COMP 3008 Homework #4 Krista Miller

2/9/2021

Part 1

1. Prove that there exist irrational numbers a and b so that a^b is rational. Hint: Use $a=\sqrt{3}$ and $b= \log_3 4$. You must show each of these is irrational by contradiction (one of these was part of last week's homework), and then show that a^b is rational.

   Proof(contradiction): By way of contradiction, assume that $a= \sqrt{3}$ is rational and $b= \log_3 4$ is also rational.

   Step 1: prove $a= \sqrt{3}$ is irrational:
   By definition of real numbers, x is called rational if there exists integers p and q such that $x=\frac{p}{q}$.
   So, we can choose integers p and q such that $\sqrt{3} = \frac{p}{q}$. We may assume that the fraction is in its simplest form, and that p and q are not both even. Squaring both sides, we see that $3= \frac{p^2}{q^2}$, so $3q^2=p^2$. This means that 3 divides p^2, which means that 3 also divides p. So, we have $p= 3k$ for some integer k and we can substitute 3k for p.

   We have $3q^2=(3k)^2$, so $3q^2= 9k^2$, and thus, $q^2=3k^2$. This means that 3 also divides q^2, which means that 3 also divides q.

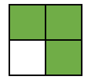
   So, 3 is a common factor for both p and q, which is a contradiction. Thus $\sqrt{3}$ must be irrational.

   Step 2: prove $b= \log_3 4$ is irrational:
   By way of contradiction, assume that $b= \log_3 4$ is also rational.
   Therefore, we can choose positive integers j and k (and k cannot equal zero) such that $\log_3 4= \frac{j}{k}$, which also means that $4= 3^{(j/k)}$, and that $3^j=4^k$. 3^j is odd, and 4^k is even, which is a contradiction. Thus, $\log_3 4$ is irrational.

   Step 3: show that a^b is rational:
   $a= \sqrt{3}$, is irrational by step 1 of this proof
   $b= \log_3 4$ is irrational by step 2 of this proof $b= \log 4/\log 3$
   $(\sqrt{3})^{\log 3(4)}$
   $=3^{1/2\log_3 4}$
   $=3^{\log_3 2}$ (Because 3^x and $\log_3 x$ are inverse)
   $=2$, which is rational. Therefore, a^b is rational. ■

2. Prove that a $2^n \times 2^n$ board with any single square removed can be tiled with L-shaped 3-unit tiles.
   Proof (by induction)
   Step 1 (base case): let n=1

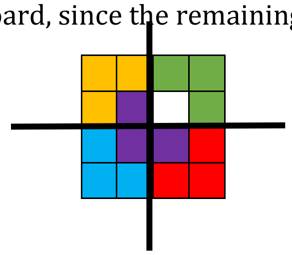
   $2^1 \times 2^1$ board has 4 units, which can be tiled with 1 L-shaped 3-unit tile, with one square removed as shown:



So, the formula holds for n=1.

Step 2 (inductive step):

Assume that for some integer k, a $2^k \times 2^k$ board can be tiled with L-shaped 3-unit tiles, with one square removed (inductive hypothesis).
Note that we want to show that tiling is also true for (k+1). In other words, a $2^{k+1} \times 2^{k+1}$ board can also be tiled.
Notice that we can split a checkerboard into units of $2^k \times 2^k$. If we let only one of the units have one square removed, by inductive hypothesis, we can tile any $2^{k+1} \times 2^{k+1}$ board, since the remaining L shape can also be covered with L-shaped tiles.



Step 3 (conclusion):

Therefore, we can tile any $2^{k+1} \times 2^{k+1}$ board ■

3. State the formula for the sum of a convergent infinite geometric series, and prove the formula is correct.
Proof:
The sum S of an infinite geometric series with -1<r<1 is given by the formula $S = \frac{a}{1-r}$, where a represents the first term in the series and r is the common ratio.
In other words, $S = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + \cancel{ar^n}$
Multiplied with r,

$$rS = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots \cancel{ar^n}$$

Subtract equation S with equation rS (all the terms cancel except a):

$$S - rS = a$$

Factor out an s: $S(1-r)=a$

Rewrite: $S = \frac{a}{1-r}$ ■

4. Use proof by induction to show that 6 is a factor of n^3-n whenever n is a non-negative integer. (Note that you have shown this same result by another method.)

Proof(by induction):
Step 1(base case): n= 0
0^3-0=0, 6 is a factor of 0 because 6|0.

Step 2(inductive):

Assume that 6 is a factor of n^3-n is true for some positive integer k (inductive hypothesis). Notice that we want to show that 6 is also a factor for (k+1).

$(k+1)^3-(k+1)= k^3+3k^2 +3k +\cancel{1}- k -\cancel{1}$

Rearranging, $= k^3-k +3k^2 +3k$

$=(k^3-k) + 3k(k+1)$

By inductive hypothesis, 6 divides (k^3-k). For 3k(k+1): if k is even, then the term (k+1) is odd. Conversely, if k is odd, then k+1 must be even. Because 2 divides k(k+1), 2 also divides 3k(k+1). Therefore, 6 divides 3k(k+1).

Step 3(conclusion):

We have shown that 6 divides the sum of (k^3-k) + 3k(k+1), which satisfies the case for k+1. Therefore, 6 is a factor of n^3-n, whenever n is a non-negative number ■

5. Consider 7-digit phone numbers. How many are there that do not start with a 0 or a 1 and have at least one repeated digit?

Set of digits: {0,1,2,3,4,5,6,7,8,9}

Condition 1: cannot start with 1 or 0:

7 spots for each digit. Spot 1 can have 8 possibilities. Spot 2 there are 10 possibilities, etc...

$=8\cdot10\cdot10\cdot10\cdot10\cdot10\cdot10\cdot10$

$= 8\cdot10^7$

Condition 2: must have at least 1 repetition. That is equal to the total number of 7 digit numbers minus the number of 7 digit numbers which don't repeat any digits:

Total number of 7 digit numbers is $= 8\cdot10^7= 80{,}000{,}000$

How many numbers have no repetition? 8*9*8*7*6*5*4 since there are 8 possibilities for the first digit, 9 for the second, 8 for the third, and so on.

$8\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4= 483{,}840$

Total number of 7 digit phone numbers that do not start with 0 or 1 and have at least one repeated digit: $= (8\cdot10^7)-(483{,}840)$

=**79,516,160**

6. We have 7 balls, each of a different color, that we are placing in 3 different boxes, each of a different size. How many ways are there to do this so that none of the boxes are empty? Inclusion/exclusion principle, where j represents the number of boxes, and k represents the number of non-empty boxes

$$\sum_{j=0}^{3}(-1)^{j-0}\binom{j}{k}(3-j)^7$$

$=(-1)^0\binom{3}{0}(3-0)^7 + (-1)^1\binom{3}{1}(3-1)^7+ (-1)^2\binom{3}{2}(3-2)^7+ (-1)^3\binom{3}{3}(3-3)^7$

$= 3^7 -3\cdot2^7+3\cdot1^7 -1\cdot0^7$

= **1806**

Part 2

1. Fill in the blank in the following statement. Using just 4¢ and 7¢ stamps, any value of postage **18¢** or greater can be produced. Then use strong induction to prove the statement. Take particular care with the number of base cases needed.

Stamp Value	Stamps Needed	
14	7+7	
15	4+4+7	
16	4+4+4+4	
18	4+7+7	14+4
19	4+4+4+7	12+7
20	4+4+4+4+4	16+4
21	7+7+7	14+7
22	4+4+7+7	14+4+4

Proof (by strong induction):

Step 1(base case): We can make 18, 19, 20, and 21¢ postage the following ways:

18¢ = 4+7+7
19¢= 4+4+4+7
20¢= 4+4+4+4+4
21¢= 7+7+7

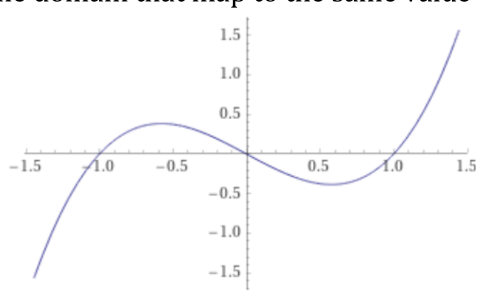
Step 2(inductive step): We'll assume that we can make postage for any postage value k, with $21 \leq k < n$. Now, we'll make k+1 postage.

Notice that k+1= (k-3)+4.

$21 \leq (k-3)+4 < n$. Since the term k-3 exists from the table above, so can k+1. Therefore, we can get any postage can be obtained as postage and so can k+1.

Step 3(conclusion): Therefore, we can obtain any postage ≥ 18¢ ■

2. Give an example of a function f: ℝ → ℝ that is surjective but not injective.
f(x): ℝ → ℝ by f(x)= x^3-x. This function is not injective because there are several values in the domain that map to the same value in the co-domain.



3. Consider sets A and B, with |A|=7 and |B|=3.
   a. How many relations are there from A to B? $\mathbf{2^{21}}$
   The cartesian product has 3·7 relations. There are $\mathbf{2^{21}}$ relations, remembering that the power set is 2^n.
   b. How many functions are there from A to B? $\mathbf{3^7}$

There are **3^7** functions from A to B. (A function from a set A into a set B (f:A →B) is a relation function f subset of A × B satisfying the property that for every a ∈ A, the relation f contains exactly one ordered pair of the form (a,b)).

c. How many injective functions are there from A to B? **0**
There are **0** injective functions from A to B. A function f:A →B is injective if for every x, y ∈ A, x≠y implies f(x) ≠ f(y).

d. How many surjective functions are there from A to B? **$_7C_3 \cdot 3^4$ = 35(81)= 2835**
surjective functions from A to B. A function f:A →B is surjective if for every b∈ B there exists an a ∈ A with f(a)= b.

Now repeat the above four questions, but with |A|=3 and |B|=7.

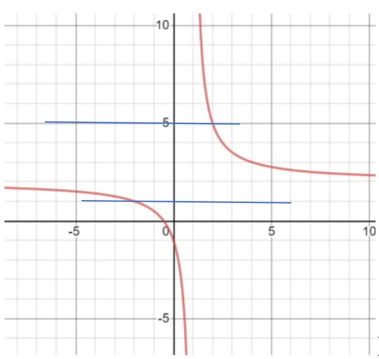
a. How many relations are there from A to B? **2^{21}**
Any relation is a subset of the cartesian product. The cartesian product has 3·7 relations. There are **2^{21}** relations, remembering that the power set is 2^n.

b. How many functions are there from A to B? **7^3**
There are **7^3** functions from A to B. (A function from a set A into a set B (f:A →B) is a relation function f subset of A × B satisfying the property that for every a ∈ A, the relation f contains exactly one ordered pair of the form (a,b).

c. How many injective functions are there from A to B? **P(7,3)= 210**.
P(7,3)= 210. There are 7 choices for image of the first element of the domain, then only 16 choices for the second, and so on.
There are 7*6*5 injective functions from A to B. A function f:A →B is injective if for every x, y ∈ A, x≠y implies f(x) ≠ f(y).

d. How many surjective functions are there from A to B? **0**
There are **0** surjective functions from A to B. A function f:A →B is surjective (or onto) if for every b∈ B there exists an a ∈ A with f(a)= b.

4. If f:ℝ\{1}→ ℝ is defined by $f(x) = \frac{(2x+1)}{(x-1)}$, find f((2,3]) and f^{-1}([1,5)).



Horizontal asymptote: f(x)=2
Vertical asymptote: x=1, excluded from set f

f((2,3]) = **[3.5,5]**

$f^{-1} = \frac{(x+1)}{(x-2)}$

f^{-1}([1,5))= **[-2, 1) ∪ (1,2]**

5. Three dice are rolled, a green one, a red one, and an orange one. What is the size of the sample space? What is the probability that the number of spots on the green die is larger than the sum of the number of spots on the red and orange dice?
Sample space= 6·6·6=**216**
Probability= $\frac{\textit{number of outcomes}}{\textit{sample space}}$
The green die achieves its goal when it is 3 or higher

Orange + Red	Orange + Red	Orange + Red	Orange + Red	Green, number of ways to beat it
1+1				**4** (b/c outcomes 3, 4, 5, 6)
1+2	2+1			**3** (b/c outcomes 4, 5, 6)
1+3	3+1	2+2		**2** (b/c outcomes 5, 6)
1+4	4+1	2+3	3+2	**1** (b/c outcome 6)

Probability= $\frac{(4\cdot1)+(3\cdot2)+(2\cdot3)+(1\cdot4)}{216}$
$= \frac{\mathbf{20}}{\mathbf{216}}$

6. Consider a standard deck of 52 cards, with 13 possible ranks (Ace, 2,3,4,5,6,7,8,9,10, Jack, Queen, and King) each in 4 different suits (spades, hearts, diamonds, clubs). A single hand of 52 cards is dealt. What is the probability that the 5 cards are all of different ranks?
The number of ways to produce a hand with no pairs is equal to the product of the number of ways to make each independent choice.
$_{13}C_5$ = choose the rank of each card in the hand
$_4C_1$ = choose a suit for the first card
$_4C_1$ = choose a suit for the second card
$_4C_1$ = choose a suit for the third card
$_4C_1$ = choose a suit for the fourth card
$_4C_1$ = choose a suit for the fifth card

$(_{13}C_5\cdot(_4C_1)^5)/(_{52}C_5)= \frac{(1287)\cdot(4)\cdot(4)\cdot(4)\cdot(4)\cdot(4)}{2{,}598{,}960} = \frac{\mathbf{1{,}317{,}888}}{\mathbf{2{,}598{,}960}} \approx \mathbf{0.507}$

7. If two events A and B are such that P(A)= 0.5, P(B)= 0.3 and P (A∩B)= 0.1, find P(A|B), P(B|A), P(A|(A∪B)), P(A|(A∩B)), and P(A∩B|A∪B).

P(A∪B)= P(A)+P(B)-P(A∩B)= (0.5)+(0.3)-(0.1)= 0.7
P(A∩B)=P(A|B)P(B)= 1/3*0.3= 0.1

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.1}{0.3} = \mathbf{\frac{1}{3}}$

$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.1}{0.5} = \mathbf{\frac{1}{5}}$

$P(A|(A\cup B)) = \frac{P(A\cap(A\cup B))}{P(A\cup B)} = \frac{P(A)}{P(A\cup B)} = \frac{0.5}{0.7} = \mathbf{\frac{5}{7}}$

$P(A|(A\cap B)) = \frac{P(A\cap(A\cap B))}{P(A\cap B)} = \frac{P(A\cap B)}{P(A\cap B)} = \frac{0.1}{0.1} = \mathbf{1}$

$P(A\cap B|A\cup B) = \frac{P((A\cap B)\cap(A\cup B))}{P(A\cup B)} = \frac{P(A\cap B)}{P(A\cup B)} = \frac{0.1}{0.7} = \mathbf{\frac{1}{7}}$

8. Computer chips are manufactured on two different fabrication lines, line A and line B. The following table shows numbers of the chips from each line that pass or fail quality-assurance (QA) tests. Let P be the event that a randomly-selected chip passes QA, and let F be the event that it fails. Let A be the event that a randomly-selected chip comes from fab line A, and B be the event that it comes from fab line B. Are the events A and P independent? Are the events B and F independent?

Outcome	Line A	Line B	Total
Pass P	24	36	60
Fail F	156	24	180
Total	180	60	240

$P(A)= 180/240= 3/4$
$P(B)= 60/240= 1/4$
$P(P)= 60/240= 1/4$
$P(F)= 180/240= 3/4$
$P(A\cap P)= 24/180= 2/15$
$P(B\cap F)= 24/60= 2/5$

Two events A and P are independent if $P(A \cap P) = P(A)P(P)$

$2/15 \neq (3/4)(1/4)$, therefore A and P are dependent

Two events B and F are independent if $P(B \cap F) = P(B)P(F)$

$2/5 \neq (3/4)(1/4)$, therefore B and F are dependent