COMP 3008 Midterm Krista Miller

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1. For each of the following statements, determine if it is a true or false statement. Circle T or F. Justify your answer.

T/F a. $247 \equiv 324 (\text{mod} 7)$

By definition, $a \equiv b (\text{mod } n)$ if $n|(a-b)$

$7|(247-324)$

$7 \nmid -77$

T/F b. If n and m are integers and $n|m^3$ then $n|m$

Counterexample: n= 4 and m=6, which are both integers

$4|6^3$ but $4 \nmid 6$

T/F c. $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m+n=0$

By definition, natural numbers is the set of all positive integers from 1 to infinity.
Counterexample: n=1, which is in the set of natural numbers.
m+(1)=0, m=-1. -1 is not in the set of natural numbers, so this statement cannot be true.

T/F d. The sets {3,8,15,24,35} and $\{n^2-1: n \in \mathbb{N}$ and $n<7\}$ are equal.

Let A= {3,8,15,24,35} and B= $\{n^2-1: n \in \mathbb{N}$ and $n<7\}$.
A is a subset of B, since all elements satisfy the conditions of B:
$2^2-1= 3$
$3^2-1=8$
$4^2-1=15$
$5^2-1=24$
$6^2-1=35$
However, notice that B is not a subset of A. Consider n=1, which is an element of the natural numbers and <7. $1^2-1=0$. 0 is not an element in set A, so the sets A and B are not equivalent.

T/F e. When written in "if-then" form, the statement "An integer n is composite **only if** 2|n" becomes "if 2|n then n is composite".

"Only if" is the necessary condition. When written in the "if-then" form, the statement becomes:

"If an integer n is composite, then 2|n."

2. Fully negate each of the following statements.

   a. If I am taller than she is, then you are taller than she is.

      p= I am taller than she is
      q= you are taller than she is
      $\neg (p \to q) \equiv p \wedge \neg q$

      Negation: "I am taller than she is and you are not taller than she is"

   b. $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}, n>m$

      $\neg(\exists n \in \mathbb{N}, \forall m \in \mathbb{N}, n>m) \equiv \forall n \in \mathbb{N}, \exists\, m \in \mathbb{N}, n \leq m.$

   c. $(R \wedge Q) \Rightarrow (P \vee Q)$

      p= $(R \wedge Q)$

      q= $(P \vee Q)$

      $\neg(p \to q) \equiv p \wedge \neg q \equiv (R \wedge Q) \wedge \neg(P \vee Q) \equiv (R \wedge Q) \wedge (\neg P \wedge \neg Q)$

3. Suppose that P= {Ace, Queen, King, Jack} and D= {Queen, Jack, Joker}.

   a. Name an element in $(P \cup D) \times (P \setminus D)$

      $(P \cup D)$= {Ace, King, Queen, Jack, Joker} $\times$ $(P \setminus D)$= {Ace, King}
      $(P \cup D) \times (P \setminus D)$= {Ace, King, Queen, Jack, Joker} $\times$ {Ace, King}

      An example of an element in $(P \cup D) \times (P \setminus D)$ is **(Jack, Ace).**

   b. What is $|(P \cup D) \times (P \setminus D)|$?

      $|(P \cup D) \times (P \setminus D)| = |(P \cup D)| \cdot |(P \setminus D)| = 5 \cdot 2 = \mathbf{10}$

4. Use the Euclidean algorithm to find the greatest common divisor of 198 and 168. Show all your work for full credit.
   Given a, b $\in \mathbb{Z}$, not both zero such that a>b:
      gcd(a,b)= gcd (b, a mod b)
      gcd(a,0)=a

   gcd(198, 168)

198= 168(1) +30	gcd(198,168)= gcd(168,30)
168=30(5)+18	gcd(168,30)= gcd(30, 18)
30=18(1)+12	gcd(30, 18)= gcd(18,12)
18= 12(1)+6	gcd(18, 12)= gcd(12, 6)
12= 6(2)+0	gcd(12, 6)= gcd(6, 0)
	gcd(198,168)=6

5. How many Social Security numbers (9-digit codes) are there that have **at most** 4 digits that are 0? Justify your answer.
Total social security numbers= 10^9
Set of digits= {0,1,2,3,4,5,6,7,8,9}

<u>Exactly 4 digits that are 0:</u>
The possible locations of non-zero digits is 9^5
The possible arrangement of social security numbers is $\binom{9}{5}=\frac{9!}{5!(4!)}= 126$
$(9^5)(126)$
<u>Exactly 3 digits that are 0:</u>
The possible locations of non-zero digits is 9^6
The possible arrangement of social security numbers is $\binom{9}{6}=84$
$(9^6)(84)$
<u>Exactly 2 digits that are 0:</u>
The possible locations of non-zero digits is 9^7
The possible arrangement of social security numbers is $\binom{9}{7}=36$
$(9^7)(36)$
<u>Exactly 1 digit that is 0:</u>
The possible locations of non-zero digits is 9^8
The possible arrangement of social security numbers is $\binom{9}{8}=9$
$(9^8)(9)$
<u>Exactly 0 digits that are 0:</u>
The possible locations of non-zero digits is 9^9
The possible arrangement of social security numbers is $\binom{9}{9}=1$
$(9^9)(1)$

Therefore, the total number of social security numbers that have at most 4 digits that are zero is equal to: $(9^5)(126)+ (9^6)(84)+ (9^7)(36)+ (9^8)(9)+ (9^9)(1)=$ **999,109,080**

6. The parity of an integer n refers to whether n is even or odd. For example, the parity of 10 is even and the parity of 13 is odd. Consider a graph G=(V,E) and let A ⊆ V be the subset of V that contains all vertices of odd degree. Furthermore, let
$$N = \sum_{v \in A} \deg(v)$$
What is the parity of N? Justify your answer with a proof.

The parity of N is even, proof below:

<u>Proof:</u>
Let G= (V,E) and |E|= m.
Let A represent the subset of V containing all odd degree vertices: A= {v∈V|deg(v) is odd}.
$N = \sum_{v \in A} \deg(v)$

Let B represent the subset of V containing all even degree vertices: B= {v ∈V| deg (v) is even}
A∪B= V, therefore A∩B is equal to the empty set { }.

$\sum_{v \in V} \deg(v)$ is equal to the sum of all odd degrees plus the sum of all even degrees:
$\sum_{v \in V} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v) = 2m$

We can rewrite the sum of even degrees:
$= \sum_{v \in B} \deg(v) = 2k$, where $k \in \mathbb{Z}$

Substituting 2k for $\sum_{v \in B} \deg(v)$:
$\sum_{v \in V} \deg(v) = 2k + \sum_{v \in A} \deg(v) = 2m$

Rearranging:
$\sum_{v \in V} \deg(v) = \sum_{v \in A} \deg(v) = 2m - 2k$

Factor out a 2:
$\sum_{v \in V} \deg(v) = \sum_{v \in A} \deg(v) = 2(m - k)$
Notice that m-k is an integer, so by definition, 2(m-k) is an even integer.

So, when we add up all the odd degrees of our graph, we get an even number. The only way to get an even number when we add up a group of odd numbers is to add up an even number of odd numbers. |A| is even.
Therefore, $N = \sum_{v \in A} \deg(v)$ is even. ■

7. Consider the following statement:
“If a, b, and c are odd integers, then the equation $ax^2 + bx + c = 0$ has no rational solutions.”

   p= “a, b, and c are odd integers”
   q= “$ax^2 + bx + c = 0$ has no rational solutions”
   p→q

   a. If you intend to prove the above statement by **direct proof**, what would your assumptions be? Phrase your answer in the form “Suppose…”

      Assumptions:
      Suppose that a, b, and c are odd integers

   b. If you intend to prove the above statement by **contrapositive**, what would your assumptions be? Phrase your answer in the form “Suppose…”
      Contrapositive : ¬q → ¬ p. “If $ax^2 + bx + c = 0$ has rational solutions, then a, b, and c are not odd integers.” (This is logically equivalent to if p then q)

      Assumptions:
      Suppose $ax^2+bx+c=0$ has a rational solution

c. If you intend to prove the above statement by **contradiction**, what would your assumptions be? Phrase your answer in the form "Suppose…"

Assumptions:
Suppose that a, b, and c represent odd integers and the equation $ax^2+bx+c=0$ has a rational solution.

8. These two questions require induction.
   a. Consider the following sum:

$\sum_{j=1}^{n} \frac{1}{j(j+1)}$ Find an explicit, closed expression for the sum above and prove it by induction.
Here $n \in \mathbb{N}$.

n=1	$\sum_{j=1}^{1} \frac{1}{1(1+1)} = \frac{1}{2}$
n=2	$\sum_{j=1}^{2} \frac{1}{j(j+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
n=3	$\sum_{j=1}^{3} \frac{1}{j(j+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$
n=n	$= \frac{(n)}{(n+1)}$

Explicit, closed expression for this sum: $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$

Proof: Show that $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$

Base case where n=1:

	Sum: $\sum_{j=1}^{n} \frac{1}{j(j+1)}$	Closed expression: $\frac{n}{n+1}$
n=1	$\sum_{j=1}^{1} \frac{1}{1(1+1)} = \frac{1}{2}$	$\frac{1}{1+1} = \frac{1}{2}$

So, the base case is true.

Inductive hypothesis: suppose that for some $k \in \mathbb{N}$: $\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}$

(Goal: show that $\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{(k+2)}$)

$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)}$

By inductive hypothesis: $\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}$

Substituting $\frac{k}{k+1}$ for $\sum_{j=1}^{k}\frac{1}{j(j+1)}$:

$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$

Common denominator:

$=\frac{k(k+2)+1}{(k+1)(k+2)}$

Expand:

$=\frac{k^2+2k+1}{(k+1)(k+2)}$

Simplify numerator:

$=\frac{(k+1)^2}{(k+1)(k+2)}$

Cancel out (k+1) from numerator and denominator:

$=\frac{(k+1)}{(k+2)}$

So, the sum of k+1 is true, so for every $n\in\mathbb{N}$, $\sum_{j=1}^{n}\frac{1}{j(j+1)}=\frac{n}{n+1}$ ■

b. A sequence is defined recursively by $a_1 = 3$, $a_2 = 5$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for integers $n>2$. Prove that $a_n = 2^n + 1$ for all integers $n \geq 1$.

Series: 3, 5, 9, 17, 33…

<u>Proof:</u>

$a_n = 2^n + 1$ for all integers $n \geq 1$.

Demonstrate for base cases where (n=1, 2, 3, 4, 5):

a_n	$a_n = 2^n + 1$ for all integers $n\geq1$	$a_n = 3a_{n-1} - 2a_{n-2}$ for integers $n>2$
a_1	$2^1+1 = 3$	
a_2	$2^2+1 = 5$	
a_3	$2^3+1= 9$	3(5)-2(3)= 9
a_4	$2^4+1= 17$	3(9)-2(5)= 17
a_5	$2^5+1= 33$	3(17)-2(9)= 33

Inductive hypothesis: suppose that for some integer k: $a_k = 2^k + 1$

$a_{k-1} = 2^{(k-1)}+1$

$a_{k-2} = 2^{(k-2)}+1$

(Goal: show that $a_{k+1} = 2^{(k+1)}+1$)

$a_{k+1} = 3a_{k-1} - 2a_{k-2}$

$=3(2^{(k-1)}+1)-2(2^{(k-2)}+1)$

Expand:

$=3 \cdot 2^{(k-1)}+3-2^{(k-1)}-2$

Simplify:

$=3 \cdot 2^{(k-1)}+3-2^{(k-1)}-2$

$=2 \cdot 2^{(k-1)}+3-2$

$=2^{k}+1$

By inductive hypothesis, $a_k= 2^k +1$

Therefore, $a_n = 2^n +1$ for all integers $n \geq 1$ ■

9. A fair red die and a fair orange die are rolled. Let the random variable X be the sum of the spots. Let A be the event that X is greater than 4. Let B be the event that the numbers on the dice are both odd. Determine if A and B are independent events. Clearly show all intermediate work and numerical reasoning.

Sample Space S:

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Mapping for random variable X :

Green shading represents event A that X is >4

2	3	4	5	6	7
3	4	6	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Orange shading represents event B that the numbers on the dice are both odd

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

<u>Definition of Independent Events</u>: Two events A and B are independent if $P(A \cap B)=P(A)P(B)$. This definition is equivalent to stating that two events A and B are independent if $P(A|B)=P(A)$, or $P(B|A)= P(B)$.

$P(A)= \frac{|A|}{|S|}= \frac{30}{36}=\frac{5}{6}$

$P(B)= \frac{|B|}{|S|}= \frac{9}{36}=\frac{1}{4}$

$P(A \cap B)= \frac{|A \cap B)|}{|S|} = \frac{6}{72}= \frac{1}{12}$

$P(A)P(B)= (\frac{5}{6}) (\frac{1}{4})= \frac{5}{24}$

$P(A|B)= \frac{P(A \cap B)}{P(B)}=\frac{1/12}{1/4}= \frac{1}{3}$

Event A and event B are not independent events, since $P(A \cap B) \neq P(A)P(B)$. Equivalently, $P(A|B) \neq P(A)$.

10. Find the expected value and variance of each of the following random variables. You do not need to explain or show any work, and you do not need to simplify any numerical answers.

   a. Say X is a random variable **uniformly distributed** on the integers from 1 to 50.
   E[X]= **25.5**
   Var[X]=**208.25**

   b. Say that $\frac{1}{3}$ of units produced in a factory are defective, and the rest are good. **A random sample of 5 units are selected.** Let Y be the number of defective ones in that sample.

   Y= number of defective units in 5 unit sample
   P(defective unit)= $\frac{1}{3}$
   E[Y]=np= (5)($\frac{1}{3}$)= $\mathbf{\frac{5}{3}}$**= 1.67**
   Var[Y]=np(1-p)= 5($\frac{1}{3}$)(1-$\frac{1}{3}$)= $\mathbf{\frac{10}{9}}$**= 1.11**

11. Approximately 5 out of every 10,000 people have a genetic condition called thalassemia. One test for the disease has a sensitivity (the probability that a person who has the condition tests positive) of about 93% and a specificity of about 84%. Let D be the event that the person has the disease, and let T be the event that the person tests positive for the disease.

Sensitivity= P(positive test T| person has the condition D)
93%= P(T|D)
Specificity = P(negative test T| person does not have the condition D)
84%= P(T^C|D^C)

	Has Thalassemia?		
	Event D: Yes	Event D^c : No	Total
Event T: Tests Positive	5(.93)=4.65 Sensitivity rate=0.93	1,599.2 False positive rate=0.16	1,603.85
Event T^c: Tests Negative	.35 False negative rate= 0.07	9,995(.84)=8,395.8 Specificity rate= 0.84	8,396.15
Total	5	9,995	10,000

   a. Find P(T|D^C)
   Thalassemia is detected (T) given the patient does not actually have the condition (D^C). In other words, this is the probability of a **false positive test**:
   Sensitivity(true positive)= 93%
   P(T|D^C)= $\frac{P(T \cap D^C)}{P(D^C)}$ =**0.16**

   b. Give a formula for P(D|T), first in terms of T, D, T^c, and D^c and then with the correct values in place of the variables.

P(D|T) means the probability of actually having Thalassemia (D) given that a test comes back positive for the disease(T). This is called the **positive predictive value**.

$$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{True\ Positive}{True\ Positive + False\ Positive} = \frac{4.65}{4.65 + 1{,}603.85} = \mathbf{0.00289}$$