20D HW8

9.5

④ $\det(A-\lambda I)=0$ $\quad \det\begin{vmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{vmatrix}=0$ $\quad (1-\lambda)(-3-\lambda)-5 = -8+2\lambda+\lambda^2=0$

$\cancel{(\lambda+2)}$

$(\lambda-2)(\lambda+4)=0 \quad \begin{cases}\lambda_1=2\\ \lambda_2=-4\end{cases}$

if $\lambda=2$ $\quad [A-\lambda I \mid \vec{0}]$

$\begin{vmatrix} 1-2 & 5 & 0 \\ 1 & -3-2 & 0 \end{vmatrix} = \begin{bmatrix} -1 & 5 & 0 \\ 1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -5 & 0 \end{bmatrix}$ $\quad x_1=5x_2$, $x_2=$ free $\quad x_2\begin{bmatrix}5\\1\end{bmatrix}$

$\therefore$ eigenvalues are 2 & -4

if $\lambda=-4$ $\quad \begin{vmatrix} 5 & 5 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad x_1=-x_2$, $x_2=$ free $\quad x_2\begin{bmatrix}-1\\1\end{bmatrix}$

$\therefore$ eigenvectors are $\begin{bmatrix}5\\1\end{bmatrix}$ & $\begin{bmatrix}-1\\1\end{bmatrix}$

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⑥ $\det(A-\lambda I)=0$ $\quad \det\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = -\lambda((-\lambda)^2-1)-1(-\lambda-1)+1(1+\lambda)$

$= -\lambda^3+\lambda+\lambda+1+1+\lambda = -\lambda^3+3\lambda+2=0$

try $\lambda=2$ $\quad \cancel{-\lambda(\lambda^2-3)+2=0}$

$-8+6+2=0$ $\quad -(\lambda^2+a\lambda+1)(\lambda-2)$ $\qquad -\lambda^3+3\lambda+2=0$

$(\lambda-2)$ $\quad -\lambda^3+2\lambda^2-a\lambda^2+2a\lambda+2-\lambda$ $\quad a=2$ $\quad \Rightarrow -(\lambda^2+2\lambda+1)(\lambda-2)=0$

$2a-1=3$

eigenvalue: $\lambda_1=\lambda_2=-1$, $\lambda_3=2$ $\qquad -(\lambda+1)^2(\lambda-2)=0$

$\lambda=-1$: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x_1+x_2+x_3=0$, $x_2=$ free, $x_1=-x_2$, $x_3=0$ $\quad x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}$ $\quad x_3=$ free, $x_2=0$, $x_1=-x_3$ $\quad x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\lambda=2$: $\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow[\&\ -R_1/2,\ \&\ R_3-R_2]{2R_2+R_1} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \xrightarrow[R_3+R_2]{R_2/3,\ -R_2} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+\frac{R_2}{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1=x_3$, $x_2=x_3$, $x_3=$ free $\quad x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad$ eigenvectors are: $\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

9.6 ⑥

$\det\begin{vmatrix} -2-\lambda & -2 \\ 4 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda)+8=0 \Rightarrow -4+2\lambda-2\lambda+\lambda^2+8=0$

$\lambda^2+4=0$ $\quad \begin{cases}\lambda_1=2i\\ \lambda_2=-2i\end{cases}$

$\lambda=2i$: $\begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $\quad \begin{bmatrix} -2-2i & -2 \\ 4 & 2-2i \end{bmatrix} \xrightarrow[\text{Switch Row}]{} \begin{bmatrix} 1 & \frac{1-i}{2} \\ -1-i & -1 \end{bmatrix} \xrightarrow{R_2-(-1-i)R_1} \begin{bmatrix} 1 & \frac{1-i}{2} \\ 0 & 0 \end{bmatrix}$

$u_1+\frac{1-i}{2}u_2=0$ $\quad u_1=\frac{i-1}{2}u_2$ $\quad u_2=s$ $\qquad -1-\left(\frac{1-i}{2}\right)(-1+i) = -1-\frac{-(1-(-1))}{2} = 0$

$\vec{u}=s\begin{bmatrix}\frac{-1+i}{2}\\1\end{bmatrix} = s\begin{bmatrix}-\frac{1}{2}+\frac{i}{2}\\1\end{bmatrix}$

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$\lambda_2 = -2i$ $\begin{pmatrix} -2+2i & -2 \\ 4 & 2+2i \end{pmatrix} \xrightarrow[\& \frac{R_1}{4}, \frac{R_2}{2}]{\text{switch}} \begin{vmatrix} 1 & \frac{1+i}{2} \\ -1+i & -1 \end{vmatrix} \xrightarrow{R_2-(i-1)R_1} \begin{vmatrix} 1 & \frac{1+i}{2} \\ 0 & 0 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$-1 - \frac{(1+i)}{2}[(1-i)] = -1 + \frac{(1-i^2)}{2}$

$= -1 + \frac{2}{2} = 0$

$u_1 = \frac{-i-1}{2} u_2$

$u_2 = s$

$s\begin{pmatrix} \frac{-i-1}{2} \\ 1 \end{pmatrix} = s\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + i\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$

$\alpha \pm \beta i = 0 \pm 2i$

$e^{\alpha t}[\cos\beta t \cdot \vec{a} - \sin\beta t \cdot \vec{b}]$

$\lambda = 2i$ $= s \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + i \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$
$\vec{a}$ $\vec{b}$

$u_1 = \left[\cos 2t \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}\right] =$ $\begin{vmatrix} \cancel{\frac{\cos 2t}{2}} & \cancel{\frac{\sin 2t}{2}} \\ \cancel{0} & \cancel{\sin 2t} \end{vmatrix}$

$\begin{vmatrix} -\frac{\cos 2t}{2} \\ \cos 2t \end{vmatrix} - \begin{vmatrix} \frac{\sin 2t}{2} \\ 0 \end{vmatrix}$

$\lambda = -2i$

$u_2 = \cos 2t \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{vmatrix} \frac{-\cos 2t - \sin 2t}{2} \\ \sin 2t \end{vmatrix}$

$\vec{a} = \begin{vmatrix} -\frac{1}{2} \\ 1 \end{vmatrix}, \quad \vec{b} = \begin{vmatrix} -\frac{1}{2} \\ 0 \end{vmatrix}$

fundamental matrix : $\begin{vmatrix} \frac{-\cos 2t - \sin 2t}{2} & \frac{-\cos 2t - \sin 2t}{2} \\ \cos 2t & \sin 2t \end{vmatrix}$

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8 $\det \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -13 & 4-\lambda \end{bmatrix} = 0 = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -13 & 4-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -13 & 4-\lambda \end{vmatrix}$

$= -\lambda\left[-\lambda\begin{pmatrix} -\lambda & 1 \\ -13 & 4-\lambda \end{pmatrix}\right] - 1 \cdot \left(1\begin{pmatrix} -\lambda & 1 \\ -13 & 4-\lambda \end{pmatrix}\right) = (\lambda^2 - 1)(-4\lambda + \lambda^2 + 13) = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2}$

$\lambda_1, \lambda_2 = \pm 1$ $(\lambda^2 - 4\lambda + 13) = 0$

$\lambda_3, \lambda_4 = 2 \pm 3i$

$\lambda = 1$ $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -13 & 3 \end{bmatrix}$

$u_1 = u_2$
$-u_3 + u_4 = -13u_3 + 3u_4 = 0$

$u_1 = u_1 \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}$,

$\lambda = -1$ $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -13 & 5 \end{bmatrix}$

$u_1 = -u_2$
$u_3 = u_4 = 0$

$u_2 = u_1 \begin{vmatrix} 1 \\ -1 \\ 0 \\ 0 \end{vmatrix}$

$\lambda = 2+3i$ $\begin{vmatrix} -2-3i & 1 & 0 & 0 \\ 1 & -2-3i & 0 & 0 \\ 0 & 0 & -2-3i & 1 \\ 0 & 0 & -13 & 2-3i \end{vmatrix}$

$(-2-3i)u_1 + u_2 = (-2-3i)u_2 + u_1$
$u_1 = u_2 = 0$

$\begin{vmatrix} -2-3i & 1 \\ -13 & 2-3i \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$\begin{cases} (-2-3i)u_3 - u_4 = 0 \\ -13u_3 - (2-3i)u_4 = 0 \end{cases} \Rightarrow$

$u_3 = \text{free} = s$
$u_4 = (2+3i) \cdot s \Rightarrow s\begin{vmatrix} 0 \\ 0 \\ 1 \\ 2+3i \end{vmatrix}$

$\lambda = 2 - 3i$ $\quad -2 \pm 3i \quad 2 + 3i$ $\quad e^{\alpha t}[$

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$\alpha + \beta i = 2 \pm 3i$, $\alpha = 2$, $\beta = 3$

$\vec{a} + \vec{b} i$: $\vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$

$$C_1 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} +$$

$$C_3 \cdot e^{2t} \left[\cos 3t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right] +$$

$$C_4 \cdot e^{2t} \left[\cos 3t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + \sin 3t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right]$$

fundamental Matrix,

$$\begin{pmatrix} e^{t} & e^{-t} & 0 & 0 \\ e^{t} & -e^{-t} & 0 & 0 \\ 0 & 0 & e^{2t}\cos 3t & e^{2t}\sin 3t \\ 0 & 0 & 2e^{2t}\cos 3t - 3e^{2t}\cos 3t & 3e^{2t}\cos 3t + 2e^{2t}\sin 3t \end{pmatrix}$$