PHYS114 Reference

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1 Distributions and Uncertainty

- For **nonstationary** PDFs, the moments are in terms of time.
- For **wide-sense stationary (WSS)** PDFs, only the **first two moments** are stationary, which represents most real-life phenomena.
- For **stationary** PDFs, all the moments are constant and *not* in terms of time.
- **Moment generating function (continuous)**:

$$n^{\text{th}} \text{ moment} = \int_{-\infty}^{\infty} x^n \, \text{PDF}(x) \, \partial x$$

- **Moment generating function (discrete)**:

$$n^{\text{th}} \text{ moment} = \sum_{i=1}^{\infty} x_i^n \, \text{PDF}(x_i)$$

- This class only focuses on the Gaussian, Binomial, and Poisson distributions. The rest are extra information.
- Remember that probabilities multiply. The probability of reproducing an entire dataset is

$$\prod_{i=1}^{N} \text{PDF}(x_i) = p_1 p_2 \dots p_N$$

- General process of data collection and processing:
    1. Determine PDF.
    2. Create estimators for moments from data.
    3. Determine if estimators are biased or unbiased using **expectation values**.
    4. Determine the chi-squared goodness of fit.
- From strongest to weakest: independent, uncorrelated, and no covariance.

1.1 Gaussian Distribution

Also known as the normal distribution. Represents humans and many natural phenomena.

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

- μ and σ are parameters.
- 1^{st} moment (mean): μ
- 2^{nd} moment (variance): σ^2
- Estimator for 1^{st} moment:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

- Estimator for 2^{nd} moment:

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

- 3rd moment (skewness)
- 4th moment (kurtosis)
- Uncertainty for 1^{st} moment:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

  This signals diminishing returns for increasing N.
- However, the above is assuming WSS. In a case without WSS:

$$\bar{x} = \frac{\sum_{i=1}^{N}\left(\frac{x_i}{\sigma_i^2}\right)}{\sum_{i=1}^{N}\left(\frac{1}{\sigma_i^2}\right)}$$

$$\sigma_{\bar{x}}^2 = \frac{1}{\sum_{i=1}^{N}\frac{1}{\sigma_i^2}}$$

1.2 Binomial Distribution

$$\mathrm{PDF}(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

- Trials must be **independent**.
- For p large and n large, similar to the **Gaussian distribution**.
- For p low and n large, similar to the **Poisson distribution**.
- n and p are parameters.
- 1^{st} moment: np
- 2^{nd} moment: $np(1-p)$
- 3^{rd} moment: $\frac{(1-p)-p}{\sqrt{np(1-p)}}$
- Estimator for 1^{st} moment:

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

- Estimator for 2^{nd} moment:

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

1.3 Poisson Distribution

Represents scenarios with counting and waiting for events.

$$\mathrm{PDF}(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

- λ is the parameter.
- 1^{st} moment: λ
- 2^{nd} moment: λ
- 3^{rd} moment: $\frac{1}{\sqrt{\lambda}}$
- Estimator for 1^{st} moment:

$$\bar{x} = \sum_{i=1}^{N} x_i$$

- Estimator for 2$^{\text{nd}}$ moment:

$$s^2 = \sum_{i=1}^{N} x_i$$

- Percent error:

$$\frac{1}{\sqrt{\bar{x}}} \times 100\%$$

1.4 Laplace Distribution

$$f(x; \mu, b) = \frac{1}{2b} \exp\left[-\frac{x-\mu}{b}\right]$$

- μ and b are parameters.
- μ is a location parameter.
- $b > 0$ is called "diversity."
- 1$^{\text{st}}$ moment: μ
- 2$^{\text{nd}}$ moment: $2b^2$
- 3$^{\text{rd}}$ moment: $\emptyset$

1.5 Gamma Distribution

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

where Γ is the gamma function.

- α and β are parameters.
- α is the shape parameter.
- $\frac{1}{\beta}$ is the scale parameter.
- 1$^{\text{st}}$ moment: $\frac{\alpha}{\beta}$
- 2$^{\text{nd}}$ moment: $\frac{\alpha}{\beta^2}$
- 3$^{\text{rd}}$ moment: $\frac{2}{\sqrt{\alpha}}$

1.6 Assumptions

We often assume that

1. Stationary or WSS PDF: $\sigma_i \approx \sigma$
2. Each measurement is independent and identically distributed (IID).
3. **Method of Maximum Liklihood (MML)**
   - Also known as Maximum Likelihood Method (MLM).
   - Maximum probability of reproducing the data.
   - An alternate method is **Maximum Entropy Method (MEM)** that seeks to add as much disorder as possible to the datapoints, but this class only focuses on MML.
4. An implicit assumption is the zero-mean process. We assume or subtract off the mean to make calculations much easier.

1.7 Expectation Values

$$\langle x_i \rangle = E[x_i] = \mu \quad \text{(typically)}$$

Examples:

- We expect the $\bar{x}$ estimator to yield μ: $\langle \bar{x} \rangle = \mu$
- And similarly for s^2: $\langle s^2 \rangle = \sigma^2$

A rule for expectation values is that you ignore constants:

$$\langle c \rangle = c$$

You can also ignore summations:

$$\langle \sum_{i=1}^{N} x_i \rangle = \sum_{i=1}^{N} \langle x_i \rangle$$

1.8 Uncertainty Propagation

- Uncertainty is represented by σ. For example, the uncertainty in $\bar{x}$ is denoted as $\sigma_{\bar{x}}$.
- Uncertainty can either propagate in a **worst case** or **even-steven** (informally named by our professor) scenario.
- Basics for worst case propagation: Suppose there are height measurements h_1 and h_2 with uncertainties σ_{h_1} and σ_{h_2}. Adding the two heights together also adds their uncertainties: $(h_1 \pm \sigma_{h_1}) + (h_2 \pm \sigma_{h_2}) = (h_1 + h_2) \pm (\sigma_{h_1} + \sigma_{h_2})$. This is the simplest example, but this can get much more complex, especially for operations like multiplication.

- **Even-steven formula** (Taylor series approximation):

$$\sigma_x^2 \approx \sigma_u^2 \left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2 \left(\frac{\partial x}{\partial v}\right)^2 + \cdots + 2\sigma_{uv}^2 \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \ldots$$

$$\sigma_{uv}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(u_i - \bar{u})(v_i - \bar{v})$$

  - σ_{uv}^2 is the covariance, which is 0 when both u and v are **independent**.

1.9 Chi-Squared Goodness of Fit

Given a histogram:

$$\chi^2 \equiv \sum_{j=1}^{n} \frac{(h(x_j) - NP(x_j))^2}{\sigma_j(h)^2} = \sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{\sigma_i^2}$$

where

- n is the number of bins.
- $h(x_j)$ is the value of the j-th bin.
- N is the total number of measurements.
- $P(x_j)$ is the probability of the j-th bin.
- $\sigma_j(h)^2$ is the variance in the j-th bin.

- χ^2 is a measure of the goodness of fit of a model to the data.
- A smaller χ^2 indicates a better fit. Ideal value is 0, but in practice it is around n because the numerator and denominator of the summation are around the same order of magnitude.
- **reduced chi-squared**:

$$\chi_\nu^2 = \frac{\chi^2}{\nu}$$

$$\nu = n - n_c$$

  - n_c is the number of constraints. Usually 1.
  - ν is the number of degrees of freedom. Usually $n-1$ with no outliers.