COMP 3008 Homework #2

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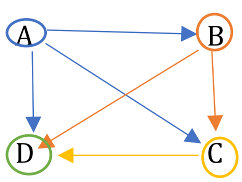
<u>Part 1</u>

1.Negate the statement $\exists x \in S, \forall y \in T, (A(x) \rightarrow B(x,y))$.

Negation: $\forall x \in S, \exists y \in T\ (A(x) \land \neg B(x,y))$.

"For all elements x in set S, there exists some y in set T such that A(x) and not B(x,y)"

2. Later in the course we'll learn about graphs and directed graphs. Here is a preview that will give you an opportunity to practice with universal and existential quantifiers. A directed graph can be represented as a set N of nodes and an open sentence $E(x,y)$ where x and y are in the set of nodes. The expression $E(x,y)$ is read "There is an edge from x to y." Determine the truth value of each of the following quantified expressions based on the graph with $N = \{a, b, c, d\}$ and $E(x,y)$ defined by setting the propositions $\forall x\ E(x,x)$, $E(a,b)$, $E(a,c)$, $E(a,d)$, $E(b,c)$, $E(b,d)$, $E(c,d)$ all to be true and by the requirement that $E(x,y)$ is false unless x=y or $E(x,y)$ is among the propositions set to be true above.



*Drawing completed with Jurgen, TA during office hours 1/22/21

(a). $\forall x \exists y E(x,y)$ **T**

(b). $\exists x \forall y E(x,y)$ **T**

(c). $\neg \exists x(\neg E(x,x))$ **T**

(d). $\forall x \forall y E(x,y)$ **F**

3. Prove or disprove that if you have an 8-liter container, a 5-liter container, a 3-liter container, a water supply, and a drain, then you can carry out a process that concludes with one container that has 4 liters of water, and no more, in it.

Step1: Fill the 5L container, transfer 3L from the 5L container to fill the 3L container

Step 2: Fill 8L container with the remaining 2L left in the 5L container

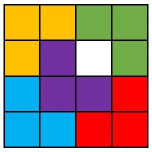
Step 3: Drain the 3L container

Step 4: Fill the 5 L container, transfer 3L from the 5L container to fill the 3L container

Step 5: Fill 8L container with the remaining 2L left in the 5L container. The 8L container now has exactly 4L of water in it.

4. Prove or disprove that a 4x4 board with any one square removed can be tiled with pieces consisting of three squares in an L-shape.

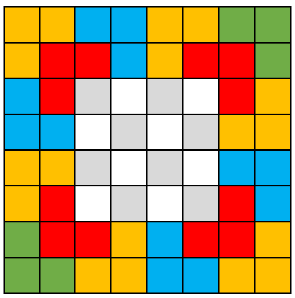
Board:



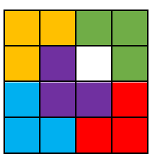
Yes, as shown above.

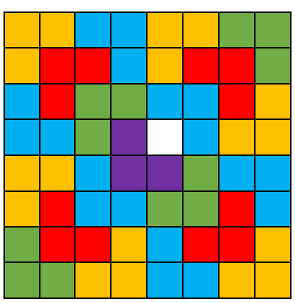
5. Prove or disprove that an 8x8 board with any one square removed can be tiled with pieces consisting of three squares in an L-shape.

Board:



Notice that the 8x8 board can be tiled above, with a 4x4 board remaining. We know that a 4x4 board can be tiled with L-shaped tiles:





Therefore, the 8x8 board with 1 tile removed can be tiled with L-shaped tiles, as shown above.

6. Determine if $(p \to (q \vee r)) \vee ((q \vee r) \to p)$ is a tautology, a contradiction, or neither.

p	q	r	$q \vee r$	$p \to (q \vee r)$	$((q \vee r) \to p)$	$(p \to (q \vee r)) \vee ((q \vee r) \to p)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	F	T	T	T

*Green shaded column shows that $(p \rightarrow (q \vee r)) \vee ((q \vee r) \rightarrow p)$ is a tautology.

7. Determine if $(\neg(p\rightarrow q) \wedge \neg(q\rightarrow p))$ is a tautology, a contradiction, or neither.

p	q	$p\rightarrow q$	$\neg(p\rightarrow q)$	$q\rightarrow p$	$\neg(q\rightarrow p)$	$(\neg(p\rightarrow q) \wedge \neg(q\rightarrow p))$
T	T	T	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	F	T	F	F

*Green shaded column shows that $(\neg(p\rightarrow q) \wedge \neg(q\rightarrow p))$ is a contradiction.

8. Use truth tables to show that $\neg p \Leftrightarrow q$ is logically equivalent to $(p\rightarrow \neg q) \wedge (\neg q\rightarrow p)$. When you're done, make a verbal argument using results we have already shown to explain why the two statement are logically equivalent.

p	q	$\neg p$	$\neg p \Leftrightarrow q$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

p	q	$\neg q$	$p\rightarrow \neg q$	$\neg q\rightarrow p$	$(p\rightarrow \neg q) \wedge (\neg q\rightarrow p)$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	F

* green shaded truth tables show that $\neg p \Leftrightarrow q$ is logically equivalent to $(p\rightarrow \neg q) \wedge (\neg q\rightarrow p)$. '

'Not p' if and only if q means that 'not p' and q have the same truth values if both 'not p' and q . The contrapositive for 'not p' then q is 'not q' then p. The contrapositive for 'not q' then p is p then 'not q'. Therefore, combining both of those contrapositive statements together results in both truth values for not p' if and only if q.

9. Prove that if n is even and m is odd, then $3nm + 5m$ is odd.

<u>Proof:</u>

Let n represent an even integer and m represent an odd integer.

By definition, there are integers x, y such that n=2x and m=2y+1.

Notice that:

$$3nm+5m= 3(2x)(2y+1) + 5(2y+1)$$

$$=12xy+6+10y+1$$

$$=12xy+10y+6+1$$

$$=2(6xy+5y+3) +1$$

Since $(6xy+5y+3) \in \mathbb{Z}$, $3nm + 5m$ is odd■

Part 2

1. Show that if n is an integer, then either n^2 or n^2 -1 is divisible by 4.

Proof by way of cases:

Let a be an even integer and b represent an odd integer.

By definition, there are integers x, y such that a= 2x and b= 2y+1.

By definition, given integers a, b with a ≠ 0, we say that a|b if there exists an integer k such that b=ak

a=2x

$a^2=(2x)^2$

$=4x^2$

Therefore, $4|x^2$

b=2y+1

$b^2=(2y+1)^2$

$=4y^2+4y+1$

$=4(y^2+y)+1$

Therefore, $4|(y^2+y)-1$

Because integers can either be even or odd, either n^2 or n^2 -1 is divisible by 4■

2. Prove that perfect squares are never congruent to 2 or 3 mod 4. What about mod 3? What are all the possibilities for the perfect squares mod 3? Prove it.

For mod 4, the possible remainders are 0,1,2,3:

0 mod 4= 0	5 mod 4= 1	10 mod 4= 2	15 mod 4= 3	20 mod 4= 0	25 mod 4= 1
1 mod 4= 1	6 mod 4= 2	11 mod 4= 3	16 mod 4= 0	21 mod 4= 1	
2 mod 4= 2	7 mod 4= 3	12 mod 4= 0	17 mod 4 = 1	22 mod 4 = 2	
3 mod 4= 3	8 mod 4= 0	13 mod 4= 1	18 mod 4= 2	23 mod 4= 3	
4 mod 4= 0	9 mod 4= 1	14 mod 4= 2	19 mod 4= 3	24 mod 4= 0	

*highlighted entries are perfect squares, which are only congruent to 0 or 1 mod 4.

Proof by cases:

Any integer n has remainder 0,1,2,3 when divided by 4.

By definition, the quotient remainder theorem states that given n, d with d>0, there exists unique integers q, r such that n=dq +r; 0≤r< d. r=n mod d

By definition, a perfect square is expressed as the product of two equal integers in the form n^2

Case 1: remainder 0, so n= 4q for some q $\in \mathbb{Z}$

So n^2= $4q^2$=2($2q^2$)

So n^2 mod 4 = 0

Case 2: remainder 1, so n= 4q+1 for some q $\in \mathbb{Z}$

So n^2= (4q+1)(4q+1)= $16q^2$+8q+1

=4($4q^2$+2q) +1

So n^2 mod 4= 1

Case 3: remainder 2, so n=4q+2 for some q $\in \mathbb{Z}$

So n^2= (4q+2)(4q+2)= $16q^2$+16q+4

= 4($4q^2$+4q+1)

So n^2 mod 4= 0

Case 4: remainder 3, so n= 4q+3 for some q $\in \mathbb{Z}$

So n^2=(4q+3)(4q+3)= $16q^2$+24q+9

=4($4q^2$+8q+2) + 1

So n^2 mod 4 = 1

After analyzing every case, n^2 mod 4 is always 0 or 1 ■

For mod 3, the possible remainders are 0,1,2:

0 mod 3= 0	5 mod 3= 2	10 mod 3= 1	15 mod 3= 0	20 mod 3= 2	25 mod 3= 1
1 mod 3= 1	6 mod 3= 0	11 mod 3= 2	16 mod 3= 1	21 mod 3= 0	
2 mod 3= 2	7 mod 3= 1	12 mod 3= 0	17 mod 3 = 2	22 mod 3 = 1	
3 mod 3= 0	8 mod 3= 2	13 mod 3= 1	18 mod 3= 0	23 mod 3= 2	
4 mod 3= 1	9 mod 3= 0	14 mod 3= 2	19 mod 3= 1	24 mod 3= 0	

*highlighted entries are perfect squares, which are only congruent to 0 or 1 mod 3.

Proof by cases:

Any integer n has remainder 0,1,2 when divided by 3.

By definition, the quotient remainder theorem states that given n, d with d>0, there exists unique integers q, r such that n=dq +r; 0≤r< d. r=n mod d

By definition, a perfect square is expressed as the product of two equal integers in the form n^2

Case 1: remainder 0, so n= 3q for some q $\in \mathbb{Z}$

So n^2= $9q^2$=3($3q^2$)

So n^2 mod 3 = 0

Case 2: remainder 1, so n= 4q+1 for some q ∈ ℤ

So n^2= (3q+1)(3q+1)= $9q^2$+6q+1

=3($3q^2$+2q) +1

So n^2 mod 3= 1

Case 3: remainder 2, so n=3q+2 for some q ∈ ℤ

So n^2= (3q+2)(3q+2)= $9q^2$+12q+4

= 3($3q^2$+4q+1)+1

So n^2 mod 3= 1

After analyzing every case, n^2 mod 3 is always 0 or 1 ■

3. Prove that an integer is odd if and only if it can be written as the sum of two consecutive integers.

Proof:

Let m + n = k such that k is odd. m,n, and k are ∈ ℤ.

n=m+1, because n is consecutive to m

m+n= m+m+1=k

so, 2m+1=k

2m+1 ∈ ℤ, therefore k is odd■

4. Use the gcd reduction theorem (the Euclidean algorithm) to find the greatest common divisor of 7854 and 4746.

gcd(7854, 4746)

7854= 4746(1) + 3108. So, gcd(7854, 4746)= gcd(4746, 1638)

4746= 1638(1) +1470. So gcd(4746, 1638)= gcd(1638, 1470)

1638= 1470(1) + 168. So gcd(1638, 1470)= gcd(1470, 168)

1470= 168(8) + 126. So gcd(1470, 168)= gcd(168, 126)

168= 126(1) + 42

126=42(3) +0. So, gcd(168, 126) = 42

So, by the Euclidean Algorithm, gcd (7854, 4746) = **42**.

5. Find [884^2 * $(-4940)^{20}$](mod 7).

Modular exponentiation (property found on Khan academy)

A^B mod C= ((A mod C$)^B$)mod C

= [(884mod 7$)^2$(mod7)*((-4940mod 7$)^{20}$(mod 7))](mod 7)

$=[\ 2^2(\text{mod } 7)*(5^{20}\ (\text{mod } 7))](\text{mod } 7)$

$=[\ 4*(5^{20})(\text{mod } 7))](\text{mod } 7)$

$=[\ 4*4](\text{mod } 7)$

$=16(\text{mod } 7)$

$=2$

6. Show that if a ≡ b(mod *n*) and c ≡ d(mod *n*), then a+c ≡ b+d (mod *n*) and ac ≡ bd(mod *n*).

<u>Proof:</u>

Suppose a ≡ b(mod *n*) and suppose c ≡ d(mod *n*). a, b,c,d are $\in \mathbb{Z}$.

By definition, this means n|(a-b) and n|(c-d).

By definition of divisibility, there is an integer q_1 for which a-b =n q_1 and there is an integer q_2 for which c-d=n q_2.

First, we want to show a+c ≡ b+d (mod *n*):

Rearranging, b= a+ nq_1 and d= c+ nq_2

$b+d= a+c+n\ q_1+ n\ q_2$

$=a+c+n(q_1+ q_2)$

Therefore, a+c= b+d(mod n).

Now we want to show that ac ≡ bd(mod *n*):

By definition, this means n|(a-b) and n|(c-d).

By definition of divisibility, there is an integer q_1 for which a-b =n q_1 and there is an integer q_2 for which c-d=n q_2.

Substituting, $ac=(b+n\ q_1)\ (d+ n\ q_2)$

Expanding, $=bd+bn\ q_2+d\ n\ q_1+n^2q_1q_2$

$=bd+n(b\ q_2+d\ q_1+n\ q_1q_2)$

Because $(b\ q_2+d\ q_1+n\ q_1q_2)$ is an integer, ac ≡ bd (mod *n*) ■

7. Let x $\in \mathbb{Z}$. Show that if x^2-6x+5 is even, then x is odd.

<u>Proof:</u>

Assume x^2-6x+5 is even. By the definition of even, $x^2-6x+5= 2k$ for some k$\in \mathbb{Z}$.

Rearranging, $x^2-6x= 2k-5$

$x^2-6x= 2(k-4) +1$

2(k-4)+1 is odd, because (k-4) $\in \mathbb{Z}$.

Case 1: let x be odd. By the definition of odd, x= 2j+1 for some j ∈ ℤ.

Substituting, $(2j+1)^2$-6(2j+1)= $4j^2$+4j+1-12j-6

Combining terms, =$4j^2$-8j+5

Factoring out a 2 =2($2j^2$-4j+2) +1

Therefore, x^2-6x

Therefore, x^2-6x= 2(k-4) +1 holds true because both sides of the statement are proven to be odd.

Case 2: let x be even. By the definition of even, x= 2m for some m∈ ℤ.

Substituting, $(2m)^2$-6m= $2m^2$-6m

Factoring out a 2 = 2(m^2-3m)

Therefore, x^2-6x= 2(k-4) +1 is false because if x were even, x^2-6x would also have to be even, which is not equal to the proven odd part of the statement from above.

Therefore, x must be odd if x^2-6x+5 is even. ■

8. Either prove or find a counter-example to the statement 3 | n if and only if 3 | n^2. (Here n is an integer).

Proof:

Since the statement is biconditional, we must prove the following two statements:

(i). If 3 | n, then 3| n^2

(ii). If 3| n^2, then 3 | n

For every n ∈ ℤ, n=3k or n=3k+1 or n=3k+2 (for some k ∈ ℤ)

Case 1:

n=3k

n^2= $9k^2$=$3(3k)^2$

Therefore, 3|n^2. Notice that this proves statement (i).

Case 2:

n=3k +1

n^2= (3k+1)(3k+1)= $9k^2$+6k+1= 3($3k^2$+2k) +1

In this case, $3 \nmid n^2$

Case 3:

n=3k+2

$n^2 = (3k+2)(3k+2) = 9k^2+12k+4 = 3(3k^2+4k+1) + 1$

In this case, $3 \nmid n^2$

Case 2 and 3 prove statement (ii), if $3 \nmid n$, then $3 \nmid n^2$. This is logically equivalent to if $3|n^2$, then $3|n$. Therefore, $3 \mid n$ if and only if $3 \mid n^2$ ■

9. Either prove or find a counter-example to the statement $m|n$ if and only if $m|n^2$. (Here n and m are integers).

<u>Counterexample:</u>

Since the statement is biconditional, we must prove the following two statements:

(i). If $m|n$ then $m|n^2$

(ii). If $m|n^2$ then $m|n$

Suppose m=4. For every $n \in \mathbb{Z}$, n=4k or n=4k+1 or n=4k+2 or n= 4k+3(for some $k \in \mathbb{Z}$)

If n= 4k+2 then

$n^2=(4k+2)(4k+2)= 16k^2+16k+4= 4(4k^2+4k+1)$,

So, for this case, $4k+2|n^2$, but $4 \nmid n$. This breaks this statement "If $m|n^2$ then $m|n$". Therefore, the biconditional statement cannot be true ■