Derivation of an actor-based crowd model

Assume that we are studying n disk-shaped actors with position vectors $\mathbf{p}_i \in \mathbb{R}^2$ and velocity vectors $\mathbf{v}_i \in \mathbb{R}^2$, for $i \in \{1, \ldots, n\}$, where all actors share the same disk radius r. The inherent idea behind a microscopic actor-based crowd model is that we individually study each actor and the forces which accelerates it. Position and velocity of an actor is then easily updated at each time step by a forward Euler method

$$\begin{aligned}\mathbf{v}_{t+1} &= \mathbf{v}_t + \mathbf{a}_t \Delta t,\\ \mathbf{p}_{t+1} &= \mathbf{p}_t + \mathbf{v}_t \Delta t,\end{aligned} \tag{1}$$

for acceleration $\mathbf{a}_t$ and time step Δt. Setting the mass of each actor equal to 1, we see that $\mathbf{F}_t = \mathbf{a}_t$.

1.1 Basic walk cycle

Each actor has a defined goal which can be used to formulate a desired (goal) velocity $\tilde{\mathbf{v}}_i$. Hence we can formulate a *willpower* driving the actor forward, which we assume to be proportional to the deviance of actual velocity to desired velocity,

$$\mathbf{F}_i = \alpha(\tilde{\mathbf{v}}_i - \mathbf{v}_i), \tag{2}$$

where $\alpha > 0$ is a proportionality constant.

In order to describe impurities in an actors walk cycle, a small *stochastic force* is applied to the actors at each time step. For simplicity the stochastic force $\mathbf{S}_i$, for actors $i \in \{1, \ldots, n\}$, is drawn from a uniform distribution,

$$\mathbf{S}_i \sim \mathrm{unif}\left(\begin{bmatrix} -\xi \\ -\xi \end{bmatrix}, \begin{bmatrix} \xi \\ \xi \end{bmatrix} \right), \tag{3}$$

for $\xi > 0$. A different choice of sampling distribution, e.g. a Gaussian distribution, might be more reasonable.

1.2 Interaction forces between actor and actor or actor and obstacle

The main idea behind the actor-based crowd model is that all actors, at all time steps, attempt to alter their velocity in such a way that they avoid collisions with obstacles or with other actors.

Let the relative position vector and relative velocity vector between two actors be $\mathbf{p}_{ij} := \mathbf{p}_i - \mathbf{p}_j$ and $\mathbf{v}_{ij} := \mathbf{v}_i - \mathbf{v}_j$. If the two actors were to continue forward with the same velocity vector for all future time steps, they would collide in time $t > 0$ if

$$||\mathbf{x}_{ij} + \mathbf{v}_{ij} t||^2 = 4r^2 \tag{4}$$

has a solution for $t \in \mathbb{R}_{>0}$. Time to collision, τ, is thus

$$\tau = \frac{-\mathbf{v}_{ij} \cdot \mathbf{x}_{ij} \pm \sqrt{(\mathbf{v}_{ij} \cdot \mathbf{x}_{ij})^2 - ||\mathbf{v}_{ij}||^2(||\mathbf{x}_{ij}||^2 - r^2)}}{||\mathbf{v}_{ij}||^2}, \tag{5}$$

where the partial derivative of time to collision with respect to relative position is

$$\frac{\partial \tau}{\partial \mathbf{x}_{ij}} = \frac{-1}{||\mathbf{v}_{ij}||^2}\left(\mathbf{v}_{ij} + \frac{||\mathbf{v}_{ij}||^2\mathbf{x}_{ij} - (\mathbf{v}_{ij} \cdot \mathbf{x}_{ij})\mathbf{v}_{ij}}{\sqrt{(\mathbf{v}_{ij} \cdot \mathbf{x}_{ij})^2 - ||\mathbf{v}_{ij}||^2(||\mathbf{x}_{ij}||^2 - r^2)}}\right). \tag{6}$$

Karamouzas et al. (2014) argues that the distribution of distances between two actors can be accurately parameterized by the time to a potential collision between the actors, and shows that the pairwise distribution function between two actors can be described by an interaction energy $E(\tau) \propto 1/\tau^2$ in situations where the average density of actors in a region is approximately constant over time. Combined with an observed truncation of the interaction energy for sufficiently small τ, Karamouzas et al. (2014) postulates that

$$E(\tau) = \frac{k}{\tau^2} e^{-\tau/\tau_0}, \tag{7}$$

such that a natural formulation of the interaction force between two prospective colliding actors is

$$\mathbf{F}_{ij} = -\frac{\partial}{\partial \mathbf{x}_{ij}} E(\tau) = \frac{k e^{-\tau/\tau_0}}{\tau^2}\left(\frac{2}{\tau} + \frac{1}{\tau_0}\right)\frac{\partial \tau}{\partial \mathbf{x}_{ij}}. \tag{8}$$

Regarding an actor's interaction force vis-à-vis wall-like obstacles, we assume that a potential collision point on a wall exposes an actor to an interaction force equivalent to that between two prospective colliding actors. For simplicity we only study straight walls, and represent each wall as a two-dimensional capsule with radius r.

Hence, let a straight wall extend between the endpoints $(\mathbf{p}_0^w, \mathbf{p}_1^w)$, where $\mathbf{p}^w := \mathbf{p}_1^w - \mathbf{p}_0^w = (p_0^w, p_1^w)^\intercal$. Actor i's centre of mass collides with the line extended through the wall's endpoints at time t if $(\mathbf{p}_i + \mathbf{v}_i t - \mathbf{p}_\mathbf{0}^\mathbf{w}) \times \mathbf{p}^w = 0$. Time to collision between the actor's centre of mass and the extended line is thus

$$\tau = \frac{\mathbf{p}^w \times (\mathbf{p}_i - \mathbf{p}_0^w)}{\mathbf{v}_i \times \mathbf{p}^w}. \tag{9}$$

A sufficient condition for the collision point between said actor and the extended line to lie between the wall's two endpoints, is that the scalar projection of the vector $(\mathbf{p}_i + \mathbf{v}_i\tau - \mathbf{p}_0^w)$ on the extended line must be contained in $[0, 1]$,

$$0 \leq \frac{(\mathbf{p}_i + \mathbf{v}_i\tau - \mathbf{p}_0^w) \cdot \mathbf{p}^w}{||\mathbf{p}^w||^2} \leq 1. \tag{10}$$

Let relative position between the actor's centre of mass and the collision point on the wall be $\mathbf{x}_{i*} := \mathbf{p}_i - \mathbf{p}^{w*}$, for collision point $\mathbf{p}^{w*} \in [\mathbf{p}_0^w, \mathbf{p}_1^w]$. If we assume that the actor maintains a constant velocity until the collision occurs, it follows from (9) that

$$\frac{\partial \tau}{\partial \mathbf{x}_{i*}} = \frac{1}{\mathbf{v}_i \times \mathbf{p}^w}(-p_1^w, p_0^w)^\intercal. \tag{11}$$

When checking if a collision occurs between an actor and the wall capsule's lower (or equivalently upper) semicircle, let the relative position between said actor and said wall capsule's lower semicircle be $\mathbf{x}_{i*} := \mathbf{p}_0^w - \mathbf{p}_i$. A collision occurs in time τ if $||\mathbf{x}_{i*} + \mathbf{v}_i \tau||^2 = r^2$ has a solution for $\tau \in \mathbb{R}_{>0}$, in which time to collision is

$$\tau = \frac{-\mathbf{v}_i \cdot \mathbf{x}_{i*} \pm \sqrt{(\mathbf{v}_i \cdot \mathbf{x}_{i*})^2 - ||\mathbf{v}_i||^2(||\mathbf{x}_{i*}||^2 - r^2)}}{||\mathbf{v}_i||^2}, \tag{12}$$

and

$$\frac{\partial \tau}{\partial \mathbf{x}_{i*}} = \frac{-1}{||\mathbf{v}_i||^2}\left(\mathbf{v}_i + \frac{||\mathbf{v}_i||^2 \mathbf{x}_{i*} - (\mathbf{v}_i \cdot \mathbf{x}_{i*})\mathbf{v}_i}{\sqrt{(\mathbf{v}_i \cdot \mathbf{x}_{i*})^2 - ||\mathbf{v}_i||^2(||\mathbf{x}_{i*}||^2 - r^2)}}\right). \tag{13}$$

1.3 Coffee stands

An important dynamic on Stripa, NTNU, is the presence of *coffee stands* which, when attempting to attract students, hand out free coffee. Some actors want coffee, others do not. If, however, many want coffee from the same stand, the queue grows, and some coffee thirsty actors might refrain from satisfying their coffee needs. For simplicity we model this psychological trait by a simple preference function

$$U(n_c) = \begin{cases} \text{actor is queuing up for coffee,} & n_c \leq C, \\ \text{actor is not queuing up for coffee,} & \text{otherwise,} \end{cases} \tag{14}$$

where n_c is the number of actors already in queue at the given stand, and $C > 0$ is a tuning parameter. A generalisation would allow for individual preference functions, e.g. by letting each actor have its own tuning parameter $C_i \sim N(\mu_C, \sigma_C^2)$.

1.4 Emitting

To dynamically create new actors entering the domain through different doors, define a set of emitters. In each time step emitter number i, E_i, creates a given number of actors as drawn from a Poisson process with rate $\lambda_i > 0$, $|E_i^{(t)}| \sim \text{Pois}(\lambda_i)$.

1.5 Emergency situation

When modelling an emergency situation, we assume that all actors change their destinations to one of several possible emergency exits $\{\mathbf{g}_1, \ldots, \mathbf{g}_m\}$. Assume that most actors will approach the closest emergency exit, but if two exits are sufficiently close and sufficiently many more actors are using one exit rather than the other, some actors might choose to approach the less visited exit – even if that exit is further away.

Hence, let the neighbourhood of exit k be $\mathcal{N}(\mathbf{g}_k)$, and let the number of actors within the neighbourhood of exit k be given by $N(\mathbf{g}_k) = \sum_{j=1}^{n} I(\mathbf{p}_j \in \mathcal{N}(\mathbf{g}_k))$. The emergency exit which actor i chooses to use, $\mathbf{g}^{(i)}$, is then modelled by the optimisation function

$$\mathbf{g}^{(i)} = \begin{cases} \min_{\mathbf{g}_k} ||\mathbf{g}_k - \mathbf{p}_i||_2, & \text{if } \min_{\mathbf{g}_k} ||\mathbf{g}_k - \mathbf{p}_i||_2 < \delta, \\ \min_{\mathbf{g}_k} \left(||\mathbf{g}_k - \mathbf{p}_i||_2^3 + \alpha \cdot N(\mathbf{g}_i)^2\right), & \text{otherwise,} \end{cases} \tag{15}$$

for a cut-off variable $\delta > 0$, which states that if an actor is sufficiently close to an exit door, the actor will always choose this exit. As discussed above, the second case defines a loss function which increases for larger distance between actor and emergency exit, as well as increasing for a larger actor-density in the emergency exit's neighbourhood.

References

Ioannis Karamouzas, Brian Skinner, and Stephen J Guy. Universal power law governing pedestrian interactions. *Physical review letters*, 113(23):238701, 2014.