\frac{AE}{L} C_y C_z \quad (9.9l)$$

9.5.3 Translation along Coordinate *k*

Similarly, when a unit translation is given to joint O along coordinate k, the contraction of member OA is equal to C_z. The compressive force required to produce this contraction of member OA is $\frac{AE}{L} C_z$. Hence, the forces at coordinates i, j, k, l, m and n due to a unit displacement at coordinate k are

$$k_{ik} = \sum \frac{AE}{L} C_z C_x \quad (9.9m)$$

$$k_{jk} = \sum \frac{AE}{L} C_z C_y \quad (9.9n)$$

$$k_{kk} = \sum \frac{AE}{L} C_z^2 \quad (9.9o)$$

$$k_{lk} = -\frac{AE}{L} C_z C_x \quad (9.9p)$$

$$k_{mk} = -\frac{AE}{L} C_z C_y \quad (9.9q)$$

$$k_{nk} = -\frac{AE}{L} C_z^2 \quad (9.9r)$$

It may be noted that the algebraic sums of the forces at the coordinates along the three coordinate axes due to a unit displacement along the three respective axes vanish, thereby satisfying the three conditions of static equilibrium, Eq. (1.1).

From Eq. (9.9) the following inferences with regard to an element k_{pq} of the stiffness matrix can be drawn. These are useful in the computation of stiffness elements.

(i) When coordinates p and q are located at the same joint,

$$k_{pq} = \sum \frac{AE}{L} C_p C_q \quad (9.10a)$$

(ii) When coordinates p and q are located at the two ends of a member,

$$k_{pq} = -\frac{AE}{L} C_p C_q \quad (9.10b)$$

where $C_p = C_x$, C_y or C_z depending upon whether coordinate p is directed along x-, y– or z-axes respectively

$C_q = C_x$, C_y or C_z depending upon whether coordinate q is directed along x-, y- or z-axes respectively.

It may be noted that the above expressions are based on the assumptions that coordinates p and q are taken along the positive directions of the cartesian axes.

Example 9.8

Develop the stiffness matrix for the pin-jointed space frame with reference to coordinates 1, 2 and 3 shown in Fig. 9.4. The numbers in parentheses by the sides of the members are the cross-sectional areas of the members in mm². Take E = 200 kN/mm²

Solution

The geometrical properties of the members of the frame have been listed in Table 9.7. The cross-sectional areas of the members are shown in Fig. 9.4 and the lengths of the members are given by Eq. (a) of Ex. 9.1. These are listed in columns 2 and 3 of Table 9.7.

Table 9.7

Member	*L* (*m*)	*A* (*mm²*)	C_x	C_y	C_z
1	2	3	4	5	6
DA	15.620	10000	−0.768	−0.384	−0.512
DB	9.000	5000	−0.444	−0.111	−0.889
DC	14.967	8000	−0.802	0.267	−0.534

The direction cosines of member DA are

$$C_x = \frac{x_A - x_D}{L_{DA}} = \frac{0-12}{15.620} = -0.768$$

$$C_y = \frac{y_A - y_D}{L_{DA}} = \frac{0-6}{15.620} = -0.384$$

$$C_z = \frac{z_A - z_D}{L_{DA}} = \frac{0-8}{15.620} = -0.512$$

The direction cosines of member DB are

$$C_x = \frac{x_B - x_D}{L_{DB}} = \frac{8-12}{9.000} = -0.444$$

$$C_y = \frac{y_B - y_D}{L_{DB}} = \frac{5-6}{9.000} = -0.111$$

$$C_z = \frac{z_B - z_D}{L_{DB}} = \frac{0-8}{9.000} = -0.889$$

The direction cosines of member DC are

$$C_x = \frac{x_C - x_D}{L_{DC}} = \frac{0-12}{14.967} = -0.802$$

$$C_y = \frac{y_C - y_D}{L_{DC}} = \frac{10-6}{14.967} = 0.267$$

$$C_z = \frac{z_C - z_D}{L_{DC}} = \frac{0-8}{14.967} = -0.534$$

The direction cosines of the members are listed in columns 4, 5 and 6 of Table 9.7.

The elements of the stiffness matrix may be computed by using Eq. (9.10). The computations necessary for this purpose have been carried out in Table 9.8.

Table 9.8

Member	$\frac{AE}{L}C_x^2$ (kN/mm)	$\frac{AE}{L}C_y^2$ (kN/mm)	$\frac{AE}{L}C_z^2$ (kN/mm)	$\frac{AE}{L}C_xC_y$ (kN/mm)	$\frac{AE}{L}C_yC_z$ (kN/mm)	$\frac{AE}{L}C_zC_x$ (kN/mm)
1	2	3	4	5	6	7
DA	75.56	18.90	33.58	37.78	25.18	50.38
DB	21.94	1.38	87.80	5.48	10.98	43.90
DC	68.70	7.64	30.54	–22.90	–15.26	45.80
Σ	166.20	27.92	151.92	20.36	20.90	140.08

$$k_{11} = \sum \frac{AE}{L} C_x^2 = 166.20$$

$$k_{22} = \sum \frac{AE}{L} C_y^2 = 27.92$$

$$k_{33} = \sum \frac{AE}{L} C_z^2 = 151.92$$

$$k_{12} = k_{21} = \sum \frac{AE}{L} C_x C_y = 20.36$$

$$k_{23} = k_{32} = \sum \frac{AE}{L} C_y C_z = 20.90$$

$$k_{31} = k_{13} = \sum \frac{AE}{L} C_z C_x = 140.08$$

Hence, required stiffness matrix $[k]$ is given by the equation

$$[k] = \begin{bmatrix} 166.20 & 20.36 & 140.08 \\ 20.36 & 27.92 & 20.90 \\ 140.08 & 20.90 & 151.92 \end{bmatrix}$$

9.6 MEMBER FORCES

Figure 9.9 shows a typical member AB connecting joints A and B of a pin-jointed space frame. The force in member AB can be calculated if the displacements at the two ends of the member are known. The components of the displacement at joint A along the x-, y- and z-axes are Δ_{Ax}, Δ_{Ay}, and Δ_{Az} respectively. Similarly, Δ_{Bx}, Δ_{By}, and Δ_{Bz} are the components of the displacement of joint B along the x-, y- and z-axes respectively. The shortening of the member due to the displacement of joint A is $(\Delta_{Ax}C_x + \Delta_{Ay}C_y + \Delta_{Az}C_z)$. Also, the elongation of the member due to the displacement of joint B is $(\Delta_{Bx}C_x + \Delta_{By}C_y + \Delta_{Bz}C_z)$. Hence, the net shortening of the member is $[(\Delta_{Ax} - \Delta_{Bx})\, C_x + (\Delta_{Ay} - \Delta_{By})\, C_y + (\Delta_{Az} - \Delta_{Bz})\, C_z]$. Consequently, the force in member AB is given by the equation

$$S_{AB} = -\frac{AE}{L}\left[\left(\Delta_{Ax} - \Delta_{Bx}\right)C_x + \left(\Delta_{Ay} - \Delta_{By}\right)C_y + \left(\Delta_{Az} - \Delta_{Bz}\right)C_z\right] \quad (9.11a)$$

where C_x, C_y, C_z = direction cosines of the line AB with x-, y- and z-axes respectively.

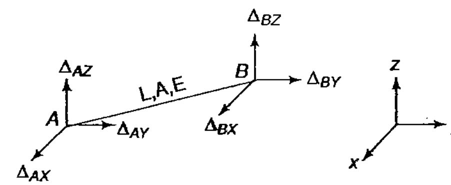


Fig. 9.9

If the member is taken as BA instead of AB, then the force in the member is given by the equation

$$S_{BA} = -\frac{AE}{L}\left[\left(\Delta_{Bx} - \Delta_{Ax}\right)C_x + \left(\Delta_{By} - \Delta_{Ay}\right)C_y + \left(\Delta_{Bz} - \Delta_{Az}\right)C_z\right] \quad (9.11b)$$

where C_x, C_y, C_z = direction cosines of line BA with x-y- and z-axes respectively.

It may be noted that S_{AB} and S_{BA} are equal because the axial force in a member of a pin-jointed frame is always constant. The forces in other members of the pin-jointed frame may be computed in a similar manner.

Example 9.9

Calculate the force in member AB of a pin-jointed space frame if the co-ordinates of joints A and B in m are (–3, 5, 2) and (4, –3, 5) respectively. The displacement components of joint A in mm are 2.0, –1.5 and 2.5 and those of joint B are –1.2, 2.4 and 1.1 along the x-, y- and z-axes respectively. The axial stiffness of the member is 100 kN/mm.

Solution

The force in the member can be calculated by using Eq. (9.11). From the given data,

$$\Delta_{Ax} = 2.0 \text{ mm} \qquad \Delta_{Ay} = -1.5 \text{ mm} \qquad \Delta_{Az} = 2.5\text{mm}$$

$$\Delta_{Bx} = -1.2 \text{ mm} \qquad \Delta_{By} = 2.4 \text{ mm} \qquad \Delta_{Bz} = 1.1 \text{ mm}$$

$$\frac{AE}{L} = 100 \text{ kN/mm}$$

$$L_{AB} = \sqrt{(-3-4)^2 + (5+3)^2 + (2-5)^2} = 11.045 \text{ m}$$

Direction cosines of line AB are

$$C_x = \frac{x_B - x_A}{L_{AB}} = \frac{4+3}{11.045} = 0.634$$

$$C_y = \frac{y_B - y_A}{L_{AB}} = \frac{-3-5}{11.045} = -0.724$$

$$C_z = \frac{z_B - z_A}{L_{AB}} = \frac{5-2}{11.045} = 0.272$$

Substituting into Eq. (9.11a),

$$S_{AB} = -100[2.0 + 1.2)(0.634) + (-1.5 - 2.4)(-0.724) + (2.5 - 1.1)(0.272)] = -523.32 \text{ kN}$$

Minus sign shows that the force in member AB is compressive. It may be checked that force S_{BA} computed by using Eq. (9.11b) is equal to force S_{AB}.

9.7 DISPLACEMENT METHOD

The displacement method of analysis of pin-jointed space frames begins with the determination of the degree of freedom of the structure. The degree of freedom has been discussed in Sec. 1.7. It may be noted that as per Eq. (1.20d) the degree of freedom of a pin-jointed space frame is $(3j - r)$ where j is the number of joints and r is the number of independent external reaction components. After the independent displacement components have been identified, a coordinate is assigned to each of them. The stiffness matrix with reference to the chosen coordinates is then developed. The elements of he stiffness matrix can be computed by using Eq. (9.10). Thereafter, as in the case of pin-jointed plane frames, the displacement components may be determined by using the equation

$$[\Delta] = [k]^{-1}[P] \qquad (9.12)$$

After the displacement components are known, the member forces may be computed by using Eq. (9.11).

Example 9.10

Using the displacement method, calculate the forces in the members of the pin-jointed space frame shown in Fig. 9.4 due to the applied loads P_1 = 40 kN, P_2 = 30 kN and P_3 = −80 kN. The numbers in parentheses by the sides of the members are the cross-sectional areas of the members of the frame in mm². Take E = 200 kN/mm².

Solution

The degree of freedom of the frame is three because joint D can move along the three cartesian axes. Hence, coordinates 1, 2 and 3 may be chosen as shown in Fig. 9.4. The stiffness matrix with reference to the coordinates has already been developed in Ex. 9.8 and is given by the equation

$$[k] = \begin{bmatrix} 166.20 & 20.36 & 140.08 \\ 20.36 & 27.92 & 20.90 \\ 140.08 & 20.90 & 151.92 \end{bmatrix}$$

The external loads acting at the coordinates 1, 2 and 3 are

$$P_1 = 40 \text{ kN} \qquad P_2 = 30 \text{ kN} \qquad P_3 = -80 \text{ kN}$$

Displacements Δ_1, Δ_2 and Δ_3 may be obtained by substituting into Eq. (9.12).

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 166.20 & 20.36 & 140.08 \\ 20.36 & 27.92 & 20.90 \\ 140.08 & 20.90 & 151.92 \end{bmatrix}^{-1} \begin{bmatrix} 40 \\ 30 \\ -80 \end{bmatrix} = \begin{bmatrix} 3.03 \\ 1.51 \\ -3.52 \end{bmatrix}$$

Knowing the displacement components at joint D, the forces in the members of the frame may be computed by using Eq. (9.11). The geometrical properties of the members of the frame required for the computation of member forces are listed in Table 9.7 of Ex. 9.8. Substituting into Eq. (9.11).

$$S_{DA} = -\frac{10000 \times 200}{15620}[3.03(-0.768) + 1.51(-0.384) - 3.52(-0.512)]$$

$$= 140.59 \text{ kN}$$

$$S_{DB} = -\frac{5000 \times 200}{9000}[3.03(-0.444) + 1.51(-0.111) - 3.52(-0.889)]$$

$$= -180.24 \text{ kN}$$

$$S_{DC} = -\frac{8000 \times 200}{14967}[3.03(-0.802) + 1.51(0.267) - 3.52(-0.534)]$$

$$= 15.34 \text{ kN}$$

These forces are practically the same as obtained in Ex. 9.1.

9.8 COMPARISON OF METHODS

In the preceding sections, the force and displacement methods for the analysis of pin-jointed space frames have been discussed. Examples to illustrate the two main methods have been given. It is evident that in the case of the pin-jointed space frames, the development of the stiffness matrix is simpler as compared to the development of the flexibility matrix but the degree of kinematic indeterminacy of these frames is generally much larger than the degree of static indeterminacy. For example, the degrees of static and kinematic indeterminacies of the frame of Ex. 9.6 are 3 and 12 respectively. Hence, in general, it may be stated that for the analysis of pin-jointed space frames the force method is preferable as compared to the displacement method.

PROBLEMS

9.1 The three-wire system shown in Fig. 9.10 carries a vertical load of 10 kN at joint O. Using the tension coefficient method, determine the forces in the wires. Verify the result by the displacement method.

9.2 Using the tension-coefficient method, determine the forces in members OB, OC and AB of the pin-jointed space frame shown in Fig. 9.11. The frame is resting on a spherical seating at C which is capable of exerting only a vertical reaction. Roller supports are provided at A, B and D which permit the movement along x, y, and x directions respectively.

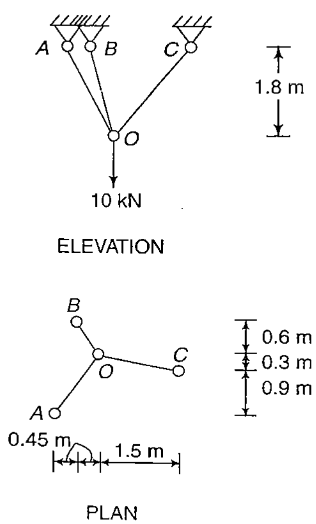


Fig. 9.10

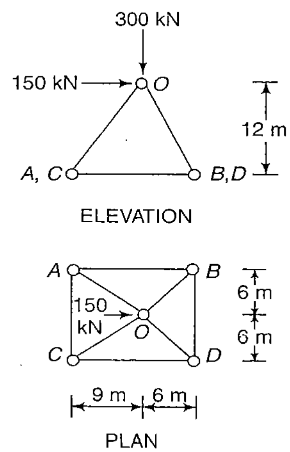


Fig. 9.11

9.3 The pin-jointed space frame of Fig. 9.12 is resting on hinge supports at A, B, C and D. Determine the forces in members EF, FC and FB by the tension-coefficient method.

9.4 The pin-jointed space frame of Fig. 9.13 rests on a hinge support at C. A spherical seating, capable of giving only a vertical reaction, is provided at A. The roller supports at B and D permit free movement along x and y directions respectively. Analyse the frame by the tension-coefficient method. Hence determine the forces in members CD and GH.

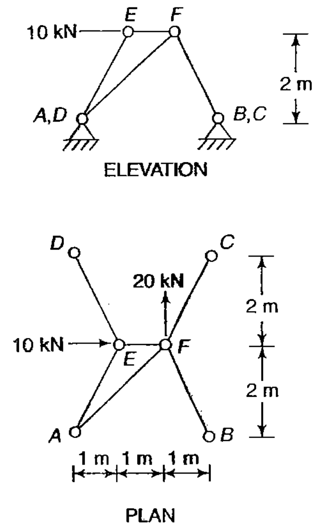


Fig. 9.12

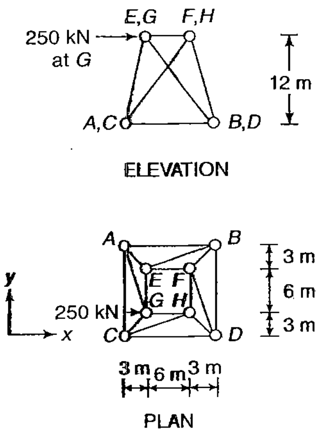


Fig. 9.13

9.5 Determine the degrees of static and kinematic indeterminacies of the space frame shown in Fig. 9.14. The frame is provided with hinge supports at A, B, C and D. Analyse the frame by the force method. Hence determine the forces in members OA, OB, OC and OD. Verify the result by the displacement method. All members have the same value of AE.

9.6 What are the degrees of static and kinematic indeterminacies of the pin-jointed space frame of Fig. 9.15? Using the force method, analyse the frame and determine the forces in members CD, CE and DE. The frame is provided with hinge supports at A, B and E. All members have the same cross-sectional area. E is constant.

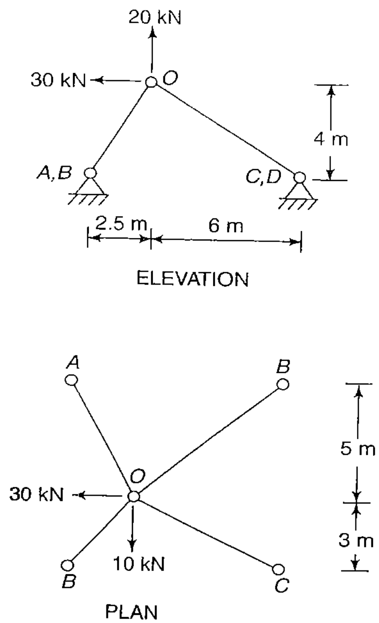


Fig. 9.14

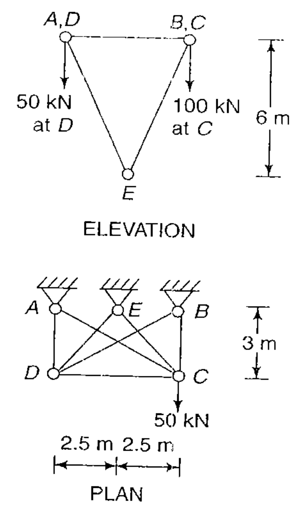


Fig. 9.15

9.7 Determine the degree of static indeterminacy for the pin-jointed space frame shown in Fig. 9.16. Analyse the frame by the force method. Hence determine the force in member *DF*. The frame is provided with hinge supports at *A*, *B*, *C* and *D*. The axial flexibility *L/AE* is the same for all the members.

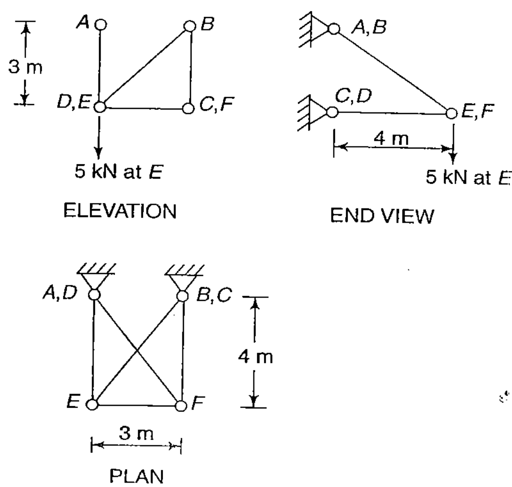


Fig. 9.16

9.8 Treating the force in member *DF* as the redundant, analyse the pin-jointed space frame shown in Fig. 9.17. Hence determine the force in member *EF*. The frame is provided with hinge supports at *A*, *B*, *C* and *D*. Verify the result by an alternative solution in which the horizontal reaction in the *x* direction at hinge support *D* is chosen as the redundant. All members of the frame have the same value of *AE*.

9.9 Determine the degrees of static and kinematic indeterminacies of the pin-jointed space frame shown in Fig. 9.18. The frame is provided with hinge supports at *A*, *B*, and *C*. Analyse the frame by the force method. Hence determine the force in member *AE*. All members of the frame have same value of axial flexibility *L/AE*.

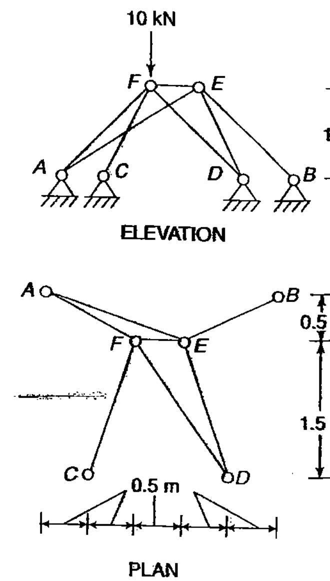


Fig. 9.17

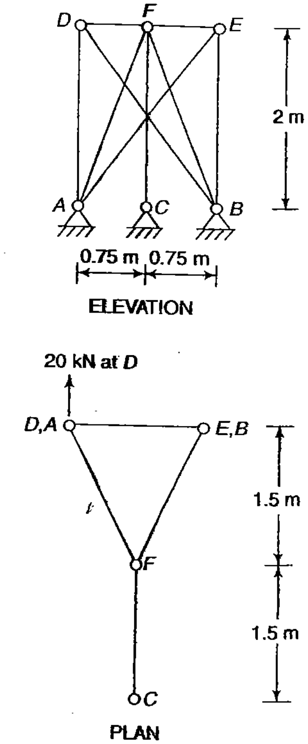


Fig. 9.18

9.10 Find the degrees of static and kinematic indeterminacies of the pin-jointed space frame shown in Fig. 9.19. Adopting the force in member *AG* as the redundant, analyse the frame by the force method. Hence determine the force in member *AD*. The frame is provided with hinge supports at *B*, *C*, *E*, *F* and *G*.

9.11 Choosing the force in members *AF* and *BE* as the redundants, analyse the pin-jointed space frame of Fig. 9.20. The frame is provided with hinge supports at *A*, *B*, *C* and *D*. Determine the forces in the members *EF*, *EC* and *EG*. Verify the result by an alternative choice of the redundants. All the members of the frame have the same value of L^3/AE.

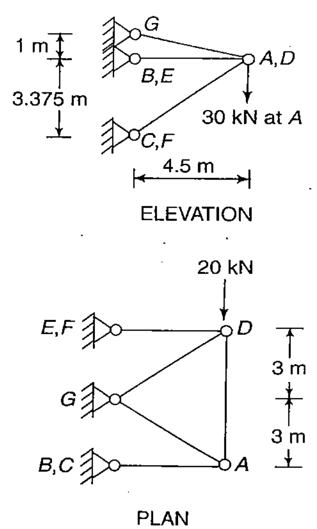


Fig. 9.19

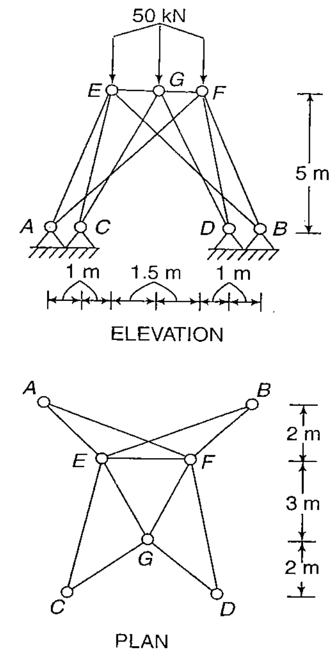


Fig. 9.20

9.12 Determine the degrees of static and kinematic indeterminacies of the pin-jointed space frame of Fig. 9.21. Analyse the frame by the force method with alternative choices of redundants. Hence determine the forces in members *AF*, *BE* and *CD*. All members of the frame have the same cross-sectional area.

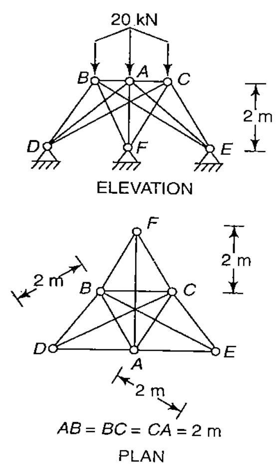


Fig. 9.21

10 COMPARISON OF FORCE AND DISPLACEMENT METHODS

10.1 INTRODUCTION

The force and displacement methods have been discussed in the preceding chapters. The two methods have many similarities but there are also important dissimilarities. The suitability or otherwise of the methods depends upon the ease with which the main operations of the two methods can be carried out for the solution of the problem at hand. The main steps in the two methods of analysis are listed in Table 10.1

Table 10.1

Step	*Force method (flexibility or compatibility method)*	*Displacement method (stiffness or equilibrium method)*
1.	Determine the degree of static indeterminacy (degree of redundancy), choose the redundants and obtain the released structure.	Determine the degree of kinematic indeterminacy (degree of freedom), identify the independent displacement components and obtain the restrained structure.
2.	Assign coordinates to the redundants and develop the flexibility matrix wtih reference to the chosen coordinates.	Assign coordinates to the independent displacement components and develop the stiffness matrix with reference to the chosen coordinates.
3.	In the released structure, compute the displacements at the coordinates due to the applied loads.	In the restrained structure, compute the forces at the coordinates due to the loads other than those acting at the coordinates.
4.	In the released structure, compute the displacement at the coordinates due to redundants.	Compute the forces required at the coordinates in the unrestrained structure to cause the independent displacement components.
5.	Use compatibility conditions at the coordinates to determine the redundants.	Use equilibrium conditions at the coordinates to determine independent displacement components.
6.	Compute member forces by statics.	Compute member forces by slope-deflection equations.

The choice of the method for the solution of a particular problem will depend upon the convenience with which the steps listed in Table 10.1 can be performed. The choice of the method is also influenced by the following considerations:

(i) familiarity and expertise of the analyst and
(ii) method of computation: by hand, semi-automatic or computerised.

The choice of the method, in light of these factors and the convenience in performing the steps mentioned in Table 10.1, is examined in the following section.

10.2 STEPWISE COMPARISON

It has been stated in the preceding section that the suitability of the two methods depends upon the ease with which the steps listed in Table 10.1 can be performed. In this section, consideration is given to each step listed in the table.

Step 1: The degree of static and kinematic indeterminacies can be calculated with almost equal ease with the help of the formulae given in Secs 1.6 and 1.7. It may, however, be mentioned that generally, students and structural engineers are more familiar with static indeterminacy as compared to kinematic indeterminacy. In the force method it is necessary to choose the redundants and obtain the released structure. This is by no means a simple matter because several alternatives are generally available. The choice of the redundants may greatly influence the accuracy and the amount of computational work involved in the force method. The choice of the released structure should serve the following objectives:

(i) minimum computational effort
(ii) maximum accuracy
(iii) simplicity.

The released structure which meets these requirements to the maximum extent is seldom evident, particularly in the case of a large structure. The factors influencing the choice of the best released structure are discussed in greater detail in Sec. 10.3. On the other hand, the restrained structure in the displacement method is self-evident because, in general, there is only one restrained structure, there being no other alternatives. Hence, for performing the first step the displacement method is simpler than the force method.

Step 2: In general, the computational effort required to develop the flexibility matrix is considerably more than for the stiffness matrix due to the following reasons:

(i) The computation of displacements, which constitute the elements of the flexibility matrix, is generally more difficult and time consuming as compared to the computation of restraining forces, which constitute the elements of the stiffness matrix.

(ii) Most of the elements of the flexibility matrix are non-zero unless the released structure has been chosen to ensure a localised phenomenon. The selection of such a released structure is not easy and may not even be feasible in some cases. On the other hand, most of the elements of the stiffness matrix of a large structure are zero. It follows that only a few of the elements of the stiffness matrix require computation.

(iii) The computation of displacements requires the use of a large number of formulae. On the other hand, the computation of the restraining forces can be carried out with the help of only a few standard formulae. In the force method, it becomes necessary to assign coordinates to internal as well as external redundants. Except in continuous beams and very elementary frames, the coordinates assigned to internal redundants are represented by double arrows which are confusing at least to a beginner. On the other hand, in the case of displacement method the use of double arrows for the coordinates is generally unnecessary.

Steps 3 and 4: In these steps, the displacements have to be computed in the force method and the restraining forces in the displacement method. For the reasons given in the discussion of step 2, the computation of displacements is more difficult and time consuming as compared to the computation of the restraining forces.

Step 5: In this step, the determination of redundants chosen in the force method requires the inversion of the flexibility matrix. Similarly, the determination of the displacements in the displacement method requires the inversion of the stiffness matrix. It has been shown in the discussion of steps 1 and 2 that for a large structure, most of the elements of the stiffness matrix are zero which is generally not so in the case of flexibility matrix. By a proper numbering of coordinates it is possible to express the stiffness matrix as a banded matrix with non-zero elements located in the vicinity of the main diagonal. As the inversion of the banded matrix is generally quicker, it would appear that the inversion of the stiffness matrix is faster than that of the flexibility matrix of the same order. As the inversion of the matrix constitutes a major portion of the total computational effort, the time saved in carrying out the inversion of the matrix forms an important consideration in the choice of the two methods.

Step 6: When the force method is adopted, the member forces are calculated by statics as soon as the chosen redundants have been determined. In this method the displacements are not computed. Hence additional computations become necessary for the complete analysis of the structure including the displacements. On the other hand, in the case of the displacement method, the computation of displacements precedes the computation of member forces. Hence no additional computations are generally necessary for the complete analysis of the structure. It may be mentioned that sometimes the displacement of the structure, rather than the internal stresses, may govern the design of the structure. From this point of view the displacement method which provides the complete analysis of the structure may appear to be preferable as compared to the force method.

10.3 CHOICE OF RELEASED STRUCTURE

It has been pointed out in Sec. 10.2 that among the large number of possible released structures, the one which leads to the minimum computational effort, maximum accuracy and simplicity should be chosen for the solution of the problem by the force method. It was also mentioned that the selection of the best released structure is not a simple problem as it requires considerable care and judgement on the part of the analyst. The choice of the released structure will now be discussed in light of the triple requirements of minimum computation, maximum accuracy and simplicity.

10.3.1 Computational Effort

To minimise the computational effort, the released structure should be chosen in such a manner that the development of internal forces in the members of the structure becomes a localised phenomenon, i.e., when a unit load is applied at any one of the coordinates, only a few members in the immediate vicinity of the coordinate are deformed and consequently develop internal forces whereas all other members of the structure remain unstressed. When the released structure is chosen in this manner and the numbering of coordinates is done properly, the resulting flexibility matrix is banded or strongly diagonal, i.e., the non-zero elements lie at or in the immediate neighbourhood of the main diagonal of the flexibility matrix. The remaining elements of the flexibility matrix are zero. Consequently, the computational effort required for the development of the flexibility matrix is considerably reduced because only a few elements need be determined. Similarly, the computation of the displacements due to the applied loads is also reduced if the released structure is chosen to ensure a localised phenomenon. In addition to the time saved in the development of the flexibility matrix, there is a further saving of time in carrying out the matrix inversion. This is so because the inversion of a banded matrix is quicker. Besides, special methods are available for the inversion of a banded matrix. The choice of the released structure which results in a localised phenomenon is discussed below.

Consider a continuous beam with n spans. Figure 10.1(a) shows two consecutive spans PQ and QR resting on supports P, Q and R. If the beam has simple supports at the ends, all the intermediate support reactions may be treated as redundants and released. The released structure thus obtained is a

simply supported beam. Assigning coordinates to the redundants as indicated in Fig. 10.1(b), it is evident that when a unit force is applied at any one of the coordinates, the displacements occur at all the coordinates. For instance, displacements occur at all the coordinates when a unit force is applied at coordinate j. Similarly, displacements occur at all the coordinates due to an external load acting on any one of the spans. Thus considerable computations are required to develop the flexibility matrix and for the determination of the displacements at the coordinates due to the applied loads. It may be noted that when intermediate support reactions are treated as redundants, the development of internal forces is not a localised phenomenon because a unit force applied at any one of the coordinates produces displacements at all the coordinates. On the other hand, the deformation of the structure and the resulting internal forces are localised if the bending moments at the supports are treated as redundants and released. This is equivalent to inserting internal hinges in the continuous beam at the supports as shown in Fig. 10.1(c). In this case, when a unit force is applied at coordinate j, displacements occur only at three coordinates, viz., $(j - 1)$, j and $(j + 1)$. Displacements at all other coordinates are zero. Similarly, when external loads act on any one of the spans, displacements occur at only two coordinates assigned to the moment releases at either end of the span under consideration. For instance, if a load acts on span PQ, displacements occur at coordinates $(j - 1)$ and j only. When the bending moments at the supports are treated as redundants and released, the continuous beam is converted to a series of simply supported beams. Hence, the deformation of any one of the spans does not get propagated to other spans and thus results in a localised phenomenon. Consequently, the computational effort is greatly reduced. It may also be noted that when the released structure is obtained by releasing the bending moments at the supports, the resulting flexibility matrix is a tridiagonal matrix. It has been noted that a banded matrix requires less time for development and inversion.

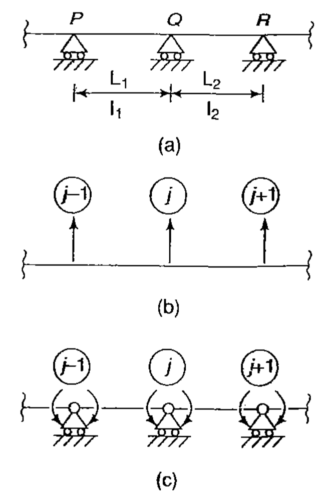


Fig 10.1

Consider next, the rigid-jointed plane frame shown in Fig. 10.2(a). The structure is statically indeterminate to the 12th degree. Among the large number of released structures possible in this case, those shown in Fig. 10.2(b) to (e) may be considered. These released structures have been obtained by introducing

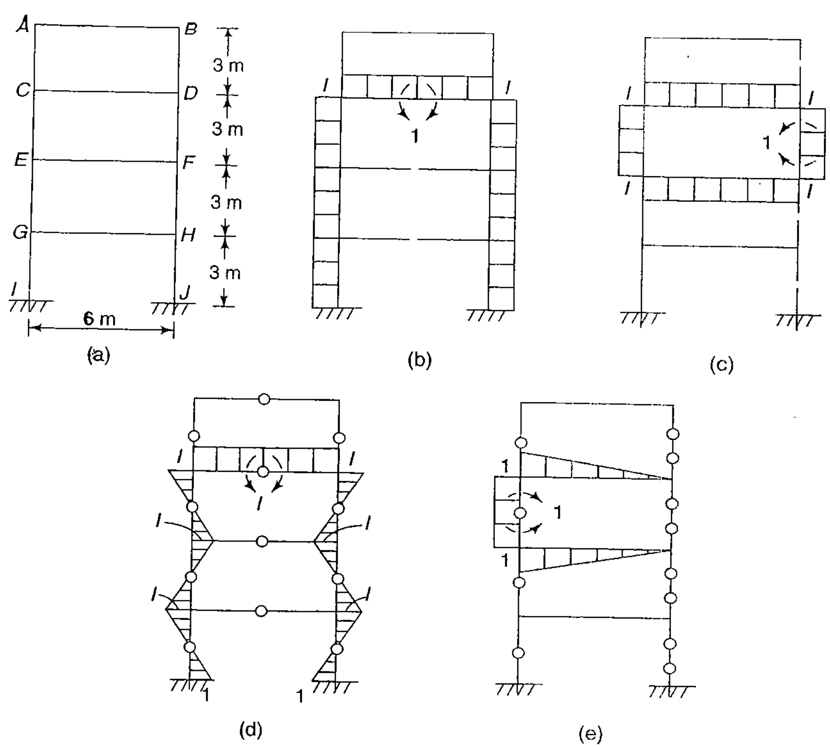


Fig. 10.2

(i) cuts at the centres of all the beams as shown in Fig. 10.2(b)
(ii) cuts in the right columns at the midheight of each storey as shown in Fig. 10.2(c)
(iii) hinges at the centres of all the beams and columns as shown in Fig. 10.2(d)
(iv) a hinge at the midheight of each storey in the left columns and at one-third points in the right columns as shown in Fig. 10.2(e).

In Fig. 10.2(b), unit couples corresponding to moment release have been applied at the cut in beam *CD*. The resulting bending-moment diagram for the released structure can be obtained by statics and is shown in the same figure. Similarly, the bending-moment diagram due to unit couples applied at the cut in column *DF* is shown in Fig. 10.2(c). It may be noted that in this case the members of cell *CDFE* alone develop internal forces. The members of all other cells do not carry internal forces. On the other hand, in the case of the released structure of Fig. 10.2(b), some of the members of all the cells below cell *CDFE* carry internal forces. It follows that among the two released structures, the one shown in Fig. 10.2(c) is superior because it leads to a localised distribution of internal forces and consequently to lesser computational effort. A similar conclusion can be drawn when unit axial forces or unit shear forces (corresponding to the other two releases at the cut, viz., the axial force release and shear force release) instead of unit couples are applied at the cuts. Thus it may be stated that the released structure of Fig. 10.2(c) is preferable because it leads to a localised phenomenon in which only the members of the cell under consideration are subjected to internal forces whereas in the other case the internal forces are propagated to all the cells below the cell under consideration.

In Fig. 10.2(d), the released structure has been obtained by introducing hinges at the midpoints of all the beams and columns. The bending-moment diagram due to unit couples corresponding to the moment release at the hinge in beam *CD* is also shown in the same figure. The bending moments can be computed by considering the free bodies of the members of the frame. It will be noted that for this released structure, the internal forces are propagated to a large portion of the frame. On the other hand, if the released structure shown in Fig. 10.2(e) is adopted, the development of internal forces becomes a localised phenomenon in which the members of only one cell carry internal forces. The bending-moment diagram due to unit couples at the hinge in column *CE* shown in Fig. 10.2(e) can be drawn by considering the free bodies of the members of the frame. Hence, it may be concluded that among the two released structures shown in Fig. 10.2(d) and (e), the one shown in Fig. 10.2(e) with hinges in the columns only would be preferable because it would lead to a localised distribution of internal forces and consequently to lesser computational effort.

10.3.2 Accuracy

By the principle of superposition, bending moment M at any point of a statically indeterminate structure is the sum of static moment M_S in the released structure due to the applied loads and bending moment M_R due to the redundants.

$$M = M_S + M_R \tag{10.1}$$

Static moment M_S can be calculated accurately by statics. Hence the inaccuracy in the computation of bending moment M arises mostly from the errors in the evaluation of bending moment M_R. This is so because bending moment M_R depends upon the redundants whose evaluation involves the development of the flexibility matrix and its inversion. Both of these are liable to computational errors. It follows that in order to reduce computational errors, it is desirable to select the released structure in such a manner that bending moment M_R is negligible. In other words, it is desirable for accuracy to choose the released structure such that static moment M_S, which is not prone to computational errors, is as close as possible to the bending moment in the statically indeterminate structure. To achieve this objective, the releases should be selected in such a manner that the released structure has approximately the same stiffness as the actual structure and consequently the deflected shape of the released structure is as close as possible to that of the actual structure. In short, the released structure should be so chosen as to minimize the redundants.

In general, the members of any skeletal structure may carry one or more of the four types of the internal forces, viz., axial force, shear force, bending moment and twisting moment. The released structure can be obtained by releasing one or more of these internal forces. From the discussion of the previous paragraph, it is evident that the releases should be chosen in such a manner that the forces thus released are insignificant. For instance, when a released structure is obtained by using moment releases, the internal hinges should be provided at the inflexion points. However, as the exact positions of inflexion points are not known, the internal hinges should be provided at the probable locations of the inflexion points. Consider, for instance, a rigid-jointed building frame subjected to lateral loads. In this case it is known that the inflexion points are located close to the midpoints of all the beams and columns. It would, therefore, appear desirable to insert internal hinges at the midpoints of the members. If the frame shown in Fig 10.2(a) is subjected to lateral loads, the released structure shown in Fig. 10.2(d) should be chosen. It may, however, be noted that the released structure of Fig. 10.2(d) does not lead to a localised phenomenon. Hence, for localised phenomenon it may become necessary to choose the released structure shown in Fig. 10.2(e). In this released structure, the hinges in the right columns have been provided at one-third points. The accuracy of computations will be reduced if the two hinges are shifted away from each other towards the ends of the column. On the other hand, the two hinges cannot be brought too close to the midpoint of the column because it leads to instability.

When the released structure is obtained by introducing cuts, their positions should be selected so that the released forces are insignificant. In a plane frame three internal forces, viz., an axial force, a shear force and a bending moment, are released at each cut. Hence, it is not possible to find locations for the cuts at which all the three internal forces are small. Therefore, the cuts may normally be introduced at the midpoints of the members for simplicity. In certain instances the central locations for the cuts may also be suitable for accuracy. For instance, in a building frame subjected to lateral loads, the axial forces in the members are constant all over the lengths and the shear forces are nearly constant. Thus for axial force and shear force releases there are no preferred locations. For bending moment releases the central locations are evidently the best because the inflexion points are located close to the midpoints.

The foregoing discussion shows that, in general, the moment releases, by insertion of hinges, are preferable to cuts or total separations. When the released structure is obtained by inserting hinges, only one internal force, viz., a bending moment, is released at each hinge. It is, therefore, possible to manoeuvre the position of the internal hinges so that all the released forces are insignificant. Evidently, this objective is achieved by placing the internal hinges at the probable locations of the inflexion points. On the other hand, three internal forces are released simultaneously at each cut, thereby making it impossible to find locations for the cuts at which all the three internal forces are the least. Hence, the objective of minimising the released forces cannot be served effectively if the released structure is obtained by the introduction of cuts. It is for this reason that while it is possible to make the stiffness of the released structure obtained by the insertion of hinges approximately the same as that of the actual structure, it is not possible to do so when the released structure is obtained by the introduction of cuts.

The above discussion on the choice of releases for maximum accuracy has been devoted to rigid-jointed frames. In the case of pin-jointed frames, the bending moment and shear force releases are irrelevant because the members of these frames carry only axial forces. To achieve greater accuracy it is desirable to choose the axial force releases which are relatively small. For instance, a greater accuracy may be obtained by choosing the web members which carry relatively smaller forces as redundants instead of the top and bottom boom members of a lattice girder. Similarly, when the external reaction components are chosen as the redundants, it would seem desirable for accuracy to release those which are relatively small in magnitude.

In the foregoing discussion, the released structures which reduce the redundants have been recommended because the evaluation of the redundants is liable to error. The redundants are computed from a set of simultaneous equations derived from the compatibility conditions. Unless these simultaneous equations are well conditioned, accurate determination of the redundants may become difficult. As the coefficients of the simultaneous equations are the elements of the flexibility matrix, it is evident that for the sake of accuracy the flexibility matrix must be well conditioned. If the matrix is not well conditioned, its inversion, which is equivalent to solving the corresponding set of simultaneous equations, is prone to significant errors. As discussed in Sec. 3.7, a common test to see whether a matrix is well conditioned or ill conditioned is to evaluate the determinant of the normalized matrix. The conditioning of the matrix deteriorates as the determinant of the normalized matrix decreases.

10.3.3 Simplicity

Simplicity of the released structure is a consideration to be kept in view while choosing the redundants. Frequently, the simplest released structure may not lead to minimum computational effort and maximum accuracy. For instance, the released structure obtained by introducing cuts at the midpoints of beams of a building frame may appear to be simplest but it neither leads to a localised phenomenon nor minimises the redundants. Hence, cuts in the columns which lead to a localised phenomenon, Fig. 10.2(c), may be preferred. Regarding the positions of the cuts in the columns it may be noted that a central location tends to minimise the released moment whereas a cut near the end of the column simplifies the bending-moment diagram. Although the released structure obtained by inserting hinges in the columns is not the simplest, it generally leads to minimum computational effort and maximum accuracy.

10.4 RESTRAINED STRUCTURE

Unlike a released structure, which has to be chosen from the large number of possible alternatives, there is only one restrained structure because it is obtained by preventing all the independent displacement components in the structure. Further, it may be noted that a localised phenomenon is automatically ensured in the case of a restrained structure. This is so because in developing the stiffness matrix only one displacement is permitted at a time, all other displacements being prevented. Consequently, the effect of any single displacement component is of localised nature. If the coordinates assigned to the displacement components are numbered appropriately, the stiffness matrix becomes a banded matrix. Such a matrix is well conditioned and consequently its inversion can be carried out precisely.

Figure 10.3 shows a continuous beam having eight spans. Coordinates 1 to 9 have been assigned to the rotations at the nine supports of the continuous beam. When a unit displacement is given at any one of the coordinates, the effect of the displacement is felt in the adjacent spans only. For example, if a unit displacement is given at coordinate 5, bending moments are produced in spans DE and EF only. The bending-moment diagram due to a unit displacement at coordinate 5 is shown in Fig. 10.3(b). It follows that when a unit displacement is given at coordinate 5, forces are produced at coordinates 4, 5 and 6 only. In general, when a unit displacement is given at coordinate j, forces are generated at coordinates $(j-1)$, j and $(j+1)$. As the jth column of the stiffness matrix is generated by giving a unit displacemnt at coordinate j, it is evident that the $(j-1)$th, jth and $(j+1)$th elements are non-zero and the rest of the elements are zero. The same applies to all other columns except the first and the last columns

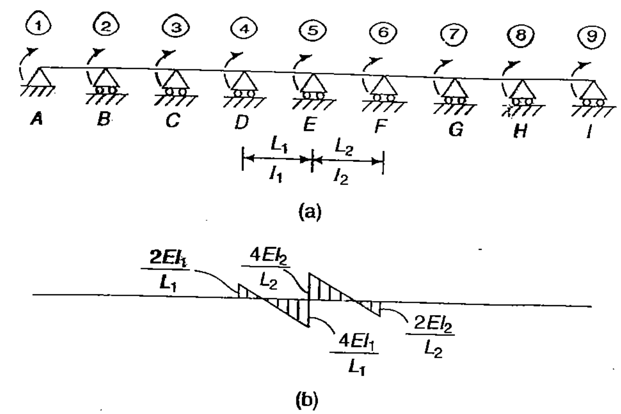


Fig. 10.3

in which only two elements are non-zero. As a result, the stiffness matrix for the continuous beam becomes tridiagonal.

Figure 10.4 shows a rigid-jointed building frame having three bays and five storeys. The independent displacement components are the rotations at all the joints and the horizontal displacement (sway) at each floor level. Ignoring changes in the lengths of the beams, all the joints at the same level have same horizontal displacement. Thus only a single coordinate is required for the horizontal displacement at each floor level. The remaining coordinates correspond to the rotations at the joints. As the changes in the lengths of the columns are ignored, no coordinate in the vertical direction need be taken. When a unit displacement is given at any one of the coordinates corresponding to a joint rotation, bending moments are produced in those members only which meet at the joint. For instance, Fig. 10.4(b) shows the bending-moment diagram due to a unit displacement at coordinate 12 corresponding to the rotation at an interior joint. Consequently, forces are generated at coordinates 7, 8, 11, 12, 13, 14, 17 and 18. It follows that in the 12th column of the stiffness matrix eight elements are non-zero and the remaining seventeen elements are zero. The position is similar when a unit rotation is given at other joints of the frame. Figure 10.4(c) shows the bending-moment diagram on account of a unit displacement at coordinate 13 which corresponds to the sway at the third floor level. It may be noted that only those columns which belong to the third and fourth storeys are subjected to bending moment. All the beams and the remaining columns do not carry any bending moment. Consequently, forces are generated at coordinates 6 to 20. The 13th column of the stiffness matrix, therefore, comprises 15 non-zero elements and 10 zero elements. It may be noted that in each column the non-zero elements are located in the immediate vicinity on either side of the main diagonal. Consequently, the stiffness matrix is strongly diagonal and well conditioned. It may be further noted that in the case of a large structure, the number of non-zero elements is small compared to the zero elements. Consequently, the stiffness matrix for a large structure is a banded matrix whose band-width is small compared to the order of the matrix.

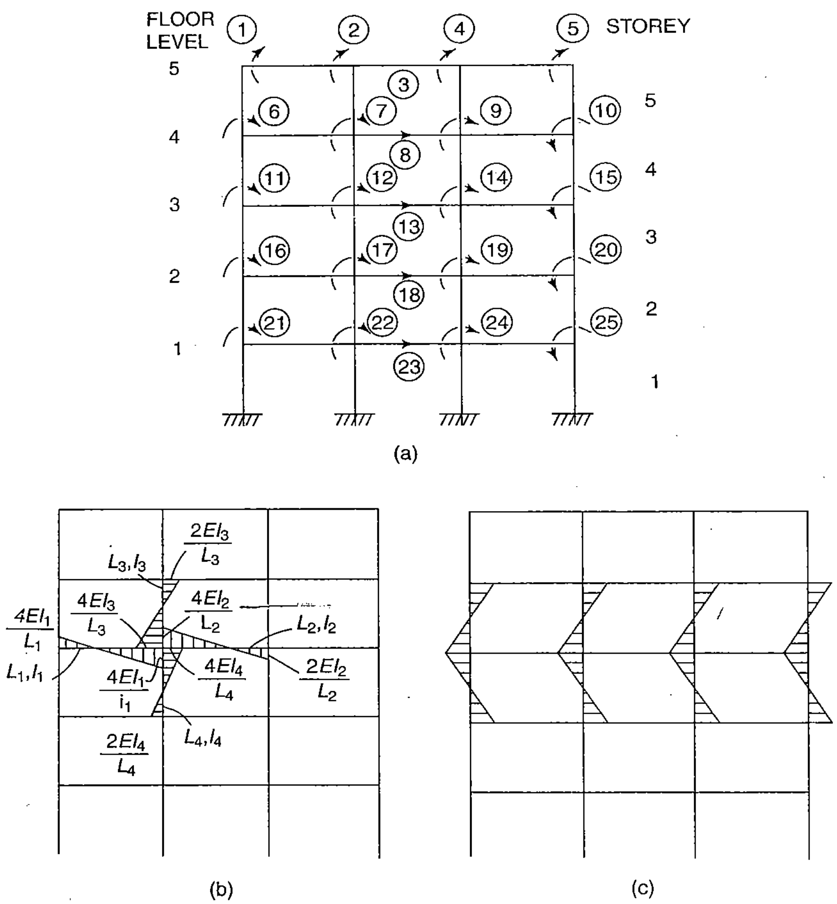


Fig. 10.4

Consider next, the pin-jointed plane frame shown in Fig. 10.5. The frame has 4 bays and 10 stages. The total number of joints is 50. As each joint has two displacement components, the degree of freedom of the structure is 100. Coordinates 1 to 100 may be assigned to the independent displacement components as shown in the figure. When a unit displacement is given at any one of the coordinates corresponding either to a horizontal or to a vertical

displacement at the joint, only those members which meet at the joint are subjected to internal forces. Consequently, forces are generated only at those joints which are directly connected to the joint under consideration. It follows that when a unit displacement is given at any coordinate, forces are generated at the neighbouring coordinates only. If the coordinates are numbered properly, the stiffness matrix becomes a banded matrix with the non-zero elements located in the vicinity of the main diagonal. When a unit displacement is given at coordinate 1, forces are generated at coordinates 1, 2, 3, 4, 11, 12, 13 and

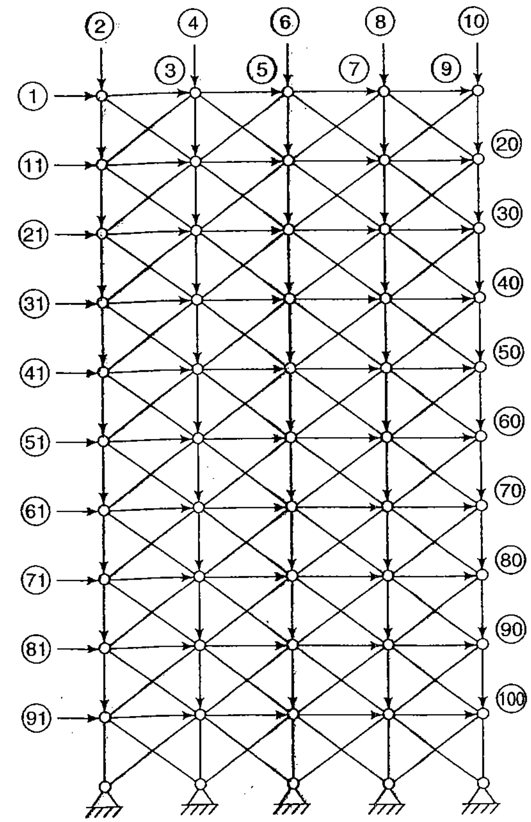


Fig. 10.5

14. Hence, in the first column of the stiffness matrix, 8 elements are non-zero and the remaining 92 elements are zero. The highest number of coordinate at which force is generated is 14, the band-width is, therefore, 14. When a unit

displacement is given at any interior joint, for instance at coordinate 25, forces are produced at coordinates 13 to 18, 23 to 28 and 33 to 38. The lowest and the highest coordinates at which forces are generated are 13 and 38 respectively. Thus the non-zero elements are spread up to the 12th element above the diagonal and 13th element below the diagonal. Consequently, the half band-width = 13 + 1 = 14.

It is evident from the foregoing examples that the independent displacement components of a structure are self-evident. Consequently, the selection of the restrained structure is automatic. Further, the localised phenomenon is automatically ensured because in a totally restrained structure the influence of any local disturbance is only felt in a small part of the structure. If the coordinates are numbered properly, the stiffness matrix becomes a strongly diagonal banded matrix. As such a matrix is well conditioned, its inversion can be carried out with precision. Besides, a banded matrix requires a smaller storage space in a digital computer. The numbering of coordinates which leads to a banded matrix is discussed in the next section.

10.5 NUMBERING OF COORDINATES

It has been seen in the preceding section that a localised phenomenon is automatically ensured in the case of a restrained structure. When a unit displacement is given at any joint of a restrained structure, forces are produced at the joints in the neighbourhood of the joint under consideration. Consequently, in each column of the stiffness matrix there are only a few non-zero elements. If the coordinates are numbered in a proper manner, the non-zero elements are located near the main diagonal. It follows that the stiffness matrix becomes a banded matrix if the numbering of coordinates follows the correct sequence. The objective in numbering the coordinates should be to make the band-width a minimum, so that the demand on the computer storage space is minimised. As the band-width is reduced, the number of elements to be stored in the computer is correspondingly reduced. The reduction of band-width is of particular importance for the analysis of a large structure which leads to a stiffness matrix too large for inversion by even a modern digital computer unless the demand on storage space is reduced. This is achieved by fully exploiting the fact that the stiffness matrix is banded and symmetrical. Hence, it is necessary to store the elements of only half the band-width instead of all the elements of the stiffness matrix.

10.5.1 Continuous Beams

In the case of continuous beams, the coordinates which correspond to the rotations at the supports should be numbered consecutively from one end of the beam to the other end. Figure 10.6 shows a continuous beam resting on n

simple supports. Coordinates 1, 2, ..., n have been assigned to the rotations at the supports. When a unit displacement is given at coordinate j, forces are produced at coordinates $(j-1)$, j and $(j+1)$. Consequently, in the jth column of the stiffness matrix, the $(j-1)$th, jth and $(j+1)$th elements are nonzero and the remaining elements are zero. It follows that the stiffness matrix is tridiagonal. It is also evident that the minimum band-width in the case of continuous beams is 3.

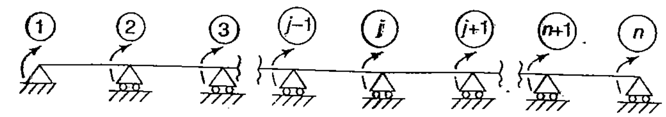


Fig. 10.6

10.5.2 Pin-jointed Frames

Consider the pin-jointed plane frame shown in Fig. 10.7. The frame has B bays and S stages. Each panel of the frame is cross-braced. The total number of joints j, excluding the supports, is $S(B + 1)$. As each joint has two

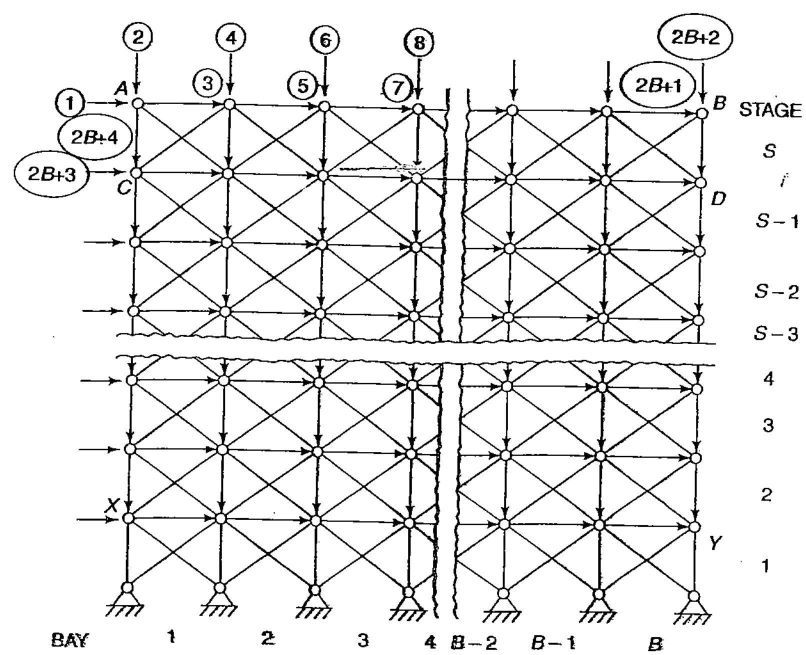


Fig. 10.7

displacement components, the degree of freedom of the structure is $2j = 2S(B + 1)$. Assigning a coordinate to each displacement component, the total number of coordinates required and consequently the order of the stiffness matrix is $2S(B+1)$. Consider first, the case in which the number of joints in the horizontal direction is less than the number of joints in the vertical direction, i.e., $(B + 1) < S$. In this case the minimum band-width is obtained by starting the numbering of coordinates from joint A and proceeding to the right to joint B. Next, the coordinates at the joints in line CD may be numbered in a similar manner. The numbering of coordinates may be continued until the coordinates at the joints in the last line XY have been numbered. When a unit displacement is given at coordinate 1, the highest number of coordinate at which force is produced is $(2B + 6)$. Therefore, the half band-width is $(2B + 6)$. Same result is obtained by considering any intermediate joint. Consider, for example, the frame shown in Fig. 10.5 having 4 bays and 10 stages. As $(B + 1)$ is less than S in this case, the coordinates for minimum band-width should be numbered as shown in the figure. In the first column of the stiffness matrix the last non-zero element is 14th. Consequently, the half band-width is 14. It may be noted that this system of numbering produces the minimum band-width. As the order of the stiffness matrix is 100, the total number of elements is $100 \times 100 = 10{,}000$. Recognising that the stiffness matrix is a symmetrical banded matrix, the number of elements to be stored in the computer is $14 \times 100 = 1400$ instead of 10,000. Thus the demand on computer storage space is considerably reduced. It may be noted that if the coordinates are numbered successively along vertical lines instead of horizontal lines, the half band-width would be 24. Consequently, the number of elements to be stored in the computer would be $24 \times 100 = 2400$. If the number of joints in a vertical line is less than the number of joints in a horizontal line, the coordinates should be numbered consecutively along vertical lines to obtain the minimum band-width. Referring to Fig. 10.7, the half band-width in this case is $(2S + 4)$. Consider, for example, a pin-jointed frame with 9 bays and 6 stages. The structure has 60 joints and 120 independent displacement components. Hence, the number of elements in the stiffness matrix is $120 \times 120 = 14{,}400$. The half band-width is $2 \times 6 + 4 = 16$. Hence, the number of elements to be stored in the computer is $16 \times 120 = 1920$. In this case if the coordinates are numbered consecutively along the horizontal lines, the half band-width will be 24 requiring the computer to store $24 \times 120 = 2880$ elements.

The system of numbering of coordinates for minimum band-width, discussed above for pin-jointed plane frames, can be extended easily for the case of pin-jointed space frames. Consider a pin-jointed space frame with B_x bays in x direction, B_y bays in y direction and S stages in z direction. All the panels of the plane frames normal to the x-, y- and z-axes are cross-braced. The space frame can be considered as an assembly of a series of plane frames parallel to each of the three coordinate planes. The numbering of coordinates for the minimum band-width should be started by taking up the plane frame which has the minimum number of joints. The numbering of coordinates of this frame may be completed in the same manner as in the case of a plane frame except that three orthogonal coordinates have to be assigned to each joint because every joint of a pin-jointed space frame has three independent displacement components. After completing the numbering of coordinates of the first frame, the other frames parallel to the previous frame may be taken up successively until the numbering of the entire space frame has been completed. The number of joints in the frames normal to the x-, y- and z-axes are $S(B_y + 1)$, $S(B_x + 1)$ and $(B_x + 1)(B_y + 1)$ respectively. The lowest of these three terms gives the number of joints in the frame which should be taken up first for the numbering of coordinates. Consider, for example, a pin-jointed space frame with $B_x = 3$, $B_y = 4$ and $S = 10$. The total number of joints in the frame is equal to $4 \times 5 \times 10 = 200$. As each joint has three independent displacement components, the order of stiffness matrix is $200 \times 3 = 600$. The number of joints in the frames normal to the x-, y- and z-axes are 50, 40 and 20 respectively. As the horizontal frame has the minimum number of joints, the numbering of coordinates should commence by taking up either the topmost or the lowermost horizontal frame first. Figure 10.8(a) shows the numbering of coordinates for the top frame and Fig. 10.8(b) for the second frame from top. The numbering may continue in a similar manner for other frames until the entire space frame has been numbered.

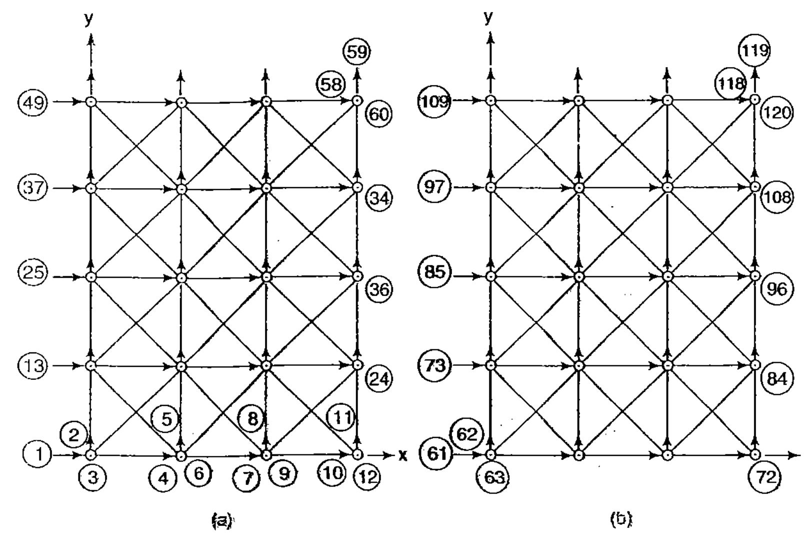


Fig. 10.8

To evaluate the half band-width, note that when a unit displacement is given at coordinate 1, the highest coordinate at which a force is generated is 75. Consequently, the half band-width is 75. The number of elements to be stored in the computer is $75 \times 600 = 45{,}000$ out of the total number of $600 \times 600 =$ 3,60,000 elements in the stiffness matrix.

10.5.3 Rigid-jointed Frames

Consider the rigid-jointed plane frame shown in Fig. 10.9. The frame has 3 bays and 5 storeys. The total number of joints, excluding the supports, is $4 \times 5 = 20$. If the axial deformations of the members are ignored, none of the joints can move in the vertical direction. Also, all the joints at the same level have the same horizontal displacement. Consequently, there are 20 rotations corresponding to the 20 joints and 5 horizontal displacements corresponding to the five floors. The degree of freedom of the structure is 25. For minimum band-width, the system for the numbering of coordinates should be as shown in the figure.

To compute the half band-width, note that when a unit displacement is given at coordinate 1, the highest coordinate at which the force is produced is 8. When a unit displacement is given at coordinate 8, the lowest and highest coordinates at which forces are generated are 1 and 15 respectively. Thus the band-width in the stiffness matrix is 15. It may be verified that the system of numbering shown in Fig. 10.9 produces the minimum band-width. In the system of numbering adopted in Fig. 10.9, the numbering starts from the left top corner and proceeds to the right until the top line is completed. Coordinates 1 to 5 have been assigned to the displacements at the topmost level, with the central number 3 of the series being assigned to the horizontal displacement

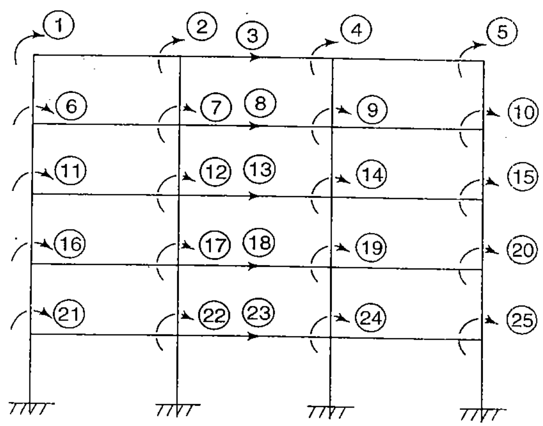


Fig. 10.9

(sway). Similarly, coordinates 6 to 10 have been assigned to the displacements in the next horizontal line and so on until the entire frame has been numbered. It may be vertified that if the coordinate number assigned to the horizontal displacement is not the central number, the band-width is increased. For example, if coordinates are numbered as shown in Fig. 10.10(a), the half band-width is 10. It may also be noted that if the coordinates are numbered along vertical lines as shown in Fig. 10.10(b), the half band-width is 12. From the

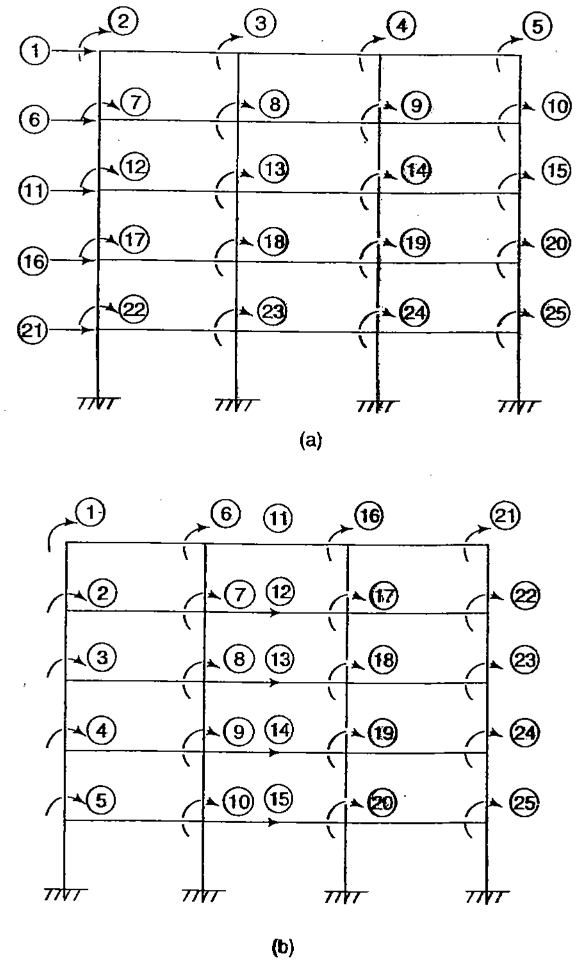


Fig. 10.10

comparison of the band-widths it may be concluded that the system of numbering shown in Fig. 10.9 is the best, because it gives the minimum band-width. The coordinate number assigned to the horizontal displacement can be made exactly central if the number of bays is odd. If the number of bays is even, the coordinate number for sway at any level should be either immediately before or after the coordinate number assigned to the rotation of the central joint. For instance, the system of numbering for the rigid-jointed plane frame shown in Fig. 10.11 for the minimum band-width is shown in the same figure. The minimum half band-width in this case is 13. From the foregoing discussion it is clear that the following two cases have to be considered for a rigid-jointed plane frame having B bays and S storeys:

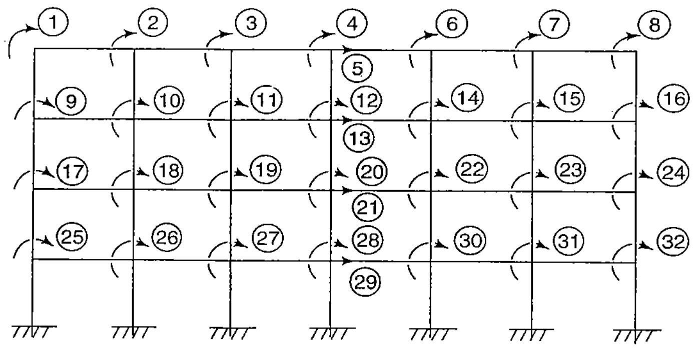


Fig. 10.11

(a) *B Odd:* In this case the number of joints at each level are $(B + 1)$ which is an even number. Hence, the coordinate number to be assigned to the horizontal displacement at the top level should be $\frac{(B+1)}{2} + 1$. It may be verified that the half band-width in this case is $(1.5B+3.5)$.

(b) *B Even:* In this case the number of joints at each level are $(B + 1)$ which is an odd number. Hence, the coordinate number to be assigned to the horizontal displacement at the top level should be $\left(\frac{B}{2} + 1\right)$ or $\left(\frac{B}{2} + 2\right)$. It may be verified that the half band-width in this case is $(1.5B + 4)$.

It may be noted that irrespective of the number of bays and storeys, the numbering of coordinates in a rigid-jointed plane frame for minimum band-width should always be carried out along horizontal lines. The effect of a rotational displacement at a joint is limited only to the members meeting at the joint. On the other hand, the effect of sway at any floor level is propagated to

all the columns belonging to the storeys immediately below and above the floor-level under consideration. As the effect of the sway extends from the extreme left column to the extreme right column, it is evident that the band-width will be much wider if the coordinates are numbered along vertical lines. This point is illustrated by the example shown in Fig. 10.10(b).

The system of numbering of coordinates for minimum band-width discussed above for rigid-jointed plane frames can be extended for the case of rigid-jointed space frames. Consider a rigid-jointed space frame with B_x bays in x direction, B_y bays in y direction and S storeys in z direction. The space frame can be considered as an assembly of plane frames parallel to each of the three coordinate planes. The total number of joints, excluding the supports, is $S(B_x + 1)(B_y + 1)$. As each joint can rotate about the three coordinate axes, there are three rotations at each joint. Therefore, the total joint rotations in the frame are $3S(B_x + 1)(B_y + 1)$. In addition, there is a horizontal displacement (sway) at each floor level in each vertical frame. As there are $(B_x + B_y + 2)$ vertical frames, the total horizontal displacements are $S(B_x + B_y + 2)$. Hence, the degree of freedom of the rigid-jointed space frame may be taken as

$$D_k = 3S(B_x + 1)(B_y + 1) + S(B_x + B_y + 2)$$
$$= S(3B_xB_y + 4B_x + 4B_y + 5) \tag{10.2}$$

It has been seen in the discussion of rigid-jointed plane frames that the numbering of coordinates for minimum band-width should be carried out in the direction of the sway. As in the case of a rigid-jointed space frame the sway occurs along both the horizontal axes. It follows that the numbering of coordinates for minimum band-width should be started by taking up the top horizontal frame and numbering the displacements of all the joints in this frame. The numbering should proceed downward in an identical manner by taking up horizontal frames successively from top to bottom. In numbering the top horizontal frame, it may be noted that if the coordinates are numbered along the x-axis, the sway along the y-axis governs the band-width; whereas if the coordinates are numbered along the y-axis, the sway along the x-axis governs the band-width. Both the alternatives should be explored to determine which one produces a smaller band-width. This alternative should be adopted for the numbering of coordinates for minimum band-width of the stiffness matrix.

Consider a rigid-jointed space frame with 2 bays in x direction, 3 bays in y direction and 10 storeys in the z direction. As the number of joints in each horizontal plane is 12, the total number of joints is $10 \times 12 = 120$. As each joint has 3 rotations, the total joint rotations are $120 \times 3 = 360$. Besides, there are $(2 + 1 + 3 + 1) = 7$ sways at each floor level. The total number of sways is $7 \times 10 = 70$. The degree of freedom of the structure may be taken as $360 + 70 = 430$. Consequently, the stiffness matrix whose order is 430 has $430 \times 430 = 1, 84, 900$ elements.

Figure 10.12 shows the numbering of the coordinates in the top horizontal frame for minimum band-width. The rotations in the x-z and y-z planes have been indicated by double-headed arrows. It may be verified that the half band-

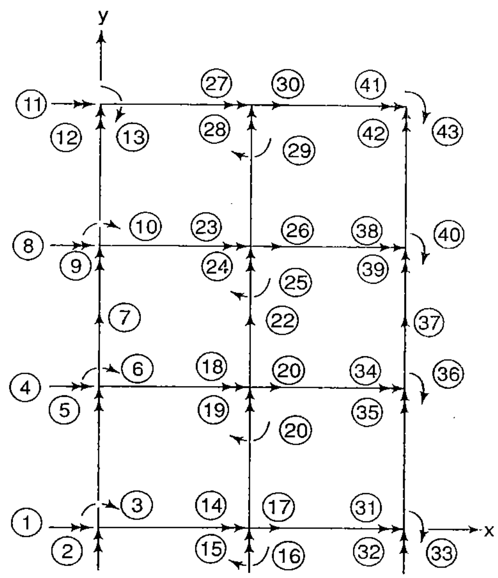


Fig. 10.12

width for the proposed system of numbering is 59. The number of elements to be stored in the computer is $59 \times 430 = 25370$ out of the total of 1,84,900 elements in the stiffness matrix. It may be further verified that the numbering system shown in Fig. 10.12 produces the minimum band-width. The band-width will be more if the coordinates are numbered in any other manner. In particular, if the coordinates are numbered along the x-axis, in which case sway along the y-axis governs the band-width, the half band-width is found to be 62.

10.6 MATRIX SIZE

The size of the matrix is an important consideration in the choice between the two methods. It has been noted that in the force method the order of the flexibility matrix is equal to the degree of static indeterminacy. On the other hand, the order of the stiffness matrix is equal to the degree of kinematic indeterminacy. It follows that in a structure with a high degree of static indeterminacy and a low degree of kinematic indeterminacy, the use of the

force method will lead to a large flexibility matrix. On the other hand, if the displacement method is adopted for the analysis of the structure, the resulting stiffness matrix will be small in size. Purely from the point of view of the size of the matrix it would appear that the force method is preferable for a structure in which the degree of static indeterminacy is lower than the degree of kinematic indeterminacy. On the other hand, the displacement method would appear to be preferable if the degree of kinematic indeterminacy is lower than the degree of static indeterminacy. The relative values of the two types of indeterminacies, generally, depend upon the type of structure.

10.6.1 Continuous Beam

Consider a continuous beam with n spans. The degrees of static indeterminacy are $(n + 1)$, n and $(n - 1)$ depending upon whether both, one or none of the end supports are fixed supports. Similarly, the degrees of kinematic indeterminacy are $(n - 1)$, n and $(n + 1)$ depending upon whether both, one or none of the end supports are fixed supports. It follows that the order of the flexibility matrix is greater than that of the stiffness matrix in the case of a fixed continuous beam. On the other hand, for a continuous beam with simple supports, the order of the stiffness matrix is greater than that of the flexibility matrix. It may, however, be noted that for a continuous beam with several spans, the difference in the order of the two matrices is only marginal. Consequently, either of the two methods appear to be suitable purely from the point of view of the size of the matrix.

10.6.2 Rigid-jointed Frames

The degree of static indeterminacy of rigid-jointed frames is, generally, higher than the degree of kinematic indeterminacy. Consider, for example, a rigid-jointed plane frame having B bays and S storeys. If the column bases are assumed to be fixed, the degree of static indeterminacy is $3BS$. From Eq. (1.22), the degree of kinematic indeterminacy is $S(B + 2)$. Comparing the degree of static and kinematic indeterminacies, it may be noted that they are equal for B equal to 1. For frames with multiple bays, the degree of static indeterminacy is higher than the degree of kinematic indeterminacy. For instance, for a building frame with 10 bays and 5 storeys, the degrees of static and kinematic indeterminacies are 150 and 60 respectively. Consequently, the size of the flexibility matrix in the force method is much bigger than the size of the stiffness matrix in the displacement method.

In the case of rigid-jointed space frames also, the degree of static indeterminacy is, generally, higher than the degree of kinematic indeterminacy. Consider a building frame having S storeys and B_x and B_y bays in the x and y directions respectively. Assuming that the column bases are fixed, the structure may be made statically determinate by making cuts at the centres of all the beams. As the total number of beams is $S(2B_xB_y + B_x + B_y)$ and six internal forces are released at each cut; the degree of static indeterminacy is $6S(2B_xB_y + B_x + B_y)$. The degree of kinematic indeterminacy as given by Eq. (10.2) is $S(3B_xB_y + 4B_x + 4B_y + 5)$. Comparing the two types of indeterminacies, it is evident that the degree of static indeterminacy is always higher than the degree of kinematic indeterminacy. For a single storey rigid frame with only one bay in each of the two directions, the degrees of static and kinematic indeterminacies are 24 and 16 respectively. As the numbers of storeys and bays increase, the degree of static indeterminacy increases at a much faster rate as compared to the degree of kinematic indeterminacy. For a ten-storeyed building frame with 3 bays in the x direction and 4 bays in the y direction, the degrees of static and kinematic indeterminacies are 1860 and 690 respectively.

From the foregoing discussion, it is evident that the degree of static indeterminacy is, generally, higher than the degree of kinematic indeterminacy. Consequently, the displacement method leads to a smaller matrix as compared to the force method. There can, however, be exceptions to this general conclusion, particularly in the case of small frames. For example, the degree of static indeterminacy of a rigid-jointed portal frame with hinged supports is only 1 whereas the degree of kinematic indeterminacy is 5.

10.6.3 Pin-jointed Frames

Unlike rigid-jointed frames, the pin-jointed frames, generally, have a higher degree of kinematic indeterminacy as compared to the degree of static indeterminacy. Consider, for example, a pin-jointed plane frame having B bays and S stages as shown in Fig. 10.7. If the supports are assumed to be hinged, the degree of kinematic indeterminacy is $2S(B + 1)$. From Eq. (1.16), the degree of static indeterminacy is $(m + r - 2j)$. Substituting $m = S(B + 1) + BS + 2BS = 4BS + S$, $r = 2(B + 1)$ and $j = (B + 1)(S + 1)$ into Eq. (1.16), the degree of static indeterminacy is $(2BS - S)$. Comparing the degrees of static and kinematic indeterminacies, it may be noted that the former is always lower than the latter. For example, in the case of a pin-jointed plane frame having 4 bays and 10 stages as shown in Fig. 10.5, the degrees of static and kinematic indeterminacies are 70 and 100 respectively. If the frame has only 1 bay and 10 stages, the degrees of static and kinematic indeterminacies are 10 and 40 respectively. For the lattice girder shown in Fig. 10.13, the degrees of static and kinematic indeterminacies are 10 and 35 respectively.

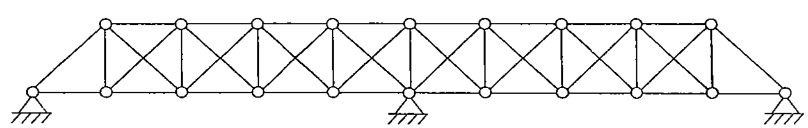


Fig. 10.13

In the case of pin-jointed space frames also, the degree of kinematic indeterminacy is, generally, higher than the degree of static indeterminacy. Consider a pin-jointed space frame having S stages and B_x and B_y bays in x and y directions respectively. The column bases are assumed to be hinged and every panel is assumed to be cross-braced. The number of joints, excluding the column bases, $j = S(B_x + 1)(B_y + 1)$. As each joint has 3 displacement components, the degree of kinematic indeterminacy is $3j = 3S(B_x + 1)(B_y + 1)$. From Eq. (1.17), the degree of static indeterminacy is $D_s = (m + r - 3j)$. Substituting $m = S[9B_xB_y + 4B_x + 4B_y + 1]$, $r = 3(B_x + 1)(B_y + 1)$ and $j = (B_x + 1)(B_y + 1)(S + 1)$ into Eq. (1.17), the degree of static indeterminacy is $D_s = S[6B_x B_y + B_x + B_y - 2]$. Comparing the degrees of kinematic and static indeterminacies, it is noted that the former is higher than the latter if $(2B_x + 2B_y + 5) > 3B_xB_y$. For example, in the case of a radio tower with $B_x = B_y = 1$ and $S = 10$, the degrees of kinematic and static indeterminacies are 120 and 60 respectively.

From the foregoing discussion it is evident that the degree of kinematic indeterminacy is, generally, higher than the degree of static indeterminacy. Consequently, the force method leads to a smaller matrix as compared to the displacement method. There can, however, be exceptions to this general conclusion. For instance, in a pin-jointed space frame, discussed in the previous paragraph, if $B_x = B_y = 3$ and $S = 10$, the degrees of kinematic and static indeterminacies are 480 and 580 respectively.

10.7 CONCLUSIONS

In the preceding sections of this chapter, the relative merits and demerits of the two main methods of matrix analysis of structures, viz., the force method and the displacement method for different types of structures have been discussed. It has been noted that each method may have certain advantages over the other method for certain types of problems. Hence, it is not possible to say that either one of the two methods is advantageous under all situations. The following are the main points in a comparative study of the two methods of matrix analysis of structures:

(i) The suitability of the matrix approach increases as the structure becomes larger. For elementary problems, the use of the matrix approach may appear to be awkward and more time consuming. For a large structure with a high degree of indeterminacy, the matrix approach provides a systematic and generalized solution particularly amenable to computer application. It follows that the analysis of a large structure by the matrix approach will almost invariably be carried out with the help of a digital computer. From this point of view, the displacement method is preferable for the analysis of large structures because all the steps of this method can be executed by a computer more conveniently as compared to those of the force method.

(ii) The selection of the released structure in the force method is not a simple problem. Usually several alternatives are available, each one having its own advantages. In general, it is difficult to select a released structure which meets the triple requirements of minimum computational effort, accuracy and simplicity. On the other hand, the restrained structure in the displacement method is self-evident and follows automatically as soon as the independent displacement components are recognised. In this regard, the superiority of the displacement method is, therefore, obvious.

(iii) In general, the development of the flexibility matrix in the force method is more difficult and time consuming as compared to the development of the stiffness matrix in the displacement method. The elements of the flexibility matrix represent the displacement components of the released structure. The computation of these displacements in a large structure requires extensive calculations unless the released structure has been chosen to ensure a localised phenomenon. On the other hand, the elements of the stiffness matrix, which represent the forces in the restrained structure due to a unit displacement, can be computed easily with the help of a few standard formulae. Also, a predominant number of elements of the stiffness matrix are zero because a localized phenomenon is automatically ensured in the restrained structure.

(iv) The flexibility matrix in the force method is not necessarily well conditioned unless great care is exercised in the selection of the released structure and numbering of the coordinates. Consequently, the inversion of the flexibility matrix may lead to significant inaccuracy. On the other hand, the stiffness matrix in the displacement method is, generally, strongly diagonal and therefore well conditioned, provided the coordinates are numbered properly. Guidelines for the proper numbering of coordinates in a restrained structure have been discussed in Sec. 10.5.

(v) The capacity of the computer may become a constraint in the analysis of a large structure. Hence, the size of the matrix may become an important consideration in the choice between the two methods. In the case of rigid-jointed frames, the degree of static indeterminacy is, generally, higher than the degree of kinematic indeterminacy. Hence, for these frames, the displacement method may appear to be preferable. On the other hand, the degree of kinematic indeterminacy is, generally, higher than the degree of static indeterminacy in the case of pin-jointed frames. Consequently, the force method may appear to be preferable for the analysis of these frames. In instances where the computer capacity becomes a constraint, advantage should be taken of the symmetry of the matrix and the bandwidth should be minimised through proper numbering of coordinates. In general, this is more easily accomplished in the case of the stiffness matrix.

(vi) The displacement method provides a complete analysis of the structure including displacements which may sometimes govern the design. In the case of the force method, the displacements are generally not computed. Separate computations are necessary for their evaluation.

Considering the points enumerated above, the suitability of the two methods for the different types of structures and under different situations may be summarised in the following table.

Table 10.2

S.No.	Point Under Consideration	Suitbale Method
1.	Type of computation	
	(a) By computer	Displacement method
	(b) By hand	
	(i) High degree of static indeterminacy and low degree of kinematic indeterminacy	Displacement method
	(ii) Low degree of static indeterminacy and high degree of kinematic indeterminacy	Force method
	(iii) Low degree of static and kinematic indeterminacies	Both methods
	(iv) High degree of static and kinematic indeterminacies	None
2.	Choice of released or restrained structure	Displacement method
3.	Development of the flexibility and stiffnes matrices and their inversion	
	(a) Computational effort	Displacement method
	(b) Accuracy	Displacement method
	(c) Simplicity	Displacement method
4.	Localised phenomenon, banded and well-conditioned matrix	Displacement method
5.	Size of matrix	
	(a) Continuous beams	Both
	(b) Rigid-jointed frames	Displacement method
	(c) Pin-jointed frames	Force method
6.	Completeness of analysis including displacements	Displacement method

PROBLEMS

10.1 Discuss critically the relative merits and demerits of the two main methods of matrix analysis for different types of structures.

10.2 Discuss the main considerations regarding the choice of the released structures for analysing (a) continuous beams, (b) rigid-jointed frames and (c) pin-jointed frames.

10.3 Discuss the relative conditioning of the flexibility and stiffness matrices and its effect on the accuracy of the solution.

10.4 (a) What decides the sizes of the flexibility and stiffness matrices for a structure?
(b) How important is the size of the matrix in deciding the suitability of the two methods?

10.5 Discuss why the released structure which minimises the magnitudes of the redundants generally leads to maximum accuracy.

11 TRANSFORMATION MATRICES—ELEMENT APPROACH

11.1 INTRODUCTION

In the discussion of the force method and displacement method in the preceding chapters, the flexibility and stiffness matrices have been developed by considering the structure as a whole. As an alternative approach, these matrices can be developed from the flexibilities and stiffnesses of the constituent elements through the use of certain matrices known as *transformation matrices*. Thus the matrices $[\delta]$ and $[\Delta_L]$ for the structural system can be developed from the flexibilities of the constituent elements by using *force-transformation matrice*. Similarly, stiffness matrix $[k]$ for the structural system can be developed by using the *displacement-transformation matrix*. This approach in which the matrices for the entire structure are obtained from the respective matrices for the constituent elements, is known as the *element* approach. This approach makes the procedure more formal and is, therefore, particularly sutiable for an automatic analysis by a digital computer.

11.2 FORCE METHOD

In the preceding chapters the structure as a whole was considered for the development of the flexibility matrix. The development of the flexibility matrix for the enitre structural system entails the computation of the displacements of the structure at all the coordinates. This is the unattractive part of the force method of analysis because the computation of system displacements by considering the structure as a whole is tedious and liable to computational errors. To overcome this difficulty, the force-transformation matrix may be used which permits the development of the flexibility matrix for the structural system from the flexibility of the constituents elements.

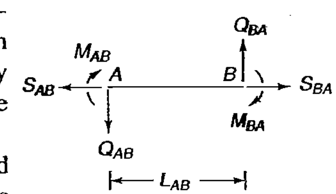


Fig. 11.1

In general, a member of a rigid-jointed plane frame carries an axial force, a shear force and a bending couple. Figure 11.1 shows a typical end-loaded member AB of a plane frame. For the equilibrium of the member it is evident that

$$S_{AB} = S_{BA}$$
$$Q_{AB} = Q_{BA}$$
$$Q_{AB} = \frac{M_{AB} + M_{BA}}{L_{AB}}$$

It follows that of the six end forces shown in Fig. 11.1, only three are independent. Generally, the axial force in the member and the two end couples are treated as the independent forces. Hence, in general, three coordinates known as *element coordinates*, are required for each member or element of a rigid-jointed plane frame. However, if the axial deformation of the member is ignored, only two coordinates are enough. Thus the total number of element coordinates required is equal to twice the number of constituent elements of the frame. Although other alternatives regarding the choice of the two coordinates for each element are available, it is generally found convenient to use the two end rotations as the element coordinates.

Consider the rigid-jointed plane frame shown in Fig. 11.2(a). The degree of static indeterminacy of the frame is six. Of the several possible released structures, the one shown in Fig. 11.2(b) has been obtained by removing the support at F and introducing a cut at joint D. Coordinates 1 to 6 assigned to the redundants thus released are known as *system coordinates*. These are shown in Fig. 11.2(b). For this released structure, twelve element coordinates are required because the frame has six members. The element coordinates 1^* to 12^* assigned to the end rotations of the members of the frame are shown in Fig. 11.2(c). The released structure shown in Fig. 11.2(d) has been obtained by removing the support at F and introducing a cut at some intermediate point G in beam BE. In this case portions BG and GE should be treated as separate elements. Consequently, the number of element coordinates increases by two. The element coordinates 1^* to 14^* are shown in Fig. 11.2(e). The released structure shown in Fig. 11.2(f) has been obtained by inserting hinges at both ends of columns AB and BC and at some intermediate point H and I in columns DE and EF. In this case portions DH, HE, EI and IF should be treated as separate elements. The element coordinates 1^* to 16^* are shown in Fig 11.2(g). From the foregoing examples it is evident that the number of element coordinates is equal to twice the number of the members if the released structure is obtained by giving releases only at the joints. If releases are given at intermediate points (other than joints), the number of element coordinates increases correspondingly.

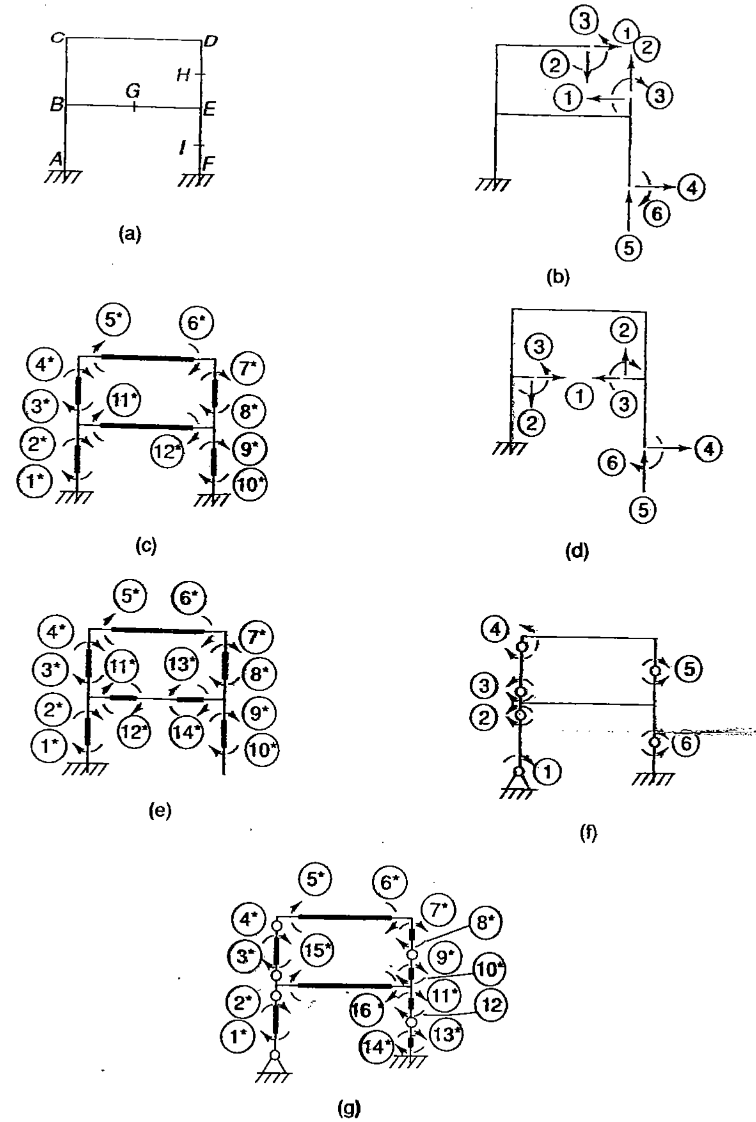


Fig. 11.2

In using the element approach for rigid-jointed frames, it is necessary to replace the applied loads by their statically *equivalent joint-loads*. For this purpose every element of the structure is considered as a fixed-ended member and the two ends of the element are considered as joints. The loads acting on the elements are replaced by the fixed-end forces imposed by the loads at two joints which the element is assumed to connect. Thus the equivalent joint loads for any element are equal and opposite to the fixed-end reactions. Consider, for example, an element AB of a rigid-jointed plane frame acted upon by a load of 200 kN as shown in Fig. 11.3(a). Ends A and B are considered to be joints. The free-body diagram of the element considered as a fixed-ended member, is shown in Fig. 11.3(b). The equivalent joint-loads shown in Fig. 11.3(c) are equal and opposite to the fixed-end reactions. It may be noted that the displacements at the joints of a structure remain unchanged when the actual loads are replaced by their equivalent joint-loads. Hence, the values of the chosen redundants remain unaffected when the actual loads are replaced by their equivalent joint-loads. After the redundants have been evaluated, the net bending-moment diagram for the structure may be obtained as usual by combining the bending-moment diagram due to the redundants with static bending-moment diagram on account of the actual applied loads.

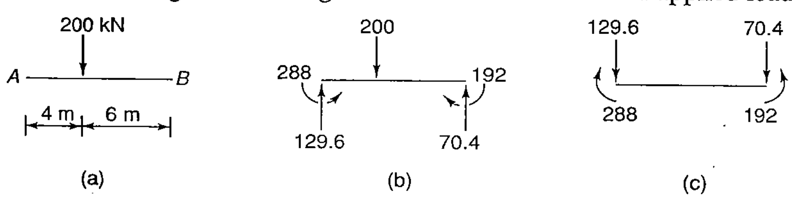


Fig. 11.3

Next, consider the pin-jointed plane frame shown in Fig. 11.4(a). The degree of static indeterminacy of the frame is three. The released structure shown in Fig. 11.4(b) has been obtained by replacing the hinge at L_4 by a roller and introducing cuts in members U_1L_2 and U_2L_3. The system coordinates 1, 2 and 3 are shown in Fig. 11.4(b). As the members of a pin-jointed frame carry only axial forces, only one element coordinate is required for each member of the frame. Thus the total number of element coordinates is equal to the number of members of the frame. Coordinates 1^* to 15^* shown in Fig. 11.4(c) are the element coordinates.

Consider a statically indeterminate structure. If the degree of static indeterminacy is n, the system coordinates assigned to the redundants may be designed as 1, 2, ..., n. Let the element coordinates be designated as $1^*, 2^*, \ldots, m^*$. Redundant forces $[P]$ which have been released to obtain the basic determinate structure and displacements $[\Delta_R]$ and $[\Delta_L]$ at the system coordinates due to the redundants and the applied loads respectively, are defined by the equations

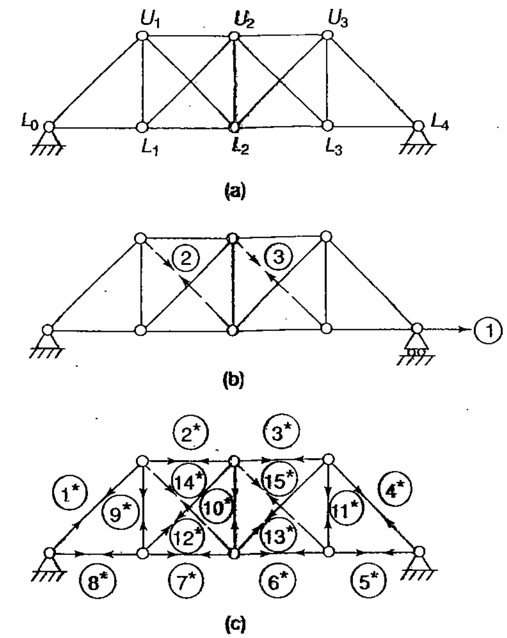


Fig. 11.4

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \qquad [\Delta_R] = \begin{bmatrix} \Delta_{1R} \\ \Delta_{2R} \\ \vdots \\ \Delta_{nR} \end{bmatrix} \qquad [\Delta_L] = \begin{bmatrix} \Delta_{1L} \\ \Delta_{2L} \\ \vdots \\ \Delta_{nL} \end{bmatrix} \tag{a}$$

Similarly, the forces and displacements at the element coordinates due to redundants and applied loads are defined by the equation

$$[P_R^*] = \begin{bmatrix} P_{1^*R} \\ P_{2^*R} \\ \vdots \\ P_{m^*R} \end{bmatrix} \qquad [\Delta_R^*] = \begin{bmatrix} \Delta_{1^*R} \\ \Delta_{2^*R} \\ \vdots \\ \Delta_{m^*R} \end{bmatrix}$$

$$[P_L^*] = \begin{bmatrix} P_{1^*L} \\ P_{2^*L} \\ \vdots \\ P_{m^*L} \end{bmatrix} \qquad [\Delta_L^*] = \begin{bmatrix} \Delta_{1^*L} \\ \Delta_{2^*L} \\ \vdots \\ \Delta_{m^*L} \end{bmatrix} \tag{b}$$

Forces $[P^*_R]$ in the released structure at the element coordinates are related to redundant forces $[P]$ at the system coordinates by the following relationship:

$$
\begin{aligned}
P_{1*R} &= f_{1*1}P_1 + f_{1*2}P_2 + \dots + f_{1*j}P_j + \dots + f_{1*n}P_n \\
P_{2*R} &= f_{2*1}P_1 + f_{2*2}P_2 + \dots + f_{2*j}P_j + \dots + f_{2*n}P_n \\
&\vdots \\
P_{i*R} &= f_{i*1}P_1 + f_{i*2}P_2 + \dots + f_{i*j}P_j + \dots + f_{i*n}P_n \\
&\vdots \\
P_{m*R} &= f_{m*1}P_1 + f_{m*2}P_2 + \dots + f_{m*j}P_j + \dots + f_{m*n}P_n
\end{aligned}
\tag{c}
$$

where f_{i*j} = force at element coordinate i^* in the released structure due to a unit redundant force at system coordinate j.

Equation (c) may be expressed in the matrix form

$$
\begin{bmatrix} P_{1*R} \\ P_{2*R} \\ \vdots \\ P_{i*R} \\ \vdots \\ P_{m*R} \end{bmatrix}
=
\begin{bmatrix}
f_{1*1} & f_{1*2} & \cdots & f_{1*j} & \cdots & f_{1*n} \\
f_{2*1} & f_{2*2} & \cdots & f_{2*j} & \cdots & f_{2*n} \\
\vdots & & & & & \\
f_{i*1} & f_{i*2} & \cdots & f_{1*j} & \cdots & f_{i*n} \\
\vdots & & & & & \\
f_{m*1} & f_{m*2} & \cdots & f_{m*j} & \cdots & f_{m*n}
\end{bmatrix}
\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \\ \vdots \\ P_n \end{bmatrix}
\tag{d}
$$

Equation (d) can be written in the compact form

$$[P^*_R] = [f][P] \tag{e}$$

Matrix $[f]$ in Eq. (e) is known as the *force-transformation matrix*. It may be noted that the elements of the jth column of the force-transformation matrix $[f]$ are obtained by applying a unit force at system coordinate j and computing the forces at all the element coordinates. The forces at the element coordinates may be determined by considering the free bodies of the elements, Sec. 1.5.

If the flexibility matrix with reference to the element coordinates is known, it is possible to determine the flexibility matrix with reference to the system coordinates by using the *principle of contragredience* defined by the following equation:

$$[\Delta_R] = [f]^T[\Delta^*_R] \tag{f}$$

where $[f]^T$ = transpose of matrix $[f]$

The principle of contragredience can be established by equating the external work done by redundants $P_1, P_2, \dots, P_n$ and the internal work done by forces $P_{1*R}, P_{2*R}, \dots, P_{m*R}$.

$$P_1\Delta_{1R} + P_2\Delta_{2R} + \dots + P_n\Delta_{nR} = P_{1*R}\Delta_{1*R} + P_{2*R}\Delta_{2*R} + \dots + P_{m*R}\Delta_{m*R} \tag{g}$$

Equation (g) can be written in the matrix form

$$[P]^T[\Delta_R] = [P^*_R]^T\ [\Delta^*_R] \tag{h}$$

Transposing both sides of Eq. (e),

$$[P^*_R]^T = [P]^T[f]^T \tag{i}$$

Substituting Eq. (i) into Eq. (h),

$$[P]^T[\Delta_R] = [P]^T[f]^T[\Delta^*_R]$$

or

$$[\Delta_R] = [f]^T[\Delta^*_R]$$

Hence, Eq. (f) is established.

Putting $[\Delta_R] = [\delta][P]$ and $[\Delta^*_R] = [\delta^*][P^*_R]$ into Eq. (f),

$$[\delta][P] = [f]^T[\delta^*][P^*_R]$$

Substituting for $[P^*_R]$ from Eq. (e) into Eq. (j),

$$[\delta][P] = [f]^T\ [\delta^*][f][P]$$

or

$$[\delta] = [f]^T[\delta^*][f] \tag{11.1}$$

where $[\delta]$ and $[\delta^*]$ = flexibility matrices with reference to system coordinates and element coordinates respectively.

It should be noted that the orders of matrices $[f]^T$, $[\delta^*]$ and $[f]$ are $n \times m^*$, $m^* \times m^*$ and $m^* \times n$ respectively. Hence, the order of product matrix $[f]^T[\delta^*][f]$ is $n \times n$ which is the order of matrix $[\delta]$.

Flexibility matrix $[\delta^*]$ may be developed by considerng individual members deforming independently from the rest of the structure. Hence, in order to develop matrix $[\delta^*]$, the flexibility matrices of individual members have to be computed. As the element coordinates correspond to the end couples in the case of rigid-jointed plane frames, the flexibility matrix for any member AB is given by the equation

$$[\delta]_{AB} = \frac{L_{AB}}{6EI_{AB}}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

In the case of pin-jointed frames, the element coordinates correspond to the axial forces. Hence, the flexibility matrix for any member AB is given by the equation

$$[\delta]_{AB} = \left[\frac{L}{AE}\right]_{AB}$$

The flexibility matrices of the members thus obtained should be placed along the main diagonal of $[\delta^*]$. This procedure becomes evident if it is noted that an element δ_{i*j*} is nonzero only if coordinates i^* and j^* belong to the same member. Element δ_{i*j*} is evidently zero if coordinates i^* and j^* belong to different members because in a dismembered structure, forces acting on any one member do not produce displacement in other members. As only two coordinates have been assigned to each member in a rigid-jointed plane frame, the member flexibility matrices are of order 2×2. Consequently, matrix $[\delta^*]$ for the entire unassembled structure is a tridiagonal matrix of order $m^* \times m^*$. In the case of pin-jointed frame, only one coordinate has been assigned to each member. The member flexibility matrices are of order 1×1 and consequently matrix $[\delta^*]$ for the entire unassembled structure is a diagonal matrix of order $m^* \times m^*$.

To illustrate the procedure for developing $[\delta^*]$, consider the frame shown in Fig. 11.2(c).

$$[\delta^*] = \begin{bmatrix} [\delta]_{AB} & & & & & \\ & [\delta]_{BC} & & & & \\ & & [\delta]_{CD} & & & 0 \\ & & & [\delta]_{DE} & & \\ & & & & [\delta]_{EF} & \\ & & 0 & & & [\delta]_{BE} \end{bmatrix} \tag{11.2}$$

Equation (f) was derived for the displacements at the system coordinates due to the redundants. In a similar manner, the displacements in the released structure at the system coordinates due to the applied loads are given by the equation

$$[\Delta_L] = [f]^T[\Delta_L^*] \tag{k}$$

Putting $[\Delta_L^*] = [\delta^*][P_L^*]$ into Eq. (k),

$$[\Delta_L] = [f]^T[\delta^*][P_L^*] \tag{11.3}$$

where $[P_L^*]$ is the column matrix whose elements are the forces at the element coordinates in the released structure due to the applied loads. In order to compute these forces, the loads acting between the joints have to be replaced by their equivalent joint loads. After replacing all intermediate loads by their equivalent joint loads, the released structure should be analysed to compute the forces at the element coordianates.

The foregoing discussion explains the procedure for a systematic development of matrices $[\delta]$ and $[\Delta_L]$ for the structural system from unassembled element flexibility matrix $[\delta^*]$ through the use of transformation matrices $[f]$ and $[P_L^*]$. Thereafter, the force method may be used in the usual way for the analysis of the structural system. The procedure may be described by the following steps:

(i) Determine the degree of static indeterminacy of the structural system. Identify the redundants and the released structure. Assign coordinates $1, 2, \ldots, n$ to the chosen redundants.

(ii) Identify the elements of the structural system and assign element coordinates $1^*, 2^*, \ldots, m^*$.

(iii) Develop force-transformation matrix $[f]$ by applying on the released structure a unit force successively at coordinates $1, 2, \ldots, n$ and determining the forces at the coordinates $1^*, 2^*, \ldots, m^*$. These forces constitute the elements of matrix $[f]$. The elements of the jth column of matrix $[f]$ are the forces at coordinates $1^*, 2^*, \ldots m^*$ due to a unit force at coordinate j.

(iv) Develop matrix $[P_L^*]$ by first replacing all the given loads on the structure by equivalent joint loads and then determining the forces at the element coordinates in the released structure by considering free bodies of the elements.

(v) Develop unassembled element flexibility matrix $[\delta^*]$. For this purpose, develop the flexibility matrices for the individual elements and place them along the main diagonal. The remaining elements of matrix $[\delta^*]$ may be taken to be zero.

(vi) Develop system flexibility matrix $[\delta]$ by using Eq. (11.1).

(vii) Develop matrix $[\Delta_L]$ for the structural system by using Eq. (11.3).

(viii) Having developed matrices $[\delta]$ and $[\Delta_L]$ of the structural system, redundants $P_1, P_2, \ldots, P_n$ in the given statically indeterminate structure may be computed in the usual manner by using the force method.

Example 11.1

Analyse the continuous beam shown in Fig. 11.5(a).

Solution

The released structure shown in Fig. 11.5(b) has been obtained by releasing the bending moments at supports B and C. The system coordinates 1 and 2 assigned to the redundant bending couples at B and C are also shown in Fig. 11.5(b). The element coordinates 1^* to 6^* for the three members AB, BC and CD are shown in Fig. 11.5(c). Force-transformation matrix $[f]$ may be developed by applying a unit force successively at system coordinates 1 and 2 and computing the forces at the element coordinates 1^* to 6^* by considering the free bodies of the elements. Thus to generate the first column of matrix $[f]$, apply a unit force at coordinate 1 and compute the forces at the element coordinates. These forces are

$$f_{1*1} = 0$$
$$f_{2*1} = -1$$
$$f_{3*1} = 1$$
$$f_{4*1} = f_{5*1} = f_{6*1} = 0$$

Similarly, to generate the second column of matrix $[f]$, apply a unit force at coordinate 2 and compute the forces at elements coordinates 1^* to 6^*. These forces are

$$f_{1*2} = f_{2*2} = f_{3*2} = 0$$
$$f_{4*2} = -1 \qquad f_{5*2} = 1 \qquad f_{6*2} = 0$$

Hence, force transformation matrix $[f]$ is given by the equation

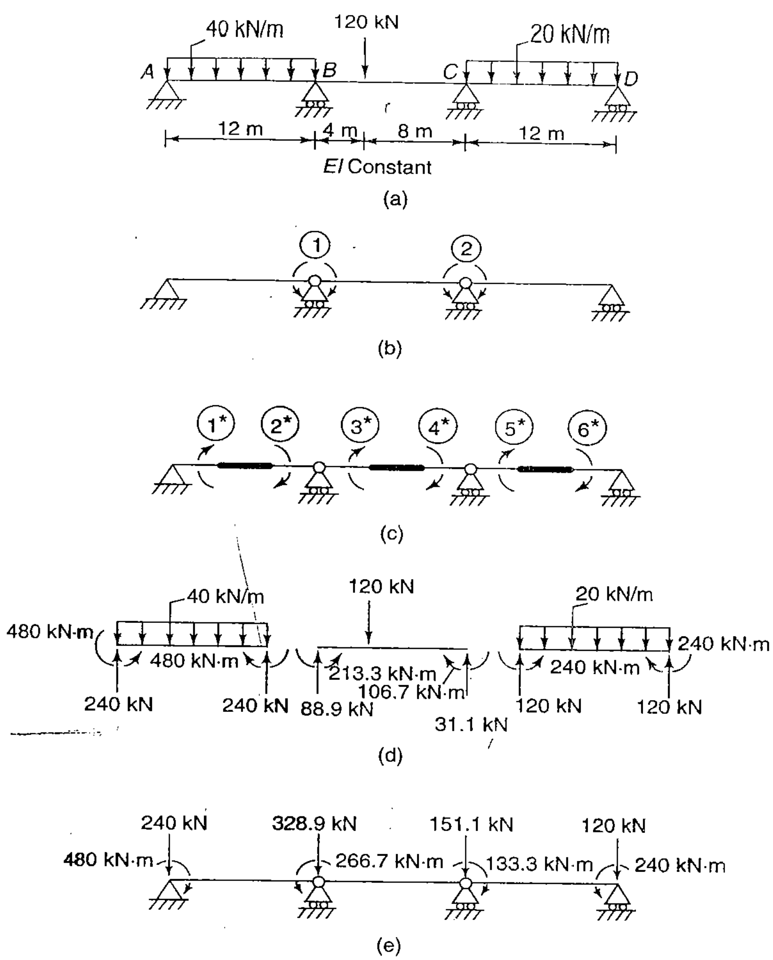


Fig. 11.5

$$[f] = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{(a)}$$

The flexibility matrices of members *AB*, *BC* and *CD* with reference to their respective element coordinates are

$$[\delta]_{AB} = [\delta]_{BC} = [\delta]_{CD} = \frac{12}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Unassembled flexibility matrix $[\delta^*]$ is obtained by placing the element matrices along the main diagonal.

$$[\delta^*] = \frac{2}{EI}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad \text{(b)}$$

Substituting from Eqs. (a) and (b) into Eq. (11.1), system flexibility matrix (δ) may be obtained.

$$[\delta] = \frac{2}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{2}{EI}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \qquad \text{(c)}$$

To develop matrix $[\Delta_L]$, the applied loads have to be replaced by their equivalent joint loads. For this purpose each element of the frame may first be considered as a fixed-ended member and the end forces shown in Fig. 11.5(d) may be computed. The equivalent joint loads have the same magnitude as these end forces but have opposite directions. Thus the net equivalent joint loads may be obtained by combining the end forces on the elements and reversing their directions. The equivalent joint loads in the released structure are shown in Fig. 11.5(e). The forces at the element coordinates may now be calculated by considering the free bodies of the elements. These forces are

$$\begin{aligned} P_{1^*L} &= 480 \text{ kN·m} & P_{2^*L} &= -480 \text{ kN·m} \\ P_{3^*L} &= 213.3 \text{ kN·m} & P_{4^*L} &= -106.7 \text{ kN·m} \\ P_{5^*L} &= 240 \text{ kN·m} & P_{6^*L} &= -240 \text{ kN·m} \end{aligned} \qquad \text{(d)}$$

Substituting from Eqs (a), (b) and (d) into Eq. (11.3), matrix $[\Delta_L]$ may be obtained.

$$[\Delta_L] = \frac{2}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 480 \\ -480 \\ 213.3 \\ -106.7 \\ 240 \\ -240 \end{bmatrix}$$

$$= \frac{1}{3EI}\begin{bmatrix} 11840 \\ 6880 \end{bmatrix}$$

It may be noted that matrices $[\delta]$ and $[\Delta_L]$ are the same as those developed earlier in Ex. 5.3. Hence, the redundant bending couples at supports B and C may be computed as in Ex. 5.3.

Example 11.2

Analyse the rigid-jointed plane frame shown in Fig. 11.6(a).

Solution

The released structure and the system coordinates assigned to the redundant reactions at D are shown in Fig. 11.6(b). The element coordinates are shown in Fig. 11.6(c). The force-transformation matrix $[f]$ may be developed by applying a unit force successively at the system coordinates and computing the forces at the element coordinates by considering the free bodies of elements AB, BC and CD. Thus force-transformation matrix $[f]$ is found to be

$$[f] = \begin{bmatrix} -5 & 10 & -1 \\ -5 & -10 & 1 \\ 5 & 10 & -1 \\ -5 & 0 & 1 \\ 5 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(a)}$$

Unassembled flexibility matrix $[\delta^*]$ is given by the equation

$$[\delta^*] = \frac{5}{12EI}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix} \qquad \text{(b)}$$

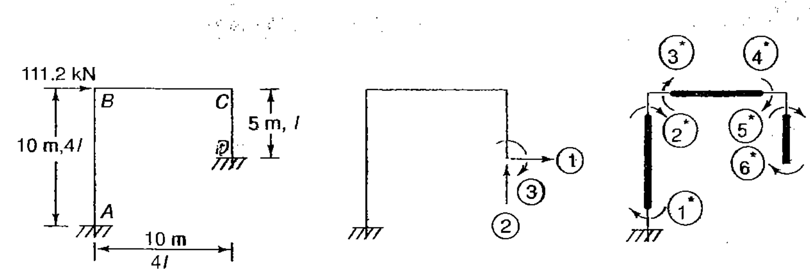


Fig. 11.6

Substituting from Eqs (a) and (b) into Eq. (11.1), the system flexibility matrix may be obtained.

$$[\delta] = \frac{5}{12EI}\begin{bmatrix} 300 & 150 & -60 \\ 150 & 800 & -90 \\ -60 & -90 & 24 \end{bmatrix} \qquad \text{(c)}$$

The forces at the element coordinates due to the applied load in the released structure are obtained by considering the free bodies of the elements. These forces are

$$P_{1^*L} = -1112 \text{ kN·m} \qquad \text{(d)}$$

$$P_{2^*L} = P_{3^*L} = P_{4^*L} = P_{5^*L} = P_{6^*L} = 0$$

Substituting from Eqs (a), (b) and (d) into Eq. (11.3), matrix $[\Delta_L]$ may be obtained.

$$[\Delta_L] = \frac{5}{12EI}\begin{bmatrix} 5560 \\ -33360 \\ 3336 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 2316.7 \\ -13900 \\ 1390 \end{bmatrix}$$

It may be noted that matrices $[\delta]$ and $[\Delta_L]$ are the same as obtained earlier in Ex. 6.1. Hence the redundants at support D may be computed as in Ex. 6.1.

Example 11.3

Analyse the portal frame shown in Fig. 11.7(a).

Solution

The released structure shown in Fig. 11.7(b) has been obtained by introducing a cut at the centre of beam BC. The system coordinates 1, 2 and 3 assigned to the redundant reactions are also shown in Fig. 11.7(b). As the cut has been introduced at an intermediate point, portions BE and EC have to be treated as separate elements. Consequently, the number of element coordinates required is $2 \times 4 = 8$. Element coordinates 1^* to 8^* are shown in Fig. 11.7(c). Force-transformation matrix $[f]$ may be developed by applying a unit force successively at the system coordinates 1, 2 and 3 and computing the forces at element coordinates 1^* to 8^* by considering the free bodies of the elements. Force-transformation matrix $[f]$ is found to be

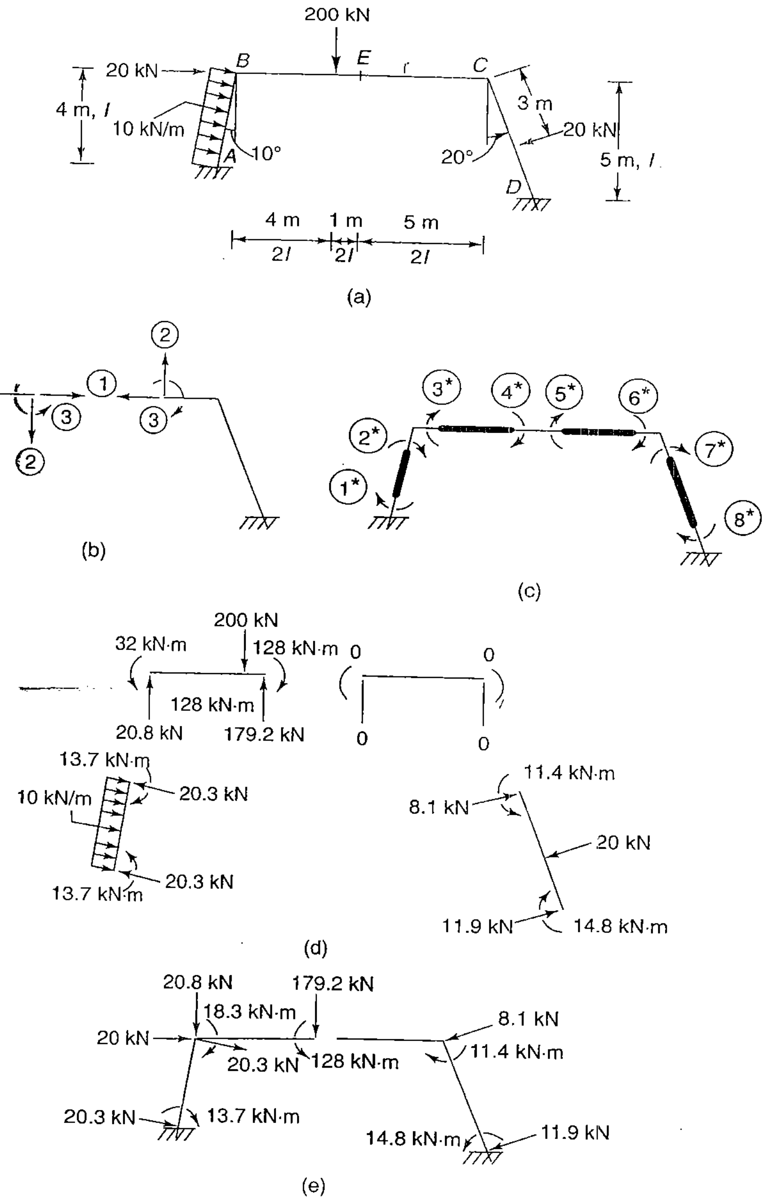


Fig. 11.7

$$[f] = \begin{bmatrix} -4 & -5.705 & 1 \\ 0 & 5 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -5 & -1 \\ 0 & 5 & 1 \\ 5 & -6.82 & -1 \end{bmatrix} \qquad \text{(a)}$$

Unassembled flexibility matrix $[\delta^*]$ is given by the equation

$$[\delta^*] = \frac{1}{EI}\begin{bmatrix} \frac{4.06}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & & & 0 \\ & \frac{5}{6\times 2EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & & \\ & & \frac{5}{6\times 2EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & \\ 0 & & & \frac{5.32}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \end{bmatrix} \qquad \text{(b)}$$

Substituting from Eqs (a) and (b) into Eq. (11.1), system flexibility matrix $[\delta]$ may be obtained.

$$[\delta] = \begin{bmatrix} 65.99 & -38.22 & -21.42 \\ -38.22 & 345.44 & 9.71 \\ -21.42 & 9.71 & 14.38 \end{bmatrix} \qquad \text{(c)}$$

To develop matrix $[\Delta_L]$, the applied loads on the released structure have to be replaced by their equivalent joint loads. For this purpose, each element of the frame may first be considered as a fixed-ended member and the end forces shown in Fig. 11.7(d) may be computed. The equivalent joint loads have the same magnitude as these end forces but have opposite directions. Thus the net equivalent joint loads may be obtained by combining the end forces on the elements and reversing their directions. The equivalent joint loads are shown in Fig. 11.7(e). The forces at the element coordinates may now be calculated by considering the free bodies of the elements. These forces are

$$\begin{aligned} P_{1^*L} &= -1089.7 \text{ kN·m} \\ P_{2^*L} &= 786.3 \text{ kN·m} \\ P_{3^*L} &= -768 \text{ kN·m} \\ P_{4^*L} &= -128 \text{ kN·m} \end{aligned} \qquad \text{(d)}$$

$$P_{5^*L} = P_{6^*L} = 0$$
$$P_{7^*L} = 11.4 \text{ kN·m}$$
$$P_{8^*L} = 31.7 \text{ kN·m}$$

Substituting from Eqs (a), (b) and (d) into Eq. (11.3), matrix $[\Delta_L]$ may be obtained.

$$[\Delta_L] = \frac{1}{EI}\begin{bmatrix} 8257 \\ 23036 \\ -4662 \end{bmatrix}$$

It may be noted that matrices $[\delta]$ and $[\Delta_L]$ are the same as obtained earlier in Ex. 6.4. Hence redundants, P_1, P_2 and P_3 may be computed as in Ex. 6.4.

Example 11.4

Analyse the pin-jointed plane frame shown in Fig. 11.8(a). The numbers in parentheses are the cross-sectional areas of the members in mm².

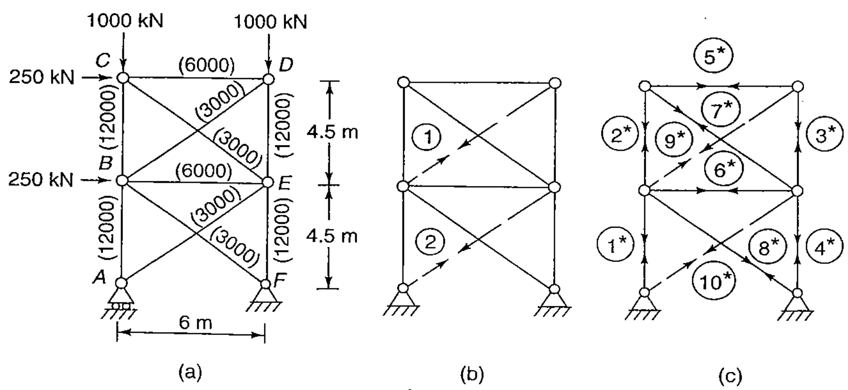


Fig. 11.8

Solution

The released structure obtained by introducing cuts in members BD and AE is shown in Fig. 11.8(b). The system coordinates 1 and 2 assigned to the redundant forces in members BD and AE are also shown in Fig. 11.8(b). Element coordinates 1^* to 10^* are shown in Fig. 11.8(c).

Force-transformation matrix $[f]$ may be developed by applying a unit force successively at the system coordinates and computing the forces at the element coordinates. Thus matrix $[f]$ is found to be

$$[f] = \begin{bmatrix} 0 & -0.6 \\ -0.6 & 0 \\ -0.6 & 0 \\ 0 & -0.6 \\ -0.8 & 0 \\ -0.8 & -0.8 \\ 1.0 & 0 \\ 0 & 1.0 \\ 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} \qquad \text{(a)}$$

Unassembled flexibility matrix $[\delta^*]$ is a diagonal matrix of order 10×10. The elements on the main diagonal are equal to the respective values of L/AE for the 10 members of the frame. The unassembled flexibility matrix may be written as

$$[\delta^*] = \begin{bmatrix} 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(b)}$$

Substituting from Eqs (a) and (b) into Eq. (11.1), the system flexibility matrix may be obtained.

$$[\delta] = \frac{1}{E}\begin{bmatrix} 6.55 & 0.64 \\ 0.64 & 5.91 \end{bmatrix} \qquad \text{(c)}$$

The forces at the element coordinates due to the applied loads are

$$\begin{aligned} P_{1^*L} &= -437.5 \text{ kN} & P_{2^*L} &= -812.5 \text{ kN} \\ P_{3^*L} &= -1000 \text{ kN} & P_{4^*L} &= -1187.5 \text{ kN} \\ P_{5^*L} &= 0 & P_{6^*L} &= 250 \text{ kN} \\ P_{7^*L} &= -312.5 \text{ kN} & P_{8^*L} &= -625 \text{ kN} \\ P_{9^*L} &= P_{10^*L} = 0 \end{aligned} \qquad \text{(d)}$$

Substituting from Eqs (a), (b) and (d) into Eq. (11.3),

$$[\Delta_L] = \frac{1}{E}\begin{bmatrix} -573.44 \\ -1396.88 \end{bmatrix}$$

It may be noted that matrices $[\delta]$ and $[\Delta_L]$ are the same as obtained earlier in Ex. 7.6. Hence redundant forces P_1 and P_2 may be computed as in Ex. 7.6.

11.3 STATIC ANALYSIS BY METHOD OF JOINTS

In Sec. 11.2, the forces at the element coordinates which constitute the elements of matrices $[f]$ and $[P_L^*]$ have been computed by starting at a convenient point and considering the interaction of the constituent members. As an alternative, the method of joints may be used. This approach, which is more formal, is particularly amenable for the analysis of large structures by a digital computer. In this method, the applied loads are replaced by their static equivalent joint loads and the equations of equilibrium for all the joints are solved as simultaneous equations. The method may be used for the analysis of statically determinate frames with rigid or pin joints. It can also be used for analysing hybrid structures.

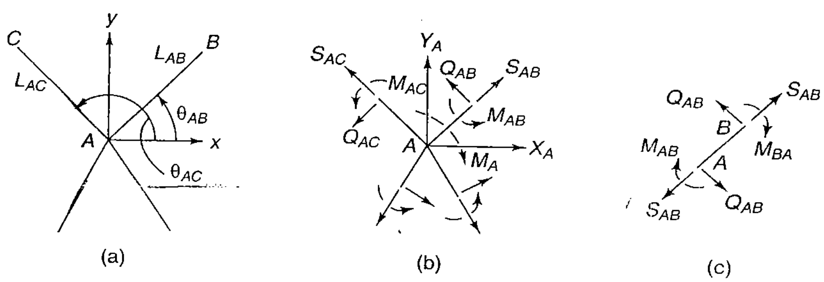


Fig. 11.9

Figure 11.9(a) shows a typical joint A of a rigid-jointed plane frame at which several members meet. Figure 11.9(b) shows the free-body diagram of joint A. In general, each member meeting at joint A carries an axial force, a shear force and a bending moment. Hence, it transmits an axial force, a transverse force and an end couple to joint A. Consider, for example, the free-body diagram of member AB shown in Fig. 11.9(c). This member transmits an axial force S_{AB}, a transverse force Q_{AB} and a bending couple M_{AB} to joint A. Similar forces are transmitted to joint A by other members. The components of the equivalent joint loads at joint A are represented by X_A, Y_A and M_A. The equations of static equilibrium for joint A may be written as

$$X_{AB} + X_{AC} + \ldots + X_A = 0 \tag{11.4a}$$

$$Y_{AB} + Y_{AC} + \ldots + Y_A = 0 \tag{11.4b}$$

$$-M_{AB} - M_{AC} - \ldots + M_A = 0 \tag{11.4c}$$

where

X_{AB} = component in x direction of the forces transmitted to joint A by member AB

$= S_{AB}\cos\theta_{AB} - Q_{AB}\sin\theta_{AB}$

$$= S_{AB}\cos\theta_{AB} - \frac{M_{AB}+M_{BA}}{L_{AB}}\sin\theta_{AB}$$

and Y_{AB} = component in y direction of the forces transmitted to joint A by member AB

$= S_{AB}\sin\theta_{AB} + Q_{AB}\cos\theta_{AB}$

$$= S_{AB}\sin\theta_{AB} + \frac{M_{AB}+M_{BA}}{L_{AB}}\cos\theta_{AB}$$

Similar equations can be written for all the joints of the frame. This procedure provides sufficient number of equations for the determination of the member forces in a statically determinate plane frame.

Consider, for example, the rigid-jointed plane frame shown in Fig. 11.10(a). The degree of static indeterminacy of the structure is three. The released structure shown in Fig. 11.10(b) has been obtained by inserting hinges at A, B and C. Element coordinates 1^* to 4^* are shown in Fig. 11.10(b) in which rotational coordinate has been assigned at each end of elements AB and BC. The static analysis of the released structure involves the determination of six internal forces, viz., forces P_1^* to P_4^* and axial forces in members AB and BC. Six equations are required for the determination of these internal forces. Using Eq. (11.4), three equations of equilibrium may be written at joint B. Two equations corresponding to the equilibrium of moments are written at joints A and C. One more equation is obtained by assigning an arbitrary value of one of the end couples P_2^* or P_3^* in the members meeting at inserted hinge B. In this way six equations are obtained which are sufficient for the determination of all the unknown forces. It may, however, be noted that if the static analysis has to be carried out for the determination of the elements of matrices $[f]$ and $[P_L^*]$, it is necessary to determine only forces P_1^* or P_4^*. The four equations required for the determination of these forces are obtained by writing Eq. (11.4c) for joints A, B and C and assigning an arbitrary value for either P_2^* or P_3^*.

Consider next, the released structure shown in Fig. 11.10(c) in which hinges have been inserted at A and C and at some intermediate point D of member BC. Treating portions BD and DC as separate elements, element coordinates 1^* to 6^* may be chosen as represented in Fig. 11.10(c). The static analysis of the released structure involves the determination of nine internal forces, viz., forces P_1^* or P_6^* and axial forces in elements AB, BD and DC. Nine equations

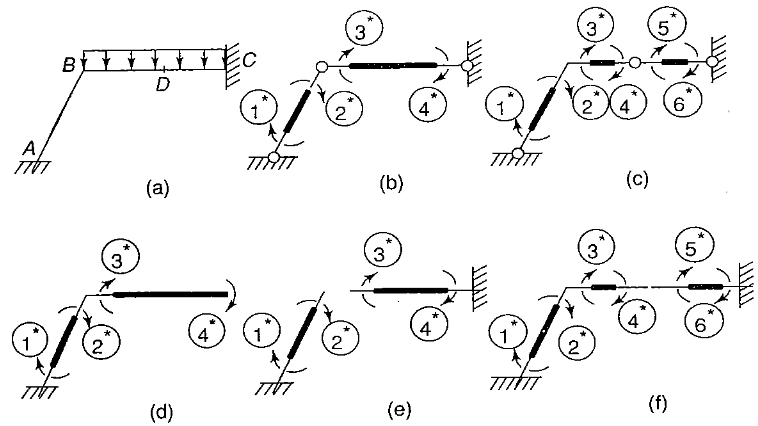


Fig. 11.10

are required for the determination of these internal forces. Using Eq. (11.4), six equations of equilibrium may be written at joints B and D. Two equations corresponding to the equilibrium of moments are written at joints A and C. One more equation is obtained by assigning an arbitrary value to one of the end couples P_4^* or P_5^* in the members meeting at inserted hinge D. In this way nine equations are obtained which are sufficient for the determination of all the unknown forces.

The released structure shown in Fig. 11.10(d) has been obtained by removing the support at C. The unknown internal forces comprise P_1^* to P_4^* and the axial forces in members AB and BC. These forces can be determined by writing three equations of equilibrium for each of the joints B and C.

Figure 11.10(e) shows the released structure obtained by introducing a cut at joint B. The unknwon internal forces comprise P_1^* to P_4^* and the axial forces in members AB and BC. These forces can be determined by writing three equations of equilibrium for joint B of member BA and joint B of member BC.

If the released structure is obtained by introducing a cut at some intermediate point D in member BC as shown in Fig. 11.10(f), portions BD and DC should be treated as separate members. Hence, the unknown internal forces are P_1^* to P_6^* and the axial forces in members AB, BD and DC. The nine equations required for the determination of these unknowns are obtained by writing three equations of equilibrium at each of the joints B, D of member BD and D of member DC.

Example 11.5

For the rigid-jointed plane frame shown in Fig. 11.11(a), determine the element forces in the released structure if the released structure is obtained by (i) inserting hinges at A, B and C (Fig. 11.11b) and (ii) introducing a cut at D, the centre of BC (Fig. 11.11c). The element forces are to be calculated for a unit force applied successively at the system coordinates 1, 2 and 3 and also on account of the applied load.

Solution

(i) The two end couples and the axial force are the three unknown forces in each of the members AB and BC. Hence, the total number of unknown element forces is six.

(a) Unit force at coordinate 1: Six equations are required for the determination of the element forces. Using Eq. (11.4), three equations of equilibrium may be written at joint B. Two equations corresponding to equilibrium of moments are written at joints A and C. One more equation is obtained by assigning an arbitrary value to one of the end couples in members AB and BC at joint B. The six equations indicated above may be written in the form

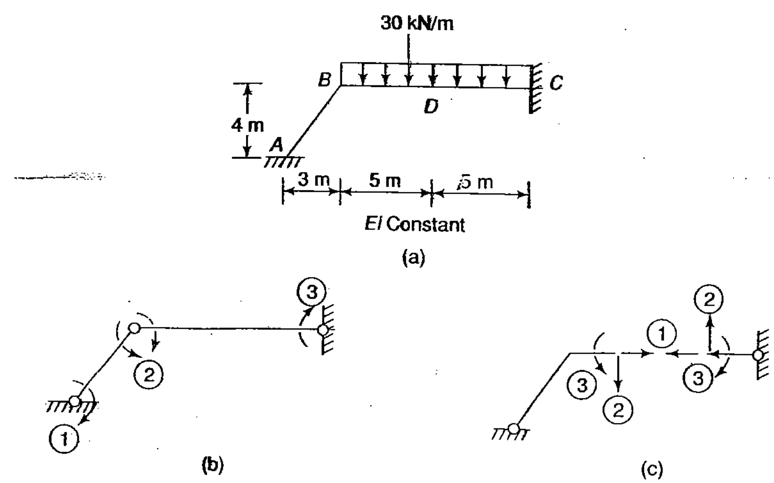


Fig. 11.11

$$S_{BA}\cos\theta_{BA} - \frac{M_{AB}+M_{BA}}{L_{AB}}\sin\theta_{BA} + S_{BC}\cos\theta_{BC}$$

$$-\frac{M_{BC}+M_{CB}}{L_{BC}}\sin\theta_{BC} + X_B = 0 \qquad \text{(a)}$$

$$S_{BA}\sin\theta_{BA} + \frac{M_{AB}+M_{BA}}{L_{AB}}\cos\theta_{BA} + S_{BC}\sin\theta_{BC}$$

$$+\frac{M_{BC}+M_{CB}}{L_{BC}}\cos\theta_{BC} + Y_B = 0 \quad \text{(b)}$$

$$-M_{BA} - M_{BC} + M_B = 0 \quad \text{(c)}$$

$$-M_{AB} + M_A = 0 \quad \text{(d)}$$

$$-M_{CB} + M_C = 0 \quad \text{(e)}$$

The sixth equation is obtained by assuming arbitrarily,

$$M_{BA} = 0 \quad \text{(f)}$$

From the given data,

$$L_{AB} = 5 \text{ m} \qquad L_{BC} = 10 \text{ m} \qquad \cos\theta_{BA} = -0.6$$
$$\sin\theta_{BA} = -0.8 \qquad \cos\theta_{BC} = 1 \qquad \sin\theta_{BC} = 0 \quad \text{(g)}$$

Also, in this case,

$$M_A = 1 \qquad X_B = Y_B = M_B = M_C = 0 \quad \text{(h)}$$

Substituting from Eqs (g) and (h) into Eqs. (a) to (f) and solving them,

$$M_{AB} = 1.0 \qquad M_{BA} = M_{BC} = M_{CB} = 0$$
$$S_{BA} = -0.15 \qquad S_{BC} = -0.25$$

(b) Unit force at coordinate 2: In this case,

$$M_{BA} = \cancel{1} \qquad M_{BC} = 1 \quad \text{(i)}$$
$$X_B = Y_B = M_B = M_A = M_C = 0$$

Substituting from Eqs (g) and (i) into Eqs (a) to (e) and solving them,

$$M_{AB} = 0 \qquad M_{BA} = -1 \qquad M_{BC} = 1$$
$$M_{CB} = 0 \qquad S_{BA} = 0.275 \qquad S_{BC} = 0.325$$

(c) Unit force at coordinate 3: In this case,

$$M_C = 1 \qquad M_A = X_B = Y_B = M_B = 0 \quad \text{(j)}$$

Substituting from Eqs (g) and (j) into Eqs (a) to (f) and solving them,

$$M_{AB} = M_{BA} = M_{BC} = 0 \qquad M_{CB} = 1$$
$$S_{BA} = 0.125 \qquad S_{BC} = 0.075$$

(d) Applied load: the load on member BC has to be replaced by its equivalent joint loads. The equivalent load at B comprises a clockwise couple of 250 kN·m and a downward force of 150 kN. Similarly, the equivalent joint load at C comprises a counter-clockwise couple of 250 kN·m and a downward force of 150 kN. Hence, in this case,

$$X_B = 0 \qquad Y_B = -150 \text{ kN} \qquad M_A = 0 \quad \text{(k)}$$
$$M_B = 250 \text{ kN·m} \qquad M_C = -250 \text{ kN·m}$$

Substituting from Eqs (g) and (k) into Eqs (a) to (f) and solving them,

$$M_{AB} = M_{BA} = 0 \qquad M_{BC} = 250 \text{ kN·m}$$
$$M_{CB} = -250 \text{ kN·m} \qquad S_{BA} = -187.5 \text{ kN}$$
$$S_{BC} = 112.5 \text{ kN}$$

(ii) In the released structure shown in Fig. 11.11(c), portions BD and DC have to be treated as separate elements. As each of the three elements AB, BD and DC, has three element forces, the total number of element forces is nine.

(a) Unit force at coordinate 1: Nine equations are required for the determination of the element forces. Using Eq. (11.4), three equations of equilibrium may be written at each of the joints B and D of element BD and D of element DC. Thus the following set of nine equations may be written:

$$S_{BA}\cos\theta_{BA} - \frac{M_{AB}+M_{BA}}{L_{AB}}\sin\theta_{BA} + S_{BD}\cos\theta_{BD}$$

$$-\frac{M_{BD}+M_{DB}}{L_{BD}}\sin\theta_{BD} + X_B = 0 \quad \text{(l)}$$

$$S_{BA}\sin\theta_{BA} + \frac{M_{AB}+M_{BA}}{L_{AB}}\cos\theta_{BA} + S_{BD}\sin\theta_{BD}$$

$$+\frac{M_{BD}+M_{DB}}{L_{BD}}\cos\theta_{BD} + Y_B = 0 \quad \text{(m)}$$

$$-M_{BA} - M_{BD} + M_B = 0 \quad \text{(n)}$$

$$S_{BD}\cos\theta_{DB} - \frac{M_{BD}+M_{DB}}{L_{BD}}\sin\theta_{DB} + X_{DL} = 0 \quad \text{(o)}$$

$$S_{BD}\sin\theta_{DB} + \frac{M_{BD}+M_{DB}}{L_{BD}}\cos\theta_{DB} + Y_{DL} = 0 \quad \text{(p)}$$

$$-M_{DB} + M_{DL} = 0 \quad \text{(q)}$$

$$S_{DC}\cos\theta_{DC} - \frac{M_{DC}+M_{CD}}{L_{CD}}\sin\theta_{DC} + X_{DR} = 0 \quad \text{(r)}$$

$$S_{DC}\sin\theta_{DC} + \frac{M_{DC}+M_{CD}}{L_{CD}}\cos\theta_{DC} + Y_{DR} = 0 \quad \text{(s)}$$

$$-M_{DC} + M_{DR} = 0 \quad \text{(t)}$$

where subscripts L and R have been added to distinguish between forces X_D, Y_D and M_D acting at joint D for the portions to the left of D and to the right of D respectively.

From the give data,

$$L_{AB} = L_{BD} = L_{DC} = 5 \text{ m}$$
$$\cos\theta_{BA} = -0.6 \quad \sin\theta_{BA} = -0.8 \quad \text{(u)}$$
$$\cos\theta_{BD} = 1 \quad \sin\theta_{BD} = 0 \quad \cos\theta_{DB} = -1$$
$$\sin\theta_{DB} = 0$$

Also, in this case,

$$X_{DL} = 1 \qquad X_{DR} = -1 \tag{v}$$

$$X_B = Y_B = M_B = Y_{DL} = Y_{DR} = M_{DL} = M_{DR} = 0$$

Substituting from Eqs (u) and (v) into Eqs (1) to (t) and solving them,

$$M_{AB} = -4 \qquad M_{BA} = 0 \qquad S_{BA} = 0.60$$

$$M_{BD} = M_{DB} = 0 \qquad S_{BD} = 1 \qquad M_{DC} = M_{CD} = S_{DC} = 0$$

(b) Unit force at coordinate 2: In this case,

$$Y_{DL} = -1 \qquad Y_{DR} = 1 \tag{w}$$

$$X_B = Y_B = M_B = X_{DL} = X_{DR} = M_{DL} = M_{DR} = 0$$

Substituting from Eqs (u) and (w) into Eqs (1) to (t) and solving them,

$$M_{AB} = -8 \qquad M_{BA} = 5 \qquad M_{BD} = -5$$

$$M_{DB} = M_{DC} = 0 \qquad M_{CD} = -5 \qquad S_{BA} = -0.8$$

$$S_{BD} = S_{DC} = 0$$

(c) Unit force at coordinate 3: In this case,

$$M_{DL} = -1 \qquad M_{DR} = 1$$

$$X_B = Y_B = M_B = X_{DL} = X_{DR} = Y_{DL} = Y_{DR} = 0 \tag{x}$$

Substituting from Eqs (u) and (x) into Eqs (1) to (t) and solving them,

$$M_{AB} = -M_{BA} = M_{BD} = -M_{DB} = M_{DC} = -M_{CD} = 1$$

$$S_{BA} = S_{BD} = S_{DC} = 0$$

(d) Applied Load: The load on elements BD and DC has to be replaced by its equivalent joint loads. The equivalent joint load at B comprises a clockwise couple of 62.5 kN·m and a downward force of 75 kN. The equivalent joint load at D in element BD comprises a counter-clockwise couple of 62.5 kN·m and a downward force of 75 kN. The equivalent joint load at D in element DC comprises a clockwise couple of 62.5 kN·m and a downward force of 75 kN. Similarly, the equivalent joint load at C comprises a counter-clockwise couple of 62.5 kN·m and a downward force of 75 kN. Thus in this case,

$$Y_B = Y_{DL} = Y_{DR} = -75 \text{ kN}$$

$$M_B = -M_{DL} = M_{DR} = 62.5 \text{ kN·m}$$

$$X_B = X_{DL} = X_{DR} = 0 \tag{y}$$

Substituting from Eqs (u) and (y) into Eqs (1) to (t) and solving them,

$$M_{AB} = -825 \text{ kN·m} \qquad M_{BA} = 375 \text{ kN·m}$$

$$M_{BD} = -312.5 \text{ kN·m} \qquad M_{DB} = -62.5 \text{ kN·m}$$

$$M_{DC} = 62.5 \text{ kN·m} \qquad M_{CD} = 312.5 \text{ kN·m}$$

$$S_{BA} = S_{BD} = S_{DC} = 0$$

11.4 DISPLACEMENT METHOD

In Sec. 11.2 a systematic procedure for the development of the system flexibility matrix has been discussed. A similar procedure may be adopted for the development of the system stiffness matrix from the element stiffness matrices. This objective is achieved by using a transformation matrix known as the displacement transformation matrix. This approach leads to a more systematic procedure which is particularly suitable for a digital computer.

Consider the rigid-jointed plane frame shown in Fig. 11.12(a). The degree of freedom of the frame is six. The system coordinates 1 to 6 are shown in Fig. 11.12(b). Element coordinates 1^* to 12^* are shown in Fig. 11.12(c). It may be noted that each member of the frame has been assigned only two coordinates corresponding to the end rotations of the member. In order to understand why only two element coordinates for each member are sufficient, consider a member PQ of a rigid-jointed plane frame shown in Fig. 11.13. If the axial deformation of the member is ignored, the axial movements of the ends P and Q are equal. As this movement represents a rigid body motion of the member, it has not been shown in the figure. Let the lateral displacements of ends P and Q be Δ_P and Δ_Q. These lateral displacements of the ends give rise to member rotation, $\phi = \dfrac{\Delta_P - \Delta_Q}{L}$. In addition to axial and lateral displacements, ends P and Q rotate through angle α_P and α_Q respectively. The net rotations at the ends of the member, including the effect of lateral displacement, are given by the equations

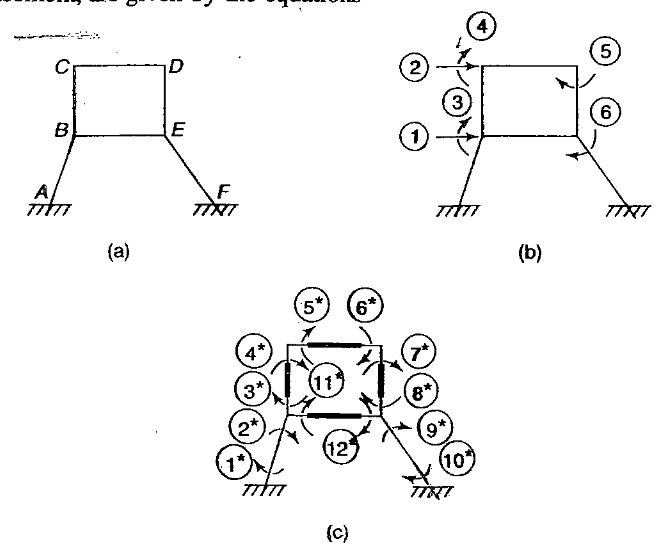


Fig. 11.12

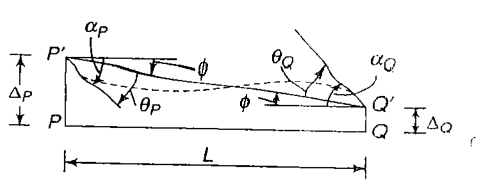


Fig. 11.13

$$\theta_P = \alpha_P - \phi = \alpha_P - \frac{\Delta_P - \Delta_Q}{L}$$

$$\theta_Q = \alpha_Q - \phi = \alpha_Q - \frac{\Delta_P - \Delta_Q}{L}$$

It follows that all the displacement components at the ends of the member are allowed for if the end rotations θ_P and θ_Q are measured from the displaced position $P'Q'$ of member PQ.

With the notations of Fig. 11.13, the slope-deflection Eq. (2.48) may be rewritten as

$$M_{PQ} = M_{PQ}^F + \frac{2EI}{L}\left[2\alpha_P + \alpha_Q - 3\phi\right]$$

$$= M_{PQ}^F + \frac{2EI}{L}\left[2(\alpha_P - \phi) + (\alpha_Q - \phi)\right]$$

$$= M_{PQ}^F + \frac{2EI}{L}\left[2\theta_P + \theta_Q\right] \tag{11.5a}$$

$$M_{QP} = M_{QP}^F + \frac{2EI}{L}\left[2\alpha_Q + \alpha_P - 3\phi\right]$$

$$= M_{QP}^F + \frac{2EI}{L}\left[2(\alpha_Q - \phi) + (\alpha_P - \phi)\right]$$

$$= M_{QP}^F + \frac{2EI}{L}\left[2\theta_Q + \theta_P\right] \tag{11.5b}$$

Next consider the pin-jointed plane frame shown in Fig. 11.14(a). The degree of freedom of the frame is eight. System coordinates 1 to 8 are shown in Fig. 11.14(b). As the members of a pin-jointed frame have only axial deformations, only one element coordinate is required for each member of the frame. Thus the total number of element coordinates is equal to the number of members of the frame. Coordinates 1^* to 10^* shown in Fig. 11.14(c) are the element coordinates.

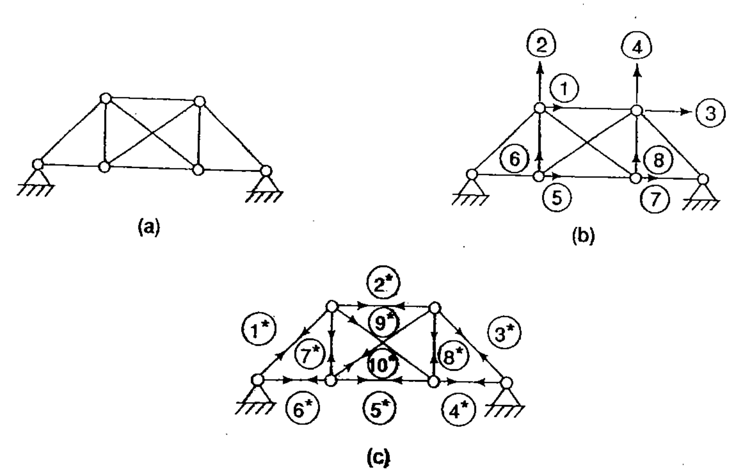


Fig. 11.14

Consider a structure with a degree of freedom equal to n. The system coordinates assigned to the independent displacement components may be designated as 1, 2, ..., n. Let the element coordinates be designated as 1^*, 2^*, ..., m^*. Forces $P'_1, P'_2, \ldots, P'_n$ in the restrained structure, forces $P_{1\Delta}, P_{2\Delta}, \ldots, P_{n\Delta}$ due to displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ and net forces $P_1, P_2, \ldots, P_n$ at the system coordinates 1, 2, ..., n may be expressed by the equations

$$[P'] = \begin{bmatrix} P'_1 \\ P'_2 \\ \vdots \\ P'_n \end{bmatrix} \qquad [P_\Delta] = \begin{bmatrix} P_{1\Delta} \\ P_{2\Delta} \\ \vdots \\ P_{n\Delta} \end{bmatrix}$$

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \qquad [\Delta] = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} \tag{a}$$

For the equilibrium of the structure,

$$[P] = [P'] + [P_\Delta]$$

The forces and displacements at the element coordinates due to forces $[P_\Delta]$ are defined by the equations

$$[P^*] = \begin{bmatrix} P_{1*} \\ P_{2*} \\ \vdots \\ P_{m*} \end{bmatrix} \qquad [\Delta^*] = \begin{bmatrix} \Delta_{1*} \\ \Delta_{2*} \\ \vdots \\ \Delta_{m*} \end{bmatrix} \tag{b}$$

Displacements $[\Delta^*]$ at the element coordinates are related to displacements $[\Delta]$ at the system coordinates by the following relationships:

$$\begin{aligned} \Delta_{1*} &= d_{1*1}\Delta_1 + d_{1*2}\Delta_2 + \ldots + d_{1*j}\Delta_j + \ldots + d_{1*n}\Delta_n \\ \Delta_{2*} &= d_{2*1}\Delta_1 + d_{2*2}\Delta_2 + \ldots + d_{2*j}\Delta_j + \ldots + d_{2*n}\Delta_n \\ &\vdots \\ \Delta_{i*} &= d_{i*1}\Delta_1 + d_{i*2}\Delta_2 + \ldots + d_{i*j}\Delta_j + \ldots + d_{i*n}\Delta_n \\ &\vdots \\ \Delta_{m*} &= d_{m*1}\Delta_1 + d_{m*2}\Delta_2 + \ldots + d_{m*j}\Delta_j + \ldots + d_{m*n}\Delta_n \end{aligned} \tag{c}$$

where d_{i*j} = displacement at element coordinate i^* due to a unit displacement at system coordinate j.

Equation (c) may be expressed in the matrix form

$$\begin{bmatrix} \Delta_{1*} \\ \Delta_{2*} \\ \vdots \\ \Delta_{i*} \\ \vdots \\ \Delta_{m*} \end{bmatrix} = \begin{bmatrix} d_{1*1} & d_{1*2} & \cdots & d_{1*j} & \cdots & d_{1*n} \\ d_{2*1} & d_{2*2} & \cdots & d_{2*j} & \cdots & d_{2*n} \\ \vdots \\ d_{i*1} & d_{i*2} & \cdots & d_{i*j} & \cdots & d_{i*n} \\ \vdots \\ d_{m*1} & d_{m*2} & \cdots & d_{m*j} & \cdots & d_{m*n} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \vdots \\ \Delta_n \end{bmatrix} \tag{d}$$

Equation (d) may be written in the compact form

$$[\Delta^*] = [d][\Delta] \tag{e}$$

If the stiffness matrix with reference to the element coordinates is known, it is possible to determine the stiffness matrix with reference to the system coordinates by using the principle of contragredience defined by the following equation:

$$[P_\Delta] = [d]^T[P^*] \tag{f}$$

The principle of contragredience can be established by equating the external work done by the forces $P_{1\Delta}, P_{2\Delta}, \ldots, P_{n\Delta}$ and the internal work done by force $P_{1*}, P_{2*}, \ldots, P_{m*}$.

$$P_{1\Delta}\Delta_1 + P_{2\Delta}\Delta_2 + \ldots + P_{n\Delta}\Delta_n = P_{1*}\Delta_{1*} + P_{2*}\Delta_{2*} + \ldots + P_{m*}\Delta_{m*} \tag{g}$$

Equation (g) may be written in the matrix form

$$[\Delta]^T[P_\Delta] = [\Delta^*]^T[P^*] \tag{h}$$

Transposing both sides of Eq. (e),

$$[\Delta^*]^T = [\Delta]^T[d]^T \tag{i}$$

Substituting Eq. (i) into Eq. (h),

$$[\Delta]^T[P_\Delta] = [\Delta]^T[d]^T[P^*]$$

or

$$[P_\Delta] = [d]^T[P^*]$$

Hence, Eq. (f) is established.

Putting $[P_\Delta] = [k][\Delta]$ and $[P^*] = [k^*][\Delta^*]$ into Eq. (f),

$$[k][\Delta] = [d]^T[k^*][\Delta^*] \tag{j}$$

Substituting for $[\Delta^*]$ from Eq. (e) into Eq. (j),

$$[k][\Delta] = [d]^T[k^*][d][\Delta]$$

or

$$[k] = [d]^T[k^*][d] \tag{11.6}$$

where $[k]$ and $[k^*]$ = stiffness matrices with reference to system coordinates and element coordinates respectively.

It should be noted that the orders of matrices $[d]^T$, $[k^*]$ and $[d]$ and $n \times m^*$, $m^* \times m^*$ and $m^* \times n$ respectively. Hence, the order of product matrix $[d]^T[k^*][d]$ is $n \times n$ which is the order of $[k]$.

Stiffness matrix $[k^*]$ may be developed by considering individual members deforming independently from the rest of the structure. Hence, in order to develop $[k^*]$, the stiffness matrices of individual members have to be computed. As the element coordinates correspond to the end rotations in the case of rigid-jointed plane frames, the stiffness matrix for any member AB is given by the equation

$$[k]_{AB} = \frac{2EI_{AB}}{L_{AB}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

In the case of pin-jointed frames, the element coordinates correspond to the axial deformations. Hence, the stiffness matrix for any member AB is given by the equation

$$[k]_{AB} = \left[\frac{AE}{L}\right]_{AB}$$

The stiffness matrices of the members thus obtained should be placed along the main diagonal of $[k^*]$. This procedure becomes evident if it is noted that element k_{i*j*} is non-zero only if the coordinates i^* and j^* belong to the same member. Element k_{i*j*} is evidently zero if coordinates i^* and j^* belong to different members because in a dismembered structure, displacements given in any member do not produce forces in other members. As only two coordinates have been assigned to each member in a rigid-jointed plane frame, the member stiffness matrices are of order 2×2. Consequently, matrix $[k^*]$ for the entire unassembled structure is a tridiagonal matrix of order $m^* \times m^*$. In the case of pin-jointed frames, only one coordinate has been assigned to each member. Hence, the order of member stiffness matrix is 1×1 and consequently, matrix $[k^*]$ for the entire unassembled structure is a diagonal matrix of order $m^* \times m^*$.

To illustrate the procedure for developing $[k^*]$, consider the frame shown in Fig. 11.12(a).

$$[k^*] = \begin{bmatrix} [k]_{AB} & & & & & \\ & [k]_{BC} & & & 0 & \\ & & [k]_{CD} & & & \\ & & & [k]_{DE} & & \\ & 0 & & & [k]_{EF} & \\ & & & & & [k]_{BE} \end{bmatrix} \tag{11.7}$$

Restraining forces $P'_1, P'_2, \ldots, P'_n$ acting at the system coordinates $1, 2, \ldots, n$ on account of the applied loads other than those acting at the system coordinates in the restrained structure, may be computed by considering the free-bodies of the constituent members and the joints as discussed in Sec. 6.7. Alternatively, the principle of virtual work, Sec. 2.9, may be utilized. For this purpose, the loads other than those acting at the system coordinates may be replaced by their equivalent joint loads as explained in Sec. 11.2. As forces $P'_1, P'_2, \ldots, P'_n$ and the equivalent joint loads are equilibrants of each other, the net work done by them must vanish in accordance with the principle of virtual work. In order to compute force P'_j at system coordinate j, a unit displacement should be given at coordinate j without any displacement at other coordinates. Then, from the principle of virtual work, the net work done by forces $P'_1, P'_2, \ldots, P'_n$ and by the equivalent joint loads should vanish. Thus forces $P'_1, P'_2, \ldots, P'_n$ may be computed by giving a unit displacement successively at coordinates $1, 2, \ldots, n$ and equating the net work done to zero.

After matrices $[k]$ and $[P']$ have been evaluated, the structure may be analysed by the displacement method in the usual manner. The procedure may be described by the following steps:

(i) Determine the degree of kinematic indeterminacy of the structural system. Identify the independent displacement components. Assign coordinates $1, 2, \ldots, n$ to the independent displacement components.

(ii) Identify the elements of the structural system and assign element coordinates $1^*, 2^*, \ldots, m^*$.

(iii) Develop displacement-transformation matrix $[d]$ by giving a unit displacement successively at coordinates $1, 2, \ldots, n$ and determining the displacements at the coordinates $1^*, 2^*, \ldots, m^*$. These displacements constitute the elements of matrix $[d]$. The elements of the jth column of matrix $[d]$ are the displacements at coordinates $1^*, 2^*, \ldots, m^*$ due to a unit displacement at coordinate j.

(iv) Develop the unassembled element stiffness matrix $[k^*]$. For this purpose, develop the stiffness matrices for the individual elements and place them along the main diagonal. The remaining elements of matrix $[k^*]$ may be taken to be zero.

(v) Develop system stiffness matrix $[k]$ by using Eq. (11.6).

(vi) Develop matrix $[P']$ for the structural system by considering the free bodies of the constituent members and the joints or by using the principle of virtual work.

(vii) Having developed matrices $[k]$ and $[P']$ of the structural system, the unknown displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ in the given structure may be computed in the usual manner by using the displacement method.

Example 11.6

Analyse the continuous beam shown in Fig. 11.15(a).

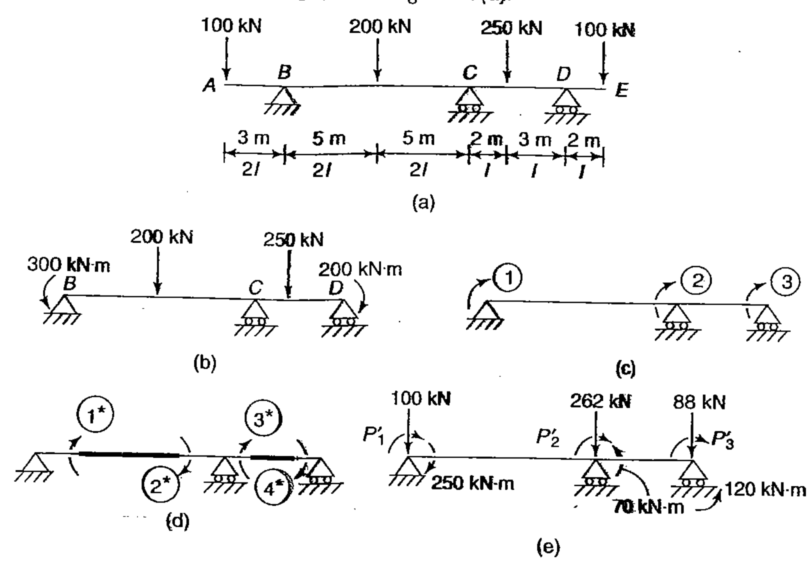


Fig. 11.15

Solution

The given structure is equivalent to the structure shown in Fig. 11.15(b). Let system coordinates 1, 2 and 3 be assigned to the rotations at supports B, C and D as shown in Fig. 11.15(c). Figure 11.15(d) shows element coordinates 1^* to 4^*.

Displacement-transformation matrix $[d]$ may be developed by giving a unit displacement successively at system coordinates 1, 2 and 3 and determining the displacements at element coordinates 1^* to 4^*. Thus to generate the first column of matrix $[d]$, give a unit displacement at system coordinate 1 without any displacement at other system coordinates and determine the displacements at element coordinates 1^* to 4^*. These displacements are

$$d_1{}^*{}_1 = 1 \quad d_2{}^*{}_1 = d_3{}^*{}_1 = d_4{}^*{}_1 = 0$$

Similarly, the second and third columns of matrix $[d]$ may be generated. The displacement-transformation matrix is found to be

$$[d] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{a}$$

The stiffness matrices of members BC and CD with reference to their respective element coordinates are

$$[k]_{BC} = \frac{2E(2I)}{10}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.4EI\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[k]_{CD} = \frac{2EI}{5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.4EI\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Unassembled stiffness matrix $[k^*]$ is obtained by placing the element stiffness matrices along the main diagonal

$$[k^*] = 0.4EI\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{b}$$

Substituting from Eqs (a) and (b) into Eq. (11.6), system stiffness matrix $[k]$ may be obtaind.

$$[k] = 0.4EI\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.8 & 0.4 & 0 \\ 0.4 & 1.6 & 0.4 \\ 0 & 0.4 & 0.8 \end{bmatrix} \tag{c}$$

Restraining forces P_1', P_2' and P_3' and the equivalent joint loads due to the loads other than those acting at the system coordinates are shown in Fig. 11.15(e). In order to determine forces P_1', P_2' and P_3', a unit displacement should be given successively at system coordinates 1, 2 and 3 and the net work done should be equated to zero.

$$P_1' \times 1 + 250 \times 1 = 0$$
$$P_2' \times 1 - 70 \times 1 = 0$$
$$P_3' \times 1 - 120 \times 1 = 0$$

Solving the above equations,

$$P_1' = -250 \text{ kN·m} \qquad P_2' = 70 \text{ kN·m} \qquad P_3' = 120 \text{ kN·m}$$

It may be noted that stiffness matrix $[k]$ and restraining forces P_1', P_2' and P_3' are the same as obtained earlier in Ex. 5.9. Hence, the unknown displacement Δ_1, Δ_2 and Δ_3 may be obtained as in Ex. 5.9.

Example 11.7

Analyse the rigid-jointed plane frame shown in Fig. 11.16(a).

Solution

Let system coordinates 1 and 2 be assigned to the horizontal displacement and the rotation of joint B and coordinate 3 be assigned to the rotation of joint C as shown in Fig. 11.16(b). Figure 11.16(c) shows element coordinates 1^* to 6^*.

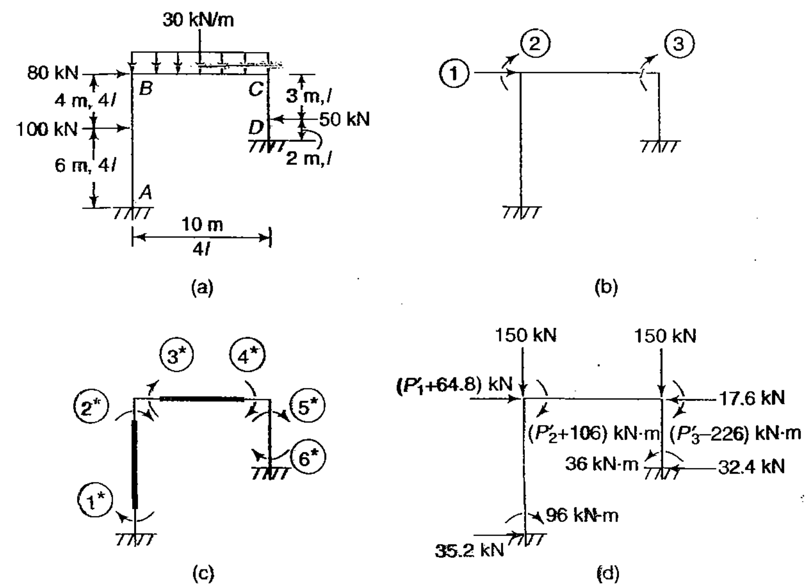


Fig. 11.16

Displacement-transformation matrix $[d]$ may be developed by giving a unit displacement successively at system coordinates 1, 2 and 3 and determining the displacements at element coordinates 1^* to 6^*. Matrix $[d]$ is found to be

$$[d] = \begin{bmatrix} -0.1 & 0 & 0 \\ -0.1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & 0 & 1 \\ -0.2 & 0 & 0 \end{bmatrix} \qquad \text{(a)}$$

Unassembled stiffness matrix $[k^*]$ is given by the equation

$$[k^*] = \begin{bmatrix} \frac{2E(4I)}{10}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & & 0 \\ & \frac{2E(4I)}{10}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \\ 0 & & \frac{2EI}{5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{bmatrix} \qquad \text{(b)}$$

Substituting from Eqs (a) and (b) into Eq. (11.6), the system stiffness matrix may be obtained.

$$[k] = EI\begin{bmatrix} 0.144 & -0.24 & -0.24 \\ -0.24 & 3.2 & 0.8 \\ -0.24 & 0.8 & 2.4 \end{bmatrix} \qquad \text{(c)}$$

Restraining forces P'_1, P'_2 and P'_3 and the equivalent joint loads due to loads other than those acting at the system coordinates, are shown in Fig. 11.16(d). In order to determine forces P'_1, P'_2 and P'_3 a unit displacement should be given successively at system coordinates 1, 2 and 3 and the net work done should be equated to zero.

$$(P'_1 + 64.8) \times 1 - 17.6 \times 1 = 0$$
$$(P'_2 + 106) \times 1 = 0$$
$$(P'_3 - 226) \times 1 = 0$$

Solving the above equations,

$$P'_1 = -47.2 \text{ kN} \qquad P'_2 = -106 \text{ kN·m}$$
$$P'_3 = 226 \text{ kN·m}$$

It may be noted that stiffness matrix $[k]$ and restraining forces P'_1, P'_2 and P'_3 are the same as obtained earlier in Ex. 6.17. Hence, the unknown displacements Δ_1, Δ_2 and Δ_3 may be obtained as in Ex. 6.17.

Example 11.8

Analyse the rigid-jointed plane frame shown in Fig. 11.17(a).

Solution

Let system coordinates 1 and 2 be assigned respectively to the horizontal displacement and rotation of joint B and coordinate 3 be assigned to the rotation of joint C as shown in Fig. 11.17(b). Figure 11.17(c) shows element coordinates 1^* to 6^*.

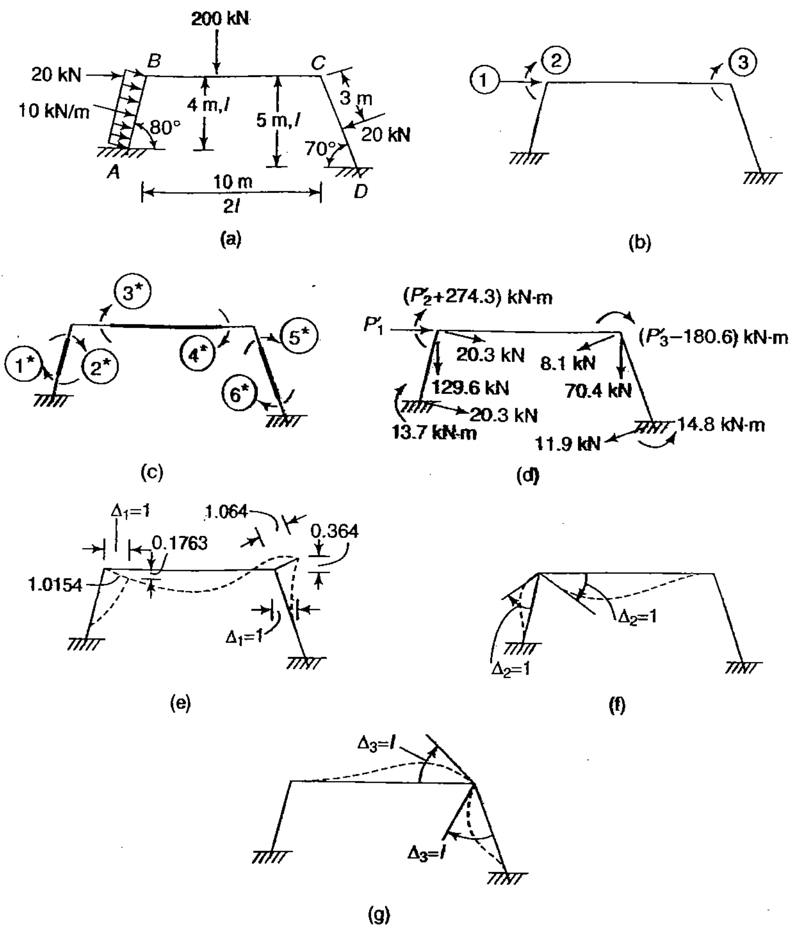


Fig. 11.17

Displacement-transformation matrix $[d]$ may be developed by giving a unit displacement successively at the system coordinates 1, 2 and 3 and determining the displacements at element coordinates 1* to 6*. Referring to Sec. 6.7, the deflected shapes of the structure due to a unit displacement given successively at coordinates 1, 2 and 3 are shown in Fig. 11.17(e), (f) and (g) respectively. Matrix $[d]$ is found to be

$$[d] = \begin{bmatrix} -0.25 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.054 & 1 & 0 \\ 0.054 & 0 & 1 \\ -0.20 & 0 & 1 \\ -0.20 & 0 & 0 \end{bmatrix} \qquad \text{(a)}$$

Unassembled stiffness matrix $[k^*]$ is given by the equation

$$[k^*] = \begin{bmatrix} \frac{2EI}{4.06}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & & 0 \\ & \frac{2E(2I)}{10}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \\ 0 & & \frac{2EI}{5.32}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{bmatrix} \qquad \text{(b)}$$

Substituting from Eqs (a) and (b) into Eq. (11.6), system stiffness matrix $[k]$ may be obtained.

$$[k] = EI \begin{bmatrix} 0.282 & -0.305 & -0.161 \\ -0.305 & 1.785 & 0.400 \\ -0.161 & 0.400 & 1.552 \end{bmatrix} \qquad \text{(c)}$$

Restraining forces P'_1, P'_2 and P'_3 and the equivalent joint loads due to the loads other than those acting at the system coordinates are shown in Fig. 11.17(d). In order to obtain forces P'_1, P'_2 and P'_3, a unit displacement should be given successively at the system coordinates 1, 2 and 3 as shown in Fig. 11.17(e), (f) and (g) respectively and the net work done should be equated to zero.

$$P'_1 \times 1 + 20.3 \times 1.0154 + 129.6 \times 0.1763 - 8.1 \times 1.064 - 70.4 \times 0.364 = 0$$

$$(P'_2 + 274.3) \times 1 = 0$$

$$(P'_3 - 180.6 \times 1) = 0$$

Solving the preceding equations,

$$P'_1 = -9.21 \text{ kN}$$

$$P'_2 = -274.3 \text{ kN·m}$$

$$P'_3 = 180.6 \text{ kN·m}$$

It may be noted that stiffness matrix $[k]$ and restraining forces P'_1, P'_2 and P'_3 are the same as obtained earlier in Ex. 6.22. Hence, the unknown displacements Δ_1, Δ_2 and Δ_3 may be obtained as in Ex. 6.22.

Example 11.9

Analyse the pin-jointed plane frame shown in Fig. 11.18(a). Axial stiffness for each member is 40 kN/mm.

Solution

Let system coordinates 1 to 5 be assigned to the independent displacement components at joints U_1, U_2 and L_2 as shown in Fig. 11.18(b). Figure 11.18(c) shows the element coordinates 1* to 6*.

Displacement-transformation matrix $[d]$ may be developed by giving a unit displacement successively at system coordinates 1 to 5 and determining the displacements at the element coordinates 1* to 6*. Thus to generate the first column of matrix $[d]$, give a unit displacement at system coordinate 1 and determine the displacements at element coordinates 1* to 6*. Referring to Sec. 7.4, the displacements at the element coordinates are

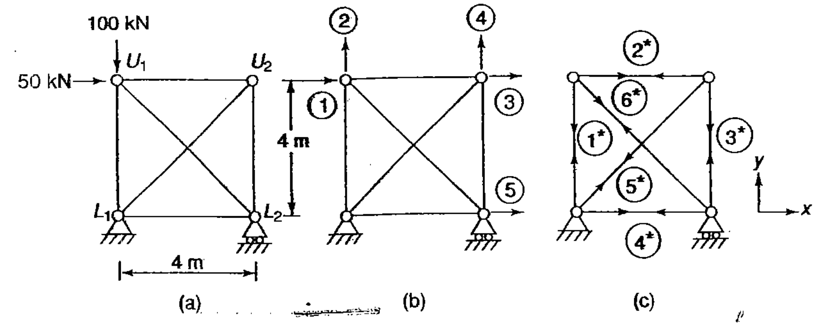


Fig. 11.18

$$d_{1^*1} = -\cos\theta = -\cos 270° = 0$$

$$d_{2^*1} = -\cos 0° = -1$$

$$d_{3^*1} = d_{4^*1} = d_{5^*1} = 0$$

$$d_{6^*1} = -\cos 315° = -0.707$$

It may be noted that the values of angle θ for members U_1U_2, U_1L_1 and U_1L_2 are 0°, 270° and 315° respectively, as it is measured counter-clockwise from the positive direction of the x-axis at joint U_1.

To generate the second column of matrix $[d]$, give a unit displacement at system coordinate 2 and determine the displacements at element coordinates 1* to 6*. These displacements are

$$d_{1^*2} = -\sin\theta = -\sin 270° = 1$$

$$d_{2^*2} = -\sin 0° = 0$$

$$d_{3^*2} = d_{4^*2} = d_{5^*2} = 0$$

$$d_{6^*2} = -\sin 315° = 0.707$$

Similarly, the other columns of matrix $[d]$ may be generated.

$$[d] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 & 0 & 0.707 \end{bmatrix} \qquad \text{(a)}$$

Unassembled stiffness matrix $[k^*]$ is a diagonal matrix of order 6×6. The elements on the main diagonal are equal to the respective values of AE/L for the six members of the frame. The unassembled stiffness matrix may be written as

$$[k^*] = 40 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(b)}$$

Substituting from Eqs (a) and (b) into Eq. (11.6), system stiffness matrix $[k]$ may be obtained.

$$[k] = \begin{bmatrix} 60 & -20 & -40 & 0 & -20 \\ -20 & 60 & 0 & 0 & 20 \\ -40 & 0 & 60 & 20 & 0 \\ 0 & 0 & 20 & 60 & 0 \\ -20 & 20 & 0 & 0 & 60 \end{bmatrix} \qquad \text{(c)}$$

It may be noted that stiffness matrix $[k]$ is the same as obtained earlier in Ex. 7.16. Hence, the unknown displacements Δ_1 to Δ_5 may be obtained as in Ex. 7.16.

11.5 EFFECT OF AXIAL DEFORMATIONS OF MEMBERS

In the analysis of rigid-jointed frames, it is assumed usually that the axial deformations of the members, i.e., the changes in the lengths of the members are small compared to the displacements caused by bending and twisting couples. Thus in the analysis of rigid-jointed frames, the members are assumed to be inextensible or in other words, the axial stiffness of the members are assumed to be infinitely large. In a majority of rigid-jointed structures, this assumption creates only a negligible error in the analysis. However, in certain hybrid structures, having a combination of rigid and pin-connected joints and in certain types of tall structures, the error due to the neglect of axial deformations

of the members may not be negligible. If the effect of axial deformations of the members has to be included in the analysis, the actual values of the axial stiffnesses of the members must be considered. In the element approach discussed in Secs. 11.2 and 11.4, only two coordinates for each element, corresponding to the end couples, are found to be sufficient if the axial deformation of the members are ignored. A third coordinate corresponding to the axial force in the member is necessary if the effect of the changes in lengths of the members has to be considered.

11.5.1 Force Method

As pointed out in the preceding paragraph, three coordinates corresponding to the end couples and the axial force in the element (Fig. 11.19) are necessary for including the effect of axial deformation of the element.

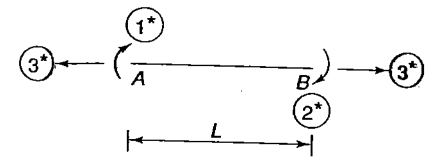


Fig. 11.19

The flexibility matrix with reference to element coordinates 1*, 2* and 3* is given by the equation

$$[\delta]_{AB} = \begin{bmatrix} \dfrac{L}{3EI} & -\dfrac{L}{6EI} & 0 \\ -\dfrac{L}{6EI} & \dfrac{L}{3EI} & 0 \\ 0 & 0 & \dfrac{L}{AE} \end{bmatrix}_{AB} \qquad (11.8)$$

Flexibility matrix $[\delta]$ with reference to the system coordinates is obtained by using Eq. (11.1). The elements of force-transformation matrix $[f]$ may be obtained by using the static analysis discussed in Sec. 11.3. Unassembled flexibility matrix $[\delta^*]$ remains the same as in Eq. (11.2) except that Eq. (11.8) has to be used for flexibility matrices of individual elements. In a similar manner, matrix $[\Delta_L]$ may be determined by using Eq. (11.3) in which the static analysis of Sec. 11.3 may be utilized for computing the elements of matrix $[P_L^*]$.

11.5.2 Displacement Method

When the effect of axial deformations of the members has to be included in the analysis of a rigid-jointed plane frame, three coordinates should be assigned to each joint of the frame. These system coordinates may comprise a horizontal, a vertical and a rotational coordinates. The deformation of an element of the frame can be described in terms of the displacements at the system coordinates located at the two ends of the element. Figure 11.20 shows an element AB of a rigid-jointed plane frame. The displacements at the element coordinates 1*, 2* and 3*, [Fig. 11.20(a)] may be expressed in terms of the displacements at the system coordinates 1 to 6 [Fig. 11.20(b)] by the equation

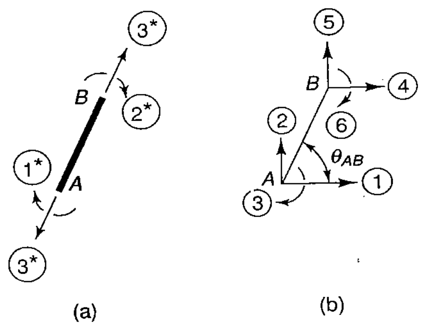


Fig. 11.20

$$\begin{bmatrix}\Delta_{1*}\\ \Delta_{2*}\\ \Delta_{3*}\end{bmatrix}=\begin{bmatrix}\dfrac{\sin\theta_{AB}}{L} & \dfrac{-\cos\theta_{AB}}{L} & 1 & \dfrac{\sin\theta_{BA}}{L} & \dfrac{-\cos\theta_{BA}}{L} & 0\\ \dfrac{\sin\theta_{AB}}{L} & \dfrac{-\cos\theta_{AB}}{L} & 0 & \dfrac{\sin\theta_{BA}}{L} & \dfrac{-\cos\theta_{BA}}{L} & 1\\ -\sin\theta_{AB} & -\sin\theta_{AB} & 0 & -\cos\theta_{BA} & -\sin\theta_{BA} & 0\end{bmatrix}\begin{bmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\\ \Delta_4\\ \Delta_5\\ \Delta_6\end{bmatrix} \qquad (11.9)$$

Similar equations can be written for all the elements of the frame. Thus the displacements at all the element coordinates may be expressed in terms of the displacements at the system coordinates. Hence, Eq. (11.9) is useful in computing the elements of displacement-transformation matrix $[d]$. It may be noted that when a unit displacement is given at system coordinate j located at any joint A, the displacements occur only at those element coordinates which belong to the elements meeting at joint A. Hence, the non-zero elements in the jth column of matrix $[d]$ occur only at those element coordinates which belong to the elements meeting at the joint under consideration.

After displacement-transformation matrix $[d]$ has been developed, Eq. (11.6) may be used to determine system stiffness matrix $[k]$. It may be noted that in using Eq. (11.6), the elements of unassembled stiffness matrix $[k^*]$ have to be written so that the effect of axial deformation of each element is included. Referring to Fig. 11.19, the stiffness matrix for any element AB may be expressed by the equation

$$[k]_{AB}=\begin{bmatrix}\dfrac{4EI}{L} & \dfrac{2EI}{L} & 0\\ \dfrac{2EI}{L} & \dfrac{4EI}{L} & 0\\ 0 & 0 & \dfrac{AE}{L}\end{bmatrix}_{AB} \qquad (11.10)$$

From the foregoing discussion it is evident that the procedure, which includes the effect of axial deformations of the members, leads to a significant proliferation of the element and system coordinates. Consider, for example, the rigid-jointed plane frame shown in Fig. 11.21. In Fig. 11.21(a) the system coordinates have been chosen in the usual manner (Chapter 6) in which the effect of the axial deformations of the members is ignored. In this case only eighteen system coordinates are sufficient for the analysis of the frame. Figure 11.21(b) shows the system coordinates which are necessary if the effect of the axial deformations of the members is considered. As three coordinates are required at each joint in this case, the total number of system coordinates is 45. Thus the system coordinates have to be increased from 18 to 45 in order to include the effect of the axial deformations of the members. The frame has 27 members. Hence the number of element coordinates required would be

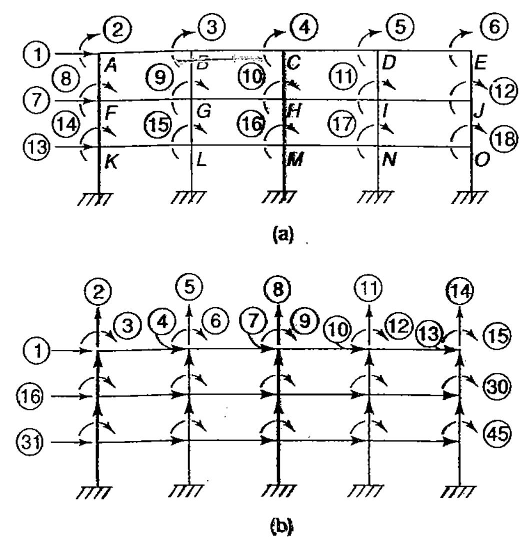


Fig. 11.21

$27 \times 2 = 54$ if the effect of axial deformations of members is ignored. The number of element coordinates would increase to $27 \times 3 = 81$ if the effect of axial deformations of the members has to be included. It follows that the orders of the matrices required in the displacement method such as $[d]$, $[k^*]$ and $[k]$ increase significantly leading to the corresponding increase in the computation effort.

Although the system of coordinates shown in Fig. 11.21(b) entails a greater computational effort, there are certain advantages associated with this coordinate system. The development of matrix $[k]$ becomes simpler when three coordinates are assigned to each joint. In this case the effect of a disturbance applied at any joint is felt only at those joints which are directly connected to the joint under consideration. Consequently, the effect of any disturbance becomes more localised. Also, it is not necessary to consider the displaced configuration of the entire structure if the coordinate system shown in Fig. 11.21(b) is adopted. As in this case three coordinates are assigned to each joint, all the joints, except the one at which the disturbance is applied, are completely immobilised. Hence, when a disturbance is applied at any one of the joints, the displaced configuration of only those members which meet at the joint rather than the displaced configuration of the entire structure has to be taken into account. Consequently, the elements of matrix $[d]$ can be evaluated more easily by using Eq. (11.9). Another advantage of the coordinate system shown in Fig. 11.21(b), is that matrix $[P']$ need not be determined if the applied loads are replaced by their equivalent joint loads so that the frame is loaded only at the system coordinates. As matrix $[P']$ vanishes in this case, Eq. (6.11) takes the simpler form

$$[\Delta] = [k]^{-1}\,[P] \tag{11.11}$$

where $[P]$ is a matrix whose elements are the net forces at the system coordinates when the applied loads are replaced by their equivalent joint loads. It may be mentioned that a static analysis of the entire structure is necessary for determining $[P']$ if the system of coordinates shown in Fig. 11.21(a) is adopted. This is unnecessary if the coordinate system shown in Fig. 11.21(b) is adopted.

Example 11.10

Analyse the rigid-jointed plane frame shown in Fig. 11.22(a) including the effect of the axial deformations of the members. A, I and E are constant. $A = 15000\ mm^2$, $I = 600 \times 10^6\ mm^4$ *and* $E = 200\ kN/mm^2$.

Solution

(i) *Force Method*

The released structure shown in Fig. 11.22(b) has been obtained by inserting hinges at A, B and C. System coordinates 1, 2 and 3 are also shown in the same figure. Element coordinates 1* to 6* are shown in Fig. 11.22(c).

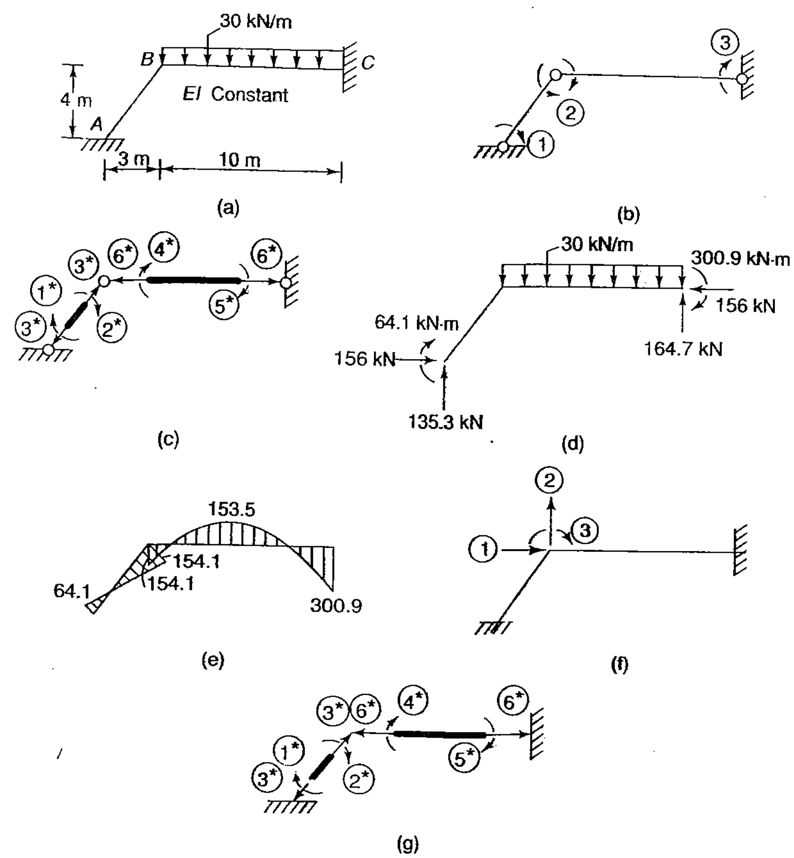


Fig. 11.22

The elements of matrices $[f]$ and $[P_L^*]$ are obtained by determining the forces at the element coordinates due to a unit force applied successively at system coordinates 1, 2 and 3 and due to applied loads. For this purpose, the static analysis may be carried out by using the method of joints discussed in Sec. 11.3. The element forces have already been calculated in Ex. 11.5. These element forces constitute the elements of matrices $[f]$ and $[P_L^*]$. Hence,

$$[f] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -0.15 & 0.275 & 0.125 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.25 & 0.325 & 0.075 \end{bmatrix} \tag{a}$$

$$[P_L^*] = \begin{bmatrix} 0 \\ 0 \\ -187.5 \\ 250 \\ -250 \\ -112.5 \end{bmatrix} \quad \text{(b)}$$

From the given data,

$$A = 15000 \text{ mm}^2 = 0.015 \text{ m}^2$$
$$I = 600 \times 10^6 \text{ mm}^4 = 0.0006 \text{ m}^4$$
$$E = 200 \text{ kN/mm}^2 = 200 \times 10^6 \text{ kN/m}^2$$

Numerically, $\dfrac{L}{AE} = \dfrac{L}{25EI}$

Unassembled flexibility matrix $[\delta^*]$, making allowance for the axial deformations of the members, may be written as

$$[\delta^*] = \begin{bmatrix} \frac{5}{6EI}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0.24 \end{bmatrix} & 0 \\ 0 & \frac{10}{6EI}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0.24 \end{bmatrix} \end{bmatrix}$$

$$= \frac{5}{6EI}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.48 \end{bmatrix} \quad \text{(c)}$$

Substituting from Eqs (a), (b) and (c) into Eqs (11.1) and (11.3),

$$[\delta] = \frac{5}{6EI}\begin{bmatrix} 2.03540 & 0.95110 & -0.01350 \\ 0.95110 & 6.06885 & -1.98005 \\ -0.01350 & -1.98005 & 4.00645 \end{bmatrix} \quad \text{(d)}$$

$$[\Delta_L] = \frac{5}{6EI}\begin{bmatrix} 2.0250 \\ 147.0075 \\ -150.9675 \end{bmatrix} \quad \text{(e)}$$

Substituting from Eqs (d) and (e) into Eq. (6.2) and solving for the redundants,

$$P_1 = 64.06 \text{ kN·m}$$
$$P_2 = -154.11 \text{ kN·m}$$
$$P_3 = 300.86 \text{ kN·m}$$

Hence, the free-body diagram and the bending-moment diagram as shown in Fig. 11.22(d) and (e) may be drawn.

This problem has been solved earlier in Ex. 6.3, ignoring the effect of axial deformations of the members. The bending moments shown in Table 11.1 indicate that there is only a small change on account of the axial deformations of the members.

Table 11.1

Moments	M_{AB}	M_{BA}	M_{CB}
(a) Including the effect of axial deformations	64.06	154.11	300.86
(b) Ignoring the effect of axial deformations	83.33	166.67	291.67

(ii) *Displacement Method*

If the effect of the axial deformations of the members has to be included in the analysis, the degree of freedom of the structure may be taken as 3. System coordinates 1, 2 and 3 assigned to the independent displacement components at joint *B* are shown in Fig. 11.22(f). Element coordinates 1* to 6* have been assigned as shown in Fig. 11.22(g). The elements of displacement-transformation matrix $[d]$ may be determined by using Eq. (11.9). For element *AB*,

$$\begin{bmatrix} \Delta_{1*} \\ \Delta_{2*} \\ \Delta_{3*} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -0.16 & 0.12 & 0 \\ 0 & 0 & 0 & -0.16 & 0.12 & 1 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} \quad \text{(f)}$$

For element *BC*,

$$\begin{bmatrix} \Delta_{4*} \\ \Delta_{5*} \\ \Delta_{6*} \end{bmatrix} = \begin{bmatrix} 0 & -0.1 & 1 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(g)}$$

Combining Eqs (f) and (g),

$$\begin{bmatrix} \Delta_{1*} \\ \Delta_{2*} \\ \Delta_{3*} \\ \Delta_{4*} \\ \Delta_{5*} \\ \Delta_{6*} \end{bmatrix} = \begin{bmatrix} -0.16 & 0.12 & 0 \\ -0.16 & 0.12 & 1 \\ 0.6 & 0.8 & 0 \\ 0 & -0.1 & 1 \\ 0 & -0.1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} \tag{h}$$

In Eq. (h), the first matrix on the right-hand side is displacement-tansformation matrix $[d]$.

From the given data,

$$\frac{AE}{L} = \frac{25EI}{L}$$

Unassembled stiffness matrix $[k^*]$, making allowance for the axial deformations of the members, may be written as

$$[k^*] = \begin{bmatrix} \frac{EI}{5}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 25 \end{bmatrix} & 0 \\ 0 & \frac{EI}{10}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 25 \end{bmatrix} \end{bmatrix}$$

$$= \frac{EI}{10}\begin{bmatrix} 8 & 4 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25 \end{bmatrix} \tag{i}$$

Substituting from Eqs (h) and (i) into Eq. (11.6),

$$[k] = \frac{EI}{10}\begin{bmatrix} 43.6144 & 23.5392 & -1.9200 \\ 23.5392 & 32.4656 & 0.8400 \\ -1.9200 & 0.8400 & 12.0000 \end{bmatrix} \tag{j}$$

Replacing the applied load by its equivalent joints loads, the forces at the system coordinates are

$$P_1 = 0 \qquad P_2 = -150 \text{ kN} \qquad P_3 = 250 \text{ kN·m} \tag{k}$$

Substituting from Eqs (j) and (k) into Eq. (11.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 62.415 \\ -97.282 \\ 235.132 \end{bmatrix}$$

Substituting from Eq. (1) and into Eq. (h),

$$\Delta_{1*} = -\frac{21.660}{EI} \qquad \Delta_{2*} = \frac{203.472}{EI}$$

$$\Delta_{3*} = -\frac{40.377}{EI} \qquad \Delta_{4*} = \frac{234.86}{EI}$$

$$\Delta_5^* = \frac{9.728}{EI} \qquad \Delta_{6*} = -\frac{62.415}{EI}$$

Using slope-deflection Eq. (2.47)

$$M_{AB} = 64.06 \text{ kN·m} \qquad M_{BA} = 154.11 \text{ kN·m}$$

$$M_{BC} = -154.11 \text{ kN·m} \qquad M_{CB} = 300.86 \text{ kN·m}$$

Also,

$$S_{AB} = \frac{AE}{L}\,\Delta_{3*} = \frac{25EI}{5}\left(-\frac{40.377}{EI}\right) = -201.89 \text{ kN}$$

$$S_{BC} = \frac{AE}{L}\,\Delta_{6*} = \frac{25EI}{10}\left(-\frac{62.415}{EI}\right) = -156.04 \text{ kN}$$

Knowing the member forces, the free-body diagram and the bending-moment diagram may be drawn as shown in Fig. 11.22(d) and (e) respectively.

PROBLEMS

11.1 Discuss the element approach and its suitability for the automatic analysis of structures by a digital computer.

11.2 Using the force method and element approach, analyse the structures of the examples and problems given in Chapters 5, 6 and 7.

11.3 Using the displacement method and element approach, analyse the structures of the examples and problems given in Chapters 5, 6 and 7.

11.4 Analyse the portal frame of Fig. 11.23(a) ignoring the effect of the axial forces in the members and (b) including the effect of the axial forces in the members. Hence determine the fixed-end moments at A and D. Numerically, $A = 3I$.

11.5 Analyse the right angled bent shown in Fig. 11.24(a) ignoring the effect of the axial forces in the members and (b) including the effect of the axial forces in the members. Hence determine the fixed-end moments at A and C. Numerically, $A = 2I$.

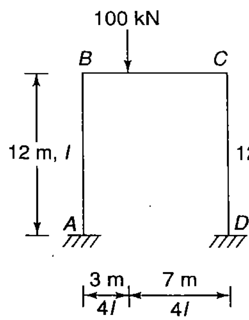


Fig. 11.23

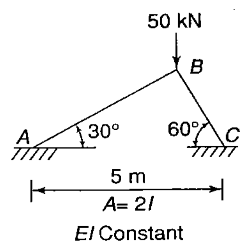


Fig. 11.24

12 SPECIAL PROBLEMS AND TECHNIQUES

12.1 INTRODUCTION

In the preceding chapters, the two main methods, viz., the force method and the displacement method, have been presented in the form in which they are commonly used for the analysis of structures. However, several alternative approaches and special techniques have been developed to achieve greater simplicity and accuracy and lesser computational effort. Some of these alternative approaches and special techniques have been discussed in the following sections.

12.2 CHOICE OF COORDINATES IN FORCE METHOD

There can be several variations of the force method described in the preceding chapters. Two of them which are of particular interest are discussed below:

(A) In the discussion of the force method in the preceding chapters, the sufficient number of redundant forces have been released so that the released structure is statically determinate. Coordinates are assigned to all the redundant forces thus released. This procedure leads to a flexibility matrix whose order is the same as the degree of static indeterminacy of the structure. As the development of the flexibility matrix and its inversion constitute the major computational effort in the force method, it would appear desirable to reduce the order of the flexibility matrix. This objective can be achieved by selecting the number of redundants to be released lesser than the degree of static indeterminacy of the structure. In this case the released structure is statically indeterminate. This procedure can be adopted when the released structure although statically indeterminate, is simple enough for analysis or else it has already been analysed previously. This approach, which is particularly amenable for hand calculations, is illustrated by Ex. 12.1.

(B) In the design of structures, the designer is interested not only in the member forces but also in the displacements at the load points. Besides, he is often required to analyse the structure for several levels of loads such as working loads, cracking loads and ultimate loads and also for different combinations of loads such as gravity loads and lateral loads. For a complete solutions of the problem, coordinates should be assigned to all the loads in addition to the redundants. For this purpose, the distributed loads, if any, may be replaced by their equivalent concentrated loads. If the coordinates are chosen in this manner, the flexibility matrix has to be developed only once. To obtain solution for different levels and combinations of loads, only the load matrix need be revised. The flexibility matrix and its inverse remain unchanged. Thus it is not necessary to formulate the problem afresh for each level and combination of loads. This procedure entails a larger flexibility matrix but this may not be a handicap when the designer has access to a digital computer. The procedure may be described by the following steps:

(i) Assign coordinates 1, 2, ..., j to the redundants and $j+1, j+2, ..., n$ to the applied loads. Let $P_1, P_2, ..., P_j$ be the redundants and $\Delta_{j+1}, \Delta_{j+2}, ..., \Delta_n$ be the displacements along the applied loads $P_{j+1}, P_{j+2}, ..., P_n$.

(ii) The force-displacement relationship may be expressed as

$$\left[\begin{array}{cccc:ccc} \delta_{11} & \delta_{12} & \cdots & \delta_{1j} & \delta_{1,j+1} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2j} & \delta_{2,j+1} & \cdots & \delta_{2n} \\ \vdots & & & & \vdots & & \\ \delta_{j1} & \delta_{j2} & \cdots & \delta_{jj} & \delta_{j,j+1} & \cdots & \delta_{jn} \\ \hdashline \delta_{j+1,1} & \delta_{j+1,2} & \cdots & \delta_{j+1,j} & \delta_{j+1,j+1} & \cdots & \delta_{j+1,n} \\ \delta_{j+2,1} & \delta_{j+2,2} & \cdots & \delta_{j+2,j} & \delta_{j+2,j+1} & & \delta_{j+2,n} \\ \vdots & & & & \vdots & & \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nj} & \delta_{n,j+1} & \cdots & \delta_{nn} \end{array}\right] \left[\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_j \\ \hdashline P_{j+1} \\ P_{j+2} \\ \vdots \\ P_n \end{array}\right] = \left[\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \hdashline \Delta_{j+1} \\ \Delta_{j+2} \\ \vdots \\ \Delta_n \end{array}\right] \qquad (12.1)$$

(iii) Partitioning the matrices as indicated by the dotted lines, Eq. (12.1) may be split up into two matrix equations. The first matrix equation to be utilized for the evaluation of the redundants takes the form

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1j} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2j} \\ \vdots & & & \\ \delta_{j1} & \delta_{j2} & \cdots & \delta_{jj} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \end{bmatrix} + \begin{bmatrix} \delta_{1,j+1} & \delta_{1,j+2} & \cdots & \delta_{1n} \\ \delta_{2,j+1} & \delta_{2,j+2} & \cdots & \delta_{2n} \\ \vdots & & & \\ \delta_{j,j+1} & \delta_{j,j+2} & \cdots & \delta_{jn} \end{bmatrix} \begin{bmatrix} P_{j+1} \\ P_{j+2} \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \end{bmatrix}$$

or

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1j} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2j} \\ \vdots & & & \\ \delta_{j1} & \delta_{j2} & \cdots & \delta_{jj} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \end{bmatrix} - \begin{bmatrix} \delta_{1,j+1} & \delta_{1,j+2} & \cdots & \delta_{1n} \\ \delta_{2,j+1} & \delta_{2,j+2} & \cdots & \delta_{2n} \\ \vdots & & & \\ \delta_{j,j+1} & \delta_{j,j+2} & \cdots & \delta_{jn} \end{bmatrix} \begin{bmatrix} P_{j+1} \\ P_{j+2} \\ \vdots \\ P_n \end{bmatrix} \right\} \qquad (12.2)$$

The second matrix equation to be used for determination of the displacements along the applied loads takes the form

$$\begin{bmatrix} \Delta_{j+1} \\ \Delta_{j+2} \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} \delta_{j+1,1} & \delta_{j+1,2} \cdots & \delta_{j+1,j} \\ \delta_{j+2,1} & \delta_{j+2,2} \cdots & \delta_{j+2,j} \\ \vdots & & \\ \delta_{n1} & \delta_{n2} \cdots & \delta_{nj} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \end{bmatrix} +$$

$$\begin{bmatrix} \delta_{j+1,j+1} & \delta_{j+1,j+2} \cdots & \delta_{j+1,n} \\ \delta_{j+2,j+1} & \delta_{j+2,j+2} \cdots & \delta_{j+2,n} \\ \vdots & & \\ \delta_{n,j+1} & \delta_{n,j+2} \cdots & \delta_{nn} \end{bmatrix} \begin{bmatrix} P_{j+1} \\ P_{j+2} \\ \vdots \\ P_n \end{bmatrix} \qquad (12.3)$$

In this equation, the elements of all the matrices on right hand side are known. It may be noted that redundants $P_1, P_2, ..., P_j$ have already been computed from Eq. (12.2).

The manner in which coordinates are chosen is illustrated by considering the continuous beam shown in Fig. 12.1(a). The deflections and rotations at points F and G have to be computed.

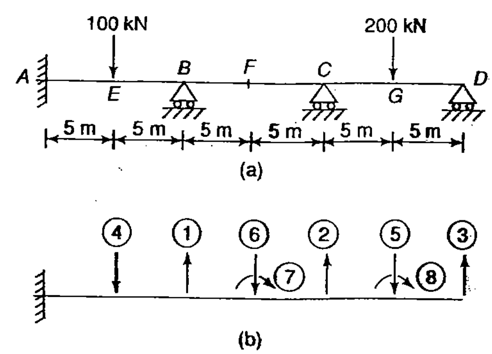


Fig. 12.1

In this problem the coordinates have been chosen in the following manner as shown in Fig. 12.1(b):

(i) Assign coordinates to all the redundants. In this case coordinates 1, 2 and 3 correspond to the redundants.

(ii) Assign coordinates to all the loads. In this case coordinates 4 and 5 correspond to the loads.

(iii) Assign coordinates to all the displacements to be computed. Coordinates 6 and 7 correspond to the deflection and rotation at pont F. Coordinate 5, already assigned to one of the applied loads, corresponds to the deflection at point G also. Coordinate 8 corresponds to the rotation at point G.

Hence the force-displacement relationship takes the from

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} & \delta_{15} & \delta_{16} & \delta_{17} & \delta_{18} \\ \delta_{21} & \delta_{22} & \delta_{23} & \delta_{24} & \delta_{25} & \delta_{26} & \delta_{27} & \delta_{28} \\ \delta_{31} & \delta_{32} & \delta_{33} & \delta_{34} & \delta_{35} & \delta_{36} & \delta_{37} & \delta_{38} \\ \delta_{41} & \delta_{42} & \delta_{43} & \delta_{44} & \delta_{45} & \delta_{46} & \delta_{47} & \delta_{48} \\ \delta_{51} & \delta_{52} & \delta_{53} & \delta_{54} & \delta_{55} & \delta_{56} & \delta_{57} & \delta_{58} \\ \delta_{61} & \delta_{62} & \delta_{63} & \delta_{64} & \delta_{65} & \delta_{66} & \delta_{67} & \delta_{68} \\ \delta_{71} & \delta_{72} & \delta_{73} & \delta_{74} & \delta_{75} & \delta_{76} & \delta_{77} & \delta_{78} \\ \delta_{81} & \delta_{82} & \delta_{83} & \delta_{84} & \delta_{85} & \delta_{86} & \delta_{87} & \delta_{88} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 100 \\ 200 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \end{bmatrix}$$

The procedure described above is illustrated by Ex. 12.2.

Example 12.1

Analyse the continuous beam shown in Fig. 12.2(a).

Solution

The continuous beam shown in Fig. 12.2(a) requires four releases in order to obtain basic determinate structure, thereby requiring the development of a 4 × 4 matrix. Alternatively, it might appear simpler to release the redundant at roller-supports B and C so that the released structure is a fixed beam as shown in Fig. 12.2(b). Although the fixed beam is a statically indeterminate structure, it is simple enough to be analysed. Hence, assign coordinates 1 and 2 as shown in Fig. 12.2(c). The displacements Δ_{1L} and Δ_{2L} in the released structure at coordinates 1 and 2 on account of the applied loads, may be computed by using the conjugate-beam method. The load on the conjugate beam is as shown in Fig. 12.2(d).

$$\Delta_{1L} = M'_B = -\frac{1010.3}{EI}$$

$$\Delta_{2L} = M'_C = -\frac{1056.4}{EI}$$

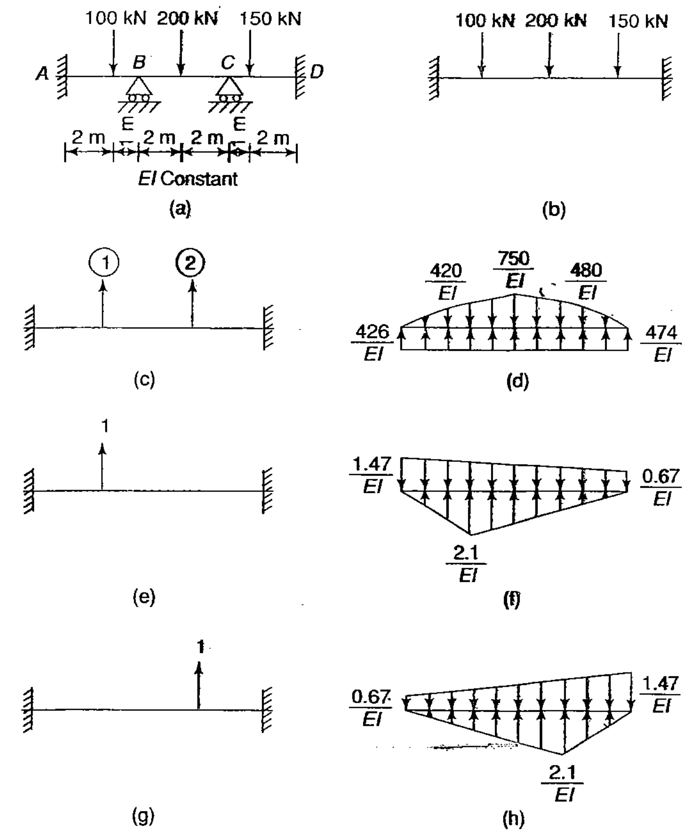


Fig. 12.2

To develop the flexibility matrix, the fixed beam must be analysed with a unit force successively at coordinates 1 and 2. The conjugate-beam method may be used again for this purpose. To generate the first column of the flexibility matrix, apply a unit force at coordinate 1 as shown in Fig. 12.2(e). The corresponding conjugate beam is shown in Fig. 12.2(f).

$$\delta_{11} = M'_B = \frac{3.087}{EI}$$

$$\delta_{21} = M'_C = \frac{1.863}{EI}$$

Similarly, to generate the second column of the flexibility matrix, apply a unit force at coordinate 2 as shown in Fig. 12.2(g). The corresponding conjugate beam is shown in Fig. 12.2(h). As the supports are symmetrically placed,

$$\delta_{22} = \delta_{11} = \frac{3.087}{EI}$$

$$\delta_{12} = \delta_{21} = \frac{1.863}{EI}$$

Hence, using Eq. (5.3),

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -EI \begin{bmatrix} 3.087 & 1.863 \\ 1.863 & 3.087 \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{1010.3}{EI} \\ -\dfrac{1056.4}{EI} \end{bmatrix}$$

Solving for the redundants,

$$P_1 = 189.9 \text{ kN} \qquad P_2 = 227.6 \text{ kN}$$

Having known the redundants, the other forces can be calculated from statics. The free-body diagram and the bending-moment diagram as shown in Fig. 5.17(e) and (f) may be drawn.

Example 12.2

In continuous beam ABCD shown in Fig. 12.3(a), vertical loads are applied at points E and F. Analyse the beam considering the support reactions at B, C and D as redundants. Choosing the coordinates as shown in Fig. 12.3(b), compute the redundants and the displacements at the load points for the following three levels of loads:

(a) $P_4 = -100$ *kN*, $P_5 = -100$ *kN*

(b) $P_4 = -100$ *kN*, $P_5 = -200$ *kN*

(c) $P_4 = -200$ *kN*, $P_5 = -100$ *kN*

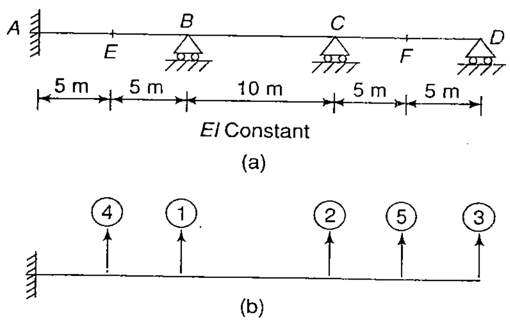


Fig. 12.3

Solution

The flexibility matrix may be developed by using any one of the methods discussed in Chapter 2. In this problem the flexibility matrix $[\delta]$ is given by the equation

$$[\delta] = \frac{1}{6EI}\begin{bmatrix} 2000 & 5000 & 8000 & 625 & 6500 \\ 5000 & 16000 & 28000 & 1375 & 22000 \\ 8000 & 28000 & 54000 & 2125 & 40625 \\ 625 & 1375 & 2125 & 250 & 1750 \\ 6500 & 22000 & 40625 & 1750 & 31250 \end{bmatrix}$$

Nothing that the displacements at coordinates 1, 2 and 3 are zero, force-displacement relationship may be written as

$$\frac{1}{6EI}\left[\begin{array}{ccc|cc} 2000 & 5000 & 8000 & 625 & 6500 \\ 5000 & 16000 & 28000 & 1375 & 22000 \\ 8000 & 28000 & 54000 & 2125 & 40625 \\ \hline 625 & 1375 & 2125 & 250 & 1750 \\ 6500 & 22000 & 40625 & 1750 & 31250 \end{array}\right]\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \hline P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline \Delta_4 \\ \Delta_5 \end{bmatrix}$$

Redundants P_1, P_2 and P_3 may be computed by using Eq. (12.2).

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 6EI \begin{bmatrix} 2000 & 5000 & 8000 \\ 5000 & 16000 & 28000 \\ 8000 & 28000 & 54000 \end{bmatrix}^{-1}$$

$$\times \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6EI}\begin{bmatrix} 625 & 6500 \\ 1375 & 22000 \\ 2125 & 40625 \end{bmatrix}\begin{bmatrix} P_4 \\ P_5 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} -0.473 & 0.173 \\ 0.0865 & -0.731 \\ -0.0144 & -0.399 \end{bmatrix}\begin{bmatrix} P_4 \\ P_5 \end{bmatrix} \qquad \text{(i)}$$

Displacements Δ_4 and Δ_5 at load points may be computed by using Eq. (12.3).

$$\begin{bmatrix} \Delta_4 \\ \Delta_5 \end{bmatrix} = \frac{1}{6EI}\left\{\begin{bmatrix} 625 & 1375 & 2125 \\ 6500 & 22000 & 40625 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} + \begin{bmatrix} 250 & 1750 \\ 1750 & 31250 \end{bmatrix}\begin{bmatrix} P_4 \\ P_5 \end{bmatrix}\right\}$$

$$= \frac{1}{EI}\begin{bmatrix} 7.119 & 0.857 \\ -1.083 & 13.850 \end{bmatrix}\begin{bmatrix} P_4 \\ P_5 \end{bmatrix} \qquad \text{(ii)}$$

Case (a)

$$P_4 = -100 \text{ kN} \qquad P_5 = -100 \text{ kN}$$

Substituting these values into Eqs (i) and (ii),

$P_1 = 30$ kN $\quad P_2 = 64.5$ kN $\quad P_3 = 41.3$ kN

$$\Delta_4 = -\frac{797.6}{EI} \qquad \Delta_5 = -\frac{1276.7}{EI}$$

Case (b)

$P_4 = -100$ kN $\quad P_5 = -200$ kN

Substituting these values into Eqs (i) and (ii),

$P_1 = 12.7$ kN $\quad P_2 = 137.6$ kN $\quad P_3 = 81.2$ kN

$$\Delta_4 = -\frac{883.3}{EI} \qquad \Delta_5 = -\frac{2661.7}{EI}$$

Case (c)

$P_4 = -200$ kN $\quad P_5 = -100$ kN

Substituting these values into Eqs (i) and (ii),

$P_1 = 77.3$ kN $\quad P_2 = 55.8$ kN $\quad P_3 = 42.8$ kN

$$\Delta_4 = -\frac{1509.5}{EI} \qquad \Delta_5 = -\frac{1168.4}{EI}$$

12.3 MIXED RELEASE SYSTEM

It has been shown in Sec. 10.3 that there are several possible released structures for any given statically indeterminate structure. It was also pointed out that the released structure should be so chosen as to achieve the objectives of minimum computational effort, maximum accuracy and simplicity. In general, it may not be possible to achieve the three objectives if a single release system is utilized. The objectives may be served better if two or three released structures are employed. These approaches give rise to the double and triple mixed release systems for the analysis of statically indeterminate structures by the force method.

12.3.1 Double Release System

For a structure, statically indeterminate to the nth degree, consider, among several possible alternatives, any two released structures. Let $1, 2, \ldots, j, \ldots, n$ be the coordinates assigned to redundants, $P_1, P_2, \ldots, P_j, \ldots, P_n$ in the first released structure. Using the principle of superposition, the bending moment M at any cross-section in the given structure may be written as

$$M = M_S + M_R = M_S + m_1P_1 + m_2P_2 + \ldots + m_jP_j + \ldots + m_nP_n \quad (12.4)$$

where M_S = static bending moment in the first released structure due to the applied loads

and m_j = bending moment in the first released structure due to a unit force at coordinate j.

Similarly, if $1', 2', \ldots j', \ldots, n'$ are the coordinates assigned to the redundants $P_{1'}, P_{2'}, \ldots, P_{j'}, \ldots, P_{n'}$ in the second released structure, the bending moment M at any cross-section may be written as

$$M = M_S' + m_{1'}P_{1'} + m_{2'}P_{2'} + \ldots + m_{j'}P_{j'} + \ldots + m_{n'}P_{n'} \quad (12.5)$$

where M_S' = static bending moment in the second released structure due to the applied loads

and $m_{j'}$ = bending moment in the second released structure due to a unit force at coordiante j'.

It may be noted that in Eqs (12.4) and (12.5), terms M_S and M_S', known as the static bending moments, reflect the intrinsic shape of the bending moment diagram for the given statically indeterminate structure due to the applied loads. For instance, if the given load is uniformly distributed, the intrinsic shape of M, M_S and M_S' diagrams is the second degree parabola. Terms M_S and M_S' are analogous to the particular integral in the general solution of a differential equation. The remaining terms on the right-hand sides of Eqs (12.4) and (12.5) are analogous to the complementary function. As the intrinsic shape of all the static moment diagrams is the same, M in the given statically indeterminate structure may be obtained by choosing any one of the static moment diagrams and adding to it the necessary corrective terms by the appropriate choice of the redundants. Thus static moment M'_S of the second released structure may be inserted in Eq. (12.4) relating to the first released structure provided the redundants $P_1, P_2, \ldots, P_j, \ldots, P_n$ at coordinates $1, 2, \ldots j, \ldots n$ are replaced by the appropriate virtual forces $F_1, F_2, \ldots, F_j, \ldots, F_n$.

$$M = M_S' + m_1F_1 + m_2F_2 + \ldots + m_jF_j + \ldots m_nF_n \quad (12.6)$$

The appropriate values of forces $F_1, F_2, \ldots, F_j, \ldots, F_n$ are evidently those which satisfy the force-displacement relationships.

$$\begin{aligned}
\Delta_1 &= \Delta_{1L}' + \delta_{11}F_1 + \delta_{12}F_2 + \ldots + \delta_{1j}F_j + \ldots + \delta_{1n}F_n \\
\Delta_2 &= \Delta_{2L}' + \delta_{21}F_1 + \delta_{22}F_2 + \ldots + \delta_{2j}F_j + \ldots + \delta_{2n}F_n \\
&\vdots \\
\Delta_i &= \Delta_{iL}' + \delta_{i1}F_1 + \delta_{i2}F_2 + \ldots + \delta_{ij}F_j + \ldots + \delta_{in}F_n \\
&\vdots \\
\Delta_n &= \Delta_{nL}' + \delta_{n1}F_1 + \delta_{n2}F_2 + \ldots + \delta_{nj}F_j + \ldots + \delta_{nn}F_n
\end{aligned} \quad \text{(a)}$$

where Δ_i = net displacement at coordinate i in the given statically indeterminate structure

$$\Delta_{iL}' = \int \frac{M_S' m_i d_s}{EI} = \text{displacement at coordinate } i \text{ due to } M_S'$$

and $\delta_{ij} = \int \frac{m_j m_i ds}{EI}$ = displacement at coordinate i due to m_j.

Forces $F_1, F_2, \ldots, F_j, \ldots, F_n$ may be determined by solving Eq. (a). The solution may be expressed in the following matrix form:

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} \ldots & \delta_{1n} \\ \delta_{21} & \delta_{22} \ldots & \delta_{2n} \\ \vdots & & \\ \delta_{n1} & \delta_{n2} \ldots & \delta_{nn} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} - \begin{bmatrix} \Delta'_{1L} \\ \Delta'_{2L} \\ \vdots \\ \Delta'_{nL} \end{bmatrix} \right\} \tag{12.7}$$

Knowing forces $F_1, F_2, \ldots F_j, \ldots, F_n$, the bending moment M in the given statically indeterminate structure may be computed by using Eq. (12.6).

In an alternative approach, using the double release system, the force displacement relationships may be written as

$$\begin{aligned} \Delta_1 &= \Delta'_{1L} + \delta_{11'}P_{1'} + \delta_{12'}P_{2'} + \ldots + \delta_{1j'}P_{j'} + \ldots + \delta_{1n'}P_{n'} \\ \Delta_2 &= \Delta'_{2L} + \delta_{21'}P_{1'} + \delta_{22'}P_{2'} + \ldots + \delta_{2j'}P_{j'} + \ldots + \delta_{2n'}P_{n'} \\ &\vdots \\ \Delta_i &= \Delta'_{iL} + \delta_{i1'}P_{1'} + \delta_{i2'}P_{2'} + \ldots + \delta_{ij'}P_{j'} + \ldots + \delta_{in'}P_{n'} \\ &\vdots \\ \Delta_n &= \Delta'_{nL} + \delta_{n1'}P_{1'} + \delta_{n2'}P_{2'} + \ldots + \delta_{nj'}P_{j'} + \ldots \delta_{nn'}P_{n'} \end{aligned} \tag{b}$$

where $\delta_{ij'} = \int \frac{m_{j'}, m_i ds}{EI}$ = displacement at coordinate i due to $m_{j'}$.

It may be noted that in writing Eq. (b), the second released structure has been considered but the displacements have been computed at coordinates 1, 2, ..., n belonging to the first released structure. Redundants $P_{1'}, P_{2'}, \ldots, P_{n'}$ may be determined by solving Eq. (b).

$$\begin{bmatrix} P_{1'} \\ P_{2'} \\ \vdots \\ P_{n'} \end{bmatrix} = \begin{bmatrix} \delta_{11'} & \delta_{12'} & \ldots & \delta_{1n'} \\ \delta_{21'} & \delta_{22'} & \ldots & \delta_{2n'} \\ \vdots & & & \\ \delta_{n1'} & \delta_{n2'} & \ldots & \delta_{nn'} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} - \begin{bmatrix} \Delta'_{1L} \\ \Delta'_{2L} \\ \vdots \\ \Delta'_{nL} \end{bmatrix} \right\} \tag{12.8}$$

Knowing redundants $P_{1'}, P_{2'}, \ldots, P_{n'}$, bending moment M in the given statically inderminate structure may be computed by using Eq. (12.5).

It may be noted that whereas the flexibility matrix is symmetrical in the first approach, Eq. (12.7), it is unsymmetrical in the second approach, Eq. (12.8).

12.3.2 Triple Release System

In this approach three different released structures are utilized for the analysis of a statically indeterminate structure by the force method. Let coordinates 1″, 2″, ..., n'' be assigned to redundants $P_{1''}, P_{2''}, \ldots, P_{n''}$ in the third released structure. The coordinates and forces of the first and second released structures have already been introduced in the preceding discussion of the double release system. Using the principle of superposition, the bending moment M at any cross-section in the given statically indeterminate structure may be written as

$$M = M''_S + m_{1''}P_{1''} + m_{2''}P_{2''} + \ldots + m_{j''}P_{j''} + \ldots + m_{n''}P_{n''} \tag{12.9}$$

Considering the second released structure, bending moment M may also be expressed as

$$M = M''_S + m_{1'}F_{1'} + m_{2'}F_{2'} + \ldots + m_{j'}F_{j'} + \ldots + m_{n'}F_{n'} \tag{12.10}$$

The appropriate values of forces $F_{1'}, F_{2'}, \ldots, F_{j'}, \ldots, F_{n'}$ are evidently those which satisfy the foce-displacement relationships. For these relationships the displacement at coordinates 1, 2, ..., n belonging to the first system may be considered.

$$\begin{aligned} \Delta_1 &= \Delta''_{1L} + \delta_{11'}F_{1'} + \delta_{12'}F_{2'} + \ldots + \delta_{1j'}F_{j'} + \ldots + \delta_{1n'}F_{n'} \\ \Delta_2 &= \Delta''_{2L} + \delta_{21'}F_{1'} + \delta_{22'}F_{2'} + \ldots + \delta_{2j'}F_{j'} + \ldots + \delta_{2n'}F_{n'} \\ &\vdots \\ \Delta_i &= \Delta''_{iL} + \delta_{i1'}F_{1'} + \delta_{i2'}F_{2'} + \ldots + \delta_{ij'}F_{j'} + \ldots + \delta_{in'}F_{n'} \\ &\vdots \\ \Delta_n &= \Delta'_{nL} + \delta_{n1'}F_{1'} + \delta_{n2'}F_{2'} + \ldots + \delta_{nj'}F_{j'} + \ldots + \delta_{nn'}F_{n'} \end{aligned}$$

where $\Delta''_{iL} = \int \frac{M''_s m_i d_s}{EI}$ = displacement at coordinate i due to M''_S.

Forces $F_{1'}, F_{2'}, \ldots, F_{j'} \ldots F_{n'}$ may be determined by solving Eq. (c). The soluton may be expressed in the following matrix form:

$$\begin{bmatrix} F_{1'} \\ F_{2'} \\ \vdots \\ F_{n'} \end{bmatrix} = \begin{bmatrix} \delta_{11'} & \delta_{12'} \ldots & \delta_{1n'} \\ \delta_{21'} & \delta_{22'} \ldots & \delta_{2n'} \\ \vdots & & \\ \delta_{n1'} & \delta_{n2'} \ldots & \delta_{nn'} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} - \begin{bmatrix} \Delta''_{1L} \\ \Delta''_{2L} \\ \vdots \\ \Delta''_{nL} \end{bmatrix} \right\} \tag{12.11}$$

Knowing forces $F_{1'}, F_{2'}, \ldots, F_{n'}$, bending moment M in the given statically indeterminate structure may be computed by using Eq. (12.10).

It may be noted that Eqs (12.8) and (12.11) are similar. Both of them involve unsymmetrical flexibility matrices. The approach utilizing the triple release system is similar to the approach utilizing the double release system except that in former approach, the static moment diagram is borrowed from a third released structure.

Example 12.3

Analyse the continuous beam shown in Fig. 12.4(a).

Solution

Figure 12.4(b) shows the *first* released structure along with coordinates 1 and 2 assigned to the redundants. Figure 12.4(c) and (d) show m_1 and m_2 diagrams respectively relevant to the first released structure. Using the method of diagram-multiplication (Table 2.11), the flexibility matrix with reference to coordinates 1 and 2 may be obtained

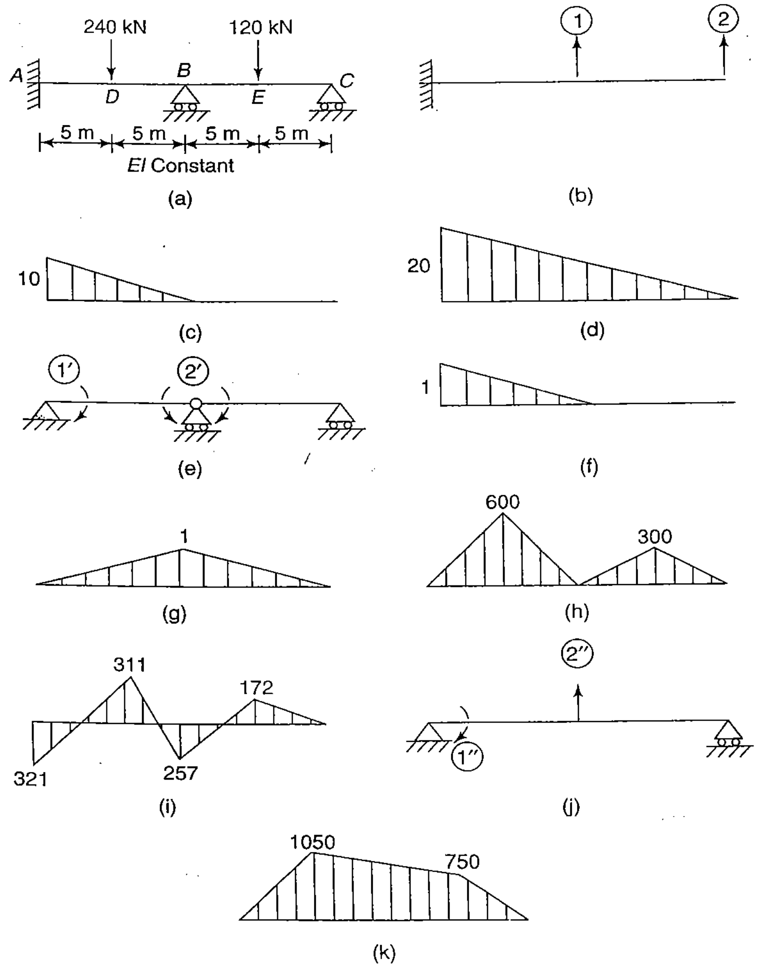


Fig. 12.4

$$\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{500}{3EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix} \tag{a}$$

Figure 12.4(e) shows the *second* released structure along with coordinates 1′ and 2′ assigned to the redundants. Figure 12.4(f) and (g) show $m_{1'}$, and $m_{2'}$ diagrams respectively relevant to the second released structure. Using the method of diagram multiplication, the mixed unsymmetrical flexibility matrix with reference to the coordinates of first and second structures, may be obtained.

$$\begin{bmatrix} \delta_{11'} & \delta_{12'} \\ \delta_{21'} & \delta_{22'} \end{bmatrix} = \frac{50}{3EI}\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix} \tag{b}$$

Also from the given data,

$$\Delta_1 = \Delta_2 = 0 \tag{c}$$

(i) *Double Release System (Symmetrical Matrix)*

In this approach the solution will be obtained by using Eq. (12.7). Figure 12.4(h) shows the M'_S diagram which is the static moment diagram due to the applied loads acting on the second released structure. Combining M'_S diagram with m_1 and m_2 diagrams and using the method of diagram-multiplication,

$$\Delta'_{1L} = \frac{15000}{EI} \qquad \Delta'_{2L} = \frac{52500}{EI} \tag{d}$$

Substituting from Eqs (a), (c) and (d) into Eq. (12.7),

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{3EI}{500}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{15000}{EI} \\ \dfrac{52500}{EI} \end{bmatrix} \right\} \tag{e}$$

Solving the preceding equation,

$$F_1 = 19.28 \text{ kN} \qquad F_2 = -25.71 \text{ kN}$$

Substituting these values into Eq. (12.6),

$$M = M'_S + 19.28m_1 - 25.71m_2$$

Hence, the bending-moment diagram shown in Fig. 12.4(i) is obtained.

(ii) *Double Release System (Unsymmetrical Matrix)*

In this approach the solution will be obtained by using Eq. (12.8). Substituting from Eqs (b), (c) and (d) into Eq. (12.8),

$$\begin{bmatrix} P_{1'} \\ P_{2'} \end{bmatrix} = \frac{3EI}{50}\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{15000}{EI} \\ \dfrac{52500}{EI} \end{bmatrix} \right\} \tag{f}$$

Solving the preceeding equation,

$$P_{1'} = -321.43 \text{ kN·m} \qquad P_{2'} = -257.10 \text{ kN·m}$$

Substituting these values into Eq. (12.5),

$$M = M_S' - 321.43\, m_{1'} - 257.10\, m_{2'}$$

Hence, the bending-moment diagram shown in Fig. 12.4(i) is obtained.

(iii) *Triple Release System*

In this approach the solution will be obtained by using Eq. (12.11). Figure 12.4(j) shows the third released structure. Static moment diagram M_S'' due to the applied loads acting on the third released structure is shown in Fig. 12.4(k). Combining M_S'' diagram with m_1 and m_2 diagrams and using the method of diagram-multiplication,

$$\Delta_{1L}'' = \frac{3000}{EI} \qquad \Delta_{2L}'' = \frac{142500}{EI} \tag{g}$$

Substituting from Eqs (b), (c) and (g) into Eq. (12.11),

$$\begin{bmatrix} F_{1'} \\ F_{2'} \end{bmatrix} = \frac{3EI}{50}\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}^{-1}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{30000}{EI} \\ \dfrac{142500}{EI} \end{bmatrix}\right\} \tag{h}$$

Solving the preceding equation,

$$F_{1'} = -321.43\ \text{kN}\cdot\text{m} \qquad F_{2'} = -1157.14\ \text{kN}\cdot\text{m}$$

Substituting these values into Eq. (12.10),

$$M = M_S'' - 321.43\, m_{1'} - 1157.14\, m_{2'}$$

Hence, the bending-moment diagram as shown in Fig. 12.4(i) obtained.

Example 12.4

Analyse the continuous beam shown in Fig. 12.5(a). The downward displacements of supports B and C in kN-m units are 2000/EI and 1000/EI respectively.

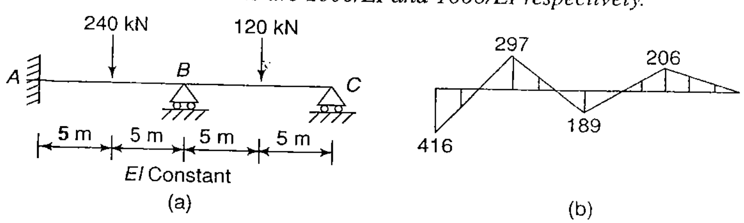


Fig. 12.5

Solution

Example 12.3 has been solved by using three alternative approaches. The same can be used in this example also. If the three released structures are the same as in Ex. 12.3,

$$\Delta_1 = -\frac{2000}{EI} \qquad \Delta_2 = -\frac{1000}{EI} \tag{i}$$

(i) *Double Release System (Symmetrical Matrix)*

In the present case, Eq. (e) of Ex. 12.3 is replaced by the following equation:

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{3EI}{500}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}^{-1}\left\{\begin{bmatrix} -\dfrac{2000}{EI} \\ -\dfrac{1000}{EI} \end{bmatrix} - \begin{bmatrix} \dfrac{15000}{EI} \\ \dfrac{52500}{EI} \end{bmatrix}\right\}$$

Solving the preceding equation,

$$F_1 = -3.86\ \text{kN} \qquad F_2 = -18.86\ \text{kN}$$

Substituting these values into Eq. (12.6).

$$M = M_S' - 3.86 m_1 - 18.86 m_2$$

Hence, the bending-moment diagram shown in Fig. 12.5(b) is obtained.

(ii) *Double Release System (Unsymmetrical Matrix)*

In the present case, Eq. (f) of Ex. 12.3 is replaced by the following equation:

$$\begin{bmatrix} P_{1'} \\ P_{2'} \end{bmatrix} = \frac{3EI}{50}\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}^{-1}\left\{\begin{bmatrix} -\dfrac{2000}{EI} \\ -\dfrac{1000}{EI} \end{bmatrix} - \begin{bmatrix} \dfrac{15000}{EI} \\ \dfrac{52500}{EI} \end{bmatrix}\right\}$$

Solving the preceding equation,

$$P_{1'} = -415.7\ \text{kN}\cdot\text{m} \qquad P_{2'} = -188.6\ \text{kN}\cdot\text{m}$$

Substituting these values into Eq. (12.5),

$$M = M_S' - 415.7 m_{1'} - 188.6 m_{2'}$$

Hence, the bending-moment diagram shown in Fig. 12.5(b) is obtained.

(iii) *Triple Release System*

In the present case, Eq. (h) of Ex. 12.3 is replaced by the following equation:

$$\begin{bmatrix} F_{1'} \\ F_{2'} \end{bmatrix} = \frac{3EI}{50}\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}^{-1}\left\{\begin{bmatrix} -\dfrac{2000}{EI} \\ -\dfrac{1000}{EI} \end{bmatrix} - \begin{bmatrix} \dfrac{30000}{EI} \\ \dfrac{142500}{EI} \end{bmatrix}\right\}$$

Solving the preceding equation,

$$F_{1'} = -415.71\ \text{kN}\cdot\text{m} \qquad F_{2'} = -1088.57\ \text{kN}\cdot\text{m}$$

Substituting these values into Eq. (12.10),

$$M = M_S'' - 415.7 m_{1'} - 1088.57 m_{2'}$$

Hence, the bending-moment diagram shown in Fig. 12.5(b) is obtained.

Example 12.5

Analyse the rigid-jointed plane frame shown in Fig. 12.6(a).

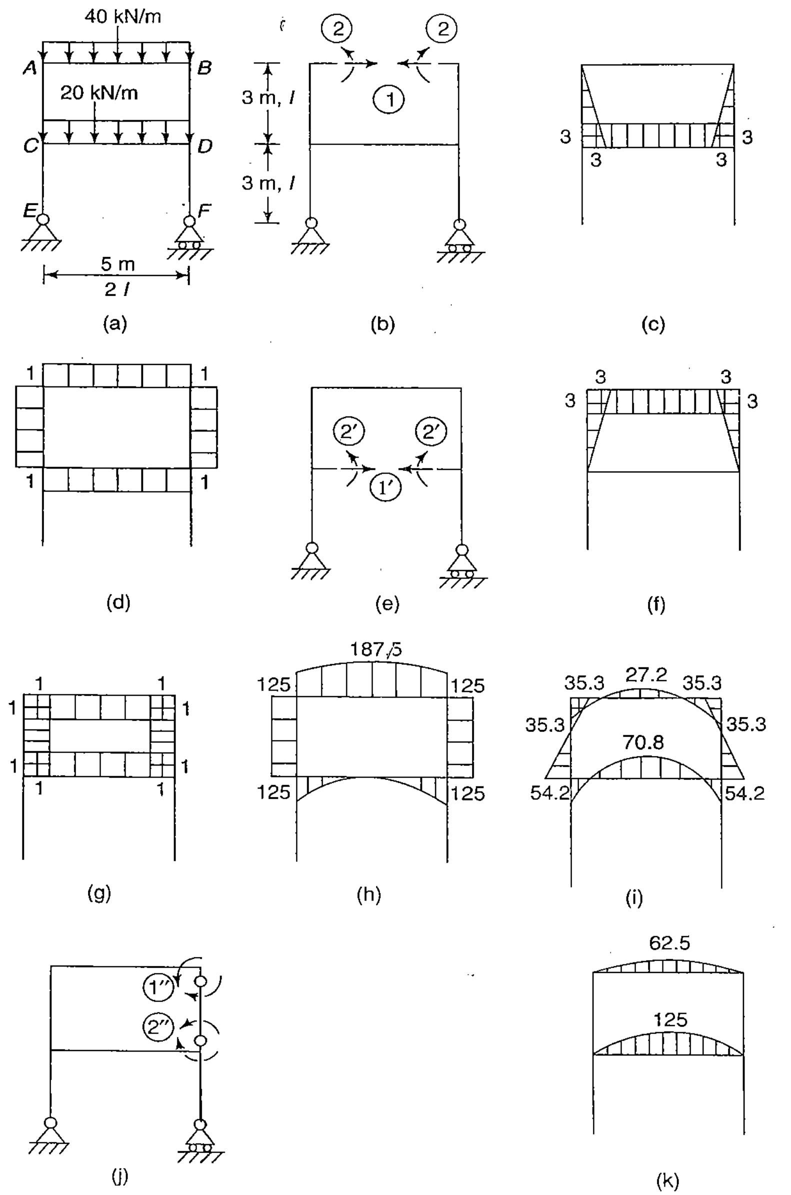


Fig. 12.6

Solution

This problem has been solved earlier in Ex. 6.6. Here the solution is presented using double and triple release systems. The frame is statically determinate externally. The degree of internal indeterminancy would appear to be three because there is one closed cell. However, taking into account the symmetry of the structure and that of the loads, the real degree of static indeterminacy is only two. Consequently, only two releases are necessary to obtain the basic determinate structure.

Figure 12.6(b) shows the first released structure which has been obtained by introducing a cut at the centre of beam AB. Due to symmetry, shear force is absent at the cut. Hence, only two reaction components, viz., an axial force and a bending moment are released. Coordinates 1 and 2 have been assigned to these reaction components.

Figure 12.6(c) and (d) show m_1 and m_2 diagrams respectively relevant to the first released structure. Using the method of diagram-multiplication (Table 2.11), the flexibility matrix with reference to coordinates 1 and 2 may be obtained.

$$\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 40.5 & -16.5 \\ -16.5 & 11 \end{bmatrix} \quad \text{(a)}$$

Figure 12.6(e) shows the *second* released structure with coordinates 1′ and 2′ assigned to the axial force and the bending moment at the centre of beam CD. Figure 12.6(f) and (g) show $m_{1'}$ and $m_{2'}$ diagrams respectively relevant to the second released structure. Using the method of diagram-multiplication, the mixed unsymmetrical flexibility matrix with reference to the coordinates of first and second released structures may be obtained.

$$\begin{bmatrix} \delta_{11'} & \delta_{12'} \\ \delta_{21'} & \delta_{22'} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 9 & 16.5 \\ -16.5 & -11 \end{bmatrix} \quad \text{(b)}$$

Also, for the continuity of the structure,

$$\Delta_1 = \Delta_2 = 0 \quad \text{(c)}$$

(i) *Double Release System (Symmetrical Matrix)*

In this approach the solution will be obtained by using Eq. (12.7). Figure 12.6(h) shows the M_S' diagram which is the static moment diagram due to the applied loads acting on the second released structure. Combining M_S' diagram with m_1 and m_2 diagram and using the method of diagram multiplication,

$$\Delta_{1L}' = -\frac{1437.5}{EI} \qquad \Delta_{2L}' = \frac{1270.8}{EI} \quad \text{(d)}$$

Substituting form Eqs (a), (c) and (d) into Eq. (12.7),

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = EI\begin{bmatrix} 40.5 & -16.5 \\ -16.5 & 11 \end{bmatrix}^{-1}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -\dfrac{1437.5}{EI} \\ \dfrac{1270.8}{EI} \end{bmatrix}\right\}$$

Solving the preceding equation,

$$F_1 = -29.76 \text{ kN} \qquad F_2 = -160.2 \text{ kN}\cdot\text{m}$$

Substituting these values into Eq. (12.6),

$$M = M'_S - 29.76m_1 - 160.2m_2$$

Hence, the bending-moment diagram shown in Fig. 12.6(i) is obtained.

(ii) *Double Release System (Unsymmetrical Matrix)*

In this approach, the solution will be obtained by using Eq. (12.8). Substituting from Eqs (b), (c) and (d) into Eq. (12.8),

$$\begin{bmatrix} P_{1'} \\ P_{2'} \end{bmatrix} = EI\begin{bmatrix} 9 & 16.5 \\ -16.5 & -11 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -\dfrac{1437.5}{EI} \\ \dfrac{1270.8}{EI} \end{bmatrix} \right\}$$

Solving the preceding equation

$$P_{1'} = 29.76 \text{ kN} \qquad P_{2'} = 70.89 \text{ kN}\cdot\text{m}$$

Substituting these value into Eq. (12.5),

$$M = M'_S + 29.76m_{1'} + 70.89m_{2'}$$

Hence, the bending-moment diagram shown in Fig. 12.6(i) is obtained.

(iii) *Triple Release System*

In this approach, the solution will be obtained by using Eq. (12.11). Figure 12.6(j) shows the third released structure. Static moment diagram M''_S due to the applied loads acting on the third released structure is shown in Fig. 12.6(k). Combining M''_S diagram with m_1 and m_2 diagrams and using the method of diagram-multiplication,

$$\Delta''_{1L} = \frac{625}{EI} \qquad \Delta''_{2L} = -\frac{104.2}{EI} \tag{e}$$

Substituting from Eqs (b), (c) and (e) into Eq. (12.11),

$$\begin{bmatrix} F_{1'} \\ F_{2'} \end{bmatrix} = EI\begin{bmatrix} 9 & 16.5 \\ -16.5 & -11 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{625}{EI} \\ -\dfrac{104.2}{EI} \end{bmatrix} \right\}$$

Solving the preceding equation

$$F_{1'} = 29.8 \text{ kN} \qquad F_{2'} = -54.2 \text{ kN}\cdot\text{m}$$

Substituting these values into Eq. (12.10),

$$M = M''_S + 29.8m'_1 - 54.2m'_2$$

Hence the bending-moment diagram as shown in Fig. 12.6(i), may be obtained.

12.4 CHOICE OF COORDINATES IN DISPLACEMENT METHOD

In the discussion of the displacement method in the preceding chapters, the restrained structure has been obtained by preventing all the independent displacement components at the joints. As alternative approaches, it is possible

to consider a restrained structure in which the displacement components to be prevented are either less than or greater than the number of independent displacement components at the joints. These alternative approaches either decrease or increase the order of the resulting stiffness matrix. These approaches, which are analogous to the alternative approaches (A) and (B) of the force method, discussed in Sec. 12.2, are given below.

(A) When the computations are carried out by hand, it might appear desirable to reduce the order of stiffness matrix. This objective can be achieved by considering a partially restrained structure provided it is simple enough for analysis. This approach can also be used when the stiffness of a part of the structure is already known or can be calculated easily. For example, if the stiffness of a three-span continuous beam is known, the four-span continuous beam may be analysed by utilising the information already available for the three-span continuous beam. The procedure is illustrated by Exs. 12.6, 12.7 and 12.8.

(B) When the computations are carried out with the help of a digital computer, a larger matrix may not be a handicap. In this case, a simple procedure can be evolved in which coordinates are assigned to the displacements at the load points in addition to those at the joints. The procedure is illustrated by Ex. 12.9.

Example 12.6

Analyse the continuous beam shown in Fig. 12.7(a).

Solution

The rotations at B and C are the two independent displacement components. The fully restrained structure may be obtained by preventing these rotations as in Ex. 5.6. As an alternative approach, a partially restrained structure, in which only the rotation at B is prevented, may be considered. In this case only one coordinate as shown in Fig. 12.7(b) is required. For the partially restrained structure shown in Fig. 12.7(b),

$$k_{11} = \frac{4EI}{10} + \frac{3EI}{10} = 0.7EI$$

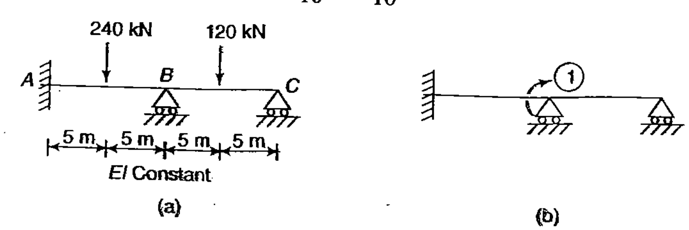


Fig. 12.7

If the spans AB and BC are considered as fixed ended, the fixed-end moments are

$$-M^F_{AB} = M^F_{BA} = \frac{240 \times 10}{8} = 300 \text{ kN}\cdot\text{m}$$

$$-M^F_{BC} = M^F_{CB} = \frac{120 \times 10}{8} = 150 \text{ kN}\cdot\text{m}$$

To make end C permanently free, a counter-clockwise moment of 150 kN·m has to be applied. As half of this moment is carred over to B, the net moment at B in span BC increases to 225 kN·m (counter-clockwise). Hence,

$$P'_1 = 300 - 225 = 75 \text{ kN}\cdot\text{m}$$

The net external moment at B is zero. Hence,

$$P_1 = 0$$

Substituting into Eq. (5.5),

$$[\Delta_1] = -[0.7EI]^{-1}[75] = \left[-\frac{107.14}{EI}\right]$$

Using the slope-deflection Eq. (2.47),

$$M_{AB} = -300 + \frac{2EI}{10}\left(-\frac{107.14}{EI}\right) = -321 \text{ kN}\cdot\text{m}$$

$$M_{BA} = 300 + \frac{2EI}{10}\left[2\left(-\frac{107.14}{EI}\right)\right] = 257 \text{ kN}\cdot\text{m}$$

These bending moments are the same as obtained earlier in Ex. 5.6.

Example 12.7

Analyse the continuous beam shown in Fig. 12.8(a).

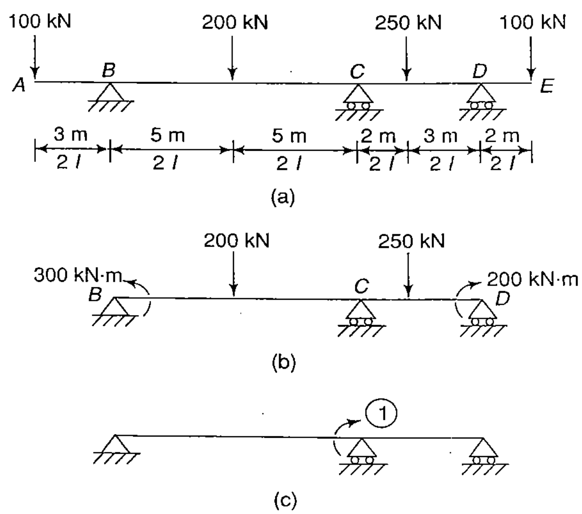


Fig. 12.8

Solution

The given structure is equivalent to that shown in Fig. 12.8(b). This problem has been solved earlier in Ex. 5.9 in which three coordinates have been chosen leading to a stiffness matrix of order 3. As an alternative approach, the problem can be solved by selecting only one coordinate as shown in Fig. 12.8(c). This procedure leads to a single element stiffness matrix.

$$k_{11} = \frac{3E(2I)}{10} + \frac{3EI}{5} = 1.2EI$$

Considering spans BC and CD as fixed ended, the fixed-end moments are

$$-M^F_{BC} = M^F_{CB} = \frac{200 \times 10}{8} = 250 \text{ kN}\cdot\text{m}$$

$$M^F_{CD} = -\frac{250 \times 2 \times 3^2}{5^2} = -180 \text{ kN}\cdot\text{m}$$

$$M^F_{DC} = \frac{250 \times 3 \times 2^2}{5^2} = 120 \text{ kN}\cdot\text{m}$$

As in Ex. 12.6, half the fixed-end moments at B and D, with their signs changed, are carried over to C when ends B and D are released. Besides, half of the external moments at B and D are also carried over to C. Hence,

$$P'_1 = 250 + \frac{250}{2} - \frac{300}{2} - 180 - \frac{120}{2} + \frac{200}{2} = 85 \text{ kN}\cdot\text{m}$$

The net external moment at C is zero. Hence,

$$P_1 = 0$$

Substituting into Eq. (5.5),

$$[\Delta_1] = -[1.2EI]^{-1}[85] = \left[-\frac{70.83}{EI}\right]$$

Knowing Δ_1, the rotation at C, the rotations at B and D may be calculated by using the slope-deflection Eq. (2.47).

$$-300 = -250 + \frac{2E(2I)}{10}\left[2\theta_B - \frac{70.83}{EI}\right]$$

Solving the preceding equation,

$$\theta_B = -\frac{27.08}{EI}$$

Using slope-deflection equation again,

$$M_{CB} = 250 + \frac{2E(2I)}{10}\left[2\left(-\frac{70.83}{EI}\right) - \frac{27.08}{EI}\right] = 182.5 \text{ kN}\cdot\text{m}$$

This moment is the same as obtained earlier in Ex. 5.9.

Example 12.8

Calculate the flexural stiffnesses at points D and E of the continuous beam shown in Fig. 12.9(a).

Solution

It may be seen that the first three spans of the beam are identical to those of Ex. 5.7. The flexural stiffness at point D was found to be $0.6628EI$. Hence, in the present problem coordinates are assigned only at points D and E as shown in Fig. 12.9(b).

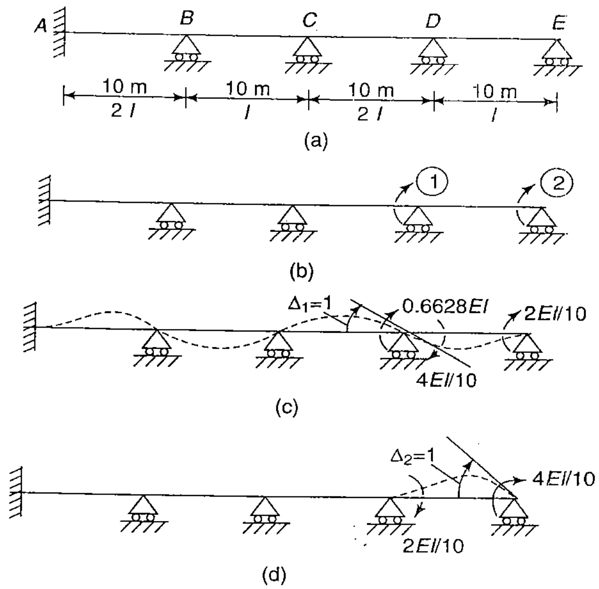


Fig. 12.9

To find out the flexural stiffness at point E, unit couple is applied at point E and the rotation at E is calculated. Using Eq. (5.11),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} - \begin{bmatrix} P_1' \\ P_2' \end{bmatrix} \right\} \qquad \text{(a)}$$

In the present problem, $P_1 = 0$ and $P_2 = 1$ since unit couple is applied only at coordinate 2. Also, $P_1' = P_2' = 0$ since there are no intermediate loads and no settlement of supports to produce fixed-end moments. Hence, substituting these values into Eq. (a),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(b)}$$

The stiffness matrix can be developed by giving unit displacements successively at coordinates 1 and 2 as shown in Fig. 12.9(c) and (d). The elements of the stiffness matrix are

$$k_{11} = 0.6628EI + 0.4EI = 1.0628EI \qquad k_{21} = 0.2EI$$
$$k_{12} = 0.2EI \qquad k_{22} = 0.4EI \qquad \text{(c)}$$

Substituting these values into Eq. (b),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 1.0628 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving the preceding equation,

$$\Delta_2 = \frac{2.76}{EI}$$

Hence, the moment required at E to produce unit rotation at E

$$= \frac{1}{\dfrac{2.76}{EI}} = 0.3623EI$$

The stiffness at point $E = 0.3623EI$

To find out the stiffness at point D in the four-span continuous beam, unit couple is applied at point D and the rotation at D is calculated. Hence, putting $P_1 = 1$, $P_2 = 0$, $P_1' = P_2' = 0$, and the elements of the stiffness matrix from Eq. (c) into Eq. (a),

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 1.0628 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving the preceding equation,

$$\Delta_1 = \frac{1.039}{EI}$$

Hence, the moment required at D to produce unit rotation at D

$$= \frac{1}{\dfrac{1.039}{EI}} = 0.9624EI$$

Thus stiffness at point $D = 0.9624EI$.

Example 12.9

Analyse the continuous beam shown in Fig. 12.10(a). The downward displacement of supports B and C in kN-m units are 2000/EI and 1000/EI respectively.

Solution

The coordinates shown in Fig. 12.10(b) are assigned in the following manner:

(i) Assign coordinates 1 and 2 to unknown displacements at simple supports B and C.

(ii) Assign coordinates 3 to 6 to unknown displacements at load points D and E.

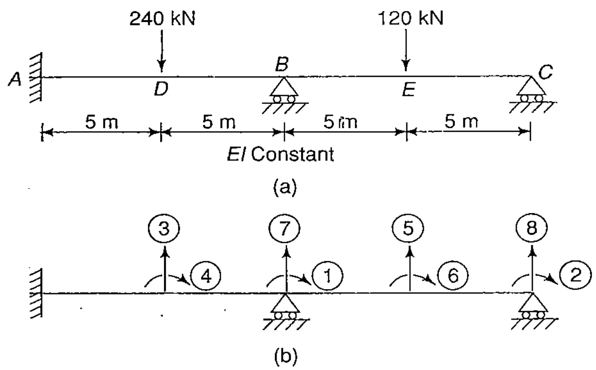


Fig. 12.10

(iii) Assign coordinates 7 and 8 to known displacements (yielding of supports) at supports B and C.

The stiffness matrix with reference to the chosen coordinates may be developed in the usual manner.

$$[k] = \begin{bmatrix} 1.60 & -0.2 & -0.24 & 0.40 & 0.24 & 0.40 & 0 & 0 \\ 0 & 0.80 & 0 & 0 & -0.24 & 0.40 & 0 & 0.24 \\ -0.24 & 0 & 0.192 & 0 & 0 & 0 & -0.096 & 0 \\ 0.4 & 0 & 0 & 1.60 & 0 & 0 & 0.24 & 0 \\ 0.24 & -0.24 & 0 & 0 & 0.192 & 0 & -0.096 & -0.096 \\ 0.40 & 0.40 & 0 & 0 & 0 & 1.60 & -0.24 & 0.24 \\ 0 & 0 & -0.096 & 0.24 & -0.096 & -0.24 & 0.192 & 0 \\ 0 & 0.24 & 0 & 0 & -0.096 & 0.24 & 0 & 0.096 \end{bmatrix}$$

Hence, the load-displacement relationship may be written as

$$\begin{bmatrix} 0 \\ 0 \\ -240 \\ 0 \\ -120 \\ 0 \\ \hline P_7 \\ P_8 \end{bmatrix} = [k][\Delta]$$

$$= EI \left[\begin{array}{cccccc|cc} 1.60 & 0 & -0.24 & 0.40 & 0.24 & 0.40 & 0 & 0 \\ 0 & 0.80 & 0 & 0 & -0.24 & 0.40 & 0 & 0.24 \\ -0.24 & 0 & 0.192 & 0 & 0 & 0 & -0.096 & 0 \\ 0.40 & 0 & 0 & 1.60 & 0 & 0 & 0.24 & 0 \\ 0.24 & -0.24 & 0 & 0 & 0.192 & 0 & -0.096 & -0.096 \\ 0.40 & 0.40 & 0 & 0 & 0 & 1.60 & -0.24 & 0.24 \\ \hline 0 & 0 & -0.096 & 0.24 & -0.096 & -0.24 & 0.192 & 0 \\ 0 & 0.24 & 0 & 0 & -0.096 & 0.24 & 0 & 0.096 \end{array}\right] \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \hline -2000/EI \\ -1000/EI \end{bmatrix} \quad \text{(a)}$$

Partitioning the matrices as indicated by the dotted lines, Eq. (a) may be split up into two matrix equations. The first equation can be written as

$$\begin{bmatrix} 0 \\ 0 \\ -240 \\ 0 \\ -120 \\ 0 \end{bmatrix} = EI \begin{bmatrix} 1.60 & 0 & -0.24 & 0.40 & 0.24 & 0.40 \\ 0 & 0.80 & 0 & 0 & -0.24 & 0.40 \\ -0.24 & 0 & 0.192 & 0 & 0 & 0 \\ 0.40 & 0 & 0 & 1.60 & 0 & 0 \\ 0.24 & -0.24 & 0 & 0 & 0.192 & 0 \\ 0.40 & 0.40 & 0 & 0 & 0 & 1.60 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{bmatrix}$$

$$+ EI \begin{bmatrix} 0 & 0 \\ 0 & 0.24 \\ -0.096 & 0 \\ 0.24 & 0 \\ -0.096 & -0.096 \\ -0.24 & 0.24 \end{bmatrix} \begin{bmatrix} -\dfrac{2000}{EI} \\ -\dfrac{1000}{EI} \end{bmatrix} \quad \text{(b)}$$

Solving for the unknown displacements,

$$\Delta_1 = \frac{21.476}{EI} \qquad \Delta_2 = -\frac{535.603}{EI}$$

$$\Delta_3 = -\frac{2222.688}{EI} \qquad \Delta_4 = \frac{294.658}{EI}$$

$$\Delta_5 = -\frac{2822.88}{EI} \qquad \Delta_6 = -\frac{21.452}{EI} \quad \text{(c)}$$

The second equation may be written as

$$\begin{bmatrix} P_7 \\ P_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.096 & 0.24 & -0.096 & -0.24 \\ 0 & 0.24 & 0 & 0 & -0.096 & 0.24 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{bmatrix}$$

$$+ \begin{bmatrix} 0.192 & 0 \\ 0 & 0.096 \end{bmatrix} \begin{bmatrix} -2000 \\ -1000 \end{bmatrix} \tag{d}$$

Substituting the values of Δ_1 to Δ_6 from Eq. (c) into Eq. (d) and solving for the unknown forces,

$$P_7 = 176.2 \text{ kN} \qquad P_8 = 41.3 \text{ kN}$$

The values coincide with those found earlier in Ex. 5.2.

12.5 ELASTIC SUPPORTS

In the preceding chapters it has been assumed that all supports are either unyielding or the settlements at the supports are prespecified. If the structure rests on elastic supports, the settlements at the supports are not known in advance. The settlements at the supports are proportional to the respective reaction components and occur in directions opposite to those of the reaction components. Thus, if coordinate j is assigned to the reaction component at an elastic support, settlement Δ_j at coordinate j is given by the equation

$$\Delta_j = -a_j P_j = -\frac{P_j}{b_j} \tag{12.12}$$

where P_j = reaction component due to the elastic support at coordinate j

a_j = flexibility of the elastic support along coordinate j

b_j = stiffness of the elastic support along coordinate j.

The minus sign shows that the direction of P_j is opposite to that of Δ_j. Using Eq. (12.12) and the force-displacement relationship, the problem of a structure resting on elastic supports may be solved either by the force method or the displacement method.

12.5.1 Force Method

In solving a problem of elastic supports, the released structure should be chosen in such a manner that all the elastic support reactions are treated as redundants. Consequently, all the elastic reaction components have coordinates assigned

to them. Let coordinates 1, 2, ..., j be assigned to elastic support reaction components and coordinates $j+1, j+2, ..., n$ to the unyielding support reaction components and the internal reaction components, if any. As the displacement at an unyielding support is zero and due to the continuity of the structure, the displacements at coordinates $j+1$, $j+2$, ..., n are zero. Hence, using Eq. (12.12), the force-displacement equations may be written as

$$\Delta_1 = -a_1P_1 = \Delta_{1L} + \delta_{11}P_1 + \delta_{12}P_2 + \ldots + \delta_{1j}P_j + \delta_{1,j+1}P_{j+1} + \ldots + \delta_{1n}P_n$$

$$\Delta_2 = -a_2P_2 = \Delta_{2L} + \delta_{21}P_1 + \delta_{22}P_2 + \ldots + \delta_{2j}P_j + \delta_{2,j+1}P_{j+1} + \ldots + \delta_{2n}P_n$$

$$\vdots$$

$$\Delta_j = -a_jP_j = \Delta_{jL} + \delta_{j1}P_1 + \delta_{j2}P_2 + \ldots + \delta_{jj}P_j + \delta_{j,j+1}P_{j+1} + \ldots + \delta_{jn}P_n$$

$$\Delta_{j+1} = 0 = \Delta_{j+1,L} + \delta_{j+1,1}P_1 + \delta_{j+1,2}P_2 + \ldots + \delta_{j+1,j}P_j + \delta_{j+1,j+1}P_{j+1} + \ldots + \delta_{j+1,n}P_n$$

$$\vdots$$

$$\Delta_n = 0 = \Delta_{nL} + \delta_{n1}P_1 + \delta_{n2}P_2 + \ldots + \delta_{nj}P_j + \delta_{n,j+1}P_{j+1} + \ldots + \delta_{nn}P_n$$

Rearranging the terms, the preceding equations can be written in the following matrix form:

$$[P] = -[\delta_M]^{-1}[\Delta_L] \tag{12.13a}$$

where modified flexibility matrix $[\delta_M]$ is given by the equation

$$[\delta_M] = \begin{bmatrix} (\delta_{11} + a_1) & \delta_{12} & \cdots & \delta_{1j} & \delta_{1,j+1} & \cdots & \delta_{1n} \\ \delta_{21} & (\delta_{22} + a_2) & \cdots & \delta_{2j} & \delta_{2,j+1} & \cdots & \delta_{2n} \\ \vdots & & & & & & \\ \delta_{j1} & \delta_{j2} & \cdots & (\delta_{jj} + a_j) & \delta_{j,j+1} & \cdots & \delta_{jn} \\ \delta_{j+1,1} & \delta_{j+1,2} & \cdots & \delta_{j+1,j} & \delta_{j+1,j+1} & \cdots & \delta_{j+1,n} \\ \vdots & & & & & & \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nj} & \delta_{n,j+1} & \cdots & \delta_{nn} \end{bmatrix} \tag{12.13b}$$

It may be noted that modified flexibility matrix $[\delta_M]$ is obtained by adding the flexibilities of the elastic supports $a_1, a_2, ..., a_j$ to the respective flexibility element $\delta_{11}, \delta_{22}, ...\delta_{jj}$ belonging to the main diagonal of the flexibility matrix $[\delta]$. Equation (12.13) may be used to determine the chosen redundants.

12.5.2 Displacement Method

In using the displacement method, displacements at the elastic supports may be treated as the independent displacement components. Hence, coordinates may also be assigned to them. Let coordinates 1, 2, ..., j be assigned to the displacement components at the elastic supports and coordinates $j+1$, $j+2$, ..., n to the other displacement components. Using Eq. (12.12), the force-displacement equations may be written as

$$
\begin{aligned}
P_1 &= -b_1\Delta_1 = P_1' + k_{11}\Delta_1 + k_{12}\Delta_2 + \ldots + k_{1j}\Delta_j + k_{1,j+1}\Delta_{j+1} + \ldots + k_{1n}\Delta_n \\
P_2 &= -b_2\Delta_2 = P_2' + k_{21}\Delta_1 + k_{22}\Delta_2 + \ldots + k_{2j}\Delta_j + k_{2,j+1}\Delta_{j+1} \ldots + k_{2n}\Delta_n \\
&\vdots \\
P_j &= -b_j\Delta_j = P_j' + k_{j1}\Delta_1 + k_{j2}\Delta_2 + \ldots + k_{jj}\Delta_j + k_{j,j+1}\Delta_{j+1} + \ldots + k_{jn}\Delta_n \\
P_{j+1} &= P_{j+1}' + k_{j+1,1}\Delta_1 + k_{j+1,2}\Delta_2 + \ldots + k_{j+1,j}\Delta_j + k_{j+1,j+1}\Delta_{j+1} + \ldots + k_{j+1,n}\Delta_n \\
&\vdots \\
P_n &= P_n' + k_{n1}\Delta_1 + k_{n2}\Delta_2 + \ldots + k_{nj}\Delta_j + k_{n,j+1}\Delta_{j+1} + \ldots + k_{nn}\Delta_n
\end{aligned}
$$

Rearranging the terms, the preceding equations can be written in the following matrix form:

$$[\Delta] = [k_M]^{-1}\{[P_M] - [P']\} \tag{12.14a}$$

where the modified stiffness matrix $[k_M]$ is given by the equation

$$[k_M] = \begin{bmatrix} (k_{11}+b_1) & k_{12} & \ldots & k_{1j} & k_{1,j+1} & \ldots & k_{1n} \\ k_{21} & (k_{22}+b_2) & \ldots & k_{2j} & k_{2,j+1} & \ldots & k_{2n} \\ \vdots & & & & & & \\ k_{j1} & k_{j2} & \ldots & (k_{jj}+b_j) & k_{j,j+1} & \ldots & k_{jn} \\ k_{j+1,1} & k_{j+1,2} & \ldots & k_{j+1,j} & k_{j+1,j+1} & \ldots & k_{j+1,n} \\ \vdots & & & & & & \\ k_{n1} & k_{n2} & \ldots & k_{nj} & k_{n,j+1} & \ldots & k_{nn} \end{bmatrix} \tag{12.14b}$$

It may be noted that the modified stiffness matrix $[k_M]$ is obtained by adding the stiffnesses of the elastic supports $b_1, b_2, \ldots, b_j$ to the respective stiffness elements $k_{11}, k_{22}, \ldots, k_{jj}$ belonging to the main diagonal of stiffness matrix $[k]$.

Matrix $[P_M]$ is obtained by modifying matrix $[P]$. It is given by the equation

$$[P_M] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ P_{j+1} \\ \vdots \\ P_n \end{bmatrix} \tag{12.14c}$$

Matrix $[P_M]$ is the same as matrix $[P]$ except that the first j elements of matrix $[P_M]$ are zero.

Example 12.10

Analyse the continuous beam shown in Fig. 12.11(a) by the force method. The beam rests on elastic supports at B and C. The flexibilities of supports B and C in kN-m units are 10/EI and 25/EI respectively.

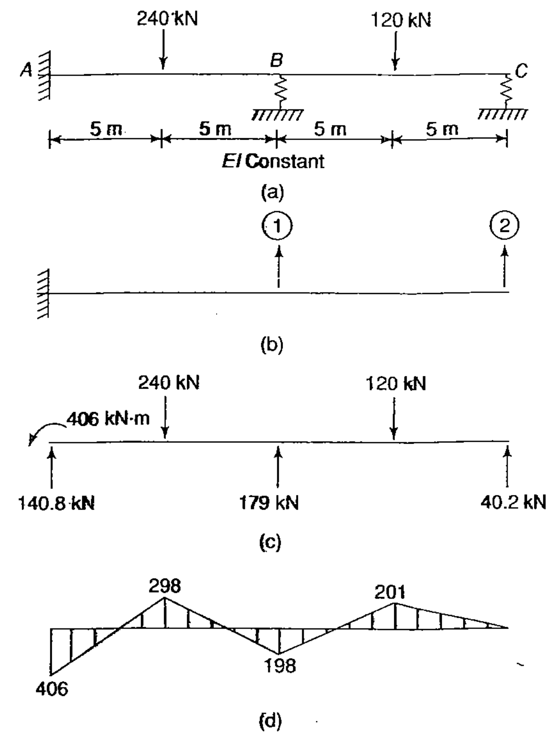


Fig. 12.11

Solution

For the analysis of the beam resting on elastic supports, coordinates 1 and 2 should be chosen along the reactions of the elastic supports as shown in Fig. 12.11(b). The released structure thus obtained is the same as in Ex. 5.1(i). Hence, matrices $[\delta]$ and $[\Delta_L]$ are given by the equations

$$[\delta] = \begin{bmatrix} \dfrac{1000}{3EI} & \dfrac{2500}{3EI} \\ \dfrac{2500}{3EI} & \dfrac{8000}{3EI} \end{bmatrix}$$

$$[\Delta_L] = \begin{bmatrix} -\dfrac{95000}{EI} \\ -\dfrac{257500}{EI} \end{bmatrix}$$

From the given data,

$$a_1 = \frac{10}{EI} \qquad a_2 = \frac{25}{EI}$$

Hence, modified flexibility matrix $[\delta_M]$ is given by the equation

$$[\delta_M] = \begin{bmatrix} \left(\dfrac{1000}{3EI} + \dfrac{10}{EI}\right) & \dfrac{2500}{3EI} \\ \dfrac{2500}{3EI} & \left(\dfrac{8000}{3EI} + \dfrac{25}{EI}\right) \end{bmatrix} = \frac{1}{3EI}\begin{bmatrix} 1030 & 2500 \\ 2500 & 8075 \end{bmatrix}$$

Substituting into Eq. (12.13a) and solving for the redundants,

$$P_1 = 179 \text{ kN} \qquad P_2 = 40.2 \text{ kN}$$

Knowing the redundants, the other reaction components can be calculated from statics. The free-body diagram and the bending-moment diagram may be drawn as shown in Fig. 12.11(c) and (d) respectively.

Example 12.11

Analyse the continuous beam of Ex. 12.10 by the displacement method.

Solution

The continuous beam resting on elastic supports at B and C is shown in Fig. 12.12(a). In solving the problem by the displacement method, the settlements at B and C may be treated as independent displacement components. Hence, coordinates 1 to 4 may be chosen as shown in Fig. 12.12(b). Proceeding in the usual manner, the stiffness matrix with reference to the chosen coordinates may be developed. It is given by the equation

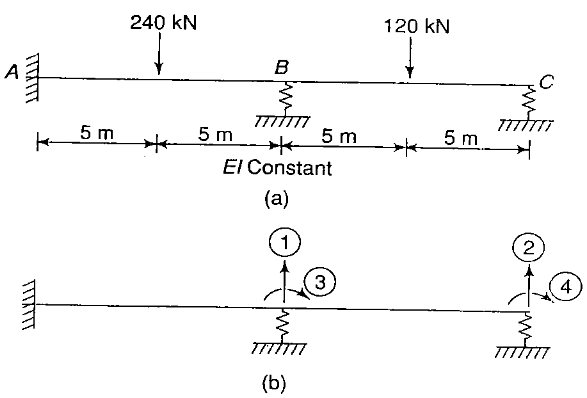


Fig. 12.12

$$[k] = EI\begin{bmatrix} 0.024 & -0.012 & 0 & -0.06 \\ -0.012 & 0.012 & 0.06 & 0.06 \\ 0 & 0.06 & 0.80 & 0.20 \\ -0.06 & 0.06 & 0.20 & 0.40 \end{bmatrix}$$

The stiffness of the elastic supports at B and C are

$$b_1 = \frac{1}{a_1} = \frac{EI}{10} = 0.1EI$$

$$b_2 = \frac{1}{a_2} = \frac{EI}{25} = 0.04EI$$

Hence, modified stiffness matrix $[k_M]$ is given by the equation

$$[k_M] = EI\begin{bmatrix} (0.024 + 0.1) & -0.012 & 0 & -0.06 \\ -0.012 & (0.012 + 0.04) & 0.06 & 0.06 \\ 0 & 0.06 & 0.80 & 0.20 \\ -0.06 & 0.06 & 0.20 & 0.40 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.124 & -0.012 & 0 & -0.06 \\ -0.012 & 0.052 & 0.06 & 0.06 \\ 0 & 0.06 & 0.80 & 0.20 \\ -0.06 & 0.06 & 0.20 & 0.40 \end{bmatrix} \qquad \text{(a)}$$

Matrix $[P']$ may be computed next by considering the spans AB and BC as fixed ended. It is given by the equation

$$[P'] = \begin{bmatrix} 180 \\ 60 \\ 150 \\ 150 \end{bmatrix} \qquad \text{(b)}$$

Using the given data and Eq. (12.14c), modified matrix $[P_M]$ is given by the equation

$$[P_M] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(c)}$$

Substituting from Eqs (a), (b) and (c) into Eq. (12.14a) and solving for the displacement components,

$$\Delta_1 = -\frac{1790}{EI} \qquad \Delta_2 = -\frac{1005}{EI}$$

$$\Delta_3 = \frac{12.6}{EI} \qquad \Delta_4 = -\frac{499.1}{EI}$$

Using Eq. (12.12),

$$P_1 = -b_1\Delta_1 = -01EI\left(-\frac{1790}{EI}\right) = 179 \text{ kN}$$

$$P_2 = -b_2\Delta_2 = -0.04EI\left(-\frac{1005}{EI}\right) = 40.2 \text{ kN}$$

These values coincide with those obtained earlier in Ex. 12.10.

As an alternative approach, the problem may be solved by partitioning the matrices. From the given data,

$$\Delta_1 = -\frac{10}{EI}P_1 \qquad \Delta_2 = -\frac{25}{EI}P_2$$

$$P_3 = P_4 = 0$$

Stiffness matrix $[k]$ and matrix $[P']$ have already been developed.

Rearranging the terms, Eq. (5.4) may be written as

$$[P] - [P'] = [k][\Delta] \tag{d}$$

Substituting into Eq. (d),

$$\begin{bmatrix} P_1 \\ P_1 \\ \hline 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 180 \\ 60 \\ \hline 150 \\ 150 \end{bmatrix} = EI\left[\begin{array}{cc|cc} 0.024 & -0.012 & 0 & -0.06 \\ -0.012 & 0.012 & 0.06 & 0.06 \\ \hline 0 & 0.06 & 0.80 & 0.20 \\ -0.06 & 0.06 & 0.20 & 0.40 \end{array}\right]\begin{bmatrix} -\frac{10}{EI}P_1 \\ -\frac{25}{EI}P_2 \\ \hline \Delta_3 \\ \Delta_4 \end{bmatrix}$$

Partitioning the matrices as indicated by the dotted lines, the preceding equation may be split up into the following two equations:

$$\begin{bmatrix} (P_1 - 180) \\ (P_2 - 60) \end{bmatrix} = EI\begin{bmatrix} 0.024 & -0.012 \\ -0.012 & 0.012 \end{bmatrix}\begin{bmatrix} -\frac{10}{EI}P_1 \\ -\frac{25}{EI}P_2 \end{bmatrix} + EI\begin{bmatrix} 0 & -0.06 \\ 0.06 & 0.06 \end{bmatrix}\begin{bmatrix} \Delta_3 \\ \Delta_4 \end{bmatrix} \tag{e}$$

$$\begin{bmatrix} -150 \\ -150 \end{bmatrix} = EI\begin{bmatrix} 0 & 0.06 \\ -0.06 & 0.06 \end{bmatrix}\begin{bmatrix} -\frac{10}{EI}P_1 \\ -\frac{25}{EI}P_2 \end{bmatrix} + EI\begin{bmatrix} 0.80 & 0.20 \\ 0.20 & 0.40 \end{bmatrix}\begin{bmatrix} \Delta_3 \\ \Delta_4 \end{bmatrix} \tag{f}$$

Solving Eqs (e) and (f),

$$P_1 = 179 \text{ kN} \qquad P_2 = 40.2 \text{ kN}$$

$$\Delta_3 = \frac{12.6}{EI} \qquad \Delta_4 = -\frac{499.1}{EI}$$

These values are the same as obtained earlier.

Example 12.12

Analyse the portal frame shown in Fig. 12.13(a) by the force method. The flexibilities of support D for horizontal and vertical displacements in kN-m units are 10/EI and 20/EI respectively.

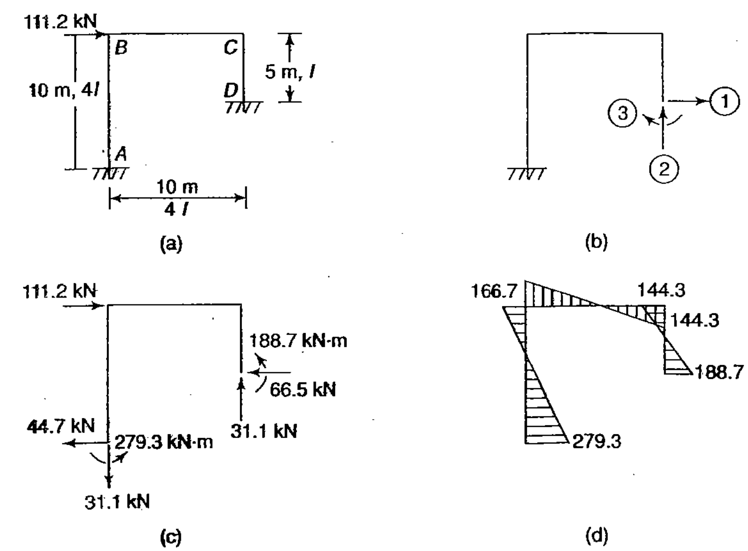


Fig. 12.13

Solution

To analyse the structure by the force method, the coordinates should be chosen as shown in Fig. 12.13(b). Coordinates 1 and 2 have been assigned to the elastic support reactions at D. The matrices $[\delta]$ and $[\Delta_L]$ have already been developed in Ex. 6.1.

$$[\delta] = \frac{1}{6EI}\begin{bmatrix} 750 & 375 & -150 \\ 375 & 2000 & -225 \\ -150 & -225 & 60 \end{bmatrix}$$

$$[\Delta_L] = \frac{1}{EI}\begin{bmatrix} 2316.7 \\ -13900 \\ 1390 \end{bmatrix} \tag{a}$$

Form the given data,

$$a_1 = \frac{10}{EI} \qquad a_2 = \frac{20}{EI}$$

Hence, modified flexibility matrix $[\delta_M]$ is given by the equation

$$[\delta_M] = \frac{1}{6EI}\begin{bmatrix} 810 & 375 & -150 \\ 375 & 2120 & -225 \\ -150 & -225 & 60 \end{bmatrix} \qquad \text{(b)}$$

Substituting into Eq. (12.13a) and solving for the redundants,

$$P_1 = -66.5 \text{ kN} \qquad P_2 = 31.1 \text{ kN} \qquad P_3 = -188.7 \text{ kN}\cdot\text{m}$$

Knowing the reaction components at D, the reaction components at A can be calculated by statics. Hence, the free-body diagram and the bending-moment diagram may be drawn as shown in Fig. 12.13(c) and (d) respectively.

Example 12.13

Analyse the portal frame of Ex. 12.12 by the displacement method.

Solution

Load acting on the portal frame is shown in Fig. 12.14(a). In solving the problem by the displacement method, the horizontal and vertical settlements at support D may be treated as the independent displacement components. Hence, coordinates 1 to 5 may be chosen as shown in Fig. 12.14(b). Proceeding in the usual manner, stiffness matrix $[k]$ with reference to the chosen coordinates may be developed. It is given by the equation

$$[k] = EI\begin{bmatrix} 0.096 & 0 & -0.096 & 0 & 0.24 \\ 0 & 0.048 & 0 & 0.24 & 0.24 \\ -0.096 & 0 & 0.144 & -0.24 & -0.24 \\ 0 & 0.24 & -0.24 & 3.20 & 0.80 \\ 0.24 & 0.24 & -0.24 & 0.80 & 2.40 \end{bmatrix}$$

From the given data, the stiffnesses relative to the horizontal and vertical displacements at support D are

$$b_1 = \frac{1}{a_1} = \frac{EI}{10} = 0.1EI$$

$$b_2 = \frac{1}{a_2} = \frac{EI}{20} = 0.05EI$$

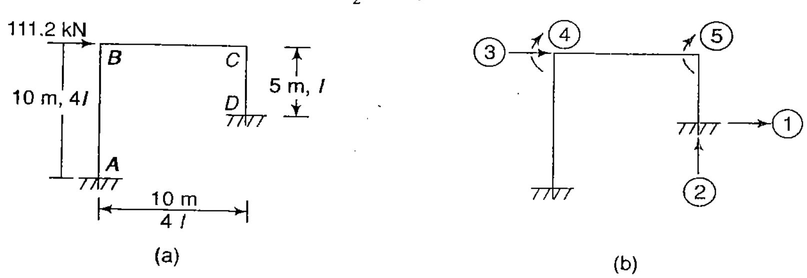


Fig. 12.14

Hence, modified stiffness matrix $[k_M]$ is given by the equation

$$[k_M] = EI\begin{bmatrix} 0.196 & 0 & -0.096 & 0 & 0.24 \\ 0 & 0.098 & 0 & 0.24 & 0.24 \\ -0.096 & 0 & 0.144 & -0.24 & -0.24 \\ 0 & 0.24 & -0.24 & 3.20 & 0.80 \\ 0.24 & 0.24 & -0.24 & 0.80 & 2.40 \end{bmatrix} \qquad \text{(a)}$$

As there are no loads other than those acting at the coordinates, matrix $[P']$ is a null matrix.

$$[P'] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(b)}$$

Using the given data and Eq. (12.14c), modified matrix $[P_M]$ is given by the equation

$$[P_M] = \begin{bmatrix} 0 \\ 0 \\ 111.2 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(c)}$$

Substituting from Eqs (a), (b) and (c) into Eq. (12.14a) and solving for the independent displacement components,

$$\Delta_1 = \frac{665.2}{EI} \qquad \Delta_2 = -\frac{621.6}{EI} \qquad \Delta_3 = \frac{1638.6}{EI}$$

$$\Delta_4 = \frac{141.4}{EI} \qquad \Delta_5 = \frac{112.4}{EI}$$

Using Eq. (12.12),

$$P_1 = -b_1\Delta_1 = -0.1EI\left(\frac{665.2}{EI}\right) = -66.5 \text{ kN}$$

$$P_2 = -b_2\Delta_2 = -0.05EI\left(-\frac{621.6}{EI}\right) = 31.1 \text{ kN}$$

Using slope-deflection Eq. (2.47), the moment at support D,

$$M_{DC} = P_3 = \frac{2EI}{5}\left[\frac{112.4}{EI} - \frac{3}{5}\left(\frac{1638.6}{EI} - \frac{665.2}{EI}\right)\right]$$

$$= -188.7 \text{ kN}\cdot\text{m}$$

Values of the reactive forces at support D are the same as obtained earlier in Ex. 12.12.

As an alternative approach, the problem may be solved by partitioning the matrices. From the given data,

$$\Delta_1 = -\frac{10}{EI}P_1 \qquad \Delta_2 = -\frac{20}{EI}P_2$$

$$P_3 = 111.2 \text{ kN} \qquad P_4 = P_5 = 0$$

Suffness matrix $[k]$ and matrix $[P']$ have already been developed.

Rearranging the terms, Eq. (5.4) may be written as

$$[P] - [P'] = [k][\Delta] \tag{d}$$

Substituting into Eq. (d),

$$\begin{bmatrix} P_1 \\ P_2 \\ \cdots\cdots \\ 111.2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = EI \left[\begin{array}{cc|ccc} 0.096 & 0 & -0.096 & 0 & 0.24 \\ 0 & 0.048 & 0 & 0.24 & 0.24 \\ \hline -0.096 & 0 & 0.144 & -0.24 & -0.24 \\ 0 & 0.24 & -0.24 & 3.20 & 0.80 \\ 0.24 & 0.24 & -0.24 & 0.80 & 2.40 \end{array}\right] \begin{bmatrix} -\frac{10}{EI}P_1 \\ -\frac{20}{EI}P_2 \\ \cdots\cdots\cdots \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \end{bmatrix}$$

Partitioning the matrices as indicated by the dotted lines, the preceding equation may be split up into the following two equations:

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = EI \begin{bmatrix} 0.096 & 0 \\ 0 & 0.048 \end{bmatrix} \begin{bmatrix} -\frac{10}{EI}P_1 \\ -\frac{20}{EI}P_2 \end{bmatrix} + \begin{bmatrix} -0.096 & 0 & 0.24 \\ 0 & 0.24 & 0.24 \end{bmatrix} \begin{bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \end{bmatrix} \tag{e}$$

$$\begin{bmatrix} 111.2 \\ 0 \\ 0 \end{bmatrix} = EI \begin{bmatrix} -0.096 & 0 \\ 0 & 0.24 \\ 0.24 & 0.24 \end{bmatrix} \begin{bmatrix} -\frac{10}{EI}P_1 \\ -\frac{20}{EI}P_2 \end{bmatrix} + \begin{bmatrix} 0.144 & -0.24 & -0.24 \\ -0.24 & 3.20 & 0.80 \\ -0.24 & 0.80 & 2.40 \end{bmatrix} \begin{bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \end{bmatrix} \tag{f}$$

Solving Eqs (e) and (f),

$$P_1 = -66.5 \text{ kN} \qquad P_2 = 31.1 \text{ kN}$$

$$\Delta_3 = \frac{1638.6}{EI} \qquad \Delta_4 = \frac{141.4}{EI} \qquad \Delta_5 = \frac{112.4}{EI}$$

Using Eq. (12.12),

$$\Delta_1 = -a_1P_1 = -\frac{10}{EI}(-66.5) = \frac{665}{EI}$$

$$\Delta_2 = -a_2P_2 = -\frac{20}{EI}(31.1) = -\frac{622}{EI}$$

It may be noted that the displacement components Δ_1 to Δ_5 are the same as obtained earlier.

Example 12.14

Analyse the pin-jointed plane frame shown in Fig. 12.15(a) by the force method. The horizontal displacement at support L_4 is 0.01 mm/kN. The axial flexibility of each member of the frame is 0.02 mm/kN.

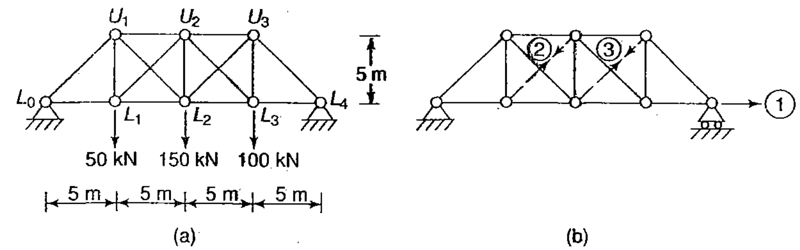


Fig. 12.15

Solution

To analyse the frame by the force method, the coordinates should be chosen as shown in Fig. 12.15(b). Coordinate 1 has been assigned to the elastic horizontal reaction at L_4. Matrices $[\delta]$ and $[\Delta_L]$ have already been developed in Exs. 7.2 and 7.8 in which coordinates have been numbered in a different sequence. For the numbering of coordinates adopted in the present example, these matrices are given by

$$[\delta] = \frac{1}{50}\begin{bmatrix} 4.0 & -0.707 & -0.707 \\ -0.707 & 4.0 & 0.5 \\ -0.707 & 0.5 & 4.0 \end{bmatrix} \tag{a}$$

$$[\Delta_L] = \frac{1}{50}\begin{bmatrix} 600.0 \\ 150.2 \\ 61.9 \end{bmatrix} \tag{b}$$

From the given data,

$$a_1 = 0.01 = \frac{1}{100}$$

Hence, modified flexibility matrix $[\delta_M]$ is given by the equation

$$[\delta_M] = \frac{1}{50}\begin{bmatrix} 4.5 & -0.707 & -0.707 \\ -0.707 & 4.0 & 0.5 \\ -0.707 & 0.5 & 4.0 \end{bmatrix} \tag{c}$$

Substituting into Eq. (12.13a) and solving for the redundants,

$$P_1 = -148.1 \text{ kN} \qquad P_2 = -59.5 \text{ kN} \qquad P_3 = -34.2 \text{ kN}$$

Knowing redundant forces P_1, P_2 and P_3, the forces in the other members of the frame may be calculated by statics.

12.6 CONCLUDING REMARKS

There are several possible variations or alternative approaches in the two main methods of the matrix analysis of structures. Some of them which are of particular interest for greater simplicity, precision and lesser computational effort, have been discussed in this chapter.

Most of the alternative approaches in the force method are related in one way or the other with the selection of the released structure. As there are several possible released structures, a variety of approaches are available for the analysis of a statically indeterminate structure by the force method. In the common approach, discussed in Chapters 5 to 9, the released structure is statically determinate so that the order of the flexibility matrix is equal to the degree of static indeterminacy of the structure. In Sec. 12.2 it has been shown that the released structure may be hyperstatic resulting in the reduction of the size of the flexibility matrix. If the computations are to be carried out by a digital computer and the size of the matrix is not a constraint, the coordinates can also be assigned to the applied loads. This approach is of particular interest when the same structure is to be analysed for different levels, and combinations of applied loads and the displacements at the load points have also to be computed.

In the common approach, only a single released structure is considered in the analysis. However, it is possible to reduce the computational effort and to achieve greater simplicity and precision, if more than one released structure are employed. This objective is achieved by using the mixed release systems presented in Sec. 12.3.

In the displacement method, the common approach, discussed in Chapters 5 to 9, is to consider the restrained structure in which the degree of freedom is governed by the joints alone. This approach leads to the stiffness matrix whose order is equal to the degree of freedom of the structure. If the computations are carried out by a computer, and the size of the matrix is not a constraint, the load points may also be considered as joints. This alternative approach, discussed in Sec. 12.4, may be used when the structure has to be analysed for several levels and combinations of loads and for the displacements at the load points. Section 12.4 also shows how it is possible to reduce the size of the stiffness matrix by using a partially restrained structure.

In Chapters 5 to 7, it has been assumed that either the supports are unyielding or they undergo prespecified settlements. The elastic supports, in which the settlements at the supports are proportional to the respective reaction components, represent a more realistic situation. The structures resting on elastic supports may be analysed by the two main methods in which the flexibility and stiffness matrices are modified so as to include the effect of elastic supports. An alternative approach in the displacement method for dealing with the problem of a structure resting on elastic supports is to use the partitioning of matrices. Both the approaches have been discussed in Sec. 12.5.

PROBLEMS

12.1 Analyse the continuous beam of Fig. 12.16 by the force method in which the partially released structure is obtained by removing the support at B. Hence detemine the bending moments at B and C. Verify the result by an alternative solution in which a partially released structure is obtained by inserting a hinge at support B.

12.2 Using the force method, analyse the continuous beam of Fig. 12.17 by considering a partially released structure obtained by introducing a hinge at support B. Hence determine the fixed-end moments at A and C. Verify the result by abopting another partially released structure obtained by the removal of the support at B.

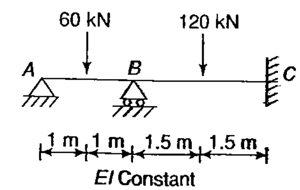


Fig. 12.16

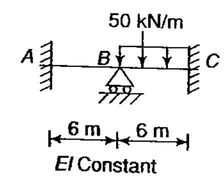


Fig. 12.17

12.3 Analyse the three-span continuous beam of Ex. 12.1 by the force method. Adopt a partially released structure obtained by inserting hinges at B and C. Hence determine the fixed-end moments at A and D.

12.4 Analyse the propped cantilever of Fig. 12.18 by the force method adopting the coordinates shown in the figure. Hence calculate the rotation and deflection at B.

12.5 Using the force method, analyse the fixed beam of Fig. 12.19 with the coordinates chosen as indicated in the figure. Hence determine the rotation and deflection at point B.

12.6 Analyse the two span continuous beam of Fig. 12.20 by the force method with the choice of the coordinates indicated in the figure. Hence determine the bending moment at B.

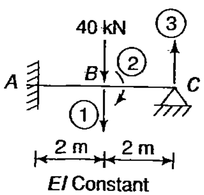


Fig. 12.18

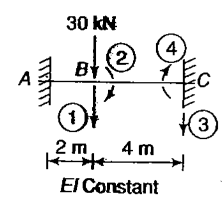


Fig. 12.19

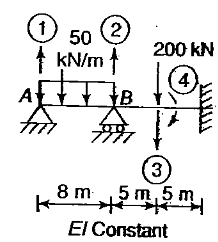


Fig. 12.20

12.7 Analyse the continuous beam of Fig. 12.21 by the force method using double and triple mixed release systems. The released systems may be obtained as follows:

(i) by inserting hinges at B and C.

(ii) by removing the supports at B and C.
(iii) by removing the support at B and inserting hinge at C.
Hence determine the bending moments at B and C.

12.8 Using the force method and the double release system, analyse the continuous beam of Fig. 12.17. Adopt the following two released structures:
(i) a fully released structure obtained by inserting hinges at A, B and C.
(ii) a partially released structure obtained by removing the support at B.
Hence determine the bending moment at B.

12.9 Using the triple release system in the force method, analyse the continuous beam of Fig. 12.22 adopting the following released structures:
(i) hinges inserted at B and C
(ii) supports at A and B removed
(iii) support at B removed and hinge inserted at C.
Hence determine the fixed-end moment at C. Verify the result by adopting the double release system using released structures (i) and (ii).

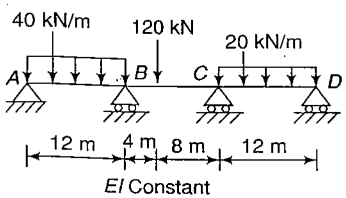


Fig. 12.21

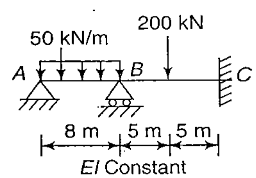


Fig. 12.22

12.10 Analyse the three-span continuous beam of Ex. 12.1 by the force method adopting double and triple release systems. The released structures may be obtained by
(i) inserting hinges at A, B, C and D
(ii) removing the supports at B and C
(iii) inserting hinges at B and C.
Hence determinate the bending moments at B and C.

12.11 Using the double release system in the force method, analyse the continuous beam of Fig. 12.23 adopting the released structures obtained by
(i) removing the support at C
(ii) inserting hinges at A and C.
Hence determinate the bending moment at B.

12.12 Using the double and triple release systems, analyse the right-angled bent shown in Fig. 12.24 adopting the released structures obtained by
(i) inserting hinges at A, B and C
(ii) removing the support at C
(iii) introducing at cut at B..
Hence calculate the support reactions at C.

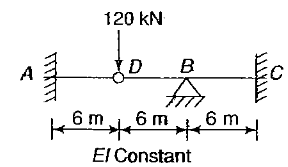


Fig. 12.23

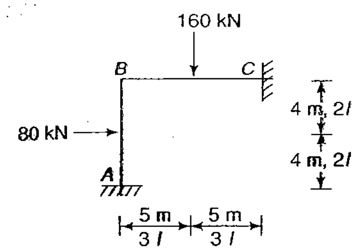


Fig. 12.24

12.13 Analyse the portal frame of Fig. 12.25 by the force method using double or triple release systems obtained by
(i) inserting hinges at A, B and C (ii) removing the support at D
(iii) introducing at cut at B.
Hence determine the fixed-end moments at A and D.

12.14 Analyse the continuous beam of Fig. 12.26 by the displacement method in which a partially restrained structure indicated by the choice of the coordinates shown in the figure is used. Hence determine the support reactions at B and C.

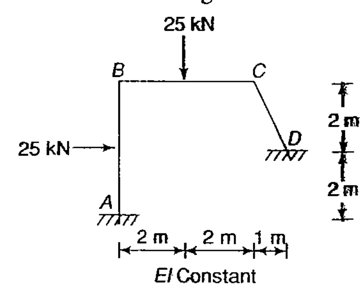


Fig. 12.25

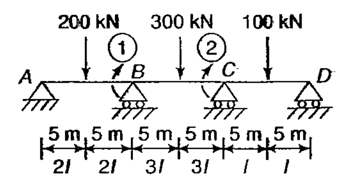


Fig. 12.26

12.15 Analyse the right-angled bent of Fig. 12.27 by the displacement method. Adopt a partially restrained structure in which only the rotation at B is prevented. Hence determine the bending moment at B.

12.16 Analyse the rigid-joined frame of Fig. 12.28 using the displacement method. For the solution adopt a partially restrained structure obtained by preventing the rotation at joint O. Hence determine the fixed-end moments at B, C and D.

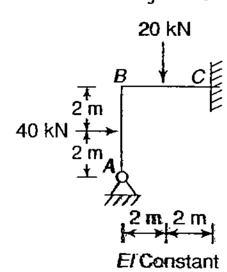


Fig. 12.27

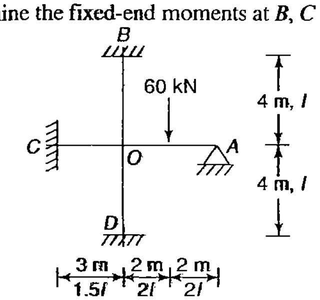


Fig. 12.28

12.17 Using the displacement method, calculate the rotation and the deflection at point B in the fixed beam of Fig. 12.29. The coordinates may be chosen as indicated in the figure.

12.18 Using the displacement method, analyse the right-angled bent of Fig. 12.30. The coordinates should be chosen as indicated in the figure. Hence calculate the bending moment at C.

12.19 Analyse the rigid-jointed frame of Fig. 12.31 by the displacement method with the coordinates chosen as indicated in the figure. Hence calculate the bending moment at O.

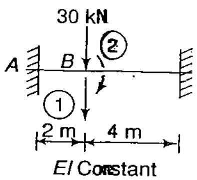


Fig. 12.29

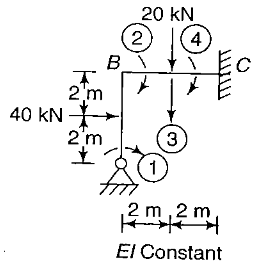


Fig. 12.30

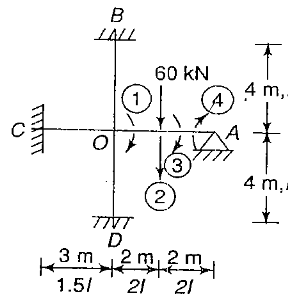


Fig. 12.31

12.20 A cantilever beam AB of length 5 m is fixed at A and supported by the elastic prop at B. The beam carries a uniformly distributed load of 4 kN/m. The stiffness of the prop is 1.8 kN/mm. Analyse the beam by the force method. Hence determine the fixed-end moment at A. Verify the result by the displacement method. Take $EI = 32000$ kN·m^2

12.21 If the cantilever beam of Prob. 12.20 carries a central concentrated load of 20 kN instead of the uniformly distributed load, determine the fixed-end moment at A by the force method. Verify the result by the displacement method.

12.22 A simply supported beam AB of length 5 m has an elastic restraint against rotation at end A which develops a bending couple of 2000 kN·m for a unit rotation. The beam carries a central load of 50 kN. Analyse the beam by the force method. Hence determine the fixed-end moment at A. Verify the result by the displacement method. Take $EI = 1000$ kN·m^2.

12.23 The two-span continuous beam shown in Fig. 12.32 has an elastic support at B with a flexibility of 1.0 mm/kN. Analyse the beam by the force method. Hence determine the fixed-end moment at A. Verify the result by the displacement method.

12.24 The continuous beam of Fig. 12.33 has an elastic support at C. The stiffness of the elastic support is 1.2 kN/mm. Analyse the beam by the force method. Hence determine the bending moment at C. Varify the result by the displacement method.

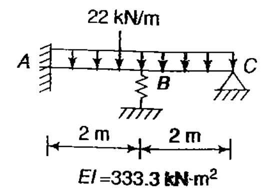


Fig. 12.32

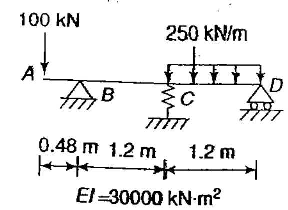


Fig. 12.33

12.25 The continuous beam of Fig. 12.34 rests on elastic supports at B and C. The elastic supports have a flexibility of 1.0 mm/kN. Analyse the beam by the force method if the beam carries a uniformly distributed load of 15 kN/m over the entire length. Hence determine the reactions at supports B and C. Verify the result by the displacement method.

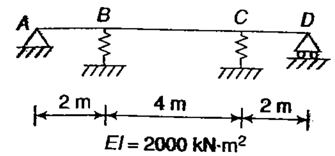
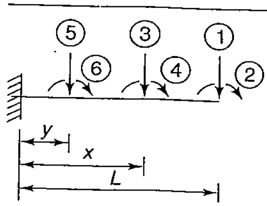


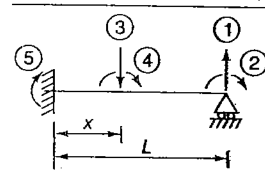
Fig. 12.34

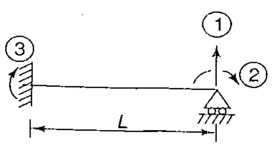
12.26 A three-span continuous beam $ABCD$ rests on simple supports at A and D and on elastic supports at B and C. $AB = BC = CD = L$. A vertical downward load P acts at B. Calculate the support reactions at B and C if the axial flexibility of the elastic supports is $L^3/6EI$ where EI is the flexural rigidity of the beam.

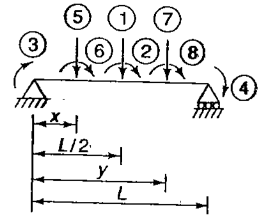
APPENDIX A
Comprehensive List of Standard Results

Expressions for the forces and displacements in structural members commonly encountered in structural analysis are given in this Appendix. All forces and displacements are expressed with reference to the generalized system of coordinates introduced in Sec. 2.3. Column 1 of the table shows the structural member along with the support conditions and the coordinates. Column 2 shows the force applied to the structural member or the displacement given at any one of the coordinates. The resulting forces and displacements are shown in column 3. It has been assumed throughout that the structural member is prismatic so that EI is constant.

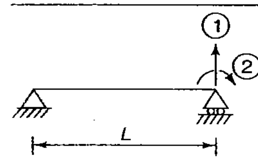
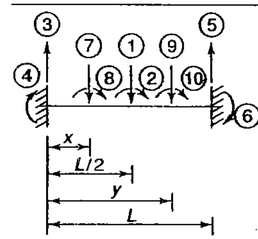
Member and coordinate	*Applied load or displacement*	*Expression*	*Equation*
1	2	3	4
	P_1	$\Delta_1 = \dfrac{P_1L^3}{3EI}$	A.1
		$\Delta_2 = \dfrac{P_1L^2}{2EI}$	A.2
		$\Delta_3 = \dfrac{P_1x^2}{6EI}(3L - x)$	A.3
		$\Delta_4 = \dfrac{P_1x}{2EI}(2L - x)$	A.4
	P_2	$\Delta_1 = \dfrac{P_2L^2}{2EI}$	A.5
		$\Delta_2 = \dfrac{P_2L}{EI}$	A.6
		$\Delta_3 = \dfrac{P_2x^2}{2EI}$	A.7
		$\Delta_4 = \dfrac{P_2x}{EI}$	A.8

1	2	3	4
	P_3	$\Delta_1 = \dfrac{P_3x^2}{6EI}(3L - x)$	A.9
		$\Delta_2 = \Delta_4 = \dfrac{P_3x^2}{2EI}$	A.10
		$\Delta_3 = \dfrac{P_3x^3}{3EI}$	A.11
		$\Delta_5 = \dfrac{P_3y^2}{6EI}(3x - y)$	A.12
		$\Delta_6 = \dfrac{P_3y}{2EI}(2x - y)$	A.13
	P_4	$\Delta_1 = \dfrac{P_4x}{2EI}(2L - x)$	A.14
		$\Delta_2 = \Delta_4 = \dfrac{P_4x}{EI}$	A.15
		$\Delta_3 = \dfrac{P_4x^2}{2EI}$	A.16
		$\Delta_5 = \dfrac{P_4y^2}{2EI}$	A.17
		$\Delta_6 = \dfrac{P_4y}{EI}$	A.18
	Uniform load p per unit length	$\Delta_1 = \dfrac{pL^4}{8EI}$	A.19
		$\Delta_2 = \dfrac{pL^3}{6EI}$	A.20
		$\Delta_3 = \dfrac{px^2}{24EI}(6L^2 - 4Lx + x^2)$	A.21
		$\Delta_4 = \dfrac{px}{6EI}(3L^2 - 3Lx + x^2)$	A.22
	P_2	$P_1 = 1.5\,\dfrac{P_2}{L}$	A.23
		$\Delta_2 = \dfrac{P_2L}{4EI}$	A.24
		$\Delta_3 = -\dfrac{P_2x^2}{4LEI}(L - x)$	A.25

1	2	3	4
		$\Delta_4 = \frac{P_2 x}{4LEI}(3x - 2L)$	A.26
		$P_5 = 0.5P_2$	A.27
	P_3	$P_1 = \frac{P_3 x^2}{2L^3}(3L - x)$	A.28
		$\Delta_2 = -\frac{P_3 x^2}{4LEI}(L - x)$	A.29
		$\Delta_3 = \frac{P_3 x^3}{12L^3 EI} \times (4L^3 - 9L^2 x + 6Lx^2 - x^3)$	A.30
		$\Delta_4 = \frac{P_3 x^2}{4L^3 EI} \times (2L^3 - 6L^2 x + 5Lx^2 - x^3)$	A.31
		$P_5 = -\frac{P_3 x}{2L^2}(2L^2 - 3Lx + x^2)$	A.32
	P_4	$P_1 = \frac{1.5P_4 x}{L^3}(2L - x)$	A.33
		$\Delta_2 = \frac{P_4 x}{4LEI}(3x - 2L)$	A.34
		$\Delta_3 = \frac{P_4 x^2}{4L^3 EI} \times (2L^3 - 6L^2 x + 5Lx^2 - x^3)$	A.35
		$\Delta_4 = \frac{P_4 x}{4L^3 EI} \times (4L^3 - 12L^2 x + 12Lx^2 - 3x^3)$	A.36
		$P_5 = \frac{P_4}{2L^2}(6Lx - 3x^2 - 2L^2)$	A.37
	Δ_1	$P_1 = \frac{12EI\Delta_1}{L^3}$	A.38
		$P_2 = P_3 = \frac{6EI\Delta_1}{L^2}$	A.39
	Δ_2	$P_1 = \frac{6EI\Delta_2}{L^2}$	A.40
		$P_2 = \frac{4EI\Delta_2}{L}$	A.41

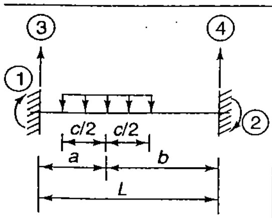
1	2	3	4
		$P_3 = \frac{2EI\Delta_2}{L}$	A.42
	P_1	$\Delta_1 = \frac{P_1 L^3}{48EI}$	A.43
		$\Delta_2 = 0$	A.44
		$\Delta_3 = -\Delta_4 = \frac{P_1 L^2}{16EI}$	A.45
		$\Delta_5 = \frac{P_1 x}{48EI}(3L^2 - 4x^2)$	A.46
		$\Delta_6 = \frac{P_1}{16EI}(L^2 - 4x^2)$	A.47
	P_2	$\Delta_1 = 0$	A.48
		$\Delta_2 = \frac{P_2 L}{12EI}$	A.49
		$\Delta_3 = \Delta_4 = -\frac{P_2 L}{24EI}$	A.50
		$\Delta_5 = \frac{P_2 x}{24LEI}(4x^2 - L^2)$	A.51
		$\Delta_6 = \frac{P_2}{24LEI}(12x^2 - L^2)$	A.52
	P_3	$\Delta_1 = \frac{P_3 L^2}{16EI}$	A.53
		$\Delta_2 = -\frac{P_3 L}{24EI}$	A.54
		$\Delta_3 = \frac{P_3 L}{3EI}$	A.55
		$\Delta_4 = -\frac{P_3 L}{6EI}$	A.56
		$\Delta_5 = \frac{P_3 x}{6LEI}(L - x)(2L - x)$	A.57
		$\Delta_6 = \frac{P_3}{6LEI}(2L^2 - 3x^2 - 6Lx)$	A.58
	P_5	$\Delta_1 = \frac{P_5 x}{48EI}(3L^2 - 4x^2)$	A.59
		$\Delta_2 = \frac{P_5 x}{24EI}(4x^2 - L^2)$	A.60

1	2	3	4
		$\Delta_3 = \dfrac{P_5 x}{6LEI}(L - x)(2L - x)$	A.61
		$\Delta_4 = -\dfrac{P_5 x}{6LEI}(L^2 - x^2)$	A.62
		$\Delta_5 = \dfrac{P_5 x^2}{3LEI}(L - x)^2$	A.63
		$\Delta_6 = \dfrac{P_5 x}{3LEI}(L - x)(L - 2x)$	A.64
		$\Delta_7 = \dfrac{P_5}{6LEI}[L(y - x)^3 + y(L - x)(2Lx - x^2 - y^2)]$	A.65
		$\Delta_8 = \dfrac{P_5}{6LEI}[3L(y - x)^2 + (L - x)(2Lx - x^2 - 3y^2)]$	A.66
	P_6	$\Delta_1 = \dfrac{P_6}{16EI}(L^2 - 4x^2)$	A.67
		$\Delta_2 = \dfrac{P_6}{24LEI}(12x^2 - L^2)$	A.68
		$\Delta_3 = \dfrac{P_6}{6LEI}(2L^2 + 3x^2 - 6Lx)$	A.69
		$\Delta_4 = \dfrac{P_6}{6LEI}(3x^2 - L^2)$	A.70
		$\Delta_5 = \dfrac{P_6 x}{3LEI}(L - x)(L - 2x)$	A.71
		$\Delta_6 = \dfrac{P_6}{3LEI}(3x^2 - 3xL + L^2)$	A.72
		$\Delta_7 = \dfrac{P_6}{6LEI}[y^3 + y(2L^2 - 6xL + 3x^2) - 3L(y - x)^2]$	A.73
		$\Delta_8 = \dfrac{P_6}{6LEI} \times (3y^2 + 2L^2 + 3x^2 - 6yL)$	A.74
	Uniform load p per unit length	$\Delta_1 = \dfrac{5pL^4}{384EI}$	A.75
		$\Delta_2 = 0$	A.76
		$\Delta_3 = -\Delta_4 = \dfrac{pL^3}{24EI}$	A.77

1	2	3	4
		$\Delta_5 = \dfrac{Px}{24EI}(x^3 + L^3 - 2Lx^2)$	A.78
		$\Delta_6 = \dfrac{P}{24EI}(4x^3 + L^3 - 6Lx^2)$	A.79
	Δ_1	$P_1 = \dfrac{3EI\Delta_1}{L^3}$	A.80
		$P_2 = \dfrac{3EI\Delta_1}{L^2}$	A.81
	Δ_2	$P_1 = \dfrac{3EI\Delta_2}{L^2}$	A.82
		$P_2 = \dfrac{3EI\Delta_2}{L}$	A.83
	P_1	$\Delta_1 = \dfrac{P_1 L^3}{192EI}$	A.84
		$\Delta_2 = 0$	A.85
		$P_3 = P_5 = \dfrac{P_1}{2}$	A.86
		$-P_4 = P_6 = \dfrac{P_1 L}{8}$	A.87
		$\Delta_7 = \dfrac{P_1 x^2}{48EI}(3L - 4x)$	A.88
		$\Delta_8 = \dfrac{P_1 x}{8EI}(L - 2x)$	A.89
		$\Delta_9 = \dfrac{P_1}{48EI}(4y - L)(L - y)^2$	A.90
		$\Delta_{10} = \dfrac{P_1}{8EI}(L - y)(L - 2y)$	A.91
	P_2	$\Delta_1 = 0$	A.92
		$\Delta_2 = \dfrac{P_2 L}{16EI}$	A.93
		$-P_3 = P_5 = \dfrac{1.5P_2}{L}$	A.94
		$P_4 = P_6 = \dfrac{P_2}{4}$	A.95
		$\Delta_7 = \dfrac{P_2 x^2}{8LEI}(2x - L)$	A.96

1	2	3	4
		$\Delta_8 = \dfrac{P_2 x}{4LEI}(3x - L)$	A.97
	P_7	$\Delta_1 = \dfrac{P_7}{48EI}(3L - 4x)x^2$	A.98
		$\Delta_2 = \dfrac{P_7 x^2}{8LEI}(2x - L)$	A.99
		$P_3 = \dfrac{P_7}{L^3}(L - x)^2(L + 2x)$	A.100
		$P_4 = \dfrac{-P_7 x}{L^2}(L - x)^2$	A.101
		$P_5 = \dfrac{P_7 x^2}{L^3}(3L - 2x)$	A.102
		$P_6 = \dfrac{P_7 x^2}{L^2}(L - x)$	A.103
		$\Delta_7 = \dfrac{P_7 x^3}{3L^3 EI}(L - x)^3$	A.104
		$\Delta_8 = \dfrac{P_7 x^2}{2L^3 EI}(L - x)^2(L - 2x)$	A.105
		$\Delta_9 = \dfrac{P_7}{6L^3 EI}[3Lxy^2(L - x)^2 - (L - x)^2 \times (L + 2x)y^3 + L^3(y - x)^3]$	A.106
		$\Delta_{10} = \dfrac{P_7}{2L^3 EI}[2Lxy(L - x)^2 - (L - x)^2(L + 2x)y^2 + L^3(y - x)^2]$	A.107
	P_8	$\Delta_1 = \dfrac{P_8 x}{8EI}(L - 2x)$	A.108
		$\Delta_2 = \dfrac{P_8 x}{4LEI}(3x - L)$	A.109
		$-P_3 = P_5 = \dfrac{6P_8 x}{L^3}(L - x)$	A.110
		$P_4 = -\dfrac{P_8}{L^2}(L - x)(L - 3x)$	A.111
		$P_6 = \dfrac{P_8 x}{L^2}(2L - 3x)$	A.112

1	2	3	4
		$\Delta_7 = \dfrac{P_8 x^2}{2L^3 EI}(L - x)^2(L - 2x)$	A.113
		$\Delta_8 = \dfrac{P_8 x}{L^3 EI}(L - x)(L^2 - 3Lx + 3x^2)$	A.114
		$\Delta_9 = \dfrac{P_8}{2L^3 EI}[(L - x) \times (L^2 - 3Lx + 2xy)y^2 - L^3(y - x)^2]$	A.115
		$\Delta_{10} = \dfrac{P_8}{L^3 EI}[(L - x)(L^2 - 3Lx + 3xy)y - L^3(y - x)]$	A.116
	P_9	$\Delta_1 = \dfrac{P_9}{48EI}(4y - L)(L - y)^2$	A.117
		$\Delta_2 = \dfrac{P_9}{8LEI}(L - y)^2(2y - L)$	A.118
		$P_3 = \dfrac{P_7}{L^3}(L + 2y)(L - y)^2$	A.119
		$P_4 = -\dfrac{P_9 y}{L^2}(L - y)^2$	A.120
		$P_5 = \dfrac{P_9 y^2}{L^3}(3L - 2y)$	A.121
		$P_6 = \dfrac{P_9 y^2}{L^2}(L - y)$	A.122
		$\Delta_7 = \dfrac{P_9}{6L^3 EI}[3Lyx^2(L - y)^2 - (L - y)^2(L + 2y)x^3]$	A.123
		$\Delta_8 = \dfrac{P_9}{2L^3 EI}[2Lxy(L - y)^2 - (L - y)^2(L + 2y)x^2]$	A.124
		$\Delta_9 = \dfrac{P_9 y^3}{3L^3 EI}(L - y)^3$	A.125
		$\Delta_{10} = \dfrac{P_9 y^2}{2L^3 EI}(L - y)^2(L - 2y)$	A.126
	P_{10}	$\Delta_1 = \dfrac{P_{10}}{8EI}(L - y)(L - 2y)$	A.127
		$\Delta_2 = \dfrac{P_{10}}{4LEI}(L - y)(2L - 3y)$	A.128

1	2	3	4
		$-P_3 = P_5 = \dfrac{6P_{10}y}{L^3}(L-y)$	A.129
		$P_4 = \dfrac{-P_{10}}{L^2}(L-y)(L-3y)$	A.130
		$P_6 = \dfrac{P_{10}y}{L^2}(2L-3y)$	A.131
		$\Delta_7 = \dfrac{P_{10}x^2}{2L^3EI}(L-y)\ (L^2 - 3Ly + 2xy)$	A.132
		$\Delta_8 = \dfrac{P_{10}}{L^3EI}(L-y)\,(L^2x - 3Lxy + 3yx^2)$	A.133
		$\Delta_9 = \dfrac{P_{10}}{2L^3EI}(L-y)^2(L-2y)$	A.134
		$\Delta_{10} = \dfrac{P_{10}}{L^2EI}[y(L-y)(L-3y) + 6L^3]$	A.135
	Uniform load p per unit length	$\Delta_1 = \dfrac{pL^4}{384EI}$	A.136
		$\Delta_2 = 0$	A.137
		$P_3 = P_5 = \dfrac{pL}{2}$	A.138
		$-P_4 = P_6 = \dfrac{pL^2}{12}$	A.139
		$\Delta_7 = \dfrac{px^2}{24EI}(L-x)^2$	A.140
		$\Delta_8 = \dfrac{px}{12EI}(L-x)(L-2x)$	A.141
	Uniform load p per unit length over part of the length	$P_1 = \dfrac{-pc}{12L^2}[12ab^2 + c^2(L-3b)]$	A.142
		$P_2 = \dfrac{pc}{12L^2}[12a^2b + c^2(L-3a)]$	A.143
		$P_3 = \dfrac{pcb}{L} - \dfrac{P_1 + P_2}{L}$	A.144
		$P_4 = \dfrac{pca}{L} + \dfrac{P_1 + P_2}{L}$	A.145

APPENDIX B
Answers to Problems

CHAPTER 1

1.1 $S_{DC} = 10$ kN (comp.), $Q_D = 20$ kN ↑↓, $M_D = 50$ kN·m (hogging)
1.2 $S_{DC} = 10$ kN (comp.), $Q_D = 34$ kN ↑↓, $M_D = 86$ kN·m (hogging)
1.3 $Q_D = 14$ kN ↑↓, $M_D = 36$ kN·m (hogging), $T_D = 5$ kN·m
1.4 (a), (c), (g)
1.5 (b) $D_k = 6$; (d) $D_k = 10$
1.6 (e) $D_s = 3, D_k = 1$; (f) $D_s = 1, D_k = 6$
1.7 (b), (d), (f), (g), (i), (j), (o), (q)
1.8 (a) $D_k = 2$; (e) $D_k = 9$; (h) $D_k = 9$; (l) $D_k = 8$; (r) $D_k = 9$
1.9 (c) $D_s = 1, D_k = 8$; (k) $D_s = 1, D_k = 13$;
(m) $D_s = 1, D_k = 15$; (n) $D_s = 1, D_k = 17$;
(p) $D_s = 1, D_k = 12$
1.10 (b), (i), (k)
1.11 (a) $D_k = 3$
1.12 (c) $D_s = 2, D_k = 4$; (d) $D_s = 1, D_k = 6$; (e) $D_s = 6, D_k = 5$; (f) $D_s = 3, D_k = 13$;
(g) $D_s = 3, D_k = 6$; (h) $D_s = 5, D_k = 5$; (j) $D_s = 1, D_k = 9$; (l) $D_s = 15$, $D_k = 9$; (m) $D_s = 9, D_k = 9$; (n) $D_s = 11, D_k = 9$
1.13 (a) $D_s = 1, D_k = 6$; (b) $D_s = 0, D_k = 9$; (c) $D_s = 0, D_k = 21$; (d) $D_s = 2, D_k = 9$
1.14 (a) $D_s = 18, D_k = 18$; (b) $D_s = 18, D_k = 27$;
(c) $D_s = 75, D_k = 30$; (d) $D_s = 36, D_k = 21$
1.15 (i) $D_s = 5580, D_k = 1770$
(ii) $D_s = 5418, D_k = 1932$

CHAPTER 2

2.1 (a) $V_A = 20$ kN ↑, $V_B = 10$ kN ↑
(b) $V_A = 8$ kN ↑, $H_A = 10$ kN ←, $V_C = 12$ kN ↑
(c) $V_A = 15$ kN ↑, $H_A = 60$ kN ←, $V_D = 105$ kN
2.2 (a) $\Delta_{1(\text{bending})} = \dfrac{WL^3}{3EI}$, $\Delta_{1(\text{shear})} = \dfrac{1.2WL}{bdG}$
(b) $\Delta_{1(\text{bending})} = \dfrac{wL^4}{8EI}$, $\Delta_{1(\text{shear})} = \dfrac{0.6wL^2}{bdG}$

(c) $\Delta_{1(\text{bending})} = \dfrac{wL^3}{48EI}$, $\Delta_{1(\text{shear})} = \dfrac{0.3wL^2}{bdG}$

(d) $\Delta_{1(\text{bending})} = \dfrac{5wL^4}{384EI}$, $\Delta_{1(\text{shear})} = \dfrac{0.15wL^2}{bdG}$

2.3 (a) $\Delta_{1(\text{bending})} = \dfrac{ML^2}{3EI}$, $\Delta_{1(\text{shear})} = \dfrac{M}{A_r G}$, $\Delta_{1(\text{axial})} = \dfrac{M}{AE}$

$\Delta_{2(\text{bending})} = \dfrac{ML}{6EI}$, $\Delta_{2(\text{shear})} = \dfrac{-M}{A_r GL}$, $\Delta_{2(\text{axial})} = \dfrac{-M}{AEL}$

(b) $\Delta_{1(\text{bending})} = \dfrac{-1.729PL^3}{EI}$, $\Delta_{1(\text{shear})} = \dfrac{-0.5PL}{A_r G}$, $\Delta_{1(\text{axial})} = \dfrac{-PL}{AE}$

$\Delta_{2(\text{bending})} = \dfrac{-PL^3}{48EI}$, $\Delta_{2(\text{shear})} = \dfrac{0.5PL}{A_r G}$, $\Delta_{2(\text{axial})} = 0$

$\Delta_{3(\text{bending})} = \dfrac{1.75PL^2}{EI}$, $\Delta_{3(\text{shear})} = \Delta_{3(\text{axial})} = 0$

2.4 $\Delta_B = 0.008$ m $\uparrow$

2.5 $\Delta'_4 = 0.006$

2.6 $\Delta_B = 0.012$ m $\rightarrow$

2.7 $V_B = 6$ kN $\uparrow$, $V_D = 16$ kN $\uparrow$

2.8 $M_A = 11.25$ kN·m

2.9 (a) $\Delta_1 = 10$ mm; (b) $\Delta_1 = 7.22$ mm
(c) $\Delta_1 = 3.6$ mm, $\Delta_2 = 0.33$ mm; (d) $\Delta_1 = 15.01$ mm

2.10 (a) $\Delta_1 = 0.3$ mm; $\Delta_2 = 1.266$ mm; (b) $\Delta_1 = 1.363$ mm

2.11 2.6 m from A, $\Delta_{max} = 7.52$ mm

2.12 3.17 m from A, $\Delta_{max} = 26.04$ mm

2.13 $\Delta_B = 20$ mm $\downarrow$

2.14 (a) $\Delta_1 = 1.47$ mm (b) $\Delta_1 = 5$ mm (c) $\Delta_1 = 20$ mm
(d) $\Delta_1 = 0.1$ radian (e) $\Delta_1 = -10$ mm (f) $\Delta_1 = 0.0875$ radian
(g) $\Delta_1 = 0.1$ radian (h) $\Delta_1 = 2$ mm, $\Delta_2 = 1.15$ mm

2.15 (a) $V_A = \dfrac{5wL}{8}\uparrow$, $M_A = \dfrac{wL^2}{8}$

(b) $V_A = 60$ kN $\uparrow$, $H_A = 10$ kN$\rightarrow$, $M_A = 10$ kN·m

2.16 $M_A = 22.4$ kN·m, $V_A = 30.5$ kN

2.17 $\Delta_D = 3.75$ mm, $\theta_A = 0.0059$ radian

2.18 $M_{BA} = -M_{BC} = 68$ kN·m, $M_{CB} = 86$ kN·m

2.19 $V_A = 50$ kN $\uparrow$, $H_A = 3.3$ kN $\rightarrow$, $M_{AB} = 0$,
$V_D = 70$ kN $\uparrow$, $H_D = 53.3$ kN $\leftarrow$, $M_{DC} = -53.3$ kN·m

CHAPTER 3

3.4 $|A| = 0$ $|B| = 0.928$ $|C| = 88$ $|D| = 64$

3.5 (i) Not conformable (ii) $\begin{bmatrix} 10 & 2 & 10 \\ 11 & 3 & 7 \end{bmatrix}$

(iii) Not conformable (iv) $\begin{bmatrix} 10 & 11 \\ 2 & 3 \\ 10 & 7 \end{bmatrix}$

(v) $\begin{bmatrix} 43 & 11 & 31 \\ 11 & 3 & 7 \\ 31 & 7 & 27 \end{bmatrix}$ (vi) Not conformable

3.6 $[A] = \begin{bmatrix} 12.6 \\ -4.6 \\ 28.1 \end{bmatrix}$

3.7 $\text{Adj }[A] = \begin{bmatrix} 6 & -2 \\ -4 & 3 \end{bmatrix}$, $\text{Adj }[B] = \begin{bmatrix} 6 & -6 & 3 \\ -1 & 2 & -1 \\ -7 & 5 & -1 \end{bmatrix}$

3.8 $[A]^{-1} = \dfrac{1}{18}\begin{bmatrix} 12 & 1 & -3 \\ 0 & 6 & 0 \\ -6 & -2 & 6 \end{bmatrix}$

$[B]^{-1} = \begin{bmatrix} 0.75 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \\ 0.25 & 0.50 & 0.75 \end{bmatrix}$

$[C]^{-1} = -\dfrac{1}{60}\begin{bmatrix} 3 & -18 & -24 & 24 \\ -1 & -34 & 8 & 12 \\ -8 & 28 & 4 & -24 \\ 16 & 4 & -8 & -12 \end{bmatrix}$

$[D]^{-1} = \dfrac{1}{2.15}\begin{bmatrix} 2.721 & 0.960 & 1.850 & -0.653 \\ 0.960 & 1.920 & 0.653 & -0.230 \\ 1.850 & 0.653 & 2.721 & -0.960 \\ -0.653 & -0.230 & -0.960 & 1.296 \end{bmatrix}$

3.9 (i) $x = -1, y = 1, z = -2$ (ii) $x = 1, y = 2, z = 3$
(iii) $x = 0, y = -1, z = 2$

3.10 (i) $x_1 = 1, x_2 = 3, x_3 = 2, x_4 = 4$
(ii) $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$

3.11 $P_1 = 17.9,\ P_2 = 4.02,\ \Delta_3 = \dfrac{1.26}{EI},\ \Delta_4 = -\dfrac{49.91}{EI}$

3.12 $P_1 = -6.65,\ P_2 = 3.11,\ \Delta_3 = \dfrac{163.86}{EI}$

$\Delta_4 = \dfrac{14.14}{EI},\ \Delta_5 = \dfrac{11.24}{EI}$

3.13 $|A|_n = 0.0775,\ |B|_n = 0.1115,\ |C|_n = 0.7593$

3.14 $[A]$

CHAPTER 4

4.2 $[\delta] = \dfrac{L}{6EI}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, stiffness matrix does not exist

4.3 $[\delta] = \begin{bmatrix} \dfrac{L}{AE} & 0 & 0 \\ 0 & \dfrac{L}{4EI} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, stiffness matrix does not exist

4.4 $[\delta] = \begin{bmatrix} \dfrac{L}{AE} & 0 & 0 \\ 0 & \dfrac{L}{EI} & -\dfrac{L^2}{2EI} \\ 0 & \dfrac{L^2}{2EI} & \dfrac{L^3}{3EI} \end{bmatrix}$, $[k] = \begin{bmatrix} \dfrac{AE}{L} & 0 & 0 \\ 0 & \dfrac{4EI}{L} & \dfrac{6EI}{L^2} \\ 0 & \dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} \end{bmatrix}$

4.5 $[k] = \begin{bmatrix} \dfrac{AE}{L} & 0 & 0 \\ 0 & \dfrac{3EI}{L} & \dfrac{3EI}{L^2} \\ 0 & \dfrac{3EI}{L^2} & \dfrac{3EI}{L^3} \end{bmatrix}$

4.6 (a) $[\delta] = \dfrac{1}{6EI}\begin{bmatrix} 250 & 75 & 52 & 48 \\ 75 & 30 & 12 & 12 \\ 52 & 12 & 16 & 12 \\ 48 & 12 & 12 & 12 \end{bmatrix}$

(b) $[\delta] = \dfrac{1}{500EI}\begin{bmatrix} 625 & -300 & -200 \\ -300 & 432 & 168 \\ -200 & 168 & 232 \end{bmatrix}$ (c) $[\delta] = \dfrac{1}{125EI}\begin{bmatrix} 108 & 42 \\ 42 & 58 \end{bmatrix}$

(d) $[\delta] = \dfrac{1}{EI}\begin{bmatrix} 288 & 0 & 36 & -36 \\ 0 & 2 & -1 & -1 \\ 36 & -1 & 8 & -4 \\ -36 & -1 & -4 & 8 \end{bmatrix}$ (e) $[\delta] = \dfrac{1}{EI}\begin{bmatrix} 288 & 0 & -36 \\ 0 & 2 & -1 \\ -36 & -1 & 8 \end{bmatrix}$

(f) $[\delta] = \dfrac{1}{EI}\begin{bmatrix} 96 & 8 & 20 & -16 \\ 8 & 2 & 1 & -2 \\ 20 & 1 & 6 & -3 \\ -16 & -2 & -3 & 6 \end{bmatrix}$ (g) $[\delta] = \dfrac{1}{EI}\begin{bmatrix} 96 & 8 & -16 \\ 8 & 2 & -2 \\ -16 & -2 & 6 \end{bmatrix}$

(h) $[\delta] = \dfrac{1}{60EI}\begin{bmatrix} 512 & 192 & 640 & -96 \\ 192 & 104 & 288 & -40 \\ 640 & 288 & 1152 & -96 \\ -96 & -4 & -96 & 56 \end{bmatrix}$

(i) $[k] = \dfrac{EI}{3}\begin{bmatrix} 4 & -2 & -4 & 2 \\ -2 & 12 & 4 & 2 \\ -4 & 4 & 8 & 0 \\ 2 & 2 & 0 & 4 \end{bmatrix}$

(j) $[k] = EI\begin{bmatrix} 0.192 & 0 & -0.096 & 0.240 \\ 0 & 1.600 & -0.240 & 0.400 \\ -0.096 & -0.240 & 0.192 & 0 \\ 0.240 & 0.400 & 0 & 1.600 \end{bmatrix}$

(k) $[\delta] = \dfrac{1}{81EI}\begin{bmatrix} 1000 & 150 \\ 150 & 90 \end{bmatrix}$ (l) $[k] = \dfrac{EI}{144}\begin{bmatrix} 43 & -6 \\ -6 & 336 \end{bmatrix}$

4.7 (a) $[\delta] = \dfrac{1}{6EI}\begin{bmatrix} 250 & 75 & 52 \\ 75 & 30 & 12 \\ 52 & 12 & 16 \end{bmatrix}$ (b) $[\delta] = \dfrac{1}{6EI}\begin{bmatrix} 250 & 52 \\ 52 & 16 \end{bmatrix}$

(c) $[\delta] = \dfrac{1}{EI}\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ (d) $[\delta] = \dfrac{1}{3EI}\begin{bmatrix} 76 & 13 \\ 13 & 4 \end{bmatrix}$

(e) $[\delta] = \dfrac{1}{500EI}\begin{bmatrix} 625 & -200 \\ -200 & 232 \end{bmatrix}$

(f) $[\delta] = \dfrac{1}{60EI}\begin{bmatrix} 512 & 192 & 640 \\ 192 & 104 & 288 \\ 640 & 288 & 1152 \end{bmatrix}$

(g) $[\delta] = \dfrac{1}{60EI}\begin{bmatrix} 512 & 192 & -96 \\ 192 & 104 & -40 \\ -96 & -40 & 56 \end{bmatrix}$ (h) $[\delta] = \dfrac{1}{60EI}\begin{bmatrix} 512 & 640 \\ 640 & 1152 \end{bmatrix}$

(i) $[\delta] = \dfrac{1}{60EI}\begin{bmatrix} 104 & -40 \\ -40 & 56 \end{bmatrix}$

(j) $[\delta] = \dfrac{1}{810EI}\begin{bmatrix} 10000 & 1500 & 6875 \\ 1500 & 900 & 1875 \\ 6875 & 1875 & 10000 \end{bmatrix}$

(k) $[k] = EI\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}$ (l) $[k] = \dfrac{EI}{30}\begin{bmatrix} 27 & 6 & 0 \\ 6 & 22 & 5 \\ 0 & 5 & 10 \end{bmatrix}$

(m) $[k] = \dfrac{EI}{30}\begin{bmatrix} 30 & 15 & 0 & 0 \\ 15 & 42 & 6 & 0 \\ 0 & 6 & 32 & 10 \\ 0 & 0 & 10 & 20 \end{bmatrix}$

(n)

$$[k] = \frac{EI}{18000}\begin{bmatrix} 18000 & -6750 & 9000 & 0 & 0 & 0 & 0 \\ -6750 & 5375 & -750 & 6000 & 0 & 0 & 0 \\ 9000 & -750 & 42000 & 12000 & 0 & 0 & 0 \\ 0 & 6000 & 12000 & 52800 & -8640 & 14400 & 0 \\ 0 & 0 & 0 & -8640 & 5184 & -4320 & 4320 \\ 0 & 0 & 0 & 14400 & -4320 & 43200 & 7200 \\ 0 & 0 & 0 & 0 & 4320 & 7200 & 14400 \end{bmatrix}$$

(o) $[\delta] = \dfrac{1}{3EI}\begin{bmatrix} 56 & 30 & 24 \\ 30 & 18 & 12 \\ 24 & 12 & 12 \end{bmatrix}$ (p) $[k] = EI\begin{bmatrix} 3 & 0 & -1.5 \\ 0 & 4 & 1 \\ -1.5 & 1 & 5 \end{bmatrix}$

(q) $[k] = \dfrac{EI}{36}\begin{bmatrix} 70 & -30 & 24 & 0 & 0 \\ -30 & 120 & 24 & 0 & 0 \\ 24 & 24 & 192 & -108 & 72 \\ 0 & 0 & -108 & 135 & -54 \\ 0 & 0 & 72 & -54 & 252 \end{bmatrix}$

(r) $[k] = EI\begin{bmatrix} 0.072 & -0.240 & -0.120 \\ -0.240 & 3.200 & 0.800 \\ -0.120 & 0.800 & 2.200 \end{bmatrix}$

(s) $[k] = EI\begin{bmatrix} 0.144 & -0.240 & -0.240 & -0.240 \\ -0.240 & 3.200 & 0.800 & 0 \\ -0.240 & 0.800 & 2.400 & 0.400 \\ -0.240 & 0 & 0.400 & 0.800 \end{bmatrix}$

4.8 $M_B = 34.4$ kN·m (hogging), $M_C = 50.3$ kN·m (hogging)

4.9 $M_{BA} = -M_{BC} = \dfrac{150}{7}$ kN·m, $M_{CB} = \dfrac{30}{7}$ kN·m

CHAPTER 5

5.1 $M_C = 128.8$ kN·m (sagging), $V_B = 74.4$ kN ↑

5.2 $M_B = 291.7$ kN·m (hogging)

5.3 $V_A = 70.8$ kN ↑

5.4 $V_A = V_C = 70.8$ kN ↑

5.5 $V_B = 258.4$ kN ↑

5.6 $V_B = 102.875$ kN ↑

5.7 $V_C = 129.675$ kN ↑, $V_D = 7.45$ kN ↑

5.8 $M_C = 87.75$ kN·m (hogging)

5.9 $M_A = \dfrac{Pab^2}{L^2}$ (hogging), $M_B = \dfrac{Pa^2b}{L^2}$ (hogging)

5.10 $M_A = 31.25$ kN·m (sagging), $M_B = 18.75$ kN·m (sagging)

5.11 $M_C = 80$ kN·m (sagging)

5.12 $M_B = 328$ kN·m (hogging)

5.13 $V_A = 159$ kN ↑, $V_B = 352.5$ kN ↑, $V_C = 88.5$ kN ↑, $M_C = 212$ kN·m (hogging)

5.16 $M_B = 48$ kN·m (hogging), $M_C = 144$ kN·m (hogging)

5.17 $V_B = 35.4$ kN ↑, $V_C = 149$ kN ↑

5.18 $V_A = 55.2$ kN ↑, $V_D = 110.6$ kN ↑

5.19 $D_k = 4$

5.20 $M_A = 37.5$ kN·m (sagging), $M_C = 187.5$ kN·m (hogging)

5.21 $V_B = 150$ kN ↑, $M_B = 75$ kN·m (hogging)

5.22 $M_B = 0.03\dfrac{EI}{L}$ (hogging)

5.23 $M_B = M_D = \dfrac{3pL^2}{28}$ (hogging), $M_C = \dfrac{pL^2}{14}$ (hogging)

5.24 $V_B = \dfrac{69}{7}\dfrac{EI}{L^3}\downarrow$, $V_C = \dfrac{66}{7}\dfrac{EI}{L^3}\uparrow$, $V_D = \dfrac{27}{7}\dfrac{EI}{L^3}\downarrow$

5.25 $V_A = 11.78$ kN ↑, $M_A = 13.55$ kN·m (hogging)

5.26 $V_D = 46.4$ kN ↑, $M_D = 83.7$ kN·m (hogging)

5.27 $M_B = 0.0944\,pL^2$ (hogging), $M_C = 0.0444\,pL^2$ (hogging)

5.28 $M_A = 4.4\dfrac{EI\Delta}{L^2}$ (sagging), $M_D = 0.4\dfrac{EI\Delta}{L^2}$ (hogging)

5.29 $M = 7.5\dfrac{EI}{L}$

5.30 59.08 kN·m (sagging), $M_C = 33.07$ kN·m (hogging)

5.31 $M_D = 45$ kN·m (hogging)

5.32 (i) $P_1 = -100$ kN·m $P_2 = -160$ kN·m
(ii) $P_1 = -70$ kN·m $P_2 = 50$ kN·m
(iii) $P_1 = 50$ kN·m $P_2 = -10$ kN·m

CHAPTER 6

6.1 $H_A = 9.65$ kN →

6.2 $M_{BA} = -M_{BC} = -0.15\,pL$, $M_{CB} = -M_{CD} = 0.35\,pL$

6.3 $V_A = 33.75$ kN ↑, $V_D = 11.25$ kN ↑

6.4 $M_{BA} = -M_{BC} = 9.6$ kN·m, $M_{CB} = -M_{CD} = 24$ kN·m

6.5 $M_{BA} = -M_{BC} = 23.4$ kN·m, $M_{CB} = -M_{CD} = 26.6$ kN·m

6.6 $H_C = 160.7$ kN →, $V_C = 42.8$ kN ↓

6.7 $H_A = 160.7$ kN ←, $V_A = 142.8$ kN ↑, $M_{AB} = -267.5$ kN·m

6.8 $M_{AB} = -267.5$ kN·m, $M_{BA} = -536$ kN·m, $M_{BC} = -464$ kN·m

6.9 $M_{BA} = -M_{BC} = -150.9$ kN·m, $M_{CB} = -M_{CD} = -121.8$ kN·m,
$M_{DC} = -157$ kN·m

6.10 (a) $M_{CB} = -M_{CD} = 70$ kN·m
(b) $M_{CB} = -M_{CD} = 55.1$ kN·m

6.11 $M_{BA} = 12.5$ kN·m, $M_{BC} = M_{BD} = -6.25$ kN·m

6.12 $H_D = 1.25$ kN ←, $V_D = 13.75$ kN ↑

6.13 $M_{BA} = 22.7$ kN·m, $M_{BD} = 3.8$ kN·m, $M_{BC} = -26.5$ kN·m

6.14 $V_C = 89.8$ ↑, $H_C = 50.2$ kN ←, $M_{CB} = 232.7$ kN·m

6.15 $V_C = 198$ kN ↑, $H_C = 9$ kN ←, $M_{CB} = 216$ kN·m

6.16 (a) $M_{AB} = 1.37$ kN·m, $M_{DC} = -3.25$ kN·m
(b) $M_{AB} = 9$ kN·m, $M_{DC} = -27$ kN·m
(c) $M_{AB} = 6.55$ kN·m, $M_{DC} = -11.05$ kN·m

6.17 (a) $M_{AB} = -12.3$ kN·m, $M_{DC} = -5.1$ kN·m
(b) $M_{AB} = 31.35$ kN·m, $M_{DC} = -3.3$ kN·m

6.18 $M_{BC} = -14.3$ kN·m, $M_{CB} = -14.6$ kN·m

6.19 $M_{AB} = 8.92$ kN·m, $M_{CB} = -1.43$ kN·m

6.20 $M_{AB} = -6.72$ kN·m, $M_{DC} = 17.68$ kN·m

6.21 $M_{AB} = -24.02$ kN·m, $M_{DC} = 27.58$ kN·m

6.22 $D_S = 3$, $D_k = 7$, $V_A = 44.3$ kN ↑,
$H_A = 11.2$ kN →, $V_F = 19.3$ kN ↑, $H_F = 4.9$ kN ←

6.23 $D_s = 5$, $D_k = 3$, $V_A = 115$ kN ↑, $H_A = 18.5$ kN →, $M_{AB} = 36.45$ kN·m

6.24 $V_E = 180.7$ kN ↑, $H_E = 21.4$ kN ←, $V_F = 299.3$ kN ↑, $H_F = 53.6$ kN ←

6.25 (a) $M_{BC} = -10.08$ kN·m
(b) $M_{BC} = 81.2$ kN·m
(c) $M_{BA} = 3$ kN·m

6.26 $M_{BA} = -808$ kN·m, $M_{FE} = -590$ kN·m

6.27 $V_F = 413.7$ kN ↑, $H_F = 130$ kN ←, $M_{FD} = -258.7$ kN·m

6.28 $V_D = 115$ kN ↑, $H_D = 54.2$ kN ←

6.29 $D_s = 2$, $D_k = 6$, $M_{AB} = 4.725$ kN·m, $M_{DC} = -5.175$ kN·m

CHAPTER 7

7.1 $D_s = 1$, $D_k = 2$, $S_{AB} = 53.5$ kN (tensile)

7.2 $S_{AE} = 45.5$ kN (tensile)

7.3 $S_{AC} = 35.4$ kN (tensile)

7.4 $S_{CD} = 40.5$ kN (comp.)

7.5 $S_{CD} = 26$ kN (tensile)

7.6 $S_{U1U2} = 14.55$ kN (comp.), $S_{U2U3} = 8.9$ kN (comp.)

7.7 $S_{AD} = 61.5$ kN (tensile), $S_{BC} = 58.5$ kN (comp.)

7.8 $S_{CD} = 22.5$ kN (tensile)

7.9 $S_{BG} = 78.2$ kN (comp.)

7.10 $S_{U1L2} = 21.3$ kN (tensile)

7.11 $S_{AC} = 4$ kN (comp.), $S_{BD} = 37.6$ kN (comp.), $S_{CD} = 4.96$ kN (tensile)

7.12 $S_{U1L1} = s_{U3L3} = 8.07$ kN (comp.), $S_{U1L1} = Su_{3L3} = 51.78$ kN (tensile)

7.13 $S_{U1L2} = 10.82$ kN

7.14 $S_{BC} = 44$ kN (comp.), $S_{CF} = 67$ kN (tensile)

7.15 $S_{U2U3} = 13.3$ kN (comp.)

7.16 $D_s = 2$, $D_k = 6$, $S_{AE} = 11$ kN (comp.), $S_{DG} = 4$ kN (tensile)

7.17 $S_{U1L2} = 14.3$ kN (comp.)

7.18 $$[k] = \begin{bmatrix} 170.7 & 0 & -100 & 0 \\ 0 & 270.7 & 0 & 0 \\ -100 & 0 & 170.7 & 0 \\ 0 & 0 & 0 & 270.7 \end{bmatrix}$$

7.19 $S_{BC} = 8.378$ kN (comp.)

7.20 $S_{U2L1} = 99$ kN (comp.), $S_{U3L2} = 68.5$ kN (comp.)
$S_{U3L4} = 24.35$ kN (comp.)

7.21 $D_{si} = 2$, $D_{se} = 2$, $S_{U2L3} = 0.75$ kN (comp.),
$S_{U4L3} = 0.885$ kN (tensile)

CHAPTER 8

8.1 $M_{Ay} = 37.5$ kN·m, $M_{Cy} = 175$ kN·m, $T_{Dy} = -25$ kN·m

8.2 $T_{Dy} = -47.4$ kN·m

8.3 $M_{Ay} = 37.5$ kN·m, $T_{Az} = 45$ kN·m, $M_{Cy} = 175$ kN·m
$M_{Cx} = -15$ kN·m, $T_{CB} = 0$, $M_{Dz} = -97.5$ kN·m,
$T_{Dy} = -25$ kN·m

8.4 $V_C = 12.75$ kN ↑

8.5 $V_C = 31.5$ kN ↑

8.6 Downward deflection $= \dfrac{12.16}{EI}$ m

Rotation about the x-axis $= \dfrac{17.6}{EI}$ radian

Rotation about the y-axis $= \dfrac{38.4}{EI}$ radian

EI is in kN·m^2

8.7 $V_C = 19$ kN ↑, $M_{Cx} = -18$ kN·m, $T_{Cy} = 118$ kN·m

8.8 $M_E = 45$ kN·m

8.9 $\Delta_E = \dfrac{3975}{EI}$ m, EI is in kN·m^2

8.10 $T_{DA} = T_{DB} = 10$ kN·m

8.11 $T_{AB} = 7$ kN·m

8.12 $T_A = 4.14$ kN·m

8.13 $V_B = 15.68$ kN ↑, $M_B = 62.5$ kN·m, $T_B = 11.7$ kN·m

8.14 $V_A = 37.8$ kN ↑, $M_A = 49.5$ kN·m, $T_A = 1.22$ kN·m

8.15 $$\begin{bmatrix} 480 & 0 & 0 & -120 & 0 & 300 & -120 & 300 & 0 & 0 & 0 & 0 \\ 0 & 2250 & 0 & 0 & -125 & 0 & -300 & 500 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8.16 $$[k] = 10^4 \begin{bmatrix} 57.013 & 0 & -13.684 \\ 0 & 46.636 & 8.758 \\ -13.684 & 8.758 & 10.344 \end{bmatrix} \text{ in kN-m units}$$

$M_A = 239.75$ kN·m
$T_A = 17.85$ kN·m

8.17 $$[k] = 10^4 \begin{bmatrix} 47.891 & 0 & -6.842 \\ 0 & 39.338 & 4.379 \\ -6.842 & 4.379 & 2.586 \end{bmatrix} \text{ in kN-m units}$$

$T_A = 56$ kN·m
$T_E = 53.6$ kN·m

8.18 $M_D = 263.5$ kN·m
$T_D = 101.5$ kN·m

CHAPTER 9

9.1 $S_{OA} = 2.54$ kN (tensile), $S_{OB} = 5.73$ kN (tensile),
$S_{OC} = 3.62$ kN (tenslie)

9.2 $S_{OB} = 184$ kN (comp.), $S_{OC} = 0$, $S_{AB} = 75$ kN (tensile)

9.3 $S_{EF} = 5$ kN (tensile), $S_{FC} = 7.5$ kN (comp.)
$S_{FB} = 2.5$ (tensile)

9.4 $S_{CD} = 62.5$ kN (tensile), $S_{GH} = 250$ kN (comp.)

9.5 $S_{OA} = 12.2$ kN (tensile), $S_{OB} = 29.53$ kN (tensile),
$S_{OC} = 7.09$ kN (comp.), $S_{OD} = 10.11$ kN (comp.)

9.6 $D_s = 1$, $D_k = 6$, $S_{CD} = 14.7$ kN (tensile),
$S_{CE} = 119$ kN (comp.), $S_{DE} = 59.5$ kN (comp.)

9.7 $D_s = 1$, $S_{DF} = 1.91$ kN (tensile)

9.8 $S_{EF} = 5.46$ kN (comp.)

9.9 $D_s = 1$, $D_k = 9$, $S_{AE} = 7.52$ kN (tensile)

9.10 $D_s = 1$, $D_k = 6$, $S_{AD} = 13.8$ kN (comp.)

9.11 $S_{EF} = 7$ kN (comp.), $S_{EC} = 10.2$ kN (comp.)
$S_{EG} = 11.2$ kN (comp.)

9.12 $D_s = 3$, $S_{AF} = S_{BE} = S_{CD} = 5.9$ kN (comp.)

CHAPTER 11

11.4 (a) $M_{AB} = 13.9$ kN·m, $M_{DC} = -16.7$ kN·m
(b) $M_{AB} = 15.6$ kN·m, $M_{DC} = -15.9$ kN·m

11.5 (a) $M_{AB} = M_{CB} = 0$
(b) $M_{AB} = -23.5$ kN·m, $M_{CB} = 46$ kN·m

CHAPTER 12

12.1 $M_B = 34.4$ kN·m (hogging), $M_C = 50.3$ kN·m (hogging)

12.2 $M_A = 37.5$ kN·m (sagging), $M_C = 187.5$ kN·m (hogging)

12.3 $M_A = 3.4$ kN·m (hogging), $M_D = 19.8$ kN·m (hogging)

12.4 $\theta_B = \dfrac{1}{2EI}$ ↻, $\Delta_B = \dfrac{9}{EI}$ ↓

12.5 $\theta_B = \dfrac{8}{9EI}$, $\Delta_B = \dfrac{4}{27EI} \downarrow$

12.6 $M_B = 328$ kN·m (hogging)

12.7 $M_B = 450$ kN·m (hogging), $M_C = 174$ kN·m (hogging)

12.8 $M_B = 75$ kN·m (hogging)

12.9 $M_C = 212$ kN·m (hogging)

12.10 $M_B = 81.9$ kN·m (hogging), $M_C = 93.8$ kN·m (hogging)

12.11 $M_B = 262$ kN·m (hogging)

12.12 $V_C = 89.8$ kN $\uparrow$, $H_C = 50.2$ kN $\leftarrow$, $M_{CB} = 232.7$ kN·m

12.13 $M_{AB} = -12.3$ kN·m, $M_{DC} = -5.1$ kN·m

12.14 $V_B = 290.4$ kN $\uparrow$, $V_C = 215.8$ kN $\uparrow$

12.15 $M_{BA} = -M_{BC} = 21.4$ kN·m

12.16 $M_{BO} = 4.09$ kN·m, $M_{CO} = 8.18$ kN·m, $M_{DO} = 4.09$ kN·m

12.17 $\theta_B = \dfrac{8}{9EI}$, $\Delta_B = \dfrac{4}{27EI} \downarrow$

12.18 $M_{CB} = 4.3$ kN·m

12.19 $M_{OA} = -32.72$ kN·m, $M_{OB} = 8.18$ kN·m,
$M_{OC} = 16.36$ kN·m, $M_{OD} = 8.18$ kN·m

12.20 $M_A = 23.72$ kN·m (hogging)

12.21 $M_A = 28.09$ kN·m (hogging)

12.22 $M_A = 36.05$ kN·m (hogging)

12.23 $M_A = 20$ kN·m (hogging)

12.24 $M_C = 59.3$ kN·m (hogging)

12.25 $V_B = V_C = 45$ kN $\uparrow$

12.26 $V_B = 0.542\ P \uparrow$, $V_C = 0.292\ P \uparrow$

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