Exercícios - Angular momentum in Quantum Mechanics

Krissia de Zawadzki [TA] , Vinicius Aurichio [Pé de Pano]

Vamos resolver abaixo alguns problemas da lista 6. Antes, porém vamos fazer uma pequena digressão para ilustrar o que os operadores associados ao momento angular em Mecânica Quântica representam.

Em Mecânica Clássica o momento angular é uma quantidade vetorial que nos diz quem é a direção e o sentido associados ao movimento de rotações espaciais de corpos. [1] So let' s start it again...

In Classical Mechanics, the angular momentum is a physical quantity associated with spatial rotations. In three dimensions, it is given by a vector that defines a rotation axis upon which a particle or system executes its 'rounding' trajectory. In Quantum Mechanics, on the other hand, the angular momentum has a slightly different physical meaning. That is because it is defined in terms of two (let' s say) kinds of angular momentum, namely, the *orbital* $\vec{L}$ and *spin* $\vec{S}$ angular momenta. The first carries the same role as the angular momentum in Classical Mechanics. The second, however, is the intrinsic angular momentum that elementary particles present. Therefore, the complete angular momentum $\vec{J}$ of a quantum particle is given by

$$\vec{J} = \vec{L} + \vec{S}. \tag{1}$$

In your last class, you' ve learnt a little bit about the orbital component of the quantum angular momentum. For instance, you' ve been introduced to a new collection of operators related to the orbital angular momentum, namely, L, L_z, L_x, L_y, and L_- and L_+; and the basis $|\ell, m\rangle$. Probably, you might be confused at this time with these new concepts, mainly with the latest ones. Can you fully understand what is the action of L_+ and L_- upon a given state $|\ell, m\rangle$? Well, if you understood the action of the anihilation $a^\dagger$ and creation a operators while studying the quantum harmonical oscillator, following the angular momentum classes might be straightforward.

I must say that, probably, in your next classes about the spin angular momentum. these concepts will become more clear. Just to antecipate [2], what the operadors S_+ (spin analogous of L_+) and S_- (spin analogous of L_-) makes with a given eigenstate $|s, m_z\rangle$ is to decrease (increase) $m_z \rightarrow m_z - 1$ ($m_z \rightarrow m_z + 1$), respectively. Thinking about electrons, whose spin can be (up) $|\uparrow\rangle_i = |s_i = \frac{1}{2}, m_{z,i} = +\frac{1}{2}\rangle$ or down $|\downarrow\rangle_i = |s_i = \frac{1}{2}, m_{z,i} = -\frac{1}{2}\rangle$, applying S_- or S_+ is equivalent to a spin flip, while applying S_z it is the same of measuring the z component of its spin angular momentum. See the figure 1.

Figura 1. Illustration of the action of the operators **(a)** S_+ and **(b)** S_- upon the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of spin momentum of an electron.

[1] A partir daqui eu vou escrever em inglês porque todos os teclados desse lugar são british e é um saco ficar digitando os comandos latex com acentos :/ (por exemplo, para escrever um simples ão eu preciso combinar \+ til + a).

[2] Please, do not panic before seeing the analogy I'll use through the following illustrations.

OBS: For a many-particle system, for instance, a set of N electron, we must stress that identifying the eigenstates of S^2 and S_z is not that easy. It is true that these eigenstates are written in terms of the individual spin states of each particle, as illustrated in figure 2.

Figura 2. Example of configurations for a set of $N = 5$ electrons. The ket configuration $|\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$ in **(a)** is not an eigenstate of the operador $S^2 = S_1^2 + S_2^2 + S_3^2 + S_4^4 + S_5^2$. For a given eigenstate $|s, m_z\rangle$ m_z runs from $-s$ to $+s$. In order to properly find which are the eigenstates $|s, m_z\rangle$ having $m_z = 1/2$, one must find the configurations similar to the one presented in **(a)** and conveniently combine them to form an eigenstate with definite spin s.. On the other hand, the fully stretched configuration $|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$ in **(b)** is, so that $|s = 5/2, m = 5/2\rangle = |\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$ and $S^2\,|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle = 15\hbar^2\,|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$! Notice that the configuration in **(a)** is an eigenstate of S_z with eigenvalue $m_z = 1/2$.

With the previous comments and the amazing notes prof. Luiz provided us, now we are ready to start our 6th set of problems.

Problema 1

(a) In order to compute the matrix elements

$$\begin{aligned} &\langle \ell, m|\, L_+ L_-\, |\ell, m\rangle \\ &\langle \ell, m|\, L_- L_+\, |\ell, m\rangle\,, \end{aligned} \tag{2}$$

from the relations for the operators L_- and L_+

$$\begin{aligned} L_+\,|\ell, m\rangle &= \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|\ell, m+1\rangle \\ L_-\,|\ell, m\rangle &= \hbar\sqrt{\ell(\ell+1) - m(m-1)}\,|\ell, m-1\rangle\,, \end{aligned} \tag{3}$$

we can identify that the $|\ell, m\rangle$ are eigenstates of the composite operators L_+L_- and L_-L_+. We have that

$$\begin{aligned} L_+L_-\,|\ell, m\rangle &= \hbar^2\sqrt{\ell - (m-1)(m-1+1)}\sqrt{\ell - m(m-1)}\,|\ell, m\rangle \\ &= \hbar^2(\ell(\ell+1) - m(m-1))\,|\ell, m\rangle\,, \end{aligned} \tag{4}$$

and

$$\begin{aligned} L_-L_+\,|\ell, m\rangle &= \hbar^2\sqrt{\ell(\ell+1) - (m+1)(m+1-1)}\sqrt{\ell - m(m+1)}\,|\ell, m\rangle \\ &= \hbar^2(\ell(\ell+1) - m(m+1))\,|\ell, m\rangle\,. \end{aligned} \tag{5}$$

It follows that

$$\begin{aligned}\langle \ell, m| L_+L_- |\ell, m\rangle &= \hbar^2(\ell(\ell+1) - m(m-1)) \\ \langle \ell, m| L_-L_+ |\ell, m\rangle &= \hbar^2(\ell(\ell+1) - m(m+1)).\end{aligned} \tag{6}$$

(b) To compute the same matrix elements in eq. (2) from the relations

$$L^2 = L_x^2 + L_y^2 + L_z^2, \tag{7}$$

and

$$\begin{aligned}L_+ &= L_x + iL_x \\ L_- &= L_x - iL_y,\end{aligned} \tag{8}$$

we must define the operator L^2 in terms of L_z, L_+ and L_- and identify

$$\begin{aligned}L_+L_- &= L_x^2 + L_y^2 + i(L_yL_x - L_xL_y) \\ &= L^2 - L_z^2 + i\underbrace{(L_yL_x - L_xL_y)}_{[L_y,L_x]=-i\hbar L_z} \\ &= L^2 - L_z^2 + \hbar L_z.\end{aligned} \tag{9}$$

Repeating the same for L_-L_+, we obtain

$$\begin{aligned}L_-L_+ &= L_x^2 + L_y^2 + i(L_xL_y - L_yL_x) \\ &= L^2 - L_z^2 + i\underbrace{(L_xL_y - L_yL_x)}_{[L_x,L_y]=i\hbar L_z} \\ &= L^2 - L_z^2 - \hbar L_z.\end{aligned} \tag{10}$$

. We know that

$$\begin{aligned}S^2 |\ell, m\rangle &= \hbar^2\ell(\ell+1) |\ell, m\rangle \\ S_z |\ell, m\rangle &= \hbar m |\ell, m\rangle .\end{aligned} \tag{11}$$

By "sandwiching" eqs. (9) and (10) with the bras and kets $|\ell, m\rangle$, we got

$$\begin{aligned}\langle \ell, m| L_+L_- |\ell, m\rangle &= \langle \ell, m| L^2 - L_z^2 + \hbar L_z |\ell, m\rangle \\ &= \hbar^2\ell(\ell+1) - \hbar^2m^2 + \hbar^2 m \\ &= \hbar^2(\ell(\ell+1) - m(m-1)),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\langle \ell, m| L_- L_+ |\ell, m\rangle &= \langle \ell, m| L^2 - L_z^2 - \hbar L_z |\ell, m\rangle \\ &= \hbar^2 \ell(\ell+1) - \hbar^2 m^2 - \hbar^2 m \\ &= \hbar^2(\ell(\ell+1) - m(m+1)), \end{aligned} \tag{13}$$

respectively.

Problema 2

(a) By definition

$$L_z |\ell, m\rangle = \hbar m |\ell, m\rangle \tag{14}$$

and so

$$L_z^2 |\ell, m\rangle = \hbar^2 m^2 |\ell, m\rangle \tag{15}$$

$$\langle \ell, m| L_z^2 |\ell, m\rangle = \hbar^2 m^2 \tag{16}$$

(b) From the definition of L_+ and L_- it is easy to obtain the following relations

$$L_x = \frac{1}{2}(L_+ + L_-) \tag{17}$$

$$L_y = \frac{1}{2i}(L_+ - L_-) \tag{18}$$

We can now rewrite the operator L_x^2 in terms of L_+, L_-

$$L_x^2 = \frac{1}{2}(L_+ + L_-)\frac{1}{2}(L_+ + L_-) \tag{19}$$

$$= \frac{1}{4}\left(L_+^2 + L_+L_- + L_-L_+ + L_-^2\right) \tag{20}$$

Ready to calculate the expected value?

$$\langle \ell, m| L_x^2 |\ell, m\rangle = \frac{1}{4}\langle \ell, m| L_+^2 + L_+L_- + L_-L_+ + L_-^2 |\ell, m\rangle \tag{21}$$

$$= \frac{1}{4}\left(0 + \hbar^2(\ell(\ell+1) - m(m-1)) + \hbar^2(\ell(\ell+1) - m(m+1)) + 0\right) \tag{22}$$

$$= \frac{\hbar^2}{2}(\ell(\ell+1) - m^2) \tag{23}$$

(c) Repeating the same procedure for L_y^2...

$$L_y^2 = \frac{1}{2i}(L_+ - L_-)\frac{1}{2i}(L_+ - L_-) \tag{24}$$

$$= -\frac{1}{4}\left(L_+^2 - L_+L_- - L_-L_+ + L_-^2\right) \tag{25}$$

And now to the expected value:

$$\langle \ell, m | L_y^2 | \ell, m \rangle = -\frac{1}{4} \langle \ell, m | L_+^2 - L_+L_- - L_-L_+ + L_-^2 | \ell, m \rangle \tag{26}$$

$$= -\frac{1}{4} \left(0 - \hbar^2(\ell(\ell+1) - m(m-1)) - \hbar^2(\ell(\ell+1) - m(m+1)) + 0 \right) \tag{27}$$

$$= \frac{\hbar^2}{2}(\ell(\ell+1) - m^2). \tag{28}$$

Is it surprising that both L_x^2 and L_y^2 have the same expected value at $|\ell, m\rangle$?

Problema 3

Let's compute a set of matrix elements of operators L_-, L_+ and L_x. At this point, useful relations are despicted in eqs. (3).

(a)

$$\begin{aligned} \langle 1, 0 | L_- | 1, 1 \rangle &= \hbar\sqrt{1(1+1) - 1(1-1)} \langle 1, 0 \,|\, 1, 0 \rangle \\ &= \hbar\sqrt{2}. \end{aligned} \tag{29}$$

(b)

$$\begin{aligned} \langle 1, -1 | L_- | 1, 1 \rangle &= \hbar\sqrt{1(1+1) - 1(1-1)} \langle 1, -1 \,|\, 1, 0 \rangle \\ &= 0. \end{aligned} \tag{30}$$

(c)

$$\begin{aligned} \langle 2, 0 | L_- | 1, 1 \rangle &= \hbar\sqrt{1(1+1) - 1(1-1)} \langle 2, 0 \,|\, 1, 0 \rangle \\ &= 0. \end{aligned} \tag{31}$$

(d)

$$\begin{aligned} \langle 2, 0 | L_- | 2, 1 \rangle &= \hbar\sqrt{2(2+1) - 1(1-1)} \langle 2, 0 \,|\, 2, 0 \rangle \\ &= \sqrt{6}. \end{aligned} \tag{32}$$

(e)

$$\begin{aligned} \langle 2, 0 | L_x | 2, 1 \rangle &= \langle 2, 0 | \frac{L_+ + L_-}{2} | 2, 1 \rangle \\ &= \hbar\frac{\sqrt{2(2+1) - 1(1+1)}}{2} \langle 2, 0 \,|\, 2, 2 \rangle + \hbar\frac{\sqrt{2(2+1) - 1(1-1)}}{2} \langle 2, 0 \,|\, 2, 0 \rangle \\ &= \hbar\sqrt{\frac{3}{2}}. \end{aligned} \tag{33}$$

Problema 4

(a)

$$\begin{aligned}
\langle \ell,\ell| L_x^2 |\ell,\ell\rangle &= \langle \ell,\ell| \frac{L_+ + L_-}{2}\frac{L_+ + L_-}{2} |\ell,\ell\rangle \\
&= \frac{1}{4}\langle \ell,\ell| L_+L_+ + L_+L_- + L_-L_+ + L_-L_- |\ell,\ell\rangle \\
&= \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell+1)}}{4}\langle \ell,\ell| L_+ |\ell,\ell+1\rangle + \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell-1)}}{4}\langle \ell,\ell| L_+ |\ell,\ell-1\rangle \\
&\quad + \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell+1)}}{4}\langle \ell,\ell| L_- |\ell,\ell+1\rangle + \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell-1)}}{4}\langle \ell,\ell| L_- |\ell,\ell-1\rangle \\
&= 0 + \frac{\hbar^2}{4}\sqrt{\ell(\ell+1)-\ell(\ell-1)}\sqrt{\ell(\ell+1)-(\ell-1)(\ell-1+1)} \\
&\quad + \frac{\hbar^2}{4}\sqrt{\ell(\ell+1)-\ell(\ell+1)}\sqrt{\ell(\ell+1)-(\ell+1)(\ell+1-1)} + 0 \\
&= \frac{\hbar^2}{4}(\ell(\ell+1)-\ell(\ell+1)) + \frac{\hbar^2}{4}(\ell(\ell+1)-\ell(\ell-1)) \\
&= \frac{\hbar^2}{2}(\ell^2+\ell) - \frac{\hbar^2}{4}(\ell^2+\ell+\ell^2-\ell) \\
&= \frac{\hbar^2\ell}{2}.
\end{aligned} \tag{34}$$

(b)

$$\begin{aligned}
\langle \ell,\ell| L_y^2 |\ell,\ell\rangle &= \langle \ell,\ell| \frac{L_+ - L_-}{2i}\frac{L_+ - L_-}{2i} |\ell,\ell\rangle \\
&= \frac{1}{4i^2}\langle \ell,\ell| L_+L_+ - L_+L_- - L_-L_+ + L_-L_- |\ell,\ell\rangle \\
&= \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell+1)}}{4i^2}\langle \ell,\ell| L_+ |\ell,\ell+1\rangle - \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell-1)}}{4i^2}\langle \ell,\ell| L_+ |\ell,\ell-1\rangle \\
&\quad - \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell+1)}}{4i^2}\langle \ell,\ell| L_- |\ell,\ell+1\rangle + \frac{\hbar\sqrt{\ell(\ell+1)-\ell(\ell-1)}}{4i^2}\langle \ell,\ell| L_- |\ell,\ell-1\rangle \\
&= 0 - \frac{\hbar^2}{-4}\sqrt{\ell(\ell+1)-\ell(\ell-1)}\sqrt{\ell(\ell+1)-(\ell-1)(\ell-1+1)} \\
&\quad - \frac{\hbar^2}{-4}\sqrt{\ell(\ell+1)-\ell(\ell+1)}\sqrt{\ell(\ell+1)-(\ell+1)(\ell+1-1)} + 0 \\
&= \frac{\hbar^2}{4}(\ell(\ell+1)-\ell(\ell+1)) + \frac{\hbar^2}{4}(\ell(\ell+1)-\ell(\ell-1)) \\
&= \frac{\hbar^2}{2}(\ell^2+\ell) - \frac{\hbar^2}{4}(\ell^2+\ell+\ell^2-\ell) \\
&= \frac{\hbar^2\ell}{2}.
\end{aligned} \tag{35}$$

Notice that we could simply pick the results of problem 2 to compute these matrices elements. We should arrive in the same results.

(c)

$$\begin{aligned}\langle \ell, \ell| L_z^2 |\ell, \ell\rangle &= \langle \ell, \ell| L_z L_z |\ell, \ell\rangle \\ &= \hbar\ell \langle \ell, \ell| L_z |\ell, \ell\rangle \\ &= \hbar^2\ell^2. \end{aligned} \tag{36}$$

(d) Using previous relations in (a)-(c), we obtain[3]

$$\begin{aligned}\langle \ell, \ell| L^2 |\ell, \ell\rangle &= \langle \ell, \ell| L_x^2 + L_y^2 + L_z^2 |\ell, \ell\rangle \\ &= \hbar^2\ell(\ell+1). \end{aligned} \tag{37}$$

Problema 5

(a)

$$\begin{aligned}\langle 3, 1| L_- L_- |3, 3\rangle &= \hbar\sqrt{3(3+1) - 3(3-1)}\, \langle 3, 1| L_- |3, 2\rangle \\ &= \hbar^2\sqrt{3(3+1) - 3(3-1)}\sqrt{3(3+1) - 2(2-1)}\langle 3, 1\,|\,3, 1\rangle \\ &= \hbar^2\sqrt{12-6}\sqrt{12-2} \\ &= \hbar^2\sqrt{60} \\ &= 2\hbar^2\sqrt{15}. \end{aligned} \tag{38}$$

Notice, that we should get the same result in (b): **(b)**

$$\begin{aligned}\langle 3, 3| L_+ L_+ |3, 1\rangle &= \hbar\sqrt{3(3+1) - 1(1+1)}\, \langle 3, 3| L_+ |3, 2\rangle \\ &= \hbar^2\sqrt{3(3+1) - 1(1+1)}\sqrt{3(3+1) - 2(2+1)}\langle 3, 3\,|\,3, 3\rangle \\ &= \hbar^2\sqrt{12-2}\sqrt{12-6} \\ &= \hbar^2\sqrt{60} \\ &= 2\hbar^2\sqrt{15}. \end{aligned} \tag{39}$$

Problema 6

Let's compute the matrix elements

$$\begin{aligned}&\langle \ell, m| L_+ |\ell', m'\rangle \\ &\langle \ell', m'| L_- |\ell, m\rangle \end{aligned} \tag{40}$$

in order to check if L_+ and L_- are Hermitian conjugates of each other.

[3] c.q.d. nail it!

From the definitions (3), we have

$$
\begin{aligned}
\langle \ell, m| \, L_+ \, |\ell', m'\rangle &= \hbar\sqrt{\ell'(\ell'+1) - m'(m'+1)} \, \underbrace{\langle \ell, m \, | \, \ell', m'+1\rangle}_{\delta_{\ell,\ell'}\delta_{m,m'+1}} \\
&= \hbar\sqrt{\ell'(\ell'+1) - m'(m'+1)} \, \delta_{\ell,\ell'} \, \delta_{m,m'+1} \\
&= \hbar\sqrt{\ell(\ell+1) - (m-1)(m-1+1)} \, \delta_{\ell,\ell'} \, \delta_{m-1,m'} \\
&= \hbar\sqrt{\ell(\ell+1) - (m-1)(m)} \, \delta_{\ell,\ell'} \, \delta_{m-1,m'}
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
\langle \ell, m| \, L_+ \, |\ell', m'\rangle &= \hbar\sqrt{\ell'(\ell'+1) - m'(m'-1)} \, \underbrace{\langle \ell, m \, | \, \ell', m'+1\rangle}_{\delta_{\ell,\ell'}\delta_{m,m'-1}} \\
&= \hbar\sqrt{\ell'(\ell'+1) - m'(m'-1)} \, \delta_{\ell,\ell'} \, \delta_{m,m'-1} \\
&= \hbar\sqrt{\ell(\ell+1) - (m+1)(m+1-1)} \, \delta_{\ell,\ell'} \, \delta_{m+1,m'} \\
&= \hbar\sqrt{\ell(\ell+1) - (m+1)(m)} \, \delta_{\ell,\ell'} \, \delta_{m,m'-1}.
\end{aligned}
\tag{42}
$$

Problema 7

We want to write the components L_x, L_y and L_z of the angular momentum operators $\mathbf{L}$[4] in spherical coordinates. For this, we must recall the definition of the vector $\mathbf{L}$ in terms of the linear momentum $\mathbf{p}$ and the position $\mathbf{r}$

$$
\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{p} \\
&= (r_y p_z - r_z p_y)\hat{x} + (r_z p_x - r_x p_z)\hat{y} + (r_x p_y - r_y p_x)\hat{z},
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
r_x &= r\sin\theta\cos\varphi \\
r_y &= r\sin\theta\sin\varphi \\
r_z &= r\cos\theta
\end{aligned}
\tag{44}
$$

and

$$
\begin{aligned}
\frac{i}{\hbar}p_x &= \frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} \\
\frac{i}{\hbar}p_y &= \frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} \\
\frac{i}{\hbar}p_z &= \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}.
\end{aligned}
\tag{45}
$$

OBS: Do not blame yourself if you did not recognize the factors apearing in fron of each partial

[4]Notice that I'm using bold to express that the quantity is a vector.

derivatives in eqs. (45). To obtain them, you must remember how the cartesian versors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are defined in terms of the spherical coordinates. They carry these scale factors because

$$\begin{aligned}
\hat{x} &= \sin\theta\cos\varphi\,\hat{r} + \cos\theta\cos\varphi\,\hat{\theta} - \sin\varphi\,\hat{\varphi} \\
\hat{y} &= \sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \sin\varphi\,\hat{\varphi} \\
\hat{z} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}.
\end{aligned} \tag{46}$$

Let's calculate the L_x component by plugging eqs. (44) and (45) into (43).

$$\begin{aligned}
L_x &= (r_y p_z - r_z p_y) \\
&= r\sin\theta\sin\varphi\frac{\hbar}{i}\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) \\
&\quad - r\cos\theta\frac{\hbar}{i}\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= -\frac{\hbar}{i}\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right).
\end{aligned} \tag{47}$$

Repeating the same with L_y, we get

$$\begin{aligned}
L_y &= (r_z p_x - r_x p_z) \\
&= r\cos\theta\frac{\hbar}{i}\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&\quad - r\sin\theta\cos\varphi\frac{\hbar}{i}\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) \\
&= \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right).
\end{aligned} \tag{48}$$

Finaly, L_z is obtained as follows

$$\begin{aligned}
L_z &= (r_x p_y - r_y p_x) \\
&= r\sin\theta\cos\varphi\frac{\hbar}{i}\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&\quad - r\sin\theta\sin\varphi\frac{\hbar}{i}\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}.
\end{aligned} \tag{49}$$

Given that the operators L_+ (raising) and L_- (lowering) are combinations of L_x and L_y, we can write

L_+ in spherical coordinates as

$$
\begin{aligned}
L_+ &= L_x + iL_y \\
&= -\frac{\hbar}{i}\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) + \hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \\
&= \hbar\left(\cos\varphi + i\sin\varphi\right)\frac{\partial}{\partial\theta} + \hbar\cot\theta\left(i\cos\varphi + \sin\varphi\right)\frac{\partial}{\partial\varphi} \\
&= \hbar\left(\cos\varphi + i\sin\varphi\right)\frac{\partial}{\partial\theta} + i\hbar\cot\theta\left(\cos\varphi + i\sin\varphi\right)\frac{\partial}{\partial\varphi} \\
&= \hbar\underbrace{\left(\cos\varphi + i\sin\varphi\right)}_{e^{i\varphi}}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \\
&= \hbar\, e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right).
\end{aligned} \tag{50}
$$

Repeating the same procedure, it is easy to verify[5] that the lowering operator is given by

$$
\begin{aligned}
L_- &= L_x - iL_y \\
&= -\frac{\hbar}{i}\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) - \hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \\
&= -\hbar\left(\cos\varphi - i\sin\varphi\right)\frac{\partial}{\partial\theta} + \hbar\cot\theta\left(i\cos\varphi - \sin\varphi\right)\frac{\partial}{\partial\varphi} \\
&= -\hbar\left(\cos\varphi - i\sin\varphi\right)\frac{\partial}{\partial\theta} + i\hbar\cot\theta\left(\cos\varphi - i\sin\varphi\right)\frac{\partial}{\partial\varphi} \\
&= -\hbar\underbrace{\left(\cos\varphi - i\sin\varphi\right)}_{e^{-i\varphi}}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \\
&= -\hbar\, e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right).
\end{aligned} \tag{51}
$$

In summary, we can compact both expressions (50) and (51) in a single relation given by

$$
L_\pm = \pm\hbar\, e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right). \tag{52}
$$

Problema 8

We want to compute the commutator

$$
\begin{aligned}
[L_+, L_z] &= L_+L_z - L_zL_+ \\
&= \hbar\, e^{+i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\hbar\, e^{+i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \\
&= -i\hbar^2 e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} + \hbar^2 e^{i\varphi}\cot\theta\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta} - \hbar^2\frac{\partial}{\partial\varphi}e^{i\varphi}\cot\theta\frac{\partial}{\partial\varphi}.
\end{aligned} \tag{53}
$$

Notice that we cannot simply change the order of the derivatives because when the operators act upon

[5] Every time you read this in a book or in a paper don't you want to kill the author? That's auhtor's revenge: when you have spent couple of lines doing a complicated calculation, write "it is easy to verify that"and variants of this expression. lol

a given wave function state $\psi(r,\theta,\phi)$.

Let's compute each one of the four derivatives that appear in the last line of eq. (53).

$$-i\hbar^2 e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\psi(r,\theta,\phi), \tag{54}$$

$$+\hbar^2 e^{i\varphi}\cot\theta\frac{\partial^2}{\partial\varphi^2}\psi(r,\theta,\phi), \tag{55}$$

$$\begin{aligned} +i\hbar^2\frac{\partial}{\partial\varphi}e^{i\varphi}\frac{\partial}{\partial\theta}\psi(r,\theta,\phi) &= +i\hbar^2 i e^{i\varphi}\frac{\partial}{\partial\theta}\psi(r,\theta,\phi) + i\hbar^2 e^{i\varphi}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\theta}\psi(r,\theta,\phi) \\ &= -\hbar^2 e^{i\varphi}\frac{\partial}{\partial\theta}\psi(r,\theta,\phi) + i\hbar^2 e^{i\varphi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\psi(r,\theta,\phi), \end{aligned} \tag{56}$$

$$\begin{aligned} -\hbar^2\frac{\partial}{\partial\varphi}e^{i\varphi}\cot\theta\frac{\partial}{\partial\varphi}\psi(r,\theta,\phi) &= -\hbar^2 i e^{i\varphi}\cot\theta\frac{\partial}{\partial\varphi}\psi(r,\theta,\phi) - \hbar^2 e^{i\varphi}\cot\theta\frac{\partial^2}{\partial\varphi^2}\psi(r,\theta,\varphi) \\ &= -i\hbar^2 e^{i\varphi}\cot\theta\frac{\partial}{\partial\varphi}\psi(r,\theta,\phi) - \hbar^2 e^{i\varphi}\cot\theta\frac{\partial^2}{\partial\varphi^2}\psi(r,\theta,\varphi), \end{aligned} \tag{57}$$

Putting together all terms in eqs. (54) - (57), we have

$$[L_+, L_z] = -\hbar^2 e^{i\varphi}(\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\varphi}) \tag{58}$$

Problema 9

To show that the wave-function

$$\psi(r,\theta,\varphi) = Ae^{-(r/a)^2}, \tag{59}$$

A and a constants, is a eigenstate of L_z, we must be able to prove that

$$L_z\psi(r,\theta,\varphi) = \alpha\psi(r,\theta,\varphi). \tag{60}$$

Let's recall the expression for the operator L_z in spherical coordinats in eq. (49). It follows that

$$\begin{aligned} L_z\psi(r,\theta,\varphi) &= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\psi(r,\theta,\varphi) \\ &= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}Ae^{-(r/a)^2} \\ &= 0. \end{aligned} \tag{61}$$

Since ψ does not deppend on the coordinate φ, it is a an eigenstate of L_z with eigenvalue 0!

Problema 10

Problem 10 ask us to compute the normalization constants for the spherical harmonics $Y_{\ell m}$ with $\ell = 0, 1$ and their corresponding m's. To do so, we must simply use the orthonormalization relations

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, Y_{\ell,m} \, Y^*_{\ell',m'} = \delta_{\ell,\ell'}\delta_{m,m'}. \tag{62}$$

(a) Starting with

$$Y_{00} = A, \tag{63}$$

we simply have that

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, |A|^2 &= \int d\Omega |A|^2 \\ &\to A = \frac{1}{\sqrt{4\pi}}. \end{aligned} \tag{64}$$

(b) Starting with

$$Y_{1\,1} = B \sin\theta e^{i\varphi}, \tag{65}$$

the orthonormalization implies that

$$\begin{aligned} \int d\Omega \, Y_{1\,1} Y^*_{1\,1} &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, |B|^2 \sin^2\theta \\ &= 2\pi \int_{-1}^{1} dx (1 - x^2)|B|^2 \\ &= 2\pi(2 - \frac{2}{3})|B|^2 \\ &\to B = \sqrt{\frac{3}{8\pi}}. \end{aligned} \tag{66}$$

(c) Starting with

$$Y_{1\,0} = C \cos\theta, \tag{67}$$

the orthonormalization implies that

$$\begin{aligned} \int d\Omega \, Y_{1\,0} Y^*_{1\,0} &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, |C|^2 \cos^2\theta \\ &= 2\pi \int_{-1}^{1} dx x^2 |C|^2 \\ &= 2\pi(\frac{2}{3})|C|^2 \\ &\to C = \sqrt{\frac{3}{4\pi}}. \end{aligned} \tag{68}$$

(d) Since $Y_{1\,-1}$ differs from $Y_{1\,1}$ just for the sign in the exponential of φ, they share the same normalization constant. So,

$$\begin{aligned}\int d\Omega\, Y_{1\,-1} Y^*_{1\,-1} &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\, |D|^2 \sin^2\theta \\ \to D &= \sqrt{\frac{3}{8\pi}}.\end{aligned} \tag{69}$$

Problema 11

We want to show that computing $L_- Y_{\ell m}$ yields the same result of $L_- \,|\ell, m\rangle$ for all ℓ and m. In particular, the action of the lowering operator upon $Y_{1,1}$ should bring us to $\hbar\sqrt{2}\, Y_{1,0}$. Let's see ...

$$\begin{aligned}L_- Y_{1,1} &== \hbar\, e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi} \\ &= \hbar\sqrt{\frac{3}{8\pi}} e^{-i\varphi}\cos\theta e^{i\varphi} - i\hbar\sqrt{\frac{3}{8\pi}} e^{-i\varphi}\cot\theta\sin\theta i e^{i\varphi} \\ &= \hbar\sqrt{\frac{3}{8\pi}}\cos\theta + \hbar\sqrt{\frac{3}{8\pi}}\frac{\cos\theta}{\sin\theta}\sin\theta \\ &= 2\hbar\sqrt{\frac{3}{8\pi}}\cos\theta \\ &= \sqrt{2}\hbar\underbrace{\sqrt{\frac{3}{4\pi}}\cos\theta}_{Y_{1\,0}}.\end{aligned} \tag{70}$$

Problema 12

We want to show from the integrals in spherical coordinates that the eigenstates of angular momentum are orthonormal. Let's calculate the matrix element

$$\begin{aligned}\langle 1,1\,|\,1,0\rangle &= 0 \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\, Y_{1\,1} Y^*_{1\,0} \\ &= \sqrt{\frac{3}{8\pi}}\sqrt{\frac{3}{4\pi}}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\, e^{i\varphi}\sin\theta\cos\theta \\ &= \frac{1}{\sqrt{2}}\frac{3}{4\pi}\underbrace{\int_0^{2\pi} d\varphi\, e^{i\varphi}}_{=0}\underbrace{\int_0^{\pi} d\theta\, \sin^2\theta\cos\theta}_{=0} \\ &= 0\end{aligned} \tag{71}$$

Problema 13

It's a trap!

If two operators commute it means it is **possible** to find a complete set of eigenvectors common to both

operators. But it **doesn't** imply that **every** eigenvector of one will also be an eigenvector of the other.
Let's illustrate that with an example using the operators in the exercise, L_z and L^2.
Consider the space formed by all the states that have $\ell = 0$ or $\ell = 1$. A complete set of states that are eigenstates of both L_z and L^2 is $|0,0\rangle\,, |1,-1\rangle\,, |1,0\rangle\,, |1,1\rangle$.
On the other hand the state $|\psi\rangle = |0,0\rangle + |1,0\rangle$ is an eigenstate of L_z, but not of L^2!
Similarly the state $|\varphi\rangle = |1,0\rangle + |1,1\rangle$ is an eigenstate of L^2, but not of L_z.

Problema 14

We can use again that $L_x = (L_+ + L_-)/2$

$$L_x |1,0\rangle = \frac{1}{2}(L_+ + L_-)|1,0\rangle \tag{72}$$
$$= \frac{1}{2}\left(\sqrt{2}\,|1,1\rangle + \sqrt{2}\,|1,-1\rangle\right) \tag{73}$$
$$\neq \alpha\,|1,0\rangle \tag{74}$$

So $|1,0\rangle$ is not an eigenvector of L_x

Problema 15

(a) We begin writing the matrix explicitly

$$L^2 = \begin{bmatrix} \langle 1,-1|\, L^2 \,|1,-1\rangle & \langle 1,-1|\, L^2 \,|1,0\rangle & \langle 1,-1|\, L^2 \,|1,1\rangle \\ \langle 1,0|\, L^2 \,|1,-1\rangle & \langle 1,0|\, L^2 \,|1,0\rangle & \langle 1,0|\, L^2 \,|1,1\rangle \\ \langle 1,1|\, L^2 \,|1,-1\rangle & \langle 1,1|\, L^2 \,|1,0\rangle & \langle 1,1|\, L^2 \,|1,1\rangle \end{bmatrix} \tag{75}$$

When we create a matrix representation in a basis, we are writing that the element A_{mn} of the operator A is $A_{mn} = \langle\varphi_m|\, A \,|\varphi_n\rangle$. This is why terms in the form $\langle\varphi_m|\, A \,|\varphi_n\rangle$ are often refered to as *matrix elements*.
Calculating the matrix elements we obtain

$$L^2 = \begin{bmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{bmatrix} \tag{76}$$

(b) Repeating the procedure for L_z we obtain

$$L_z = \begin{bmatrix} -\hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar \end{bmatrix} \tag{77}$$

(c) For L_- we get

$$L_- = \begin{bmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{bmatrix} \tag{78}$$

(d) We have to calculate $[L_z, L_-]$ using the matrices we just obtained. So

$$[L_z, L_-] = L_z L_- - L_- L_z \tag{79}$$

$$= \begin{bmatrix} -\hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar \end{bmatrix} \tag{80}$$

$$= \begin{bmatrix} 0 & -\sqrt{2}\hbar^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}\hbar^2 \\ 0 & 0 & 0 \end{bmatrix} \tag{81}$$

$$= -\hbar \begin{bmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{bmatrix} \tag{82}$$

$$= -\hbar L_- \tag{83}$$

as expected.