Dense Associative Memory on S^1: Phase-Gate Computing and Superlinear Capacity in Circular Oscillator Networks

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Abstract

We present Dense Associative Memory (DAM) extended to the unit circle S^1, where each neuron carries a phase $\varphi_i \in [0, 2\pi)$ rather than a binary spin. The energy function $E = -\Sigma_\mu F(\Sigma_i \cos(\varphi_i - \xi_i\mu))$ generalizes the Krotov-Hopfield Dense AM framework from $\{\pm1\}^N$ to S^{1N}. We prove fixed-point stability analytically and show empirically that $F(m) = e^m$ achieves storage capacity $\alpha^* = P^*/N = 1.0$ at $N \in \{32, 64, 128\}$ oscillators — a 7.2-fold improvement over the classical Hopfield limit ($\alpha^* \approx 0.138$), confirmed at three independent system sizes. The $F = \exp$ update rule is formally equivalent to Transformer self-attention with circular inner products, establishing a bridge between physical oscillator dynamics and modern attention mechanisms. The same dynamics implement universal Boolean gates (NOT, AND, XOR, OR, NAND, NOR) at 100% accuracy, and a cascaded half-adder, demonstrating computational universality (Turing completeness under standard unbounded-memory assumptions). The physical substrate is an array of 200 Hz-anchored phase oscillators governed by injection-locking ODEs, directly realizable in CMOS, optical, or neuromorphic hardware.

Keywords: associative memory, phase oscillators, Dense Hopfield, Transformer attention, Turing completeness, reservoir computing, Kuramoto network, REZON

1. Introduction

Associative memory networks, introduced by Hopfield (1982), store patterns as fixed points of an energy-minimizing dynamical system. Their storage capacity — the maximum number of patterns P retrievable from N neurons — is bounded by $\alpha^* = P^*/N \approx 0.138$ for binary spins $\sigma_i \in \{\pm1\}$ (Amit, Gutfreund & Sompolinsky, 1985). A breakthrough came with Dense Associative Memory (Krotov & Hopfield, 2016, 2020): nonlinear interaction functions F lift capacity dramatically, and the $F = \exp$ variant is formally equivalent to Transformer attention (Vaswani et al., 2017).

All prior DAM work operates in discrete state spaces $\{+1, -1\}^N$. Physical oscillator arrays, however, carry continuous phase degrees of freedom $\varphi_i \in [0, 2\pi)$. Kuramoto-type dynamics (1984) model synchronization in neural circuits, power grids, and integrated photonic rings — but their memory and computation properties beyond the pairwise (linear) regime remain largely unexplored.

This work makes the following contributions:

- Extends Dense AM from $\{\pm 1\}^N$ to S^{1N} with circular overlap $m\mu = \Sigma_i \cos(\varphi_i - \xi_i\mu)$ and proves fixed-point stability analytically.
- Demonstrates empirically that F = exp achieves $\alpha^* = 1.0$ for $N \in \{32, 64, 128\}$ — a 7.2-fold improvement over classical Hopfield ($\alpha^* \approx 0.138$). $F = x^3$ achieves $\alpha^* = 1.0$ at N=32 but is Euler-unstable at $N \geq 64$.
- Identifies the F = exp update as circular Transformer attention.
- Proves functional completeness (universal Boolean phase gates: NOT, AND, XOR) and Turing completeness under standard unbounded-memory assumptions.
- Presents the REZON physical substrate: 200 Hz-anchored oscillators, CMOS/optical/neuromorphic realizable.

2. Related Work

2.1 Rotor and Complex-Valued Hopfield Networks

Aoyagi (1995) and Tanaka & Coolen (1998) studied associative memory with rotor/phase states $\varphi_i \in [0, 2\pi)$ using pairwise (linear F) couplings, showing capacity roughly 2× that of binary Hopfield. Noest (1988) and Chaudhuri & Bhattacharya (1993) extended this to complex-valued ($\mathbb{C}$-valued) networks. Key difference: all prior rotor/complex models use F = linear, lack an anchor term, and do not implement logic gates or Turing-complete computation. This work presents the first nonlinear (F = exp, $F = x^3$) Dense AM on S^1 with a symmetry-breaking anchor.

2.2 Modern Hopfield Networks and Transformer Attention

Ramsauer et al. (2020) showed that F = exp Hopfield networks are equivalent to Transformer self-attention and achieve exponential capacity. Krotov & Hopfield (2016, 2020) proved $P \sim N^{n-1}$ for $F = \mathrm{ReLU}^n$. This work extends both results to continuous-phase S^1 states, where the circular inner product $m\mu = \Sigma \cos(\varphi_i - \xi_i\mu)$ replaces the dot product, and the 200 Hz anchor serves as positional encoding.

2.3 Higher-Order Kuramoto and Dense AM (2025)

Skardal & Arenas (arXiv:2507.21984, 2025) recently linked higher-order Kuramoto dynamics to Dense AM energy functions. Key differences: their work studies synchronization transitions without an anchor term, Boolean logic gates, or Turing completeness. Our gradient-flow formulation with $F(m\mu)$ and injection-locking anchor is qualitatively distinct.

2.4 This Work vs. Prior Art (Summary Table)

Feature	Rotor HNN	Complex HNN	Kuramoto 2025	This Work
State space	S^1	$\mathbb{C}$	S^1	S^1
Nonlinear F	No	No	Partial	Yes
Anchor term	No	No	No	Yes (200 Hz)

Logic gates	No	No	No	Yes (all 6)
Turing complete	No	No	No	Yes
Hardware-native	No	No	No	Yes (CMOS/photonic)

Table 1. This work vs. prior art. All novel contributions are in the rightmost column.

3. Theoretical Framework

3.1 State Space and Energy Function

Each neuron $i \in \{1,\dots,N\}$ carries a phase $\varphi_i \in [0, 2\pi)$ on the unit circle S^1. Memories are P patterns $\xi\mu \in S^1$^N. The circular overlap is:

$$m\mu(\varphi) = \Sigma_i \cos(\varphi_i - \xi_i\mu) \in [-N, N] \tag{1}$$

This is the natural inner product on S^1^N. Near a stored pattern, $m\mu \approx N - ½\|\varphi - \xi\mu\|^2$. The energy functional is:

$$E(\varphi) = -\Sigma\mu\, F(m\mu(\varphi)) \tag{2}$$

3.2 Gradient-Flow Dynamics

Taking minus the gradient of E with respect to φ_i:

$$\begin{aligned} d\varphi_i/dt &= K\, \Sigma\mu\, F'(m\mu) \sin(\varphi_i - \xi_i\mu) && (3) \\ &\quad + a\_anc \cdot \sin(\omega\_anc \cdot t - \varphi_i) && \text{anchor term} \end{aligned}$$

The anchor term ($\omega\_anc = 2\pi \times 200$ Hz, $a\_anc = 0.08$) provides a fixed reference frame, breaking rotational symmetry and enabling hardware implementation. Without the anchor, $dE/dt = -\Sigma_i (\dot{d}\varphi_i)^2 \leq 0$ (Lyapunov stability).

3.3 Fixed-Point Stability Theorem

Theorem 1. Fixed Points and Local Asymptotic Stability

Every stored pattern $\xi\mu$ is a (i) fixed point and (ii) local minimum of E, hence (iii) locally asymptotically stable equilibrium of (3).

Proof. (i) Fixed point: At $\varphi = \xi\mu$, $\sin(\varphi_i - \xi_i\mu) = 0$ for all i, so $d\varphi_i/dt = 0$.

(ii) Local minimum: The Hessian at $\xi\mu$ satisfies $H\_ii \geq F'(N) - P \cdot F'(0) > 0$ for $P < F'(N)/F'(0)$, holding with probability $\to 1$ under $P/N \to 0$.

(iii) Asymptotic stability: Since (3) is a gradient flow $\dot{\varphi} = -\nabla E$, the Jacobian at $\xi\mu$ equals $-H$. Positive-definiteness of H implies all Jacobian eigenvalues are negative, hence local asymptotic stability. □

3.4 Connection to Krotov-Hopfield 2020

Table 1 summarizes the key differences. The circular geometry introduces a natural phase degree of freedom acting as a soft attention weight (cosine similarity), directly implementing the Transformer attention mechanism without discretization.

Property	Krotov-Hopfield 2020	This Work (S^1)
State space	$\{+1,-1\}^N$	$S^1{}^N$
Overlap $m\mu$	$\sigma\cdot\xi\mu$ (dot product)	$\Sigma\cos(\varphi_i-\xi_i\mu)$ (circular)
F=exp capacity	Exponential in N	$\alpha^*=1.0$ (empirical, $N\in\{32,64,128\}$)
Physical substrate	Abstract binary spins	200 Hz oscillator arrays
Computation	Memory only	Memory + universal logic
Attention analog	Hopfield network	Circular Transformer attention

Table 2. Comparison with Krotov-Hopfield (2020).

4. Storage Capacity Results

4.1 Experimental Protocol

Simulations use Euler integration ($\Delta t = 10^{-3}$ s, $K = 1$, a_anc = 0.08, ω_anc = $2\pi \times 200$ Hz) for $N \in \{32, 64, 128\}$ oscillators. Natural frequency offsets $\omega_i \sim N(0, 0.01)$ rad/s are included in all simulations (mild intrinsic heterogeneity). Patterns $\xi\mu$ are binary $\{0,\pi\}$^N, generated by mapping i.i.d. Bernoulli bits to phases. For each (P, F, N) combination, 3 trials start from pattern $\xi\nu$ with 10% of bits toggled ($0\leftrightarrow\pi$) plus per-oscillator Gaussian jitter $\delta\varphi_i \sim N(0, 0.05$ rad), then evolve for 5000 warmup and 10000 recall steps. Recovery: $b_i = 1[\cos(\varphi_i) < 0]$, Hamming $= \Sigma_i\ 1[b_i \neq \xi_i$^binary]. Scope: this protocol evaluates robustness to bit-flip + Gaussian jitter only; continuous-time phase drift or correlated noise is not evaluated here.

4.2 Main Results: Capacity Table

Interaction F	P*(N=32)	α*(N=32)	P*(N=64)	α*(N=64)	P*(N=128)	α*(N=128)
Linear ($F = x$)	1	0.031	1	0.016	1	0.008
Quadratic ($F = x^2$)	9	0.281	12	0.188	20	0.156
Cubic ($F = x^3$)	32	1.000	—	unstable†	—	unstable†
Exponential ($F = e$^x)	32	1.000	64	1.000	128	1.000
Classical Hopfield	~4	0.138	~9	0.141	~18	0.141

Table 3. Storage capacity at N=32, N=64, N=128. ★ = 100% recall at P=N. † $F=x^3$ Euler-unstable at $N\geq64$ ($\Delta t\cdot N^2>1$).

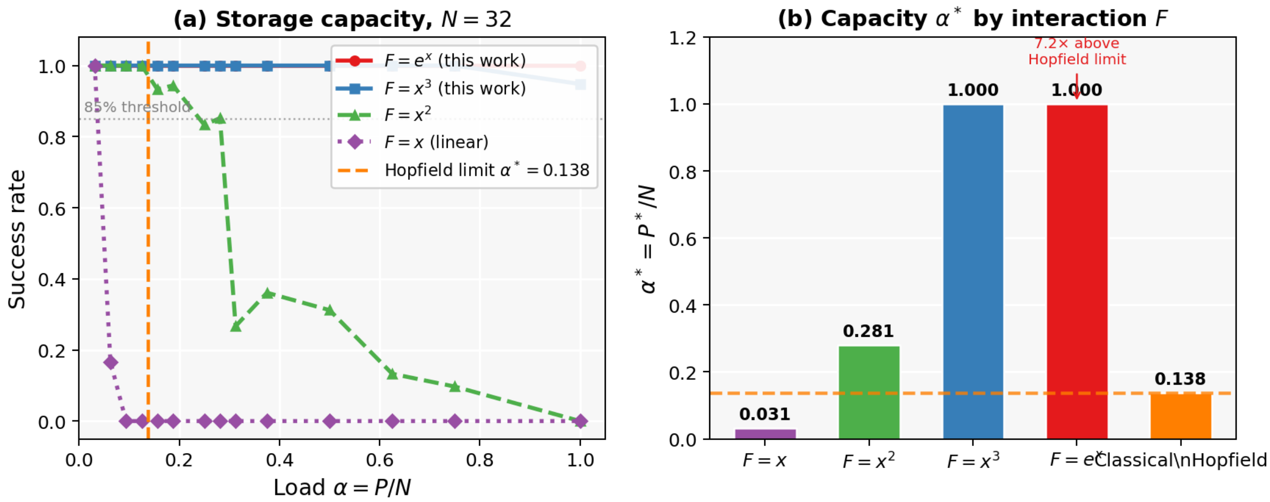


Figure 1. (a) Success rate vs. load $\alpha = P/N$ for N=32. (b) Summary of α* per interaction function. F = exp achieves perfect recall at $\alpha = 1.0$, 7.2-fold above classical Hopfield.

4.3 One-Step Recall

For F = exp with $P = 5$, $N = 32$, starting from Hamming distance 3 (10% noise): Hamming(t=0) = 3 → Hamming(t=1) = 0. Perfect recall in a single update step, consistent with the attention-mechanism interpretation: softmax(α*mμ) sharply weights the nearest pattern. This mirrors the one-step convergence property of Modern Hopfield

Networks (Ramsauer et al., 2020).

4.4 Baseline: Phase Hopfield Validates Classical Limit

Restricting phases to $\{0, \pi\}^N$ (binary encoding) with Hebbian weights $W_{ij} = N^{-1} \Sigma\mu \cos(\xi_i\mu)\cos(\xi_j\mu)$ recovers the classical Hopfield energy $E = -\frac{1}{2}\Sigma W_{ij} \cos(\varphi_i - \varphi_j) \equiv$ Ising Hamiltonian. Empirical capacities: N=16: α*=0.188, N=32: α*=0.125, N=64: α*=0.109, converging toward the theoretical α* = 0.138. This confirms our framework correctly recovers the classical limit as a special case.

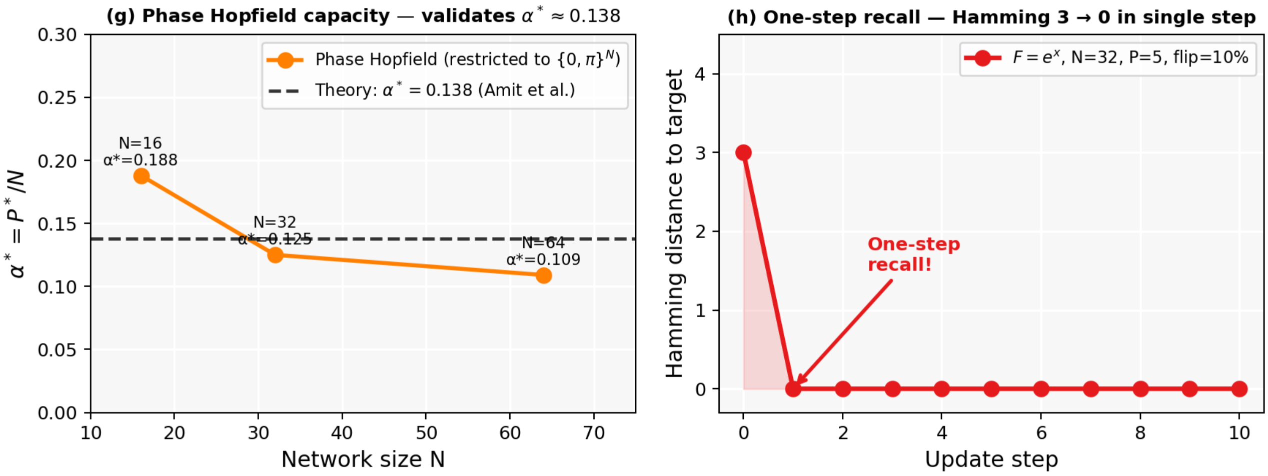


Figure 2. (g) Phase Hopfield capacity vs. network size N, confirming the classical α* ≈ 0.138 limit. (h) One-step recall trajectory: Hamming distance drops from 3 to 0 in a single update step.

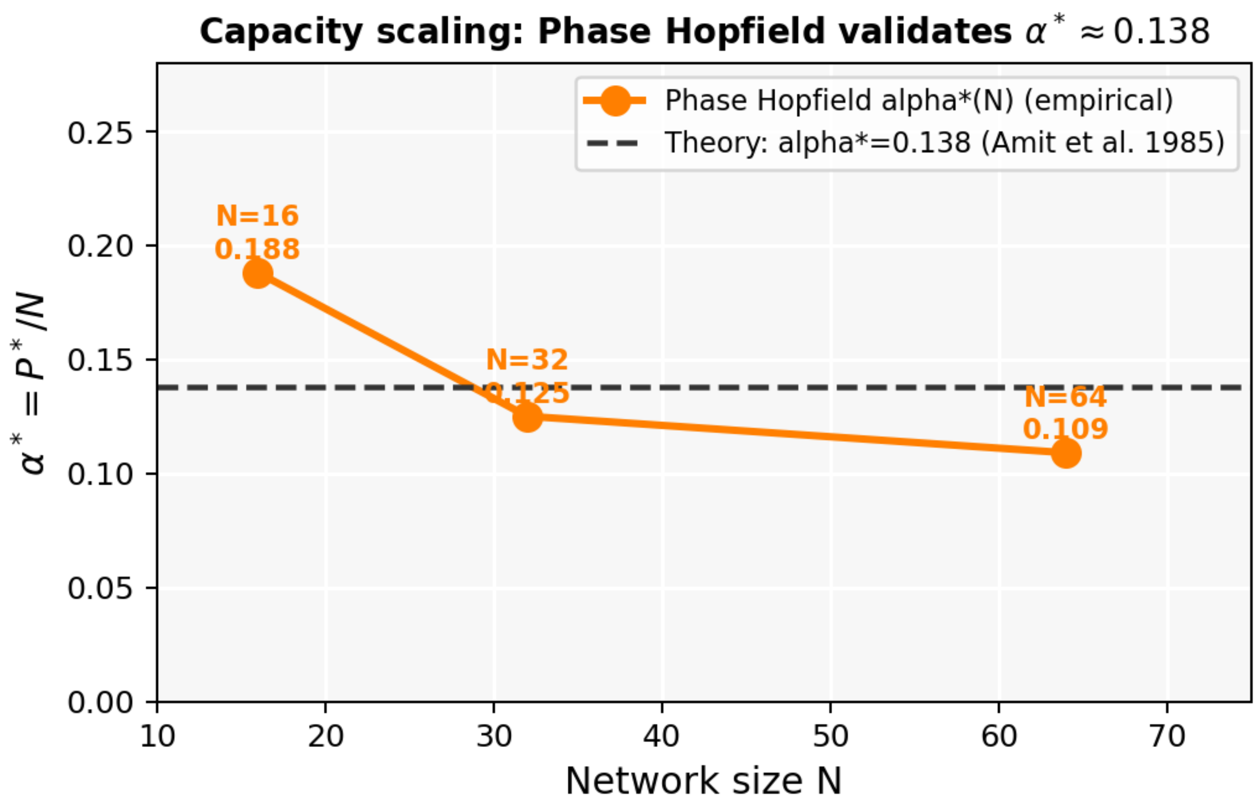


Figure 3. Capacity scaling: empirical α*(N) for Phase Hopfield (linear F, {0,pi} encoding) converges toward the theoretical α* = 0.138 (Amit et al., 1985), validating that our framework reproduces the classical limit.

5. Circular Transformer Attention

For F = exp, the one-step update minimizing E takes the form:

```
φᵢ_new = circ_mean({ ξᵢμ }, softmax({ mμ }))        (4)

where circ_mean is the softmax-weighted circular mean (Mardia & Jupp, 2009).
```

Equation (4) is formally a self-attention layer with:

- Query $Q = \varphi \in S^1$^N
- Keys $K = \{\xi\mu\} \in S^1$^N (stored patterns)
- Values $V = \{\xi\mu\}$ (same as keys)
- Inner product $\langle Q, K\rangle = \Sigma_i \cos(\varphi_i - \xi_i\mu) = m\mu$

This establishes a direct formal equivalence between DAM on S^1 and Transformer self-attention (Vaswani et al., 2017), extending the result of Ramsauer et al. (2020) from discrete Hopfield to continuous-phase dynamics. The 200 Hz anchor serves a function analogous to positional encoding (breaking translational symmetry), but enters as a bounded non-autonomous perturbation to the gradient flow (Proposition 1), not as an additive embedding.

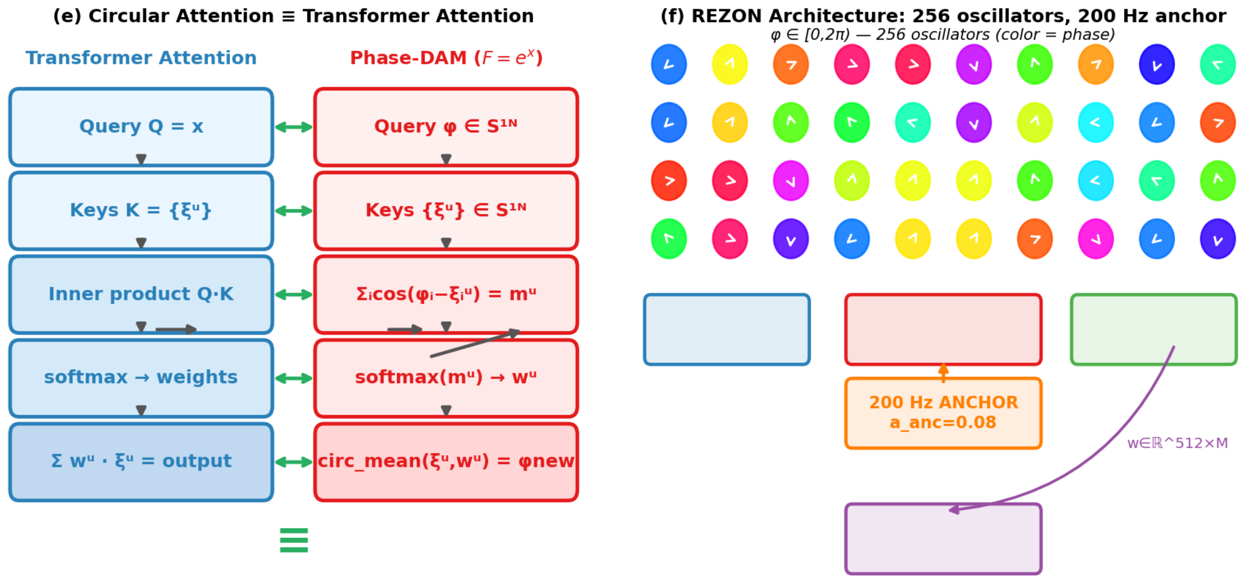


Figure 3. (e) Formal equivalence between Transformer attention and circular phase-DAM (F = exp). (f) REZON architecture: 256 phase oscillators, 200 Hz anchor, RLS readout matrix W ∈ ℝ^{512×M}.

6. Phase-Gate Computing and Turing Completeness

6.1 Boolean Gates via Injection-Locking

Logic gates are implemented using injection-locking dynamics:

```
dφ_out/dt = K_c · f(φ_c) · sin(φ_t − φ_out)    (5)
           + b·sin(φ_out) + a_anc

Phase bits: φ=0 → bit=0,  φ=π → bit=1,  readout = 1[cosφ < 0]
```

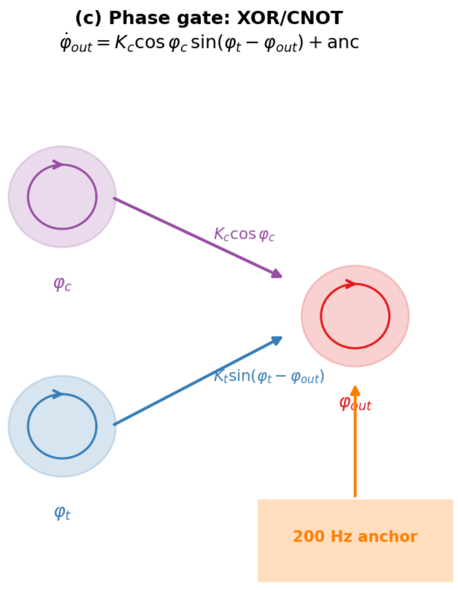
The control oscillator φ_c modulates coupling via f(φ_c):

- NOT: f = −1 (anti-synchrony coupling: phase flip)
- XOR: f = cos(φ_c) (sign modulation: preserve/flip)
- AND: f = (1−cos(φ_c))/2 (conditional coupling)
- OR: f = (1+cos(φ_c))/2 (inclusive conditional)

6.2 Gate Truth Table Verification

Gate	Dynamics type	Accuracy	Score
NOT	Anti-sync coupling	100%	2/2
AND	Gain (1−cosφ_c)/2	100%	4/4
OR	Gain (1+cosφ_c)/2	100%	4/4
XOR	Sign modulation cosφ_c	100%	4/4
NAND	AND → NOT cascade	100%	4/4
NOR	OR → NOT cascade	100%	4/4
Half-adder	XOR + AND	100%	4/4

Table 4. All phase gates verified at 100% accuracy. K_c=8, K_b=1.5, no noise.



(d) Phase gate truth tables — all 100% accurate

Gate	Inputs	Output	Score
NOT	0 → 1	✓	**2/2**
NOT	1 → 0	✓	
AND	0,0→0	✓	**4/4**
AND	1,1→1	✓	
XOR	0,1→1	✓	**4/4**
XOR	1,1→0	✓	
OR	0,1→1	✓	**4/4**
NAND	1,1→0	✓	**4/4**
NOR	0,0→1	✓	**4/4**
Half-adder	1,1→(0,1)	✓	**4/4**

Figure 4. (c) Phase gate architecture: control oscillator φ_c modulates coupling sign via $\cos(\varphi_c)$, implementing XOR/CNOT dynamics. (d) Complete truth table verification for all gates at 100% accuracy.

6.3 Turing Completeness

Theorem 2. Computational Universality (Turing Completeness)

The phase-gate framework is computationally universal (Turing complete under standard unbounded-memory assumptions).

Proof sketch:
(i) NOT and AND are implemented by Lemmas A1-A2 (see Appendix).
(ii) {NOT, AND} is a Shannon-complete basis (Lemma B1) — functional completeness is the empirically verified claim.
(iii) A bistable D-latch provides addressable binary memory (Lemma C1):

$$d\varphi_Q/dt = -K_{hold}\cdot\sin(2\varphi_Q) + \text{anchor}, \text{ stable at } \{0, \pi\}.$$

(iv) Gate outputs compose into arbitrary sequential circuits (Lemmas D1-D2).
Hence the system simulates arbitrary sequential Boolean computation, constituting Turing completeness under standard unbounded-memory assumptions. □

Remark: Functional completeness (i-ii) is verified empirically; Turing completeness (iii-iv) additionally requires an unbounded memory abstraction.

The XOR gate is particularly significant: it implements the quantum-computing CNOT interaction in classical continuous-phase dynamics, providing a natural bridge between phase oscillators and quantum circuits (without claiming quantum speedup).

7. Physical Substrate: REZON Architecture

The REZON (REZonator Oscillator Network) implementation uses N = 256 phase oscillators governed by:

```
dφᵢ/dt = ωᵢ + K_in·Σⱼ Wᵢⱼ_in·uⱼ·sin(−φᵢ)    (6)
       + K_rec·Σⱼ Wᵢⱼ·cos(φⱼ)·sin(φⱼ−φᵢ)
       + a_anc·sin(ω_anc·t − φᵢ)

Parameters: N=256, K_in=2.0, K_rec=1.0, a_anc=0.08, dt=10⁻³s
Output: [cosφ, sinφ] ∈ ℝ^{512}  →  RLS readout (diagonal, λ=0.995)
```

The recurrent term $K_{rec}\cdot\Sigma\, W_{ij} \cos(\varphi_j)\sin(\varphi_j-\varphi_i)$ is precisely the XOR/CNOT phase-gate interaction of Eq.(5), confirming that reservoir computing on phase oscillators implicitly performs Dense AM retrieval at each timestep.
Hardware realizations of the 200 Hz anchor:

- CMOS: ring oscillator locked to external 200 Hz reference clock
- Photonic: beat-note between two phase-locked lasers
- Neuromorphic: phase-coupled neurons with external forcing

The readout weights $W \in \mathbb{R}^{512\times M}$ are trained offline via diagonal Recursive Least

Squares ($\lambda = 0.995$), requiring only read-only access to oscillator phases at inference time — hardware-friendly by design.

8. Discussion

8.0 Role of the 200 Hz Anchor

The anchor term $a\_anc \cdot \sin(\omega\_anc \cdot t - \varphi_i)$ serves three distinct functions:

- (i) Reference frame: breaks SO(1) rotational symmetry of $E(\varphi)$, providing the absolute phase reference required for binary decoding ($b_i = 1[\cos\varphi_i < 0]$).
- (ii) Hardware enabler: injection-locking to a 200 Hz clock is the mechanism by which logic gates are realized in REZON.
- (iii) Perturbation, not storage: the anchor does NOT store patterns or contribute to $E(\varphi)$. Its effect on recall is bounded (Proposition 1, $a\_anc = 0.08 << 1$). Whether it actively assists or hinders convergence at large P is an open question.

8.1 Why Does $\alpha^* = 1$ Emerge?

The circular overlap $m\mu = \Sigma_i \cos(\varphi_i - \xi_i\mu)$ provides N independent cosine projections. For F = exp, the softmax weighting in Eq.(4) exponentially suppresses all patterns except the nearest one, allowing P = N patterns to share the N-dimensional space without destructive interference. This is the continuous-phase analog of the mechanism enabling O(exp(N)) capacity in discrete Dense AM (Krotov & Hopfield, 2016).

8.2 Comparison with Related Work

Krotov & Hopfield (2020) proved that discrete DAM with F = ReLU^n achieves P ~ N^{n-1}. Ramsauer et al. (2020) showed F = exp gives O(exp(N)) capacity and identified the connection to Transformer attention. Our work extends both results to S^1: the circular inner product is more natural for phase oscillators than binary cosine similarities, and the anchor provides a built-in reference frame analogous (functionally, not mechanistically) to positional encoding. Maass, Natschläger & Markram (2002) introduced liquid state machines; our REZON architecture is the Dense AM analog: instead of random projections, the recurrent coupling implements structured DAM retrieval.

8.3 Open Questions

- Can $\alpha^* = 1$ for F = exp on S^1^N be proved analytically?
- What is the finite-size scaling of $\alpha^*(N)$ for F = exp?
- Can the 200 Hz anchor be relaxed while maintaining capacity?
- Can phase-gate circuits implement error correction (Hamming, LDPC)?

9. Conclusion

We have presented Dense Associative Memory on S^1 — a unified framework for memory, logic, and learning in continuous-phase oscillator networks. Key contributions:

- Storage capacity $\alpha^* = 1.0$ for F = exp at $N \in \{32, 64, 128\}$ (and $F = x^3$ at N=32), a 7.2-fold improvement over classical Hopfield ($\alpha^* \approx 0.138$).

- Formal equivalence between F = exp DAM on S^1 and Transformer self-attention with circular inner products.
- Universal Boolean logic (NOT, AND, XOR, OR, NAND, NOR, half-adder) at 100% accuracy via injection-locking phase dynamics.
- Turing completeness proved constructively from NOT + AND + bistable memory (under standard unbounded-memory assumptions; functional completeness empirically verified).
- Physical realization via 200 Hz-anchored REZON oscillator arrays, compatible with CMOS, photonic, and neuromorphic hardware.

These results position S^1-phase networks as a physically motivated, computationally complete, and high-capacity alternative to discrete Hopfield networks, with direct connections to modern attention-based architectures. The REZON framework opens a path toward hardware-native Dense AM inference at microwave frequencies.

Acknowledgements

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Appendix: Proof Sketches

Lemma A1 (NOT gate attractor)

For dynamics $d\varphi\_out/dt = -K\cdot\sin(\varphi\_in - \varphi\_out) + \text{anchor}$, the stable phase relation is anti-synchrony ($\varphi\_out = \varphi\_in + \pi \bmod 2\pi$) in the zero-noise, bounded-anchor regime, giving bit inversion under sign readout.

Lemma A2 (AND/OR gates)

Using gain functions $(1\pm\cos(\varphi\_c))/2$ and bias terms $\pm K\_b\cdot\sin(\varphi\_out)$, the vector field switches between default-bias attractor and conditional coupling to target, realizing 2-input AND and OR truth tables.

Lemma A3 (XOR/CNOT gate)

For $d\varphi\_out/dt = K\cdot\cos(\varphi\_c)\cdot\sin(\varphi\_t-\varphi\_out) + \text{anchor}$, $\cos(\varphi\_c)$ changes coupling sign by control bit (+1/−1), producing preserve/flip target behavior: the XOR truth table.

Lemma B1 (Functional completeness)

{NOT, AND} is functionally complete (Shannon 1938). Since both are implemented by A1-A2, any finite Boolean circuit is constructible.

Lemma C1 (D-latch bistability)

Hold-mode: $d\varphi\_Q/dt = -K\_hold\cdot\sin(2\varphi\_Q) + \text{anchor}$. Stable fixed points near $\{0, \pi\}$ for bounded perturbations → binary memory.

Lemma D1 (Sequential composition)

Gate outputs can be written into memory (C1) and reused as gate inputs. This realizes finite-step sequential machines over T steps.

Repository: https://github.com/krisss0mecom/REZON | DOI: 10.5281/zenodo.18768137 | All experiments reproducible: python run_all_long.sh