The Exponential Function and Characterizations of e

Research Question: What is a rigorous definition of exponentiation and the exponential function? How are the various characterizations of the constant e (Euler's Number) equivalent and which should be assumed as a principal definition?

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Contents

1 Introduction **2**

2 The Exponential Function **3**

3 Characterizations of e **9**

3.1 A note on circular definitions 10

4 Equivalences of the characterizations of e **12**

4.1 Limit definition $\iff$ Integral definition 12

4.2 Limit definition $\iff$ Series definition 15

4.3 Integral definition $\iff$ Series definition 17

5 Conclusion **17**

6 Bibliography **19**

1 Introduction

Euler's number (often referred to as the constant $e \approx 2.71828\ldots$) is a fundamental constant with wide applications in mathematics and science. However, students are not necessarily offered one unifying definition of the constant; there are various definitions, each of which illustrates their own unique properties of the number. This motivates an investigation of each of the definitions and their equivalences to better understand this important constant.

Though named after Leonhard Euler (1707-1783), the number is known to have been discovered by Jacob Bernoulli (1655-1705) in 1683 as it relates to compound interest. His discovery of the constant was motivated by the question "If some lender were to invest [a] sum of money [at] interest, let it accumulate, so that [at] every moment [it] were to receive [a] proportional part of [its] annual interest; how much would be owing [at the] end of [the] year?" Bernoulli referred to this unknown constant as b. He eventually found an equivalent definition of the constant (that is now known as its power series) which he used to find more decimal places of the constant's expansion. It was not until one of Euler's seminal works titled "Mechanica sive Motvs scientia analytice exposita" (which focused on Newtonian dynamics through the lens of calculus) did the constant really begin to be recognized as e [1].

The essay will demonstrate the fundamental properties of each of the various definitions of e by using rigorous proof. It will first establish a formal definition of exponentiation to illustrate applications and intuition surrounding the series definition. This investigation will ultimately lead to an evaluation of each definition in terms of its utility to a mathematics learner. It will use sources such as textbooks and mathematical articles to maintain standardized notation for common techniques or definitions.

2 The Exponential Function

When the mathematical operations of addition, multiplication, and exponentiation are introduced to a first-time learner, multiplication may be analogized to repeated addition and exponentiation may be analogized to repeated multiplication. While this is a useful tool when first learning mathematics (when problems are largely framed in terms of positive integers), these quickly break down with the introduction of rational, irrational, or complex numbers.

It is thus important to settle on a concrete definition of exponents and exponentiation. This section will provide two key insights. The first will be the importance of a solid understanding of the continuity of the exponential function. This will allow limits involving exponents to be taken across real numbers instead of only integers or rational numbers. The second insight is that this section will also serve to advocate for the case of having the series definition of e as the principal definition, as we are able to watch how exponential properties will naturally emerge from the function we ultimately define.

To rigorously define exponentiation, we will first look at a specific function. While the function's definition may seem initially unmotivated, the properties of this function will resemble our understanding of exponent properties, such as the product rule or the power rule (for real numbers a, m and n, the product rule dictates that $a^m \cdot a^n = a^{mn}$ and the power rule dictates that $(a^m)^n = a^{mn}$.)

Consider the function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find when this power series converges, we must find its interval of convergence. From the ratio test,

$$\lim_{n\to\infty} \frac{\frac{x^{(n+1)}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n\to\infty} \frac{x}{n+1} = 0,$$

meaning that the power series converges for all real x. Furthermore, since the ratio test implies absolute convergence, this function will also converge for all complex numbers. Though all of the following arguments can be generalized to complex numbers, we will keep the quantities real just so arguments with limits are more straightforward.

An immediate consequence of this function is that it is its own derivative, as

$$E'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = E(x).$$

This can be thought of as 'exponential growth', or growth proportional to the current size of the function. Already, this function is beginning to show its similarity to exponents.

Another consequence is that it satisfies many of our 'exponential properties'. For example, consider

$$E(x)E(y) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right).$$

Since these series are both absolutely convergent, we can analyze a new series by collecting each of its terms after multiplication. As an aside, the knowledge that absolutely convergent series can be multiplied in this way is a nontrivial result. However, the proof of this result is unrelated to the subject matter of the exploration. If desired, a proof can be found as the proof to Theorem 3.50 on pages 74-75 of [4].

We expand the 'infinite multiplication'

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots\right)\left(\frac{y^0}{0!} + \frac{y^1}{1!} + \frac{y^2}{2!} + \dots\right)$$

and collect the terms by their degree (recall that the degree of $x^k y^l$ will be $k + l$).

For each degree n, there will be $n + 1$ terms added together from the expansion of the product. For example, in degree 2 the terms will be $\frac{x^2}{2!} + \frac{xy}{1!1!} + \frac{y^2}{2!}$. In general, for degree n, the term will be

$$\sum_{k=0}^{n} \frac{x^{(n-k)} y^k}{(n-k)!k!}.$$

Notice that this is very similar to the statement of the binomial theorem. To match it entirely we multiply the expression cleverly by 1, keeping it the same. The 1 that we choose to multiply it by will be $\frac{n!}{n!}$ so that we can take advantage of the binomial coefficient that appears in the expression.

$$\frac{n!}{n!}\sum_{k=0}^{n}\frac{x^{(n-k)}y^k}{(n-k)!k!}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}x^{(n-k)}y^k=\frac{(x+y)^n}{n!}$$

Thus, using the binomial theorem, the term for each degree n will be $\frac{(x+y)^n}{n!}$. Since we are summing this over all non-negative n, this becomes

$$E(x)E(y)=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}=E(x+y),$$

meaning we have shown that this function has one of the crucial properties of multiplying exponents.

If we were to continue only with the function $E(x)$, we would fall into a trap. The next logical property would be to show that $[E(x)]^m = E(mx)$, but we could only reasonably do so for integer m, and would not even know what to do with rational or irrational m. We must identify a way to connect this idea of exponentiation without getting trapped in circular definitions. Instead, we notice that this function is invertible. If we can identify a convenient way to compute the inverse of this function without using exponents, then we can define exponentiation across the real numbers using this inverse function. To see that $E(x)$, we show that it is increasing over all real numbers. From its definition, it is clear that it is increasing over the positive real numbers (as you are plugging larger numbers into an infinite series that sums positive quantities). Furthermore, since $E(x)E(-x) = E(x-x) = E(0) = 1$, the function is positive for all negative real quantities (as $x > 0$ implies $E(x) > 0$ and thus $E(-x) = \frac{1}{E(x)} > 0$). Thus, $0 < x < y$ implies that $E(x) < E(y)$, which is equivalent to $\frac{1}{E(-x)} < \frac{1}{E(-y)}$, implying that $E(-y) < E(-x)$. This means $E(x)$ is increasing over all real numbers. This means we can define an inverse function of $E(x)$ which is also increasing and differentiable. This inverse function would have a domain of all positive numbers as the range of $E(x)$ is $(0, \infty)$ from its end behavior.

We can refer to this inverse function as $L(x)$. We want to use the properties and values of $E(x)$ that we have thus far to find an easier way to compute values of $L(x)$. The only value of $E(x)$ that we have so far is at $x = 0$, namely that $E(0) = 1$ and thus $L(1) = 0$. Furthermore, we know that $E'(x) = E(x)$, so we can utilize both the fact that E and L are inverses and that E has a derivative we know. This is best motivated by the notion that when working with inverse functions, we should try to use the statement that the composition of a function and its inverse yields the input.

Thus, differentiating

$$L(E(x)) = x$$

on both sides with respect to x yields

$$L'(E(x)) \cdot E(x) = 1$$

by the chain rule. Substituting $t = E(x)$ for $t > 0$ yields

$$L'(t) = \frac{1}{t}.$$

We now use our other fact and the fundamental theorem of calculus to find a more convenient form of $L(x)$. Namely,

$$L(x) = L(x) - L(1) = \int_1^x L'(t)\ dt = \int_1^x \frac{dt}{t}.$$

This expression will be analyzed at length for the "Integral Definition" characterization. For now, this expression tells us that there is an alternative way to compute the inverse of $E(x)$. We know that this function takes on unique values as $\frac{1}{x}$ is integrable for $x > 1$. Though the concept of integrability is outside of the scope of this extended essay, this fact is used to communicate that using this integral is a new way to find values of the inverse that is necessary before moving forward.

We are now in a position to define exponentiation for real exponents and positive bases. For $x > 0$ and real a,

$$x^a := E(aL(x))$$

where $E(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $L(x) := \int_1^x \frac{dt}{t}$. It is important to note that these are definitions taken as a choice of this section of the essay. This is one of many possible ways to extend exponentiation to the real numbers. The reason that these ideas are now complete is that L and E are functions that we know take on unique values due to the integrability of $\frac{1}{x}$ and the convergence of our power series. This shows that $E(aL(x))$ is a tangible quantity that satisfies the exponent properties. Thus, it makes sense to separate exponentiation from its original definition of repeated multiplication and utilize this definition in terms of power series and integrals. Though it may seem more convoluted, this treats the repeated multiplication as a consequence of this more strong and general definition of exponentiation.

From this definition, we can now put meaning to our 2nd property of exponents. For real r, $[E(x)]^r = E(rL(E(x)))$ by definition, and because E and L are inverses,

$$[E(x)]^r = E(rx).$$

Furthermore, $E(r) = E(1 \cdot r) = [E(1)]^r$, meaning it is natural to have a great interest in what $E(1) = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots$ may be. We can show that it must be a number between 2.5 and 3. To show it is greater than 2.5, we use that $1 + 1 + \frac{1}{2!} + \cdots > 1 + 1 + \frac{1}{2!} = 2.5$. To show that $E(1)$ is less than 3, consider comparing each of the terms of the series with $\frac{1}{2^n}$ for

n a non-negative integer.

$$\begin{aligned} E(1) &= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \dots \\ &< 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 2} + \dots \\ &= 1 + \frac{1}{1-\frac{1}{2}} \\ &= 3 \implies 2.5 < E(1) < 3 \end{aligned}$$

Using this infinite series, we can get infinitely many digits of $E(1)$. We know that this is a unique number (since E is a one-to-one function) that is foundational to this function, so follows to give it a name– e. This is one of many choices that can lead to a first definition of e. While this differs from Bernoulli's original discovery of the constant in terms of interest, this is often a path taken through many textbooks teaching Real Analysis. As stated by Rudin "... quite frequently, [these are] taken as the starting point of the theory of the logarithm and the exponential function" [4]. The functions $E(x)$ and $L(x)$ are the foundations of the much more commonly known e^x and $\ln(x)$.

This section rigorously defines exponentiation in terms of an infinite series and an integral of a hyperbola. While this definition may seem convoluted, it lends valuable insight to the utility of defining e in terms of its infinite series. This path allows the exponent properties to follow naturally from multiplying power series and using its definition. A consideration to note is that while this will end up being the most rigorous and reinforced of our definitions, it requires the most background– both the calculus of infinite series and integral calculus. Thus, the series definition of e and this method of rigorously defining e simplifies the process of proving important exponent properties but requires the most background knowledge.

3 Characterizations of e

A “characterization” in mathematics is a description of an object that is different from those mentioned in its definition but is equivalent to them [2]. It can be thought of as an ‘alternative definition’ of sorts that elucidates different properties of an object than those of its original definition. This concept is especially applicable to Euler’s number and the exponential function, which have several characterizations. The characterizations of focus in this essay will be referred to as the limit definition, the series definition, and the integral definition. This essay seeks to determine which characterization can act best as a ‘first definition’ of e. In showing each equivalence, it will be made clear which characterization is being used as ‘the definition’ and how that definition naturally leads itself to the properties found in other characterizations.

As mentioned earlier, the limit definition of e was how Jacob Bernoulli discovered the constant. It is the precise statement that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ converges to the constant e. This characterization sheds the most light on the appearance of e in various limits related to compounding interest. Students most concerned with quantifying the maximum amount of money from continually compounding interest will be most satisfied with this definition. However, this definition is also treated the least rigorously of the three– the knowledge that this limit even converges requires further theory that is often skipped over for the sake of practical application.

Another characterization of e can be defining at as a sum of a certain infinite series. Namely,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots,$$

where $n! = n(n-1)\dots(2)(1)$ for positive integers n. This characterization naturally follows if the exponential function is first defined in terms of a power series, as we did in the previous section. To be specific, this definition is equivalent to saying that e is defined to be $E(1)$.

This definition can be of great use to those seeking a rigorous treatment of calculus and by extension mathematical analysis. This definition requires the most background knowledge but motivates various properties of exponents quickly.

The third characterization of e that will be studied in this essay is the integral definition of e. Namely, define e to be the unique number $t > 0$ such that $\int_1^t \frac{dx}{x} = 1$. I.e, e is the number for which the area between the hyperbola $y = \frac{1}{x}$ and the x-axis from 1 to that number is 1. With reference to the function L that was previously defined in terms of this integral, e would be defined as the x value for which $L(x) = 1$. This definition is also useful in a rigorous treatment of calculus but requires more work in deriving various properties of the exponents (as we must invert the function L to get properties of E). Nonetheless, according to [3], integrals are conventionally taught before the calculus of infinite series. This means that this definition may fall into place better for students learning calculus in this sequence.

3.1 A note on circular definitions

Before the equivalences of the characterizations of e are shown, it is important to make a few observations on certain information that may already assume the establishment of the constant. For example, any usage of the natural logarithm or the fact that $\int_a^b \frac{dx}{x} = \ln \frac{b}{a}$ for $0 < a < b$ is already contingent on e having already been established. It is important to make careful note about information that can or cannot be used to avoid circular reasoning. We want to assume *only* the chosen definition at the beginning of each proof of the equivalences.

This essay initially defined the natural logarithm as the inverse function of the exponential function e^x. This already shows why using it in any proofs will be troublesome, as using the natural log requires some prior knowledge of e and the exponential function which must be avoided. We reduced the natural log to the area under a hyperbola. This essay will avoid using the familiar natural log function by investigating various properties of the area under the hyperbola. These properties will be derived by only using techniques of integration, namely substitution. It will be evident that the properties of this integral are properties of the logarithm in disguise.

Let $\displaystyle\int_1^a \frac{dx}{x} = A$ and $\displaystyle\int_1^b \frac{dx}{x} = B$ for real numbers $a, b > 1$. It is of interest to express $\displaystyle\int_1^{ab} \frac{dx}{x}$ in terms of A and B.

First, note that

$$\int_1^{ab} \frac{dx}{x} = \int_1^a \frac{dx}{x} + \int_a^{ab} \frac{dx}{x}$$

The last integral can be simplified with the substitution $x = au$. Substitution yields

$$\int_a^{ab} \frac{dx}{x} = \int_1^b \frac{a\ du}{(au)} = \int_1^b \frac{du}{u} = B,$$

meaning that $\displaystyle\int_1^{ab} \frac{dx}{x} = A + B$. Similarly, it is of interest to express $\displaystyle\int_1^{a^n} \frac{dx}{x}$ in terms of n and A. Using the substitution $x = u^n$ yields

$$\int_1^{a^n} \frac{dx}{x} = \int_1^a \frac{nu^{n-1}\ du}{u^n} = n\int_1^a \frac{du}{u} = nA.$$

In terms of our previous notation, if $L(x) = \displaystyle\int_1^x \frac{dt}{t}$ then these properties can be consolidated into two facts:

1. $L(ab) = L(a) + L(b)$ for $a, b > 1$

2. $L(a^n) = nL(a)$ for $n > 0$ and $a > 1$

These are essential properties of logarithms which have been derived solely through the manipulation of the area under the hyperbola. These properties of this integral will be vital in later proofs. Being careful about invoking circular definitions clarifies what is and is not in the 'toolbox' of ideas that can be used in proofs. Additionally, this exercise also emphasizes the importance of the hyperbola and how its geometry lends itself to these interesting properties. By using this definition we can approach the exponent properties through using the logarithm first. This only requires background knowledge of basic integral calculus and substitution.

4 Equivalences of the characterizations of e

The goal of this essay is to identify the equivalences of each of the selected characterizations of Euler's number. This will help us ultimately answer the research question of which should be ultimately assumed as the principal definition of e based on the essential properties discerned from each characterization. This ultimately leads to a greater understanding of the constant and how it naturally arises in these very different contexts. We would like to show the passage from each characterization to each other, elucidating how each could be used as a starting point (i.e. a principal definition). This passage will take shape in the form of mathematical proof, showing that by accepting a certain characterization as a principal definition, through logical steps that characterization can be shown equivalent to another.

4.1 Limit definition $\iff$ Integral definition

The first equivalence to be shown is that the limit definition of e is equivalent to the integral definition. This means that by using only the definition $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, we need to show that e is the number for which $\int_1^e \frac{dx}{x} = 1$.

First, analyze $\int_1^{\left(1+\frac{1}{n}\right)^n} \frac{dx}{x}$ without taking any limits. As shown in the previous section, we can move the exponent to a coefficient of the integral. Specifically,

$$\int_1^{\left(1+\frac{1}{n}\right)^n} \frac{dx}{x} = n \int_1^{\left(1+\frac{1}{n}\right)} \frac{dx}{x}.$$

It is of interest to bound the area of $\int_1^{\left(1+\frac{1}{n}\right)} \frac{dx}{x}$, By analysis of Figure 1, a natural bounding of this area could be the left and right Riemann sums of one partition, which are an overestimate and an underestimate, respectively. The overestimate for the area is a rectangle with base $\frac{1}{n}$ and height 1, giving an area of $\frac{1}{n}$. The underestimate rectangle has base $\frac{1}{n}$ and height $\left(1+\frac{1}{n}\right)^{-1} = \frac{n}{n+1}$, giving an area of $\frac{1}{n} \times \frac{n}{n+1} = \frac{1}{n+1}$. Thus, we have the inequality

$$\frac{1}{n+1} < \int_1^{\left(1+\frac{1}{n}\right)} \frac{dx}{x} < \frac{1}{n}. \tag{1}$$

We can multiply on all sides by n (as it is a positive quantity) and place it back as the exponent of the upper bound of the integral. This yields

$$\frac{n}{n+1} < \int_1^{\left(1+\frac{1}{n}\right)^n} \frac{dx}{x} < 1.$$

Finally, we use the squeeze theorem and take the limit of all sides, showing that $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ is a number for which $\int_1^e \frac{dx}{x} = 1$. Furthermore, this number is the only number for which this is true as $f(x) = \int_1^x \frac{dt}{t}$ for $x > 1$ is a one-to-one function. This is justified by the fact that it is an area that is increasing as x grows. Since this is the unique number satisfying this property, this shows both directions of the equivalence (as in the integral definition, there is only one such number, and the number in question was shown to satisfy the integral).

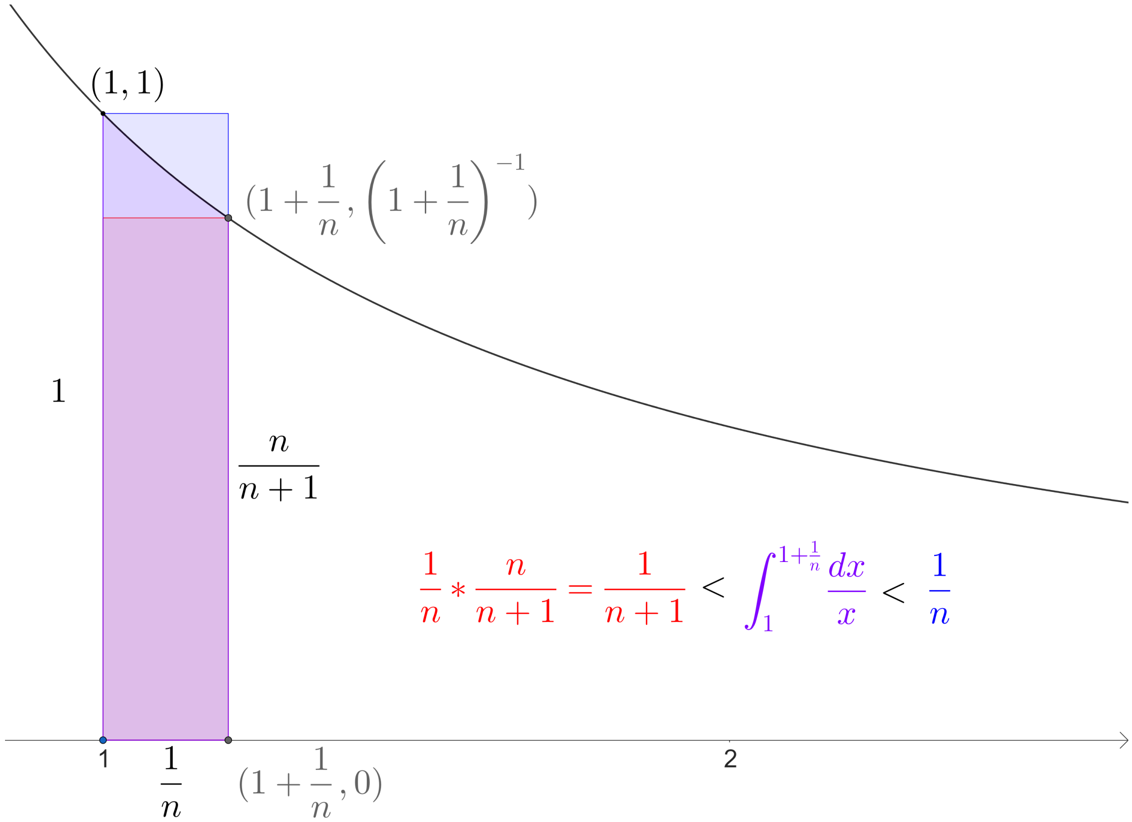

Figure 1: Limit to Integral equivalence diagram

Figure 1 provides a useful visualization for Equation 1 used in the proof. We are bounding the integral (or the purple area) between a larger rectangle (in blue) and a smaller rectangle (in red). To fully visualize the final inequality that is taken in the limit, we must extend the areas into three-dimensional prisms with width n. The prism with the area of the hyperbola as cross sections would be squeezed between the two rectangular prisms, having a volume of 1 in the limit.

This proof shows the connection of integral properties and areas to the limit definition. By squeezing our desired value between two known values, we were able to show that only one number (which we defined to be e) will make our desired integral 1. Working with the integral definition of e also provides a useful visual for the size of the quantity in terms of the hyperbola.

4.2 Limit definition $\iff$ Series definition

The following is an argument adapted from Rudin's "Principles of Mathematical Analysis" found in Theorem 3.31 in pages 64-65 ([4]). Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. I.e, s_n are the partial sums for the infinite series for e. If we assume the series definition, this means that $\lim_{n\to\infty} s_n = e$. Furthermore, let $t_n = \left(1 + \frac{1}{n}\right)^n$. Our goal is to show that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = e$. To show this, we will show that t_n is both a lower and upper bound of s_n in the limit, forcing the two to be equal in the limit.

We first notice that the limit definition includes a binomial. Using the binomial theorem,

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= \binom{n}{0}\frac{1}{n^0} + \binom{n}{1}\frac{1}{n^1} + \binom{n}{2}\frac{1}{n^2} + \dots + \binom{n}{n}\frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n^2} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n^3} + \dots \\
&\quad + \frac{1}{n!}\frac{n(n-1)\dots(2)(1)}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots \\
&\quad + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right) \\
&\le 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} \\
&= s_n
\end{aligned}$$

This means that for all n, $t_n \le s_n$. Since n is a positive quantity, we are multiplying each term by a positive number less than 1. This means that we are adding up numbers that are term by term lesser or equal to s_n, meaning the entire sum is lesser than or equal to s_n. Taking limits on both sides yields that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le \lim_{n\to\infty} s_n = e.$$

Thus, for the remainder of our proof, it will suffice to show that $\lim_{n\to\infty} t_n \geq \lim_{n\to\infty} s_n$.

We will proceed by taking a partial sum of our partial sum, which we know to be less than the whole. Let $2 < m < n$ be some fixed integer. Then

$$t_m = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{m-1}{n}\right) \leq t_n,$$

as we are simply adding less positive quantities in t_m than t_n (the sum on the left-hand side is entirely contained in the sum on the right-hand side). However, if we take the limit as $n \to \infty$, each of the terms involving n in the denominator goes to 0. This means that the amount we are decreasing each term becomes lesser and lesser until they are essentially 1. This means our inequality now is that

$$\begin{aligned} \lim_{n\to\infty} t_m &\leq \lim_{n\to\infty} t_n \\ 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m &\leq \lim_{n\to\infty} t_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \\ s_m &\leq \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \end{aligned}$$

However, since the right side does not depend on m, if we take the limit as $m \to \infty$ of both sides we get

$$\lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} = \lim_{m\to\infty} s_m \leq \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n,$$

This is exactly what we needed to finish the proof. Namely,

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} \geq \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \text{ and } \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} \leq \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

implies that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n,$$

proving that the limit definition and the series definition are equivalent.

This equivalence shows the importance of the binomial theorem and binomial coefficient to e. The prevalence of the factorials can both be attributed to the binomial coefficients in the binomial expansion of the limit or the definition of the power series. However, this proof was the less intuitive of the two, involving many bounding arguments without a convenient visual.

4.3 Integral definition $\iff$ Series definition

It is important to note that at this point, the integral definition and the series definition are necessarily equivalent as they are connected by the limit definition. This is due to the transitive property of equivalence– since the Integral is equivalent to the Limit, and the Limit is equivalent to the Series, the Integral and Series are thus equivalent.

5 Conclusion

With all three characterizations considered, an argument can be made for the integral definition as the clearest. With only the properties of integrals and substitution, the theory of logarithms can be created visually and intuitively. However, the limit definition will often be the first definition offered as it is the most effective for math teachers to have something tangible for the quantity in discussions of compound interest. Additionally, the series definition of e requires the greatest investment but pays off immensely, making all properties follow quickly. The best understanding of e ultimately stems from considering each as a definition, as the level of rigor and necessary applications of e vary throughout one's mathematical journey.

A limitation of this essay could be a lack of proof between the integral and series definitions. This essay aims to build intuition about the main tendons connecting each of the definitions to build intuition about Euler's constant and its properties. Though they are equivalent due to transitivity, only with another proof will the fundamental properties of those definitions come out. Further explorations could be taken through the lens of the logarithm and the integral definition, qualifying the difficulty of developing exponent and logarithm properties.

This essay serves to assist students in their understanding and interpretation of Euler's number. The constant e has far-reaching applications and it is thus important to understand its history and its various interpretations. By showing the equivalences between each characterization, not only did this essay replicate important mathematical findings used in better understanding the concept, but it also lent deeper insight into the important properties and facets of each characterization.

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